1 Introduction to polarization effects in plasma

Physical explanation, first of the *classical* polarization drift, due to gyromotion.

Then, the *neoclassical* polarization.

Galeev Sagdeev book for *classical*.

Robertson Hinton for *neoclassical polarization*.

Hahm Fong for bounce-averaged kinetic equation, also **Hinton Rosenbluth** for NBI and *alphas*.

Honda for polarization from NBI.

The **thesis** for the connection between divergences of the polarization (perpendicular) and parallel currents, with dissipation of the latter.

Related to polarization

our idea that the AXIAL ANOMALY is the model for the ion polarization drift in tokamak

A sheared velocity of plasma (poloidal or toroidal velocity) means vorticity

$$\omega \sim \frac{\partial v}{\partial r}$$

And

$$\omega \sim \Delta \phi$$

vorticity means laplacian of the electrostatic potential, but

$$\Delta \phi = -\frac{1}{\varepsilon_0} \rho_q$$

The Laplacian of electrostatic potential is the electric charge density.

Therefore the sheared velocity means electric charge separation.*

This implies polarization.

Possibly this means that the "nonlocal" effect involves fast sheared rotation of plasma.

*There is a paper on the generation of *magnetic field* (dynamo) by a sheared velocity flow in plasma. But neutrals are necessary.

Generalities.

Classical polarization (**Baek Chang**)

$$v_{cp} = \frac{m}{eB^2} \frac{\partial E}{\partial t}$$

classical construction of the polarization dielectric

$$\varepsilon_0 \left(1 + \frac{c^2}{v_A^2} \right)$$

see *rotation.tex*, about **Novakovskii**.

In **Novakovskii** and in many other papers it is discussed the *decay of the poloidal rotation*, due to the Transit Time Magnetic Pumping, or the parallel viscosity (Shaing).

Then it is assumed that an initial rotation is present and it decays with a rate that is calculated from the solution of the equation of the kinetic distribution function.

A similar formulation of this problem starts with an initial radial current (i.e. different displacements of electrons and ions) which is shown to produce fast oscillations that decay to a stationary rotation velocity (in poloidal direction $\sim \nabla T$).

Situations when the neoclassical polarization is important

- the *spatial* variation of the radial electric field is on a spatial interval comparable with the *banana width.* [squeezing factor]
- the *time* variation of the radial electric field is on a time interval comparable with the bounce period.

In these cases, the two halves of a banana have different properties.

When the banana is large, the two parts of orbit traverse regions with *different temperature*. Also, different magnitudes of B (**Fong Hahm).**

There is a different meaning that is used for "halves": here means left and right parts of a banana, and in **Roberston Hinton** is upper and lower part relative to the equatorial plane.

2 Basic picture of polarization Galeev Sagdeev

It is the book.

Remark a difference of formulating the problem.

1. there is an external electric field that is applied to plasma. What source? it may be the field $-\boldsymbol{\nabla}\phi$ that results from the necessity to compensate the non-ambipolarity. It will have variation on surface. But, this field will make plasma to respond by a polarization accumulation of charge and creation of an internal field div $E_{int} = \frac{\sigma}{\varepsilon_0}$. This will compensate quasi-totally the external electric field, and what remains is very small, $E_p \sim \frac{1}{\varepsilon_0 \varepsilon_\perp}$.

2. the "polarization" is an intrisic process, as it is the case at the motion of the center of charges after ionization for NBI.

Notes are in *density enhanced confinement.* Here is that part

An estimation of the absolute value of the electric field can only be obtained if we adopt the assumption that the time derivative is due to the rapid variation of the rate of ionization $S(x,t)$.

We cannot attempt to find the value of the electric field $E(x,t)$ because the reaction of the bulk plasma to the charge imbalance created by ion motion (neoclassical *i.e.* geometrical origin) is very fast, it creates the necessary surface density of charge (E^{σ} which practically cancels E in plasma, reducing it to $E^{p}=\frac{E}{1+c^{2}/v_{A}^{2}}$) in a time of the order Ω_{ci}^{-1}, which is the time to change a the Larmor gyration circle onto a cycloid.

