1 Introduction

A series of Notes about the effect of the **Neutral Beam Injection**.

The NBI ions are very energetic and they transfer their energy and their momentum to the target plasma. The energy transfer leads to higher temperature of both electron and ion components of the background plasma. And the transfer of momentum leads to plasma motion (toroidal rotation) and to an electric current since the transfers to electrons and to ions and the collisional effects are different.

An important problem is the loss of trapped NBI ions from the plasma since their orbit is so large that can be greater than the plasma radius. This is a radial current and must be compensated by a return current flowing through the bulk plasma producing a torque and therefore rotation, both poloidal and toroidal.

The effect of the NBI ions is studied using their kinetic distribution function.

This is solution to a Fokker Planck equation. The particularity is the necessity to use the complete form of the collision operator in order to represent the multitude of the processes determined by the NBI.

First the fast NBI ions are slowed down by collisions with electrons of the background plasma.

When the velocity of the NBI ions has been reduced to a particular critical threshold v_c the role of ions of the background plasma becomes important with two effects on the NBI ions: (1) slowing down by collisional transfer of energy to the background ions, and (2) pitch angle scattering of the NBI ions, which means particular evolution of the kinetic distribution in velocity space at the boundary between trapped and circulating new NBI ions.

The electric current determined by the NBI is due to the momentum transfer to the background ions, which sustains a current in the direction of the injection. If the transfer of momentum from the fast NBI ions to the background electrons would be equal then there would be an equal flow of electrons in the direction of the injection and therefore no new electric current. Actually both background ions and background electrons that have received momentum in the direction of injection undergo collisional friction with all plasma and this is different for the two species. Electrons collide with background ions and with impurity ions and lose the momentum obtained from the fast NBI ions at a rate that is different of the similar loss (friction) rate of the ions. In addition the trapped electrons have an effect on the flow of electrons. Then the flow (current) of electrons in the direction of injection is different of that of ions and the relative magnitude is not unique in the space of parameters. If the electron flow (in the direction of injection) is less affected by collisions then they will move faster than the ions (since these are more affected by friction) and the total current is *negative*. In the opposite case the ion flow is predominant and the current is *positive*. The resulting current must be combined with the main electric current in the plasma in the absence of NBI.

The problem is similar with that of *alpha* particles, which are fast ions too. There are however differences between the case of NBI ions and of the alpha particles. The first have an anisotropic distribution of velocity (they are produced by a beam) while the alphas have *isotropic* distribution of velocity at birth since they are created by nuclear reaction in the volume.

1.1 List of increasingly complex approaches

Ohkawa 1970

Basic idea of current drive by NBI.

Cordey Houghton 1973. The FP equation has no *drift* convective term, only $\partial/\partial t$ and *collisions*. The Rosenbluth potentials are used in the collision operator. With their explicit form the collision operator reduces to : (1) slowing down, and (2) pitch angle scattering. The solution is written as a series of terms with *separation of variable* factors, of v and μ.

Connor Cordey 1974 equilibria. This time the drift convection of the zeroth distribution function is included. Variables are $w = v^2/2$ and $\mu = v_\perp^2/(2B)$. Expansion in the ratio of bounce to collision $\omega_{bounce}/\nu_{\text{collision}}$ (slowing down) times $\tau_{bounce}/\tau_{slowing-down} \ll 1$ (very fast bounce and rare collisions). The zeroth order f_0 of the fast ions does NOT depend on θ. Then

- write the equation for the next order f_1, the neoclassical correction of the distribution of the fast ions
- use the periodicity on θ, which introduces the averages on surface

The averages of quantities like $\langle v_\parallel/B \rangle$ and $\langle 1/v_\parallel \rangle$ are approximated. Later they will be better calculated.

Then, to find the solution, *separation of variables*, in w and $\eta \equiv \sqrt{1 - \frac{\mu B_0}{w}}$ ($B_0 \equiv$the minimum value of the magnetic field, at the outermost point on the equatorial plane; typical to **Cordey**).

The current generated by NBI is calculated.

Cordey Core 1974 Fokker Planck. The Rosenbluth potentials h and g. The variables are $\left(v, \xi \equiv \frac{v_{\parallel}}{v}\right)$. Later the variables are $\left(u \equiv \frac{v}{v_0}, \xi = \frac{v_{\parallel}}{v}\right)$ for separation of variables by series of products. The derivative of the distribution function at the *birth velocity*, $v = v_0$, or $u = 1$ has *discontinuity* given by the angular spreading of the beam new ions.

Cox Start NBI + ICRH

Fowler code
details of the collision operator are in **Gaffey**

Cordey Jones Start Curtis Jones 1979, kinetic theory NB Current Drive

Equations for the collisional effects on the electrons.

$$\begin{aligned} & C_{e,beam}\left(F_{Me}, f_{beam}\right) + C_{e,i}\left(f_e' \ , \ F_{Mi}\right) \\ & + C_{e,e}\left(f_e' \ , \ F_{Me}\right) + C_{e,e}\left(F_{Me} \ , \ f_e'\right) \\ = \ & 0 \end{aligned}$$

The unknown is f_e', since we want to calculate the current (carried by electrons after interaction with fast ions). It is expressed by sum over terms with *separated variables*.

$$j_e = -e \int d^3v \ v_{\parallel} \ f_e'$$

The total current is "beam+electrons", where "beam" means fast ions

$$\begin{aligned} j &= K_1 \ en_{beam} \ v_{beam} \\ & \quad -\frac{4}{3}\frac{e \ v_{th,e} \ n}{\sqrt{\pi}} I_3\left(\infty\right) \end{aligned}$$

and can be positive or negative.

The factor

$$\begin{aligned} F &= \frac{j}{K_1 \ en_{beam} v_{beam}} \\ &= 1 - \frac{16}{3\sqrt{\pi}} \frac{I_3\left(\infty\right)}{v_b^* B} \end{aligned}$$

where

$$v_b^* = \frac{v_b}{v_e}$$

can become negative.

$$\begin{aligned} F &= 1 - \frac{1}{Z} - \frac{v_b^*}{Z} \\ &\text{for small } v_b^* \end{aligned}$$

The net current is in the opposited direction relative to the beam for

$$\begin{aligned} Z &= 1 \\ v_b^* &\sim 1.3 \end{aligned}$$

Start Cordey Jones effect of trapped electrons on beam-driven *current* 1980

The objective is the distribution function of the electrons, after interaction with the fast ions, from wich the electron current is calculated.

[*Later* there will be **Lin Liu Hinton** for NBCD]. There is neoclassical drift ($\mathbf{v}_D$) convection of f_{beam} (or *fast*).

Collision operator is already separated into *slowing-down* and the *pitch angle*.

Collisions are

- fast ions with electrons (the NB-induced electron current)
- electrons with electrons (friction)
- electrons with background ions (loss of the moment gain from NB, by friction with cold ions)
- fast ions with background ions

If the electrons benefit of the full momentum transfer from the fast ions, then their motion accompanies the motion of the fast ions and these two currents almost cancel each other.

One current, of the fast ions, and of the background ions that have received momentum from the fast ions, is in the direction of the injection.

The electron flow is in the same direction, as they have received momentum through collisions with fast ions. Then their current is opposite (*reversed*) and tends to cancel the current of the fast+background ions.

There is trapping for the electrons.

The trapping inhibits the *reversed current* of the electrons:

- reduce the number of current carrying particles
- introduce a new friction force, collisions between circulating electrons and trapped ones

There is expansion in

$$\frac{\tau_{bounce}}{\tau_{slowing-down}} \ll 1$$

(very fast bounce)

The zeroth order does not depend on θ. For the first order one uses periodicity which becomes a constraint for the zeroth order, an equation to be solved. It is calculated $\langle v_{\parallel}/v\rangle$ and $\langle v/v_{\parallel}\rangle$. Very useful figures of these averages, showing the singularity at $\xi/\sqrt{2\varepsilon} = 1$, where $\xi = \sqrt{1 - \frac{\mu B_0}{\epsilon}}$. The discontinuity of the derivative to ξ of the function f at the boundary ξ_t. Solution obtained with *separation of variables*, (v, ξ) after f_0 is represented as a series of terms-product of factors.

See below.

Hsu Catto Sigmar *alphas*
Hsu Shaing Gromley Sigmar *alpha* bootstrap

1.2 Notes

The loss of *counter-injected NBI hot ions* is used by **Yushmanov Horton** in **electrostatic potential formation at the edge due to hot ions lost to LIMITER.**

This is in *ITB, Notes.tex*.

The loss of hot ions from NBI due to large orbits is in *ion_loss*.

The collisions are essential.
See **fokker_planck_quasilinear_karney1986**. Review.
It is detailed in my notes in *collisions.tex*.
See for collisions **Gaffey**, in the notes *collisions.tex*.

From the paper **parallel velocity shear instabilities Catto Rosenbluth Liu PF 16 (1973) 1719**.

There is a critical velocity:

$$v_c \sim v_{th,e} \left(\frac{m_e}{m_i}\right)^{1/3}$$

For NBI ion velocities higher than v_c the beam ions are slowed down by collision with background *electrons*.

For velocities smaller than v_c the collisions with background *ions* and electrons consists of both slowing down and of pitch angle scattering.

2 Fokker Plank equation Cordey Houghton 1973

This part is also in *collisions.tex*.

The Fokker Plank eq. is written for the *hot* (fast) ions of the NBI.

- in uniform magnetic field (no toroidality, no trapped particles). Local in space, the process is the velocity space
- axisymmetry in velocity space, isotropy around the direction $\mathbf{v}$ parallel with $\mathbf{B}$;
- use of the Rosenbluth potentials, g and h;
- the background ions and electrons are cold Maxwellians

The eq. of collisional balance is written for the FAST IONS generated by the neutral beam

$$\begin{aligned}
& \frac{1}{C_{hot}}\frac{\partial f}{\partial t} \\
= \; & \frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2 \; f \; \frac{\partial h}{\partial v}\right) \\
& +\frac{1}{2}\frac{1}{v^2}\frac{\partial}{\partial\left(v^2\right)}\left(v^2 \; f \; \frac{\partial^2 g}{\partial v^2}\right) \\
& +\frac{1}{2}\frac{1}{v^2}\frac{\partial}{\partial \xi^2}\left[f \; \frac{1}{v}\left(1-\xi^2\right)\frac{\partial g}{\partial v}\right] \\
& +\frac{1}{2}\frac{1}{v^2}\frac{\partial}{\partial v}\left(-2f \; \frac{\partial g}{\partial v}\right) \\
& +\frac{1}{2}\frac{1}{v^2}\frac{\partial}{\partial \xi}\left(2f \; \xi\frac{1}{v}\frac{\partial g}{\partial v}\right)
\end{aligned}$$

The Rosenbluth potentials

$$
\begin{aligned}
h &= \frac{4}{\sqrt{\pi}} \sum_{j=e,i} n_j \left(\frac{m_{hot} + m_j}{m_j v_j^3} \right) \\
&\times \left[\int_0^v dv' \; v'^2 \frac{1}{v} \exp\left(-\frac{v'^2}{v_{th,j}^2} \right) + \int_v^\infty dv' \; v' \; \exp\left(-\frac{v'^2}{v_{th,j}^2} \right) \right]
\end{aligned}
$$

$$
\begin{aligned}
g &= \frac{4}{\sqrt{\pi}} \sum_{j=i,e} \frac{n_j}{v_{th,j}^3} \left[v \int_0^v dv' \exp\left(-\frac{v'^2}{v_{th,j}^2} \right) v'^2 \left(1 + \frac{v'^2}{3v^2} \right) \right. \\
&\qquad \left. + \int_v^\infty dv' \; \exp\left(-\frac{v'^2}{v_{th,j}^2} \right) v'^3 \left(1 + \frac{v^2}{3v'^2} \right) \right]
\end{aligned}
$$

with the coefficient

$$C_{hot} = 4\pi \frac{Z_{hot}^2 e^4}{m_{hot}^2} \log \Lambda$$

Range of velocities

$$v_{th,i} \ll v_{hot} \ll v_{th,e}$$

This will simplify the expressions

$$
\begin{aligned}
\tau_s \frac{\partial f}{\partial t} &= \frac{1}{v^2} \frac{\partial}{\partial v} \left[\left(v_c^3 + v^3 \right) \; f \right] \\
&+ \frac{1}{2} \frac{m_i}{m_{hot}} \frac{v_c^3}{v^3} \frac{\partial}{\partial \xi} \left[\left(1 - \xi^2 \right) \frac{\partial f}{\partial \xi} \right] \\
&+ \widetilde{S} \left(v - v_0 \right) \; \times \delta \left(\xi - \xi_0 \right) \; \tau_s
\end{aligned}
$$

where

$$
\begin{aligned}
v_c &= \left(\frac{3\sqrt{\pi}}{4} \frac{m_e}{m_i} \right)^{1/3} \; v_{th,e} \\
&= \text{the value of the hot ion velocity} \\
&\quad \text{at which the rate of transfer of energy} \\
&\quad \text{from the hot ions to the background electrons} \\
&\quad \text{equals the rate of transfer of energy} \\
&\quad \text{from the hot ions to the background ions}
\end{aligned}
$$

NOTE in **Connor Cordey 1974** it is used the *critical energy*,

$$w_c = \left(\frac{3\sqrt{\pi}}{4} Z_i \right)^{2/3} \left(\frac{m_i}{m_e} \right)^{1/3} \frac{m_{hot}}{m_i} T_e$$

End

and

$$v_0 = \text{peak of the injection velocity}$$

$$\tau_s = \frac{m_i}{m_{hot}} v_c^3 \; n \; C_{hot}$$

The source is constant and all particles are injected at only one angle with the magnetic field which is

$$\arccos \xi_0 = \arccos \left(\frac{v_{\parallel 0}}{v_0} \right)$$

NOTE
the absence of the space-dependent part (convective derivative) in the FP equation.

This is because the NBI are directed along an initial, particular, direction relative to the magnetic field, they are NOT isotropic.

By contrast, for the fast *alpha* particles, the FP must contain a term

$$\left(v_{\parallel} \widehat{\mathbf{n}} + \mathbf{v}_D \right) \cdot \boldsymbol{\nabla} f_{\alpha}$$

and this leads to the first neoclassical correction (**Hsu Shaing Gromley Sigmar**)

$$f_{\alpha} = -I \frac{v_{\parallel}}{\Omega_c} \frac{\partial f_{\alpha 0}}{\partial \psi} + P \left(\lambda, w, \psi \right)$$

For the α particles the initial distribution is isotropic.

This is the difference:

- the NBI ions are NOT isotropic, are directed; then there is no neoclassical correction
- the α particles are isotropic; then there is a neoclassical correction.

END

NOTE
that in **Fowler code** the term of slowing down (the first, not the pitch angle scattering) is explicitely separated into two contributions, from electrons and from ions.

The equation in **Fowler** is (see **below**)

$$f \equiv f^{fast-ions}$$

and the variable x is NOT space, it is the variable in the velocity space

$$\begin{aligned}
\tau_s \frac{\partial f}{\partial t} &= -\frac{\tau_s}{\tau_{cx}} f \qquad \text{charge exchange} \\
&+ \frac{1}{x^2}\frac{\partial}{\partial x}\left[\left(x^3 - 2Bx + x_c^3 + \frac{C}{x^2}\right) f\right] \qquad \text{drag} \\
&+ \frac{1}{x^2}\frac{\partial^2}{\partial x^2}\left[\left(Bx^2 + \frac{C}{x}\right) f\right] \qquad \text{diffusion in velocity} \\
&+ \frac{D}{x^3}\left(1 - \frac{D_1}{x^2} + D_2 x\right)\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) \text{ angular scattering} \\
&+ \tau_s \sum_l \dot{n}_{f_l} S_l(x,\theta) \quad \text{source of injected ions}
\end{aligned}$$

without electric field E^* and without compression $\dot{R}$ of the plasma column. Here

$$\begin{aligned}
x &= \frac{v}{v_0} \\
v_0 &= \text{the speed of the NBI ions}
\end{aligned}$$

$$x_e = \frac{v_{th,e}}{v_0} \;,\; x_i = \frac{v_{th,i}}{v_0}$$

$$x_c = \text{critical velocity} = \frac{v_c}{v_0}$$

The terms that we discuss are the "drag" and the "diffusion in velocity". The coefficients in these terms are

$$\begin{aligned}
B &= \frac{1}{2}\frac{m_e}{m_{fast}} x_e^2 \\
C &= \frac{1}{2}\frac{m_i}{m_{fast}} x_i^2 x_c^3
\end{aligned}$$

and this shows the separation between electrons and ions.

Remark that in **Cordey Houghton** (and later in other works, like **Rosenbluth Hinton 1996**) the *diffusion* of f in the velocity space

$$\frac{\partial^2 f}{\partial x^2} \text{ or } \frac{\partial^2 f}{\partial v^2}$$

is *ABSENT*. Then the distribution function is exactly zero for velocities *higher* than the initial, injection, one: v_0,

$$\begin{aligned}
f &\sim \mathbf{H}(v_b - v) \\
\mathbf{H} &\equiv \text{Heaviside}
\end{aligned}$$

Also, in **Gaffey** the solutions obtained for:

- (1) stationary, no source;
- (2) stationary, with source;
- (3) with time variation (Laplace transform);
- (4) with electric field;

- are all *without diffusion* in velocity.
END of the NOTE

The solution of the equation for f.
The form of the *pitch angle scattering* operator suggests to use Legendre polynomials.
It suggests also to use *separation of variables* v and ξ.

It is adopted the series expansion

$$\begin{aligned} f &= \sum_{n=0}^{\infty} A_n(v) \ P_n(\xi) \\ P_n(\xi) &\equiv \text{Legendre polynomials} \end{aligned}$$

Approximation adopted by **Cordey Houghton** for the range of velocities

$$v > v_c$$

Here is the domain where the NBI ions are slowed down collisionally by the *electrons*. Then the term of *pitch angle scattering* can be neglected. There is mainly *slowing down*.

The remaining equation is solved for the following adopted form of the source term

$$\begin{aligned} \widetilde{S}(v - v_0) &= S \frac{\exp\left[-\frac{(v-v_0)^2}{\delta^2}\right]}{v^2 \delta \sqrt{\pi}} \\ S &\equiv \text{number of particles injected per second} \\ &\quad \text{per unit volume} \end{aligned}$$

We note that the source is *smeared out* with extension in velocity space around the central NBI-ions velocity v_0 of width δ. A Gaussian shape.

Define

$$v^*(v,t) = \left[\left(v^3 + v_c^3\right) \exp\left(\frac{3t}{\tau_s}\right) - v_c^3\right]^{1/3}$$

The solution is

$$
\begin{aligned}
f &= \tau_s \frac{1}{v^3+v_c^3}\delta\left(\xi-\xi_0\right)\int_v^{v^*} dv' \; v'^2 \; \widetilde{S}\left(v'-v_0\right) \\
&= \tau_s \frac{1}{v^3+v_c^3}\delta\left(\xi-\xi_0\right) \; S \\
&\qquad\qquad\qquad \times\frac{1}{2}\left[\operatorname{erf} c\left(\frac{v-v_0}{\delta}\right)-\operatorname{erf} c\left(\frac{v^*-v_0}{\delta}\right)\right]
\end{aligned}
$$

where

$$
\begin{aligned}
\operatorname{erf} c\left(x\right) &= 1-\operatorname{erf}\left(x\right) \\
&= \text{complimentary error function}
\end{aligned}
$$

The last expression is written

$$
f \equiv r\left(v\right) \; \delta\left(\xi-\xi_0\right)
$$

(the distribution function remains strictly localized on the *pitch angle* ξ_0 of initial condition).

For large time

$$
\begin{aligned}
t &\rightarrow \infty \\
v^* &\rightarrow \infty
\end{aligned}
$$

the second erf function is zero and for

$$
\begin{aligned}
\frac{v-v_0}{\delta} &\gg 1 \\
(\delta &\equiv \text{half width of the source)} \\
&\quad \text{(the spreading in velocity } v \text{ is large)}
\end{aligned}
$$

Then

$$
f = \tau_s \frac{1}{v^3+v_c^3} \; \delta\left(\xi-\xi_0\right) \; S
$$

This is a solution obtained after the approximations have reduced the problem to *one dimension.*

Exact solution by series: little angular spreading for

$$
v > v_c
$$

This means that the scattering and slowing down of the hot ions *on electrons* does not produce spreading in

$$\xi = \frac{v_{\|}}{v}$$

This is because the ions are heavy and when they collide with *electrons* (since $v > v_c$) there is little pitch angle scattering and more transfer of energy, *i.e. slowing down*.

The spreading begins when the hot ions slow down and pitch-angle-scatter on background ions, equally heavy.

The paper also discusses the formation of an electric field due to the separation of charges after ionization.

Summary

The Rosenbluth potentials g and h.

The derivation of a *simple* form of the equation FP, with clear separation in *slowing down* and *pitch angle scattering.*

Approximation in the high NBI-ions-velocity limit, compared with critical v_c, $\rightarrow$ neglect of pitch angle. Eq. becomes one-dimensional, it is integrated giving ERFC functions. They can be approximated in certain regimes and give a function

$$f \sim \frac{S}{v^3 + v_c^3}$$

The solution based on the usual expansion and separation of vaiables $f = \sum a_n(v) P_n(\xi)$ confirms the result in the regime where the approximations have been made.

3 NBI effects on toroidal equilibria Connor Cordey 1974

The equation for NBI *hot ions* with anisotropic source

It is cited **Ohkawa 1970 NF**, the origin of the idea to produce current by NBI.

Remark the parallel convection which turns out to only be poloidal convection since $f \sim f(\theta)$

$$\begin{aligned}
& \frac{\partial f_h}{\partial t} - \frac{B_\theta}{B} v_\parallel \frac{\partial f_h}{r \partial \theta} \\
= \; & \frac{1}{\tau_s} \left[\frac{2}{\sqrt{w}} \frac{\partial}{\partial w} \left[\left(w_c^{3/2} + w^{3/2} \right) f_h \right] \right. \\
& \quad + 2\mu \frac{w_c^{3/2} + w^{3/2}}{w^{3/2}} \frac{\partial f_h}{\partial \mu} \\
& \quad \left. + \frac{m_i}{m_h} \left(\frac{w_c}{w} \right)^{3/2} v_\parallel \frac{1}{B} \frac{\partial}{\partial \mu} \left(\mu v_\parallel \frac{\partial f_h}{\partial \mu} \right) \right] \\
& + S
\end{aligned}$$

where

$$\begin{aligned}
w &= \frac{v^2}{2} \\
\mu &= \frac{v_\perp^2}{2B} \\
v_\parallel &= \sqrt{2(w - \mu B)}
\end{aligned}$$

$$\tau_s = \frac{3}{4\sqrt{2\pi}} \frac{m_h}{\sqrt{m_i}} \frac{1}{e^4 Z_h^2} \frac{1}{\ln \Lambda} \frac{T_e^{3/2}}{n}$$

The critical energy

$$w_c = \left(\frac{3\sqrt{\pi}}{4} Z_i \right)^{2/3} \left(\frac{m_i}{m_h} \right)^{1/3} \frac{m_h}{m_i} T_e$$

The ratio

$$\begin{aligned}
\frac{B_\theta}{B} &\equiv \Theta(r) \\
& \quad \text{independent of } \theta
\end{aligned}$$

Expansion in small parameter

$$\frac{\tau_{bounce}^{hot-ion}}{\tau_s} = \frac{\text{time of bounce of hot ion}}{\text{time of slowing down}} \ll 1$$

fast bouncing compared with rare collisions.

Particular choice of expression for parameters

$$B_0 = \text{minimum of the magnetic field on a surface}$$

is the magnetic field at the farthest point on the equatorial plane, in plasma.

$$B = B_0 \frac{1-\varepsilon\cos\theta}{1-\varepsilon}$$

In the expansion that will provide the solution.

