Notes on equilibrium neoclassical flows in tokamak and spontaneous poloidal spin-up

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Abstract

This is the basic level of organization of the plasma at equilibrium in tokamak. It corresponds to the neoclassical flows.

Possible application to helical entrainement of tungsten toward the center, carried by poloidal rotation of Stringer effect.

This text is prepared for a meeting of the **Work Session of Plasma Theory**. It is a ground for discussions, never to be taken as final.

1 Introduction

Stringer effect.

First it is the remark of Stringer that there is an intrinsic reason for poloidal rotation. It is because in the plasma it is generated a radial electric field that ensures ambipolarity.

Second, the nonuniformity of the radial fluxes or of sources on θ leads to rotation.

The subject is related to *rotation.tex*.

Possible application to: Mechanism for accumulation of heavy impurities in the center of the tokamak. The text is in *collaboration, cea*. There is a folder *impurity accumulation / plasma models*.

2 The physical picture of the Stringer effect

The qualitative picture is actually for the variation on the magnetic surface (i.e. with θ) of the parameters of the plasma.

Start by recalling that the diamagnetic flows of electrons and ions are the result of the gradient of pressure combined with the magnetic field. These flows $\sim \widehat{\mathbf{n}} \times \boldsymbol{\nabla} p$ are perpendicular on the magnetic field line, which means that their main component is poloidal with just a small projection on the toroidal direction. The two flows, of electron and ions, are added into a current, the diamagnetic current

$$j_{\perp} \equiv j^{dia}$$

The toroidal geometry makes that an element of plasma in the diamagnetic flow undergoes variations of volume along the poloidal direction: contraction when the element moves from the low-field side of the tokamak toward the high-field side (from outboard to inboard) and then dilation when it makes the other half of the circumference. The diamagnetic current cannot have a zero-divergence $\boldsymbol{\nabla} \cdot j^{dia} \neq 0$, due to this variations.

In order to preserve the zero-divergence (equation of continuity of charge/current) one needs to accept the existence of another current, parallel with the magnetic field lines, with a non-zero divergence that compensate exactly the non-zero divergence of the diamagnetic current. It is in this way that the divergence of the *total* current is made equal to zero.

The current that we find necessary to compensate the non-zero divergence of the diamagnetic current is the Pfirsch Schluter current. We note that this argument for the existence of the Pfirsch Schluter current is based on a conservation constraint, $\boldsymbol{\nabla} \cdot \mathbf{j} = 0$ and at this moment the dynamical factors that produce the flow of charges along the magnetic line, *i.e.* the PS current have not been examined. Starting from the expression of the diamagnetic current, taking into account the toroidal geometry in the operators and imposing the condition that the new (PS) current compensate the non-zero divergence of the diamagnetic current, one derives the expression of the PS current density. It has harmonic $(\cos\theta)$ variation of the amplitude on the poloidal angle and is in the direction of the magnetic field line. On the low field side it flows in one direction and on the high field side it flows in the opposite direction, just as $\cos\theta$ says.

Now a quantity with variation on the magnetic surface (variation with the poloidal angle θ) has been introduced in the plasma equations. This current is no more a function of only the surface label, r or ψ. It is also a function of θ. The variation on the magnetic surface will be present in everything that we will further consider.

Necessarily, all the other quantities involved in the equations of balance must contain a variation in the magnetic surface, *i.e.* with θ. This may be small and can be seen as a correction to the value of a variable which is only a function of the surface label r or ψ. We are then led to work with plasma variables that have a main part that is a function constant on surfaces and in addition a small component which has also variation with θ.

Now with this new dependence $\sim (r, \theta)$ of any plasma variable: density, electric potential (but not temperature because we have not yet involved the heat flow) we return to the equations of moments of plasma.

The equation of continuity will contain, for density, $n_0(r)$ and $n_1(r, \theta)$. Also the velocity, which is supposed to be $E \times B$, *i.e.* produced by an electric field will be $v_0(r) = -\frac{1}{B_0}\frac{\partial \phi_0}{\partial r}$ plus the correction for the variation over the surface $\mathbf{v}_1(r,\theta) = \frac{\mathbf{E}_1 \times \mathbf{B}}{B^2}$ where $\mathbf{E}_1 = -\boldsymbol{\nabla}\phi_1$ and $\phi_1 = \phi_1(r,\theta)$. [The gradient of ϕ_1 has a component that is radial, as for $\phi_0 \equiv \overline{\phi}$ (constant on surface) and a component that is poloidal, since ϕ_1 has variation on surface, i.e. with θ. This part of $-\boldsymbol{\nabla}\phi_1$ which is E_θ combines with $\mathbf{B}$ and produces toroidal flow (with B_θ) or a radial flow (with B_T)].

The equation for momentum balance is an equation for the velocity $\mathbf{v}_0+\mathbf{v}_1$. Here we can see more clearly that it really was necessary to return to the equation *with* the assumption that the plasma variables contain a part that depends on θ. We see that by the occurence of the term

$$e_j n \mathbf{v}_1 \times \mathbf{B}$$

which is $j_1 \times B$ and now j_1 contains the Pfirsch Schluter current density which has a dependence of θ as $\cos\theta$. All variables that are involved in the equation of momentum and equally in the equation of continuity must have, we conclude, a part that has a trigonometric dependence on θ.

Indeed the solution for $n_1(r,\theta)$ can be expressed as terms that are coefficients of $\cos\theta$ and $\sin\theta$.

The correction $\phi_1(r,\theta)$ is obtained from neutrality.

The next step is to include the solutions for $n_1(r,\theta)$ and $\phi_1(r,\theta)$, beside the constant-on-surface functions $n_0(r)$, $\phi_0(r)$, to calculate the fluxes of particles across the magnetic surface. The flux is

$$\left.\boldsymbol{\Gamma}\right|_r = \left. n\mathbf{v}^{drift}\right|_r$$

and these factors can be formally calculated for each species, ions and electrons.

It is better to average over surface these fluxes (simply by integrating on θ between 0 and 2π) since we have in this way the flux of ions and of electrons that flow out from the volume enclosed by a magnetic surface. This will bring the problem of neutrality if the two fluxes are not equal. In this process we *note* that there are products of trigonometric functions, like $\cos\theta\sin\theta$ and $(\sin\theta)^2$. From the integration over θ some of them will cancel by periodicity $(\cos\theta\sin\theta \to 0)$ and some of them will give finite effect $\left(\sin^2\theta \to \frac{1}{2}\right)$. This effect results from the combination of the trigonometric variation of

- the plasma variable on surface, like $n_1(r,\theta) \sim \cos\theta$ and $\sin\theta$; and
- the variation of the *drift velocity* projected on the radial direction (perpendicular on the magnetic surface) $v_r^{drift} \sim \sin\theta$

and we note that the second factor is a pure neoclassical effect.

The neoclassical drifts of ions and respectively electrons are very different in magnitude.

Now these (averaged on θ) radial fluxes of electrons and ions are compared. They are not equal.

The ambipolarity is however a constraint: one has to have equal fluxes of electrons and of ions leaving the volume bounded by a magnetic surface. If not, the plasma inside the volume bounded by the magnetic surface will not remain neutral.

To reach ambipolarity a certain parameter in the expression of fluxes must be found. The ambipolarity becomes an equation for this parameter. This is v_0 or the gradient of the potential Φ_0 constant on surfaces, or equivalently, the radial electric field $-\boldsymbol{\nabla}\Phi_0$. When this gradient has a certain value, the two fluxes are equal and the ambipolarity is verified.

And now we examine the conclusion: in order the fluxes to be ambipolar, one has to admit the existence of a radial electric field with a particular magnitude. This means that the plasma rotates $E \times B$.

There is poloidal rotation.

This is the main conclusion of Stringer analysis.

We also note that there is no external factor except for the toroidality. The plasma spontaneously rotates, by the simple existence of the neoclassical consequences of the toroidality.

It is an intrinsic effect.

There is another explanation of the Pfirsch Schluter effect, which describes it as a dynamical effect. This is in contrast with the one presented above which just makes use of the conservation $\boldsymbol{\nabla} \cdot \mathbf{j} = 0$ of the current. The Pfirsch Schluter current appears as a necessity to obtain the conservation of charge. The explanation given by **Stringer PRL** is more physical.

Later Hassam and Drake and Kleva have introduced the external factor, $\Gamma_r(\theta)$, possibly due to the turbulence. The poloidal rotation is driven in this case.

Regarding the magnitude of variation of the electrostatic potential on the magnetic surface, in **poloidalpotentialvariation Stringer** it is mentioned the estimation made by Hinton Rosenbluth

$$
\begin{aligned}
& e\left(\frac{1}{T_{i0}} - \frac{1}{T_{e0}}\right)(\Phi - \langle\Phi\rangle) \\
\sim \quad & -\varepsilon\, \frac{\sqrt{\pi}}{2}\, \frac{\rho_{i\theta}}{L_T} \sin\theta
\end{aligned}
$$

and in the banana regime

$$
\begin{aligned}
& e\left(\frac{1}{T_{i0}} - \frac{1}{T_{e0}}\right)(\Phi - \langle\Phi\rangle) \\
\sim \quad & -1.81\ \varepsilon\ \left[\frac{1}{\varepsilon^{3/2}} \frac{\nu_i}{(v_{th,i}/qR)}\right] \sin\theta
\end{aligned}
$$

3 The equilibrium flows

There is a text on this, in plasma, general, plasma theory, **equilibrium flows**.

For a physical picture of flows (intended for bootstrap) see **Hsu Shaing Gormley bootstrap from** α.

3.1 The Pfirsch Schluter current

driftwithradialelectric field Stringer1969.

The drift motions of electrons and ions $v_{drift,j}$ in *toroidal* field lead to *charge separation*.

$$\mathbf{v}_D = \frac{1}{\Omega_{e,i}} \widehat{\mathbf{n}} \times \left(\mu \boldsymbol{\nabla} B + v_\parallel^2 \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} + \frac{e_{e,i} \Phi}{m_{e,i}} \right)$$

The e, i drifts have opposite signs due to $\Omega_{e,i}$.

And the drifts are very different in magnitude due to $m_i \gg m_e$.

In order to suppress this charge separation a *current flows along magnetic field lines.*

An instantaneous reaction of charges along the magnetic field line would cancel the difference of charges.

However when there is resistivity (collisions) the neutralization of the charge separation by the parallel current is *incomplete*.

Then there is a residual electric field which still remains. This is $E_\parallel$ and is connected with $j_\parallel$ by $\eta \neq 0$.

This electric field induces an *enhancement* of the diffusion. The enhancement comes from the *radial velocity* v_r that exists due to the coupling of the parallel electric field $E_\parallel = -\nabla_\parallel \phi$ with the poloidal magnetic field B_θ via the radial velocity v_r in the Ohm's law $-\nabla_\parallel \phi + v_r B_\theta = \eta j_\parallel$. The *radial velocity* v_r produces a radial flux $\Gamma_r = v_r n$. This is the factor q^2 which multiplies the classical diffusion.

We **NOTE** that the enhancement of the diffusion rate beyond the *classical collisional* rate, by the factor

$$\sim \left(1 + 0.4 q^2\right)$$

found by Pfirsch Schluter does NOT involve in any way the trapped particles. A still higher enhancement of the diffusion rate comes from collisions of trapped particles, where the spatial step is the width of the banana $\sim \rho_\theta$. See **Galeev Sagdeev**.

Now, in regimes where collisions are rare, instead of collisional resistivity, it is the *Landau damping* that is invoked. **Hammet**.

Stringer PRL mentions the fact that the variation of ion's density on the magnetic surface is related to finite *ion inertia*. This means $(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}$. Indeed this is stationary and $\partial/\partial t = 0$.

The parallel current arising from the non-zero divergence of the *diamagnetic current* (Pfirsch Schluter)

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{j} &= 0\\ \boldsymbol{\nabla}_{\perp}\cdot\mathbf{j}_{\perp}+\boldsymbol{\nabla}_{\parallel}\cdot\mathbf{j}_{\parallel} &= 0\end{aligned}$$

Now taking the perpendicular current as resulting from the *diamagnetic* flows of electrons and ions, the parallel gradient can be written as

$$\begin{aligned}\boldsymbol{\nabla}_{\parallel}\cdot\mathbf{j}_{\parallel} &= \frac{1}{qR}\frac{\partial}{\partial\theta}j_{\parallel}\\ &= -\boldsymbol{\nabla}_{\perp}\cdot\mathbf{j}_{\perp} = -\boldsymbol{\nabla}_{\perp}\cdot\left(e\frac{1}{m\Omega}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}p\right)\\ &= -\boldsymbol{\nabla}_{\perp}\cdot\left(e\frac{1}{|e|\,B}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}p\right)\end{aligned}$$

Let us look to the last term. It is the perpendicular divergence of the *diamagnetic* flow.

Note that the operator of parallel derivative is

$$\nabla_{\parallel}\sim\frac{1}{qR}\frac{\partial}{\partial\theta}$$

and that the perpendicular current $\mathbf{j}_{\perp}$ is the *diamagnetic current, of ions + electrons*. **End.**

This is a *neoclassical* effect.

It is the magnetic field that has a space variation in the perpendicular direction. First we have

$$\begin{aligned}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}p &= \left|\frac{dp}{dr}\right|(\widehat{\mathbf{n}}\times\widehat{\mathbf{e}}_r)\\ &= -\widehat{\mathbf{e}}_{\theta}\left|\frac{dp}{dr}\right|\end{aligned}$$

Then, restricting the gradient to the part that contains B, we use the expression of the gradient operator expressed in the geometry of the toroidal region.

This part is repeated later in this text.

We have the *magnitude* of the magnetic field

$$B=\frac{B_0}{1+\varepsilon\cos\theta}$$

and we must calculate the perpendicular divergence of the perpendicular current, which means

$$\boldsymbol{\nabla}\cdot\left(-\widehat{\mathbf{e}}_{\theta}\frac{1}{B}\left|\frac{dp}{dr}\right|\right)$$

and this is approximated by (B_0 is constant)

$$\boldsymbol{\nabla}\cdot\left(\widehat{\mathbf{e}}_\theta\frac{B_0}{B}\right)=\boldsymbol{\nabla}\cdot\left[\widehat{\mathbf{e}}_\theta\left(1+\varepsilon\cos\theta\right)\right]$$

Here is the essential part of the calculation: there is a divergence of the diamagnetic "flow" that is exclusively due to the *geometry*. This has consequences in the balance of flows.

Here it is explained how this divergence is calculated.

In the orthogonal coordinates (r,θ,φ) we have the element of distance:

$$dl^2=(dr)^2+r^2\,(d\theta)^2+(R_0+r\cos\theta)^2\,d\varphi^2$$

which gives the coefficients

$$\begin{aligned}h_1&=1\\h_2&=r\\h_3&=R_0+r\cos\theta\end{aligned}$$

Then the divergence of a vector $\mathbf{a}$ is written

$$\boldsymbol{\nabla}\cdot\mathbf{a}=\frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial r}(h_2h_3a_1)+\frac{\partial}{\partial\theta}(h_1h_3a_2)+\frac{\partial}{\partial\varphi}(h_1h_2a_3)\right)$$

which gives

$$\begin{aligned}\boldsymbol{\nabla}\cdot\left[\widehat{\mathbf{e}}_\theta\left(1+\varepsilon\cos\theta\right)\right]&=\frac{1}{r\left(R_0+r\cos\theta\right)}\frac{\partial}{\partial\theta}\left(\left(R_0+r\cos\theta\right)\left(1+\varepsilon\cos\theta\right)\right)\\&=\frac{1}{r\left(R_0+r\cos\theta\right)}R_0\frac{\partial}{\partial\theta}\left[\left(1+\varepsilon\cos\theta\right)^2\right]\\&=\varepsilon\frac{(-2\sin\theta)}{r}\end{aligned}$$

or

$$\boldsymbol{\nabla}\cdot\left(h\;\widehat{\mathbf{e}}_\theta\right)=\varepsilon\frac{(-2\sin\theta)}{r}$$

From this result we get

$$\begin{aligned}-\boldsymbol{\nabla}_\perp\cdot\mathbf{j}_\perp&=-\boldsymbol{\nabla}_\perp\left(dia\right)=\\&=-\boldsymbol{\nabla}_\perp\cdot\left(e\frac{1}{m\Omega}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}p\right)\\&=-\boldsymbol{\nabla}_\perp\left[\left(e\frac{1}{m\Omega}\right)\left(-\widehat{\mathbf{e}}_\theta\right)\left|\frac{dp}{dr}\right|\right]\\&=\boldsymbol{\nabla}_\perp\cdot\left(\widehat{\mathbf{e}}_\theta\frac{B_0}{B}\right)\frac{1}{B_0}\left|\frac{dp}{dr}\right|\\&=\varepsilon\frac{(-2\sin\theta)}{r}\frac{1}{B_0}\left|\frac{dp}{dr}\right|\\&=\frac{r}{RB_0}\left|\frac{dp}{dr}\right|\frac{\partial}{r\partial\theta}\left(2\cos\theta\right)\end{aligned}$$

NOTE ON An alternative calculation

$$-\boldsymbol{\nabla}_{\perp}\cdot\left(e\frac{1}{m\Omega}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}p\right)=-\boldsymbol{\nabla}_{\perp}\cdot\left[\left(e\frac{1}{m\Omega}\right)\left(-\widehat{\mathbf{e}}_{\theta}\left|\frac{dp}{dr}\right|\right)\right]$$

Taking factor $|dp/dr|$ we have to calculate

$$\begin{aligned}\boldsymbol{\nabla}_{\perp}\cdot\left(e\frac{1}{m\Omega}\widehat{\mathbf{e}}_{\theta}\right) &= \boldsymbol{\nabla}_{\perp}\cdot\left(\frac{1}{B}\widehat{\mathbf{e}}_{\theta}\right)\\ &= \boldsymbol{\nabla}_{\perp}\left(\frac{1}{B}\right)\cdot\widehat{\mathbf{e}}_{\theta}+\frac{1}{B}\left(\boldsymbol{\nabla}_{\perp}\cdot\widehat{\mathbf{e}}_{\theta}\right)\end{aligned}$$

The first term is

$$\begin{aligned}\boldsymbol{\nabla}_{\perp}\left(\frac{1}{B}\right) &= -\frac{1}{B^2}\boldsymbol{\nabla}_{\perp}B\\ &= -\frac{1}{B^2}\boldsymbol{\nabla}_{\perp}\left(B_0\frac{R_0}{R}\right)\\ &= -\frac{1}{B^2}B_0R_0\left(-\frac{1}{R^2}\boldsymbol{\nabla}_{\perp}R\right)\\ &= \frac{B_0}{B^2}\frac{R_0}{R^2}\widehat{\mathbf{e}}_R\end{aligned}$$

We take separately

$$\begin{aligned}\boldsymbol{\nabla}_{\perp}R &= \left(\widehat{\mathbf{e}}_r\frac{1}{h_r}\frac{\partial}{\partial r}+\widehat{\mathbf{e}}_{\theta}\frac{1}{h_{\theta}}\frac{\partial}{\partial\theta}+\widehat{\mathbf{e}}_{\varphi}\frac{1}{h_{\varphi}}\frac{\partial}{\partial\varphi}\right)(R_0+r\cos\theta)\\ &= \left(\widehat{\mathbf{e}}_r\frac{\partial}{\partial r}+\widehat{\mathbf{e}}_{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}+\widehat{\mathbf{e}}_{\varphi}\frac{1}{R_0+r\cos\theta}\frac{\partial}{\partial\varphi}\right)(R_0+r\cos\theta)\\ &= \widehat{\mathbf{e}}_r\cos\theta+\widehat{\mathbf{e}}_{\theta}\left(-\sin\theta\right)\\ &= \widehat{\mathbf{e}}_R\end{aligned}$$

Here we should decide if the angle θ is measured from the equatorial plane or from the symmetry axis of the torus. Above it was considered that θ is measured from the equatorial plane towards the higher z direction.

$$\frac{1}{B^2}\frac{B_0R_0}{R^2}=B_0R_0\frac{(1+\varepsilon\cos\theta)^2}{B_0^2}\frac{1}{(R_0+r\cos\theta)^2}=\frac{1}{B_0R_0}$$

The first term is then

$$\boldsymbol{\nabla}_{\perp}\left(\frac{1}{B}\right)\cdot\widehat{\mathbf{e}}_{\theta}=\frac{1}{B_0R_0}\widehat{\mathbf{e}}_R\cdot\widehat{\mathbf{e}}_{\theta}=\frac{1}{B_0R_0}\left(-\sin\theta\right)=\frac{1}{B_0R_0}\frac{\partial}{\partial\theta}\left(\cos\theta\right)$$

The second term is $\frac{1}{B}\left(\boldsymbol{\nabla}_{\perp}\cdot\widehat{\mathbf{e}}_{\theta}\right)$ and contains the *divergence* of the versor

$$\begin{aligned}
& \boldsymbol{\nabla}_{\perp}\cdot\widehat{\mathbf{e}}_{\theta} \\
= \; & \frac{1}{h_r h_\theta h_\varphi}\left\{\frac{\partial}{\partial r}\left[h_\theta h_\varphi\left(\widehat{\mathbf{e}}_\theta\right)_r\right]+\frac{\partial}{\partial\theta}\left[h_r h_\varphi\left(\widehat{\mathbf{e}}_\theta\right)_\theta\right]+\frac{\partial}{\partial\varphi}\left[h_r h_\theta\left(\widehat{\mathbf{e}}_\theta\right)_\varphi\right]\right\} \\
= \; & \frac{1}{r\left(R_0+r\cos\theta\right)}\left\{\frac{\partial}{\partial\theta}\left[h_r h_\varphi\left(\widehat{\mathbf{e}}_\theta\right)_\theta\right]\right\} \\
= \; & \frac{1}{r\left(R_0+r\cos\theta\right)}\frac{\partial}{\partial\theta}\left[\left(R_0+r\cos\theta\right)\right] \\
= \; & \frac{1}{R_0+r\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\right)
\end{aligned}$$

The perpendicular divergence of the diamagnetic current is

$$\begin{aligned}
-\boldsymbol{\nabla}_{\perp}\cdot\mathbf{j}_{\perp} &= -\boldsymbol{\nabla}_{\perp}\cdot\left(e\frac{1}{m\Omega}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}p\right) \\
&= -\boldsymbol{\nabla}_{\perp}\left(e\frac{1}{m\Omega}\right)\cdot\left(-\widehat{\mathbf{e}}_\theta\left|\frac{dp}{dr}\right|\right) \\
&= \left|\frac{dp}{dr}\right|\left[\frac{1}{B_0R_0}\frac{\partial}{\partial\theta}\left(\cos\theta\right)+\right. \\
&\quad \left.+\frac{1}{B}\frac{1}{R_0+r\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\right)\right] \\
&= \left|\frac{dp}{dr}\right|\frac{1}{B_0R_0}\frac{\partial}{\partial\theta}\left(2\cos\theta\right)
\end{aligned}$$

and obtain the same result.

END OF NOTE on the alternative calculation

Equating the two sides of the *current conservation* equation

$$\frac{1}{qR}\frac{\partial}{\partial\theta}j_{\parallel}=-\frac{r}{RB}e\left(\frac{dp}{dr}\right)\frac{\partial}{r\partial\theta}\left(2\cos\theta\right)$$

Integrating on the poloidal angle θ:

$$J_{\parallel}=-\varepsilon\frac{2}{B_\theta}\frac{dp}{dr}\cos\theta$$

This is the **Pfirsch Schluter current**.

We note

$$\begin{aligned}
\varepsilon\frac{1}{B_\theta} &= \frac{r}{RB_\theta}\frac{B}{B}=q\frac{1}{B} \\
\text{or}\quad \frac{\varepsilon}{q} &= \frac{B_\theta}{B}\ll 1
\end{aligned}$$

and the combination

$$\frac{1}{B}\frac{dp}{dr}$$

is clearly coming from the diamagnetic flow

$$n\mathbf{v}^{dia} = \frac{1}{m\Omega}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p$$

and remark that the Pfirsch Schluter current is

- parallel with B
- *proportional with* q
- harmonic on θ
- proportional with the diamagnetic current
- however, while the individual diamagnetic flow is *signed* by the charge sign of the species (electrons , ions) the Pfirsch Schluter current has a unique sign. This is given by the sign of the *diamagnetic current* which is the one of the ions.

There is a poloidal electric field related to this current

$$E_\theta = \frac{1}{\sigma_\parallel}\frac{B}{B_\theta}\left(-\varepsilon\frac{2}{B_\theta}\frac{dp}{dr}\cos\theta\right)$$

It is the projection on θ (poloidal) of the relationship

$$E_\parallel = \frac{J_\parallel \ \text{(Pfirsch Schluter)}}{\sigma_\parallel}$$

with the factor of projection

$$\begin{aligned} E_\parallel\,(B/B_\theta) &= E_\theta \\ \text{or} \ \ E_\parallel &= E_\theta \times \frac{\varepsilon}{q} \\ &\ll E_\theta \end{aligned}$$

As mentioned above it is this *electric field* that still exists after the parallel current $j_\parallel$ has tried to neutralize the charge separation produced by the non-zero divergence of the current of diamagnetic origin. This electric field is due to either

- finite resistivity $\eta = \sigma^{-1}$, or
- Landau damping

An important explanation: the diamagnetic motion of the electrons and ions is perpendicular on **B**. This motion is constrained and is acted upon by the modulation of the magnetic field *outer/inner*. Then there arise differences between electrons and ions. To compensate the poloidal motion, there is a parallel flow, as

$$u_{\parallel} - \frac{B_{\theta}}{B_T} v_E \approx 0$$

or use notation $\Theta = \frac{B_{\theta}}{B_T}$. This parallel flow is NOT neutral, it is a current.