Then

$$E(x,t)\sim\delta t\left(\frac{\partial E}{\partial t}\right)$$

with

$$\delta t\sim\Omega_{ci}^{-1}\sim10^{-8}\ s$$

then

$$\begin{aligned} E &\sim 5\times10^{9}\left(\frac{V}{ms}\right)\times10^{-8}(s)\\ &= 50\left(\frac{V}{m}\right)\end{aligned}$$

The factor

$$\begin{aligned}\frac{\varepsilon}{\varepsilon_0} &= 1+\frac{c^2}{v_A^2}\\ &\sim \left(\frac{1}{7.28\times\mu^{-1/2}\left(n_i\left[cm^{-3}\right]\right)^{-1/2}B\left[Gs\right]}\right)^2\\ &\sim \left(\frac{\left(10^{14}\right)^{1/2}}{7.28\times3\times10^{4}}\right)^2\sim\left(\frac{10^{7-4}}{22}\right)^2=\left(\frac{1000}{22}\right)^2\sim50^2\\ &\sim 2500\end{aligned}$$

and further

$$\begin{aligned}E_p &= \frac{E}{\varepsilon/\varepsilon_0}\\ &\sim \frac{50\ \ (V/m)}{2500}\\ &\sim 0.02\ \ (V/m)\end{aligned}$$

According to the notations of **book page 275 Galeev Sagdeev** this E_p is the electric field that remains inside the plasma when we apply an external electric field E_0 and the plasma responds by charge accumulations at the two edges of the interval of ionization

$$\begin{aligned}E_p &= E_0-\frac{1}{\varepsilon_0}\sigma\\ &= E_0-\frac{\mu_0 n m_i c^2}{B^2}E_p\end{aligned}$$

and this gives

$$E_p = \frac{E_0}{\varepsilon_0 \left(1 + \frac{c^2}{v_A^2}\right)}$$

and it is a very small quantity: the plasma has reacted by displacement of charges and accumulation at the two surfaces precisely to suppress the external electric field. Inside plasma the electric field is practically zero.

This is in contradiction with the observations.
The electric field is not $E_p \ll E$. It is actually very large, $20\ (kV/m)$.
Then the polarization does not suppress actually the field E in plasma.
Then v_D^{pol} is not small.

The estimation of the velocity of drift is much higher than the speed of polarization.

NOTE

Below there will be a more severe estimation of the speed of drift for ions

$$\begin{aligned} T_i &= 2000\ (eV) \\ R &\sim 6\ (m) \\ B &\sim 3\ (T) \end{aligned}$$

$$\begin{aligned} v_{i,th} &\sim \sqrt{\frac{2T_i}{m_i}} \text{ (thermal speed of ions)} \\ &= 9.79 \times 10^3 \sqrt{T_i\,(eV)} \quad (m/s) \\ &\sim 10^4 \times 4.5 \times 10^1 = 4.5 \times 10^5\ (m/s) \end{aligned}$$

and

$$\begin{aligned} \Omega_{ci} &= 9.58 \times 10^3\ B\,(Gs) \quad \left(s^{-1}\right) \\ &\sim 10^4 \times 3 \times 10^4 \\ &= 3 \times 10^8\ \left(s^{-1}\right) \end{aligned}$$

$$v_{Di} \sim \frac{v_{th,i}^2}{2\Omega_{ci} R}$$

(note the factor 1/2)

$$\begin{aligned} v_{Di} &\sim \frac{1}{2}\frac{1}{3 \times 10^8}\frac{1}{6}\left(10^4 \times \sqrt{2000}\right)^2 \\ &= \frac{10^8 \times 2 \times 10^3}{40 \times 10^8} \sim 50\ (m/s) \end{aligned}$$

According to this range of parameters we re-estimate the polarization drift speed as

$$\begin{aligned} v_{Di}^{(pol)} &= \frac{1}{\Omega_{ci} B} \frac{dE}{dt} \\ &\sim \frac{1}{3 \times 10^8 \times 3} 5 \times 10^9 \\ &\sim 5 \ (m/s) \end{aligned}$$

Then the ratio of the zeroth-order drift to the polarization drift is

$$\frac{v_{Di}^{(0)}}{v_{Di}^{(pol)}} \sim \frac{50}{5} \sim 10$$

not so different but still higher.