The lowest order for f_{h0} is *independent on* θ.

$$\frac{\partial f_{h0}}{\partial\theta} = 0$$

or

$$f_h = f_h(\mu, w, r)$$

Procedure:

one writes the next order in $\tau_{bounce}^{hot-ion}/\tau_s$ and imposes the constraint of periodicity (which means the eq. is multiplied by B then averaged over the magnetic surface): this gives zero. The constraint becomes an equation for the zeroth order f_{h0}.

$$\begin{aligned}\tau_s \frac{\partial f_{h0}}{\partial t} &= \frac{2}{\sqrt{w}}\frac{\partial}{\partial w}\left[\left(w_c^{3/2}+w^{3/2}\right) f_{h0}\right] \\ &+ \frac{1}{4}\frac{m_i}{m_h}\left(\frac{w_c}{w}\right)^{3/2} \frac{B_0}{\eta w \left\langle \frac{1}{v_\parallel}\right\rangle}\frac{\partial}{\partial\eta}\left[\frac{(1-\eta^2)\left\langle\frac{v_\parallel}{B}\right\rangle}{\eta}\frac{\partial f_{h0}}{\partial\eta}\right] \\ &+ \tau_s S\end{aligned}$$

where

$$\eta = \sqrt{1-\frac{\mu B_0}{w}} = \sqrt{1-\frac{v_\perp^2}{v^2}\frac{B_{\min}}{B}}$$

Since at $\theta = 0$ the magnetic field is

$$B(\theta=0) = B_0\frac{1-\varepsilon}{1-\varepsilon} = B_0 = B_{\min}$$

we see that they define θ to be measured from the equatorial plane and we have

$$\begin{aligned}\frac{B_0}{B} &= \frac{B_{\min}}{B_{\min}\frac{1-\varepsilon\cos\theta}{1-\varepsilon}} = \frac{1-\varepsilon}{1-\varepsilon\cos\theta} = (1-\varepsilon)\left(1+\varepsilon\cos\theta+\varepsilon^2\cos^2\theta+...\right) \\ &= 1-\varepsilon+\varepsilon\cos\theta-\varepsilon^2\cos\theta+\varepsilon^2\cos^2\theta-\varepsilon^3\cos^2\theta+...\end{aligned}$$

which means

$$\frac{B_{\min}}{B} = h - \varepsilon + O\left(\varepsilon^2\right)$$

and

$$\eta = \sqrt{1 - \frac{v_\perp^2}{v^2}\left(h - \varepsilon\right)}$$

then η does not use λ, the conventional variable in velocity space

$$\lambda^{conventional} = \frac{v_\perp^2}{v^2} h$$

Note that

$$\begin{aligned} h - \varepsilon &= 1 + \varepsilon\cos\theta - \varepsilon \\ &= \frac{B_0}{B} \quad (\text{in } \textbf{Cordey trapped}) \\ &= \frac{1-\varepsilon}{1-\varepsilon\cos\theta} \\ &\approx (1-\varepsilon)(1+\varepsilon\cos\theta) = 1 - \varepsilon + \varepsilon\cos\theta \quad \text{OK} \end{aligned}$$

Then

$$\left\langle \frac{v_\parallel}{B} \right\rangle = \begin{cases} \oint \frac{d\theta}{B}\sqrt{2\left(w - \mu B\right)} & \text{for} \qquad \text{passing hot ions} \\ \int_A^B \frac{d\theta}{B}\sqrt{2\left(w - \mu B\right)} & \text{for} \qquad \begin{array}{c}\text{trapped ions} \\ A, B \text{ are turning points}\end{array} \end{cases}$$

The calculation of an *analytic* expression for the averages $\left\langle \frac{v_\parallel}{B} \right\rangle$ and $\left\langle \frac{1}{v_\parallel} \right\rangle$ can be done near

$$\eta \approx 0$$

where

$$\begin{aligned} v^2 &= v_\perp^2\left(h - \varepsilon\right) \\ &= v_\perp^2\left[1 + \varepsilon\left(\cos\theta - 1\right)\right] \end{aligned}$$

If we assume that ε is small this is close to deep trapped.

$$\left\langle \frac{v_\parallel}{B} \right\rangle \approx \frac{\pi}{2B_{\min}}\sqrt{\frac{w}{\varepsilon}}\,\eta^2$$

$$\left\langle \frac{1}{v_\parallel} \right\rangle \approx \frac{\pi}{\sqrt{w\varepsilon}}$$

The opposite limit is

$$\begin{aligned} \eta & \rightarrow 1 \\ & \text{hot ions are passing} \end{aligned}$$

Solution in the case of all hot ions deep trapped

$$\eta \ll 1$$

Better expressions are in **Cordey trapped**.

The operator that includes two η-derivations becomes operator for Bessel functions.

$$f_{h0} = \sum_{n=0}^{\infty} a_n (w) \;\; J_0 (j_n \eta)$$

J_0 is Bessel function and j_n are zeros of $\frac{dJ_0}{dx}$.

Explanation

The differential equation for the eigenfunction $\Lambda_n (\xi)$ is written in **Cordey**

End

The equation

$$\begin{aligned} & \frac{2}{\sqrt{w}} \frac{d}{dw} \left[\left(w_c^{3/2} + w^{3/2} \right) a_n \right] \\ & - \frac{1}{4} \frac{m_i}{m_h} \left(\frac{w_c}{w} \right)^{3/2} j_n^2 \;\; a_n \\ = & \frac{\int_0^1 S \;\; J_0 (j_n \eta) \;\; \eta d\eta}{\int_0^1 J_0^2 (j_n \eta) \;\; \eta d\eta} \end{aligned}$$

The solution is obtained analytically.

$$\begin{aligned} a_n = & \frac{1}{2} \frac{w_c^{\frac{3}{2}\rho}}{\left(w_c^{3/2} + w^{3/2} \right)^{1+\rho}} \\ & \times \int_w^{\infty} dw \; w^{\frac{1}{2} - \frac{3}{2}\rho} \;\; \left(w^{3/2} + w_c^{3/2} \right)^{\rho} \; \frac{\int_0^1 S \;\; J_0 (j_n \eta) \;\; \eta d\eta}{\int_0^1 \eta J_0^2 (j_n \eta) \;\; \eta d\eta} \end{aligned}$$

$$\rho = \frac{1}{12} \frac{m_i}{m_h} j_n^2$$

The idea of Ohkawa is considered: to use two neutral beams

- one is high energetic, intended to introduce energy in the target plasma; but it also introduces momentum

- the other beam is opposite, and is low energy but is high current.

This combination will mainatin balance of momentum, not total displacement of the plasma.

The momentum that is obtained by the background ions is the momentum lost by the hot ions

$$m_i n_i u_i + m_{hot} n_{hot} u_{hot} = 0$$

The current is also calculated.

The current carried by the ions

$$\begin{aligned} j_i &= eZ_{hot}\ n_{hot} u_{hot} - eZ_i\ n_i u_i \\ &= eZ_{hot} n_{hot} u_{hot} \left(1 - \frac{m_{hot} Z_i}{m_i Z_{hot}}\right) \end{aligned}$$

For the current carried by the electrons, one has to solve the kinetic equation

$$\begin{aligned} & v_\parallel \frac{B_\theta}{B} \frac{\partial f_e^{(1)}}{r\partial\theta} - \frac{m_e}{e} v_\parallel \frac{\partial}{r\partial\theta}\left(\frac{v_\parallel}{B}\right) \frac{\partial}{\partial r} F_{Me} \\ = \ & C_{ei}\left[f_e^{(1)}\right] + C_{ee}\left[f_e^{(1)}\right] + C_{e,hot}\left[F_{Me}\right] \end{aligned}$$

where

$$C_{ei}\left[f_e^{(1)}\right] = \text{Lorentz operator corrected for moving ion}$$

$$C_{ee}\left[f_e^{(1)}\right] = \text{operator pitch-angle plus compensating momentum-conservation (Kovrizhnykh)}$$

$$C_{e,hot}\left[F_{Me}\right] = \text{collision operator between electrons and hot ions}$$

To calculate $C_{e,hot}$ one must replace the hot distribution function

$$f_{hot,0}$$

in the Rosenbluth potentials.

Use

$$\eta = \sqrt{1 - \frac{v_\perp^2}{v^2}\left(h - \varepsilon\right)}$$

and in domains

$$\eta \ll 1$$

$$\begin{aligned} f_{hot,0} &= \sum_{n=0}^{\infty} a_n\left(w\right) J_0\left(j_n \eta\right) \\ \text{where} \quad j_n &\equiv \text{zeros of } \frac{dJ_0}{dx} = 0 \\ w &\equiv \frac{v^2}{2} \end{aligned}$$

and

$$\eta \sim 1$$

$$f_{hot,0} = \sum_{n=0}^{\infty} a_n\left(w\right) \; P_n\left(\eta\right)$$

After replacing in the Rosenbluth potentials, and

$$w > w_c$$

$$C_{e,hot}\left[f_e\right] = -\frac{2\pi \; Z_{hot}^2 e^4}{m_e^2} \ln \Lambda \; u_{hot} \cos \xi \; \frac{n_{hot}}{w} \frac{\partial f_e}{\partial w}$$

where

$$\begin{aligned} \xi &\equiv \text{angle between injection line} \\ &\quad\;\text{and magnetic line} \\ &= \{0 \;\; \text{or} \;\; \pi\} \end{aligned}$$

It is obtained, total current

$$\begin{aligned} j &= e \; n_{hot} \; u_{hot} Z_{hot} \\ &\quad \times \left\{1 - \frac{Z_{hot}}{Z_i} + 1.46\sqrt{\varepsilon}\left(\frac{Z_{hot}}{Z_i} - \frac{Z_i}{Z_{hot}} \frac{m_{hot}}{m_i}\right) \; A\left(Z_i\right)\right\} \end{aligned}$$

where A is a complicated expression, $\sim$ 1,..., 1.6

$$A\left(Z_i\right) = 1 + \frac{2.12}{3\sqrt{\pi} Z_i} \frac{\int_0^\infty \frac{h \exp(-x) \; x^{3/2}}{h + Z_i} dx}{\int_0^\infty \frac{h \exp(-x)}{h + Z_i} dx}$$

where

$$h(x) = \left(1 - \frac{1}{2x}\right) \eta(x) + \frac{d\eta(x)}{dx}$$

$$\eta(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t) \ \sqrt{t} \ dt$$

Return for comment of

$$\begin{aligned} j &= e \ n_{hot} \ u_{hot} Z_{hot} \\ &\quad \times \left\{1 - \frac{Z_{hot}}{Z_i} + 1.46\sqrt{\varepsilon} \left(\frac{Z_{hot}}{Z_i} - \frac{Z_i}{Z_{hot}} \frac{m_{hot}}{m_i}\right) \ A(Z_i)\right\} \end{aligned}$$

The first two terms are from Ohkawa. This is zero if

$$Z_{hot} = Z_i$$

There is a contribution from trapped electrons. It vanishes if

$$\frac{Z_{hot}^2}{Z_i^2} = \frac{m_{hot}}{m_i}$$

Comments regarding the effect of the beam on the
- particle pinch, and
- particle diffusion

The beam has the same effect as a toroidal electric field.
Then one can expect an effect similar to the Ware pinch.

The plasma diffusion caused by the beam is

$$\Gamma = 1.46 \ \frac{\sqrt{\varepsilon} \ m_e}{eB_\theta} \ n_{hot} \ u_{hot} \ \nu \ \left(1 + \frac{0.53}{Z_i}\right) \left(1 - \frac{m_{hot}}{m_i} \frac{Z_i^2}{Z_{hot}^2}\right)$$

$$\nu = \frac{4}{3} \frac{\sqrt{2\pi} \ Z_{hot}^2 e^4}{\sqrt{m_e}} \ln \Lambda \frac{n_e}{T_e^{3/2}}$$

It is taken

$$\begin{aligned} Z_i &= 1 \\ Z_{hot} &= Z \\ \frac{m_{hot}}{m_i} &= Z \end{aligned}$$

It is evaluated $A(1)$ and results

$$j = e\, n_{hot}\, u_{hot}\, Z\,(1-Z)\left(1-2.54\,\sqrt{\varepsilon}\right)$$

and the particle flux caused by the beam

$$\Gamma = 2.24\,\sqrt{\varepsilon}\frac{m_e}{eB_\theta}\, n_{hot}\, u_{hot}\, \nu\, \left(1-\frac{1}{Z}\right)$$

which may be comparable with the diffusion flux due to the gradient of density.

intersting part: the decay of toroidal rotation due to the ripple.

4 Fokker Planck equation for NBI ions Cordey Core

The next level.
(it is also in *collisions.tex*)
The equation
only velocity space and charge-exchange
but electric field.

The equation is for the *hot (fast) ions* resulted from Neutral Beam

$$\begin{aligned}
&\frac{\partial f}{\partial t}+\frac{Z_{hot}eE^*}{m_{hot}}\left(\frac{\left(1-\xi^2\right)}{v}\frac{\partial f}{\partial \xi}+\xi\frac{\partial f}{\partial v}\right)\\
= \;& C\,\left\{\frac{1}{2v^2}\frac{\partial}{\partial v^2}\left(v^2\frac{\partial^2 g}{\partial v^2}f\right)-\frac{1}{v^2}\frac{\partial}{\partial v}\left[f\left(v^2\frac{\partial h}{\partial v}+\frac{\partial g}{\partial v}\right)\right]\right.\\
&\qquad\qquad\left.-\frac{1}{2v^3}\frac{\partial g}{\partial v}\frac{\partial}{\partial \xi}\left(1-\xi^2\right)\frac{\partial f}{\partial \xi}\right\}\\
&-\frac{1}{\tau_{cx}}f\\
&+S\delta\left(v-v_0\right)K\left(\xi\right)
\end{aligned}$$

where

$$g,h \equiv \text{Rosenbluth potentials}$$

$$C=\frac{4\pi e^4 Z_{hot}^2}{m_{hot}^2}\log\Lambda$$

The modified electric field

$$E^{*} = E\left(1 - \frac{1}{Z}\right)$$

$$\xi = \frac{v_{\parallel}}{v}$$

$$Z = \sum \frac{n_i Z_i^2}{n_e}$$

NOTE

again the absence of the convective derivative, which is justified by the anisotropy of the beam, in contrast with the case of α particles.

Therefore no *drift* hence no neoclassical term.

END

Explanation regarding the term with the electric field.

There are two parts.

One is the *acceleration* of the new hot ions by the electric field.

The other is the *drag exerced against the new ions* by the electrons in their motion in the electric field. Electrons move in opposite direction than the (fast) ions and exert a friction.

To obtain the drag

$$\begin{aligned} h &= h_0 + h_1(v)\,\xi \ \times E \\ g &= g_0 + g_1(v)\,\xi \ \times E \end{aligned}$$

The two functions are determined through the method for Spitzer Harm.

The *spread* of the source in $\xi \equiv v_{\parallel}/v$ is represented by $K(\xi)$.

The Rosenbluth potentials.

Are simplified by assuming

$$v_i \ll v_{hot} \ll v_e$$

The theory is extended in the *energy of the hot ions* subjected to slowing down, up to

$$1.5 \times v_{th,i}$$

(probably in **Fowler** to $2 \times v_{th,i}$).

$$v^2 \frac{\partial h}{\partial v} + \frac{\partial g}{\partial v} = -\frac{4}{3\sqrt{\pi}} \frac{m_{hot}}{m_e} v^3 \frac{n_e}{v_{th,e}^3} - n = m_{hot} \sum_{j(ions)} \frac{n_j Z_j^2}{m_j}$$

$$\frac{\partial g}{\partial v} = \sum n_j Z_j^2 = n_e Z_{eff}$$

$$v^2 \frac{\partial^2 g}{\partial v^2} = \frac{4}{3\sqrt{\pi}} v^2 \frac{n_e}{v_{th,e}} + \frac{1}{v} \sum_{j(ions)} n_j Z_j^2 v_{th,j}^2$$

An approximation based on neutrality

$$\sum_{j(ions)} \frac{n_j Z_j^2}{m_j} \approx \frac{n_e}{m_i}$$

$$\sum_{j(ions)} n_j Z_j^2 v_{th,j}^2 = n_e v_{th,i}^2$$

and $Z_1 = 1$ (basic ions are hydrogen). Notation

$$\begin{aligned} u &\equiv \frac{v}{v_0} \\ v_0 &\equiv \text{velocity at birth} \end{aligned}$$

Then (remember everything is velocity space)

$$\begin{aligned} & \delta a \frac{\partial^2 f}{\partial u^2} \qquad \text{(diffusion)} \\ & + b \frac{\partial f}{\partial u} \qquad \text{(drag)} \\ & + d\ f \\ & + \delta a r\ \mathcal{L}(f) - \frac{\eta}{u}\left(1 - \xi^2\right) \frac{\partial f}{\partial \xi} \\ = & -\tau_s S\ \delta(u-1) K(\xi) \end{aligned}$$

where

$$\begin{aligned} a(u) &= 1 + \frac{2\beta}{u^3} \\ b(u) &= \frac{u_c^3}{u^3} + u - \eta\xi - 4\delta \frac{\beta}{u^3} + 4\delta \frac{1}{u} \end{aligned}$$

$$\begin{aligned} d(u) &= 3 - \frac{\tau_s}{\tau_{cx}} + \frac{4\delta\beta}{u^5} + \frac{2\delta}{u^2} \\ r(u) &= \frac{m_i Z_{eff}}{2 m_{hot}} \frac{u_c^3}{u^3} \frac{1}{\delta a} \end{aligned}$$

$$u_c^3 = \frac{v_c^3}{v_0^3}$$

$$v_c = \left(\frac{3\sqrt{\pi}}{4} \frac{m_e}{m_i} \right)^{1/3} v_{th,e}$$

$$\eta = \frac{Z_{hot} e E^*}{m_{hot} v_0} \tau_s$$

$$\mathcal{L} \equiv \frac{\partial}{\partial \xi} \left(1 - \xi^2\right) \frac{\partial}{\partial \xi}$$

$$\delta = \frac{1}{2} \frac{m_e v_{th,e}^2}{m_{hot} v_0^2} = \frac{1}{2} \frac{T_e}{\epsilon_0}$$

$$\beta = 0.66 \frac{v_{th,e} \ v_{th,i}^2}{v_0^3}$$

$$\tau_s = \frac{3}{16\sqrt{\pi}} \frac{m_e m_{hot}}{e^4 Z_h^2 \ln \Lambda} \frac{v_{th,e}^3}{n_e}$$

The expression contains

$$\frac{\partial^2 f}{\partial u^2} \ , \ \frac{\partial f}{\partial u} \ , \ f \ , \ \text{source}$$

They correspond to

- diffusion in velocity space $\frac{\partial^2 f}{\partial u^2}$
- drag (slowing down) $\frac{\partial f}{\partial u}$

The solution is obtained with the expansion and separation of variables.

Much more details in *collisions.tex*.

5 Fokker Planck equation for NBI ions + ICRH Cox Start

Also in *collisions*.

The generation of current by ICRH. The increase of the energy of the *minority* (heated) ions leads to reduction of the collisionality of these ions with the electrons. The background ions preserve the collisionality, so there will be a difference - a flux - resulting for the two ion types.

Corrected scenario: start with NBI but with different ions, like He^3 or heavy ions. Then ICRH.

The Fokker Planck

$$\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial t}\right)_{wave} + \left(\frac{\partial f}{\partial t}\right)_{C} + S_f\left(v,\xi\right)$$

The collisions between the hot ions and a *Maxwellian* plasma.
The magnetic field is uniform.

5.1 Collisions

Coulomb collision operator for

$$v_{th,i} \ll v_{hot} \ll v_{th,e}$$

Assumptions:

- there is no *diffusion* $\frac{\partial^2 f}{\partial u^2}$ in velocity space caused by electron collisions.
- the magnetic field is zero
- in the collision operator it remains the *slowing down* of fast ions on both background electrons and ions + pitch angle scattering (it is as if the ions were under the critical velocity v_c)

Then

$$\left(\frac{\partial f}{\partial t}\right)_{C} = \frac{1}{\tau_s}\left[b\left(u\right)\frac{\partial f}{\partial u} + d\left(u\right)\ f + r\left(u\right)\ Lf\right]$$

where

$$\begin{aligned} u &\equiv \frac{v}{v_0} \\ v_0 &\equiv \text{injection velocity} \end{aligned}$$

$$b\left(u\right) = \frac{u_c^3}{u^2} + u$$

$$\begin{aligned} u_c^3 &= \frac{v_c^3}{v_0^3} \\ &= \frac{3\sqrt{\pi}}{4}\ \frac{m_e}{m_i}\overline{Z}\ \frac{v_{th,e}^3}{v_0^3} \end{aligned}$$

$$d = 3 - \frac{\tau_s}{\tau_{cx}}$$

$$\tau_s = \frac{1}{4}\frac{3}{4\sqrt{\pi}}\; m_e m_h \frac{1}{Z_h^2 e^2}\frac{1}{\ln\Lambda}\;\frac{v_{th,e}^3}{n_e}$$

$$r(u) = \beta\frac{u_c^3}{u^3}$$

$$\beta = \frac{1}{2}\frac{m_i}{m_h}\frac{Z_{eff}}{\overline{Z}}$$

$$Z_{eff} = \sum_i \frac{n_i Z_i^2}{n_e}$$

$$\overline{Z} = \sum_i \frac{m_h}{m_i}\frac{n_i Z_i^2}{n_e}$$

$$L \equiv \frac{\partial}{\partial\xi}\left(1-\xi^2\right)\frac{\partial}{\partial\xi}$$

$$\xi \equiv \frac{v_\parallel}{v}$$

REMARK
No diffusion, since no second order derivation in the operator.

5.2 Wave

Now there is *diffusion* in velocity space, due to the wave.

The effect of the wave is a diffusion in the space of the velocity

$$\left(\frac{\partial f}{\partial t}\right)_W = D_c \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\left[v_\perp \frac{\partial f}{\partial v_\perp}\right]\delta\left(\omega - \Omega_c\right)$$

After change of variables

$$\left(v_\parallel, v_\perp\right) \;\rightarrow\; (u,\xi)$$
$$\left(u = \frac{v}{v_0}\right)$$

$$\begin{aligned}\left(\frac{\partial f}{\partial t}\right)_w &= \frac{\gamma}{\tau_s}\frac{1}{u} \\ &\times\left[\left(1-\xi^2\right)u\frac{\partial^2 f}{\partial u^2} + \left(1+\xi^2\right)\frac{\partial f}{\partial u} - 2\xi\left(1-\xi^2\right)\frac{\partial^2 f}{\partial u\partial\xi}\right. \\ &\left. + \frac{\xi}{u}\frac{\partial}{\partial\xi}\left(1-\xi^2\right)\xi\frac{\partial f}{\partial\xi}\right]\end{aligned}$$

$$\gamma = \frac{D_c}{v_0^2}\tau_s$$

5.3 In the absence of the wave

Without the wave,

no diffusion in u, but possibly in ξ

$$\begin{aligned} & b(u)\frac{\partial F}{\partial u} + dF \\ & +\beta\frac{u_c^3}{u^3}\frac{\partial}{\partial \xi}\left(1-\xi^2\right)\frac{\partial F}{\partial \xi} \\ & +\tau_s S\frac{K(\xi)}{2\pi K_0 v_0^3}\ \delta(u-1) \\ = & \ 0 \end{aligned}$$

The solution, *separation of variables*, u and ξ

$$F(u,\xi) = \sum_{n=0}^{\infty} a_n^0(u)\ \ P_n(\xi)$$

The equation becomes a system

$$\begin{aligned} & b\frac{da_n^0}{du} + da_n^0 - \beta\frac{u_c^3}{u^3}n(n+1)\,a_n^0 \\ & +\frac{(2n+1)\,\tau_s S\ K_n}{4\pi v_0^3 K_0}\delta(u-1) \\ = & \ 0 \end{aligned}$$

where

$$K(\xi) = \text{angular distribution of the beam}$$

$$K_n = \int_{-1}^{1} d\xi\ K(\xi)\,P_n(\xi)$$

The integration of the eq. gives a_n^0 as

$$a_n^0(u) = \begin{cases} C_+A_n(u) & \text{for} \quad u>1 \\ C_-A_n(u) & \text{for} \quad u<1 \end{cases}$$

where

$$A_n(u) = u^{n(n+1)\beta}\ \left[\frac{1+u_c^3}{u^3+u_c^3}\right]^{\frac{d+n(n+1)\beta}{3}}$$

and

$$C_+ \text{ and } C_- \text{ are constants}$$

It results that the velocity of injection v_0 is an important limit.

The equation for a_n^0 is integrated from just below 1^- to just above 1^+ this limit.

$$a_n^0\left(1^-\right) - a_n^0\left(1^+\right) = \frac{(2n+1)\ \tau_s\ K_n}{4\pi\ v_0^3\left(1+u_c^3\right)\ K_0}$$

The asymptotic condition on f is a constraint that should determine the constants $C_\pm$.

The condition is

$$fu^3 \to 0 \text{ for } u \to \infty$$

(see **Cordey Houghton**).

Then

$$C_+ = 0$$

The final form of the functions a is

$$a_n^0(u) = \begin{cases} \frac{(2n+1)\ \tau_s\ K_n\ S}{4\pi v_0^3(1+u_c^3)\ K_0} A_n(u) & \text{for } u < 1 \\ 0 & \text{for } u > 1 \end{cases}$$

The function F is zero above the velocity of injection, v_0.

This is because there is NO diffusion of velocity.

Collisions just decrease the energy of the fast ions.

6 Cordey Jones Start Curtis Jones 1979, kinetic theory NBCD

See **Connor Cordey 1974**.

Equations for the collisional effects on the electrons.

$$\begin{aligned} & C_{e,beam}\left(F_{Me}, f_{beam}\right) + C_{e,i}\left(f_e'\ ,\ F_{Mi}\right) \\ & + C_{e,e}\left(f_e'\ ,\ F_{Me}\right) + C_{e,e}\left(F_{Me}\ ,\ f_e'\right) \\ = \ & 0 \end{aligned}$$

The unknown is f_e'. It is expressed by sum over terms with *separated variables.*

$$j_e = -e \int d^3v\ v_\parallel\ f_e'$$

The total current is "beam+electrons"

$$
\begin{aligned}
j &= K_1 \; e n_{beam} \; v_{beam} \\
&\quad -\frac{4}{3}\frac{e \; v_{th,e} \; n}{\sqrt{\pi}} I_3(\infty)
\end{aligned}
$$

and can be positive or negative.

The factor

$$
\begin{aligned}
F &= \frac{j}{K_1 \; e n_{beam} v_{beam}} \\
&= 1 - \frac{16}{3\sqrt{\pi}} \frac{I_3(\infty)}{v_b^* B}
\end{aligned}
$$

where

$$
v_b^* = \frac{v_b}{v_e}
$$

can become negative.

$$
\begin{aligned}
F &= 1 - \frac{1}{Z} - \frac{v_b^*}{Z} \\
&\quad \text{for small } v_b^*
\end{aligned}
$$

The net current is in the opposited direction relative to the beam for

$$
\begin{aligned}
Z &= 1 \\
v_b^* &\sim 1.3
\end{aligned}
$$

7 Start Cordey Jones trapped electrons 1980

"The effect of trapped electrons on beam driven currents in toroidal plasmas"

This paper is very useful because it separates the distribution function of the electrons in parts that do not depend on θ so they do not contain the trapped particles and parts that are dependent on trapped particles and depend on *magnetic mirror* $\sim \theta$.

There is neoclassical drift ($\mathbf{v}_D$) convection of f_{beam} (or *fast*).

Collision operator is already separated into *slowing-down* and the *pitch angle*.