The absence of neutrality is avoided by generation of a parallel electric field $E_{\parallel}$.

It is related to the parallel flow of charges $j_{\parallel}$ through the resistivity η.

The existence of a parallel electric field is the existence of a poloidal projection E_{θ}

3.2 Resistive plasma equilibrium of flows and currents (Stringer PRL)

See **driftwithradialelectricfield Stringer1969**.

3.2.1 Equations

The equations involve in this approach the *neoclassical drifts* of the particles

$$\begin{aligned} \mathbf{v} &= \widehat{\mathbf{n}} v_{\parallel} + \mathbf{v}_{Dj} \\ &\quad + \mathbf{V}_E^{(0)} + \mathbf{V}_E^{(1)} + ... \end{aligned}$$

where the *drift velocity* is approximated

$$\begin{aligned} \mathbf{v}_{Dj} &= \frac{1}{\Omega_j} \widehat{\mathbf{n}} \times \frac{v_{\perp}^2/2 + v_{\parallel}^2}{R} \left(-\widehat{\mathbf{e}}_R \right) \\ &\approx -\frac{1}{e_j B} \frac{2T_j}{R} \widehat{\mathbf{e}}_{vert} \end{aligned}$$

it has been taken

$$\frac{2T_j}{m_j} = v^2$$

The other velocity is *electric*

$$\begin{aligned} \mathbf{V}_E^{(0)} (r) &= \frac{-\boldsymbol{\nabla} \phi^{(0)} \times \widehat{\mathbf{n}}}{B} \\ \mathbf{V}_E^{(1)} (r, \theta) &= \frac{-\boldsymbol{\nabla} \phi^{(1)} \times \widehat{\mathbf{n}}}{B} \end{aligned}$$

In the equation of continuity we have the divergence of the flow of the particles caused by their *neoclassical drift*.

This is particularly interesting, in this approach.

$$\boldsymbol{\nabla}\cdot\left(n\mathbf{v}_{Dj}\right)=-\frac{1}{e_jB}\frac{2T_j}{R}\frac{dn_0}{dr}\sin\theta$$

The divergence of the flow of particles moving with the electric velocity is

$$\begin{aligned}\boldsymbol{\nabla}\cdot\left(\mathbf{V}_E^{(0)}\right) &= \boldsymbol{\nabla}\cdot\left[\frac{-\boldsymbol{\nabla}\phi^{(0)}\times\mathbf{B}}{B^2}\right] \\ &= -\boldsymbol{\nabla}\left(\frac{1}{B^2}\right)\left[-\boldsymbol{\nabla}\phi^{(0)}\times\mathbf{B}\right] \\ &\quad +\frac{1}{B^2}\left(\boldsymbol{\nabla}\times\mathbf{B}\right)\cdot\boldsymbol{\nabla}\phi^{(0)}\end{aligned}$$

The last term is zero since the gradient of the potential (constant on surface) is almost radial and $\perp$ on $\mathbf{B}$.

The first term is purely geometrical, comes from the variation of the magnitude of the magnetic field. It is

$$-\boldsymbol{\nabla}\left(\frac{1}{B^2}\right)\left[-\boldsymbol{\nabla}\phi^{(0)}\times\mathbf{B}\right]=-\frac{2}{R}v_{E\theta}^{(0)}\sin\theta$$

The equation of continuity is

$$\begin{aligned}&\left(\mathbf{V}_E^{(0)}\cdot\boldsymbol{\nabla}\right)n_1+\left(\mathbf{V}_E^{(1)}\cdot\boldsymbol{\nabla}\right)n_0 \\ &+n_0\frac{\partial v_{\parallel j}}{\partial s} \\ =\;&\frac{2}{R}\left(\frac{1}{e_jB}\frac{dp_j}{dr}+n_0V_{E\theta}^{(0)}\right)\sin\theta\end{aligned}$$

NOTE

This equation is familiar. In **Hassam Drake** we find for the equation of continuity

$$\begin{aligned}&\frac{\partial n_1}{\partial t}+V_E\frac{\partial n_1}{r\partial\theta}+n_0V_E\left(-2\varepsilon\frac{\sin\theta}{r}\right) \\ &+n_0\nabla_{\parallel}u_{\parallel} \\ =\;&S-\frac{1}{r}\frac{\partial}{\partial r}\left(r\Gamma_r\right)\end{aligned}$$

and the differences are clear:

Stringer does not expect a variation $\partial/\partial t$ of the perturbation of the density profile $n_1(r,\theta)$ in *time*. Or, **Hassam Drake** are looking for spin-up and the time variation must be kept.

Stringer does not include *sources*, S. Indeed for spin-up the sources are not stricly needed. The spin-up is spontaneous and is due to the Pfirsch-Schluter current that introduces a non-uniformity on the poloidal direction θ. Or, the Pfirsch Schluter current is a geometrical consequence. **Hassam Drake** take into account the variation of the sources on θ, for spin-up.

The terms

$$\left(\mathbf{V}_E^{(0)} \cdot \boldsymbol{\nabla}\right) n_1 = V_E \frac{\partial n_1}{r \partial \theta}$$

are the same.

the parallel derivatives are the same

$$n_0 \frac{\partial v_{\parallel j}}{\partial s} = n_0 \nabla_{\parallel} u_{\parallel}$$

Now, the terms

$$\frac{2}{R}\left(\frac{1}{e_j B}\frac{dp_j}{dr} + n_0 V_{E\theta}^{(0)}\right)\sin\theta$$

and (after transfering the term in the same side for comparison, the sign changes) respectively

$$n_0 V_E \left(2\varepsilon \frac{\sin\theta}{r}\right)$$

We can recognize the last term of **Stringer**

$$\begin{aligned} & \frac{2}{R} n_0 V_{E\theta}^{(0)} \sin\theta \quad , \text{ and} \\ n_0 V_E \left(2\varepsilon \frac{\sin\theta}{r}\right) &= \frac{2}{R} n_0 V_E \sin\theta \end{aligned}$$

But the first term

$$\frac{2}{R}\frac{1}{e_j B}\frac{dp_j}{dr}\sin\theta$$

is *not present at Hassam Drake*. Its presence at Stringer is the result of :

- using the neoclassical drift of the particles, $v_{drift,j}$. These drifts are *unbalanced* and lead to a charge separation and are responsible for a flow which compensates
- approximation of the square-velocity part in the $v_{drift,j}$ which is $v_{\perp}^2/2 + v_{\parallel}^2 \sim \frac{2T_j}{m_j}$.

By this approximation, the expression of the velocity appears similar with that of the *diamagnetic* flow.

The point of departure in the Stringer's equations is NOT the diamagnetic flow.

It is the *particle drifts* $v_{drift,j}$ and the flows (electrons, ions) created by these *neoclassical drifts* produce the Pfirsch Schluter parallel current in the end.

$[-\phi]$

END

Using this form of the *continuity equation* Stringer obtains the Pfirsch Schluter current, and this results precisely from the term where the *neoclassical drift* flows have been expressed in terms of the gradient of the pressure (as if they would come from diamagnetic: they do not come from diamagnetic).

The derivation starts from the equation of continuity

$$\begin{aligned} & \left(\mathbf{V}_E^{(0)}\cdot\boldsymbol{\nabla}\right)n_1+\left(\mathbf{V}_E^{(1)}\cdot\boldsymbol{\nabla}\right)n_0 \\ & +n_0\frac{\partial v_{\parallel j}}{\partial s} \\ = & \ \frac{2}{R}\left(\frac{1}{e_jB}\frac{dp_j}{dr}+n_0V_{E\theta}^{(0)}\right)\sin\theta \end{aligned}$$

written for $j=e$ and $j=i$, then substract them.

The first two termsin the LHS are the same (electric drift and also the neutrality)

$$\begin{aligned} & n_0\left(\frac{\partial v_{\parallel i}}{\partial s}-\frac{\partial v_{\parallel e}}{\partial s}\right) \\ = & \ \frac{2}{R}\left(\frac{1}{eB}\frac{dp_i}{dr}-\frac{1}{-eB}\frac{dp_e}{dr}\right)\sin\theta \end{aligned}$$

It is adopted

$$e=|e|$$

We note

$$p=p_i+p_e$$

and take into account that (s =parallel)

$$\frac{\partial}{\partial s}=\frac{B_\theta}{B}\frac{\partial}{r\partial\theta}$$

then

$$\begin{aligned} & \frac{B_\theta}{B}\frac{\partial}{r\partial\theta}\frac{1}{e}\left(en_0v_{\parallel i}-en_0v_{\parallel e}\right) \\ = & \ \frac{2}{R}\frac{1}{eB}\frac{dp}{dr}\sin\theta \end{aligned}$$

$$\frac{\partial}{\partial\theta}j_\parallel=2\frac{r}{R}\frac{1}{B_\theta}\frac{dp}{dr}\sin\theta$$

or

$$j_\parallel=-2q\frac{1}{B_T}\frac{dp}{dr}\cos\theta$$

This is the Pfirsch Schluter current.

The part that contains the *diamagnetic drifts* is the source of the gradient of pressure dp/dr.

The next equation is the momentum conservation where the basic flow is the electric velocity $V_{E\theta}^{(0)}$. The equation is of the type

$$nm\left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{v}=-\boldsymbol{\nabla}p$$

projected on the *parallel* direction and with the temperatures taken constants.

$$nm_i\left(v_{E\theta}^{(0)}\frac{\partial}{r\partial\theta}\right)v_{\parallel,i}=-\left(T_i+T_e\right)\frac{\varepsilon}{q}\frac{\partial n_1}{r\partial\theta}$$

This balance of momenta involves the nonlinear static advection of the parallel velocity and the gradient of the pressure along the magnetic field line.

3.2.2 Perturbation of the density on surface

The equation of momentum

$$nm_i\left(v_{E\theta}^{(0)}\frac{\partial}{r\partial\theta}\right)v_{\parallel,i}=-\left(T_i+T_e\right)\frac{\varepsilon}{q}\frac{\partial n_1}{r\partial\theta}$$

The Ohm's law

$$\eta j_{\parallel}=\frac{\varepsilon}{q}\frac{\partial}{\partial\theta}\left[-\phi^{(1)}+\frac{T_e}{|e|}\frac{n_1}{n_0}\right]$$

The Pfirsch Schluter current $j_{\parallel}$ allows now to write the system of equations for n_1 and $\phi^{(1)}$, perturbations on surface.

The Ohm's law, after replacing $j_{\parallel}$

$$\eta\ \left(-2q\frac{1}{B_0}\frac{dp}{dr}\cos\theta\right)=\frac{\varepsilon}{q}\frac{\partial}{\partial\theta}\left[-\phi^{(1)}+\frac{T_e}{|e|}\frac{n_1}{n_0}\right]$$

From this equation we extract $\partial n_1/\partial\theta$,

$$\eta\ \left(-2q\frac{1}{B_0}\frac{dp}{dr}\cos\theta\right)-\frac{\varepsilon}{q}\frac{\partial\left(-\phi^{(1)}\right)}{\partial\theta}=\frac{T_e}{|e|}\frac{1}{n_0}\frac{\varepsilon}{q}\frac{\partial n_1}{\partial\theta}$$

$$\frac{\varepsilon}{q}\frac{\partial n_1}{r\partial\theta}=-\eta\ 2q\frac{1}{B_0}\frac{1}{r}\frac{e}{T_e}n_0\frac{dp}{dr}\cos\theta+\frac{e\ n_0}{T_e}\frac{\partial\phi^{(1)}}{r\partial\theta}$$

and this is replaced in $nm_i\left(v_{E\theta}^{(0)}\frac{\partial}{r\partial\theta}\right)v_{\parallel,i}=-\left(T_i+T_e\right)\frac{\varepsilon}{q}\frac{\partial n_1}{r\partial\theta}$,

$$\begin{aligned}&-\frac{nm_i}{T_i+T_e}\left(v_{E\theta}^{(0)}\frac{\partial}{r\partial\theta}\right)v_{\parallel,i}\\=&\quad-\eta\ 2q\frac{1}{B_0}\frac{1}{r}\frac{e}{T_e}n_0\frac{dp}{dr}\cos\theta+\frac{e\ n_0}{T_e}\frac{\partial\phi^{(1)}}{r\partial\theta}\end{aligned}$$

which we can now integrate on θ, since $\frac{\cos\theta}{r} = -\frac{\partial}{r\partial\theta}\sin\theta$

$$\begin{aligned} & -\frac{nm_i}{T_i+T_e} v_{E\theta}^{(0)}\ v_{\parallel,i} \\ = & \ \ \eta\ 2q \frac{1}{B_0}\frac{e}{T_e} n_0 \frac{dp}{dr}\sin\theta + \frac{en_0}{T_e}\phi^{(1)} \end{aligned}$$

we have

$$\begin{aligned} & \phi^{(1)} \\ = & \ \ \frac{T_e}{e}\left(\frac{m_i}{T_i+T_e}\right) v_{E0}^{(0)}\ v_{\parallel,i} - \eta\ 2q\frac{1}{B_0}\frac{dp}{dr}\sin\theta \end{aligned}$$

this gives the expression of $\phi^{(1)}(r,\theta)$ if we can find $v_{\parallel,i}$.

Now we will need the parallel velocity, $v_{\parallel,i}$ and this can be obtained from the equation of continuity

$$\begin{aligned} & \left(\mathbf{V}_E^{(0)}\cdot\boldsymbol{\nabla}\right)n_1 + \left(\mathbf{V}_E^{(1)}\cdot\boldsymbol{\nabla}\right)n_0 \\ & + n_0\frac{\partial v_{\parallel j}}{\partial s} \\ = & \ \ \frac{2}{R}\left(\frac{1}{e_j B}\frac{dp_j}{dr} + n_0 V_{E\theta}^{(0)}\right)\sin\theta \end{aligned}$$

where

$$\left(\mathbf{V}_E^{(0)}\cdot\boldsymbol{\nabla}\right)n_1 = v_{E\theta}^{(0)}\frac{\partial}{r\partial\theta}n_1$$

and we have the expression for $\frac{\partial}{r\partial\theta}n_1$. Further,

$$\begin{aligned} & \left(\mathbf{V}_E^{(1)}\cdot\boldsymbol{\nabla}\right)n_0 \\ = & \ \ v_{E\theta}^{(1)}\frac{\partial}{r\partial\theta}n_0 + v_{Er}^{(1)}\frac{\partial}{\partial r}n_0 \\ = & \ \ \left[\frac{-\boldsymbol{\nabla}\phi^{(1)}\times\widehat{\mathbf{n}}}{B}\cdot\widehat{\mathbf{e}}_r\right]\ \frac{dn_0}{dr} \\ = & \ \ -\frac{1}{B_0}\frac{\partial\phi^{(1)}}{r\partial\theta}\frac{dn_0}{dr} \end{aligned}$$

The next term

$$n_0\frac{\partial v_{\parallel j}}{\partial s} = n_0\frac{B_\theta}{B}\frac{\partial v_{\parallel,j}}{r\partial\theta}$$

allows to find $\frac{\partial v_{\parallel i}}{r\partial\theta}$ then $v_{\parallel i}$ for the formulas above.

The result is

$$\begin{aligned} n_1 = & \ \ n_0\frac{1}{D}2\varepsilon\frac{1}{V_{E\theta}^{(0)}}\left[-\left(v_i^{dia}+V_{E\theta}^{(0)}\right)\cos\theta\right. \\ & \left. +\eta\left(\frac{1}{n_0}\frac{dn_0}{dr}\right)\left(\frac{1}{B}\frac{dp}{dr}\right)\left(\frac{q^2}{\varepsilon^2}\right)\frac{r\sin\theta}{B}\right] \end{aligned}$$

where

$$v_j^{dia} = \frac{T_j}{e_j B}\left(\frac{1}{n_0}\frac{dn_0}{dr}\right)$$

$$c_s^2 = \frac{T_i + T_e}{m_i}$$

and the denominator

$$\begin{aligned} D &= 1 + \frac{v_e^{dia}}{V_{E\theta}^{(0)}} \\ &\quad - \frac{c_s^2}{\left(V_{E\theta}^{(0)}\right)^2}\frac{\varepsilon^2}{q^2} \end{aligned}$$

We **Note** that the combination

$$1 + \frac{v_i^{dia}}{V_{E\theta}^{(0)}}$$

and the combination

$$1 + \frac{v_e^{dia}}{V_{E\theta}^{(0)}} = 1 - \frac{\left|v_e^{dia}\right|}{V_{E\theta}^{(0)}}$$

occur in the expression of n_1. But, in D, the posible resonance $1 - \frac{\left|v_e^{dia}\right|}{V_{E\theta}^{(0)}} = 0$ only involves the *electrons*. It is avoided by the quantity

$$\begin{aligned} & - \frac{c_s^2}{\left(V_{E\theta}^{(0)}\right)^2}\frac{\varepsilon^2}{q^2} \\ = & - \frac{c_{s\theta}^2}{\left(V_{E\theta}^{(0)}\right)^2} \end{aligned}$$

which compares, $c_s \frac{B_\theta}{B_{tor}}$, the poloidally projected sound velocity

$$c_s^2 \frac{\varepsilon^2}{\left(\frac{rB_{tor}}{RB_\theta}\right)^2} = \left(c_s \frac{B_\theta}{B_{tor}}\right)^2$$

to $V_{E\theta}^{(0)}$ the electric poloidal velocity.

NOTE

that this parameter is also used by **Rosenbluth hose** as

$$c_s \frac{\varepsilon}{q} = c_s \frac{B_\theta}{B_T} = c_s^\theta$$

in normalizations. See *rotation.tex* in subsection *hose-like.*

END

The latter will be the *critical* value of the parameter

$$M_p^2 = \frac{c_s^2 \left(\varepsilon/q\right)^2}{\left(V_{E\theta}^0\right)^2}$$

to look for shocks (**Rosenbluth Hazeltine Lee,** Sanuki)

We also **NOTE** the presence of the combination of terms

$$1 - \frac{\left|v_e^{dia}\right|}{V_{E\theta}^{(0)}} \sim 1 - \frac{\left|v_e^{dia}\right|}{u}$$

that is also present in the expression of a drift vortex moving in plasma. This is the factor that defines the *effective Larmor radius* (see our works on the role of this effective ρ_s)

$$\frac{1}{\left(\rho_s^{eff}\right)^2} = \frac{1}{\rho_s^2}\left(1 - \frac{v^{dia}}{u}\right)$$

It is then useful to reflect to the explanation given by **Stringer** to this term.

3.2.3 The dielectric response of plasma in rotation

comment on the possibility of resonance at the denominator D.

The resonance would correspond to infinite effective Larmore radius as

$$1 - \frac{v_{dia}}{v_E} \to 0$$

but this is only possible if in D the supplementary term is small

$$\begin{aligned} D &= 1 + \frac{v_e^{dia}}{V_{E\theta}^{(0)}} \\ &\quad - \frac{c_s^2}{\left(V_{E\theta}^{(0)}\right)^2} \frac{\varepsilon^2}{q^2} \end{aligned}$$

The "deffect" is the term

$$M_p^2 = \frac{c_s^2 \left(\varepsilon/q\right)^2}{\left(V_{E\theta}^0\right)^2}$$

and resonance is only possible if $M_p \to 0$ which means that the electric poloidal velocity is much higher than the poloidal projection of the sound speed. Actually

$$\frac{\varepsilon}{q}$$

is already small, then we also need c_s to be small but this does not happen in the $H-$mode layer.

End comment.

The explanation offered by **Stringer** on the denominator D.

It is regarded from the frame that is co-moving with the plasma that rotates.

Then here the plasma is static and the magnetic field rotates.

Then the *toroidal magnetic field* appear as a perturbation with harmonic poloidal number

$$m = 1$$

which rotates with poloidal *angular frequency*

$$\omega \sim -\frac{v_{E\theta}}{r}$$

$(v_0 \equiv v_E)$.

The perturbation has along the field lines the wavenumber

$$\begin{aligned} k_{\parallel} &= \frac{\varepsilon}{rq} \\ &= \frac{1}{qR} \quad \text{(so called connection length)} \end{aligned}$$

The variation (rotation) of the magnetic field produces a perturbation of the density.

This density perturbation drives a *forced oscillation* of the plasma.

This forced oscillation of the density travels *in phase* with the magnetic perturbation.

The response of plasma must be inverse proportional with the *dielectric constant.*

In a rest frame the dielectric constant is

$$D\left(\omega, k_{\parallel}\right)$$

Then, when it is in rotation the dielectric constant is

$$D\left(-\frac{v_{E\theta}}{r}, \frac{\varepsilon}{eq}\right)$$

The factor D is the dielectric constant for the ion-acoustic drift waves in an inhomogeneous plasma.

3.2.4 No electric poloidal rotation

When

$$v_E \to 0$$

the diffusion is that of Pfirsch Schluter

3.2.5 The dielectric D is suppressed

When

the wavelength

and

the Doppler-shifted frequency seen by the plasma

coincide with those of a natural mode of plasma,

we have

$$D \rightarrow 0$$

then the diffusion is much higher than that of Pfirsch Schluter.

This possibility, $D \rightarrow 0$ is restricted by the inclusion of new effects

- finite Larmor radius
- ion-ion collisions

When they are included, D cannot become zero. D will acquire a dissipative component.

There will not be exact resonance between the rotation and a natural mode.

3.2.6 Energy balance

The Pfirsch Schluter diffusion verifies the energy conservation

- the amount of decrease of energy content due to PS diffusion

is equal to

- the ohmic dissipation

$$\eta \ \left(j_{\parallel}^{PS}\right)^2$$

The Ohmic dissipation ηj^2 must be corrected when there is *variation of density* on surface.

This is because

$E_{\parallel} j_{\parallel}$

can be greater than

$\eta j_{\parallel}^2$

due to the pressure gradient $-\nabla_{\theta} p$

The zero-order electric drift transfers energy to the plasma by working against, - or for, the gradient of pressure

$$\int dr \int dS \ \mathbf{v}_E \cdot \boldsymbol{\nabla} p$$

The sign is given by D

- for

$$D < 0$$

the plasma injects energy into the electric rotation (in the electrostatic field Φ)

- for

$$D > 0$$

energy is extracted from the rotation by the plasma. See below what follows when $D > 0$, in particular the *reversed diffusion*.

3.2.7 Radial flux of particles

The radial flux of particles is calculated as an average over the magnetic surface, $\theta \in (0, 2\pi)$. The quantity that is averaged is the local radial flux obtained as product between the density (zero-order n_0 and correction for variation in the surface, $n_1(\theta)$) and the radial velocity. The radial velocity is the *neoclassical drifts* $v_{drift,j}|_r$ plus the electric contribution produced by the variation of the potential in the surface

$$\begin{aligned}
\Gamma_{rj} &= n v_{rj} \\
&= \int_0^{2\pi} \frac{d\theta}{2\pi} (n_0 + n_1) \left(\frac{1}{B_0} \frac{\partial \phi^{(1)}}{r \partial \theta} + \frac{T_j}{e_j B_0} \varepsilon \frac{2 \sin\theta}{r} \right) (1 + \varepsilon \cos\theta)^2
\end{aligned}$$

The second term in the second parantheis is

$$\frac{T_j}{e_j B_0} \varepsilon \frac{2 \sin\theta}{r} \approx v_{drift,j}|_{radial}$$

consistent with the approximation adopted by Stringer for the neoclassical drift expressing $v_\perp^2/2 + v_\parallel^2$ in terms of Temperature.

The result

$$\begin{aligned}
\Gamma_{rj} &= n v_{rj} \\
&= q^2 \eta \; 2 n_0 \frac{1}{D} \left(\frac{1}{B_0} \frac{dp}{dr} \right) \frac{1}{B_0} \left[\frac{c_s^2 \frac{\varepsilon^2}{q^2}}{\left(V_{E\theta}^{(0)} \right)^2} + \frac{v_i^{dia} - v_j^{dia}}{V_{E\theta}^{(0)}} \right]
\end{aligned}$$

Following the discussion about the sign of D.

When $D > 0$ the diffusion is *inwards*. [since $dp/dr < 0$]

But when $D > 0$ some mechanism is needed to support the electric rotation. Otherwise this will decay.

3.2.8 The Landau damping

When the collisons are very rare, there is a situation when there are particles that have zero azimuthal velocity. They are in resonance with the static modulation of the magnetic field.

3.3 The variation of density and potential on the surface Stringer 1991

This is also in *variation on surfaces*.

This is **pfb3 1991**.

It is a detailed form of the PRL of 1969.

Stringer calculates the correction to the distribution function that is associated with the variation of n and Φ on magnetic surfaces. This variation is a result of toroidality. The input is therefore the drift of the particles.