But the density of BULK ions is much higher that that of the ions created by ionization and can compensate this difference between the two velocities.

However for pellet it is not so.

END

This means that the ionization produces new ions that are transported in the region of charge accumulation with a velocity much higher than the velocity with which the BULK ions reacts to $\partial E/\partial t$. The charge is accumulated while the inhibition of this process by the return current due to polarization of the bulk ions is much slower.

However the explanation given by **book Particle Motion** means that the response of the bulk plasma consists of just a small (ρ) radial displacement of the ions and that this is sufficient to create a surface charge density such that the electric field produced in this way to almost cancel the electric field produced by the accumulation of charge via ion drifts. Then, what they derive, the speed of response is insignifiant, since it can be taken almost simultaneous: the accumulation of charge is instantly quasi-cancelled by polarization surface charge. The field inside plasma results almost zero, $E_p = E/(\varepsilon/\varepsilon_0)$.

But our problem is: what about the current?

NOTE that Rosenbluth avoids this problem and go further without answering to this. For him, the current in the bulk is derived by simply asking equality and opposition to the fast ions current. The condition adopted by **Rosenbluth and Hinton** is equivalent to $(\nabla \times \mathbf{B})_{radial} = 0$. There is also the paper by **Rosenbluth and Taylor**.

And in **Hinton Wong**, according to the citation of **Rosenbluth alpha**.

END.

3 Polarization in the presence of a magnetic island

Poli.

The island

$$\mathbf{B} = B_{tor}\ R\boldsymbol{\nabla}\varphi + \boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\left(\psi + \widetilde{\psi}\right)$$

where

$$\begin{aligned} \widetilde{\psi} &= \alpha \cos \xi \\ &= -RA_{\parallel} \\ \xi &= m\theta - n\varphi - \omega t \end{aligned}$$

Define

$$\Omega = 2\frac{(\psi - \psi_s)^2}{W_\psi^2} - \cos \xi$$

Then Ω is a label for a magnetic surface affected by magnetic island perturbation

$$W_\psi^2 = 4\alpha \frac{1}{\left(\frac{d}{d\psi}\ln q\right)_s}$$

$$s \equiv \text{resonant surface}$$

We have

$$\nabla_{\parallel}\Omega = 0$$

and

$$\begin{aligned} \Omega &= -1 \quad \text{the } O \text{ point of the island} \\ \Omega &= 1 \quad \text{the separatrix of the island} \end{aligned}$$

NOTE
We mention the alternative choices **Connor Waelbroek**.

$$\psi = \psi_0 + \widetilde{\psi}\cos \xi$$

$$w = \sqrt{\frac{4L_s\widetilde{\psi}}{B_0}}$$

$$\begin{aligned} L_s &= \frac{Rq}{s} \\ s &= \frac{rq'}{q} = r\frac{d}{dr}\ln q \end{aligned}$$

Normalized flux-surface label

$$\begin{aligned} \chi^2 &= \frac{\widetilde{\psi} - \psi}{2\widetilde{\psi}} \\ &= x^2 + \sin^2\left(\frac{\xi}{2}\right) \\ \text{where } x &= \frac{X}{w} \end{aligned}$$

It results

$$\begin{aligned}
\chi &= 0 \ \text{ the } O \ \text{ point of the island} \\
\chi &= 1 \text{ the island separatrix}
\end{aligned}$$

$$\xi = m\theta - n\varphi - \int^{t} dt' \ \omega\left(t'\right)$$

END

The new operator is the *poloidally averaged* parallel gradient.
It can be replaced by

$$\nabla_{\|} = k_{\|} \left. \frac{\partial}{\partial \xi} \right|_{\Omega}$$

i.e. restricted to a surface of the island-deformed field.