Collisions are

- fast ions with electrons
- electrons with electrons
- electrons with background ions
- further, fast ions with background ions

There is trapping for the electrons.
The trapping inhibits the *reversed current* of the electrons:

- reduce the number of current carrying particles
- introduce a new friction force, collisions between circulating electrons and trapped ones

Another effect of trapping:

- due to the *injection of momentum* from the fast ions, the thermal ions begin to rotate, in the same direction
- this should be an electric current too
- normally (with full mobility of electrons, no trapping) the electrons follow the moving ions (collisionally) and so there is NO electric current
- but now, we have trapped electrons. Then some electrons cannot follow the rotating ions and so there is umbalanced flow ions/electrons and then an electric current

There is expansion in

$$\frac{\tau_{bounce}}{\tau_{slowing-down}} \ll 1$$

(very fast bounce)

The zeroth order does not depend on θ. For the first order one uses periodicity which becomes a constraint for the zeroth order, an equation to be solved. It is calculated $\left\langle v_{\|}/v \right\rangle$ and $\left\langle v/v_{\|} \right\rangle$. Very useful figures of these averages, showing the singularity at $\xi/\sqrt{2\varepsilon} = 1$, where $\xi = \sqrt{1 - \frac{\mu B_0}{\epsilon}}$. The discontinuity of the derivative to ξ of the function f at the boundary ξ_t. Solution obtained with *separation of variables*, (v, ξ) after f_0 is represented as a series of terms-product of factors.

The distribution function for *ELECTRONS*
It is expanded

$$f_e = F_{Me} + f_e^{(1)}$$

The Fokker Planck equation

$$\begin{aligned} & C_{e-fast}\left[f_{Me}, f_{fast}\right] \\ & +C_{e-i}\left[f_e^{(1)}, F_{Mi}\right] \\ & +C_{e-e}\left[f_e^{(1)}, F_{Me}\right] + C_{e-e}\left[F_{Me}, f_e^{(1)}\right] \\ = \; & v_{\parallel}\frac{B_\theta}{B_T}\frac{\partial f_e^{(1)}}{r\partial\theta} \end{aligned}$$

$$C \equiv \text{linearized operator}$$

where

$$\begin{aligned} f_{fast} &= \sum_n a_{n,fast} P_n\left(\xi\right) \\ \xi &\equiv \frac{v_{\parallel}}{v} \end{aligned}$$

where

$$a_{n,fast} = \left(n + \frac{1}{2}\right) n_{fast} \frac{1}{2\pi v_b^2} K_n \delta\left(v - v_b\right)$$

the coefficient K_n is obtained by projecting on the set of Legendre plynomials $\left(P_n\left(\xi\right)\right)$ the function $K_n\left(\xi\right)$.

$$K_n = \int_{-1}^{+1} d\xi \; K_n\left(\xi\right) P_n\left(\xi\right)$$

To solve the equation.
Expansion in

$$\sqrt{\varepsilon}$$

The distribution function is composed

$$\begin{aligned} f_e^{(1)} &= f_e^{(1)(0)} \left(\begin{array}{c} \text{solution of FP without trapped electrons} \\ \text{equiv. without } \theta \text{ variation} \end{array}\right) \\ & +h^{(0)} \quad \text{(localized part of the dist. function)} \\ & +f^{*} \quad \text{(non-localized part of dist. function)} \end{aligned}$$

The first part $f_e^{(1)(0)}$ is the solution of the FP equation without the neoclassical convective term (RHS), which introduces the θ variation through the dependence $\mathbf{v}_D(\theta)$:

$$\begin{aligned} & C_{e-fast}\left[f_{Me}, f_{fast}\right] \\ & +C_{e-i}\left[f_e^{(1)(0)}, F_{Mi}\right] \\ & +C_{e-e}\left[f_e^{(1)(0)}, F_{Me}\right]+C_{e-e}\left[F_{Me}, f_e^{(1)(0)}\right] \\ = & \ 0 \end{aligned}$$

which means that the *trapped electrons* (equiv. the *magnetic mirror effect*) are neglected. This results in the absence of the variation with θ of

$$\begin{aligned} f_e^{(1)(0)} & \not\sim \theta \\ \frac{\partial f_e^{(1)(0)}}{\partial \theta} & = 0 \end{aligned}$$

since the variation with θ comes from the trapped particles.

The part $h^{(0)}$ is the *localized* part of the distribution function. It depneds on θ.

The part f^* is the non-localized part, to the order $\sqrt{\varepsilon}$.

The expansion in series of products of Legendre polynomials of ξ and coefficients a_n must be adapted to the three components of the distribution function

$$f_e^{(1)(0)}=F_{Me}\sum_n f_n^{(1)(0)}\ P_n(\xi)$$

$$h^{(0)}=F_{Me}\sum_n h_n^{(0)}\ (v,\theta)\ P_1(\xi)$$

$$f^*=F_{Me}\sum_n f_n^*(v,\theta)\ P_1(\xi)$$

The electron current depends on the following coefficients from these expansions

$$\begin{aligned} & f_1^{(1)(0)} \\ & h_1^{(0)} \\ & f_1^* \end{aligned}$$

We note that in the expansions for the components $h^{(0)}$ and f^{*} the Legendre functions $P_1(\xi)$ will provide the dependence on θ.

But, in addition, their coefficients

$$
\begin{aligned}
& h_1^{(0)}(v,\theta) \\
& f_1^{*}(v,\theta)
\end{aligned}
$$

also have some θ dependence.

To deal with *this* θ dependence, one has to take the surface averaging.

The surface averaging will only be acted upon these coefficients, not the Legendre functions.

$$
\begin{aligned}
\left\langle h_1^{(0)} \right\rangle &= \int_0^{2\pi} \frac{d\theta}{2\pi}\ h_1(v,\theta) \\
\left\langle f_1^{*} \right\rangle &= \int_0^{2\pi} \frac{d\theta}{2\pi}\ f_1^{*}(v,\theta)
\end{aligned}
$$

It exists the connection

$$
\begin{aligned}
& C_e\left[f^{*}\right] \quad (\text{def.}) \\
\equiv\ & C_{ee}\left[F_{Me}, f^{*}\right] + C_{ee}\left[f^{*}, F_{Me}\right] \\
& + C_{ei}\left[f^{*}, F_{Mi}\right] \\
=\ & -\left(C_e - C_e^{(0)pitch-angle}\right)\left[h^{(0)}\right]
\end{aligned}
$$

The pitch angle scattering part of the operator is separated from $C_e\left[h^{(0)}\right]$. This is

$$
\begin{aligned}
& C_e^{(0)pitch-angle} \\
=\ & C_{ei}\left[F_{Mi}, h^{(0)}\right] \\
& + C_{ee}\left[F_{Me}, h^{(0)}\right]
\end{aligned}
$$

This has been calculated

$$
\begin{aligned}
& C_e^{(0)pitch-angle}\left[h^{(0)}\right] \\
=\ & F_{Me}\ n_e\ \Gamma_e\ \frac{1}{4v_{the}^3}\frac{1}{x^5} \\
& \times\left[x\frac{dE(x)}{dx} + \left(2x^2-1\right)E(x) + 2Z_{eff}\ x^2\right] \\
& \times\frac{\partial}{\partial\xi}\xi\left(1-\xi^2\right)\frac{\partial}{\partial\xi}h^{(0)}
\end{aligned}
$$

$$
x \equiv \frac{v}{v_{the}}
$$

$$\Gamma_e = \frac{4\pi e^4}{m_e^2} \ln \Lambda$$

$$E(x) = \mathrm{erf}(x)$$

$$Z_{eff} = \frac{\sum n_i Z_i^2}{n_e}$$

We have from previous calculations (**Hazeltine Rosenbluth**) the solution to the equation

$$\begin{aligned} C_e[f^*] &= -\left(C_e - C_e^{(0)pitch-angle}\right)\left[h^{(0)}\right] \\ &= C_{ee}[F_{Me}, f^*] + C_{ee}[f^*, F_{Me}] + C_{ei}[f^*, F_{Mi}] \end{aligned}$$

and here we replace the Legendre expansions of

$$f_e^{(1)(0)} = F_{Me} \sum_n f_n^{(1)(0)} \; P_n(\xi)$$

$$h^{(0)} = F_{Me} \sum_n h_n^{(0)} \; (v, \theta) \; P_1(\xi)$$

$$f^* = F_{Me} \sum_n f_n^*(v, \theta) \; P_1(\xi)$$

One obtains a set of uncoupled equation for the coefficients

$$\begin{aligned} & \frac{d^2 a_1}{dx^2} + U(x) \frac{da_1(x)}{dx} + V(x) \\ & - \frac{16}{3\sqrt{\pi}} \frac{1}{\Lambda(x)} \left[x I_3(x) - 1.2x \; I_5(x) - x^4 \left(1 - 1.2x^2\right) \left(I_0(x) - I_0 \infty\right) \right] \\ = \; & R(x) \end{aligned}$$

Here

$$a_1 = \langle f_1^* \rangle + \left\langle h_1^{(0)} \right\rangle$$

$$U(x) = -\frac{1}{x} - 2x + 2x^2 \frac{1}{\Lambda(x)} \frac{dE(x)}{dx}$$

$$V(x) = \frac{1}{x^2} - 2\frac{1}{\Lambda(x)} \left(Z_{eff} + E - 2x^2 \frac{dE(x)}{dx} \right)$$

$$\Lambda(x) \equiv E(x) - x \frac{dE(x)}{dx}$$

$$R(x) \equiv \left\langle h_1^{(0)} \right\rangle \frac{1}{\Lambda(x)} \frac{1}{x^2} \left[x \frac{dE(x)}{dx} + \left(2x^2 - 1\right) E(x) + 2Z_{eff}\ x^2 \right]$$

and

$$I_n(x) = \int_0^x dy\ y^n\ \exp\left(-y^2\right)\ a_1$$

We note that the integrand in the expression of $I_n(x)$ contains the coefficient a_1.

This depends on $h^{(0)}$ which is the *localized* part of the distribution function.

Therefore we must calculate the function $h^{(0)}$, the localized part.

This function obeys the equation

$$\frac{\partial h^{(0)}}{\partial \lambda} = \left[\frac{1}{\frac{\sqrt{1-\lambda B}}{B}} - \mathrm{H}(\lambda_c - \lambda) \frac{1}{\left\langle \frac{\sqrt{1-\lambda B}}{B} \right\rangle} \right] F_{Me}\ f_1^{(1)(0)}(x)$$

where

$$\lambda \equiv \frac{v_\perp^2}{v^2} \frac{1}{B}$$

$$\mathrm{H}(\lambda_c - \lambda) = 0 \quad \text{for trapped}$$

Orders

$$\begin{aligned} h^{(0)} &\sim \sqrt{\varepsilon} \\ \sqrt{1-\lambda B} \frac{\partial h^{(0)}}{\partial \lambda} &\sim O(1) \quad \text{for trapped} \\ &\sim O(\varepsilon) \quad \text{for circulating} \end{aligned}$$

The result

$$\left\langle h_1^{(0)} \right\rangle = -1.46\ \sqrt{\varepsilon}\ f_1^{(1)(0)}$$

The formula

$$\begin{aligned} & \frac{3}{4} \langle B \rangle \int_0^{1/B_{\max}} \frac{\lambda d\lambda}{\int_0^{2\pi} \frac{d\theta}{2\pi} \frac{\sqrt{1-\lambda B}}{B}} \\ \approx\ & 1 - 1.46\ \sqrt{\varepsilon} \end{aligned}$$

8 Fokker Planck for NBI: Fowler code FIFPC

This part has been copied in the file *Notes.tex* of project *ITB* from *Models*.

The paper is **Fowler CPC 13 (1978) 323**.

See also the Notes.tex on the numerical implementation.

The equations

$$\begin{aligned}
\tau_s \frac{\partial f}{\partial t} &= -\frac{\tau_s}{\tau_{cx}} f \qquad \text{charge exchange} \\
&+\frac{1}{x^2}\frac{\partial}{\partial x}\left[\left(x^3 - 2Bx + x_c^3 + \frac{C}{x^2}\right) f\right] \qquad \text{drag} \\
&+\frac{1}{x^2}\frac{\partial^2}{\partial x^2}\left[\left(Bx^2 + \frac{C}{x}\right) f\right] \qquad \text{diffusion in velocity} \\
&+\frac{D}{x^3}\left(1 - \frac{D_1}{x^2} + D_2 x\right)\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) \text{ angular scattering} \\
&+E\left(-\cos\theta\frac{\partial f}{\partial x} + \frac{\sin\theta}{x}\frac{\partial f}{\partial\theta}\right) \qquad \text{electric field} \\
&-\tau_s\frac{\dot{R}(t)}{R(t)}\left[-\left(1 - \frac{1}{2}\sin^2\theta\right)x\frac{\partial f}{\partial x} + \frac{1}{2}\sin\theta\cos\theta\frac{\partial f}{\partial\theta}\right] \text{ field compression} \\
&+\tau_s\sum_l \dot{n}_{f_l} S_l(x,\theta) \quad \text{source of injected ions}
\end{aligned}$$

where the constants are (adimensional)

$$\begin{aligned}
x &= \frac{v}{v_0} \\
v_0 &= \text{the speed of the NBI ions}
\end{aligned}$$

$$x_e = \frac{v_e}{v_0}\ ,\ x_i = \frac{v_i}{v_0}$$

$$x_c = \text{critical velocity} = \frac{v_c}{v_0}$$

$$\begin{aligned}
B &= \frac{1}{2}\frac{m_e}{m_{fast}} x_e^2 \\
C &= \frac{1}{2}\frac{m_i}{m_{fast}} x_i^2 x_c^3
\end{aligned}$$

$$
\begin{aligned}
D &= \frac{1}{2}\frac{m_i}{m_{fast}} x_c^3 \frac{\langle Z\rangle}{[Z]} \\
D_1 &= \frac{1}{2}\frac{T_i}{T_e}\frac{[Z]}{\langle Z\rangle} B \\
D_2 &= \frac{4}{3\sqrt{\pi}}\frac{1}{x_e\langle Z\rangle}
\end{aligned}
$$

$$
\begin{aligned}
E &= \frac{Z_{fast} e E^*}{m_{fast} v_0}\tau_s \\
E^* &= E^{ext}\left(1 - \frac{1}{\langle Z\rangle}\right)
\end{aligned}
$$

$$
[Z] = \frac{\sum_j n_j Z_j^2 \frac{m_i}{m_j}}{\sum_j n_j Z_j}
$$

$$
\langle Z\rangle = \frac{\sum_j n_j Z_j^2}{\sum_j n_j Z_j}
$$

Slowing down time

$$
\begin{aligned}
\tau_s &= 120 \times \left(T_e \times 10^{-3}\right)^{2/3} \times \left(\frac{m_{fast}}{m_p}\right) \frac{1}{(n_e/10^{13}) \times ZF^2} \\
&\quad \text{(milliseconds)}
\end{aligned}
$$

Maximum fast ion lifetime

$$
\begin{aligned}
\tau_{fast} &= \tau_s \times \frac{1}{3}\log\left[1 + \left(\frac{E0}{EC}\right)^{3/2}\right] \\
&\quad \text{(milliseconds)}
\end{aligned}
$$

and v_c is the velocity that corresponds to the energy

$$
E_c = \text{const} \times T_e \left(\frac{m_{fast}}{m_H}\right)^{1/3}\left(\frac{m_{fast}}{m_i}\right)^{2/3}[Z]^{2/3}
$$

Part I
Comments

9 Current induced by NBI

9.1 Introduction

The beam transfers momentum to both *ions* and *electrons*.

The ions are moved in the direction of the beam and this is a current.

The electrons are also moved in the direction of the beam and this is an anti-current, which must be subtracted from the current induced by the beam.

The problem of determination of the current induced by NBI consists of finding the *electron distribution function* (different of a Maxwellian due to collisions) and then calculate as usual the electron current.

The total current is a the sum

$$j = |e| \, n_b v_b - j^{elec}$$

where

$$j^{elec} = - \int d^3 v \ \ |e| \, v_{\parallel} f'_e$$

with

$$f_e = F_{Me} + f'_e$$

The current produced by the new ions is directed along the injection NBI.
The electron flow v^{elec} has the same direction, since it results from

- collisions with the NBI ions, $C_{eb}\left(F_{Me}, f_b\right)$ which imprints a motion in the direction of v_b;
- collisions with the thermal ions $C_{ei}\left(f'_e, F_{Mi}\right)$, a friction force
- collisions with the electrons $C_{ee}\left(f'_e, F_{Me}\right) + C_{ee}\left(F_{Me}, f'_e\right)$, which have both effects: transfer momentum to the electrons, which is a friction, and also induce supplementary flow

If the beam would produce by transfer the same velocity of ions and of electrons then the total current would be zero.

However the electron flow (which is opposite current) is smaller.

It is substracted from the ion's current and so the total current is smaller than the beam current.

The total current is therefore always smaller than the beam current and the quantity that is used is

$$F = \frac{j^{total}}{j_b} < 1$$

Now one can understand the very important role of the *impurities*, especially the high-Z impurities: when there are high-Z impurities, the ions produced by NBI (hot-ions) have a higher friction on the electrons while the ion current continues. Then

$$F \sim 1 - \frac{1}{Z}$$

When $Z \gg 1$ the total current is close to the beam current.

The arrival of the Tungsten ions in the center means *an increase* of the beam-induced electric current.

9.2 Kinetic theory of beam-induced current

(**Cordey Jones Start Curtis Jones**)

The NBI ions are generated with directed momentum, the geometry is *ANISOTROPIC.*

The equation *does not need space variation of the electron distribution function* (neoclassical correction to the Maxwellian $\rho_\theta \frac{\partial f_0}{\partial r}$) since there are no trapped particles, the current is sustained by *circulating electrons.*

One basic component of the current produced by NBI is the current of the fast ions.

The other component is due to the electrons that get an ordered flow after colliding with the fast ions.

This flow has the same direction as the flow (the current) of the fast ions.

Therefore the current of electrons is *negative* and it will be subtracted from that of the ions.

The theory of Ohkawa, fluid: electrons have a Maxwellian distribution, shifted. The displacement of the Maxwellian is due to the momentum gain of the electrons from Coulomb collisions with the fast ions. This flow of electrons is then saturated by the loss of momentum to the background ions.

[we recognize the picture for the *bootstrap* current, also by **Cordey**].

Compared with experiment: in experiment the current is reduced when the electron temperature grows.

And, at high electron temperature, the direction of the current produced by NBI is *opposite* to the current of fast ions.

The assumption that there is a displaced Maxwellian is not correct.

"*the velocity dependence of the*
frictional force between the fast ions and the electrons
is, in general, different from that between the thermal
ions and the electrons. This leads to the electron distribution
being distorted in such a manner that the
distribution cannot be represented by a displaced
Maxwellian."
and
"*This distortion of the electron distribution*
from Maxwellian is similar to that caused by an electric
field, which was discussed by Spitzer [3,4] and coworkers
in their fundamental papers on the calculation
of the resistivity of a plasma"
[**Codey Jones Start... NF19 (1979) 249**.]

Conclusion: the distribution function of the electrons after collision with fast ions is NOT shifted Maxwellian but it is of the same type as the Spitzer distribution, *i.e.* in the presence of an Electric field and collisions.

Assumptions

$$
\begin{array}{rcl}
 & & \text{number of fast NBI ions} \\
 & & \text{is much smaller} \\
 & & \text{than the bulk ion density} \\
n_b & \ll & n
\end{array}
$$

(**However** see **Assunta**)

and

$$\text{usually } \; v_{elect} \gg v_{fast-ions} \gg v_{th,i}$$

Then

$$f_{elec} \equiv f_e = F_{Me} + f'_e$$

The current of the electrons, under all these effects [drive by n_{fast} of NBI, friction to bulk ions, drive by bulk electron (whose thermal velocity is much higher than v_{fast}) and friction with bulk electrons], is

$$j_e = -e \int d^3v \; f'_e(v) \; v_{\parallel}$$

We **NOTE** that, when the current, in some situations (like high electron temperature), reverses its direction and the current becomes oriented opposed to the fast ions current, - then this should be due to the electrons, and precisely to the perturbed distribution function f_e'. This means that in some situations the electrons flow *faster* than the fast ions, and in the same direction with them, and this means a minus current along the direction of NBI, or, an opposed current. *Reversed* current.

9.2.1 Kinetic distribution of fast ions

The *fast ions* distribution f_{fast} is obtained separately, from a Fokker Planck equation for NBI.

See **Cordey NBI** (born anisotropically).

See **Hsu Catto Sigmar**, transport for α (born isotropically).

It is assumed to be known.

Formal expression, *separation of variables* v and ξ

$$f_{fast} = \sum a_n^{fast}(v) \ P_n(\xi)$$

where

$$\xi = \frac{v_\parallel}{v}$$

The situation

$$\begin{aligned} & \text{slowing down time of the fast ions} \\ = \ & 10 \times \\ & \text{charge exchange time} \end{aligned}$$

Then the fast ions approx. remain with the energy they came in the plasma and are monoenergetic.

$$a_n^{fast}(v) = \left(n + \frac{1}{2}\right) K_n \ n_{fast} \ \frac{1}{2\pi v_{fast}^2} \delta(v - v_{fast})$$

where

$$K_n = \int_{-1}^{+1} K(\xi) P_n(\xi) \, d\xi$$

$$K(\xi) \equiv \text{angular distribution of fast ions}$$

The toroidal current of the fast ions is the velocity-space integration of the $v_{\|} = v\xi$, with $f \equiv$ distribution function for *fast-ions*.

$$
\begin{aligned}
j_{fast-ions} &= Z_{fast-ions}\, e \iint 2\pi v^2 dv \ (v\xi) \ f(v,\xi) \\
&= S\tau_s K_1 \int dv \ \frac{v^3}{v^3+v_c^3}\left(\frac{v_0^3+v_c^3}{v^3+v_c^3}\ \frac{v^3}{v_c^3}\right)^{\frac{2}{3}\beta}
\end{aligned}
$$

where

$$\beta = \sqrt{\frac{3}{2}\sqrt{\pi}}$$

In the Legendre expansion of f only the component with $P_1(\xi)$ survives because $\xi = P_1(\xi)$.

NOTE

Regarding the possible variation of the NBI ions distribution with the poloidal angle θ, reflecting the shape of the banana (see **LH, notes**) here we cannot see any trace, since everything is solved in velocity space.

END

9.2.2 Kinetic distribution of electrons

The Fokker Planck equation for *ELECTRONS* is reduced to

$$
\begin{aligned}
& C_{eb}(F_{Me}, f_{fast}) \ \ \text{bulk electrons - beam ions} \\
& +C_{ei}(f_e', \ F_{Mi}) \ \ \text{pert. elect. - bulk ions} \\
& +C_{ee}(f_e', F_{Me}) \ \ \text{pert. elect. - bulk electrons} \\
& +C_{ee}(F_{Me}, f_e') \ \ \text{bulk elect. - pert. electrons} \\
= \ & 0
\end{aligned}
$$

The collision operator C is linearized.

The perturbed distribution function of the *electrons*, which always accompany the NBI ions, is symmetric relative to the field line. Then it can be expanded in $P_n(\xi)$ as well

$$f_e'(v) = F_{Me}\sum_n a_n(v) \ P_n(\xi)$$

Similarly, the Rosenbluth potentials that occur in the collision operators are expanded in the set of polynomials, with *separation of variables* (v, ξ)

$$\begin{aligned} h &= \sum_n h_n(v) \; P_n(\xi) \\ g &= .. \end{aligned}$$

Using these expansions in the Fokker Planck equation one obtains a system of equations for $a_n(v)$, whose solution determined the *perturbed electron distribution function*, f_e'.

It is necessary FOR THE ELECTRIC CURRENT carried by the electrons, only $a_1(v)$.

The Fokker Planck equation becomes

$$\begin{aligned} & a_1'' + P(x) \; a_1' + Q(x) \; a_1 \\ = & \frac{16}{3\sqrt{\pi}} \frac{1}{W} \\ & \times \left[x I_3(x) - 1.2x \; I_5(x) - x^4 \left(1 - 1.2x^2\right) \left(I_0(x) - I_0(\infty)\right) \right] \\ & + R(x) \quad \text{this is the drive of the electron current} \end{aligned}$$

where

$$x = \frac{v}{v_{th,e}}$$

$$P(x) = -\frac{1}{x} - 2x + 2x^2 \frac{\Phi'}{W}$$

$$Q(x) = -\frac{1}{x^2} - 2\frac{Z_{eff} + \Phi - 2x^3\Phi'}{W}$$

$$Z_{eff} = \frac{\sum Z_i^2 n_i}{n_e}$$

$$W = \Phi - x\Phi'$$

$$\Phi' \equiv \frac{d\Phi}{dx}$$

$$\Phi(x) = \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt \; \exp\left(-t^2\right)$$

$$I_n(x) = \int_0^x dt \; a_1(t) \exp\left(-t^2\right) \; t^n$$

$$\begin{aligned} v_{fast}^* &\equiv \text{normalized fast ion velocity} \\ &= \frac{v_{fast}}{v_{th,e}} \end{aligned}$$

The drive of the electron current is the Coulomb collision with the fast ions.

$$R(x) = -\frac{B}{W} \times \begin{cases} -\frac{1}{\left(v_{fast}^*\right)^2}\left(\frac{6}{5}x^6 - 2x^4\right) & \text{for} \quad \begin{array}{c} x < v_{fast}^* \\ \text{or } \frac{v}{v_{th,e}} < \frac{v_{fast}}{v_{th,e}} \end{array} \\ \left[v_{fast}^* + \frac{6}{5}\left(v_{fast}^*\right)^3\right] x & \text{for} \quad \begin{array}{c} x > v_{fast}^* \\ \text{or } \frac{v}{v_{th,e}} > \frac{v_{fast}}{v_{th,e}} \end{array} \end{cases}$$

with a constant

$$B \equiv 4K_1 \frac{n_{fast}}{n}$$

the total current: fast ions plus the electron current

$$j = eK_1 \left(n_{fast} v_{fast} - \frac{4}{3} \frac{v_{th,e}}{\sqrt{\pi} K_1} n I_3(\infty) \right)$$

The parameter defined as the ratio between the current (fast ions plus electrons) divided to the fast ion current

$$\begin{aligned} F &= \frac{j}{n_{fast}\ e\ v_{fast}\ K_1} \\ &= 1 - \frac{16}{3\sqrt{\pi} v_{fast}^*} \frac{I_3(\infty)}{B} \end{aligned}$$

From the results shown by **Cordey et al** we see that when $Z = 1$ the ratio F can become negative.