The treatment is *drift-kinetic*.

$$\begin{aligned} f_j &= f_j^{(0)}\left(r, v_{\|}, v_{\perp}^2\right) \\ &\quad + f_j^{(1)}\left(r, \theta, v_{\|}, v_{\perp}^2\right) + ... \end{aligned}$$

$$\Phi\left(r,\theta\right) = \Phi^{(0)}\left(r\right) + \Phi^{(1)}\left(r,\theta\right) + ...$$

The guiding center velocity

$$\begin{aligned} \mathbf{V}_j &= v_{\|}\widehat{\mathbf{n}} \\ &\quad + \mathbf{V}_D \\ &\quad + \mathbf{V}^{(0)} + \mathbf{V}^{(1)} + ... \end{aligned}$$

where

$$\mathbf{V}_D = -\frac{1}{\Omega_j}\frac{v_{\perp}^2/2 + v_{\|}^2}{R}\left(\frac{B_{\varphi}}{B}\widehat{\mathbf{e}}_z - \frac{B_z}{B}\widehat{\mathbf{e}}_{\varphi}\right)$$

with

$$\Omega_j = \frac{e_j B}{m_j}$$

The drift is mainly vertical,

$$\widehat{\mathbf{e}}_R \times \widehat{\mathbf{n}}$$

In any case the vertical magnetic field is very small

$$\frac{B_z}{B} \ll 1$$

and the velocities are

$$\begin{aligned} \mathbf{V}^{(0)} &= \frac{1}{B}\frac{d\Phi^{(0)}}{dr} \\ \mathbf{V}^{(1)} &= \frac{-\boldsymbol{\nabla}\Phi^{(1)} \times \widehat{\mathbf{n}}}{B} \end{aligned}$$

$$
\begin{aligned}
\Theta &\equiv \frac{B_\theta}{B_\varphi} \\
&= \frac{\varepsilon}{q} = O(\varepsilon) \\
&\ll 1
\end{aligned}
$$

The diamagnetic velocities

$$
\begin{aligned}
v_{*j} &= \frac{T_{0j}}{e_j B} \frac{d \ln n_0}{dr} \\
v_{*j}^T &= \frac{1}{e_j B} \frac{dT_{0j}}{dr}
\end{aligned}
$$

Take the parallel velocity

$$
v_\parallel = \sqrt{\frac{2}{m_j} \left(\epsilon - \mu B - e_j \Phi \right)}
$$

then

$$
\begin{aligned}
\frac{dv_\parallel}{dt} &= -\frac{1}{m_j v_\parallel} \left(\mathbf{V} \cdot \boldsymbol{\nabla} \right) \left(e_j \Phi + \mu B \right) \\
&= -\frac{1}{m_j v_\parallel} \left(V^{(0)} \frac{\partial}{r \partial \theta} + v_\parallel \frac{\partial}{\partial l_\parallel} + V_r \frac{\partial}{\partial r} \right) \left(e_j \Phi + \mu B \right) \\
&= -\frac{e_j}{m_j} \frac{B_\theta}{B_\varphi} \frac{\partial \Phi^{(1)}}{r \partial \theta} - \varepsilon \left(\frac{B_\theta}{B_\varphi} \frac{v_\perp^2}{2} - V^{(0)} v_\parallel \right) \frac{\sin \theta}{r}
\end{aligned}
$$

This comes from

$$
\begin{aligned}
\frac{dv_\parallel}{dt} &= \left(\frac{\partial}{\partial t} + \left(\mathbf{V} \cdot \boldsymbol{\nabla} \right) \right) v_\parallel = \left(\mathbf{V} \cdot \boldsymbol{\nabla} \right) v_\parallel \\
&= \left(\mathbf{V} \cdot \boldsymbol{\nabla} \right) \sqrt{\frac{2}{m_j} \left(\epsilon - \mu B - e_j \Phi \right)} \\
&= \left(\mathbf{V} \cdot \boldsymbol{\nabla} \right) \frac{1}{2} \frac{1}{\sqrt{\frac{2}{m_j} \left(\epsilon - \mu B - e_j \Phi \right)}} \frac{2}{m_j} \left(-\mu B - e_j \Phi \right) \\
&= -\frac{1}{m_j} \frac{1}{v_\parallel} \left(\mathbf{V} \cdot \boldsymbol{\nabla} \right) \left(\mu B + e_j \Phi \right)
\end{aligned}
$$

The derivation along the magnetic field line is

$$
\begin{aligned}
\frac{\partial}{\partial l_\parallel} &= \nabla_\parallel = \frac{B_\theta}{B_\varphi} \frac{\partial}{r \partial \theta} \\
&= \frac{1}{qR} \frac{\partial}{\partial \theta}
\end{aligned}
$$

The radial drift velocity takes into account the existence of the perturbation of the electric potential

$$\begin{aligned} V_r &= -\frac{1}{B}\frac{\partial \Phi^{(1)}}{r\partial\theta} \\ &\quad -\frac{1}{\Omega_j}\frac{v_\perp^2/2+v_\parallel^2}{R}\sin\theta \end{aligned}$$

The drift-kinetic equation

$$\frac{\partial f_j}{\partial t}+(\mathbf{V}_j\cdot\boldsymbol{\nabla})\,f_j+\frac{\partial f_j}{\partial v_\parallel}\frac{dv_\parallel}{dt}+\frac{\partial f_j}{\partial v_\perp^2}\frac{dv_\perp^2}{dt}=0$$

The drift kinetic equation is linearized to order ε.
the result is

$$\begin{aligned} f_j^{(1)} &= \frac{1}{V^{(0)}+\Theta v_\parallel}\left\{\left[\frac{\Phi^{(1)}}{B}-\frac{1}{\Omega_j}\varepsilon\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)\cos\theta\right]\frac{\partial f_j^{(0)}}{\partial r}\right. \\ &\quad -e_j\frac{v_\perp^2}{v_{th,j}^2}\left(V^{(0)}+\Theta v_\parallel\right)f_j^{(0)}\cos\theta \\ &\quad \left.+\left[\frac{e_j}{m_j}\Theta\Phi^{(1)}-\varepsilon\left(\Theta\frac{v_\perp^2}{2}-V^{(0)}v_\parallel\right)\right]\frac{\partial f_j^{(0)}}{\partial v_\parallel}\right\} \end{aligned}$$

We note the poloidal velocity, composed of the electric velocity $V_E^{(0)}$ and of the poloidal projection of the parallel velocity $\Theta v_\parallel$.

This combination, representing the poloidal velocity, should be almost zero

$$V_E^{(0)}+\frac{\varepsilon}{q}v_\parallel\approx 0$$

This correction to the distribution function contains

- the effect of the drift of the particles v_D.
- the effect of the presence of a potential constant on the magnetic surfaces $\Phi^{(0)}$.
- the effect of a variation of the electric potential in the surface, $\Phi^{(1)}$.

A term

$$\frac{\Phi^{(1)}}{B}\frac{\partial f_j^{(0)}}{\partial r}$$

is the radial advection due to the potential $\Phi^{(1)}$, of the equilibrium distribution function.

A term

$$\left[\Theta\frac{e_j\Phi^{(1)}}{m_j}\right]\frac{\partial f_j^{(0)}}{\partial v_\parallel}$$

is the acceleration in parallel velocity produced by the electric field correction $e_j\Phi^{(1)}$ after being projected along the parallel direction by $\Theta\equiv B_\theta/B_\varphi$.

Therefore these two terms contain the effect of $\Phi^{(1)}$ on the distribution function (necessarily of zero-order $f_j^{(0)}$ since $\Phi^{(1)}$ is itself small).

The variation of the electric potential in the surface $\Phi^{(1)}$ is determined from the condition of neutrality

$$n_e = n_i$$

We have to obatin the densities by integrating over the velocity space

$$\int dv_\parallel \int dv_\perp^2$$

The integration over $v_\perp^2$ can be done.

The integration over $v_\parallel$ is complicated by the singularities of the denominator

$$\frac{1}{V^{(0)}+\Theta v_\parallel}$$

and this integration must be treated like Landau singularity.

The distribution function that is to be integrated is the Maxwell function. Then one introduces the *definition*

$$\frac{1}{n_0}\int_{-\infty}^{\infty}\frac{F_j^{(0)}}{v_\parallel - W}\ \left(v_\parallel^s\right)\ dv_\parallel \equiv K_s\left(\frac{W}{v_{th,j}}\right)$$

These functions are expressed through the *plasma dispersion function*.

The relations are

$$\begin{aligned} K_s\left(\frac{W}{v_{th,j}}\right) &= W\ K_{s-1}\left(\frac{W}{v_{th,j}}\right) \\ &\quad +J_s \end{aligned}$$

where

$$J_s = \begin{cases} (s-2)\,(s-4)\,...1\ \left(\frac{v_{th,j}}{2}\right)^{\frac{s-1}{2}} & \text{for} \quad s \text{ odd} \\ 0 & \text{for} \quad s \text{ even} \end{cases}$$

The connection with the Plasma Dispersion Function is

$$\begin{aligned} K_0\left(\frac{W}{v_{th,j}}\right) &= \frac{1}{W}\left[I\left(\frac{W}{v_{th,j}}\right)-1\right] \\ K_1\left(\frac{W}{v_{th,j}}\right) &= I\left(\frac{W}{v_{th,j}}\right) \end{aligned}$$

where

$$
\begin{aligned}
I(z) &= 1 - 2z\exp\left(-z^2\right)\int_0^z dt\exp\left(-t^2\right) \\
&\quad + i\sqrt{\pi}\; z\exp\left(-z^2\right)
\end{aligned}
$$

NOTE that we have here the Principal value and the singularity $i\pi\delta$ which, after integration, gives the Landau term

The expression of the distribution function $f_j^{(1)}$ will be integrated over the velocity space to obtain the densities.

Then neutrality will be invoked, obtaining an equation for the potential $\Phi^{(1)}$.

Definitions

$$
\begin{aligned}
V_{*n,j} &= \frac{T_{0j}}{e_j B}\frac{1}{n_0}\frac{dn_0}{dr} \\
V_{*T,j} &= \frac{T_{0j}}{e_j B}\frac{1}{T_{0j}}\frac{dT_{0j}}{dr}
\end{aligned}
$$

Stringer finds that the supplementary velocity $V_j^{(1)}$ induced by the variation of the potential $\Phi^{(1)}$ in the surface is a fraction of the diamagnetic velocity

$$
V_j^{(1)} \sim \varepsilon V_{*n,j}
$$

The density is

$$
\begin{aligned}
\frac{n_j^{(1)}}{n_0} &= \frac{e_j\Phi^{(1)}}{T_j} \\
&\quad\times\left[\frac{1}{V^{(0)}}\left(V_{*n,j} - \frac{V_{*T,j}}{2}\right)(1 - I_j) - I_j - \frac{V^{(0)}V_{*T,j}}{v_{th,j}^2\Theta^2} I_j\right] \\
&\quad + \varepsilon\exp(i\theta)\left[\left(1 + \frac{V_{*n,j}}{V^{(0)}} + \frac{V_{*T,j}}{2V^{(0)}}\right)(I_j - 1)\right. \\
&\qquad + 2z_j^2 I_j\left(1 + \frac{V_{*n,j}}{V^{(0)}}\right) \\
&\qquad \left. + z_j^2\frac{V_{*T,j}}{V^{(0)}}\left(1 + 2z_j^2 I_j\right)\right]
\end{aligned}
$$

The new notations are

$$
\begin{aligned}
z_j &\equiv -\frac{V^{(0)}}{v_{th,j}\Theta} \\
I_j &\equiv I(z_j)
\end{aligned}
$$

Note that

$$
z_j = -\frac{V_E^{(0)}\frac{\varepsilon}{q}}{v_{th,j}} = -\frac{V^{pol}\ \text{(projected on parallel direction)}}{\text{(thermal velocity)}}
$$

After calculation of $n_e^{(1)}$ and $n_i^{(1)}$ it is invoked the neutrality.

The equation for neutrality becomes an equation for the perturbation to the uniform electric potential on the surface: $\Phi^{(1)}(r,\theta)$.

3.4 The equations for the currents and flows

The papers by **Rosenbluth Lee Hazeltine PRL** and PF71

The line

$$ds^2 = h_r^2 dr^2 + h_\theta^2 d\theta^2 + h_\varphi^2 d\varphi^2$$

The values for circular geometry are

$$\begin{aligned} h_r^C &= 1 \\ h_\theta^C &= r \\ h_\varphi^C &= R_0 + r\cos\theta \end{aligned}$$

The magnetic field will be assumed slightly more general than in circular surfaces, and we will return to this simple geometry by taking $B(r) = B_0$ =const.

$$\begin{aligned} B_r &= 0 \\ B_\theta &= \frac{b(r)}{h} = \frac{\varepsilon}{q}\frac{B_0(r)}{h} \\ B_\varphi &= \frac{B_0(r)}{h} \end{aligned}$$

where

$$h = 1 + \varepsilon\cos\theta = \frac{R}{R_0} \quad , \quad \varepsilon = \frac{r}{R}$$

and q is the safety factor. The current is (HLR)

$$J_r = 0$$

$$J_\theta = -\frac{1}{1+\varepsilon\cos\theta}\frac{dB_0(r)}{dr} = -\frac{1}{h}\frac{dB_0(r)}{dr}$$

$$\begin{aligned} J_\varphi &= -\frac{B_0}{b(r)}\frac{dB_0}{dr}\frac{1}{1+\varepsilon\cos\theta} - \frac{1}{b}\frac{dp}{dr}(1+\varepsilon\cos\theta) \\ &= -\frac{1}{h}\frac{B_0(r)/h}{b(r)/h}\frac{dB_0(r)}{dr} - \frac{1}{b(r)/h}\frac{dp}{dr} \\ &= -\frac{1}{h}\frac{B_\varphi}{B_\theta}\frac{dB_0(r)}{dr} - \frac{1}{B_\varphi}\frac{B_\varphi}{B_\theta}\frac{dp}{dr} \end{aligned}$$

Since

$$\frac{B_\varphi}{B_\theta} \equiv \Theta^{-1} = \left(\frac{\varepsilon}{q}\right)^{-1}$$

we have

$$\frac{\varepsilon}{q}J_{\varphi}=-\frac{1}{h}\frac{dB_0\left(r\right)}{dr}-\frac{h}{B_0\left(r\right)}\frac{dp}{dr}$$

which is identical with HassamKulsrud. We will find later below that the HK result is obtained from the **Grad Shafranov** equation.

We will also use $\mathbf{B}=B_{\theta}\widehat{\mathbf{e}}_{\theta}+B_{\varphi}\widehat{\mathbf{e}}_{\varphi}$ from where we have

$$\left|\mathbf{B}\right|=\sqrt{B_{\theta}^2+B_{\varphi}^2}=\frac{B_0\left(r\right)}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}}$$

The perpendicular current $j_{\perp}$ comes from

$$\mathbf{j}\times\mathbf{B}=-\boldsymbol{\nabla}p$$

where

$$\begin{aligned}\mathbf{j}\times\mathbf{B} &= j_{\perp}\left|\mathbf{B}\right|\left(-\widehat{\mathbf{e}}_r\right)=j_{\perp}\frac{B_0\left(r\right)}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}}\left(-\widehat{\mathbf{e}}_r\right)\\ &= -\frac{dp}{dr}\widehat{\mathbf{e}}_r\end{aligned}$$

from where

$$j_{\perp}=\frac{h}{B_0\left(r\right)}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}\frac{dp}{dr}$$

We notice that it is usual to work with two sets of projections of the current, (j_{θ},j_{φ}) and $\left(j_{\parallel},j_{\perp}\right)$. The connection is ensured by the expressions

$$\widehat{\mathbf{e}}_{\varphi}\cdot\widehat{\mathbf{e}}_{\perp}=-\frac{B_{\theta}}{\left|\mathbf{B}\right|}=-\frac{\frac{\varepsilon}{q}\frac{B_0(r)}{h}}{\frac{B_0(r)}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}}}=-\frac{\varepsilon}{q}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}$$

and

$$\widehat{\mathbf{e}}_{\perp}\cdot\widehat{\mathbf{e}}_{\theta}=\frac{B_{\varphi}}{\left|\mathbf{B}\right|}=\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}$$

We use the two expressions (j_{θ},j_{φ}) to obtain geometrically $j_{\parallel}$ as

$$j_{\parallel}=j_{\theta}\sin\alpha+j_{\varphi}\cos\alpha$$

where

$$\cos\alpha=\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}\ ,\ \sin\alpha=\frac{\varepsilon}{q}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}$$

and, as derived above

$$\begin{aligned}j_{\theta} &= -\frac{1}{h}\frac{dB_0\left(r\right)}{dr}\\ j_{\varphi} &= \frac{q}{\varepsilon}\left[-\frac{1}{h}\frac{dB_0\left(r\right)}{dr}-\frac{h}{B_0\left(r\right)}\frac{dp\left(r\right)}{dr}\right]\end{aligned}$$

Then

$$j_{\|} = \left[-\frac{1}{h}\frac{dB_0(r)}{dr}\right]\frac{\varepsilon}{q}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}} + \frac{q}{\varepsilon}\left[-\frac{1}{h}\frac{dB_0(r)}{dr} - \frac{h}{B_0(r)}\frac{dp(r)}{dr}\right]\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}$$

or

$$\begin{aligned} j_{\|} &= \left[-\frac{1}{h}\frac{dB_0(r)}{dr}\right]\left(\frac{\varepsilon}{q}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}} + \frac{q}{\varepsilon}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}\right) \\ &\quad + \left[-\frac{h}{B_0(r)}\frac{dp(r)}{dr}\right]\frac{q}{\varepsilon}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}} \end{aligned}$$

$$\begin{aligned} j_{\|} &= \frac{q}{\varepsilon}\left[-\frac{1}{h}\frac{dB_0(r)}{dr}\right]\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}\left(\frac{\varepsilon^2}{q^2}+1\right) \\ &\quad + \left[-\frac{h}{B_0(r)}\frac{dp(r)}{dr}\right]\frac{q}{\varepsilon}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}} \end{aligned}$$

$$j_{\|} = \frac{q}{\varepsilon}\left[-\frac{1}{h}\frac{dB_0(r)}{dr}\right]\sqrt{1+\frac{\varepsilon^2}{q^2}} + \left[-\frac{h}{B_0(r)}\frac{dp(r)}{dr}\right]\frac{q}{\varepsilon}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}$$

This is precisely the result of **Hassam Kulsrud**.

For the normal case where there is no radial variation of B_0,

$$\begin{aligned} j_\theta &= 0 \\ j_\varphi &= \frac{q}{\varepsilon}\left[-\frac{h}{B_0(r)}\frac{dp(r)}{dr}\right] \end{aligned}$$

$$j_{\|} = \left[-\frac{h}{B_0(r)}\frac{dp(r)}{dr}\right]\frac{q}{\varepsilon}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}$$

We note that the poloidal variation of $j_{\|}$ is given by h.

Compare this with **Hazeltine Rosenbluth Lee phys fluids feb. 1971**

$$J_z = \frac{q}{\varepsilon}J_{\|\theta} - \frac{\varepsilon}{q}J_{\perp\theta}$$

where

$$J_{\|\theta} \approx -2c_s^2\frac{1}{B_0}\varepsilon\cos\theta\frac{\partial}{\partial r}\left[\overline{\rho}\left(1-\frac{\overline{v}^2}{2}\right)\right]$$

and

$$J_{\perp\theta} = c_s^2\frac{1}{B_0}\rho h\left(\frac{1}{\rho}\frac{\partial\rho}{\partial r} - \frac{v^2}{h}\frac{\partial h}{\partial r}\right)$$

We recongnize the first term in $J_{\perp\theta}$ as the *diamagnetic current*.

Hazeltine Lee Rosenbluth PF71

See **rotation.tex** *hose like*.

That we must use the **Grad Shafranov** equation, which in **Hassam Kulsrud** is

$$\frac{h}{r}\frac{\varepsilon}{q}B_0(r)\frac{d}{dr}\left[\frac{r}{h}\frac{\varepsilon}{q}B_0(r)\right]=-B_0(r)\frac{dB_0(r)}{dr}-h^2\frac{dp(r)}{dr}$$

we use $B_\theta=\frac{\varepsilon}{q}\frac{B_0(r)}{h}$ and divide by B_0 and h

$$\frac{\varepsilon}{q}\frac{1}{r}\frac{d}{dr}\left[rB_\theta(r)\right]=-\frac{1}{h}\frac{dB_0(r)}{dr}-\frac{h}{B_0}\frac{dp(r)}{dr}$$

We remind that the equilibrium equation

$$0=-\nabla p+\mathbf{j}\times\mathbf{B}$$

becomes the GS equation after using the Ampere's law

$$\nabla\times\mathbf{B}=\mu\mathbf{j}$$

projected on the toroidal (φ) direction

$$\mu_0\mathbf{j}=\nabla\times\mathbf{B}|_\varphi=\frac{h_\varphi}{h_rh_\theta h_\varphi}\widehat{\mathbf{e}}_\varphi\left[\frac{\partial}{\partial r}(h_\theta B_\theta)-\frac{\partial}{\partial\theta}(h_rB_r)\right]=\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)$$

and taking units such that $\mu_0=1$, we have

$$j_\varphi=\frac{1}{r}\frac{d}{dr}\left[rB_\theta(r)\right]$$

Replacing in the equilibrium equation we find

$$-\frac{1}{h}\frac{dB_0(r)}{dr}-\frac{h}{B_0}\frac{dp(r)}{dr}=\frac{\varepsilon}{q}J_\varphi$$

and note that actually the **Grad Shafranov** equation provides us with the explicit form of the toroidal component of the current.

We also note that the components of the current obey the zero-divergence condition (charge continuity)

$$\nabla\cdot\mathbf{j}=0$$

$$\frac{1}{h_rh_\theta h_\varphi}\left[\frac{\partial}{\partial r}(h_\theta h_\varphi\ j_r)+\frac{\partial}{\partial\theta}(h_rh_\varphi\ j_\theta)+\frac{\partial}{\partial\varphi}(h_rh_\theta\ j_\varphi)\right]=0$$

Here we must insert $j_r=0$ and assume axisymmetry

$$\frac{\partial}{\partial\varphi}(h\ j_\varphi)=0$$

It results

$$\frac{1}{rh}\frac{\partial}{\partial\theta}\left(h\ j_{\theta}\right)=0$$

where we use $j_{\theta}=-\frac{1}{1+\varepsilon\cos\theta}\frac{dB_{0}(r)}{dr}$. We obtain an identity

$$\frac{\partial}{\partial\theta}\frac{dB_{0}\left(r\right)}{dr}=0$$

The order of magnitude is

$$\frac{dB_{0}}{dr}\sim\frac{dp}{dr}\sim b^{2}\sim\frac{a^{2}}{R^{2}}$$

3.4.1 The velocities

In HLR PRL there is a radial velocity

$$v_{r}$$

In the treatment of the Pfirsch Schluter current the radial velocity v_r exists ONLY if there is a resistivity $\eta\neq 0$ in the Ohm's law that, according to **Stringer PRL** prevents the exact neutralization of the charge along the magnetic field line.

In another paper the radial flux

$$\Gamma_{r}=nv_{Dr}^{transp}$$

is calculated on the base of the distribution function determined by solving the drift-kinetic equation and of the particle neoclassical drift $v_{De,i}$, integrated over the velocity space.