Assumption
the electrons are rapid so they short-circuit the parallel electric field.
The parallel electric field is

$$E_{\|} = -\nabla_{\|}\phi - \frac{\partial A_{\|}}{\partial t}$$

Since we have the connection between $A_{\|}$ and ψ, $\widetilde{\psi} = -RA_{\|} = \alpha \cos \xi$, we find

$$\frac{\partial A_{\|}}{\partial \xi} = \frac{qk_{\|}}{m} \left. \frac{\partial \psi}{\partial \xi} \right|_{\Omega}$$

The condition of short-circuit

$$E_{\|} = 0$$

is

$$\begin{aligned}
\phi &= \frac{\omega q}{m}\left[\left(\psi - \psi_s\right) - h\left(\Omega\right)\right] \\
h\left(\Omega\right) &\equiv \text{ a flux surface function}
\end{aligned}$$

For the electric field to vanish far from the island

$$\begin{aligned}
h\left(\Omega\right) &\rightarrow \left(\psi - \psi_s\right) \\
\text{for } \left|\psi - \psi_s\right| &\gg W_s
\end{aligned}$$

It is adopted

$$h\left(\Omega\right) = \frac{W_{\psi}}{\sqrt{2}}\left(\sqrt{\Omega} - 1\right) \times \Theta\left(\Omega - 1\right)$$

This formula gives 0 for

$$\Omega < 1$$

which is the interior of the island structure.

3.1 Slab

"The corresponding acceleration and
deceleration of the plasma along the flux surfaces, $\rho\frac{dv}{dt}$
(ρ is the mass density here), must be balanced by a Lorentz
force $\mathbf{j}\times\mathbf{B}$, where the current is flowing perpendicular to flux surfaces"

The velocity that has modulations around the island is the *electric velocity*, $\mathbf{v}_E$.

The change of the electric velocity is static (stationary)

$$\frac{d\mathbf{v}_E}{dt} = (\mathbf{v}_E\cdot\boldsymbol{\nabla})\,\mathbf{v}_E$$

The variation of the electric velocity is due to the variation of the *electric field* in its formula ($E\times B$).

The variation of the electric field is connected with the radial flux of particles (or, charges, j_r) through the polarization formula

$$\begin{aligned}\varepsilon_0\left(1+\frac{c^2}{v_A^2}\right)\frac{\partial E}{\partial t} &= env_r \\ &= j_r^{polar}\end{aligned}$$

since we have

$$\varepsilon_0\frac{c^2}{\frac{B^2}{mn\mu_0}}\frac{\partial E}{\partial t} = \left(\varepsilon_0\mu_0\ c^2\right)mn\frac{1}{B}\frac{\partial}{\partial t}\left(\frac{E}{B}\right) = j_r^{polar}$$

and

$$n\frac{e}{\frac{eB}{m}}\frac{\partial}{\partial t}\left(\frac{E}{B}\right) = j_r^{polar}$$

$$\begin{aligned}j_r^{polar,\ classic} &= en\frac{1}{\Omega_c}\frac{dv_E}{dt} \\ &= en\frac{1}{\Omega_c}(\mathbf{v}_E\cdot\boldsymbol{\nabla})\,\mathbf{v}_E\end{aligned}$$

This is *classical* polarization.

3.2 Toroidal

Poloidal rotation is damped by TTMP.

The mechanism of damping imposes the existence of a *parallel flow* $u_\parallel$. This is

$$u_\parallel = \frac{E_r}{B_\theta}$$

The poloidal component of this flow $\left(\frac{B}{B_\theta}\right)\times$ is such that it *compensates* the poloidal component of the electric velocity $\mathbf{v}_E = \mathbf{E}\times\mathbf{B}/B^2$.

$$j_r^{polar,\ neo} = en\frac{1}{\Omega_\theta}\left(\mathbf{v}_E\cdot\boldsymbol{\nabla}\right)u_\parallel$$

(it is

$$B^2/B_\theta^2 = \frac{q^2}{\varepsilon^2} \gg 1$$

higher than classical).