When $Z = 2$ or $Z = 4$ the ratio F remains positive (no *reversal* of current) but there is a decrease of F in a certain range of the variable $\left(v_{th,e}/v_{fast}\right)^{-2}$.

The presence of higher Z (impurities) means that it is maintained the direction of the current sustained by NBI fast ions: the electrons have much more efficient friction and they are not able to move in the direction imposed by the fast NBI ions (ion current) at a similar flux. Their *negative* curent is smaller.

9.3 Shielding current of electrons modifying the fast-ions current

The coefficient ($nbcd$ =neutral beam current drive)

$$j_{nbcd} = \eta_{nbcd} \times f_{fast-ions}$$

where

$$\eta_{nbcd} = 1 - \frac{Z_{fast-ions}}{Z_{eff}} (1 - L_{31})$$

9.4 The effect of trapped electrons on the NBI-induced current (Start Cordey Jones 1980)

The paper **trapped electron NBI start cordey jones 1980**.

The FP equation for *electrons* will contain a convective derivative,

$$\left(v_{\|}\widehat{\mathbf{n}}+\mathbf{v}_{D}\right)\cdot\boldsymbol{\nabla}f_{e}$$

and this leads to a neoclassical correction

$$\rho_{\theta}\frac{\partial f_{e}}{\partial r}$$

The distribution function for the *electrons* is perturbed by the fast ions

$$f_{e}=F_{Me}+f_{e}^{\prime}$$

and there is also the distribution function of the *beam* fast ions.

The equation for *electrons* is

$$\begin{aligned}
& C_{eb}\left(F_{Me},f_{b}\right) \text{ beam-electron momentum transfer}\\
& +C_{ei}\left(f_{e}^{\prime},F_{Mi}\right) \quad \text{accelerated electrons - friction by ions}\\
& +C_{ee}\left(f_{e}^{\prime},F_{Me}\right) \quad \text{accelerated electrons drag other electrons}\\
& +C_{ee}\left(F_{Me},f_{e}^{\prime}\right) \quad \text{accelerated electrons - friction by electrons}\\
= & \ \xi v\frac{B_{\theta}}{B_{\varphi}}\frac{\partial f_{e}^{\prime}}{r\partial\theta} \qquad \text{(parallel convection - projected on } \theta\text{)}
\end{aligned}$$

where

$$\xi\equiv\frac{v_{\|}}{v}$$

Then the factor is

$$\xi v=v_{\|}$$

and multiplied with B_{θ}/B_{φ} obtains the projection of the parallel velocity on the poloidal direction. Then actually in the right hand side we have

$$\xi v\frac{B_{\theta}}{B_{\varphi}}\frac{\partial f_{e}^{\prime}}{r\partial\theta}=v_{\|}\frac{B_{\theta}}{B_{\varphi}}\frac{\partial f_{e}^{\prime}}{r\partial\theta}=v_{\|}\nabla_{\|}f_{e}^{\prime}$$

The fact that the distribution function $f_{e}^{\prime}$ dependes on θ (equivalently on $l_{\|}$) is due to banana effects.

NOTE. So this equation expresses the basic equilibrium

$$v_{\|}\nabla_{\|}f_{e}^{\prime}=C \quad \text{(collisions)}$$

We know what to do

- divide by $v_{\|}$,
- multiply by B
- surface average

this will cancel (by periodicity) the LHS. A constraint on the collision operator will result, acting as an equation for f_0.

The radial drift of the Maxwellian of equilbium $\mathbf{v}_D \cdot \boldsymbol{\nabla} f_{Me}$ is not included. An electrostatic field with potential Φ is not included since there is no such act in the *velocity space* $|e| \boldsymbol{\nabla}\Phi/m_s \cdot \frac{\partial f_M}{\partial \mathbf{v}}$ for small Φ. Nor its time variation $\partial\Phi/\partial t$ since we do not have time variation of the radial electric field as in the case of the poloidal transit time magnetic pumping.

Here the ions are *hot* and they simply are slowing down by collisions with the background ions and electrons. **END.**

The collisions are determined in general in terms of the Rosenbluth potentials.

These potentials are expressed as integrals over functions that depend on

$$|\mathbf{v} - \mathbf{v}'|$$

a kind of "distance" in velocity space. Or, as shown in *collisions.tex* after Morse Feshbach, this "distance" can be expressed in an expansion in Legendre polynomials (something well known from electrdynamics).

It is assumed that there is a spherical system of coordinates in the velocity space (v, θ, φ), where θ is the pitch angle formed between the vector velocity $\mathbf{v}$ and the magnetic field line.

It is assumed that there is azimuthal symmetry, *i.e.* no dependence on φ.

Then the Legendre functions depend on

$$P_l (\cos\theta)$$

An equivalent explanation is in the derivation of **Gaffey**. The pitch angle scattering operator arises from the derivatives in velocity space requested by the collision operator. Next, the pitch angle scattering operator is shown to be the Legendre polynomial operator.

The fast ion distribution f_b is expanded in Legendre polynomials whose variable is the *pitch angle* ξ. This is a *separation of variables*, for v and for ξ, in every term.

$$f_b = \sum_n a_n^{(b)} (v) P_n (\xi)$$

(see **Cordey**) and the factors that depend on the velocity v are

$$a_n^{(b)}(v) = n_b \left(n + \frac{1}{2}\right) K_n \frac{1}{v_b^2} \frac{1}{2\pi} \delta(v - v_b)$$

The coefficients K_n are obtained from the *angular distribution* of the beam $K(\xi)$, ξ being the pitch angle variable, $v_{\parallel}/v$.

$$K_n = \int_{-1}^{+1} K(\xi) P_n(\xi) d\xi$$

NOTE

In the work **Hsu Catto Sigmar** on NBI the separation of variables with reduction of the Fokker Planck equation to two equations of eigenfunctions $P \sim \sum V_n C_n$ the Legendre polynomial occurs only when we adopt an approximation $\sqrt{\varepsilon} \ll 1$.

And in **Cordey NBI** too

END

To solve the equation for the perturbed electron distribution function f_e', this is shown as composed of

$$f_e' = f^{(0)} + h^{(0)} + f^*$$

$$\begin{aligned} f^{(0)} &= \text{solution of the eq.} \\ & \text{in absence of trapping} \\ \frac{\partial f^{(0)}}{\partial \theta} &= 0 \quad \text{(no poloidal variation)} \end{aligned}$$

This part of the function is due to the momentum transfer from the beam fast ions to the electrons. It is the essential aspect of generation of the flow of electrons in the same direction as the fast ions.

The second term

$$h^{(0)} \equiv \text{localized } (\sim \theta) \text{ part of the dist. function}$$

depends on θ.

$$f^* \equiv \text{non-localized part of the dist. function}$$

depends on θ.

The three functions are expanded

$$\begin{aligned}
f^{(0)} &= F_{Me} \sum f_n^{(0)}(v) P_n(\xi) \\
h^{(0)} &= F_{Me} \sum h_n^{(0)}(v,\theta) P_1(\xi) \\
f^{*} &= F_{Me} \sum f_n^{*}(v,\theta) P_1(\xi)
\end{aligned}$$

Here only the

$$n = 1$$

terms will be retained in the expansions

$$f_1^{(0)} \quad , \quad h_1^{(0)} \quad , \quad f_1^{*}$$

for the calculation of the electron current.

The dependence on θ suggests to take surface averages

$$\left\langle h_1^{(0)} \right\rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} h_1^{(0)}$$

$$\left\langle f_1^{*} \right\rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} f_1^{*}$$

The localized and non-localized distribution functions are solutions of the collisional eq

$$\begin{aligned}
& C_{ei}(f^*, F_{Mi}) \quad \text{friction by background ions} \\
& + C_{ee}(f^*, F_{Me}) + C_{ee}(F_{Me}, f^*) \ \text{drag and friction with background electrons} \\
= \ & -\left[C_{ei}\left(h^{(0)}, F_{Mi}\right) + C_{ee}\left(h^{(0)}, F_{Me}\right) + C_{ee}\left(F_{Me}, h^{(0)}\right)\right] \\
& + \left[C_{ei}\left(F_{Mi}, h^{(0)}\right) + C_{ee}\left(F_{Me}, h^{(0)}\right)\right]_{pitch-angle}
\end{aligned}$$

The last square paranthesis is the *pitch-angle part of the operators inside*

$$\begin{aligned}
& \left[C_{ei}\left(F_{Mi}, h^{(0)}\right) + C_{ee}\left(F_{Me}, h^{(0)}\right)\right]_{pitch-angle} \\
= \ & \frac{1}{4}\Gamma_e \frac{n_e}{v_{th,e}^3} \frac{1}{x^5}\left[xE'(x) + \left(2x^2 - 1\right)E(x) + 2Z_{eff}x^2\right] \\
& \times \frac{\partial}{\partial \xi}\left[\xi\left(1 - \xi^2\right)\frac{\partial h^0}{\partial \xi}\right]
\end{aligned}$$

The error function

$$E(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-x^2\right) dx$$

Using the expansions in Legendre polynomials in the equation for f^* and $h^{(0)}$, [in simple form is

$$C_e\left(f^*\right) = -C_e\left(h^{(0)}\right) + C_e^0\left(h^{(0)}\right)$$

] one obtains a system of uncoupled ordinary differential equations for the Legendre coefficients.

The system is written by **Cordey Jones Start Curtis Jones 1979**.
In that case it was *NO effect of trapped particles.*

$$\begin{aligned}
& a_1'' + P(x)\ a_1' + Q(x)\ a_1 \\
= & \frac{16}{3\sqrt{\pi}}\frac{1}{W} \\
& \times\left[xI_3(x) - 1.2x\ I_5(x) - x^4\left(1-1.2x^2\right)\left(I_0(x) - I_0(\infty)\right)\right] \\
& +R(x) \quad \text{this is the drive of the electron current}
\end{aligned}$$

where

$$a_1 = \left\langle h_1^{(0)}\right\rangle + \left\langle f_1^*\right\rangle$$

$$x = \frac{v}{v_{th,e}}$$

$$P(x) = -\frac{1}{x} - 2x + 2x^2\frac{\Phi'}{W}$$

$$Q(x) = -\frac{1}{x^2} - 2\frac{Z_{eff} + \Phi - 2x^3\Phi'}{W}$$

$$Z_{eff} = \frac{\sum Z_i^2 n_i}{n_e}$$

$$W = \Phi - x\Phi'$$

$$\Phi' \equiv \frac{d\Phi}{dx}$$

$$\Phi(x) = \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x dt\ \exp\left(-t^2\right)$$

$$I_n(x) = \int_0^x dt\ a_1(t)\exp\left(-t^2\right)\ t^n$$

$$\begin{aligned}
v_{fast}^* & \equiv \text{normalized fast ion velocity} \\
& = \frac{v_{fast}}{v_{th,e}}
\end{aligned}$$

The drive of the electron current is the Coulomb collision with the fast ions.

$$R(x) = -\frac{B}{W} \times \begin{cases} -\frac{1}{\left(v^*_{fast}\right)^2}\left(\frac{6}{5}x^6 - 2x^4\right) & \text{for} \quad \begin{array}{c} x < v^*_{fast} \\ \text{or } \frac{v}{v_{th,e}} < \frac{v_{fast}}{v_{th,e}} \end{array} \\ \left[v^*_{fast} + \frac{6}{5}\left(v^*_{fast}\right)^3\right] \; x & \text{for} \quad \begin{array}{c} x > v^*_{fast} \\ \text{or } \frac{v}{v_{th,e}} > \frac{v_{fast}}{v_{th,e}} \end{array} \end{cases}$$

with a constant

$$B \equiv 4K_1 \frac{n_{fast}}{n}$$

[discussed above].

Here there is a different expression for $R(x)$, reflecting the fact that the drive R now depends on $h_1^{(0)}$.

$$R(x) = \frac{\langle h_1^0 \rangle}{W x^2}\left[x\Phi' + \left(2x^2 - 1\right)\Phi + 2Z_{eff}\; x^2\right]$$

The function $h_1^{(0)}(x)$ is determined by **Hazeltine, Hinton, Rosenbluth 1973**

$$\frac{\partial h^{(0)}}{\partial \lambda} = \left[\frac{B}{\sqrt{1-\lambda B}} - \mathrm{H}(\lambda_c - \lambda) \; \times \frac{1}{\left\langle \frac{\sqrt{1-\lambda B}}{B} \right\rangle}\right] F_{Me}\; f_1^{(0)}(x)$$

where

$$\begin{aligned} \lambda &= \frac{v_\perp^2}{v^2}\frac{1}{B} \\ \sqrt{1-\lambda B} &= \frac{v_\parallel}{v} \equiv \xi \end{aligned}$$

the last term exists only for *circulating* particles.

The NBI modifies the distribution function of the electrons

$$F_{Me} \to F_{Me} + f_e'$$

The equation for the non-Maxwellian perturbation f_e' is

$$v_\parallel \frac{B_\theta}{B_\varphi}\frac{\partial f_e'}{r\partial\theta} = C_{eb}F_{Me} + C_{ei}f_e' + C_{ee}f_e'$$

where the collision operators are

$$\begin{aligned} C_{eb} &= \text{electron - fast-ion} \\ C_{ei} &= \text{electron - thermal ion} \\ C_{ee} &= \text{electron - electron} \end{aligned}$$

Now it is introduced the distinction between the situation with NO TRAPPED particles and the correction due to trapped particles

$$f_e' = f^0 + \widehat{f}$$

the first part verifies

$$C_{eb}F_{Me} + C_{ei}f^0 + C_{ee}f^0 = 0$$

This is the simple dynamics of the electrons, neglecting any effect of trapped electrons

- accelerated by momentum transfer from the beam ions
- friction with the ions of the background plasma
- friction with the electrons of the background plasma

The regime in which the thermal electron velocity is much higher than the fast ion velocity

$$v_{th,e} \gg v_{fast-ion}$$

there is a solution for the NON-TRAPPED part of the non-Maxwellian perturbation function

$$f^0 = \frac{2Z_b^2}{Z_{eff}} \frac{n_b v_b}{n v_{th,e}^2} v_\parallel \; K_1 \; F_{Me}$$

where

$$Z_{eff} = \frac{\sum n_i Z_i^2}{n}$$

$$\begin{aligned} K_1 \;\equiv\; & \text{first order coefficient of the Legendre expansion} \\ & \text{of the angular distribution of fast ions} \end{aligned}$$

Knowing this part of f_e' will simplify the equation

$$v_\parallel \frac{B_\theta}{B_\varphi} \frac{\partial f_e'}{r\partial\theta} = (C_{ei} + C_{ee})\, \widehat{f}$$

We adopt explicit forms for the collision operators

$$\begin{aligned}
C_{ee}\widehat{f} &= \nu_{ee}v_{\parallel}\frac{\partial}{\partial\mu}\left(\mu\frac{v_{\parallel}}{B}\frac{\partial\widehat{f}}{\partial\mu}\right) \quad \text{pitch angle } e-e \\
&\quad +\nu_{ee}v_{\parallel}\frac{F_{Me}\int d^3v \ \ \nu_{ee}v_{\parallel}\widehat{f}}{\int d^3v \ \ \nu_{ee}v_{\parallel}^2 F_{Me}} \quad \text{(Connor1973)}
\end{aligned}$$

and

$$C_{ei} = \nu_{ei}v_{\parallel}\frac{\partial}{\partial\mu}\left(\mu\frac{v_{\parallel}}{B}\frac{\partial\widehat{f}}{\partial\mu}\right) \quad \text{only pitch angle}$$

where

$$\begin{aligned}
\mu &= \frac{v_{\perp}^2}{2B} \\
\nu_{ei} &= \frac{4\pi e^4 Z_{eff}}{m_e^2}\ln\Lambda\frac{n}{v_{th,e}^3}\frac{1}{x^{3/2}} \\
\nu_{ee} &= \frac{4\pi e^4}{m_e^2}\ln\Lambda\frac{n}{v_{th,e}^3}\frac{1}{x^{3/2}}\left[\frac{d\eta}{dx}+\left(1-\frac{1}{2x}\right)\eta\right] \\
x &= \frac{v^2}{v_{th,e}^2} \\
\eta(x) &= \frac{2}{\sqrt{\pi}}\int_0^x dt \ \ \exp(-t)\sqrt{t}
\end{aligned}$$

the current is carried by the *circulating electrons.*

One must calculate the distribution function of the un-trapped (passing) particles

$$f'_{ep}$$

from the equation

$$\begin{aligned}
\frac{\partial f'_{ep}}{\partial\mu} &= -\frac{2Z_b^2}{Z_{eff}}\frac{n_b v_b}{n v_{th,e}^2}K_1\frac{1}{\langle v_{\parallel}/B\rangle}F_{Me} \\
&\quad \times\left(1-\frac{\nu_{ee}}{\nu_{ee}+\nu_{ei}}\frac{\int d^3v \ \nu_{ee}v_{\parallel}^2F_{Me}-\langle u\rangle}{\int d^3v \ \nu_{ee}v_{\parallel}^2F_{Me}}\right)
\end{aligned}$$

and

$$\langle u\rangle = \frac{\left\langle\int d^3v \ \frac{\nu_{ee}v_{\parallel}\mu \ F_{Me}}{\langle v_{\parallel}/B\rangle}\right\rangle-\left\langle\int d^3v\frac{\nu_{ee}^2}{\nu_{ee}+\nu_{ei}}\frac{v_{\parallel}\mu F_{Me}}{\langle v_{\parallel}/B\rangle}\right\rangle}{1-\left\langle\int d^3v\frac{\nu_{ee}^2}{\nu_{ee}+\nu_{ei}}\frac{v_{\parallel}\mu F_{Me}}{\langle v_{\parallel}/B\rangle}\right\rangle\left(\int d^3v \ \nu_{ee}v_{\parallel}^2F_{Me}\right)^{-1}}$$

9.4.1 Notes

the current is composed of

$$\begin{aligned}&\text{fast ion current}\\&+\text{reverse electron current}\end{aligned}$$

The reverse electron current is due to the same transfer of momentum from NBI ions as for the thermal ions. But this motion of electrons in the same direction as the thermal ions, both induced by the transfer from NBI, means a current of opposite sign: *the reverse electron current.*

The quantity that measures the current produced by NBI is defined as a ratio between the net current and the beam current

$$\begin{aligned}F &= \frac{j_{net}}{j_b}\\&= 1-\frac{\left\langle \int d^3v\ v_\parallel f'_{ep}\right\rangle}{n_b Z_b v_b}\end{aligned}$$

The expression is further written in terms of two parameters

$$\begin{aligned}F &= 1-\frac{4Z_b}{3\sqrt{\pi}Z_{eff}}\times I\\&\times\left(\int_0^\infty dx\frac{\nu_{ei}}{\nu_{ei}+\nu_{ee}}\exp\left(-x\right)x^{3/2}\right.\\&\left.+I\frac{\int_0^\infty dx\frac{\nu_{ee}}{\nu_{ee}+\nu_{ei}}\exp\left(-x\right)\ \int_0^\infty dx\frac{\nu_{ee}}{\nu_{ee}+\nu_{ei}}\exp\left(-x\right)x^{3/2}}{\int_0^\infty dx\frac{\nu_{ee}}{\nu_{ei}}\exp\left(-x\right)-I\int_0^\infty dx\frac{\nu_{ee}^2}{\nu_{ei}\left(\nu_{ee}+\nu_{ei}\right)}\exp\left(-x\right)}\right)\end{aligned}$$

with the definition

$$I=\frac{3}{4}\left\langle B\right\rangle\int_0^{1/B_{\max}}\frac{\lambda d\lambda}{\left\langle\frac{\sqrt{1-\lambda B}}{B}\right\rangle}$$

10 Trapped electron correction to the NBI current Lin-Liu Hinton

This is the paper by **Lin-Liu Hinton.**

The expression of the Neutral Beam Current

$$j_{NBCD}=j_{fast}\left[1-\frac{Z_{fast}}{Z_{eff}}\left(1-G\right)\right]$$

where

$$j_{fast} \equiv \text{current of fast ions}$$

$$Z_{fast} \equiv \text{charge of fast ions}$$

$$Z_{eff} \equiv \text{ effective charge of background ions}$$

$$G \equiv \text{trapped ELECTRON correction to Ohkawa NBCD}$$

The magnetic field

$$B = \frac{B_0}{h} = \frac{B_0}{1 + \varepsilon \cos \theta}$$

Important assumption

$$v_e \gg v_{fast}$$

The fast ions have a directed velocity

$$u_{fast}$$

Method

- Fokker Planck analysis
- electron-electron collisions
- fast ions - electrons collisions

The current

$$\begin{aligned} j_{\|} &= j_{fast\|} + j_{e\|} \\ &= n_{fast} Z_{fast} e \, u_{fast} - e \int d^3v \; v_{\|} \; f_{e1} \end{aligned}$$

Then the objective is to calculate the first order correction to the *electron* distribution function, f_{e1}, which results from

- neoclassical drifts
- neoclassical trapping (structure of distribution in velocity space)
- collisions of electrons with background ions
- collisions with fast ions
- collisions between electrons

the equation

$$v_{\|}\nabla_{\|}f_{e1}=C_{ee}^{lin}+C_{ei}-\mathbf{v}_{De}\cdot\boldsymbol{\nabla}\psi\ \frac{1}{n_e}\frac{\partial n_e}{\partial\psi}\ f_{e0}$$

where

$$2\pi\psi\equiv\text{poloidal flux function}$$

$$\begin{aligned}C_{ei} &= \nu_{ei}(v)\ \mathcal{L}\ [f_{e1}]\\ &\quad+\nu_{eff}(v)\ \frac{1}{T_e/m_e}v_{\|}\ u_{fast\|}\ f_{e0}\end{aligned}$$

$$\mathcal{L}\equiv\text{pitch angle scattering operator}$$

$$\nu_{ei}(v)\equiv\text{scattering rate}$$

$$\nu_{eff}(v)=\frac{Z_{fast}^2\ n_{fast}}{Z_{eff}\ n_e}\nu_{ei}(v)$$

One makes the substitution

$$\begin{aligned}f_{e1} &= I\frac{v_{\|}}{\Omega_e}\frac{1}{n_e}\frac{\partial n_e}{\partial\psi}f_{e0}\\ &\quad+g\end{aligned}$$

with

$$I=RB_{\varphi}$$

and use

$$\mathbf{v}_{De}\cdot\boldsymbol{\nabla}\psi=-I\ v_{\|}\nabla_{\|}\left(\frac{v_{\|}}{\Omega_e}\right)$$

The equation becomes

$$\begin{aligned}v_{\|}\nabla_{\|}g &= C_e[g]\\ &\quad+\nu_{ei}(v)\ I\frac{v_{\|}}{\Omega_e}f_{e0}\left[-\frac{1}{n_e}\frac{\partial n_e}{\partial\psi}+\frac{1}{I}\ \frac{1}{p_e}\frac{Z_{fast}}{Z_{eff}}\ j_{fast\|}\ B\right]\end{aligned}$$

The operator C_e contains the linearized electron-electron collision operator and pitch angle scattering of electrons and ions

$$\begin{aligned}C_e &= C_{ee}^{lin}\\ &\quad+\nu_{ei}(v)\,\mathcal{L}\end{aligned}$$

The bounce frequency is very highcompared with collisionality.

$$g=g_0+g_1+...$$

The small parameter is

$$\left(\frac{\omega_{e,bounce}}{\nu_e/\varepsilon}\right)^{-1} \ll 1$$

Then

$$v_{\parallel}\nabla_{\parallel}g_0 = 0$$

The equation for g_1 is written and becomes the solubility condition

$$-\oint \frac{dl}{v_{\parallel}} C_e\left[g_0\right] = \nu_{ei}\left(v\right)\ f_{e0}\ \oint dl\ I\ \frac{1}{\Omega_e}\left[-\frac{1}{n_e}\frac{\partial n_e}{\partial \psi} + \frac{1}{I}\frac{1}{p_e}\frac{Z_{fast}}{Z_{eff}}\ j_{fast\parallel}\ B\right]$$

$$dl = \frac{B}{B_\theta} dl_\theta$$

from which the function g_0 will be calculated.

The loop integration of the RHS gives zero for trapped electrons.

$$g_0 = H\ \left\langle B\frac{1}{\Omega_e}\left[-\frac{1}{n_e}\frac{\partial n_e}{\partial \psi} + \frac{1}{I}\frac{1}{p_e}\frac{Z_{fast}}{Z_{eff}}\ j_{fast\parallel}\ B\right]\right\rangle$$

where

$$H \equiv H\left(\mathbf{v}, \psi\right)$$

The average is over the flux surface

$$\langle A\rangle = \frac{\oint \frac{dl_\theta}{B_\theta} A}{\oint \frac{dl_\theta}{B_\theta}}$$

$$g_0 = I\frac{1}{e/m_e}H\ \left[-\frac{1}{n_e}\frac{\partial n_e}{\partial \psi} + \frac{1}{I}\frac{1}{p_e}\frac{Z_{fast}}{Z_{eff}}\ \left\langle j_{fast\parallel}\ B\right\rangle\right]$$

With this solution for g_0 we return to the distribution function for the electrons, f_{e1} and calculate the current of the electrons, the second term in the general expression of the full current

$$\begin{aligned}
j_{\parallel} &= j_{fast\parallel} \\
&\quad -I\frac{1}{B}n_e T_e\ \frac{1}{n_e}\frac{\partial n_e}{\partial \psi} - e\int d^3v\ v_{\parallel}\ g_0 \\
&= j_{fast\parallel} \\
&\quad -I\frac{1}{B}n_e T_e\ \frac{1}{n_e}\frac{\partial n_e}{\partial \psi} \\
&\quad -\gamma I\ n_e T_e\ \frac{B}{\langle B^2\rangle}\left[\frac{1}{n_e}\frac{\partial n_e}{\partial \psi} - \frac{1}{I}\frac{1}{p_e}\frac{Z_{fast}}{Z_{eff}}\ \left\langle j_{fast\parallel}\ B\right\rangle\right]
\end{aligned}$$

defining γ $\sim$integral of H.