4 Poloidal rotation Hassam Kulsrud

4.1 Basic equations. We take NO variation of the central magnetic field $dB_{0}\left(r\right)/dr=0$.

The version that includes the term

$$\frac{dB_{0}\left(r\right)}{dr}\neq 0$$

is in **flows on magnetic surface notes, in Plasma Theory.**

And in *old.zip.*

The equations

$$\begin{aligned} nm_{i}\left(\frac{\partial\mathbf{v}}{\partial t}+\left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{v}\right) &= -\boldsymbol{\nabla}p-\boldsymbol{\nabla}\cdot\boldsymbol{\Pi} \\ &\quad +\mathbf{j}\times\mathbf{B} \end{aligned}$$

where

$$\mathbf{\Pi} = -3\eta_0 \left(\widehat{\mathbf{n}}\, \widehat{\mathbf{n}} - \frac{1}{3} \right) \left(\widehat{\mathbf{n}}\, \widehat{\mathbf{n}} - \frac{1}{3} \right) : \boldsymbol{\nabla}\ \mathbf{v}$$

where

$$\eta_0 = \text{collisional viscosity coeff.}$$

$$p = n \left(T_e + T_i \right)$$

The heat equation is expressed in terms of the *entropy*

$$nT \left(\frac{\partial s}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\, s \right) = -\boldsymbol{\nabla} \cdot \mathbf{q} - \mathbf{\Pi} \colon \boldsymbol{\nabla}\ \mathbf{v}$$

with the flux of heat

$$\mathbf{q} = -\chi \frac{1}{B} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} T$$

note that this is *parallel*

$$q_{\parallel} = \chi_{\parallel} \frac{1}{B} \nabla_{\parallel} T$$

The Ohm's law

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j}$$

and

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{B} &= \mu_0 \mathbf{j} \\ \boldsymbol{\nabla} \times \mathbf{E} &= 0 \\ \boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \end{aligned}$$

The magnetic field is (more general than circular)

$$\mathbf{B} = \left(0, \frac{b(r)}{h}, \frac{B_0}{h} \right)$$

The following averaging operator is introduced

$$\langle f \rangle = \frac{\int \frac{dS}{|\boldsymbol{\nabla} p|} f}{\int \frac{dS}{|\boldsymbol{\nabla} p|}}$$

The equation of continuity, averaged on surface

$$\frac{\partial}{\partial t} \langle n \rangle + \frac{1}{\int \frac{dS}{|\boldsymbol{\nabla} p|}} \frac{\partial}{\partial p} \int \mathbf{dS} \cdot n \mathbf{v} = 0$$

The equation for the *circulation*, which is the *mixed helicity* $\mathbf{v}\cdot\mathbf{B}$, averaged on surface

$$\begin{aligned} & \frac{\partial}{\partial t}\left\langle \mathbf{v}\cdot\mathbf{B}\right\rangle + \frac{1}{\int \frac{dS}{|\boldsymbol{\nabla} p|}}\frac{\partial}{\partial p}\int \mathbf{dS}\cdot\mathbf{v}\ \mathbf{v}\cdot\mathbf{B} \\ = & -\mathbf{v}\cdot\left\langle \boldsymbol{\nabla}\times\left(\eta\mathbf{j}-\widehat{\mathbf{e}}_z\frac{\mathcal{E}}{h}\right)\right\rangle \\ & -\left\langle \frac{3\eta_0}{nm_i}\mathbf{B}\cdot\boldsymbol{\nabla}\ln B\ \mathbf{v}\cdot\boldsymbol{\nabla}\ln B\right\rangle \\ & +\frac{1}{m_i}\sum_{e,i}\left\langle T\mathbf{B}\cdot\boldsymbol{\nabla}s\right\rangle \end{aligned}$$

Note the flux of mixed helicity through surface

$$\int \mathbf{dS}\cdot\mathbf{v}\ H^m,\quad \text{where } H^m=\mathbf{v}\cdot\mathbf{B}$$

and the sources of mixed helicity

$$-\mathbf{v}\cdot\left\langle \boldsymbol{\nabla}\times\left(\eta\mathbf{j}-\widehat{\mathbf{e}}_z\frac{\mathcal{E}}{h}\right)\right\rangle$$

The Ohm's law

$$\eta\mathbf{j}=\mathbf{E}+\mathbf{v}\times\mathbf{B}$$

In the composition of the electric field we have

$$\mathbf{E}=-\boldsymbol{\nabla}\phi+\widehat{\mathbf{e}}_z\frac{\mathcal{E}}{h}$$

then

$$\eta\mathbf{j}=-\boldsymbol{\nabla}\phi+\widehat{\mathbf{e}}_z\frac{\mathcal{E}}{h}+\mathbf{v}\times\mathbf{B}$$

and

$$\boldsymbol{\nabla}\times\left(\eta\mathbf{j}-\widehat{\mathbf{e}}_z\frac{\mathcal{E}}{h}\right)=\boldsymbol{\nabla}\times\left(\mathbf{v}\times\mathbf{B}\right)$$

the vector rotational uses

$$\begin{aligned} & \boldsymbol{\nabla}\times\left(\mathbf{A}\times\mathbf{B}\right) \\ = & \mathbf{A}\left(\boldsymbol{\nabla}\cdot\mathbf{B}\right)-\mathbf{B}\left(\boldsymbol{\nabla}\cdot\mathbf{A}\right)+\left(\mathbf{B}\cdot\boldsymbol{\nabla}\right)\mathbf{A}-\left(\mathbf{A}\cdot\boldsymbol{\nabla}\right)\mathbf{B} \end{aligned}$$

which gives

$$\boldsymbol{\nabla}\times\left(\mathbf{v}\times\mathbf{B}\right)=\mathbf{v}\left(\boldsymbol{\nabla}\cdot\mathbf{B}\right)-\mathbf{B}\left(\boldsymbol{\nabla}\cdot\mathbf{v}\right)+\left(\mathbf{B}\cdot\boldsymbol{\nabla}\right)\mathbf{v}-\left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{B}$$

assuming incompressible velocity

$$\boldsymbol{\nabla}\times\left(\mathbf{v}\times\mathbf{B}\right)=B\nabla_{\parallel}\mathbf{v}-\left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{B}$$

and this is further averaged over surface.

Then

$$
\begin{aligned}
\frac{\partial}{\partial t}\langle \mathbf{v}\cdot\mathbf{B}\rangle &\sim -\mathbf{v}\cdot\left\langle \boldsymbol{\nabla}\times\left(\eta\mathbf{j}-\widehat{\mathbf{e}}_z\frac{\mathcal{E}}{h}\right)\right\rangle \\
&= \boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{B}) \\
&= B\nabla_{\|}\mathbf{v}-(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{B}
\end{aligned}
$$

The first term is

$$B\nabla_{\|}\mathbf{v}\sim B\boldsymbol{\nabla}_{\|}\mathbf{j}$$

is similar to the term driving the change in time of the fluid vorticity.

The equation for toroidal momentum

$$\frac{\partial}{\partial t}\langle nm_iRh\ \mathbf{v}\cdot\widehat{\mathbf{e}}_\varphi\rangle+\frac{1}{\int\frac{dS}{|\boldsymbol{\nabla}p|}}\frac{\partial}{\partial p}\int\mathbf{dS}\cdot\mathbf{v}\ \ (nm_iRh\ \mathbf{v}\cdot\widehat{\mathbf{e}}_\varphi)=0$$

The equation for the *entropy* assuming that the species are adiabatic

$$
\begin{aligned}
&\langle n\rangle\frac{\partial s}{\partial t}+\frac{1}{\int\frac{dS}{|\boldsymbol{\nabla}p|}}\int\mathbf{dS}\cdot\ n\mathbf{v}\left(\frac{\partial s}{\partial p}\right) \\
=\ &\left\langle\chi\left(\frac{\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}T}{T}\right)^2\right\rangle \\
&+\left\langle\frac{3\eta_0}{T}(\mathbf{v}\cdot\boldsymbol{\nabla}\ln B)^2\right\rangle
\end{aligned}
$$

We want to calculate the flux through surface of a scalar quantity f, averged over surface

For the averaging operator we have

$$\frac{1}{\int\frac{dS}{|\boldsymbol{\nabla}p|}}\int\mathbf{dS}\cdot\mathbf{v}\ f=\frac{dp}{dr}\langle v_r\ f\rangle$$

For different functions f the quantity $\langle v_r\ f\rangle$ is derived by averaging the toroidal component of the Ohm's law.

The radial velocity v_r that carries the scalar f through the surface is generated by the current that flows toroidally j_φ (close to $j_{\|}$) and the poloidal magnetic field B_θ.

The connection between these three quantities v_r, j_φ and B_θ is the Ohm's law.

This will introduce the resistivity η.

Start from

$$\mathbf{E}+\mathbf{v}\times\mathbf{B}=\eta\mathbf{j}$$

with

$$\mathbf{E} = -\boldsymbol{\nabla}\phi + \frac{\mathcal{E}}{h}\widehat{\mathbf{e}}_{\varphi}$$

where an external, inductive, electric field is considered, $\mathcal{E}$, toroidal.

First we multiply by $\mathbf{B}$ the Ohm's law

$$\mathbf{E}\cdot\mathbf{B} = \eta j_{\parallel}\left|\mathbf{B}\right|$$

and replace

$$\begin{aligned}\left(-\boldsymbol{\nabla}\phi + \frac{\mathcal{E}}{h}\widehat{\mathbf{e}}_{\varphi}\right)\cdot\mathbf{B} &= \eta j_{\parallel}\left|\mathbf{B}\right| \\ \mathcal{E}\frac{B_{\varphi}}{h} &= \eta j_{\parallel}\left|\mathbf{B}\right|\end{aligned}$$

The magnitude is

$$\left|\mathbf{B}\right| = \frac{B_0}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}}$$

The equation becomes

$$\mathcal{E}\frac{B_0}{h^2} = \eta j_{\parallel}\frac{B_0}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}}$$

after averaging

$$\mathcal{E} = \frac{\left\langle \eta j_{\parallel}\frac{B_0}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}}\right\rangle}{\left\langle \frac{B_0}{h^2}\right\rangle}$$

the previously derived expression of the parallel current

$$j_{\parallel} = \left[-\frac{h}{B_0\left(r\right)}\frac{dp\left(r\right)}{dr}\right]\frac{q}{\varepsilon}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}$$

$$j_{\parallel} = \frac{q}{\varepsilon}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}h\left[-\frac{1}{B_0}\frac{dp}{dr}\right]$$

(we **note** that this is a more accurate expression for the Pfirsch Schluter current, which now contains the possible non-zero derivative $dB_0/dr \neq 0$. But we do not retain this.)

The numerator of the expression of $\mathcal{E}$ is the average of

$$\begin{aligned} & \eta j_{\parallel}\frac{B_0}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}} \\ = \; & \eta\frac{B_0}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}}\frac{q}{\varepsilon}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}h\left(-\frac{1}{B_0}\frac{dp}{dr}\right) \\ = \; & \eta B_0\frac{q}{\varepsilon}\left(-\frac{1}{B_0}\frac{dp}{dr}\right)\end{aligned}$$

Then

$$
\begin{aligned}
\mathcal{E} &= \frac{\left\langle \eta j_{\parallel} \frac{B_0}{h} \sqrt{1+\frac{\varepsilon^2}{q^2}} \right\rangle}{\left\langle \frac{B_0(r)}{h^2} \right\rangle} \\
&= \frac{1}{B_0} \frac{1}{\left\langle \frac{1}{h^2} \right\rangle} \eta B_0 \frac{q}{\varepsilon} \left\langle -\frac{1}{B_0} \frac{dp}{dr} \right\rangle
\end{aligned}
$$

Consider an arbitrary function f of plasma variables.

We take the φ (toroidal) component of the Ohm's law

$$
\begin{aligned}
\frac{\mathcal{E}}{h} + (\mathbf{v} \times \mathbf{B_0})_{\varphi} &= \eta j_{\varphi} \\
\frac{\mathcal{E}}{h} + v_r B_{\theta} &= \eta j_{\varphi}
\end{aligned}
$$

and multiply by

$$
\frac{f}{B_{\theta}}
$$

$$
\eta j_{\varphi} \frac{f}{B_{\theta}} - \frac{\mathcal{E}}{h} \frac{f}{B_{\theta}} = v_r f
$$

and average over surface

$$
\left\langle \left(\eta j_{\varphi} - \frac{\mathcal{E}}{h} \right) \frac{f}{B_{\theta}} \right\rangle = \langle v_r f \rangle
$$

Now we use

$$
\frac{\varepsilon}{q} j_{\varphi} = -\frac{h}{B_0} \frac{dp(r)}{dr}
$$

We return to

$$
\begin{aligned}
\langle v_r f \rangle &= \left\langle \left(\eta j_{\varphi} - \frac{\mathcal{E}}{h} \right) \frac{f}{B_{\theta}} \right\rangle \\
&= \eta \left\langle j_{\varphi} \frac{f}{B_{\theta}} \right\rangle - \left\langle \frac{\mathcal{E}}{h} \frac{f}{B_{\theta}} \right\rangle
\end{aligned}
$$

and take into account that $\mathcal{E}$ is already averaged. Replacing j_{φ} by its expression in terms of the gradient of pressure

$$
\begin{aligned}
\langle v_r f \rangle &= \eta \left\langle f \frac{1}{B_{\theta}} \frac{q}{\varepsilon} \left[-\frac{h}{B_0} \frac{dp(r)}{dr} \right] \right\rangle \\
&\quad - \mathcal{E} \left\langle f \frac{1}{h B_{\theta}} \right\rangle
\end{aligned}
$$

The second term is

$$
-\mathcal{E} \left\langle f \frac{1}{h B_{\theta}} \right\rangle
$$

where

$$\frac{1}{hB_\theta} = \frac{1}{h\frac{b(r)}{h}} = \frac{1}{b\,(r)}$$

and is factored out from the averaging.

$$\begin{aligned} & -\mathcal{E}\frac{1}{b\,(r)}\langle f\rangle \\ = \quad & -\frac{1}{B_0}\frac{1}{\left\langle\frac{1}{h^2}\right\rangle}\eta B_0\frac{q}{\varepsilon}\left\langle -\frac{1}{B_0}\frac{dp}{dr}\right\rangle\frac{1}{b\,(r)}\langle f\rangle \end{aligned}$$

and note that

$$\frac{B_0}{b} = \frac{B_0/h}{b/h} = \frac{B_\varphi}{B_\theta} = \Theta^{-1} = \frac{q}{\varepsilon}$$

We have

$$\begin{aligned} & -\mathcal{E}\frac{1}{b\,(r)}\langle f\rangle \\ = \quad & \eta\frac{1}{B_0}\left(\frac{q}{\varepsilon}\right)^2\frac{\left\langle\frac{1}{B_0}\frac{dp}{dr}\right\rangle}{\left\langle\frac{1}{h^2}\right\rangle} \end{aligned}$$

NOTE that this almost the same as the second term in the Appendix C of **Hassam Kulsrud**, with the difference that they have $B_0\frac{dp}{dr}$ instead of $\frac{1}{B_0}\frac{dp}{dr}$. There is a problem of units in HK. To be checked more carefully. **END.**

Working the first term

$$\eta\left\langle f\frac{1}{B_\theta}\frac{q}{\varepsilon}\left[-\frac{h}{B_0}\frac{dp\,(r)}{dr}\right]\right\rangle$$

of $\langle v_r f\rangle$ we remind that

$$\frac{\varepsilon}{q} = \Theta = \frac{B_\theta}{B_\varphi}$$

$$\frac{1}{B_\theta} = \frac{q}{\varepsilon}\frac{1}{B_\varphi} = \frac{q}{\varepsilon}\frac{h}{B_0}$$

then

$$\begin{aligned} & f\frac{1}{B_\theta}\frac{q}{\varepsilon}\left[-\frac{h}{B_0}\frac{dp\,(r)}{dr}\right] \\ = \quad & f\frac{q}{\varepsilon}\frac{h}{B_0}\frac{q}{\varepsilon}\left[-\frac{h}{B_0}\frac{dp\,(r)}{dr}\right] \\ = \quad & -f\left(\frac{q}{\varepsilon}\right)^2\frac{1}{B_0^2}\frac{dp}{dr}h^2 \end{aligned}$$

One can factorize from the averaging operator all factors that only depend on ψ (*i.e.* on the radius r).

$$
\begin{aligned}
& \eta \left\langle f \frac{1}{B_\theta} \frac{q}{\varepsilon} \left[-\frac{h}{B_0} \frac{dp\left(r\right)}{dr} \right] \right\rangle \\
= & \quad - \left(\frac{q}{\varepsilon} \right)^2 \frac{1}{B_0^2} \frac{dp}{dr} \left\langle f h^2 \right\rangle \\
= & \quad -\eta \left(\frac{q}{\varepsilon} \right)^2 \frac{1}{B_0} \left[\frac{1}{B_0} \frac{dp}{dr} \left\langle f h^2 \right\rangle \right]
\end{aligned}
$$

This is indeed the first term in **HK** appendix C.

Finally, the expression of the average of the radial flux of the scalar f through surface becomes

$$
\begin{aligned}
& \left\langle f v_r \right\rangle \\
= & \quad -\eta \left(\frac{q}{\varepsilon} \right)^2 \frac{1}{B_0} \left[\frac{1}{B_0} \frac{dp}{dr} \left\langle f h^2 \right\rangle \right] \\
& \quad +\eta \frac{1}{B_0} \left(\frac{q}{\varepsilon} \right)^2 \frac{\left\langle \frac{1}{B_0} \frac{dp}{dr} \right\rangle}{\left\langle \frac{1}{h^2} \right\rangle}
\end{aligned}
$$

or

$$
\begin{aligned}
& \left\langle f v_r \right\rangle \\
= & \quad \eta \left(\frac{q}{\varepsilon} \right)^2 \frac{1}{B_0} \left\{ - \left[\frac{1}{B_0} \frac{dp}{dr} \left\langle f h^2 \right\rangle \right] + \frac{\left\langle \frac{1}{B_0} \frac{dp}{dr} \right\rangle}{\left\langle \frac{1}{h^2} \right\rangle} \right\}
\end{aligned}
$$

The components of the magnetic field $\mathbf{B}$ and of the current $\mathbf{J}$ are related by the equation

$$0 = -\boldsymbol{\nabla} p + \mathbf{J} \times \mathbf{B}$$

The other equation is

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}$$

It results the **Grad Shafranov** equation

$$\frac{1}{h_r h_\theta} \frac{\partial}{\partial r} \left[\frac{h_\theta}{h_r h_\varphi} b\left(r\right) \right] = -\frac{B}{b\left(r\right)} \frac{dB}{dr} \frac{1}{h_\varphi} - \frac{h_\varphi}{b\left(r\right)} \frac{dp}{dr}$$

The RHS is J_φ.

The LHS is $\boldsymbol{\nabla} \times \mathbf{B}$, component along φ.

4.2 Averaged equations for poloidal and toroidal rotation

The equations

$$\begin{aligned} nm_i \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla}) \mathbf{v} \right) &= -\boldsymbol{\nabla} p - \boldsymbol{\nabla} \cdot \boldsymbol{\Pi} \\ & \quad + \mathbf{j} \times \mathbf{B} \end{aligned}$$

where

$$\boldsymbol{\Pi} = -3\eta_0 \left(\widehat{\mathbf{n}}\, \widehat{\mathbf{n}} - \frac{1}{3} \right) \left(\widehat{\mathbf{n}}\, \widehat{\mathbf{n}} - \frac{1}{3} \right) : \boldsymbol{\nabla}\, \mathbf{v}$$

$$p = n \left(T_e + T_i \right)$$

The heat equation is expressed in terms of the *entropy*

$$nT \left(\frac{\partial s}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla}) s \right) = -\boldsymbol{\nabla} \cdot \mathbf{q} - \boldsymbol{\Pi} \colon \boldsymbol{\nabla}\, \mathbf{v}$$

with the flux of heat

$$\mathbf{q} = -\chi \frac{1}{B} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} T$$

The Ohm's law

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j}$$

and

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{B} &= \mu_0 \mathbf{j} \\ \boldsymbol{\nabla} \times \mathbf{E} &= 0 \\ \boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \end{aligned}$$

The magnetic field is (more general than circular)

$$\mathbf{B} = \left(0, \frac{b(r)}{h}, \frac{B_0}{h} \right)$$

The following averaging operator is introduced

$$\langle f \rangle = \frac{\int \frac{dS}{|\boldsymbol{\nabla} p|} f}{\int \frac{dS}{|\boldsymbol{\nabla} p|}}$$

The equation of continuity involves the averaging of the divergence of a flux of particles, $n\mathbf{v}$.

$$\frac{\partial}{\partial t} \langle n \rangle + \frac{1}{\int \frac{dS}{|\boldsymbol{\nabla} p|}} \frac{\partial}{\partial p} \int \mathbf{dS} \cdot n\mathbf{v} = 0$$

Let us note here that the integrand of the operator at numerator has changed and a derivation to p has been introduced instead. This form of the average will be treated in detail below.

The equation for the *circulation*

$$
\begin{aligned}
& \frac{\partial}{\partial t}\left\langle \mathbf{v}\cdot\mathbf{B}\right\rangle + \frac{1}{\int \frac{dS}{|\boldsymbol{\nabla} p|}}\frac{\partial}{\partial p}\int \mathbf{dS}\cdot\mathbf{v}\ \mathbf{v}\cdot\mathbf{B} \\
= & -\mathbf{v}\cdot\left\langle \boldsymbol{\nabla}\times\left(\eta\mathbf{j}-\widehat{\mathbf{e}}_z\frac{\mathcal{E}}{h}\right)\right\rangle \\
& -\left\langle \frac{3\eta_0}{nm_i}\mathbf{B}\cdot\boldsymbol{\nabla}\ln B\ \mathbf{v}\cdot\boldsymbol{\nabla}\ln B\right\rangle \\
& +\frac{1}{m_i}\sum_{e,i}\left\langle T\mathbf{B}\cdot\boldsymbol{\nabla}s\right\rangle
\end{aligned}
$$

The equation for toroidal momentum

$$
\frac{\partial}{\partial t}\left\langle nm_iRh\ \mathbf{v}\cdot\widehat{\mathbf{e}}_\varphi\right\rangle + \frac{1}{\int \frac{dS}{|\boldsymbol{\nabla} p|}}\frac{\partial}{\partial p}\int \mathbf{dS}\cdot\mathbf{v}\ \left(nm_iRh\ \mathbf{v}\cdot\widehat{\mathbf{e}}_\varphi\right) = 0
$$

The equation for the *entropy* assuming that the species is adiabatic

$$
\begin{aligned}
& \left\langle n\right\rangle\frac{\partial s}{\partial t} + \frac{1}{\int \frac{dS}{|\boldsymbol{\nabla} p|}}\int \mathbf{dS}\cdot\ n\mathbf{v}\left(\frac{\partial s}{\partial p}\right) \\
= & \left\langle \chi\left(\frac{\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}T}{T}\right)^2\right\rangle \\
& +\left\langle \frac{3\eta_0}{T}\left(\mathbf{v}\cdot\boldsymbol{\nabla}\ln B\right)^2\right\rangle
\end{aligned}
$$

For the averaging operator we have

$$
\frac{1}{\int \frac{dS}{|\boldsymbol{\nabla} p|}}\int \mathbf{dS}\cdot\mathbf{v}\ f = \frac{dp}{dr}\left\langle v_r\ f\right\rangle
$$

For different functions f the quantity $\left\langle v_r\ f\right\rangle$ is derived by averaging the toroidal component of the Ohm's law.

Start from

$$
\mathbf{E}+\mathbf{v}\times\mathbf{B} = \eta\mathbf{j}
$$

with

$$
\mathbf{E} = -\boldsymbol{\nabla}\phi + \frac{\mathcal{E}}{h}\widehat{\mathbf{e}}_\varphi
$$

First we multiply by $\mathbf{B}$ the Ohm's law

$$
\mathbf{E}\cdot\mathbf{B} = \eta j_\parallel\left|\mathbf{B}\right|
$$

and replace

$$
\begin{aligned}
\left(-\boldsymbol{\nabla}\phi + \frac{\mathcal{E}}{h}\widehat{\mathbf{e}}_\varphi\right)\cdot\mathbf{B} &= \eta j_\parallel\left|\mathbf{B}\right| \\
\mathcal{E}\frac{B_\varphi}{h} &= \eta j_\parallel\left|\mathbf{B}\right|
\end{aligned}
$$

The magnitude is

$$|\mathbf{B}| = \frac{B_0}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}}$$

The equation becomes

$$\mathcal{E}\frac{B_0}{h^2} = \eta j_{\|}\frac{B_0}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}}$$

after averaging

$$\mathcal{E} = \frac{\left\langle \eta j_{\|}\frac{B_0}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}}\right\rangle}{\left\langle\frac{B_0}{h^2}\right\rangle}$$

the previously derived expression of the parallel current

$$j_{\|} = \left[-\frac{h}{B_0(r)}\frac{dp(r)}{dr}\right]\frac{q}{\varepsilon}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}$$

$$j_{\|} = \frac{q}{\varepsilon}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}h\left[-\frac{1}{B_0}\frac{dp}{dr}\right]$$

The numerator of the expression of $\mathcal{E}$ is the average of

$$\begin{aligned}
&\eta j_{\|}\frac{B_0}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}}\\
= \;&\eta\frac{B_0}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}}\frac{q}{\varepsilon}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}h\left[-\frac{1}{B_0}\frac{dp}{dr}\right]\\
= \;&\eta B_0(r)\frac{q}{\varepsilon}\left[-\frac{1}{B_0}\frac{dp}{dr}\right]
\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{E} &= \frac{\left\langle \eta j_{\|}\frac{B_0}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}}\right\rangle}{\left\langle\frac{B_0}{h^2}\right\rangle}\\
&= \frac{1}{B_0}\frac{1}{\left\langle\frac{1}{h^2}\right\rangle}\eta B_0\frac{q}{\varepsilon}\left\langle-\frac{1}{B_0}\frac{dp}{dr}\right\rangle
\end{aligned}$$

Return to a previous calculation (this part starts by copy/paste)
Next we take the φ (toroidal) component

$$\begin{aligned}
\frac{\mathcal{E}}{h} + (\mathbf{v}\times\mathbf{B_0})_{\varphi} &= \eta j_{\varphi}\\
\frac{\mathcal{E}}{h} + v_r B_{\theta} &= \eta j_{\varphi}
\end{aligned}$$

which is multiplied by

$$\frac{f}{B_\theta}$$

$$\eta j_\varphi \frac{f}{B_\theta} - \frac{\mathcal{E}}{h}\frac{f}{B_\theta} = v_r f$$

and averaged

$$\left\langle \left(\eta j_\varphi - \frac{\mathcal{E}}{h}\right)\frac{f}{B_\theta}\right\rangle = \langle v_r f\rangle$$

Now we use

$$\frac{\varepsilon}{q} j_\varphi = -\frac{h}{B_0}\frac{dp\left(r\right)}{dr}$$

We return to

$$\begin{aligned} \langle v_r f\rangle &= \left\langle \left(\eta j_\varphi - \frac{\mathcal{E}}{h}\right)\frac{f}{B_\theta}\right\rangle \\ &= \eta \left\langle j_\varphi \frac{f}{B_\theta}\right\rangle - \left\langle \frac{\mathcal{E}}{h}\frac{f}{B_\theta}\right\rangle \end{aligned}$$

and take into account that $\mathcal{E}$ is already averaged.

$$\begin{aligned} \langle v_r f\rangle &= \eta \left\langle f \frac{1}{B_\theta}\frac{q}{\varepsilon}\left[-\frac{h}{B_0}\frac{dp\left(r\right)}{dr}\right]\right\rangle \\ & -\mathcal{E}\left\langle f\frac{1}{hB_\theta}\right\rangle \end{aligned}$$

The second term can be expressed as

$$-\mathcal{E}\left\langle f\frac{1}{hB_\theta}\right\rangle$$

where

$$\frac{1}{hB_\theta} = \frac{1}{h\frac{b(r)}{h}} = \frac{1}{b\left(r\right)}$$

and is factored out from the averaging.

$$\begin{aligned} & -\mathcal{E}\frac{1}{b\left(r\right)}\langle f\rangle \\ = & -\frac{1}{B_0}\frac{1}{\left\langle\frac{1}{h^2}\right\rangle}\eta B_0\frac{q}{\varepsilon}\left\langle -\frac{1}{B_0}\frac{dp}{dr}\right\rangle\frac{1}{b\left(r\right)}\langle f\rangle \end{aligned}$$

and note that

$$\frac{B_0}{b} = \frac{B_0/h}{b/h} = \frac{B_\varphi}{B_\theta} = \Theta^{-1} = \frac{q}{\varepsilon}$$

We have

$$
\begin{aligned}
& -\mathcal{E}\frac{1}{b(r)}\langle f\rangle \\
= \quad & \eta\frac{1}{B_0}\left(\frac{q}{\varepsilon}\right)^2\frac{\left\langle\frac{1}{B_0}\frac{dp}{dr}\right\rangle\langle f\rangle}{\left\langle\frac{1}{h^2}\right\rangle}
\end{aligned}
$$

Working the first term we remind that

$$\frac{\varepsilon}{q}=\Theta=\frac{B_\theta}{B_\varphi}$$

$$\frac{1}{B_\theta}=\frac{q}{\varepsilon}\frac{1}{B_\varphi}=\frac{q}{\varepsilon}\frac{h}{B_0(r)}$$

then

$$
\begin{aligned}
& f\frac{1}{B_\theta}\frac{q}{\varepsilon}\left[-\frac{h}{B_0}\frac{dp(r)}{dr}\right] \\
= \quad & f\frac{q}{\varepsilon}\frac{h}{B_0(r)}\frac{q}{\varepsilon}\left[-\frac{h}{B_0}\frac{dp(r)}{dr}\right] \\
= \quad & -f\left(\frac{q}{\varepsilon}\right)^2\frac{1}{B_0^2}\frac{dp}{dr}h^2
\end{aligned}
$$

One can factorize from the averaging operator all factors that only depend on ψ (*i.e.* on the radius r).

$$
\begin{aligned}
& \eta\left\langle f\frac{1}{B_\theta}\frac{q}{\varepsilon}\left[-\frac{h}{B_0}\frac{dp(r)}{dr}\right]\right\rangle \\
= \quad & -\left(\frac{q}{\varepsilon}\right)^2\frac{1}{B_0^2}\frac{dp}{dr}\left\langle fh^2\right\rangle \\
= \quad & -\eta\left(\frac{q}{\varepsilon}\right)^2\frac{1}{B_0}\left[\frac{1}{B_0}\frac{dp}{dr}\left\langle fh^2\right\rangle\right]
\end{aligned}
$$

This is indeed the first term in **HK** appendix C.