When the collisions are very *rare*, it is only the trapped ions that contribute to the polarization current, which changes to

$$\frac{q^2}{\sqrt{\varepsilon}}$$

amplification.

"the polarization current contributes to
the island evolution through its parallel closure, which can
be obtained by integrating the continuity equation"

$$\nabla_\parallel j_\parallel = -\nabla_\perp j_\perp$$

4 Bootstrap in island Peeters

Monte Carlo polarization Peeters.
Polarization current Peeters
The formulas for the field

$$B_{tor} = \frac{B_0}{1+\varepsilon\cos\theta}$$

$$B_{pol} = \frac{B_{\theta 0}}{1+\varepsilon\cos\theta}$$

and it is adopted

$$q\left(r\right) = q_0\left(1+br^2\right)$$

Then

$$B_{\theta 0} = \frac{\varepsilon\ B_0}{q\left(r\right)\sqrt{1-\varepsilon^2}}$$

Connect the coordinates

$$\left(r,\theta\right)$$

with the Boozer coordinates

$$(\psi, \chi)$$

as

$$r = R_c \sqrt{1 - \frac{1+b}{b} \tanh^2 \left[\sqrt{b\,(1+b)}\ \ (a_0 - q_0 \psi) \right]}$$

$$\theta = 2 \arctan \left[\sqrt{\frac{1+r}{1-r}}\ \tan \left(\frac{\chi}{2} \right) \right]$$

where

$$a_0 = \frac{\arctan h \sqrt{b\,(b+1)}}{\sqrt{b\,(b+1)}}$$

The bootstrap current is calculated in the presence of the island with

$$j_{bs} = \left\langle en\ v_{i\|}\ B \right\rangle$$

The distribution function is calculated numerically.

Probably the $v_{i\|}$ is averaged. Apparently it is NOT used the gradient of pressure for the bootstrap current.

the boostrap current is just the averaged parallel ion flow.

In other words: if there is parallel flow of ions it can only be produced by the collsional transfer of momentum to circulating ions by the trapped ions which have agradient of pressure.

There are three time scales

- bounce time

$$\tau_B = \frac{1}{\sqrt{\varepsilon}} \frac{qR}{v_{th,i}}$$

- trapped to passing scattering time

$$\tau_s = \frac{1}{\nu_i / \varepsilon}$$

- toroidal drift time of trapped particles

$$\tau_D = 4\pi\varepsilon\Omega_c \frac{R^2}{q v_{th}^2}$$

"the bootstrap current requires a few collisional times to build up (outside the

island). Inside the island, the current oscillates about zero (the poorer statistics is essentially

due to the smaller number of simulation particles in the island)."

5 The drift motion of particles according to Novakovskii Galeev Liu Sagdeev Hassam

It is also in *rotation.tex* (coming from *Neoclassics2*).

The paper discusses the poloidal damping due to magnetic pumping in the *plateau* regime, *and* the Geodesic Acoustic Modes.

This is also in **Notes density enhanced confinement**. For the GAMs see also *Zonal Flows and GAMs*.

See also *plasma, general, viscosity*.

It is considered a fast temporal variation of the radial electric field.

This is accompanied by a change in the neoclassical prolarization.