The authors note the dependence of g with θ, seen in the presence of B as factor. This is also noted if the equation for g , which is (see above)

$$\begin{aligned} v_{\parallel}\nabla_{\parallel}g &= C_e[g] \\ &+\nu_{ei}(v)\ I\frac{v_{\parallel}}{\Omega_e}f_{e0}\left[-\frac{1}{n_e}\frac{\partial n_e}{\partial\psi}+\frac{1}{I}\ \frac{1}{p_e}\frac{Z_{fast}}{Z_{eff}}\ j_{fast\parallel}\ B\right] \end{aligned}$$

is integrated over velocity space

$$\nabla_{\parallel}\left[\frac{1}{B}\int d^3v\ v_{\parallel}\ g\right]=0$$

We note that $v_{\parallel}$ integrated via g (part of the distribution function which is different of the neoclassical drift-induced $\rho_\theta/L_n\times f_{e0}$) is a *parallel current*. This is the wighted with

$$\frac{1}{B}$$

and the result is independent on θ , it does not have variation along the magnetic field line

$$\nabla_{\parallel}\rightarrow 0$$

which can be formulated: the parallel current resulting from g has precisely the same variation along the line as the magnitude of the magnetic field B. Their ratio is constant along the line.

The equation for g_0 contains H and further its integral, γ.

We have to find γ and for this it is taken the surface average of the equation

$$\begin{aligned} \left\langle j_{\parallel}B\right\rangle &= \left\langle j_{fast\parallel}\ B\right\rangle - I\ p_e\frac{1}{n_e}\frac{\partial n_e}{\partial\psi} \\ &-\gamma\ I\ p_e\left[\frac{1}{n_e}\frac{\partial n_e}{\partial\psi}-\frac{1}{I}\frac{1}{p_e}\frac{Z_{fast}}{Z_{eff}}\left\langle j_{fast\parallel}\ B\right\rangle\right] \end{aligned}$$

Now it is suppressed the part of the current that comes directly from the fast particles

$$j_{fast\parallel}=0$$

and what remains is called *bootstrap*

$$\begin{aligned} &\left\langle j_{\parallel}\ B\right\rangle_{bootstrap} \\ &= -(1-\gamma)\ I\ p_e\ \frac{1}{n_e}\frac{\partial n_e}{\partial\psi} \end{aligned}$$

and this equated to

$$= -L_{31}\ \frac{1}{n_e}\frac{\partial n_e}{\partial \psi}$$

which allows to identify

$$\gamma = \frac{L_{31}}{I\ p_e} - 1 = l_{31} - 1$$

the total NBCD

$$j_{NBCD} = j_{fast\|}\ \ - (1 - l_{31})\ \ \frac{Z_{fast}}{Z_{eff}}\frac{B}{\langle B^2\rangle}\left\langle j_{\|}B\right\rangle$$

and

$$\langle j_{NBCD}\ B\rangle = \left\langle j_{fast\|}\right\rangle \left[1 - \frac{Z_{fast}}{Z_{eff}}\left(1 - l_{31}\right)\right]$$

$$G = l_{31} = \frac{L_{31}}{I\ p_e}$$

11 Effect of trapped particles on the *slowing down* of fast ions Cordey

In a time of *slowing down* a fast ion is scattered at 90°. Therefore trapped and circulating *fast NBI ions* can change their state.

$$\tau_{scattering} < \tau_{slowing-down}$$

It is necessary to determine the *full distribution function* of NBI fast ions, in both regions : trapped and circulating.

The equation is *bounce-averaged Fokker Planck.*

The Fokker Planck equation is *separable* in

$$\xi = \sqrt{1 - \mu B_0\frac{1}{\epsilon}}$$

and

$$v$$

Then a set of eigenfunctions that reflect the geometry in ξ is adopted.

NOTE that here the symbol ξ is NOT $v_{\|}/v$ as it is sometimes defined.

It will become clear below that B_0 is measured at the furthest point on the equatorial plane.

The usual definition is

$$\lambda = \frac{\mu}{w}$$

or

$$\lambda = \frac{v_{\perp}^{2}}{v^{2}} \frac{1}{B(\mathbf{x})}$$

We have

$$B(\mathbf{x}) = \frac{B_0}{h}$$

we place the upperscript *axis* to the usual choice

$$\begin{aligned} B &= \frac{B_0^{axis}}{1 + \varepsilon \cos\theta} \\ &= B_0^{Cordey} \frac{(1 - \varepsilon \cos\theta)}{1 - \varepsilon} \end{aligned}$$

then

$$B_0^{Cordey} \approx B_0^{axis} \left(1 - \frac{r}{R}\right)$$

Further

$$\begin{aligned} \xi &= \sqrt{1 - \mu B_0^{Cordey} \frac{1}{\epsilon}} \\ &= \sqrt{1 - \mu B_0^{axis} \frac{1 - \varepsilon}{\epsilon}} \end{aligned}$$

The part inside

$$\begin{aligned} & \frac{v_{\perp}^{2}}{2B} \frac{1}{v^{2}/2} B_0^{axis} (1 - \varepsilon) \\ = & \frac{v_{\perp}^{2}}{v^{2}} \frac{h}{1 - \varepsilon} \end{aligned}$$

In the usual notation

$$\text{usual notation} \quad \frac{v_{\parallel}}{v} \equiv \xi = \sqrt{1 - \lambda B}$$

where

$$\lambda = \frac{v_{\perp}^{2}}{2B} \frac{1}{v^{2}/2}$$

In this *usual* notation

$$\begin{aligned} 1 - \lambda B &= 1 - \frac{v_{\perp}^{2}}{v^{2}} = \frac{v_{\parallel}^{2}}{v^{2}} \\ v_{\parallel} &= v\sqrt{1 - \lambda B} \end{aligned}$$

For comparison, in **Cordey**

$$\begin{aligned} \left|v_{\parallel}\right|^{Cordey} &= v\sqrt{1-\frac{B}{B_0}\left(1-\xi^2\right)} \\ &= v\sqrt{1-\frac{B}{B_0}\left[1-\left(1-\frac{v_\perp^2 B_0}{v^2 B}\right)\right]} \\ &= v\sqrt{1-\frac{B}{B_0}\frac{v_\perp^2 B_0}{v^2 B}} = v\frac{\sqrt{v_\parallel^2}}{v} \quad \text{OK} \end{aligned}$$

Formal, the choices of **Cordey** are

$$\lambda^{Cordey} = \frac{v_\perp^2 B_0}{v^2 B}$$

and in this way ξ is an invariant of the orbit

$$\xi^{Cordey} = \sqrt{1-\frac{\mu B_0}{\epsilon}}$$

We conclude that

$$\xi^{Cordey} \neq \xi^{usual} = \frac{v_\parallel}{v}$$

and ξ^{usual} is NOT an invariant of the orbit.
END.

The fast ion Fokker Planck equation in *guiding centre* approximation (see also *collisions* text for **Cordey Connor**, the preceding article) NBI ions have an anisotropic source.

$$\begin{aligned} & \frac{\partial f}{\partial t} + \sigma\Theta v_\parallel \frac{\partial f}{r\partial\theta} \qquad \text{(neoclassical correction)} \\ = \; & \frac{1}{\tau_s}\left[\frac{1}{v^2}\frac{\partial}{\partial v}\left\{\left(v^3+v_c^3\right)f\right\} \qquad \text{(slowing down)}\right. \\ & \left. +\beta\frac{B_0}{B}\frac{v_c^3}{v^3}\frac{\left|v_\parallel\right|}{\xi v}\frac{\partial}{\partial\xi}\left\{\left(1-\xi^2\right)\frac{\left|v_\parallel\right|}{\xi v}\frac{\partial f}{\partial \xi}\right\}\right] \quad \text{pitch angle} \\ & +S \qquad \text{(source of NBI)} \end{aligned}$$

where

$$f \equiv f^{fast-ions}$$

Note

This is the equation for the *fast ions* in *mirrors*, **Hinton and Rosenbluth**.

There is a solution, for the boundary and initial condition.

The solution is also used in **NBI Hinton Rosenbluth**, torque from transversal NBI in tokamak.

End.

NOTE

This equation is for *fast ions of NBI.* (see also **Hsu Catto Sigmar** for isotropic source, which can be applied to ALPHAs)

The slowing down time τ_s is attributed to *electrons.* In Hsu Catto Sigmar it is explained that this is true if the initial velocity of the fast ions is much higher than the *critical* velocity

$$v_0 \gg v_c$$

In this region

- there is *drag* (not pitch angle), *slowing down*, and
- The drag (slowing down) is dominated by electrons: after creation of the fast ion, it collides with the *electrons.* At a critical velocity the collisional effect of electrons is equal to that of ions. All background ions + impurities are compressed in collision operator by quantities like Z_{eff} and $\overline{Z}$.
- **Cordey** says that collisions with background ions lead to radial diffusion.

We see the *neoclassical correction* ρ_θ/a and the rest is collision, *pitch angle* and *slowing down.* Caution, $a \neq L_n$ since f_0 is NOT Maxwellian.

No energetic action $\sim v_\parallel \; e\left(-\nabla_\parallel \widetilde{\Phi}\right)\frac{\partial}{\partial \epsilon}$ on parallel direction. The fast ions are not obliged to make *work* against an electric field, either poloidal $E_\theta \sim -\partial\widetilde{\Phi}/r\partial\theta$ or parallel $E_\parallel$ or *radial* $E_r \sim -\partial\Phi^{(0)}/\partial r$.

It is normal: the fast ions do not have a *stationary* distribution with variation of density on the magnetic surface, as it is for *impurities* (either in equilibrium or in rotation).

END.

In this article it is adopted the formula

$$\begin{aligned} \xi &= \sqrt{1-\frac{v_\perp^2 B_0}{v^2 B}} \\ &= \sqrt{1-\mu B_0 \frac{1}{w}} \end{aligned}$$

It results that ξ^{Cordey} is an invariant of the orbit.

$$\left|v_{\parallel}\right| = v\sqrt{1 - \frac{B}{B_0}\left(1-\xi^2\right)}$$

or

$$\begin{aligned}
\left(\frac{v_{\parallel}}{v}\right)^2 &= 1 - \frac{B}{B_0}\left(1-\xi^2\right) \\
\frac{v_{\perp}^2}{v^2} &= \frac{B}{B_0}\left(1-\xi^2\right) \\
\xi^2 &= 1 - \frac{B_0}{B}\frac{v_{\perp}^2}{v^2}
\end{aligned}$$

The Spitzer slowing down

$$\begin{aligned}
\tau_s &= \frac{3}{16\sqrt{\pi}}\frac{m_e}{e^4}\frac{m_{fast}}{Z_{fast}^2}\frac{1}{\ln\Lambda}\frac{v_e^3}{n_e} \\
&\sim \frac{T^{3/2}}{n}
\end{aligned}$$

$$\beta = \frac{1}{2}\frac{Z_{eff}m_i}{\overline{Z}\;m_f}$$

$$\begin{aligned}
\overline{Z} &= \frac{\sum Z_j^2 n_j m_j}{n_e m_e} \\
&\sim 1
\end{aligned}$$

$$Z_{eff} = \sum \frac{n_j Z_j^2}{n_e}$$

The *critical velocity* is

$$v_c = \left(\frac{3\sqrt{\pi}}{4}\overline{Z}\right)^{1/3}\left(\frac{m_e}{m_i}\right)^{1/3} \times v_e$$

This velocity is the limit where the drag due to *electrons* becomes equal to the drag by *ions*.

The geometrical effects enter through

$$\begin{aligned}
&\quad\text{dependence of } B \text{ with } \theta \\
B^{Cordey} &= B_0\frac{1-\varepsilon\cos\theta}{1-\varepsilon}\quad (!)
\end{aligned}$$

$$\varepsilon = \frac{r}{R}$$

We **note** that for $\theta = 0$ (which is the point where the last magnetic surface intersects the equatorial plane) we have

$$\begin{aligned} B &= B_0 = B_{\min} \\ B_0^{Cordey} &\equiv \text{magnetic field at the furthest point} \\ & \quad \text{on equatorial plane} \end{aligned}$$

NOTE

Usually we take

$$\begin{aligned} B^{usual} &= \frac{B_0}{h} = \frac{B_0}{1 + \varepsilon \cos\theta} \\ & B_0 \text{ is USUALLY the magnetic field on the axis} \end{aligned}$$

END

Other notations

$$\Theta \equiv \frac{B_p}{B_T}$$

The authors make the observation that the only variables now are

$$(\xi, v)$$

which are invariants of the particle motion.

There was a change of variables, from local variables to *invariants*

$$\frac{v'_{\parallel}}{v'} = \sqrt{1 - \frac{B}{B_0}\left(1 - \xi^2\right)} \text{ and } v' = v$$

to

$$(\xi, v)$$

The small quantity for expansion

$$\frac{\tau_{bounce}}{\tau_s} \ll 1$$

(many bounces between two collisions)

$$f = f_0 + \frac{\tau_{bounce}}{\tau_s} f_1 + ...$$

We remind that this is the distribution function of *FAST IONS*.

It is essentially determined by a *source* S and by the drag, pitch angle scattering.

Bounce averaging

The procedure

- the zeroth order
$$\frac{\partial f_{h0}}{\partial \theta} = 0$$
corresponds to the very short time of bounce on bananas compared to the time between two collisions $\tau_B/\tau_s \ll 1$.
- one continues with the first order f_1
- next, impose the condition that the first order f_1 function is *periodic* on θ; this means to take average over θ, $\langle\rangle$.
- this results in a constraint for the *zeroth* order f_0

It is
$$\begin{aligned}
\tau_s \frac{\partial f_0}{\partial t} &= \frac{1}{v^2}\frac{1}{v^2}\frac{\partial}{\partial v}\left[\left(v^3+v_c^3\right) f_0\right] \\
&+\beta \frac{v_c^3}{v^3}\frac{1}{\xi\left\langle \frac{v}{v_\parallel}\right\rangle}\frac{\partial}{\partial \xi}\left\{\frac{\left(1-\xi^2\right)\left\langle \frac{v_\parallel}{v}\right\rangle}{\xi}\frac{\partial f_0}{\partial \xi}\right\} \\
&+\tau_s S
\end{aligned}$$

NOTE
In zero order we get no information of f_0 except that it is constant on θ. In the next order we have to consider that f_0 has time variation: source + drag (slowing down) and pitch angle scattering.
END

The new averages are
$$\begin{aligned}
\langle\rangle &\equiv \text{bounce averaging operator} \\
&\quad \text{see Rosenbluth Hinton}
\end{aligned}$$

$$\begin{aligned}
\left\langle \frac{v_\parallel}{v}\right\rangle &= \frac{1}{2\pi\sqrt{B_0}}\oint \sqrt{\xi^2 B-(B-B_0)}\, d\theta \\
&\quad \text{for circulating particles}
\end{aligned}$$

$$\begin{aligned}
\left\langle \frac{v_\parallel}{v}\right\rangle &= \frac{1}{2\pi\sqrt{B_0}}\int_A^B \sqrt{\xi^2 B-(B-B_0)}\, d\theta \\
&\quad \text{for trapped particles between } A \text{ and } B
\end{aligned}$$

These "averages" are integrations along the trajectories: full circulating trajectory, all θ's. Respectively between the two limits θ's of the banana.

Here the magnetic field can be introduced

$$B^{Cordey} = B_0 \frac{1 - \varepsilon \cos\theta}{1 - \varepsilon}$$

DIGRESSION

From the paper **NBI currents Connor Cordey.**

Here too

$$B = B_0 \frac{1 - \varepsilon \cos\theta}{1 - \varepsilon}$$

with B_0 the *minimum* magnitude of B, $\varepsilon = \frac{r}{R}$,

$$\begin{aligned}\left\langle \frac{v_\parallel}{B} \right\rangle &= \oint d\theta \, \frac{\sqrt{2(w - \mu B)}}{B} \\ &\text{for circulating particles}\end{aligned}$$

and

$$\begin{aligned}\left\langle \frac{v_\parallel}{B} \right\rangle &= \int_A^B d\theta \, \frac{\sqrt{2(w - \mu B)}}{B} \\ &\text{for trapped particles}\end{aligned}$$

and the variables that are used

$$w = \frac{v^2}{2}$$

$$\mu = \frac{v_\perp^2}{2B}$$

$$\eta = \sqrt{1 - \frac{\mu B_0}{w}} = \sqrt{1 - \frac{v_\perp^2}{v^2} h}$$

$$\lambda = \frac{v_\perp^2}{v^2} h$$

$$B = B_0 \frac{1 - \varepsilon \cos\theta}{1 - \varepsilon}$$

In the neighborhood of *perfect trapping* the parallel velocity is near *zero*, $\xi = v_\parallel / v \to 0$, and $v_\perp^2 \approx v^2$,

$$\begin{aligned}\eta &\approx 0 \\ &\text{like } \sqrt{-\varepsilon \cos\theta}\end{aligned}$$

the averages are

$$\left\langle \frac{v_\parallel}{v} \right\rangle \approx \frac{\pi}{2B_0}\sqrt{\frac{w}{\varepsilon}}\ \eta^2$$

$$\left\langle \frac{1}{v_\parallel} \right\rangle \approx \frac{\pi}{\sqrt{w\varepsilon}}$$

In the opposite limit, *perfect circulating hot ions*, along magnetic field lines, both averages are independent of η.

The two limits adopted as reference (to avoid the intermediate values of η since difficult analytically) are

- the small $\eta \ll 1 \rightarrow$ perpendicular injection
- larger $\eta \approx 1$, parallel injection

END DIGRESSION

To check, the integrand in $\oint \sqrt{\xi^2 B - (B - B_0)}\ d\theta$ is

$$B - B_0 = B_0 \left[\frac{1-\varepsilon\cos\theta}{1-\varepsilon} - 1\right] = B_0 \frac{\varepsilon - \varepsilon\cos\theta}{1-\varepsilon}$$

and

$$\begin{aligned}
\xi^2 B - (B - B_0) &= \xi^2 B_0 \frac{1-\varepsilon\cos\theta}{1-\varepsilon} - B_0 \frac{\varepsilon - \varepsilon\cos\theta}{1-\varepsilon} \\
&= \frac{B_0}{1-\varepsilon}\left[\xi^2 - \varepsilon\xi^2\cos\theta - \varepsilon + \varepsilon\cos\theta\right] \\
&= \frac{B_0}{1-\varepsilon}\left[\xi^2 - \varepsilon + \varepsilon\left(1-\xi^2\right)\cos\theta\right]
\end{aligned}$$

We have to compute

$$\begin{aligned}
\left\langle \frac{v_\parallel}{v} \right\rangle &= \frac{1}{2\pi\sqrt{B_0}} \oint \sqrt{\xi^2 B - (B - B_0)}\ d\theta \\
&= \frac{1}{2\pi\sqrt{B_0}} \sqrt{\frac{B_0}{1-\varepsilon}} \oint d\theta \sqrt{\xi^2 - \varepsilon} \left[\sqrt{1 + \frac{\varepsilon\left(1-\xi^2\right)}{\xi^2 - \varepsilon}\cos\theta}\right]
\end{aligned}$$

We have to look for

$$\oint d\theta \sqrt{1 + \alpha\cos\theta}$$

We should use **Gradshtein Ryzhik** 3.671

$$\int_0^\pi d\theta\sqrt{a+b\cos\theta} = 2\sqrt{a+b}\mathbf{E}\left(\sqrt{\frac{2b}{a+b}}\right)$$

and in our case

$$\begin{aligned} a &= 1 \\ b &= \frac{\varepsilon\left(1-\xi^2\right)}{\xi^2-\varepsilon} \end{aligned}$$

$$a+b = \frac{\varepsilon-\varepsilon\xi^2+\xi^2-\varepsilon}{\xi^2-\varepsilon} = \xi^2\frac{1-\varepsilon}{\xi^2-\varepsilon}$$

$$\frac{2b}{a+b} = \frac{2\frac{\varepsilon\left(1-\xi^2\right)}{\xi^2-\varepsilon}}{\xi^2\frac{1-\varepsilon}{\xi^2-\varepsilon}} = 2\left(\frac{1}{\xi^2}-1\right)\frac{\varepsilon}{1-\varepsilon}$$

NOTE that B_0 is the magnetic field at $\theta=0$, *i.e.* at the farthest point on the equatorial plane. Then

$$\begin{aligned} \left\langle\frac{v_\parallel}{v}\right\rangle &= \frac{2\xi}{\pi}\mathbf{E}\left(\frac{2\varepsilon}{\xi^2}\right) \\ &\quad \text{circulating ions} \\ \xi^2 &> 2\varepsilon \end{aligned}$$

$$\begin{aligned} \left\langle\frac{v_\parallel}{v}\right\rangle &= \frac{2\sqrt{2\varepsilon}}{\pi}\left[\mathbf{E}\left(\frac{\xi^2}{2\varepsilon}\right)-\left(1-\frac{\xi^2}{2\varepsilon}\right)\mathbf{K}\left(\frac{\xi^2}{2\varepsilon}\right)\right] \\ &\quad \text{trapped particles} \\ \xi^2 &< 2\varepsilon \end{aligned}$$

and, for the average of the inverse ratio

$$\begin{aligned} \left\langle\frac{v}{v_\parallel}\right\rangle &= \frac{2}{\pi}\mathbf{K}\left(\frac{2\varepsilon}{\xi^2}\right) \\ &\quad \text{circulating ions} \\ \xi^2 &> 2\varepsilon \end{aligned}$$

$$\begin{aligned} \left\langle\frac{v}{v_\parallel}\right\rangle &= \frac{2}{\pi\xi_{bound}}\mathbf{K}\left(\frac{\xi^2}{2\varepsilon}\right) \\ &\quad \text{trapped ions} \\ \xi^2 &< 2\varepsilon \end{aligned}$$

The expressions only depend on ξ which is an invariant of the orbit.

The boundary in velocity space between trapped and circulating particles is

$$\xi_{boundary} = \sqrt{2\varepsilon}$$

NOTE

that **Hsu Catto Sigmar** have the following formula

$$\xi_t = \frac{1}{\pi}\sqrt{2\varepsilon\,(1-\varepsilon)} + (1+\varepsilon)\arcsin\left[\sqrt{\frac{2\varepsilon}{1+\varepsilon}}\right]$$

and for the *trapping/passing* boundary

$$\begin{aligned}\lambda_c &= 1-\varepsilon \\ &= \frac{B_0}{B_{\max}}\end{aligned}$$

Also, in **Hsu Catto Sigmar** the limiting values

$$\begin{aligned}\lambda &\rightarrow 0 \text{ which is strongly passing } v_\perp^2 \rightarrow 0 \\ &\downarrow \\ \langle\xi\rangle &\rightarrow 1 \\ \text{and } \frac{\partial}{\partial\lambda}\langle\xi\rangle &= -\frac{1}{2}\end{aligned}$$

The other limit

$$\begin{aligned}\lambda &\rightarrow 1-\varepsilon \;\text{ which is deeply trapped } v_\perp^2 \rightarrow v^2 \text{ at } \theta = 0 \\ &\downarrow \\ \langle\xi\rangle &\rightarrow \xi_{boundary} \equiv \xi_t \\ & \quad \frac{\partial}{\partial\xi}\langle\xi\rangle \;\text{ is logarithmically singular}\end{aligned}$$

END

The equation for f_0 (resulted from the consistency condition - periodicity on θ - of the first order f_1) needs a condition at boundary

$$\begin{aligned}&\left.\frac{\left\langle\frac{v_\parallel}{v}\right\rangle}{\xi}\frac{\partial f}{\partial\xi}\right|_{\xi^+_{boundary}} - \left.\frac{\left\langle\frac{v_\parallel}{v}\right\rangle}{\xi}\frac{\partial f}{\partial\xi}\right|_{-\xi^-_{boundary}} \\ &= 2\left.\frac{\left\langle\frac{v_\parallel}{v}\right\rangle}{\xi}\frac{\partial f}{\partial\xi}\right|_{\xi^-_{boundary}}\end{aligned}$$

This is the conservation of the flux of particles between the circulating and trapped regions in velocity space.

To keep in mind

the conservation of the flux across the separatrix between the trapped and the passing regions is expressed as a DISCONTINUITY of the derivative $\partial f / \partial \xi$ *.*

At the transition point $\xi = \sqrt{2\varepsilon}$

$$\left\langle \frac{v}{v_{\parallel}} \right\rangle \text{ has a logarithmic singularity}$$

$$\text{the derivative of } \left\langle \frac{v_{\parallel}}{v} \right\rangle \text{ is singular}$$

In the trapped region there should be symmetry relative to the sign of $v_{\parallel}$,

$$f(-\xi) = f(\xi)$$

The derivative

$$\frac{\partial f}{\partial \xi} \text{ is discontinuous at } \xi_{boundary}$$

Close to the transition

$$\left\langle \frac{v}{v_{\parallel}} \right\rangle \text{ has logarithmic singularity}$$

and

$$\frac{\partial}{\partial \xi} \left\langle \frac{v_{\parallel}}{v} \right\rangle \text{ is singular}$$

and

$$\tau_{bounce} \rightarrow \infty$$

since the trapped particles become circulating.