Finally, the expression of the average is

$$
\begin{aligned}
& \langle fv_r\rangle \\
= \quad & -\eta\left(\frac{q}{\varepsilon}\right)^2\frac{1}{B_0}\left[\frac{1}{B_0}\frac{dp}{dr}\left\langle fh^2\right\rangle\right] \\
& +\eta\frac{1}{B_0}\left(\frac{q}{\varepsilon}\right)^2\frac{\left\langle\frac{1}{B_0}\frac{dp}{dr}\right\rangle\langle f\rangle}{\left\langle\frac{1}{h^2}\right\rangle}
\end{aligned}
$$

or

$$
\begin{aligned}
& \langle fv_r\rangle \\
= \quad & \eta\left(\frac{q}{\varepsilon}\right)^2\frac{1}{B_0}\left\{-\left[\frac{1}{B_0}\frac{dp}{dr}\left\langle fh^2\right\rangle\right]+\frac{\left\langle\frac{1}{B_0}\frac{dp}{dr}\right\rangle\langle f\rangle}{\left\langle\frac{1}{h^2}\right\rangle}\right\}
\end{aligned}
$$

This must be taken as a basis for the averages that will involve a function f.

$$\begin{aligned} & \langle f v_r \rangle \\ = \;& \eta \left(\frac{q}{\varepsilon}\right)^2 \frac{1}{B_0} \left\{ \frac{1}{B_0} \frac{dp}{dr} \left(- \left\langle f h^2 \right\rangle + \frac{\langle f \rangle}{\left\langle \frac{1}{h^2} \right\rangle} \right) \right\} \end{aligned}$$

The first round paranthesis is

$$\begin{aligned} - \langle f \rangle + \frac{\left\langle \frac{1}{h^2} \left(1 + \frac{\varepsilon^2}{q^2}\right) \right\rangle \langle f \rangle}{\left\langle \frac{1}{h^2} \right\rangle} &= \langle f \rangle \left(-1 + 1 + \frac{\varepsilon^2}{q^2} \right) \\ &= \langle f \rangle \frac{\varepsilon^2}{q^2} \end{aligned}$$

and the expression becomes

$$\begin{aligned} & \langle f v_r \rangle \\ = \;& \eta \left(\frac{q}{\varepsilon}\right)^2 \frac{1}{B_0} \left\{ \frac{1}{B_0} \frac{dp}{dr} \left(- \left\langle f h^2 \right\rangle + \frac{\langle f \rangle}{\left\langle \frac{1}{h^2} \right\rangle} \right) \right\} \\ = \;& \eta \frac{1}{B_0} \left\{ \frac{q^2}{\varepsilon^2} \frac{1}{B_0} \frac{dp}{dr} \left(- \left\langle f h^2 \right\rangle + \frac{\langle f \rangle}{\left\langle \frac{1}{h^2} \right\rangle} \right) \right\} \end{aligned}$$

Rearranging and factoring the sign

$$\langle f v_r \rangle = -\eta \frac{dp}{dr} \frac{1}{B_0} \frac{1}{B_0} \left\{ \left(\frac{q}{\varepsilon}\right)^2 \left(\left\langle f h^2 \right\rangle - \frac{\langle f \rangle}{\left\langle \frac{1}{h^2} \right\rangle} \right) \right\}$$

The reason to factor out the gradient of the pressure comes from the definition of the average

$$\frac{\int \mathbf{dS} \cdot \mathbf{v} \; f}{\int \frac{\mathbf{dS}}{|\boldsymbol{\nabla} p|}} = \frac{dp}{dr} \langle f v_r \rangle$$

We introduce the notations

$$v_D \equiv -\eta \frac{1}{B_0} \frac{dp}{dr}$$

It has similar parametric dependence as the diamagnetic velocity but contains the resistivity η. It is the *resistive classical flow*.Then

$$\langle f v_r \rangle = v_D \frac{1}{B_0} \left(\frac{q}{\varepsilon}\right)^2 \left(\left\langle f h^2 \right\rangle - \frac{\langle f \rangle}{\left\langle \frac{1}{h^2} \right\rangle} \right)$$

NOTE the presence of the *resistivity* as FACTOR to the entire expression, *i.e.* again we see that v_r owes its existence to the resistivity that, in the Ohm's law, introduces the imperfect neutralization via parallel currents of the charge separation induced by the different drifts of electrons and ions. (Stringer PRL). **END.**

NOTE the still persistent occurence of a factor $1/B_0$ compared with HK. **End.**

The equation of continuity

$$\frac{\partial}{\partial t}\langle n\rangle + \frac{1}{\int \frac{dS}{|\mathbf{\nabla} p|}}\frac{\partial}{\partial p}\int \mathbf{dS}\cdot n\mathbf{v} = 0$$

means

$$\frac{1}{\int \frac{dS}{|\mathbf{\nabla} p|}}\int \mathbf{dS}\cdot n\mathbf{v} = \frac{dp}{dr}\langle nv_r\rangle$$

$$\frac{\partial n}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(rn\langle v_r\rangle\right) = 0$$

To calculate this averaged over the surface we use the prefiously derived equation for

$$f = 1$$

and obtain

$$\langle v_r\rangle = v_D q^2 \frac{1}{\varepsilon^2}\left(\left\langle h^2\right\rangle - \frac{1}{\left\langle \frac{1}{h^2}\right\rangle}\right)$$

and introduce the notation

$$\alpha_1 \equiv \frac{1}{2\varepsilon^2}\left(\left\langle h^2\right\rangle - \frac{1}{\left\langle \frac{1}{h^2}\right\rangle}\right)$$

then

$$\langle v_r\rangle = v_D\left(2q^2\alpha_1\right)$$

The equation of continuity becomes

$$\frac{\partial n}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left[rnv_D\left(2q^2\alpha_1\right)\right] = 0$$

We note that α_1 is close to 1.

In a similar way, it is obtained the time evolution of the toroidal component of the flow. Define

$$v_t \equiv \langle hv_\varphi\rangle$$

We must repeat the calculation made for v_r . The velocity v_φ is obtained from the φ projection of the Ohm's law, *i.e.* after multiplying it with $\widehat{\mathbf{e}}_\varphi$ we take the average. We will need the component j_φ of the current, already derived. Finally

$$\begin{aligned} v_t &\equiv \langle hv_\varphi\rangle \\ &= \frac{q}{\varepsilon}\left(v_p - v_E\left\langle h^2\right\rangle\right) \end{aligned}$$

where

$$v_E = \frac{1}{B_0} \frac{\partial \phi}{r \partial \theta}$$

and the poloidal rotation velocity v_θ is expressed through the function v_p that only depends on the magnetic surface (ψ)

$$v_\theta = \frac{v_p (r)}{h}$$

The projection of the rotation velocity perpendicular on the magnetic line is

$$\begin{aligned} v_\perp &= v_E \frac{h}{\sqrt{1+\frac{\varepsilon^2}{q^2}}} \\ &= \frac{1}{B_0/h} \frac{\partial \phi}{r\partial \theta} \frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}} = \frac{\partial \phi}{r \partial \theta} \frac{1}{B} \end{aligned}$$

As before, together with (v_θ, v_φ) it is possible to work with $\left(v_\parallel, v_\perp\right)$.

The equation for a combination of v_p and v_t has been derived from the average of the equation for the *circulation* $\mathbf{v} \cdot \mathbf{B}$ by Hassam and Drake.

$$\begin{aligned} & \frac{\partial}{\partial t} \left[v_t + \left(\alpha_3 + 2q^2 \alpha_1\right) \frac{\varepsilon}{q} v_p \right] \\ & + \frac{1}{r} \frac{\partial}{\partial r} \left\{ r \left[v_D \left(2q^2 \alpha_1\right) \left(v_t + \left(\alpha_3 + 2q^2 \alpha_1\right) \frac{\varepsilon}{q} v_p \right) \right] - 2q^2 \alpha_1 \frac{v_p}{\varepsilon / q} \right\} \\ = & -\frac{3}{2} \alpha_4 \frac{\eta_0}{n m_i R^2} \frac{\varepsilon}{q} v_p \\ & + \Xi \end{aligned}$$

By Ξ we note the terms related to thermal diffusion of the adiabatic species of particles (electrons). The notations are

$$\begin{aligned} \alpha_3 &= \left\langle \frac{1}{h^2} \right\rangle \\ \alpha_4 &= \left(2R^2\right) \left\langle \frac{1}{h^2} \left(\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}} \frac{\partial}{r \partial \theta} \ln h \right)^2 \right\rangle \end{aligned}$$

An equation for v_t results from the momentum conservation projected along the toroidal direction then averaged

$$\frac{\partial}{\partial t} \langle n v_t \rangle + \frac{1}{r} \frac{\partial}{\partial r} \left(r n v_D \left[\left(2q^2 \alpha_1\right) v_t + 2q^2 \alpha_2 \left(v_t - v_p \frac{q}{\varepsilon} \right) \right] \right) = 0$$

where

$$\alpha_2 = \frac{1}{2\varepsilon^2} \left[\left\langle h^4 \right\rangle \frac{1}{\langle h^2 \rangle} - \left\langle h^2 \right\rangle \right]$$

4.3 Equation for poloidal rotation

The previous calculations allow to write down the equation for the evolution of the poloidaal velocity

$$\begin{aligned}\left(1+\frac{1}{2q^2}\right)\frac{\partial \ln v_p}{\partial t} &= -\frac{q^2}{\varepsilon^2}v_D\frac{1}{n}\frac{dn}{dr} \\ &\quad -\frac{3}{4}\frac{\eta_0}{nm_i q^2 R^2}+\Xi'\end{aligned}$$

The symbol Ξ' is introduced to represent the effect of the thermal conductivity of the electrons

$$\Xi' \sim \chi_e$$

Since v_t is connected with v_p we have the equation for it

$$\frac{\partial v_t}{\partial t}=\frac{1}{2n}\frac{1}{r}\frac{\partial}{\partial r}\left(rnq^2 v_D \frac{q}{\varepsilon}v_p\right)$$

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PoP 1, 1994,337.

5.1 Up-down asymmetric fueling

Plus momentum input with asymmetry on θ.

the momentum equation, projected along $\mathbf{B}$

$$\begin{aligned}\frac{\partial}{\partial t}\left(nM\ \mathbf{B}\cdot\mathbf{u}\right)+\mathbf{B}\ \boldsymbol{\nabla} \quad &: \quad \left(n\ \mathbf{u}\ \mathbf{u}\right)=-T\mathbf{B}\cdot\boldsymbol{\nabla}n-\mathbf{B}\ \boldsymbol{\nabla}:\boldsymbol{\pi} \\ &\quad +\mathbf{B}\cdot\mathbf{P}\left(\mathbf{x},t\right) \quad (\text{external momentum } \sim\theta)\end{aligned}$$

and perpendicular to $\mathbf{B}$

$$\begin{aligned}\mathbf{j}_\perp &= \frac{1}{B}\widehat{\mathbf{n}}\times\left[\frac{\partial}{\partial t}\left(nM\ \mathbf{u}\right)+\nabla\cdot\left(nM\ \mathbf{u}\ \mathbf{u}\right)\right. \\ &\quad +T\boldsymbol{\nabla}n+\boldsymbol{\nabla}\cdot\boldsymbol{\pi} \\ &\quad \left.-\mathbf{P}\left(\mathbf{x},t\right)\right]\end{aligned}$$

which is used further with the conservation of charge

$$\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{j_\parallel}{B}\right)=-\boldsymbol{\nabla}\cdot\mathbf{j}_\perp$$

The source is s

$$\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{\mathbf{B}\cdot\mathbf{u}}{B^2}\right)=-\boldsymbol{\nabla}\cdot\mathbf{u}_\perp\ -\frac{1}{n}\frac{\partial n}{\partial t}+\frac{1}{n}S$$

where

$$\mathbf{u}_\perp=\frac{\mathbf{E}\times\mathbf{B}}{B^2}$$

Now taking

$$S=\langle S\rangle+\widetilde{S}$$

we have

$$\frac{\partial n}{\partial t}=\langle S\rangle$$

and

$$\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{\mathbf{B}\cdot\mathbf{u}}{B^2}\right)=-\boldsymbol{\nabla}\cdot\mathbf{u}_\perp+\frac{1}{n}\widetilde{S}$$

The definition of K

$$\begin{aligned}\frac{\mathbf{u}\cdot\mathbf{B}_\theta}{B_\theta^2} &= \overline{K}+\widetilde{K}\\ \left\langle\widetilde{K}\right\rangle &= 0\end{aligned}$$

It results

$$\begin{aligned}\overline{K} &= \frac{1}{W^2}\,\frac{1}{\langle B\rangle^2}\left[\langle\mathbf{B}\cdot\mathbf{u}\rangle-I\frac{\left\langle R^2\boldsymbol{\nabla}\varphi\cdot\mathbf{u}\right\rangle}{\langle R^2\rangle}-\left\langle B^2\ \widetilde{K}\right\rangle\right]\\ &= \left\langle\frac{\mathbf{u}\cdot\mathbf{B}_\theta}{B_\theta^2}\right\rangle\end{aligned}$$

with

$$W^2\equiv 1-I^2\frac{1}{\langle R^2\rangle\langle B^2\rangle}$$

Since $\mathbf{u}_\perp$ is the electric velocity, we replace

$$-\boldsymbol{\nabla}\cdot\mathbf{u}_\perp=\mathbf{B}\cdot\boldsymbol{\nabla}\left(I\ \frac{1}{B^2}\frac{\partial\phi}{\partial\psi}\right)$$

and returning to the equation of continuity

$$\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{\mathbf{B}\cdot\mathbf{u}}{B^2}\right)=\mathbf{B}\cdot\boldsymbol{\nabla}\left(I\ \frac{1}{B^2}\frac{\partial\phi}{\partial\psi}\right)+\frac{1}{n}\widetilde{S}$$

which can be integrated

$$\mathbf{B}\cdot\mathbf{u}=I\frac{\partial\phi}{\partial\psi}+B^2\left(\overline{K}+\widetilde{K}\right)$$

where $\overline{K}$ is a constant of integration and

$$\mathbf{B}\cdot\boldsymbol{\nabla}\ \widetilde{K}=\frac{1}{n}\widetilde{S}$$

or $\nabla_{\parallel}\widetilde{K}=\frac{1}{nB}\widetilde{S}$ is the poloidal variation of the poloidal flow, due to the asymmetric sources.

The time evolution of the average poloidal velocity

$$\frac{\partial}{\partial t}\overline{K}=-\gamma_{DD}\ \overline{K}\ +S_K$$

here is the local drive-damping

$$\begin{aligned}\gamma_{DD} &= \frac{\langle S\rangle}{n}+\frac{1}{W^2}\frac{\langle S\ \delta_B\rangle}{n}\\ &\quad+\eta_0\frac{1}{nM}\frac{1}{W^2}\frac{\left\langle\nabla_{\parallel}B\right\rangle}{\langle B^2\rangle}\end{aligned}$$

where

$$\delta_B=\frac{B^2-\left\langle B^2\right\rangle}{\langle B^2\rangle}$$

and similar

$$\delta_R=\frac{R^2-\left\langle R^2\right\rangle}{\langle R^2\rangle}$$

The term

$$\langle S\delta_B\rangle$$

is the source-driven destabilizing term for Stringer spin-up.

S_K is the local source of poloidal rotation.

the magnetic punping damping factor

$$\gamma_{MP}=\eta_0\frac{1}{nM}\frac{1}{W^2}\frac{\left\langle\left(\nabla_{\parallel}B\right)^2\right\rangle}{\langle B^2\rangle}$$

There is a drive of the poloidal rotation, given by the input of momentum which is poloidally nonuniform (in-out asymmetry)

$$S_K=\frac{1}{nM}\frac{1}{W^2}\frac{1}{\langle B^2\rangle}\left[\langle\mathbf{B}_\theta\cdot\mathbf{P}\rangle-\langle\delta_R\ I\ \boldsymbol{\nabla}\varphi\cdot\mathbf{P}\rangle\right]$$

The drive of the poloidal rotation by an input of momentum with *in-out* asymmetry, $\mathbf{P}$, is caused by the Stringer mechanism.

5.2 Detailed treatment

The particularity of this treatment.

It defines and study a point of equilibrium for the rotation of plasma.

Then considers the perturbation of this equilibrium to show that there is spontaneous spin-up.

The equations

$$\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{nv_{\parallel}}{B}\right)=-\boldsymbol{\nabla}\cdot\left(n\mathbf{v}_{\perp}\right)-\frac{\partial n}{\partial t}$$

This arises from

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\left(n\mathbf{v}\right) &= \boldsymbol{\nabla}\cdot\left[n\left(\mathbf{v}_{\parallel}+\mathbf{v}_{\perp}\right)\right]=\boldsymbol{\nabla}\cdot\left(nv_{\parallel}\frac{\mathbf{B}}{B}+n\mathbf{v}_{\perp}\right)\\
&= \left[\boldsymbol{\nabla}\left(\frac{nv_{\parallel}}{B}\right)\cdot\mathbf{B}+\frac{nv_{\parallel}}{B}\boldsymbol{\nabla}\cdot\mathbf{B}\right]+\boldsymbol{\nabla}\cdot\left(n\mathbf{v}_{\perp}\right)\\
&= \mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{nv_{\parallel}}{B}\right)+\boldsymbol{\nabla}\cdot\left(n\mathbf{v}_{\perp}\right)
\end{aligned}$$

The equation of continuity

$$\begin{aligned}
\frac{\partial n}{\partial t}+\boldsymbol{\nabla}\cdot\left(n\mathbf{v}\right) &= 0\\
\frac{\partial n}{\partial t}+\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{nv_{\parallel}}{B}\right)+\boldsymbol{\nabla}\cdot\left(n\mathbf{v}_{\perp}\right) &= 0
\end{aligned}$$

As it is written it shows that we will calculate the parallel gradient of the parallel velocity.

The second equation

$$\begin{aligned}
T\mathbf{B}\cdot\boldsymbol{\nabla}n &= -nm_i\mathbf{B}\cdot\mathbf{v}:\boldsymbol{\nabla}\mathbf{v}\\
-\mathbf{B}\boldsymbol{\nabla}:\boldsymbol{\Pi} & \\
& -nm_i\frac{\partial\left(\mathbf{B}\cdot\mathbf{v}\right)}{\partial t}
\end{aligned}$$

We recognize here the momentum equation

$$nm_i\frac{\partial\mathbf{v}}{\partial t}+nm_i\left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{v}=-\boldsymbol{\nabla}\left(nT\right)-\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}$$

considering T constant, multiply by $\mathbf{B}$. This corresponds to the equation for the *circulation* $\mathbf{v}\cdot\mathbf{B}$.

The equation of Ohm, in the absence of resistivity

$$\eta\mathbf{j}=-\boldsymbol{\nabla}\phi+\mathbf{v}\times\mathbf{B}$$

$\eta=0$, multiplied by $\mathbf{B}$ is

$$\mathbf{B}\cdot\boldsymbol{\nabla}\phi=0$$

the potential is constant on magnetic surfaces.

$$\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{j_{\parallel}}{B}\right)=-\boldsymbol{\nabla}_{\perp}\cdot\mathbf{j}_{\perp}$$

The perpendicular current is extracted from the equation of momentum. For this, in contrast to previous multiplication by $\mathbf{B}$ we multiply vectorially by $\mathbf{B}$

$$\mathbf{j}_{\perp}=\frac{1}{B^2}\mathbf{B}\times\left(T\boldsymbol{\nabla}n+\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}+nm_i\frac{d\mathbf{v}}{dt}\right)$$

This is essentially the *diamagnetic current.*

the equilibrium is defined by the functions that are *flux-functions*

$$\begin{aligned} & & n\left(r\right) \\ V_p\left(r\right) &=& \left\langle v_{\theta}h\right\rangle \\ V_t &=& \left\langle v_{\varphi}h\right\rangle \end{aligned}$$

The average over the flux surface is

$$\left\langle f\right\rangle =\oint\frac{d\theta}{2\pi}\ h\ f$$

more detail

$$\left\langle f\right\rangle =\frac{\int\frac{dS}{\left|\boldsymbol{\nabla}\psi\right|}\ f}{\int\frac{dS}{\left|\boldsymbol{\nabla}\psi\right|}}$$

Using this surface average operator on the equation of current conservation $\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{j_{\parallel}}{B}\right)=-\boldsymbol{\nabla}_{\perp}\cdot\mathbf{j}_{\perp}$ one obtains

$$\int\mathbf{dS}\cdot\mathbf{j}_{\perp}=0$$

The exclusion of any flow of charges through the magnetic surface is the constraint that allows to derive the Stringer effect and others.

the equilibrium state means

$$\begin{aligned} \frac{\partial}{\partial t} &\equiv& 0 \\ \boldsymbol{\Pi} &=& 0 \ \ \text{pressure is isotropic} \\ \mathbf{R}_{\perp} &=& 0 \ \ \text{no friction} \end{aligned}$$

the equations under this equilibrium assumption lead to

$$v_{\theta}=\frac{V_p\left(r\right)}{h}$$

$$
\begin{aligned}
v_\varphi &\approx V_t - 2qV_p\cos\theta \\
&+\varepsilon\left[V_t\cos\theta + 2qV_p\left(1+\frac{1}{4}\cos 2\theta\right)\right]
\end{aligned}
$$

The first equation says that the poloidal rotation is the rotation uniform on surface V_p modulated by

$$h = 1+\varepsilon\cos\theta$$

Then

$$\langle nRv_\varphi\rangle = nR_0V_t$$

$$\left\langle v_\parallel\frac{B}{B_0}\right\rangle = V_t + \frac{\varepsilon}{q}\left(1+2q^2\right)V_p$$

Note the poloidal velocity V_p appears with the "inertial factor" $\left(1+2q^2\right)$. and

$$
\begin{aligned}
\frac{\varepsilon}{q} &= \frac{r}{R}\frac{RB_\theta}{rB_T} = \frac{B_\theta}{B_T} \equiv \Theta \ll 1 \\
&\quad \text{the factor of projection of } V_\theta \text{ on parallel}
\end{aligned}
$$

The equations

$$\frac{\partial n}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(rn\overline{v}_r\right) = 0$$

$$\frac{\partial}{\partial t}\left[nV_t\right] + \frac{1}{r}\frac{\partial}{\partial r}\left(r\ n\left[V_t\overline{v}_r - qV_p\widetilde{v}_r\right]\right) = 0$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left[V_t + \frac{\varepsilon}{q}\left(1+1q^2\right)V_p\right] \\
&+\overline{v}_r\frac{\partial V_t}{\partial r} - \widetilde{v}_r\frac{\partial}{\partial r}\left[qV_p\right] \\
&+\ \text{(magnetic pumping)} \\
=&\ \ 0
\end{aligned}
$$

The velocity is associated to a flux of transport of particles, across the surfaces (in radial direction). The flux is generated by the collisional friction that acts perpendicular to the magnetic field line

$$nv_r = \frac{R_\perp}{|e|\,B}$$

Then the average and the variable part

$$\overline{v}_r = \langle v_r\rangle$$

and

$$\widetilde{v}_r = \langle 2\cos\theta\ v_r\rangle$$

The origin of the poloidal spin-up : *the existence of the variation of the flux of transport with poloidal angle*

Equivalently,

$$\widetilde{v}_r \neq 0$$

The equation

$$\begin{aligned} & \frac{\varepsilon}{q}\left(1+2q^2\right)\left(\frac{\partial V_p}{\partial t}+\gamma_{MP}V_p\right) \\ & +qV_p\frac{\partial}{\partial r}\left(r\ n\widetilde{v}_r\right) \\ = \ & 0 \end{aligned}$$

The logic of the instability that consists of poloidal spin-up

Assume there is a poloidal velocity.

Due to the toroidality and

$$\boldsymbol{\nabla}\cdot\mathbf{v}=0$$

the poloidal rotation (with compression - distension of volume alternatively in low-field and high-field sides) necessarily is accompanied by toroidal flows that ensure the preservation of the incompressibility.

the toroidal flows have a spatial distribution which is harmonic in the poloidal section. It is Pfirsch Schluter flow and current.

The friction $R_{\perp}$ is modulated in the surface by these flows.

The friction generates transport fluxes Γ_r which are themselves modulated in the surface but for reasons that are independent of the Pfirsch-Schluter harmonic flows. The radial velocity they induce is also modulated, it is

$$\begin{aligned} \Gamma_r &= nv_r \\ v_r &= -D\left(1+\delta\cos\theta\right) \\ & \quad -v_0\frac{r}{a} \end{aligned}$$

From the combination between the two independent poloidal modulations

$$\begin{aligned} \Gamma_r &\sim f\left(\theta\right) \\ \text{Pfrirsch-Schluter flow} &\sim g\left(\theta\right) \end{aligned}$$

it is induced a variation of the radial velocity

$$\widetilde{v}_r=\left\langle 2\cos\theta\ v_r\right\rangle$$

This combination acts like a drive (a torque) in the equation for the poloidal velocity V_p (function of surface ψ).

The higher the angular matching between the poloidal variation of transport rate $\Gamma_r\left(\theta\right)$ with the harmonic Pfirsch Schluter flow $\cos\theta$, the higher the drive of poloidal rotation.

If the poloidal rotation is enhanced by this drive $qV_p\frac{\partial}{\partial r}\left(r\ n\widetilde{v}_r\right)$ then the amplitude of the harmonic compensatory Pfirsch Schluter flows increases

then the poloidal drive is still higher.

5.3 The spontaneous poloidal spin-up due to poloidal asymmetry of particle fluxes (Hassam Antonsen preprint)

The equations for the *equilibrium state.*

The continuity ignores the time variation of the density $\partial n/\partial t$.

$$\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{nv_{\|}}{B}\right)=-\boldsymbol{\nabla}\cdot(n\mathbf{v}_{\perp}) \tag{1}$$

The plasma momentum conservation projected along $\mathbf{B}$

$$\mathbf{B}\ \mathbf{u}:\boldsymbol{\nabla}\mathbf{u}=-\mathbf{B}\cdot\boldsymbol{\nabla}\ln n \tag{2}$$

(one still needs a temperature in the pressure term). The static convection term $(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}$ balances the variation of the density (pressure) along the magnetic field line.

Ohm's law

$$\mathbf{B}\cdot\boldsymbol{\nabla}\phi=0 \tag{3}$$

Current conservation

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{j} &= 0 \\ \mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{j_{\|}}{B}\right) &= -\boldsymbol{\nabla}\cdot\mathbf{j}_{\perp}\end{aligned} \tag{4}$$

At equilibrium

$$\boldsymbol{\nabla}\cdot(n\mathbf{v}_{\perp})=-\mathbf{B}\cdot\boldsymbol{\nabla}\left(I\frac{n}{B^2}\frac{d\phi}{d\psi}\right) \tag{5}$$

and

$$\mathbf{B}\ \mathbf{u}:\boldsymbol{\nabla}\mathbf{u}=\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{u_{\|}^2}{2}-\frac{|\mathbf{u}_{\perp}|^2}{2}-I\frac{u_{\|}}{B}\frac{d\phi}{d\psi}\right) \tag{6}$$

Now we have expression for the divergence of the perpendicular flux

$$\begin{aligned}\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{nv_{\|}}{B}\right) &= -\boldsymbol{\nabla}\cdot(n\mathbf{v}_{\perp}) \\ &= \mathbf{B}\cdot\boldsymbol{\nabla}\left(I\frac{n}{B^2}\frac{d\phi}{d\psi}\right)\end{aligned}$$

and we know that the operator is

$$\mathbf{B}\cdot\boldsymbol{\nabla}=B\nabla_{\|}=B\frac{1}{qR}\frac{\partial}{\partial\theta}$$

and since this operator is the same in the left and in the right side the equation can be integrated with introduction of a function that does not depend on θ,

$$\frac{nv_{\|}}{B}=\frac{In}{B^2}\frac{d\phi}{d\psi}+f(r)$$

6 Hassam Drake 1993. Spontaneous spin-up

See **Hassam Antonsen**.

6.1 The equation of continuity

The continuity

$$\frac{\partial n}{\partial t} + \nabla\cdot\left(n\mathbf{u}_{\perp}\right) + \mathbf{B}\cdot\nabla\left(\frac{nu_{\parallel}}{B}\right) = S - \frac{1}{r}\frac{\partial}{\partial r}\left(r\Gamma_r\right)$$

This equation is important for the derivation of the expression for the Pfirsch Schluter current.

NOTE
that here too we have that separation, for example

$$\nabla\cdot\mathbf{j} = \nabla\cdot\mathbf{j}_{\perp} + \mathbf{B}\cdot\nabla\left(\frac{j_{\parallel}}{B}\right)$$

END

This is because it introduces the *divergence of the flux of particles*, of the flow. This is where the *geometrical* poloidal compression and dilation will enter the dynamics. In the term $\nabla\cdot\left[\widehat{\mathbf{e}}_{\theta}\left(1+\varepsilon\cos\theta\right)\right]$.

The *momentum for all plasma* (the mass is taken m_i), isothermal

$$nm_i\left[\frac{\partial\mathbf{u}}{\partial t} + \left(\mathbf{u}\cdot\nabla\right)\mathbf{u}\right] = -T\nabla n + \mathbf{j}\times\mathbf{B} - m_i S\mathbf{u}$$

The *current conservation* is essential in connecting the perpendicular current (diamagnetic) with the parallel current (Pfirsch Schluter)

$$\nabla\cdot\mathbf{j} = 0$$

The *Ohm's law*. Here, without the resistivity. This means that the radial velocity v_r will be attributed to another reason for which the charge neutrality cannot be fully suppressed by the parallel currents. The reason may be the *Landau damping* which acts when the collisionality is low. This appears in **Stringer** where a kinetic treatment allows to calculate the variation of the density and of the potential on the magnetic surface, by integrating over the velocity space the distribution functions of electron and ions and imposing neutrality. During the integration, one has to traverse the singularity $v_{\parallel} - \frac{\varepsilon}{q}v_{\theta}^{E} = 0$.

$$-\nabla\phi + \mathbf{u}\times\mathbf{B} = 0$$

The magnetic field is

$$\mathbf{B} = \boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\varphi + I(\psi)\,\boldsymbol{\nabla}\varphi$$

Relative to the work **Hassam Kulsrud** here it is assumed that the electrons and ions are *isothermal*.

$$S(r,\theta) \equiv \text{particle source}$$

It is interesting to note how the *source* extracts from the momentum a part which is proportional with $m_i\mathbf{u}$ through S.

The radial flux

$$\begin{aligned}
\Gamma_r &= \langle\langle \widetilde{n}\widetilde{v}_r \rangle\rangle \\
&= -D(r,\theta)\frac{\partial n}{\partial r}
\end{aligned}$$

is the source of poloidal non-uniformity which will produce torque.

An object of study is the *circulation*.

This is obtained taking the projection in the *parallel* direction of the equation of momentum conservation. It is interesting that the variation of the density in the parallel direction (for isothermal plasma) gives the pressure that opposes to the geometrical advection of the flow, $\mathbf{B}\cdot(\mathbf{u}\cdot\boldsymbol{\nabla})\,\mathbf{u}$, which can be static. The imbalance gives $\frac{\partial}{\partial t}(\mathbf{u}\cdot\mathbf{B})$.

$$\begin{aligned}
\frac{\partial}{\partial t}(\mathbf{u}\cdot\mathbf{B}) + \mathbf{B}\cdot(\mathbf{u}\cdot\boldsymbol{\nabla})\,\mathbf{u} &= -c_s^2\mathbf{B}\cdot\boldsymbol{\nabla}\ln n \quad \text{(parallel pressure)} \\
&\quad -\frac{S}{n}\mathbf{u}\cdot\mathbf{B} \quad \text{(external source of momentum)}
\end{aligned}$$

6.2 Poloidal projection of the momentum equation

The poloidal projection of the equation for the plasma momentum

$$\mathbf{B}_{pol}\cdot\left(nm_i\frac{d\mathbf{u}}{dt} = -\boldsymbol{\nabla}p + \mathbf{j}\times\mathbf{B}\right)$$

for the last term in the RHS

$$\begin{aligned}
\mathbf{B}_{pol}\cdot(\mathbf{j}\times\mathbf{B}) &= (\mathbf{B}_{pol}\times\mathbf{j})\cdot\mathbf{B} \\
&= [(\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\varphi)\times\mathbf{j}]\cdot\mathbf{B} \\
&= -[\boldsymbol{\nabla}\psi(\mathbf{j}\cdot\boldsymbol{\nabla}\varphi) - \boldsymbol{\nabla}\varphi(\mathbf{j}\cdot\boldsymbol{\nabla}\psi)]\cdot\mathbf{B}
\end{aligned}$$

The first scalar product $\boldsymbol{\nabla}\psi\cdot\mathbf{B}(\mathbf{j}\cdot\boldsymbol{\nabla}\varphi) = 0$ since $\boldsymbol{\nabla}\psi$ is perpendicular on surface which contain $\mathbf{B}$ line. The second term

$$\boldsymbol{\nabla}\varphi\cdot\mathbf{B}\ \ (\mathbf{j}\cdot\boldsymbol{\nabla}\psi) = \frac{B_\varphi}{R}(\mathbf{j}\cdot\boldsymbol{\nabla}\psi)$$

Then

$$\mathbf{B}_{pol} \cdot \left(n m_i \frac{d\mathbf{u}}{dt} + T \boldsymbol{\nabla} n \right) = \frac{B_\varphi}{R} \mathbf{j} \cdot \boldsymbol{\nabla} \psi \quad \text{(radial current)}$$

We note that

$$\mathbf{B}_{pol} = \boldsymbol{\nabla} \psi \times \boldsymbol{\nabla} \varphi$$

and the product $\mathbf{j} \cdot \boldsymbol{\nabla} \psi$ extracts the radial current.

We remember that

$$\begin{aligned} |\boldsymbol{\nabla} \psi| &= 2\pi R B_p \\ \text{and } \boldsymbol{\nabla} \psi &\sim \widehat{\mathbf{e}}_r \end{aligned}$$

Then this equation gives an expression for a *radial component of the electric current*, j_r.

$$\mathbf{j} \cdot \boldsymbol{\nabla} \psi \sim R B_p \; j_r$$

We see that there is a radial electric current $j_r(r,\theta)$, if the poloidal projection of the expression

$$n m_i \frac{d\mathbf{u}}{dt} + T \boldsymbol{\nabla} n$$

is not zero. This expression is the time variation of the plasma velocity corrected with the gradient of the pressure. If the time variation of the velocity is NOT entirely driven by pressure gradient then there is a radial current. However we note that to the lowest order the gradient of the pressure is radial and it has zero poloidal projection. This term will be significant only to first order in ε, *i.e.* with variation of the density on the surface. We recognize the setup of the Pfisrch Schluter current: it relies on variation in the magnetic surface. Like there, there is a *radial flow* v_r or *radial current* j_r.

In the case of PS current we have found a non-zero radial flow v_r from the term $\mathbf{v} \times \mathbf{B}$ in the Ohm's law, since this was the connection with the parallel electric field $E_\|$ which exists due to the difference of velocities of electrons and ions. The direct connection was: the electric field cannot suppress the difference between the flows of electrons and ions because there is a finite resistivity η. Then the Ohm's law indicates the necessity of the term $\mathbf{v} \times \mathbf{B}$ with parallel projection, implicitely the necessity of v_r.

Here the radial current appears in the force $\mathbf{j} \times \mathbf{B}$ which is needed to compensate the un-balance between inertia $d\mathbf{u}/dt$ and the pressure. The latter are due to variations in the magnetic surfaces.

The equation will further provide the radial current

$$\frac{R}{B_\varphi} \; \mathbf{B}_{pol} \cdot \left[n m_i \frac{d\mathbf{u}}{dt} + T \boldsymbol{\nabla} n \right] = \mathbf{j} \cdot \boldsymbol{\nabla} \psi$$

6.3 The constraint of zero total current through a surface

Now comes the constraint that will provide the third equation: the total radial current traversing a magnetic surface must be zero. Integrated over a magnetic surface the zero-divergence of the current density leads to

$$\begin{aligned}\int_{flux\_surf} \mathbf{dS}\cdot\mathbf{j} &= \int d\,(area) \quad \mathbf{j}\cdot\widehat{\mathbf{e}}_r \\ &= \int d\,(area) \quad \mathbf{j}\cdot\frac{\boldsymbol{\nabla}\psi}{|\boldsymbol{\nabla}\psi|} = 0\end{aligned}$$

where $\mathbf{dS} = ds\ \widehat{\mathbf{e}}_r = d\,(area)\ \ \widehat{\mathbf{e}}_r = 2\pi R r d\theta\ \widehat{\mathbf{e}}_r$. From the previous equation we have

$$\frac{R}{B_\varphi}\ \mathbf{B}_{pol}\cdot\left[nm_i\frac{d\mathbf{u}}{dt}+T\boldsymbol{\nabla}n\right]=\mathbf{j}\cdot\boldsymbol{\nabla}\psi$$

We remark that

$$B_\varphi=\frac{B_0}{h}=\frac{B_0}{R/R_0}$$

then

$$\frac{R}{B_\varphi}=\frac{1}{R_0B_0}R^2$$

and now we divide the equation by $|\boldsymbol{\nabla}\psi|$ and integrate over surface, with ds the scalar area element

$$\begin{aligned}0 &= \int dS \quad \mathbf{j}\cdot\widehat{\mathbf{e}}_\psi \\ &= \int dS \quad \mathbf{j}\cdot\frac{\boldsymbol{\nabla}\psi}{|\boldsymbol{\nabla}\psi|} \\ &= \int_{flux\_surf} dS\ \frac{1}{|\boldsymbol{\nabla}\psi|}\frac{1}{R_0B_0}R^2\mathbf{B}_{pol}\cdot\left(nm_i\frac{d\mathbf{u}}{dt}+T\boldsymbol{\nabla}n\right)\end{aligned}$$

The last replacements uses the expression of the radial current

$$\frac{R}{B_\varphi}\ \mathbf{B}_{pol}\cdot\left[nm_i\frac{d\mathbf{u}}{dt}+T\boldsymbol{\nabla}n\right]=\mathbf{j}\cdot\boldsymbol{\nabla}\psi$$

denote $r\sim\psi$ as directions

$$\begin{aligned}j_r\ RB_\theta &= \frac{R}{B_\varphi}\ B_\theta\left[nm_i\frac{d\mathbf{u}}{dt}+T\boldsymbol{\nabla}n\right]\cdot\widehat{\mathbf{e}}_\theta \\ j_r\ B_\varphi &= \left[nm_i\frac{d\mathbf{u}}{dt}+T\boldsymbol{\nabla}n\right]_\theta\end{aligned}$$

the last equation show that the $j\times B$ torque (time variation of momentum) is *poloidal*.

By the RHS in the problem enter

$$T\boldsymbol{\nabla} n|_\theta = T\frac{\partial n}{r\partial\theta} \quad \text{poloidal variation of density}$$

$$nm_i \left.\frac{d\mathbf{u}}{dt}\right|_\theta$$

The $\langle j_r \rangle = 0$ equation, derived from current conservation $\boldsymbol{\nabla}\cdot\mathbf{j} = 0$ will be used to derive the *time variation of the poloidal velocity.*

The two terms in the integrand will be treated separately.

The first will be reduced to acceleration $\partial V_E/\partial t$. This is time dynamics of the rotation, i.e. spin-up.

The second will provide the *drive* for the acceleration, with a structure which is particular to the Stringer effect.

This equation expresses the fact that the total current that goes transversarly through a magnetic surface is zero. It also means that the integral through the surface of the inertial term minus the pressure term is zero. However the factors are essential for the integration.

6.4 Ordering in ε

It is assumed first that the plasma velocity is smaller than the sound velocity.

6.4.1 Zero order in ε

The equilibrium, in zero order, is the equilibrium without variations in the magnetic surface.

$$\frac{\partial n_0}{\partial t} = 0$$

the equilibrium density is constant in time

$$\mathbf{u}_{\perp 0}\cdot\boldsymbol{\nabla} n_0 = 0$$

The perpendicular advection of the equilibrium density is zero: the density does not vary along *perpendicular* direction.

$$\begin{aligned} 0 &= T\mathbf{B}_0\cdot\boldsymbol{\nabla} n_0 \\ \nabla_\parallel n_0 &= 0 \end{aligned}$$

The equilibrium density does not vary parallel with the magnetic field line

$$\oint d\theta\ \frac{1}{r}\frac{\partial n_0}{\partial\theta} = 0$$

If there is a poloidal variation of the density along the poloidal direction, the periodicity must be taken into account.

$$0 = -\boldsymbol{\nabla}\phi_0 + \mathbf{u}_0 \times \mathbf{B}_0$$

The Ohm's law without *resistivity*.

This means that the lowest order density is constant on the surfaces

$$n_0(r)$$

and the velocity which is perpendicular on the magnetic field is contained in the magnetic surface. It is the electric velocity

$$\begin{aligned}\mathbf{u}_{\perp 0} &= V_E \widehat{\mathbf{e}}_\theta \\ &= \frac{1}{B}\frac{d\phi_0}{dr}\widehat{\mathbf{e}}_\theta\end{aligned}$$

This velocity V_E is poloidal.

6.4.2 Order 1 in ε

the first order in ε will reveal the presence of the variation of the density on the magnetic surface, n_1.

$$n = n_0 + n_1$$

Also we will have to work with the parallel velocity $u_\parallel$.

$$\begin{aligned}& \frac{\partial n_1}{\partial t} + V_E \frac{\partial n_1}{r\partial\theta} + n_0 V_E \left(-2\varepsilon\frac{\sin\theta}{r}\right) \\ & + n_0 \nabla_\parallel u_\parallel \\ = \;& S - \frac{1}{r}\frac{\partial}{\partial r}\left(r\Gamma_r\right)\end{aligned}$$

The second and the third term come from

$$\begin{aligned}\boldsymbol{\nabla}\cdot(n\mathbf{v}) &\rightarrow V_\theta \frac{\partial\left(n_0 + n_1(\theta)\right)}{r\partial\theta} + n_0 \boldsymbol{\nabla}\cdot\left(\frac{-\boldsymbol{\nabla}\left(\phi_0 + \widetilde{\phi}(\theta)\right)\times\widehat{\mathbf{n}}}{B}\right) \\ &= V_\theta \frac{\partial n_1}{r\partial\theta} + n_0 \boldsymbol{\nabla}\cdot\left[-\frac{d\phi_0}{dr}\widehat{\mathbf{e}}_r \times \widehat{\mathbf{n}}\;\frac{1}{B}\right] \\ &\approx V_\theta \frac{\partial n_1}{r\partial\theta} + n_0\left(-\frac{d\phi_0}{dr}\right)\;\boldsymbol{\nabla}\cdot\left[\frac{\widehat{\mathbf{e}}_\theta\left(1+\varepsilon\cos\theta\right)}{B_0}\right]\end{aligned}$$

(the divergence of the current in the equation of continuity).

To understand the third term we should remember

$$\begin{aligned}\boldsymbol{\nabla}\cdot\left[\widehat{\mathbf{e}}_\theta\left(1+\varepsilon\cos\theta\right)\right] &= \frac{1}{r\left(R_0 + r\cos\theta\right)}\frac{\partial}{\partial\theta}\left(\left(R_0 + r\cos\theta\right)\left(1+\varepsilon\cos\theta\right)\right) \\ &= \varepsilon\frac{\left(-2\sin\theta\right)}{r}\end{aligned}$$

The factor $h = 1 + \varepsilon \cos\theta$ comes from the magnitude of the magnetic field $B = B_0/h$. The divergence is calculated for the poloidal flow resulting from the electric velocity V_E that carries the density $n_0 + n_1$. Both quantities do not have variation in this order but the *geometry* is essential.

Termenul $\nabla_\parallel u_\parallel$

The parallel momentum

$$n_0 m_i \left(\frac{\partial u_\parallel}{\partial t} + V_E \frac{\partial u_\parallel}{r \partial \theta} \right) = -T \nabla_\parallel n_1$$

Note that it is here that the *parallel viscosity* $\mathbf{\Pi}$ should appear to introduce the *magnetic damping*. Shaing, etc. **End.**

The parallel gradient is

$$\nabla_\parallel = \frac{1}{qR} \frac{\partial}{\partial \theta}$$

Finally the condition

$$\oint d\theta \; B_{\theta 0} \frac{\partial n_1}{r \partial \theta} = 0$$

Since $B_{\theta 0}$ is actually constant on the surface and is taken out the integral the condition is trivially satisfied in this order.

The condition satisfied trivially at the first order must be recalculated in higher order, *i.e.* two, ε^2.

The equation to be used is

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{j} &= 0 \\ \text{or, the integral form } \int_{flux\_surf} \mathbf{dS} \cdot \mathbf{j} &= \mathbf{0} \end{aligned}$$

$$\int_{flux\_surf} \frac{dS}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol} \cdot \left(n m_i \frac{d\mathbf{u}}{dt} + T \boldsymbol{\nabla} n \right) = 0$$

derived from the condition of zero-divergence of the current.

The part

$$\int_{flux\_surf} \frac{ds}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol} \cdot (T \boldsymbol{\nabla} n)$$

will be calculated as

$$\begin{aligned} R^2 &\approx \text{order 1 in } \varepsilon \\ \frac{dS}{|\boldsymbol{\nabla}\psi|} &\sim \text{order 0 in } \varepsilon \\ \mathbf{B}_{pol} \cdot \boldsymbol{\nabla} n &\sim \text{order 1 \ (intrinsic, since the variation on } \theta \text{ is } \varepsilon^1) \end{aligned}$$

An approximation

$$\mathbf{B}_{pol} \cdot \boldsymbol{\nabla} n \approx B_\theta \frac{\partial n_1}{r\partial\theta}$$

and

$$|\boldsymbol{\nabla}\psi| = 2\pi R B_\theta$$

In the product

$$\mathbf{B}_{pol} \cdot \left(n m_i \frac{d\mathbf{u}}{dt} \right)$$

we only retain

$$\mathbf{B}_{pol} \cdot \left(n m_i \frac{\partial\mathbf{u}}{\partial t} \right)$$

since $\mathbf{B}_{pol} \cdot (\mathbf{u} \cdot \boldsymbol{\nabla})\, \mathbf{u}$ is of higher order. This term is the time variation $\frac{\partial V_E}{\partial t}$ of the *poloidal velocity* $V_E(r,t)$.

We also have

$$dS = 2\pi R r d\theta$$

The integration of the first part The integration of the first part is

$$\begin{aligned} & \int_{flux\_surf} \frac{dS}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol} \cdot \left(n m_i \frac{d\mathbf{u}}{dt} \right) \\ \approx & \int_{flux\_surf} \frac{2\pi R r d\theta}{2\pi R B_\theta} B_\theta n m_i \frac{\partial V_E}{\partial t} \\ = & (2\pi)\, r n m_i \frac{\partial V_E}{\partial t} \end{aligned}$$

The integration of the second term The integration of the second term (variation of pressure $T n_1(\theta)$ on surface) is

$$\begin{aligned} = & T \int_{flux\_surf} \frac{dS}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol} \cdot \boldsymbol{\nabla} n_1(r,\theta) \\ = & T \int 2\pi r R d\theta\ \frac{1}{2\pi R B_{pol}} R^2 B_{pol} \widehat{\mathbf{e}}_\theta \cdot \boldsymbol{\nabla} n_1 \\ = & T\ r \int d\theta\ R^2 \frac{\partial n_1}{r\partial\theta} \end{aligned}$$

We make an integration by parts

$$\begin{aligned}
&= T\ r \int d\theta\ R^2 \frac{\partial n_1}{r\partial\theta} = T\left[\left(R^2 n_1\right)_{\theta=0}^{\theta=2\pi} - \int d\theta\ \frac{\partial R^2}{\partial\theta} n_1\right] \\
&= -TR_0^2 \int d\theta\ \frac{\partial}{\partial\theta}\left(1+2\varepsilon\cos\theta+\varepsilon^2\cos^2\theta\right)\ n_1 \\
&= -TR_0^2 \int d\theta\ \left(-2\varepsilon\sin\theta\right)\ n_1
\end{aligned}$$

We rewrite the two terms of the LHS of the equation

$$rnm_i R_0^2\ \frac{\partial V_E}{\partial t} \times (2\pi) + \left[-TR_0^2 \int d\theta\ \left(-2\varepsilon\sin\theta\right)\ n_1\right] = 0$$

or

$$\frac{\partial V_E}{\partial t} = -\frac{T}{m_i}\frac{1}{n_0}\frac{1}{r}2\varepsilon \int \frac{d\theta}{2\pi}\ \sin\theta\ n_1$$

This is the result for the constraint that the total current through the surface is zero.

$$\frac{\partial V_E}{\partial t} = -\frac{1}{r}\varepsilon\frac{c_s^2}{n_0}\int \frac{d\theta}{2\pi} 2\sin\theta\ n_1$$

NOTE

We can already see which is the drive of the acceleration of the poloidal velocity.