The equation obtained through expansion in $\frac{\tau_{bounce}}{\tau_s} < 1$ cannot be used.

the width of the transition layer in velocity space is

$$\begin{aligned} \delta\xi &\sim \sqrt{\frac{\tau_{bounce}}{\tau_s}} \\ &\sim 10^{-3} \end{aligned}$$

11.1 Solution of the equation for f_0

The equation is *separable* in ξ and v.

$$f_0 = \sum a_n(v)\, C_n(\xi)$$

here C_n are eignefunctions of

$$\frac{1}{\rho}\frac{d}{d\xi}\left(1-\xi^2\right)\varrho\frac{dC_n}{d\xi} + \lambda_n C_n = 0$$

$$\begin{aligned}
C_n \text{ continuous at } \xi &= \xi_{boundary} \\
\varrho\left.\frac{dC_n}{d\xi}\right|_{\xi^+_{boundary}} - \varrho\left.\frac{dC_n}{d\xi}\right|_{-\xi^-_{boundary}} &= 2\varrho\left.\frac{dC_n}{d\xi}\right|_{\xi^-_{boundary}} \\
C_n(-\xi) &= C_n(\xi) \quad \text{for} \quad -\xi_{boundary} < \xi < \xi_{boundary} \\
C_n \text{ finite at } \xi &= \pm 1 \quad , 0
\end{aligned}$$

The two functions are defined as

$$\rho(\xi) \equiv \left\langle \frac{v}{v_\parallel} \right\rangle \xi$$

$$\varrho(\xi) \equiv \left\langle \frac{v_\parallel}{v} \right\rangle \frac{1}{\xi}$$

Other forms of these definitions

$$\begin{aligned}
\rho(\xi) &= \xi \left\langle \frac{1}{\xi} \right\rangle \\
\varrho(\xi) &= \frac{1}{\xi}\langle \xi \rangle
\end{aligned}$$

For the coefficients a_n,

$$\frac{1}{v^2}\frac{d}{dv}\left[\left(v^3+v_c^3\right)a_n\right] - \lambda_n a_n = S_0\delta(v-v_0)\, K_n \tau_s$$

for the source

$$\begin{aligned}
S(\xi, v) &= S_0\delta(v-v_0)\, K(\xi) \\
&= S_0\delta(v-v_0)\sum K_n C_n
\end{aligned}$$

The restrictions on C_n lead to

$$\begin{aligned} C_n(\xi) &= C_n(-\xi) \\ & \text{even set} \end{aligned}$$

or

$$\begin{aligned} C_n\left(\pm\xi_{boundary}\right) &= 0 \\ & \text{odd set} \\ C_n(\xi) &= -C_n(-\xi) \quad \text{for } |\xi| > \xi_{boundary} \\ \text{and } C_n(\xi) &= 0 \quad \text{for } -\xi_{boundary} < \xi < \xi_{boundary} \end{aligned}$$

For the boundary condition

$$a_n(v = v_0) = 0$$

we obtain

$$\begin{aligned} a_n(v) &= S_0 \tau_s K_n \\ & \times \Theta\left(1 - \frac{v}{v_0}\right) \\ & \times \frac{\left(v_0^3 + v_c^3\right)^{\beta\lambda_n/3}}{\left(v^3 + v_c^3\right)^{1+\beta\lambda_n/3}} \left(\frac{v}{v_0}\right)^{\beta\lambda_n} \end{aligned}$$

This gives the full solution

$$\begin{aligned} f_0 &= S_0 \tau_s \sum K_n \frac{\left(v_0^3 + v_c^3\right)^{\beta\lambda_n/3}}{\left(v^3 + v_c^3\right)^{1+\beta\lambda_n/3}} \left(\frac{v}{v_0}\right)^{\beta\lambda_n} C_n(\xi) \\ \text{for } v &< v_0 \end{aligned}$$

To use this expression one has to solve numerically for C_n using exact expressions for $\langle v_\parallel / v \rangle$, ...

11.2 Approximative equations and Legendre polynomial solution

The idea of approximation is to fix the averages involving ξ (as ρ and ϱ) by adopting their limiting values for

- $\xi = 0$ which means deeply trapped, the parallel velocity is zero $v_\parallel \to 0$.

- $\xi = 1$ which means full passing, there is no perpendicular velocity since $v_\parallel \to v$.

Take the approximations

$$\begin{aligned} \tau &\equiv \left\langle \frac{v}{v_\parallel} \right\rangle \xi \to \frac{\xi}{\xi_{boundary}} \\ \varrho &\equiv \left\langle \frac{v_\parallel}{v} \right\rangle \frac{1}{\xi} \to \frac{\xi}{2\xi_{boundary}} \\ \text{both for } |\xi| &< \xi_{boundary} \text{ (trapped)} \end{aligned}$$

and

$$\begin{aligned} \rho &= 1 \\ \varrho &= 1 \\ \text{both for } |\xi| &> \xi_{boundary} \text{ (circulating)} \end{aligned}$$

The equations for circulating ions

$$\frac{d}{d\xi}\left[\left(1-\xi^2\right)\frac{dC_n}{d\xi}\right] + \lambda_n C_n = 0 \text{ for } |\xi| > \xi_{boundary}$$

For trapped ions

$$\frac{1}{2}\frac{1}{\xi}\frac{d}{d\xi}\left[\left(1-\xi^2\right)\ \xi\ \frac{dC_n}{d\xi}\right] + \lambda_n C_n = 0 \text{ for } |\xi| < \xi_{boundary}$$

11.2.1 Trapped

The substitution

$$x = 1 - 2\xi^2$$

leads for TRAPPED ions

$$2\frac{d}{dx}\left(1-x^2\right)\frac{dC_n}{dx} + \lambda_n C_n = 0$$

has solution

$$C_n\left(\xi\right) = P_{\nu_n}\left(\xi\right)$$

The indice results from

$$P_{\nu_n}\left(\xi_{boundary}\right) = 1$$

$$\nu_n = n + \frac{4\xi_{boundary}}{\pi} \frac{\Gamma^2(1+n/2)}{\Gamma^2(1/2+n/2)}$$

and

$$\begin{aligned}\lambda_n &= \nu_n(\nu_n+1)\\ &= n(n+1) + \frac{4\xi_{boundary}}{\pi}(2n+1)\frac{\Gamma^2(1+n/2)}{\Gamma^2(1/2+n/2)}\end{aligned}$$

11.2.2 Circulating

The solution of the equation for the *circulating* ions

$$C_n = P_{\nu_n}(\xi) \quad \text{for} \quad \xi > \xi_{boundary}$$

To find the indice, one has to consider the solution for *trapped* ions close to the limit

$$\begin{aligned}C_n &= A\left(1 - \nu_n \frac{\nu_n+1}{2}\xi^2\right)\\ \text{for} \quad |\xi| &< \xi_{boundary} \quad \text{(trapped)}\end{aligned}$$

Matching the two solutions at the limit $\xi_{boundary}$

$$2\frac{dP_{\nu_n}}{d\xi}\left(\xi = \xi_{boundary}\right) = -\xi_{boundary} \times \ \nu_n(\nu_n+1) \ P_{\nu_n}\left(\xi_{boundary}\right)$$

and

$$A = \frac{P_{\nu_n}\left(\xi_{boundary}\right)}{1 - \nu_n\frac{\nu_n+1}{2}\xi^2_{boundary}}$$

The order ν_n can be obtained approximately

$$\begin{aligned}\nu_n &= n + \xi_{boundary}\, n(n+1)\frac{\Gamma^2\left(\frac{1}{2}+\frac{n}{2}\right)}{2\pi\Gamma^2\left(1+\frac{n}{2}\right)}\\ & \quad n \quad \text{even}\end{aligned}$$

$$\begin{aligned}\lambda_n &= \nu_n(\nu_n+1)\\ &= n(n+1) + (2n+1)\xi_{boundary}\, n(n+1)\frac{\Gamma^2\left(\frac{1}{2}+\frac{n}{2}\right)}{2\pi\Gamma^2\left(1+\frac{n}{2}\right)}\end{aligned}$$

11.2.3 Collecting the results for C_n

The result for n ODD

$$
\begin{aligned}
C_n &= P_{\nu_n}(\xi) \quad \text{for} \quad \xi_{boundary} < \xi < 1 \\
& \quad n \;\; \text{odd} \\
\nu_n &= n + \frac{4\xi_{boundary}}{\pi} \frac{\Gamma^2(1+n/2)}{\Gamma^2(1/2+n/2)}
\end{aligned}
$$

and

$$C_n = 0 \quad \text{for} \quad -\xi_{boundary} < \xi < \xi_{boundary}$$

$$C_n = -P_{\nu_n}(-\xi) \quad \text{for} \quad -1 < \xi < -\xi_{boundary}$$

The result for n EVEN

$$C_n = P_{\nu_n}(\xi) \quad \text{for} \quad \xi_{boundary} < \xi < 1$$

$$
\begin{aligned}
C_n &= \frac{P_{\nu_n}\left(\xi_{boundary}\right) \; \left[2 - \nu_n(\nu_n+1)\xi^2\right]}{2 - \nu_n(\nu_n+1)\xi^2_{boundary}} \\
\text{for} \quad -\xi_{boundary} &< \xi < \xi_{boundary}
\end{aligned}
$$

$$C_n = P_{\nu_n}(-\xi) \quad \text{for} \quad -1 < \xi < -\xi_{boundary}$$

11.3 The distribution functions

11.3.1 Passing ions

The expression

$$
\begin{aligned}
f_0 &= S_0\tau_s \sum K_n \frac{\left(v_0^3 + v_c^3\right)^{n(n+1)\beta/3}}{\left(v^3 + v_c^3\right)^{1+n(n+1)\beta/3}} \left(\frac{v}{v_0}\right)^{n(n+1)\beta} \quad P_n(\xi) \\
&+ S_0\tau_s\sqrt{2\varepsilon} \sum
\end{aligned}
$$

See also **Gaffey**.

12 Fast NBI ions in mirrors Hinton Rosenbluth 1982

Also in *impurities.tex*.

the equation for the distribution function for the *fast ions* from **mirrors Hinton Rosenbluth** (see also **NBI**)

$$\begin{aligned}
& \frac{\partial f}{\partial t} + \left(v_{\parallel}\widehat{\mathbf{n}} + \mathbf{v}_D\right) \cdot \boldsymbol{\nabla} f \qquad \text{strangely: convective term} \\
& -\nu_s \frac{2m_i}{m_{fast}} \frac{v_c^3}{v^3} \xi \frac{1}{B} \frac{\partial}{\partial \lambda}\left(\lambda \xi \frac{\partial f}{\partial \lambda}\right) \qquad \text{pitch angle} \\
& -\nu_s \frac{1}{v^2} \frac{\partial}{\partial v}\left[\left(v^3 + v_c^3\right) f\right] \qquad \text{slowing down} \\
= \; & S \frac{\delta\left(v - v_0\right)}{v_0^2}
\end{aligned}$$

(what is strange: the neoclassical correction $\mathbf{v}_D \cdot \boldsymbol{\nabla} f$ - is-it important here ?)

Note looks like **Cordey Houghton**. **END.**

The slowing down due to the electron drag is

$$\begin{aligned}
\nu_s &= \frac{m_e}{m_{fast}} Z_{fast}^2 \frac{1}{\tau_e} \\
\frac{1}{\tau_e} &= \frac{16\sqrt{\pi}}{3} \frac{e^4}{m_e^2} \ln \Lambda \frac{n_e}{v_{th,e}^3}
\end{aligned}$$

and the critical velocity

$$v_c = \left(\frac{3\sqrt{\pi}}{4} \frac{m_e}{m_i}\right)^{1/3} v_{th,e}$$

The critical velocity v_c is, like usual, the limit where the drag by electrons (for the fast NBI ions) becomes comparable with the drag by ions. After this limit, the slowing down is accompanied by pitch angle scattering and velocity-space remodelation of f.

Technical advancements.
In the case of *mirrors* the equation can ignore the drift term.
This is similar to the anisotropic distribution of the NBI fast ions.

What remains is the variation of f along the line, $\partial f/\partial l$ balanced by the pitch angle scattering and by the friction (slowing down) collisions.

$$\begin{aligned}
& v_{\parallel}\frac{\partial f}{\partial l_{\parallel}} && \left(\text{this is } v_{\parallel}\nabla_{\parallel}\right) \\
& -\nu_s \frac{2m_i}{m_{fast}}\frac{v_c^3}{v^3}\frac{\xi}{B}\frac{\partial}{\partial\lambda}\left(\xi\lambda\frac{\partial f}{\partial\lambda}\right) && \text{(pitch angle)} \\
& -\nu_s \frac{1}{v^2}\frac{\partial}{\partial v}\left[\left(v^3+v_c^3\right)f\right] && \text{(slowing down)} \\
= \; & 0
\end{aligned}$$

Then it is *bounce averaged*: it is divided by ξ (and in the first term we have $v\xi = v_{\parallel}$) and then it is integrated over the line. The remaining factor in the first term v is invariant, being the energy

$$\oint dl_{\parallel} \times$$

and the first term is zero by periodicity.

$$\begin{aligned}
& \frac{2m_i}{m_{fast}}\frac{v_c^3}{v^3}\frac{\partial}{\partial\lambda}\left[\left(\oint dl_{\parallel}\frac{\xi}{B}\right)\lambda\frac{\partial f}{\partial\lambda}\right] \\
& +\left(\oint dl_{\parallel}\frac{1}{\xi}\right)\frac{1}{v^2}\frac{\partial}{\partial v}\left[\left(v^3+v_c^3\right)f\right] \\
= \; & 0
\end{aligned}$$

The boundary condition at the limit where the fast particles are generated, v_0, is

$$f \to \frac{1}{\nu_s\left(v_0^3+v_c^3\right)}\frac{\oint dl_{\parallel}\frac{1}{\xi}S}{\oint dl_{\parallel}\frac{1}{\xi}}$$

For the solution.

See also **Cordey Houghton**, the one-dimensional approximation when the pitch angle can be neglected, for high velocities, slowing down on the electrons.

Substitutions

$$u \equiv \frac{v}{v_c}$$

by this new variable, the physical velocity v is compared with the reference *critical* velocity v_c.

$$f = \frac{G\left(u,\lambda\right)}{v_c^3\left(u^3+1\right)}$$

Results

$$
\begin{aligned}
& \frac{2m_i}{m_{fast}} \frac{1}{u^3 \left(u^3+1\right)} \frac{\partial}{\partial \lambda} \left[I_1 \left(\lambda\right) \lambda \, \frac{\partial G}{\partial \lambda} \right] \\
& + I_2 \left(\lambda\right) \frac{1}{u^2} \frac{\partial G}{\partial u} \\
= \; & 0
\end{aligned}
$$

with the definitions

$$
\begin{aligned}
I_1 \left(\lambda\right) &= \int_0^{s_t} ds \, \frac{\xi}{B} \\
I_2 \left(\lambda\right) &= \int_0^{s_t} ds \frac{1}{\xi}
\end{aligned}
$$

where s_t is the *turning point*

$$\lambda B \left(s_t\right) = 1$$

Another transformation: a new *time-like* variable (but, of course, NOT time)

$$
\begin{aligned}
\tau &= \frac{2}{3} \frac{m_i}{m_{fast}} \ln \left(\frac{1+u^{-3}}{1+u_0^{-3}} \right) \\
\text{where } u_0 &\equiv \frac{v_0}{v_c}
\end{aligned}
$$

The notation is important u_0 is the velocity of birth compared with the critical velocity v_c.

The equation

$$\frac{\partial G}{\partial \tau} - \frac{1}{I_2 \left(\lambda\right)} \frac{\partial}{\partial \lambda} \left[I_1 \left(\lambda\right) \; \lambda \, \frac{\partial G}{\partial \lambda} \right] = 0$$

Note that the first term is what remains from the *slowing down* part of the collision operator. The second term is the *pitch angle* part.

The general form

$$\frac{\partial G}{\partial \tau} = -A \left(\lambda\right) \frac{\partial G}{\partial \lambda} + D \left(\lambda\right) \frac{\partial^2 G}{\partial \lambda^2}$$

$$A \left(\lambda\right) \equiv -\frac{\frac{d}{d\lambda} \left(\lambda I_1\right)}{I_2}$$

$$D(\lambda) \equiv \frac{\lambda I_1}{I_2}$$

In the case of mirrors

$$I_2(\lambda) = -2\frac{dI_1}{d\lambda}$$

Substitution

$$G_1(\lambda, \tau) = \exp\left[\phi(\lambda, \tau)\right]$$

with

$$\frac{\partial \phi}{\partial \tau} = -A(\lambda)\frac{\partial \phi}{\partial \lambda} + D(\lambda)\left[\frac{\partial^2 \phi}{\partial \lambda^2} + \left(\frac{\partial \phi}{\partial \lambda}\right)^2\right]$$

Further

$$\begin{aligned} \phi(\lambda, \tau) &= -\frac{w(\lambda)}{\lambda} - \frac{1}{2}\ln \tau \\ &\quad + y(\lambda, \tau) \end{aligned}$$

and

$$\begin{aligned} y(\lambda, \tau) &= y_0(\lambda) + y_1(\lambda) \times \tau \\ &\quad + ... \end{aligned}$$

After equating terms of the same power in τ it results

$$w(\lambda) = \frac{1}{4}\left[\int_{\lambda_0}^{\lambda} \frac{d\lambda}{\sqrt{D(\lambda)}} + c\right]^2$$

$$y_0(\lambda) = -\frac{1}{4}\ln\left[\lambda I_1(\lambda)\ \ I_2(\lambda)\right] + k$$

The solution

$$\begin{aligned} G_1(\lambda, \tau) &= \frac{Q}{4\pi^{3/2}}\frac{1}{\sqrt{D(\lambda_0)\ \times \tau}} \\ &\quad \times \left[\frac{\lambda_0 I_1(\lambda_0)\ \ I_2(\lambda_0)}{\lambda I_1(\lambda)\ \ I_2(\lambda)}\right]^{1/4} \\ &\quad \times \exp\left[-\frac{w_1(\lambda)}{\tau}\right] \end{aligned}$$

where w_1 is w with $c = 0$.

13 The transport of FAST *alpha (isotropic)* IONS Hsu Catto Sigmar

It is **Hsu Catto Sigmar PF-B2 (1990) 280**.

It is the basis for **Hsu Shaing Gormley Sigmar bootstrap of alphas**.

It uses much of **Cordey Houghton,** of **Connor Cordey**, of **Cordey Start** and of **Cordey Cox.**

All work Fokker - Planck equation for a *fast* ion component, with full collision operator. The other papers taken as reference, work for NBI.

Part of text is in *collisions.tex*, and in *bootstrap.tex*.

More *physics*.

The *flux surface averaged Fokker Planck eq.*

Their theory is NOT appropriate to describe the distribution function of fast ions whose width of the banana is comparable with the plasma radius.

It also is NOT appropriate for NBI fast ions whose source is *anisotropic* in the velocity space.

The collisions:

- drag (*friction*, collisions with *electrons* and then with background ions with different *velocity*)
- pitch angle

A fomulation of the first approaches.

The transport decreases

- first very hot ions are generated isotropically (*alpha*).
- second, they are slowed down by collisions with background electrons and later with ions, that induces reduction of the perpendicular velocities

"*banana collapse*" : the reduction of the width of the bananas of the fast ions due to the slowing down by *drag* of the background ions.

A fast ion undergoes

- more *drag* when the energy is high (at birth), firstly by electrons

- progressively more *pitch angle* scattering after the drag has slowed it down, from background ions. Then the velocity space structure *trapped/circulating* is modified

Three classes of fast ions

- those that suffer more *drag*
- those that have in comparable proportion *drag* and *pitch angle*
- those that suffer more *pitch angle* scattering

The contribution to transport depend on

- the inverse aspect ratio ε (toroidality)
- the ratio
$$\frac{v_0}{v_c} = \frac{\text{birth velocity}}{\text{critical velocity}}$$
or the birth velocity v_0 and the *critical velocity* v_c.

For fast ion velocities that are higher than v_c

$$v^{fast-ion} > v_c$$

the electrons produce a drag that is higher than that of the background ions.

The drag caused by *electrons* produces slowing down of the fast ions. At a limit of velocity, v_c, the drag of the electrons becomes equal to the drag by ions.

So the history consists, after generation of a fast ion, of: (1) slowing down by the *electrons* until the reduced velocity becomes comparable with the *critical velocity*; (2) then the slowing down is taken over by the background ions, together with *pitch angle* scattering.

The Fokker Planck equation for the fast ions, *bounce averaged* - is a second order differential equation in two-variables. The method is as in **Cordey**: *separate the variables* $f_0 = \sum a_n(v)\, C_n(\xi)$ and the equation into two:

- a second order eq. for pitch-angle scattering variable ξ, and
- an first order differential equation for the ratio velocity/critical-velocity v/v_c

The source from fusion

$$S = n_D n_T \langle \sigma_f \ v \rangle$$

this is the rate of generation of *alpha* ions.

The bounce averaged equation

$$\frac{\partial f}{\partial t} = C(f) + \frac{S}{4\pi v^2}\delta(v - v_0)$$

and

$$C(f) = \frac{1}{\tau_s}\frac{\partial}{\partial \mathbf{v}} \cdot \left[\left(1 + \frac{v_c^3}{v^3} \right) \mathbf{v} f + \frac{1}{2}\frac{v_b^3}{v^3}\left(v^2 \ \mathbf{I} - \mathbf{v} \ \mathbf{v} \right) \cdot \frac{\partial f}{\partial \mathbf{v}} \right]$$

NOTE NOTE NOTE

We copy from *collision.tex* the final part of the derivation made by **Gaffey** for collisions between *beam ions* and background ions. Here

$$\begin{aligned} f_b &\equiv \text{beam-ions distribution function} \\ f_i &\equiv \text{ions} \end{aligned}$$

First

$$\begin{aligned} & C(f_b, f_i) \\ = \ & \Gamma_{bi}\frac{\partial}{\partial \mathbf{v}} \cdot \left[\frac{\partial^2 v}{\partial \mathbf{v} \ \partial \mathbf{v}} \cdot \frac{\partial f_b}{\partial \mathbf{v}} + \frac{m_b}{m_i}\frac{2\mathbf{v}}{v^3} f_b \right] \end{aligned}$$

An important expression

$$\frac{\partial}{\partial \mathbf{v}_i} \cdot \left[\frac{\partial^2 v_i}{\partial \mathbf{v}_i \ \partial \mathbf{v}_i} \cdot \frac{\partial f_b}{\partial \mathbf{v}_i} \right]$$

is calculated introducing the variables in the velocity space

$$(\xi, \theta, \varphi)$$

$$\begin{aligned} \xi &= \cos\theta \\ &= \frac{\mathbf{v}_i}{v_i} \cdot \widehat{\mathbf{n}} \\ &= \frac{v_\parallel}{v} \end{aligned}$$

then

$$\begin{aligned}\frac{\partial \xi}{\partial \mathbf{v}_i} &= \frac{\partial}{\partial \mathbf{v}_i}\frac{\mathbf{v}_i}{v_i} \cdot \widehat{\mathbf{n}} \\ &= \frac{\partial^2 v}{\partial \mathbf{v}_i\, \partial \mathbf{v}_i} \cdot \widehat{\mathbf{n}}\end{aligned}$$

It is assumed that the distribution function f_b does not depend on the angle φ.

$$\frac{\partial f_b}{\partial \varphi} = 0$$

$$\begin{aligned}\frac{\partial f_b}{\partial \mathbf{v}_i} &= \frac{\partial f_b}{\partial v}\frac{\partial v}{\partial \mathbf{v}_i} \\ &\quad + \frac{\partial f_b}{\partial \xi}\frac{\partial \xi}{\partial \mathbf{v}_i}\end{aligned}$$

We multiply this formal equality by

$$\frac{\partial^2 v}{\partial \mathbf{v}_i\, \partial \mathbf{v}_i}$$

It results

$$\begin{aligned}\frac{\partial f_b}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 v}{\partial \mathbf{v}_i\, \partial \mathbf{v}_i} &= \frac{\partial f_b}{\partial v}\frac{\partial v}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 v}{\partial \mathbf{v}_i\, \partial \mathbf{v}_i} \\ &\quad + \frac{\partial f_b}{\partial \xi}\frac{\partial \xi}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 v}{\partial \mathbf{v}_i\, \partial \mathbf{v}_i}\end{aligned}$$

we take into account, for the second term

$$\frac{\partial \xi}{\partial \mathbf{v}_i} = \frac{\partial^2 v}{\partial \mathbf{v}_i\, \partial \mathbf{v}_i} \cdot \widehat{\mathbf{n}}$$

and for the first term

$$\begin{aligned}\frac{\partial x_{ij}}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 x_{ij}}{\partial \mathbf{v}_i\, \partial \mathbf{v}_i} &= \frac{1}{v_{th,j}}\frac{\mathbf{v}_i}{v_i} \cdot \left(\frac{1}{v_i}\mathbf{I} - \frac{\mathbf{v}_i\ \mathbf{v}_i}{v_i^3}\right) \\ &= 0\end{aligned}$$

where we can take the particular case, for clarity

$$\begin{aligned}\mathbf{v}_j &= 0 \\ x_{ij} &= |\mathbf{v}_i| = v_i\end{aligned}$$

Assumption that f_b is axisymmetric in velocity space relative to the direction of $\mathbf{B}$. This introduces $\xi = v_{\parallel}/v$.

$$\begin{aligned}
\frac{\partial f_b}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 v_i}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} &= \frac{\partial f_b}{\partial \xi} \frac{\partial \xi}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 v_i}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} \\
&= \frac{\partial f_b}{\partial \xi} \left(\frac{\partial^2 v_i}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} \cdot \widehat{\mathbf{n}} \right) \cdot \frac{\partial^2 v_i}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} \\
&= \frac{\partial f_b}{\partial \xi} \widehat{\mathbf{n}} \cdot \frac{1}{v_i} \frac{\partial^2 v_i}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial}{\partial \mathbf{v}} \cdot \left[\frac{\partial f_b}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 v}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} \right] \\
= & \left[\frac{\partial^2 f_b}{\partial \xi^2} \widehat{\mathbf{n}} \cdot \frac{1}{v_i} \frac{\partial^2 v_i}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} + \frac{1}{v} \frac{\partial f_b}{\partial \xi} \frac{\partial}{\partial \mathbf{v}} \right] \cdot \frac{\partial^2 v_i}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} \cdot \widehat{\mathbf{n}} \\
= & \left(1 - \xi^2\right) \frac{1}{v^3} \frac{\partial^2 f_b}{\partial \xi^2} - \frac{2\xi}{v^3} \frac{\partial f_b}{\partial \xi} \\
= & \frac{1}{v^3} \frac{\partial}{\partial \xi} \left[\left(1 - \xi^2\right) \frac{\partial f_b}{\partial \xi} \right]
\end{aligned}$$

This will lead to

$$\begin{aligned}
& C\left(f_b, f_i\right) \\
= & \Gamma_{bi} \frac{1}{v^3} \left[\frac{\partial}{\partial \xi} \left(1 - \xi^2\right) \frac{\partial f_b}{\partial \xi} + \frac{m_b}{m_i} 2v \frac{\partial f_b}{\partial v} \right]
\end{aligned}$$

END END END

NOTE
The equation consists of evolution in the *velocity space*.
There is no *spatial* variation of the SOURCE.
It is hard to see how to apply this to the transient expansion of bananas which intrude one half in the layer
END

NOTE
Apparently there is no neoclassical contribution $\mathbf{v}_D \cdot \boldsymbol{\nabla} f^{(0)}$ to the Fokker Planck equation.
Later it is introduced.
END

The two particular values of the velocity are

$$
\begin{aligned}
v_c &= \frac{3\sqrt{\pi}}{\sqrt{2}} \frac{1}{\sqrt{m_e}} \frac{\sum_j \frac{Z_j^2 n_j \ln \Lambda_j}{m_j}}{\ln \Lambda_e} \frac{T_e^{3/2}}{n_e} \\
& \text{critical velocity of the fast ions} \\
& \text{boundary from drag by electrons to drag by ions} \\
m_j &\equiv \text{mas of background ions, incl. impurities}
\end{aligned}
$$

$$
\begin{aligned}
v_b &= \frac{3\sqrt{\pi}}{\sqrt{2}} \frac{1}{\sqrt{m_e}} \frac{\sum \frac{Z_j^2 n_j \ln \Lambda_j}{m}}{\ln \Lambda_e} \frac{T_e^{3/2}}{n_e} \\
& \text{velocity for the pitch angle scattering} \\
& \text{of fast ions by the thermal ions} \\
m &\equiv \text{mass of fast ions}
\end{aligned}
$$

where $j \equiv$ ion species, $m \equiv$fast ions

$$
\begin{aligned}
\tau_s &= \frac{3}{4\sqrt{2\pi}} \frac{1}{\sqrt{m_e}} \frac{m}{Z^2 e^4} \frac{1}{\ln \Lambda_e} \\
& \times \frac{n_e}{T_e^{3/2}}
\end{aligned}
$$

where

$$
\begin{aligned}
\tau_s &\equiv \text{slowing down time by drag produced by ELECTRONS} \\
v_c &\equiv \text{speed above which electron drag dominates} \\
& \quad \text{(critical velocity)}
\end{aligned}
$$

and

$$
\begin{aligned}
v_b &\equiv \text{pitch angle scattering by ions} \\
\frac{\tau_s v^3}{2 v_b^3} &\equiv \text{deflection time}
\end{aligned}
$$

Remark the change between

$$\sum \frac{Z_j^2 n_j \ln \Lambda_j}{m_j}$$

which is in the *critical velocity* with

$$\sum \frac{Z_j^2 n_j \ln \Lambda_j}{m}$$

which is in the *pitch angle* velocity.