It is the integral over poloidal circumference (θ) of the product of two functions of θ

- one is $\sin\theta$. It comes from the poloidal projection of the momentum equation. It occurs as a geometric factor $\frac{R}{B_\varphi} \rightarrow R^2$ necessary to obtain the *radial* component of the current density. (The latter, integrated over surface would produce an important constraint, *ambipolarity*).
- the other is $n_1(\theta)$ and this can be *neoclassic* or can be anomalous.

END

NOTE

From the formula for $\frac{\partial V_E}{\partial t}$ we find that, if the poloidal variation of the density $n_1(0)$ is zero then $V_E = 0$. This will be taken as the situation of "*reference*".

END

6.5 Final set of equations

New notation

$$N \equiv \frac{n_1}{n_0}$$

the equations

$$\begin{aligned}
& \frac{\partial N}{\partial t} + V_E \frac{\partial N}{r\partial\theta} + V_E \left(-2\varepsilon \frac{\sin\theta}{r}\right) \\
& +\nabla_{\|} u_{\|} \\
= \; & \frac{S}{n_0} - \frac{1}{n_0}\frac{1}{r}\frac{\partial}{\partial r}\left(r\Gamma_r\right) \qquad \text{(continuity)}
\end{aligned}$$

$$\frac{\partial u_{\|}}{\partial t} + V_E \frac{\partial u_{\|}}{r\partial\theta} = -c_s^2 \nabla_{\|} N \qquad \text{(parallel momentum)}$$

$$\frac{\partial V_E}{\partial t} = c_s^2 \oint \frac{d\theta}{2\pi} N \left(-2\varepsilon \frac{\sin\theta}{r}\right) \quad \text{(zero current through surface)}$$

NOTE

Let us stop to make a comparison between this (**Hassam Drake**) system prepared for the spontaneous spin-up and the **Stringer PRL** system.

We note that the *time variation* in the equation for

- the density, $\partial n_1/\partial t$, and
- the velocity

$$n m_i V_{E\theta}^{(0)} \frac{\partial v_{i\|}}{r\partial\theta} = -\left(T_e + T_i\right)\frac{\varepsilon}{q}\frac{\partial n_1}{r\partial\theta}$$

  where the LHS is

$$\begin{aligned}
V_{E\theta}^{(0)} \frac{\partial v_{i\|}}{r\partial\theta} &= \left(\mathbf{V}_E \cdot \boldsymbol{\nabla}_\theta\right) v_{i\|} \quad \left(\text{where } V_{E\theta}^{(0)} \text{ is only on } \theta\right) \\
& \quad \text{poloidal convection of the parallel velocity}
\end{aligned}$$

  and the RHS is

$$\begin{aligned}
-\left(T_e + T_i\right)\frac{\varepsilon}{q}\frac{\partial n_1}{r\partial\theta} &= \frac{B_\theta}{B_T} \times \left(-T\nabla_\theta n_1\right) \\
& \quad \text{parallel projection (i.e. } B_\theta/B_T \times \text{)} \\
& \quad \text{of the poloidal gradient of pressure}
\end{aligned}$$

  this equation shows the balance of momentum carried by the "statically advected" velocity (*i.e.* space variation of the velocity, $(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}$) with the *pressure*. The projection is made on parallel direction.

is *absent* at **Stringer**.

Since Hassam Drake work for spin-up driven by external source (poloidally asymmetric) the explicit time variation must be retained.

The main term however in the formulation Hassam Drake is still $V_E \frac{\partial u_\parallel}{r\partial\theta}$ (later $\widehat{u}_E \frac{\partial \widetilde{u}_\parallel}{r\partial\theta}$) which is the same as in Stringer. This term will be the main part of the expansion around the equilibrium static state.

6.5.1 Separation of constant on surface $\overline{()}$ from nonuniform perturbations $\widetilde{()}$ on surface

The functions that must be determined $N(r,\theta,t)$, $u_\parallel(r,\theta,t)$, $V_E(r,t)$.

Each has a part that is constant n surface, $\overline{N}$, $\overline{u}_\parallel$ and $\overline{V}_E$, (which is taken as "*reference*") and a part that is variable with θ.

The global balance is obtained by integrating over the surface $\int \frac{d\theta}{2\pi}(...)$.

$$\frac{\partial \overline{N}}{\partial t} = \frac{\overline{S}}{n_0} - \frac{1}{n_0}\frac{\partial}{r\partial\theta}\left(r\overline{\Gamma}_r\right) \quad \text{and} \quad \frac{\partial \overline{u}_\parallel}{\partial t} = 0$$

After introducing the average over surfaces, the new variables are the differences that have variations in the surfaces $\widetilde{f} = f - \overline{f}$. The source in the surface is

$$F \equiv \frac{S - \frac{1}{r}\frac{\partial}{\partial r}\left(r\Gamma_r\right)}{n_0}$$

6.5.2 The state taken as reference

The state adopted as reference will include both the parts constant on surface and the parts variable with θ, however different for the variables.

The state that is taken as reference is the *absence of the poloidal rotation*

$$V_E^{ref} = 0$$

and this reduces the equations to

$$\begin{aligned} N^{ref} &= \overline{N} \\ \text{i.e.} \quad \widetilde{N}^{ref} &= 0 \quad \text{(no variation of the density in the surface)} \end{aligned}$$

$$\nabla_\parallel \widetilde{u}_\parallel^{ref} = \widetilde{F}$$

$$\nabla_\parallel \widetilde{N}^{ref} = 0 \quad \text{from where } \widetilde{N} = \text{const} = 0$$

The variation of the parallel velocity along the magnetic line (equivalently, in the magnetic surface) is obtained in terms of the source

$$\overline{u}_\parallel = 0 \quad \text{(no 'uniform on } \theta \text{' parallel flow)}$$

At this point, the assumption are

$$V_E^{ref} = 0$$

$$\begin{aligned} N^{ref} &= \overline{N} + \widetilde{N}^{ref} \\ &= \overline{N} \end{aligned}$$

$$\begin{aligned} u_{\parallel}^{ref} &= \overline{u}_{\parallel} + \widetilde{u}_{\parallel}^{ref} \\ &= \widetilde{u}_{\parallel}^{ref} \end{aligned}$$

Then the equations for the state of reference will actually consist of equations for the variable ($\sim \theta$) part of the plasma variables.

They become

$$\begin{aligned} \nabla_{\parallel}\widetilde{u}_{\parallel}^{ref} &= \widetilde{F} \\ \text{or } \frac{1}{qR}\frac{\partial}{\partial\theta}\widetilde{u}_{\parallel}^{ref} &= \widetilde{F} \text{ which can be integrated} \\ \widetilde{u}_{\parallel}^{ref} &= qR\int d\theta' \; \widetilde{F} \end{aligned}$$

(remember $F \equiv \frac{S - \frac{1}{r}\frac{\partial}{\partial r}(r\Gamma_r)}{n_0}$).

If there is parallel flow that is variable on surface ($\widetilde{u}_{\parallel}^{ref} \sim \theta$) then this comes from the SOURCE and/or from radial fluxes that are variable on surface.

6.5.3 Perturbation of the *reference* state

In order to detect the possibility of an instability that can lead to acceleration of the poloidal rotation, we must expand around the *state of reference*, with a small perturbation.

The reference state consists of the variables

$$\begin{aligned} & V_E^{ref} \\ N^{ref} &= \overline{N} + \widetilde{N} \\ u_{\parallel}^{ref} &= \widetilde{u}_{\parallel}^{ref} \end{aligned}$$

and they verify the equations of reference state

$$\begin{aligned} V_E^{ref} &= 0 \\ \nabla_{\parallel}\widetilde{N} &= 0 \\ \nabla_{\parallel}\widetilde{u}_{\parallel} &= \widetilde{F} \end{aligned}$$

or

$$
\begin{aligned}
V_E^{ref} &= 0 \\
\widetilde{N} &= 0 \\
\widetilde{u}_{\parallel} &= qR \int^{\theta} d\theta' \ \widetilde{F}
\end{aligned}
$$

Now consider a perturbation of this reference state

$$
\begin{aligned}
V_E &= V_E^{ref} + \widehat{V}_E \\
u_{\parallel} &= \widetilde{u}_{\parallel}^{ref} + \widehat{u}_{\parallel} \\
N &= \overline{N} + \widetilde{N}^{ref} + \widehat{N}
\end{aligned}
$$

This will induce a time variation of the poloidal (electric) velocity and of the density N and of the parallel velocity.

However the time variation is assumed to be slower than the sound speed

$$
\frac{\partial}{\partial t} \ll \frac{c_s}{qR}
$$

The time variation for N is neglected and the equation for density becomes a balance

$$
\widehat{V}_E \left(-2\varepsilon \frac{\sin\theta}{r} \right) + \nabla_{\parallel} \widehat{u}_{\parallel} = 0
$$

and the following equations are used

$$
\begin{aligned}
\frac{\partial u_{\parallel}}{\partial t} + V_E \frac{\partial u_{\parallel}}{r\partial\theta} &= -c_s^2 \nabla_{\parallel} N \quad \text{(parallel momentum)} \\
\frac{\partial V_E}{\partial t} &= c_s^2 \oint \frac{d\theta}{2\pi} N \left(-2\varepsilon \frac{\sin\theta}{r} \right) \quad \text{(zero current through surface)}
\end{aligned}
$$

that are written in the form

$$
\begin{aligned}
\frac{\partial \widehat{u}_{\parallel}}{\partial t} + \widehat{V}_E \frac{\partial \widetilde{u}_{\parallel}^{ref}}{r\partial\theta} &= -c_s^2 \nabla_{\parallel} \widehat{N} \\
\frac{\partial \widehat{V}_E}{\partial t} &= c_s^2 \oint \frac{d\theta}{2\pi} \widehat{N} \left(-2\varepsilon \frac{\sin\theta}{r} \right)
\end{aligned}
$$

Note the preservation of the poloidal derivative of the *reference* parallel velocity in the second equation. This reference value of the paralel velocity is fixed by the radial flux and the source of particles. It exists only because these sources and fluxes are *NOT constant on the poloidal circumference.*

The equilibrium static state at Hassam Drake is

$$
\begin{aligned}
\nabla_{\parallel} \widetilde{u}_{\parallel} &= \widetilde{F} \\
\nabla_{\parallel} \widetilde{N} &= 0
\end{aligned}
$$

and represents the static state that has taken into consideration the *variations in the magnetic surface.*

END

This set of equations can be integrated.

The operator that must be made explicit is

$$\nabla_{\|} = \frac{1}{qR}\frac{\partial}{\partial\theta}$$

Then, since $\widehat{V}_E$ is constant on magnetic surfaces, the first equation is

$$\widehat{V}_E\left(-2\varepsilon\frac{\sin\theta}{r}\right) + \nabla_{\|}\widehat{u}_{\|} = 0$$

or

$$\widehat{V}_E\left(2\varepsilon\frac{\sin\theta}{r}\right) = \frac{1}{qR}\frac{\partial}{\partial\theta}\widehat{u}_{\|}$$

and after the integration over θ,

$$\widehat{u}_{\|} = -2qR\varepsilon\frac{\cos\theta}{r}\widehat{V}_E$$

This equations shows that a parallel velocity is determined from a poloidal rotation velocity, modulated by a factor $\sim\cos\theta$.

The expression of $\widehat{u}_{\|} = -2qR\varepsilon\frac{\cos\theta}{r}\widehat{V}_E$ is introduced in the second equation, $\frac{\partial\widehat{u}_{\|}}{\partial t} + \widehat{V}_E\frac{\partial\widetilde{u}_{\|}^{ref}}{r\partial\theta} = -c_s^2\nabla_{\|}\widehat{N}$,

$$\begin{aligned}\frac{\partial}{\partial t}\left[-2qR\varepsilon\frac{\cos\theta}{r}\widehat{V}_E\right] + \widehat{V}_E\frac{\partial\widetilde{u}_{\|}^{ref}}{r\partial\theta} &= -c_s^2\nabla_{\|}\widehat{N} \\ &= -c_s^2\frac{1}{qR}\frac{\partial\widehat{N}}{\partial\theta}\end{aligned}$$

and is integrated over the poloidal angle θ,

$$\begin{aligned}-c_s^2\widehat{N} &= -2\left(qR\right)^2\varepsilon\frac{1}{r}\frac{\partial\widehat{V}_E}{\partial t}\int^{\theta}d\theta'\cos\theta' \\ &\quad +\widehat{V}_E\frac{qR}{r}\widetilde{u}_{\|}^{ref}\end{aligned}$$

we ignore for the moment the constant of integration which should be a function of surface.

We have obtained an expression for the "perturbation of the density" $\widehat{N}(r,\theta)$.

This $\widehat{N}(r,\theta)$ is introduced in the equation for the time variation of $\widehat{V}_E$, (which is $\langle j_r \rangle = 0$), the third equation

$$\begin{aligned}
\frac{\partial \widehat{V}_E}{\partial t} &= c_s^2 \oint \frac{d\theta}{2\pi} \widehat{N} \left(-2\varepsilon \frac{\sin\theta}{r}\right) \\
&= \oint \frac{d\theta}{2\pi} \left(2\varepsilon \frac{\sin\theta}{r}\right) \left\{-2(qR)^2 \varepsilon \frac{1}{r} \frac{\partial \widehat{V}_E}{\partial t} \int^{\theta} d\theta' \cos\theta' + \widehat{V}_E \frac{qR}{r} \widetilde{u}_{\parallel}^{ref}\right\}
\end{aligned}$$

or

$$\begin{aligned}
\frac{\partial \widehat{V}_E}{\partial t} &= -4(qR)^2 \varepsilon^2 \frac{\partial \widehat{V}_E}{\partial t} \frac{1}{r^2} \oint \frac{d\theta}{2\pi} \sin\theta \sin\theta \\
&\quad +2\varepsilon \frac{qR}{r^2} \widehat{V}_E \oint \frac{d\theta}{2\pi} \sin\theta \widetilde{u}_{\parallel}^{ref}
\end{aligned}$$

We **NOTICE** that in this equation there are two occurences of $\partial \widehat{V}_E / \partial t$.

This is the technical origin of the inertia factor $\left(1+2q^2\right)$.

The first term in the RHS is

$$-4(qR)^2 \varepsilon^2 \left(\frac{\partial \widehat{V}_E}{\partial t}\right) \frac{1}{r^2} \oint \frac{d\theta}{2\pi} \sin\theta \sin\theta = -4q^2 R^2 \frac{r^2}{R^2} \left(\frac{\partial \widehat{V}_E}{\partial t}\right) \frac{1}{r^2} \frac{1}{2} = -2q^2 \left(\frac{\partial \widehat{V}_E}{\partial t}\right)$$

This term will go to the LHS and will complete the "neoclassical inertia factor".

The second term contains the factors

$$\begin{aligned}
2\varepsilon \frac{qR}{r^2} \widehat{V}_E &= 2 \frac{r}{R} q \frac{R}{r} \frac{1}{r} \widehat{V}_E \\
&= \frac{2q}{r} \widehat{V}_E
\end{aligned}$$

Replacing in the equation we obtain the formor

$$\left(1+2q^2\right) \frac{\partial \widehat{V}_E}{\partial t} = \frac{2q}{r} \widehat{V}_E \oint \frac{d\theta}{2\pi} \sin\theta \widetilde{u}_{\parallel}^{ref}$$

or

$$\left(1+2q^2\right) \frac{\partial \widehat{V}_E}{\partial t} = \frac{2q}{r} \widehat{V}_E \ \overline{\sin\theta \widetilde{u}_{\parallel}^{ref}}$$

Comment on this equation.

It contains the acceleration of a perturbation of the reference poloidal velocity.

It is then an instability.

One can see the inertia of plasma to poloidal rotation.

In this equation we replace the reference state for the parallel velocity, which is fixed by the source and the flux, both these contributions being retained with their variation along the poloidal direction

$$\left(1+2q^2\right)\frac{\partial \widehat{V}_E}{\partial t} = \widehat{V}_E \quad \times \frac{1}{\varepsilon^2}2q^2\left[\frac{1}{n_0}\oint\frac{d\theta}{2\pi}S\cos\theta - \frac{1}{n_0}\frac{1}{r}\frac{\partial}{\partial r}\left(r\oint\frac{d\theta}{2\pi}\Gamma_r\cos\theta\right)\right]$$

we can easily recognize that an integration by parts have been made in the right hand side.

NOTE
How is generated this *inertia factor* $1+2q^2$.
We have seen that the first integration

$$\widehat{u}_\parallel = -2qR\varepsilon\frac{\cos\theta}{r}\widehat{V}_E$$

actually obtains the Pfirsch Schluter parallel current, with a poloidal flow given by $\widehat{V}_E(\theta)$ (instead of diamagnetic $\sim \frac{dp}{dr}$). This PS flow has a coefficient q, as usual. (And there is a factor that mediates between direction, $qR/r = \frac{B_T}{B_\theta}$).

The second equation has the RHS term $-c_s^2\frac{1}{qR}\frac{\partial \widehat{N}}{\partial\theta}$ which comes from the grad-pressure force in eq. for $u_\parallel$, and this introduces the second q factor. They now multiply the term $\partial\widehat{u}_\parallel/\partial t$, from $\frac{\partial \widehat{u}_\parallel}{\partial t} + \widehat{V}_E\frac{\partial \widetilde{u}_\parallel^{ref}}{r\partial\theta} = -c_s^2\nabla_\parallel\widehat{N}$.

The enhancement of the *radial diffusion* with a factor q^2, the known characteristics of the Pfirsch Schluter "regime" has the same origin. We note however that it is not yet clear what means PS regime.
END

6.5.4 Comments on the enhanced radial flux due to Pfirsch Schluter current

The radial velocity is determined above, with the functions

$$v_D \equiv -\eta\frac{1}{B_0}\frac{dp}{dr}$$

Then

$$\langle f v_r\rangle = v_D\left[\left(\frac{q}{\varepsilon}\right)^2\left(\langle f h^2\rangle - \frac{\langle f\rangle}{\left\langle\frac{1}{h^2}\right\rangle}\right)\right]$$

$$\langle v_r\rangle = v_D\left(\frac{q}{\varepsilon}\right)^2\left(\langle h^2\rangle - \frac{1}{\left\langle\frac{1}{h^2}\right\rangle}\right)$$

The factor $1/B_0$ has been excluded for compatibility with HK.

6.5.5 Sonic versus sub-sonic poloidal rotation

The growth rate of the spontaneous poloidal spin up is determined by

$$\frac{\partial}{\partial t} \ll \frac{c_s}{qR}$$

This means that the time variations of the process of spin-up are much slower than the time needed by a sound wave to travel a distance

$$l_{\parallel}^{-1} = \frac{1}{qR}$$

in the parallel direction. The order of magnitude is

$$\gamma \sim \frac{u_{\parallel}}{r}$$

since the assumption is

$$u_{\parallel} \sim \varepsilon c_s$$

The system from which it is derived the equation for the spin-up are

$$\begin{aligned} \widehat{V}_E \left(-2\varepsilon \frac{\sin\theta}{r} \right) + \nabla_{\parallel} \widehat{u}_{\parallel} &= 0 \\ \frac{\partial \widehat{u}_{\parallel}}{\partial t} + \widehat{V}_E \frac{\partial \widetilde{u}_{\parallel}^{ref}}{r \partial \theta} &= -c_s^2 \nabla_{\parallel} \widehat{N} \\ \frac{\partial \widehat{V}_E}{\partial t} &= c_s^2 \oint \frac{d\theta}{2\pi} \widehat{N} \left(-2\varepsilon \frac{\sin\theta}{r} \right) \end{aligned}$$

Most important: the drive of the spontaneous spin-up enters this system through

$$\widehat{V}_E \frac{\partial \widetilde{u}_{\parallel}^{ref}}{r \partial \theta}$$

where

$$\widetilde{u}_{\parallel}^{ref} = qR \int^{\theta} d\theta' \widetilde{F}$$

where

$$\begin{aligned} \widetilde{F} \quad \equiv \quad & \text{the part of the source} \\ & \text{which has poloidal } \theta \text{ variation} \end{aligned}$$

$$F = \frac{1}{n_0} \left[S - \frac{1}{r} \frac{\partial}{\partial r} \left(r \Gamma_r \right) \right]$$

with

$$\widetilde{F}(r,\theta) = F(r,\theta) - \overline{F}(r)$$

The object of discussion will be the first equation of this system. It should be modified if we want to include the time variation of the (normalized) density $\widehat{N} = \widehat{n}_1/n_0$. The new system is

$$
\begin{aligned}
\frac{\partial \widehat{N}}{\partial t} + \widehat{V}_E \left(-2\varepsilon \frac{\sin\theta}{r} \right) + \nabla_{\|} \widehat{u}_{\|} &= 0 \\
\frac{\partial \widehat{u}_{\|}}{\partial t} + \widehat{V}_E \frac{\partial \widetilde{u}_{\|}^{ref}}{r\partial\theta} &= -c_s^2 \nabla_{\|} \widehat{N} \\
\frac{\partial \widehat{V}_E}{\partial t} &= c_s^2 \oint \frac{d\theta}{2\pi} \widehat{N} \left(-2\varepsilon \frac{\sin\theta}{r} \right)
\end{aligned}
$$

It will be assumed that

$$\widetilde{F} \sim \cos\theta$$

The system of equation shows *stable sound waves*

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2 \nabla_{\|}^2 \right) \overline{\widehat{N} \cos\theta} = 0$$

In **Hassam Drake** the trigonometric components of the variables depending on θ are introduced

$$
\begin{aligned}
\widehat{N}_s &= \overline{\widehat{N} \sin\theta} \\
\widehat{u}_{\|c} &= \overline{\widehat{u}_{\|} \cos\theta}
\end{aligned}
$$

and the new system is rewritten

$$
\begin{aligned}
\frac{\partial \widehat{N}_s}{\partial t} - \frac{\varepsilon \widehat{V}_E}{r} - \frac{1}{qR} \widehat{u}_{\|c} &= 0 \\
\frac{\partial \widehat{u}_{\|c}}{\partial t} + \frac{\widehat{V}_E}{r} \widetilde{u}_s^{ref} &= -\frac{c_s^2}{qR} \widehat{N}_s \\
\frac{\partial \widehat{V}_E}{\partial t} &= -2\varepsilon \frac{c_s^2}{r} \widehat{N}_s
\end{aligned}
$$

The drive is again the θ-varying part of the parallel velocity which comes from the θ-varying part of the sorce and radial transport flux.

$$
\begin{aligned}
\widetilde{u}_{\|s}^{ref} &\equiv \overline{\widetilde{u}_{\|}^{ref} \sin\theta} \\
&= qR \ \overline{\widetilde{F} \cos\theta}
\end{aligned}
$$

From this system, adopting

$$\sim \exp(\gamma t)$$

one obtaines fof the poloidal velocity $\widehat{V}_E(t)$

$$\gamma\left[\gamma^2+\left(1+2q^2\right)\frac{c_s^2}{q^2R^2}\right]=2c_s^2\frac{1}{r}\frac{1}{qR}\widetilde{u}_{\|s}$$

This expression in the limit that was assumed before (slow rotation)

$$\gamma \ll \frac{c_s}{qR}$$

is

$$\gamma\left(1+2q^2\right)\approx\frac{2q}{r}\widetilde{u}_{\|s}$$

This is the classical spontaneous spin-up.

When the inequality is not fullfilled

$$\gamma^3\approx\frac{2}{Rr}\frac{c_s^2}{qR}\widetilde{u}_{\|s}$$

7 Stringer poloidal spin-up Hassam Antonsen Drake Guzdar Liu

Also in *rho-effective*.

The paper **Hassam Antonsen Drake Guzdar Liu**.
The source S that varies with θ.
The flux of particles that has variation with θ.

$$\frac{\partial\overline{n}}{\partial t}=-\boldsymbol{\nabla}_r\overline{\boldsymbol{\Gamma}}_r+\overline{S}$$

where the terms are averaged over surface.
The variation is resolved by *flow in the parallel direction*

$$\overline{n}\ \nabla_{\|}\widetilde{V}_{\|}\approx-\nabla_r\widetilde{\Gamma}_r+\widetilde{S}$$

the parallel flows are a necessary consequence of the poloidally varying flux $\widetilde{\Gamma}_r$. The parallel flows will drain particles that have the tendency to accumulate at particular agles θ, due to $\widetilde{\Gamma}_r$ and due to the source $\widetilde{S}$.

$\widetilde{V}_{\|}$ exists and has parallel variation,
as an equilibrium state
It is a necessary part of the equilibrium

This is an equilibrium situation.

The Stringer spin up arises as a perturbation of this equilibirum.

We assume the existence of a perturbation consisting of a *poloidal rotation*

$$V_\theta$$

The effect of V_θ is different - and interacts - with the existence of the *parallel* velocity $V_\parallel$.

These velocities will be combined in the *convective* terms in the momentum equation.