13.1 Solution of the drift kinetic equation.

Expansion in

$$\frac{\rho_\theta}{L} \ll 1 \text{ neoclassic}$$

Bounce motion is very fast compared with slowing down. Expansion in

$$\frac{\text{time of bounce}}{\text{time of slowing down}} = \frac{\tau_{bounce}}{\tau_s} \ll 1$$

The zeroth order can be written explicitely

$$f_0 = S\frac{\tau_s}{4\pi\left(v^3+v_c^3\right)}\Theta\left(v_0-v\right)$$

NOTE

In **Cordey Houghton 1973** the equation for the distribution function of the fast NBI (*anisotropic*) ions is reduced by approximation to a one-dimensional equation in the regime $v \gg v_c$ (*i.e.* when there is slowing down on the background electrons and the pitch angle scattering is negligible). Here one-dimensional means, in velocity space, that f is only dependent on v NOT on $v_\parallel$:

$$\begin{aligned}\tau_s\frac{\partial f_b}{\partial t} &= \frac{1}{v^2}\frac{\partial}{\partial v}\left[\left(v_c^3+v^3\right)f_b\right] \\ &\quad +\tau_s\widetilde{S}\left(v-v_0\right)\ \delta\left(\xi-\xi_0\right)\end{aligned}$$

(where f_b is for BEAM-ions) with solution

$$f_b = \tau_s\frac{1}{v_c^3+v^3}\delta\left(\xi-\xi_0\right)\int_v^{v^*}dv'\ v'^2\ \widetilde{S}\left(v'-v_0\right)$$

with the definition

$$v^* \equiv \left[\left(v_c^3+v^3\right)\exp\left(\frac{3t}{\tau_s}\right)-v_c^3\right]^{1/3}$$

After adopting a shape for $\widetilde{S}\left(v-v_0\right)$ as a Gaussian and after further simplification it results an time-asymptotic solution

$$f_b = S\frac{\tau_s}{v_c^3+v^3}\ \delta\left(\xi-\xi_0\right)$$

This is the explanation for the initial form adopted for f_b.

Compare with $f_0 = S\frac{\tau_s}{4\pi(v^3+v_c^3)}\Theta\left(v_0 - v\right)$.
END

NOTE about zeroth order.
If the source is *anisotropic in the velocity space* (like in NBI) then the zero order distribution function is strongly changed, by *pitch angle scattering*

$$f_0 = f_0\left(\psi, v, \lambda\right)$$

When the banana is very large

$$\Delta r_b \approx \sqrt{\varepsilon}\rho_\theta$$

the trapped particle experiences *different* friction forces on one side and the other side of the orbit, and f_0 becomes a function of also θ

$$f_0 \rightarrow f_0 = f_0\left(\psi, \theta; v, \lambda\right)$$

END

NOTE
Cordey NBI finds for the zeroth order distribution function of fast NBI ions the simple property

$$\frac{\partial f_0}{\partial \theta} = 0$$

but after writting the first order's equation f_1 and using periodicity, one obtaines an equation for f_0 in velocity space.
END

Gyroaverage first.
The equation in the first order

$$v_\parallel \nabla_\parallel \left(\overline{f}_1 + \frac{I}{\Omega} v_\parallel \left.\frac{\partial f_0}{\partial \psi}\right|_{\epsilon=\text{const}}\right) = \overline{C}\left(\overline{f}_1\right)$$

where $\overline{f}$ is gyroaveraged.

$$\mathbf{B} = \boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi + I\boldsymbol{\nabla}\varphi$$

$$\begin{aligned} J &= \frac{1}{\left|\left(\boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi\right)\cdot\boldsymbol{\nabla}\theta\right|} = \frac{1}{\frac{1}{R}RB_\theta\frac{1}{r}} = \frac{R}{\frac{RB_\theta}{rB_T}B_T} = \frac{qR}{B_T} = \frac{qR}{B_0}h \\ &\sim \left(\frac{qR_0}{B_0}\right)h^2 \end{aligned}$$

dependence on θ is $\rightarrow \quad 1 + 2\varepsilon\cos\theta$

$$\mathbf{B}\cdot\boldsymbol{\nabla}=\frac{1}{J}\frac{\partial}{\partial\theta}$$

since $\mathbf{B}\cdot\boldsymbol{\nabla}=(B_0/h)\,\nabla_\parallel=\frac{B_0}{h}\frac{B_\theta}{B}\frac{\partial}{r\partial\theta}=\frac{1}{h\frac{qR}{B_0}}\frac{\partial}{\partial\theta}=\frac{1}{J}\frac{\partial}{\partial\theta}$.

The gyrophase averaged collision operator is

$$\begin{aligned}\overline{C}\,(g) &= \frac{1}{\tau_s}\left(\frac{1}{v^2}\frac{\partial}{\partial v}\left[\left(v^3+v_c^3\right)g\right]\right.\\ &\quad\left.+2\frac{v_b^3}{v^3}h\ \xi\ \frac{\partial}{\partial\lambda}\lambda\xi\ \frac{\partial}{\partial\lambda}g\right)\end{aligned}$$

where

$$\begin{aligned} g &\equiv \text{ gyrophase averged part of the distrubution}\\ \text{function,} &= 0\text{ in the trapped region}\end{aligned}$$

where, the definitions are

$$\begin{aligned}\lambda &= \frac{v_\perp^2}{v^2}h\\ \xi &= \frac{v_\parallel}{v}=\pm\sqrt{1-\frac{\lambda}{h}}\end{aligned}$$

Therefore, the *overbar*, is *gyration-average.* No bounce average yet.
These must be compared with **Cordey** where the definitions are different.

Using the expression of the zeroth order distribution function of the *fast ions (isotropic) alphas* f_0 , the derivative is

$$\left.\frac{\partial f_0}{\partial\psi}\right|_{\epsilon=\text{const}}=\left(\frac{\partial}{\partial\psi}\ln\left(S\tau_s\right)-\frac{v_c^3}{v^3+v_c^3}\frac{\partial}{\partial\psi}\ln v_c^3+\frac{3}{2}\frac{v^3}{v^3+v_c^3}\frac{Ze\phi}{mv^2/2}\frac{\partial}{\partial\psi}\ln\phi\right)f_0$$

The solution for $\overline{f}_1$ is obtained adopting the form

$$\overline{f}_1=\left(-I\frac{1}{2\Omega_0}v\left.\frac{\partial f_0}{\partial\psi}\right|_{\epsilon=\text{const}}\right)(2h\xi+P)$$

where $\Omega_0=\frac{eB_0}{m_i}$ for $B=\frac{B_0}{h}$.The first term is

$$-I\frac{1}{2\Omega_0}v\left.\frac{\partial f_0}{\partial\psi}\right|_{\epsilon=\text{const}}\times 2h\xi=-I\frac{1}{\Omega}v_\parallel\left.\frac{\partial f_0}{\partial\psi}\right|_{\epsilon=\text{const}}$$

which is precisely the first order neoclassic correction. We **note** that this is the usual neoclassical correction, in order 1 in ρ_θ/L_n.

The first part is OK, it is the neoclassical correction to the 0 distribution function.

But the second part?

It is a factor $-I\frac{1}{2\Omega_0}v\left.\frac{\partial f_0}{\partial\psi}\right|_{\epsilon=\text{const}}$ multiplying the unknown function P.

One has, as usual

$$P \equiv 0 \text{ in trapped region}$$

NOTE

Compare with **Hsu Shaing Gormley Sigmar 1992** (bootstrap from α particles) where the *perturbation* P is introduced as

$$\begin{aligned} f_{\alpha 1} &= -I\frac{v_\parallel}{\Omega_\alpha}\frac{\partial f_{\alpha 0}}{\partial\psi} \\ &\quad +v_\parallel V_{\parallel i}^{*}\frac{\partial f_{\alpha 0}}{\partial w} \\ &\quad +P \end{aligned}$$

and is not multiplied by the neoclassical factor, as it is above. Then the expansion for the *separation of variables* is

$$P(\lambda, w, \psi) = \sum_{j=1,2,3}\left(\sum_{n=1}^{\infty}\Lambda_n(\lambda,\psi)\ V_{nj}(w,\psi)\right)A_j(w,\psi)$$

We conclude that, in the treatment of *bootstrap*, what introduces (formally) the *forces* A_j in the expression of P is the fact that it is separated from the neoclassical correction factor.

END

Returning, the solubility condition for the equation

$$v_\parallel\nabla_\parallel\left(\overline{f}_1+\frac{I}{\Omega}v_\parallel\left.\frac{\partial f_0}{\partial\psi}\right|_{\epsilon=\text{const}}\right) = \overline{C}\left(\overline{f}_1\right)$$

is, after dividing by $v_\parallel$, multiplying with B $\left(=\frac{B_0}{h}\right)$ and averaging over the surface (since this removes the left hand side, being the *annihilator* and exploits the periodicity) See **Rutherford1970**

$$\left\langle\frac{B}{v_\parallel}\overline{C}\left(\overline{f}_1\right)\right\rangle = 0$$

The average $\langle\rangle$ will lead to periodicity after first multiplication by B. We have

$$B\times| \quad \nabla_{\|}(...) = C$$

but

$$\begin{aligned}
dl_{\|} &= dl_{\theta}\frac{B_{\theta}}{B_T} = rd\theta\frac{B_{\theta}}{B_T} = R\left(\frac{rB_{\theta}}{RB_T}\right)d\theta = qR\ d\theta \\
\text{then } \frac{d}{dl_{\|}} &= \frac{1}{qR}\frac{d}{d\theta}
\end{aligned}$$

$$\begin{aligned}
\nabla_{\|} &= \frac{\partial}{\partial l_{\|}} = \frac{1}{qR}\frac{\partial}{\partial\theta} \ \text{ and} \\
B &= \frac{B_0}{h}
\end{aligned}$$

so that

$$B\nabla_{\|}() \rightarrow \frac{B_0}{h}\frac{1}{\frac{rB_T}{RB_{\theta}}R}\frac{\partial}{\partial\theta}() = B_{\theta}\frac{\partial}{r\partial\theta}()$$

Now, we have to recall the definition of the average operation

$$\langle A\rangle = \frac{\oint\frac{d\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}A(\theta)}{\oint\frac{d\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}} = \frac{\oint\frac{rd\theta}{B_{\theta}}A(\theta)}{\oint\frac{rd\theta}{B_{\theta}}} = \frac{\frac{1}{b(r)}\oint rd\theta\ h\ A(\theta)}{\frac{1}{b(r)}\oint rd\theta\ h}$$

and

$$\begin{aligned}
\left\langle B\nabla_{\|}()\right\rangle &= \left\langle B_{\theta}\frac{\partial}{r\partial\theta}()\right\rangle \\
&= \frac{\oint rd\theta\ h\ B_{\theta}\frac{\partial}{r\partial\theta}()}{\oint rd\theta\ h} \\
&= \frac{b(r)\oint rd\theta\ h\ \frac{1}{h}\frac{\partial}{r\partial\theta}()}{\oint rd\theta\ h} = \frac{b(r)}{\oint rd\theta\ h}()_{\theta=0}^{\theta=2\pi} \\
&= 0 \ \text{ for periodic}
\end{aligned}$$

This is the reason for which we must multiply (...) with B before taking the surface average, if we intend to exploit poloidal periodicity.

In detail, the periodicity constraint $\left\langle \frac{B}{v_{\|}} \overline{C} \left(\overline{f}_1 \right) \right\rangle = 0$ consists of two terms

$$
\begin{aligned}
& v_b^3 \left(\left. \frac{\partial f_0}{\partial \psi} \right|_{\epsilon=\mathrm{const}} \right) \frac{\partial}{\partial \lambda} \lambda \left(1 - \langle \xi \rangle \frac{\partial P}{\partial \lambda} \right) \qquad \text{(this part is pitch angle)} \\
= & \frac{\partial}{\partial v} \left[\left(v^3 + v_c^3 \right) v \left. \frac{\partial f_0}{\partial \psi} \right|_{\epsilon=\mathrm{const}} \left(1 - \frac{\partial \langle \xi \rangle}{\partial \lambda} P \right) \right] \qquad \text{(this part is } \underline{drag}\text{)}
\end{aligned}
$$

Solution is obtained by separation of the dependence on λ and on v,

$$
P(\psi, v, \lambda) = \sum_{n=1} \Lambda_n(\psi, \lambda) \; V_n(\psi, v)
$$

with still dependence on radial coordinate ψ.

NOTE that at **Cordey NBI** it is $f_0 = \sum a_n(v) C_n(\xi)$. **END.**

The solution for P is based on an expansion in the parameter

$$
\frac{\tau_{bounce}}{\tau_s} \ll 1
$$

which means fast bouncing and rare collisions.

The limiting cases

- drag only solution

$$
\begin{aligned}
v_b & \rightarrow 0 \\
P_{drag} & = \frac{1}{\frac{\partial \langle \xi \rangle}{\partial \lambda}}
\end{aligned}
$$

- pitch-angle-only solution

$$
\frac{\partial}{\partial \lambda} P_{pitch-ang} = \frac{1}{\langle \xi \rangle}
$$

The functions $\Lambda_n(\psi, v)$ are eigenfunctions of

$$
\frac{\partial}{\partial \lambda} \lambda \; \langle \xi \rangle \frac{\partial}{\partial \lambda} \Lambda_n = \kappa_n \frac{\partial \langle \xi \rangle}{\partial \lambda} \; \Lambda_n
$$

with

$$
\kappa_n \equiv \text{eigenvalues}
$$

NOTE
in the paper of NBI in mirrors **Hinton Rosenbluth** find the equation

$$\frac{\partial G}{\partial \tau} - \frac{1}{I_2(\lambda)} \frac{\partial}{\partial \lambda}\left[I_1(\lambda)\ \lambda\ \frac{\partial G}{\partial \lambda}\right] = 0$$

where

$$I_1(\lambda) = \int_0^{s_t} ds\ \frac{\xi}{B}$$
$$I_2(\lambda) = \int_0^{s_t} ds \frac{1}{\xi}$$

and the "time" variable is

$$\tau = \frac{2}{3}\frac{m_i}{m_{fast}} \ln\left(\frac{1+u^{-3}}{1+u_0^{-3}}\right)$$

Also **note** that in **RABBIT** there is a new parameter

$$\tau(v', v) = \frac{\tau_s}{3} \ln \frac{v'^3 + v_c^3}{v^3 + v_c^3}$$

which plays the role of time.
END

The conditions

$$\begin{aligned} \Lambda_n(\lambda = 0) &= 1 \text{ deep passing } (v_\perp^2 = 0) \\ \Lambda_n(\lambda = \lambda_c) &= 0 \text{ the trapped-passing separatrix} \end{aligned}$$

Limiting behavior

$$\begin{aligned} \lambda &\rightarrow 0 \text{ all velocity is parallel} \\ \text{means } \langle \xi \rangle &\rightarrow 1 \text{ or } v_\parallel \approx v \\ \text{and } \frac{\partial \langle \xi \rangle}{\partial \lambda} &\rightarrow -\frac{1}{2} \end{aligned}$$

The equation has a regular singular point at $\lambda = 0$ (purely passing, $v_\perp^2 = 0$). Here the equation becomes Bessel of zeroth order. Taking the derivations in LHS in the equation one finds the term of the second derivative with respect to λ multiplied by λ, like

$$\lambda \frac{\partial^2}{\partial \lambda^2} + ...$$

and this means a singularity at $\lambda = 0$.

At the separatrix passing-trapped

$$
\begin{aligned}
\lambda &\rightarrow \lambda_c = 1 - \varepsilon = \frac{B_0}{B_{\max}} \\
&\quad \text{remember that} \\
B_0 &= \min(B) \ \text{ the field at the farthest point} \\
B_{\max} &= \max(B) \ \text{ the field at the point nearest to axis}
\end{aligned}
$$

we have

$$
\langle \xi \rangle \rightarrow \xi_{boundary} = \frac{1}{\pi} \left\{ \sqrt{2\varepsilon(1-\varepsilon)} + (1+\varepsilon) \arcsin\left[\sqrt{\frac{2\varepsilon}{1+\varepsilon}}\right] \right\}
$$

The derivative

$$
\frac{\partial}{\partial \lambda} \langle \xi \rangle = -\frac{1}{2} \left\langle \frac{1}{h\xi} \right\rangle
$$

(which explains why $\frac{\partial}{\partial \lambda} \langle \xi \rangle = -\frac{1}{2}$ at the limit $\xi \rightarrow 1$, highly circulating $v_{\parallel} \rightarrow v$)

$$
\frac{\partial}{\partial \lambda} \langle \xi \rangle \ \text{ is logarithmically singular at } \lambda_c
$$

See above for **Cordey**, where $\xi_t^{Cordey} = \sqrt{2\varepsilon}$.

NOTE
Return to **Cordey trapped electron effect on NBI**
The definitions

$$
\begin{aligned}
B &= B_0 \frac{1 - \varepsilon \cos\theta}{1 - \varepsilon} \\
B_0 &\equiv \text{magnetic field at the farthest point, } R_0 + r, \theta = 0
\end{aligned}
$$

$$
\begin{aligned}
\xi &= \sqrt{1 - \frac{v_{\perp}^2}{v^2} \frac{B_0}{B}} \\
&\quad \text{this is NOT } \frac{v_{\parallel}}{v}
\end{aligned}
$$

The combination occurs

$$
\frac{q}{\xi v} \equiv \frac{|v_{\parallel}|}{\xi v}
$$

in the pitch angle part of the operator of collision

$$
\frac{1}{\tau_s} \beta \, \frac{B_0}{B} \frac{v_c^3}{v^3} \, \frac{|v_{\parallel}|}{\xi v} \frac{\partial}{\partial \xi} \left\{ \left(1 - \xi^2\right) \frac{|v_{\parallel}|}{\xi v} \frac{\partial}{\partial \xi} f \right\}
$$

Then

$$\left\langle \frac{v_\parallel}{v} \right\rangle = \frac{1}{2\pi\sqrt{B_0}} \oint d\theta \sqrt{\xi^2 B - (B - B_0)}$$
for passing electrons

and

$$\left\langle \frac{v_\parallel}{v} \right\rangle = \frac{1}{2\pi\sqrt{B_0}} \int_A^B d\theta \sqrt{\xi^2 B - (B - B_0)}$$
for trapped electrons

In these formulas the dependence on θ enters through B. Then

$$\left\langle \frac{v_\parallel}{v} \right\rangle^{circ} = \frac{2\xi}{\pi} \mathbf{E} \left(\frac{2\varepsilon}{\xi^2} \right)$$
for passing electrons
$$\xi^2 > 2\varepsilon$$

and

$$\left\langle \frac{v_\parallel}{v} \right\rangle^{trap} = \frac{2\sqrt{2}\varepsilon}{\pi} \left[\mathbf{E} \left(\frac{\xi^2}{2\varepsilon} \right) - \left(1 - \frac{\xi^2}{2\varepsilon} \right) \mathbf{K} \left(\frac{\xi^2}{2\varepsilon} \right) \right]$$
for trapped electrons
$$\xi^2 < 2\varepsilon$$

and

$$\left\langle \frac{v}{v_\parallel} \right\rangle = \frac{2}{\pi} K \left(\frac{2\varepsilon}{\xi^2} \right)$$
circulating ions
$$\xi^2 > 2\varepsilon$$

$$\left\langle \frac{v}{v_\parallel} \right\rangle = \frac{2}{\pi \xi_{bound}} K \left(\frac{\xi^2}{2\varepsilon} \right)$$
trapped ions
$$\xi^2 < 2\varepsilon$$

END of memories from Cordey NBI

The equation for $\Lambda_n(\lambda)$ is Sturm Liouville. The eigenvalues can be obtained from

$$\kappa_n = \frac{\int_0^{\lambda_c} d\lambda \ \lambda \langle \xi \rangle \left[\frac{\partial \Lambda_n}{\partial \lambda} \right]^2}{- \int_0^{\lambda_c} d\lambda \ \frac{\partial \langle \xi \rangle}{\partial \lambda} \ (\Lambda_n)^2}$$

using *trial functions.*

Now we return to the equation for P, and introduce the assumed expansion with separation of λ and v functions.

For the functions $\Lambda_n(\lambda)$: they are a set of orthogonal functions

$$\int_0^{\lambda_c} d\lambda \; \Lambda_n \Lambda_m \frac{\partial \langle \xi \rangle}{\partial \lambda} = \delta_{m,n}$$

Then the equation for P is multiplied by $\Lambda_n(\lambda)$ and integrated on the interval

$[0, \lambda_c]$
(circulating, where $P \neq 0$)

to obtain and equation for the other function V_n

$$\frac{\partial}{\partial v}\left[v\left(\frac{v^3+v_c^3}{v_b^3}\right)(\sigma_n - V_n)\left.\frac{\partial f_0}{\partial \psi}\right|_{\epsilon=\text{const}}\right] = (\sigma_n - \kappa_n V_n)\left.\frac{\partial f_0}{\partial \psi}\right|_{\epsilon=\text{const}}$$

NOTE

The equation is

$$v_b^3\left(\left.\frac{\partial f_0}{\partial \psi}\right|_{\epsilon=\text{const}}\right)\frac{\partial}{\partial \lambda}\lambda\left(1-\langle\xi\rangle\frac{\partial P}{\partial \lambda}\right) \qquad \text{(this part is pitch angle)}$$

$$= \frac{\partial}{\partial v}\left[\left(v^3+v_c^3\right)v\left.\frac{\partial f_0}{\partial \psi}\right|_{\epsilon=\text{const}}\left(1-\frac{\partial\langle\xi\rangle}{\partial\lambda}P\right)\right] \qquad \left(\text{this part is } \underline{\textit{drag}}\right)$$

END

With the conditions

$$V_n(\psi, v > v_0) = 0$$

no fast ion has velocity higher than v_0
(no velocity diffusion)

all velocities are smaller than the one at the birth.

Here is the definition of σ_n, it is $V_n(\psi, v_0)$ calculated at the *birth* velocity v_0,

$$V_n(\psi, v_0) = \sigma_n \equiv \frac{\int_0^{\lambda_c} d\lambda \; \Lambda_n}{\int_0^{\lambda_c} d\lambda \; \frac{\partial\langle\xi\rangle}{\partial\lambda} \; \Lambda_n^2}$$

the jump condition at v_0

The solution for the functions V_n is

$$
\begin{aligned}
V_n &= \sigma_n \left[1 - (\kappa_n - 1) \frac{v_b^3}{(v_c^3 + v^3)\ v\ \left.\frac{\partial f_0}{\partial \psi}\right|_{\epsilon=\mathrm{const}}} \right. \\
&\qquad \left. \times \int_v^{v_0} du \left(\frac{v^3 (v_c^3 + u^3)}{u^3 (v_c^3 + v^3)} \right)^{\kappa_n \frac{v_b^3}{3 v_c^3}} \left.\frac{\partial f_0}{\partial \psi}\right|_{\epsilon(u)} \right]
\end{aligned}
$$

When

$$\sqrt{\varepsilon} \ll 1$$

approximations are possible and lead to the Legendre functions of λ.