The inertial term of advection of the equilibrium parallel velocity $\widetilde{V}_\parallel$ by the poloidal perturbation velocity is only balanced by the parallel pressure

$$\overline{n}\ M\ V_\theta \frac{\partial}{r\partial\theta}\widetilde{V}_\parallel = -T\nabla_\parallel \widetilde{n}$$

Further, one introduces the *gravitation* effect for torus

$$\overline{n}\left(\frac{\partial V_\theta}{\partial t}\right) = -g_{eff}\ \overline{n\ \sin\theta}$$

where

$$\begin{aligned} g_{eff} &= \frac{c_s^2}{R} \\ &\quad \text{effective gravity} \end{aligned}$$

It results

$$\frac{\partial V_\theta}{\partial t} = q^2 \frac{1}{e\overline{n}}\ V_\theta\ \left(-\nabla_r \overline{\Gamma_r \cos\theta}\ +\ \overline{S\cos\theta}\right)$$

The interesting situation arises at the *saturation* of the poloidal rotation velocity

The poloidal rotation velocity V_θ saturates at the poloidal rotation SOUND speed.

If

$$\frac{V_\theta}{r} \text{ becomes greater than } \frac{c_s}{qR}$$

then the accumulation of density at particular θ regions is suppressed by

the *poloidal convection*

instead of the *parallel flow*

This situation, where the density accumulation in a region of θ is suppressed by the *poloidal convection*, is described by the poloidal convection term $(\mathbf{v}\cdot\boldsymbol{\nabla})\,\widetilde{n} \to V_\theta \frac{\partial\widetilde{n}}{r\partial\theta}$,

$$V_\theta \frac{\partial \widetilde{n}}{r\partial\theta} = -\nabla_r \widetilde{\Gamma}_r + \widetilde{S}$$

replacing n,

$$\frac{\partial V_\theta}{\partial t} = -\frac{1}{\overline{n} V_\theta}\ \varepsilon\ c_s^2\ \left(-\nabla_r \overline{\Gamma_r\ \cos\theta}\ +\ \overline{S\ \cos\theta}\right)$$

There is a cross over between the two expressions for $\partial V_\theta / \partial t$, when V_θ / r becomes greater

$$\frac{V_\theta}{r} > \frac{c_s}{qR}$$

8 Poloidal instability (Hazeltine, Lee, Rosenbluth 1970)

The most general equations, simplified, in the toroidal geometry.

$$\begin{aligned} 0 &= -\boldsymbol{\nabla} p + \mathbf{j} \times \mathbf{B} \\ \boldsymbol{\nabla} \times \mathbf{B} &= \mu_0 \mathbf{j} \\ \boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\ \mathbf{E} + \mathbf{v} \times \mathbf{B} &= \eta \mathbf{j} \\ \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) &= 0 \\ p &= c_s^2 \rho \end{aligned}$$

and the resulting equilibrium parameters (taking $\mu_0 = 1$)

$$p = p(r)$$

$$\begin{aligned} B_r &= 0 \\ B_\theta &= \frac{b(r)}{h} \\ B_\varphi &= \frac{B(r)}{h} \end{aligned}$$

$$\begin{aligned} j_r &= 0 \\ j_\theta &= -\frac{1}{h}\frac{dB(r)}{dr} \\ j_\varphi &= -\frac{1}{h}\frac{B(r)}{b(r)}\frac{dB(r)}{dr} - \frac{h}{b(r)}\frac{dp}{dr} \end{aligned}$$

The expressions for the current components is derived from the two equations $0 = -\boldsymbol{\nabla} p + \mathbf{j} \times \mathbf{B}$ and $\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j}$.

$$\begin{aligned} j_\varphi &= \frac{1}{r}\frac{\partial}{\partial r}(r B_\theta) \\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{b(r)}{h}\right) \end{aligned}$$

Then equalizing the two expressions for j_φ leads to

$$-\frac{1}{h}\frac{B(r)}{b(r)}\frac{dB(r)}{dr}-\frac{h}{b(r)}\frac{dp}{dr}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{b(r)}{h}\right)$$

The electric field has an externally induced component

$$\mathbf{E}=-\boldsymbol{\nabla}\phi+\frac{\mathcal{E}}{h}\widehat{\mathbf{e}}_\varphi$$

The next equation to be included is the Ohm's law

$$\eta j_\varphi=\frac{\mathcal{E}}{h}+v_r B_\theta$$

This results from taking the θ component of the Ohm's law

$$(\mathbf{E})_\theta+(\mathbf{v}\times\mathbf{B})_\theta=\eta(\mathbf{J})_\theta$$

where

$$(\mathbf{E})_\theta=-\frac{\partial\phi}{r\partial\theta}$$

and

$$(\mathbf{v}\times\mathbf{B})_\theta=\left(\begin{array}{ccc}\widehat{\mathbf{e}}_r & \widehat{\mathbf{e}}_\theta & \widehat{\mathbf{e}}_\varphi\\ v_r & v_\theta & v_\varphi\\ 0 & \frac{b}{h} & \frac{B}{h}\end{array}\right)_\theta=-v_r\frac{B}{h}$$

then

$$\begin{aligned}-\frac{\partial\phi}{r\partial\theta}-v_r\frac{B}{h} &= \eta j_\theta\\ &= \eta\left[-\frac{1}{h}\frac{dB(r)}{dr}\right]\end{aligned}$$

we extract the derivative of the potential to θ,

$$\frac{\partial\phi}{\partial\theta}=-rv_r\frac{B}{h}+\eta r\frac{1}{h}\frac{dB(r)}{dr}$$

and integrate over $[0,2\pi]$

$$0=\int_0^{2\pi}d\theta\frac{\partial\phi}{\partial\theta}=\int_0^{2\pi}d\theta\left[-rv_r\frac{B}{h}+\eta r\frac{1}{h}\frac{dB(r)}{dr}\right]$$

From the integrand a result with partial determination is

$$v_r=\eta\frac{1}{B}\frac{dB}{dr}+[\text{function independent of }\theta]$$

which means a function that is constant on the surface, v^H.

Now we modify the equation

$$\eta j_\varphi = \frac{\mathcal{E}}{h} + v_r B_\theta$$

by taking $\frac{\mathcal{E}}{h}$ to be of higher order and obtaining

$$\begin{aligned}
v_r^H &\approx \frac{\eta j_\varphi}{B_\theta} \\
&= \eta \frac{1}{B_\theta}\left(-\frac{1}{h}\frac{B}{b}\frac{dB}{dr} - \frac{h}{b}\frac{dp}{dr}\right) \\
&= \eta \frac{h}{b}\left(-\frac{1}{h}\frac{B}{b}\frac{dB}{dr} - \frac{h}{b}\frac{dp}{dr}\right) \\
&= -\eta\frac{B}{b^2}\frac{dB}{dr} - \eta\frac{h^2}{b^2}\frac{dp}{dr}
\end{aligned}$$

and we **NOTE** that this expression is function of only r.

$$\begin{aligned}
v_r &= \eta\frac{1}{B}\frac{dB}{dr} \\
&\quad -\eta\frac{B}{b^2}\frac{dB}{dr} - \eta\frac{h^2}{b^2}\frac{dp}{dr}
\end{aligned}$$

The last term

$$\eta\frac{h^2}{b^2}\frac{dp}{dr} = \eta\frac{1}{b^2}\left(1 + 2\varepsilon\cos\theta + \varepsilon^2\cos^2\theta\right)\frac{dp}{dr}$$

Now we have to use two equations. One is

$$-\frac{1}{h}\frac{B(r)}{b(r)}\frac{dB(r)}{dr} - \frac{h}{b(r)}\frac{dp}{dr} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{b(r)}{h}\right)$$

COMMENT

Before all calculations, we note that the radial velocity v_r only exists because there is a resistivity, which opposes the *parallel current*.

The presence of the resistivity imposes the activation of the term $\mathbf{v}\times\mathbf{B} \to v_r B_\theta$ in the Ohm's law.

This confirms the assertion of **Stringer** that the parallel current flows because it tries to neutralize the charge separation produced by the diamagnetic flow with non-zero divergence. And, when this is not possible in a complete way, due to the existence of the resistivity, then there is a *radial velocity*.

A pure radial velocity cannot exist.

There is also a poloidal velocity

$$v_\theta = \frac{u(r)}{h}$$

and

$$\begin{aligned} v_\varphi &= \frac{B}{b}\frac{u(r)}{h} - \frac{h}{b}\frac{d\phi}{dr} \\ &+O(\eta) \end{aligned}$$

$$\frac{d\phi}{dr} = Bu(r) - b\overline{v}_\varphi$$

9 Numerical study of the Stringer rotation

We have developed a numerical framework that incorporates the effect of poloidal variation of the rate of particle and energy losses. The equation for poloidal rotation under the Stringer effect is solved and the rate of rotation is compared with the *transit time magnetic pumping* damping.

10 hose-like : System of equation for spontaneous poloidal rotation and shock formation (Rosenbluth Lee Hazeltine 2)

This part is also in **rotation.tex**, more expanded.

The paper **Resistive plasma rotation shock formation Rosenbluth Lee Hazeltine 1971**.

The physical picture: in zeroth order in the dissipative mechanism (here the resistivity η) the quantities that are θ-averaged over the magnetic surface

$\overline{\rho}$ density
$\overline{u}$ poloidal rotation speed
$\overline{v}$ toroidal velocity

can be prescribed independently from surface to surface, by arbitrary functions.

The zeroth-order (in η) equations uniquely determine the *azimuthally-varying* parts of the full functions

$$\begin{aligned} &\overline{\rho} + \delta\rho(\theta) \\ &\overline{u} + \delta u(\theta) \\ &\overline{v} + \delta v(\theta) \end{aligned}$$

When there is an *interaction* due to the presence of a *resistivity* η, there is a slow

$$\tau \sim \frac{1}{\eta}$$

transition between the steady states found at the zeroth-order. The time-dependent equations resulted from the inclusion of the small η show that the rotations with small poloidal speeds have the tenddency to accelerate.

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}=-\frac{c_s^2}{\rho}\boldsymbol{\nabla}\rho+\frac{1}{\rho}\mathbf{J}\times\mathbf{B} \tag{7}$$

$$\frac{\partial \rho}{\partial t}+\boldsymbol{\nabla}\cdot(\rho\mathbf{v})=0 \tag{8}$$

$$\boldsymbol{\nabla}\cdot\mathbf{J}=0 \tag{9}$$

$$-\boldsymbol{\nabla}\phi+\mathbf{v}\times\mathbf{B}=\eta\mathbf{J} \tag{10}$$

Note the absence of anisotropy of the pressure tensor in $\parallel$ relative to $\perp$ directions, that are usually invoked to represent the magnetic pumping dumping of poloidal rotation.

End.

Note the absence of the Ampere's law

$$\boldsymbol{\nabla}\times\mathbf{B}=\mu_0\mathbf{J}$$

from the system of equations. They say that the fields are produced by external coils.

End.

11 Notes

The paper of **Stringer 1969** calculates the radial flux of particles taking into account the nonuniformity of the particle density $n_1(\theta)$ and of electric potential $\phi_1(\theta)$ on the magnetic surface. These result from the neoclassical drifts and the equations of continuity, the poloidal projected equation of momentum, in the presence of an equilibrium radial electric field represented by a potential $\phi_0(r)$. In the regime of low collisions instead of collisional resistivity that permits to use the Ohm law for the parallel current projected on poloidal direction, it is invoked the *kinetic* process of Landau damping.

The result is

$$nv_{Di}=-\frac{\sqrt{\pi}}{8}\varepsilon\frac{\rho_i^2 v_{th,i}}{r}\frac{1}{q}\left(1+\frac{v_0}{U_{in}}\right)\exp\left(-z_i^2\right)\left[1+\frac{S\,(S+\tau)}{F^2+L^2}\right]\frac{dn_0}{dr}$$

where

$$S\equiv 1+\tau+2z_i^2\left(1+\frac{U_{en}}{v_0}\right)$$

$$z_j\equiv-\frac{v_0}{v_{th,j}}\frac{q}{\varepsilon}$$

This is

$$\frac{1}{v_{th,j}}\frac{q}{\varepsilon}=\frac{1}{v_{th,j}}\frac{B_T}{B_\theta}$$

$$v_{th,j}\frac{B_\theta}{B_T}=v_{th,j}^{\theta}$$

the projection of the thermal (reasonably assumed parallel) velocity on the poloidal direction. And

$$v_0\widehat{\mathbf{e}}_\theta \quad = \quad \frac{-\boldsymbol{\nabla}\phi_0\times\widehat{\mathbf{n}}}{B}$$

the poloidal velocity due to radial electric field

$$z_j=-\frac{v_0}{v_{th,j}^{\theta}}$$

This parameter compares the electric poloidal rotation velocity to the poloidal projection of the thermal velocity. It is a kind of *poloidal Mach number* if the thermal speed projection is comparable with the *sound speed* projected on poloidal direction.

Later it is found that

$$\frac{v_0}{U_{ni}}=-1+\frac{1+\tau}{1+2z_i^2+2z_i^4}\left(\tau\frac{m_e}{m_i}\right)^{1/2}\exp\left(-z_i^2\right)$$

Then the diamagnetic and the electric rotations are almost equal in magnitude and opposite.

When

$$1+\frac{v_{dia}}{v_E}\approx 0$$

the neoclassical diffusion vanishes.

12 Appendix A

12.1 The velocities

In HLR PRL there is a radial velocity

$$v_r$$

the equation that has NOT been used is the Ohm's law

$$\mathbf{E}+\mathbf{v}\times\mathbf{B}=\eta\mathbf{J}$$

plus the definition of the electric field

$$\mathbf{E}=-\boldsymbol{\nabla}\phi+\widehat{\mathbf{e}}_z\frac{\mathcal{E}^{external}}{h}$$

The toroidal φ projection of the Ohm's law

$$\eta J_\varphi=\frac{b\left(r\right)}{h}v_r+\frac{\mathcal{E}^{external}}{h}$$

We **NOTE** that the vector product involves

$$v_r \quad \text{and} \quad B_\theta$$

which means that it is directed in the toroidal direction.

One notes that $-\boldsymbol{\nabla}\phi$ does not have a toroidal component. This is because the toroidal component is *only* imposed from exterior, as *inductive electric field.*

Another relation is obtained by using the poloidal periodicity, *i.e.* using an integral on θ of the poloidal projection of the electric field, that must result 0.

We will integrate over $\theta \in [0, 2\pi]$ potentialul electric which is obtained from the Ohm law

$$\eta \mathbf{j} = -\boldsymbol{\nabla}\phi + \mathbf{v} \times \mathbf{B}$$

The component along φ is

$$\eta j_\varphi = (\mathbf{v} \times \mathbf{B})_\varphi = v_r B_\theta = v_r \frac{b(r)}{h}$$

we stop here to remark that we get an expression for v_r using

$$\frac{\varepsilon}{q} J_\varphi = -\frac{1}{h} \frac{dB_0(r)}{dr} - \frac{h}{B_0} \frac{dp(r)}{dr}$$

we have

$$\begin{aligned} v_r &= \frac{h}{b(r)} \eta j_\varphi \\ &= \frac{h}{b(r)} \eta \frac{q}{\varepsilon} \left[-\frac{1}{h} \frac{dB_0(r)}{dr} - \frac{h}{B_0(r)} \frac{dp(r)}{dr} \right] \end{aligned}$$

we take into account that

$$\frac{h}{b(r)} \frac{B_0(r)}{h} = \frac{B_\varphi}{B_\theta} = \frac{q}{\varepsilon}$$

$$\begin{aligned} v_r = & -\frac{1}{B_0(r)} \frac{dB_0(r)}{dr} \eta \left(\frac{q}{\varepsilon} \right)^2 \\ & -\eta \frac{q}{\varepsilon} \frac{1}{B_\theta B_\varphi} \frac{dp(r)}{dr} \end{aligned}$$

in the last term

$$\begin{aligned} -\eta \frac{q}{\varepsilon} \frac{1}{B_\theta B_\varphi} \frac{dp(r)}{dr} &= -\eta \frac{B_\varphi}{B_\theta} \frac{1}{B_\theta B_\varphi} \frac{dp(r)}{dr} = -\eta \frac{h^2}{b^2(r)} \frac{dp}{dr} \\ &= -\eta \frac{1}{b^2(r)} \left(1 + 2\varepsilon \cos\theta + \varepsilon^2 \cos^2\theta \right) \frac{dp}{dr} \end{aligned}$$

It results

$$
\begin{aligned}
v_r &= -\eta \frac{1}{B_0(r)} \frac{dB_0(r)}{dr} \left(\frac{q}{\varepsilon}\right)^2 \\
&\quad -\eta \frac{1}{b^2(r)} \left(1 + 2\varepsilon\cos\theta + \varepsilon^2 \cos^2\theta\right) \frac{dp}{dr}
\end{aligned}
$$

The component along θ is

$$
\begin{aligned}
\frac{\partial\phi}{r\partial\theta} &= -\eta j_\theta + \begin{pmatrix} \widehat{\mathbf{e}}_r & \widehat{\mathbf{e}}_\theta & \widehat{\mathbf{e}}_\varphi \\ v_r & v_\theta & v_\varphi \\ 0 & B_\theta & B_\varphi \end{pmatrix}_\theta \\
&= -\eta \left[-\frac{1}{h} \frac{dB(r)}{dr} \right] - v_r B_\varphi \\
&= \eta \frac{1}{h} \frac{dB(r)}{dr} - v_r \frac{B(r)}{h}
\end{aligned}
$$

(**note** that there is a difference in the expression of $\partial\phi/\partial\theta$ after multiplication with r in the expression above, it is different from eq.13 of HLR PRL by their absence of an r in the second term).

and we integrate

$$
\begin{aligned}
0 &= \int_0^{2\pi} d\theta \frac{\partial\phi}{\partial\theta} \\
&= \int_0^{2\pi} d\theta \left[\eta \frac{dB}{dr} \frac{h_\theta}{h_r h_\varphi} - B v_r \frac{h_r}{h_\varphi} \right] \\
&= \int_0^{2\pi} d\theta \left[\eta \frac{dB}{dr} \frac{r}{h} - r \frac{B}{h} v_r \right]
\end{aligned}
$$

We replace here v_r as

$$
\begin{aligned}
v_r &= -\eta \frac{1}{B_0(r)} \frac{dB_0(r)}{dr} \left(\frac{q}{\varepsilon}\right)^2 \\
&\quad -\eta \frac{1}{b^2(r)} \left(1 + 2\varepsilon\cos\theta + \varepsilon^2 \cos^2\theta\right) \frac{dp}{dr}
\end{aligned}
$$

we have

$$
\int_0^{2\pi} d\theta \frac{1}{h} = \left\langle \frac{1}{h} \right\rangle
$$

In **Gradshtein Rijik** gasim 3.613

$$
\int_0^{\pi} \frac{d\theta}{1 + \varepsilon\cos\theta} = \frac{\pi}{\sqrt{1-\varepsilon^2}}
$$

and

$$\left\langle \frac{1}{h} \right\rangle = \int_0^{2\pi} \frac{d\theta}{1+\varepsilon\cos\theta} = \frac{2\pi}{\sqrt{1-\varepsilon^2}}$$

The first term in the integration is

$$\int_0^{2\pi} d\theta \left[\eta \frac{dB}{dr} \frac{r}{h} \right] = \eta \frac{dB}{dr} r \left\langle \frac{1}{h} \right\rangle$$

The second term is

$$\begin{aligned}
& \int_0^{2\pi} d\theta \left[-r \frac{B_0}{h} v_r \right] \\
= \quad & -rB_0 \int_0^{2\pi} d\theta \frac{1}{h} \left[-\eta \frac{1}{B_0(r)} \frac{dB_0(r)}{dr} \left(\frac{q}{\varepsilon} \right)^2 - \eta \frac{1}{b^2(r)} \left(1 + 2\varepsilon\cos\theta + \varepsilon^2\cos^2\theta \right) \frac{dp}{dr} \right] \\
= \quad & \eta r \frac{dB_0(r)}{dr} \left(\frac{q}{\varepsilon} \right)^2 \int_0^{2\pi} d\theta \frac{1}{h} \\
& + rB_0 \eta \frac{1}{b^2(r)} \frac{dp}{dr} \int_0^{2\pi} d\theta \frac{1}{h} h^2 \\
= \quad & \eta r \frac{dB_0(r)}{dr} \left(\frac{q}{\varepsilon} \right)^2 \left\langle \frac{1}{h} \right\rangle + rB_0 \eta \frac{1}{b^2(r)} \frac{dp}{dr} \langle h \rangle
\end{aligned}$$

Then

$$0 = \eta \frac{dB_0}{dr} r \left\langle \frac{1}{h} \right\rangle + r\eta \frac{dB_0(r)}{dr} \left(\frac{q}{\varepsilon} \right)^2 \left\langle \frac{1}{h} \right\rangle + rB_0 \eta \frac{1}{b^2(r)} \frac{dp}{dr} \langle h \rangle$$

$$\frac{1}{B_0} \frac{dB_0}{dr} \left\langle \frac{1}{h} \right\rangle \left[1 + \left(\frac{q}{\varepsilon} \right)^2 \right] + \frac{1}{b^2(r)} \frac{dp}{dr} \langle h \rangle = 0$$

This is an identity, reflecting the periodicity on θ.

We return to the expression of v_r,

$$\begin{aligned}
v_r \quad = \quad & -\eta \frac{1}{B_0(r)} \frac{dB_0(r)}{dr} \left(\frac{q}{\varepsilon} \right)^2 \\
& -\eta \frac{1}{b^2(r)} \left(1 + 2\varepsilon\cos\theta + \varepsilon^2\cos^2\theta \right) \frac{dp}{dr}
\end{aligned}$$

From the identity we extract

$$\frac{1}{B_0} \frac{dB_0}{dr} \left(\frac{q}{\varepsilon} \right)^2 = -\frac{1}{\left\langle \frac{1}{h} \right\rangle} \frac{1}{b^2(r)} \frac{dp}{dr} \langle h \rangle - \frac{1}{B_0} \frac{dB_0}{dr}$$

and replace in the expression of v_r

$$\begin{aligned}
v_r \quad = \quad & -\eta \left[-\frac{1}{\left\langle \frac{1}{h} \right\rangle} \frac{1}{b^2(r)} \frac{dp}{dr} \langle h \rangle - \frac{1}{B_0} \frac{dB_0}{dr} \right] \\
& -\eta \frac{1}{b^2(r)} \left(1 + 2\varepsilon\cos\theta + \varepsilon^2\cos^2\theta \right) \frac{dp}{dr} \\
= \quad & \eta \frac{1}{B_0} \frac{dB_0}{dr} + \eta \frac{1}{b^2(r)} \frac{dp}{dr} \left[\frac{1}{\left\langle \frac{1}{h} \right\rangle} \langle h \rangle - 1 - 2\varepsilon\cos\theta - \varepsilon^2\cos^2\theta \right]
\end{aligned}$$

The term

$$
\begin{aligned}
\frac{1}{\left\langle \frac{1}{h} \right\rangle} \langle h \rangle - 1 - \varepsilon^2 \cos^2 \theta &= \frac{1}{\frac{2\pi}{\sqrt{1-\varepsilon^2}}} - 1 - \varepsilon^2 \cos^2 \theta \\
&= \frac{1}{2\pi} \left(1 - \frac{1}{2} \varepsilon^2 \right) - 1 - \varepsilon^2 \cos^2 \theta \\
&= \left(\frac{1}{2\pi} - 1 \right) - \varepsilon^2 \left(\frac{1}{2} - \cos^2 \theta \right)
\end{aligned}
$$

The form is

$$
\begin{aligned}
v_r &= \eta \frac{1}{B_0} \frac{dB_0}{dr} - \eta 2 \varepsilon \cos \theta \frac{1}{b^2(r)} \frac{dp}{dr} \\
&\quad + \eta \frac{1}{b^2(r)} \frac{dp}{dr} \left[\frac{1}{\left\langle \frac{1}{h} \right\rangle} \langle h \rangle - 1 - \varepsilon^2 \cos^2 \theta \right]
\end{aligned}
$$

does not seem to reproduce the eq(14) in HLR PRL.

The result

$$
\begin{aligned}
\frac{v_r}{h_r} &= \eta \frac{1}{B} \frac{dB}{dr} \\
&\quad + \eta \frac{1}{[b(r)]^2} \frac{dp}{dr} \varepsilon^2 (X + Y - 1) \\
&\quad - \eta \frac{1}{[b(r)]^2} \frac{dp}{dr} 2 \varepsilon \cos \theta
\end{aligned}
$$

The observation from this article

$$
\frac{\partial \phi}{\partial \theta} \sim O(\eta)
$$

the poloidal variation of the electric potential is of the order of the resistivity. As much variation of potential on the magnetic surface exists, as there is resistivity.

Then we note that the TUNGSTEN produces an increase of the resistivity η and this enhances the poloidal variation of the electric potential.

The above equations define an equilibrium state.
It is then perturbed to see the evolution of a small quantity.
The equations are

$$
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla}) \mathbf{v} = -\frac{1}{\rho} \boldsymbol{\nabla} p + \frac{1}{\rho} \mathbf{J} \times \mathbf{B}
$$

$$
\frac{\partial p}{\partial t} + \boldsymbol{\nabla} \cdot (\mathbf{v} p) = c_s^2 S
$$

where S is a source of particles.

It results from pressure equation that

$$\begin{aligned} v_\theta &= \frac{u(r)}{h_r h_\varphi} \\ &\quad +O(\eta) \end{aligned}$$

and from the z projection of the Ohm's law

$$v_z = \frac{B}{b(r)} \frac{u(r)}{h_\varphi} - \frac{1}{b(r)} \frac{d\phi}{dr} h_\varphi + O(\eta)$$

where

$$\begin{aligned} \frac{d\phi}{dr} &= Bu - b(r)\,\overline{v}_z \\ &\quad +O(\eta) \\ &\quad +O\left(\varepsilon^2\right) \end{aligned}$$

We **NOTE** that this is a reduced form of the classical equation for the radial electric field E_r where the diamagnetic term has been neglected.