The parameters

$$\frac{v_0}{v_c} \equiv \text{controls the ELECTRON drag effect}$$

where $v_c \equiv$ critical velocity separating transfer of energy from fast ions to electrons from transfer to background ions,

$$\frac{v_b^3}{v_c^3} \equiv \text{controls the pitch angle scattering}$$

The neoclassical flux is driven by *forces*

$$
\begin{aligned}
A_1 &= \frac{\partial}{\partial \psi} \ln (S \tau_s) \\
A_2 &= \frac{\partial}{\partial \psi} \ln \left(v_c^3 \right)
\end{aligned}
$$

which are gradients of functions containing plasma parameters with space localization.

Perturbation theory for $\sqrt{\varepsilon} \ll 1$
The new variable

$$
\begin{aligned}
\eta &= \sqrt{1 - \frac{\lambda}{\lambda_c}} \\
&= \sqrt{1 - \frac{v_\perp^2}{v^2} h \frac{1}{1 - \varepsilon}}
\end{aligned}
$$

where

$$\lambda = \frac{v_\perp^2}{v^2} h$$

$$\begin{aligned}
\lambda_c &= 1 - \varepsilon \\
&= \text{boundary trapped/passing}
\end{aligned}$$

Then

$$\begin{aligned}
\langle \xi \rangle &= \left\langle \frac{v_\parallel}{v} \right\rangle \\
&= \frac{2}{\pi} \sqrt{\eta^2 + 2\varepsilon} \mathbf{E} \left(\frac{2(1+\eta^2)\varepsilon}{\eta^2 + 2\varepsilon} \right) + O(\varepsilon)
\end{aligned}$$

Compare

$$\frac{2(1+\eta^2)\varepsilon}{\eta^2 + 2\varepsilon} = \frac{2\varepsilon \left(\frac{\lambda}{\lambda_c} \right)}{1 - \frac{\lambda}{\lambda_c} + 2\varepsilon}$$

In **Cordey** the result of the θ integration for passing has the argument

$$2 \left(\frac{1}{\xi^2} - 1 \right) \frac{\varepsilon}{1 - \varepsilon}$$

which seems to be approximated to

$$\begin{aligned}
2 \left(\frac{1}{\xi^2} - 1 \right) \frac{\varepsilon}{1 - \varepsilon} &\approx \frac{2\varepsilon}{\xi^2} \\
\text{but at Cordey } \xi^2 &= \frac{v_\perp^2}{v^2} \frac{B_0}{B}
\end{aligned}$$

14 Attenuation of the beam and distribution of new ions Rabbit

The paper **RABBIT NF 58 (2018) 082032**.

14.1 The *fast-ion birth rate* is the *attenuation* of the neutral beam

beam emission forward model BESFM.

- reaction of neutral of the beam with fast ions of the plasma, with comparable velocities

- reaction of neutral of the beam with the background ions, with largely different velocities

Comparisons:

- center-line attenuation according to *beam emission forward model*;
- a Monte Carlo calculation of the attenuation

Result of this model of attenuation: *the flux* Γ *of neutrals still in the beam, along the thin center-line of the NBI source* in units of $1/s$.

The *shine-through power* of the beam, at the position where it leaves the plasma volume:

$$\begin{aligned} P_{shine}^{(i)} &= \Gamma^{(i)}(\infty)\, E^{(i)} \\ \text{where } E^{(i)} &= \text{energy of the neutrals} \\ i &\equiv \text{label of the beams and of energies} \\ & \quad\ \text{summation over } i \end{aligned}$$

14.2 The fast ion birth rate

The *fast-ion birth rate* or the *deposition rate* can be calculated, in $1/s$, by taking the derivative of the fluxes along the line. In discrete form

$$\widetilde{S}_{line}\left(l+\frac{\Delta l}{2}\right) = \Gamma(l+\Delta l) - \Gamma(l)$$

The source rate is

$$\begin{aligned} S &= \frac{\widetilde{S}_{line}}{\Delta V} \quad \text{(divied to the volume of the mesh)} \\ & \quad \left[\frac{part}{m^3 s}\right] \end{aligned}$$

The need to consider the *poloidal spreading of the beam profile* (not only the pencil-type, *center-line*).

Variable

$$\rho_{tor} = \sqrt{\frac{\psi}{\psi_n}}$$

and enlargement in the poloidal plane

$$r(\rho) = \rho \frac{r_b}{\rho_b}$$

Assume Gaussian broadening, with σ, of the beam

$$\widetilde{S}(\rho_i) = \frac{\widetilde{S}_{line}(l_b) \cdot w(\rho_i)}{\mathfrak{s}}$$

where

$$\begin{aligned}
w(\rho_i) &= \frac{1}{2}\left(\operatorname{erf}\frac{h_{2u}}{\sqrt{2}\sigma} - \operatorname{erf}\frac{h_{1u}}{\sqrt{2}\sigma}\right) \\
&\quad + \frac{1}{2}\left(\operatorname{erf}\frac{h_{2l}}{\sqrt{2}\sigma} - \operatorname{erf}\frac{h_{1l}}{\sqrt{2}\sigma}\right) \\
u, l &\equiv \text{upper / lower, relative to the} \\
&\quad \text{intersection of beam center-line} \\
&\quad \text{with the transversal to it in } \rho_i
\end{aligned}$$

Next, σ of the assumed Gaussian enlargement is corrected for the *elongation.*

14.3 The Fokker-Planck equation

Assumption

$$\begin{aligned}
&\text{thermal electron velocity } v_{th,e} \\
\gg\ &\\
&\text{fast ion velocity} \quad \text{(ion beam ions)} \\
\gg\ &\\
&\text{thermal ion velocity } v_{th,i}
\end{aligned}$$

The equation has in the LHS here, collisional effects

$$\begin{aligned}
& \frac{1}{\tau_s}\frac{1}{v^2}\frac{\partial}{\partial v}\left[\left(v^3 + v_c^3\right) f\right] \quad \text{(drag, friction, slowing down, drift)} \\
& + \frac{1}{\tau_s}\beta\frac{v_c^3}{v^3}\frac{\partial}{\partial \xi}\left(1-\xi^2\right)\frac{\partial f}{\partial \xi} \quad \left(\begin{array}{c}\text{pitch angle scattering} \\ \text{diffusion, because second deriv. to } v\end{array}\right) \\
& + \frac{1}{\tau_s}\frac{1}{v^2}\frac{\partial}{\partial v}\left[\left(\frac{T_e}{m_{fast-ions}}v^2 + \frac{T_i}{m_{fast-ions}}\frac{v_c^3}{v}\right)\frac{\partial f}{\partial v}\right] \quad \text{(diffusion)} \\
= \quad & \frac{\partial f}{\partial t} - \sigma^{source}
\end{aligned}$$

where

$$\sigma \equiv \text{source of ions}$$

and the Spitzer time

$$\tau_s = 6.32 \times 10^8 \frac{A_{fast-ions}}{Z_{fast-ions}^2 \ln \Lambda_e} \frac{\left[T_e\,(eV)\right]^{3/2}}{n_e\,(cm^{-3})} \quad [s]$$

The Critical velocity

$$v_c = 5.33 \times 10^4 \sqrt{T_e\,(eV)} \left\langle \frac{Z_i^2}{A_i} \right\rangle^{1/3} \quad \left[\frac{m}{s}\right]$$

The other parameters

$$\beta = \frac{1}{2} \frac{\frac{\left\langle Z_i^2 \right\rangle}{A_{fast-ions}}}{\left\langle \frac{Z_i^2}{A_i} \right\rangle}$$

$$\left\langle \frac{Z_i^2}{A_i} \right\rangle = \frac{\sum\limits_{ion-species\ i} n_i \frac{Z_i^2}{A_i} \ln \Lambda_i}{n_e \ln \Lambda_e}$$

$$\left\langle Z_i^2 \right\rangle = \frac{\sum\limits_{ion-species\ i} n_i Z_i^2 \ln \Lambda_i}{n_e \ln \Lambda_e}$$

with

$$\begin{aligned} Z_e &= -1 \\ A_e &= \frac{1}{1836.1} \end{aligned}$$

Next step: calculate the *Coulombian logarithm* of the collision between fast-ions and the bulk ions.

For

$$v_c \gg v$$

the collisions with ions dominate.

For

$$v \gg v_c$$

the collisions with the electrons dominate.

14.4 The source

It is

$$\sigma^{source} = \frac{S}{2\pi v}\delta\left(v - v_0\right) K\left(\xi\right)$$

This means to assume that the injection is *monoenergetic* with velocity v_0.

The Source is only in velocity space.

14.5 To steady state solution of the Fokker-Planck equation

The remark

the Legendre polynomials
are eigenfunctions of the operator
of pitch-angle scattering

(see **Cordey**. he takes limiting values for some averages, deep trapped and deep passing.

More correct **Hsu Catto Sigmar**)

The expansion

$$f\left(v,\xi\right) = \frac{1}{2\pi} S \frac{\tau_s}{v^3 + v_c^3} \sum_{l=0}^{\infty} \left(l + \frac{1}{2}\right) u^{l(l+1)} \ P_l\left(\xi\right) \ K_l \ \mathbf{H}\left(v - v_0\right)$$

where

$$K_l = \int K\left(\xi\right) P_l\left(\xi\right) d\xi$$

$$u = \left(\frac{v_0^3 + v_c^3}{v^3 + v_c^3} \frac{v^3}{v_c^3}\right)^{\frac{\beta}{3}}$$

this is the function of the velocity v

$\mathbf{H} \equiv$ Heaviside function

The solution to the Fokker Planck equation will be used for *velocity space integrations* to determine various quantities.

Volume

$$d^3v = 2\pi v^2 dv \ d\xi$$

The fast ion density

$$
\begin{aligned}
n_{fi} &= \int_0^{v_0}\int_{-1}^{1} 2\pi v^2 dv d\xi \ f \\
&= S\tau_s \frac{1}{3}\ln\left(\frac{v_0^3+v_c^3}{v_c^3}\right)
\end{aligned}
$$

and the fast ion current

$$
\int_0^{v_0}\int_{-1}^{1} 2\pi v^2 dv d\xi \quad (v\xi) \ f
$$

the properties of the Legendre polynomials

$$
\int_{-1}^{+1} P_n(\xi) P_m(\xi) d\xi = \frac{2}{2n+1}\delta_{nm}
$$

and

$$
\begin{aligned}
\int d\xi &= \int P_0(\xi) d\xi \\
\text{since } P_0(\xi) &= 1
\end{aligned}
$$

14.6 The orbit average of the source term

We should include the exact particle orbit. But this is impossible.

One way to do that

- the source is represented by a set of markers
- for each marker the exact orbit is calculated
- during each step a Monte Carlo collision operator is applied which changes the velocity vector

Why is necessary: the ion after its birth moves (due to the neoclassical drift) across "several" magnetic surfaces and this means that there are changes in its *pitch* angle variable, $v_\parallel/v$ since B is changing along.

14.7 The time-dependent solution of the Fokker-Planck equation

The solution

$$
\begin{aligned}
f\left(v,\xi,t\right) &= \tau_s \frac{1}{v^3+v_c^3}\sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right)\left(\frac{v^3}{v^3+v_c^3}\right)^{\frac{l}{l+1}\frac{\beta}{3}} P_l\left(\xi\right) \\
&\quad \times \int_v^{\infty} v'^2 dv'\ S_l\left(v';t-\frac{\tau_s}{3}\ln\frac{v'^3+v_c^3}{v^3+v_c^3}\right)\left(\frac{v'^3+v_c^3}{v'^3}\right)^{\frac{l}{l+1}\frac{\beta}{3}}
\end{aligned}
$$

where S_l is the projection of S on the $l-$th polynomial.

A new parameter arises

$$
\tau\left(v',v\right)=\frac{\tau_s}{3}\ln\frac{v'^3+v_c^3}{v^3+v_c^3}
$$

At each time step

$$
S_l\left(v,t\right)=S\frac{K_l}{2\pi v^2}\delta\left(v-v_0\right)\delta\left(\frac{t-t_0}{\Delta t}\right)
$$

The solution

$$
\begin{aligned}
f\left(v,\xi,t\right) &= \frac{1}{2\pi}S\frac{\tau_s}{v^3+v_c^3}\sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right)\left(\frac{v^3}{v^3+v_c^3}\right)^{\frac{l}{l+1}\frac{\beta}{3}} K_l P_l\left(\xi\right) \\
&\quad \times\delta\left(t-t_0-\frac{\tau_s}{3}\ln\frac{v_0^3+v_c^3}{v^3+v_c^3}\right) \\
&\quad \times\left(\frac{v_0^3+v_c^3}{v_0^3}\right)^{\frac{l}{l+1}\frac{\beta}{3}}
\end{aligned}
$$

it is written

$$
\begin{aligned}
f\left(v,\xi,t\right) &= f_{ss}\left(v,\xi\right) \\
&\quad \times\delta\left(t-t_0-\frac{\tau_s}{3}\ln\frac{v_0^3+v_c^3}{v^3+v_c^3}\right) \\
&\quad \times\Delta t
\end{aligned}
$$

15 Heating power

The formula

$$P_{total} = -\int d^3v \, \frac{m_{fast-ions} v^2}{2} C_{sd}(f)$$

where the collision is the divergence of a flux in the space of velocity

$$\begin{aligned}
C(f) &= -\boldsymbol{\nabla} \cdot \boldsymbol{\Gamma}_c \\
\Gamma_c &= -\frac{1}{\tau_s} \frac{v^3 + v_c^3}{v^2} f \, \widehat{\mathbf{e}}_v \\
&= -\frac{1}{\tau_s} \left(1 + \frac{v_c^3}{v^3}\right) f \quad (v \widehat{\mathbf{e}}_v)
\end{aligned}$$

16 Toroidal momentum input by NBI (Rosenbluth Hinton)

Also in *polarization.tex*.

The equation for the ion momentum

$$\begin{aligned}
m_i n_i \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla}) \mathbf{u}\right] &= -\boldsymbol{\nabla}(p_e + p_i) - \boldsymbol{\nabla} \cdot \boldsymbol{\Pi}_i \\
&\quad + \mathbf{j} \times \mathbf{B} \\
&\quad + \mathbf{F}
\end{aligned}$$

The force is the sum of the collisional momentum transfer from fast ions to (1) electrons and (2) thermal ions.

We take the toroidal projection by multiplying with

$$R \boldsymbol{\nabla} \varphi \cdot$$

and average over surface

$$\begin{aligned}
m_i n_i \frac{\partial}{\partial t} \langle u_\varphi R \rangle &= -\langle R \widehat{\mathbf{e}}_\varphi \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Pi}_i \rangle \\
&\quad + \langle \mathbf{j} \cdot \boldsymbol{\nabla} \psi \rangle \\
&\quad + \langle R \widehat{\mathbf{e}}_\varphi \cdot \mathbf{F} \rangle
\end{aligned}$$

NOTE

that in **Honda** it is multiplied by $R^2 \nabla \varphi \cdot$ because the radial derivation is made with respect to ψ instead of r.

END

The average over surface is made with the formula

$$\langle A \rangle = \frac{\oint \frac{d\theta}{\mathbf{B} \cdot \nabla \theta} A}{\oint \frac{d\theta}{\mathbf{B} \cdot \nabla \theta}}$$

The magnetic field

$$\mathbf{B} = \frac{I(\psi)}{R} \widehat{\mathbf{e}}_\varphi + \frac{1}{R} \widehat{\mathbf{e}}_\varphi \times \nabla \psi$$

RH explain that the surface average of toroidal projection of the convective term is zero

$$\langle R \widehat{\mathbf{e}}_\varphi \cdot (\mathbf{u} \cdot \nabla) \mathbf{u} \rangle = 0$$

It is referenced **Hirshman1978**.

The current.

The Maxwell equation for the rotational of the magnetic field, projected onto the direction which is perpendicular to the surface

$$\begin{aligned} \langle (\nabla \times \mathbf{B}) \cdot \nabla \psi \rangle &= 0 \\ 0 &= \left\langle \left[\mu_0 \left(\mathbf{j} + \mathbf{j}^{fast} \right) + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \right] \cdot \nabla \psi \right\rangle \end{aligned}$$

The term with time variation of the electric field is

$$\mathbf{E} \cdot \nabla \psi \approx u_\varphi R B_\theta^2$$

which comes from $(\mathbf{E} + \mathbf{v} \times \mathbf{B})_{\nabla \psi} = 0$, and the dielectric constant (**note** that it is neoclassic **Robertson Hinton**) is very high

$$1 + \frac{c^2}{B_\theta^2 / (n_i m_i)} \gg 1$$

the term with $\mathbf{E}$ is negligible.

Then

$$\langle \mathbf{j} \cdot \nabla \psi \rangle = - \left\langle \mathbf{j}^{fast} \cdot \nabla \psi \right\rangle$$

This is the equation expressing the fact that the radial current from the expansion of the orbits of the fast ions induces an opposite current of the thermal ions, as response.

The conservation of momentum in collisions

$$\mathbf{F}_{e-fast} + \mathbf{F}_{ion-fast} = -\left(\mathbf{F}_{fast-e} + \mathbf{F}_{fast-i}\right)$$

The collisional force on the fast ions will be used to calculate the collisional force on the thermal ions.

Returning to the surface average of the toroidal projection of the momentum equation, one can separate the toroidal *torque*

$$\begin{aligned} T_{\varphi} &= \langle \mathbf{j} \cdot \boldsymbol{\nabla}\psi \rangle + \langle R\widehat{\mathbf{e}}_{\varphi} \cdot (\mathbf{F}_{e-fast} + \mathbf{F}_{i-fast}) \rangle \\ &= -\left\langle \mathbf{j}^{fast} \cdot \boldsymbol{\nabla}\psi \right\rangle - \langle R\widehat{\mathbf{e}}_{\varphi} \cdot (\mathbf{F}_{fast-e} + \mathbf{F}_{fast-i}) \rangle \end{aligned}$$

This is approximated as

$$T_{\varphi} = -\left\langle \mathbf{j}^{fast} \cdot \boldsymbol{\nabla}\psi \right\rangle - I \left\langle \frac{1}{B}\left(F_{\parallel}^{fast-e} + F_{\parallel}^{fast-i}\right) \right\rangle$$

The kinetic equation for the fast ions.

$$\frac{\partial f}{\partial t} + \left(v_{\parallel}\widehat{\mathbf{n}} + \mathbf{v}_D\right) \cdot \boldsymbol{\nabla} f = C^{fast}\left(f\right) + S^{fast}$$

where

$$\mathbf{v}_D = -v_{\parallel}\widehat{\mathbf{n}} \times \boldsymbol{\nabla}\left(\frac{v_{\parallel}}{\Omega_{fast}}\right)$$

Other velocity variables

$$\begin{aligned} \xi &= \frac{v_{\parallel}}{v} = \sqrt{1-\lambda B} \\ w &= \frac{v^2}{2} \\ \mu &= \lambda w \end{aligned}$$

The collision operator for the fast ions

$$C^{fast}\left(f\right) = C^{fast-e}\left(f\right) + C^{fast-i}\left(f\right)$$

$$C^{fast-e}\left(f\right) = \nu_s \, \frac{1}{v^2}\frac{\partial}{\partial v}\left(v^3 f\right)$$

This is the slowing down part of the collisions, the fast ions are loosing momentum to *electrons*.

$$C^{fast-i}(f) = \nu_s \, \frac{2m_i}{m_{fast}} \, v_c^3 \frac{1}{v^3} \frac{\xi}{B} \frac{\partial}{\partial \lambda} \left(\lambda \xi \frac{\partial f}{\partial \lambda} \right)$$

$$\begin{aligned} \nu_s &= \frac{1}{\tau_e} \frac{m_e}{m_{fast}} Z_{fast}^2 \\ &= \text{rate of slowing down of the fast ions} \end{aligned}$$

The electron collision time

$$\frac{1}{\tau_s} = \left(\frac{16\sqrt{\pi}}{3} \right) \frac{e^4}{m_e^2} \ln \Lambda \, \frac{n_e}{v_{th,e}^3}$$

The *critical velocity*

$$v_c = v_{th,e} \, \left(\frac{3\sqrt{\pi}}{4} \frac{m_e}{m_i} \right)^{1/3}$$

The slowing down on the thermal ions is neglected.
With the thermal ions there is *pitch angle scattering*.

The source of fast ions
It is a source of momentum

$$\frac{dM_\varphi}{dt} = \int d^3 v \, \widehat{\mathbf{e}}_\varphi \cdot m_{fast} \mathbf{v} \, \mathbf{R} \, S_{fast}$$

$$\begin{aligned} S_{fast} &= \dot{n}_{fast}(\psi, \theta, t) \, \oint \frac{d\zeta}{2\pi} \, \delta(\mathbf{v} - \mathbf{v}_0) \\ &= \dot{n}_{fast}(\psi, \theta, t) \, \frac{|\xi_0|}{B} \, \delta_{\sigma\sigma_0} \, \delta(\lambda - \lambda_0) \, \frac{\delta(v - v_0)}{\pi v_0^2} \end{aligned}$$

$$\dot{n}_{fast} \equiv \text{birth rate per unit volume} \; \left(\frac{\text{ions}}{\text{time} \times \text{volume}} \right)$$

$$|\xi_0| = \frac{|v_{\|0}|}{v_0} = \left| \sqrt{1 - \lambda_0 B} \right|$$

$$\lambda_0, v_0, \; \xi_0 \equiv \text{birth values}$$

Injection of new ions into trapped orbits on a magnetic surface

$$\lambda_0 B_{\max} > 1$$

The ions that are generated close to the magnetic axis are NOT trapped. They are a source of momentum as

$$\frac{dM_\varphi}{dt} = m_{fast}\ \frac{I}{B}\dot{n}_{fast} v_{\|0}$$

The parameter

$$\begin{aligned} & \frac{\text{slowing down rate}}{\text{frequency of bounce}} \\ = & \ \frac{\text{time of bounce}}{\text{time of slowing down}} \\ \ll & \ 1 \end{aligned}$$

The bounce motion is much faster than the time of slowing down.

Other parameter

$$\begin{aligned} & \frac{\text{guiding center drift frequency}}{\text{bounce frequency}} \\ = & \ \frac{v_D / L_{fast}}{v_{th} / (qR)} \\ \ll & \ 1 \end{aligned}$$

The bounce motion is much faster than the drift motion.

Series

$$f = f_{-1} + f_0 + f_1 + ...$$

The first equation

$$\begin{aligned} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} f_{-1} &= 0 \\ \nabla_{\|} f_{-1} &= 0 \end{aligned}$$

The next order

$$v_{\|} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} f_0 = -\mathbf{v}_D \cdot \boldsymbol{\nabla} \psi\ \frac{\partial f_{-1}}{\partial \psi} + C_{fast}\ f_{-1} + S_{fast} - \frac{\partial f_{-1}}{\partial t}$$

This equation is now bounce-averaged

$$\overline{A} = \frac{1}{T} \oint \frac{d\theta}{v_{\|} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \theta} A$$

$$T = \oint \frac{d\theta}{v_{\parallel} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta}$$

where

$$\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta = \nabla_{\parallel}\theta = \frac{d\theta}{dl_{\parallel}} = \frac{dl_{\theta}}{dl_{\parallel}} \frac{d\theta}{dl_{\theta}} = \frac{B_{\theta}}{B} \frac{1}{r}$$

By this operation the first term disappears

$$\begin{aligned}
\frac{\partial f_{-1}}{\partial t} &= -\overline{(\mathbf{v}_D \cdot \boldsymbol{\nabla}\psi)} \frac{\partial f_{-1}}{\partial \psi} \\
&\quad +\overline{C}_{fast}\ f_{-1} \\
&\quad +\overline{S}_{fast}
\end{aligned}$$

The function f_{-1} is factorized with respect to the bounce average since it does not depend on θ and σ.

The bounce average of the drift-convective deviation should use

$$\mathbf{v}_D \cdot \boldsymbol{\nabla}\psi = I\ v_{\parallel} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta \ \frac{\partial}{\partial \theta} \left(\frac{v_{\parallel}}{\Omega} \right)$$

the bounce average is zero

$$\overline{\mathbf{v}_D \cdot \boldsymbol{\nabla}\psi} = 0$$

The equation is then reduced to the terms

$$\frac{\partial f_{-1}}{\partial t} = \overline{C}_{fast}\ f_{-1} + \overline{S}_{fast}$$

where

$$\begin{aligned}
\overline{S}_{fast} &= \frac{\int_{\theta_1}^{\theta_2} \frac{d\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}\ \dot{n}_{fast}}{\int_{\theta_1}^{\theta_2} \frac{d\theta}{|\xi_0|\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta}} \\
&\quad \times \delta\left(\lambda - \lambda_0\right) \ \frac{\delta\left(v - v_0\right)}{2\pi v_0^2}
\end{aligned}$$

The equation bounce-averaged is now used in the original one to replace $\partial f_{-1}/\partial t$,

$$\begin{aligned}
v_{\parallel} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} f_0 &= -\mathbf{v}_D \cdot \boldsymbol{\nabla}\psi \ \frac{\partial f_{-1}}{\partial \psi} + C_{fast}\ f_{-1} - \frac{\partial f_{-1}}{\partial t} \\
&= -\mathbf{v}_D \cdot \boldsymbol{\nabla}\psi \ \frac{\partial f_{-1}}{\partial \psi} \\
&\quad +C_{fast}\ f_{-1} - \overline{C}_{fast}\ f_{-1} \\
&\quad +S_{fast} - \overline{S}_{fast}
\end{aligned}$$