For the *barely circulating ions*, it is possible to calculate the radial polarization current (**Novakovski**). It is assumed that the radial electric field has a time variation which can be linearized

$$E_r = E_{r0} + \left(\frac{\partial E_r}{\partial t}\right) t$$

accordingly the electric (poloidal) velocity now has time variation

$$v_E = v_{E0} + \left(\frac{\partial v_E}{\partial t}\right) t \quad \text{(in the poloidal direction)}$$

The particles that are considered are

$$\begin{array}{rcl} & & \text{trapped ions} \\ \text{with } v_{\|} & \ll & v_{\perp} \end{array}$$

("deeply trapped"). The equation for the poloidal motion

$$r\frac{d\theta}{dt} = v_E + v_{\|}\frac{B_\theta}{B_T}$$

where

$$\frac{B_\theta}{B_T} = \frac{\varepsilon}{q} \equiv \Theta$$

is the factor that projects the parallel direction on the poloidal direction. Integrating on time

$$r\theta\,(t) = r\theta_0 + \left(v_{E0} + v_{\|}\frac{B_\theta}{B_T}\right) t + \frac{1}{2}\left(\frac{\partial v_E}{\partial t}\right) t^2$$

This is the (short-time) evolution of the poloidal position of an ion.

The radial velocity is the radial component of the *drift* of the guiding center

$$v_r = \mathbf{v}_D \cdot \widehat{\mathbf{e}}_r = v_D \sin\theta$$

$$v_D = \frac{1}{\Omega} \frac{v_\perp^2/2 + v_\parallel^2}{R}$$

We assume *circulating* particle but with very small *parallel* velocity, close to the boundary trapped/circulating

$$v_\parallel \ll v_\perp$$

$$v_{D,r} \approx \frac{1}{\Omega} \frac{\frac{v_\perp^2}{2}}{R} \sin\theta = \frac{1}{\Omega} \frac{1}{R} \frac{m v_\perp^2}{2B} \frac{B}{m} \sin\theta = \frac{1}{\Omega R} \frac{\mu B}{m} \sin\theta$$

(**note** that here $\mu = \frac{m v_\perp^2}{2B}$) and

$$v_r(t) = \frac{1}{\Omega R} \frac{\mu B}{m} \sin[\theta(t)]$$

and the average over a period is

$$\begin{aligned} \langle v_r \rangle &= \frac{1}{2} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} dt \; v_r(t) \\ &= \frac{1}{\Omega R} \frac{\mu B}{m} \frac{1}{2} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} dt \; \sin[\theta(t)] \end{aligned}$$

Now we replace the equation for $\theta(t)$ and expand the function $\sin\theta$ for small argument

$$\begin{aligned} \sin[\theta(t)] &= \sin\left[\theta_0 + \frac{1}{r}\left(v_{E0} + v_\parallel \frac{B_\theta}{B_T}\right) t + \frac{1}{2}\left(\frac{\partial v_E}{\partial t}\right) t^2\right] \\ &\approx \sin\theta_0 \\ &\quad + \cos\theta_0 \left[\frac{1}{r}\left(v_{E0} + v_\parallel \frac{B_\theta}{B_T}\right) t + \frac{1}{r}\frac{1}{2}\left(\frac{\partial v_E}{\partial t}\right) t^2\right] \end{aligned}$$

and the integrations over the period of poloidal circuit gives

$$\frac{1}{\tau} \int_{-\tau/2}^{\tau/2} dt \; \sin\theta_0 = \text{constant}$$

This constant must be zero since there is no constant averaged radial velocity.

$$\frac{1}{\tau} \int_{-\tau/2}^{\tau/2} dt \; \cos\theta_0 \frac{1}{r}\left(v_{E0} + v_\parallel \frac{B_\theta}{B_T}\right) t = \cos\theta_0 \frac{1}{r}\left(v_{E0} + v_\parallel \frac{B_\theta}{B_T}\right) \frac{1}{\tau} \left[\frac{\tau^2}{2}\right]_{-\tau/2}^{\tau/2} = 0$$

This reflects the periodicity, as expected if we work with closed bananas.

The difference arises if there is a change *in time* of the radial electric field $E_r(t)$, as assumed. Equivalently, there is a change in time of the electric velocity, which is poloidal, $\partial v_E / \partial t$. If the time scale of the variation of the radial electric field (equivalently of v_E) is comparable with the bounce time, then the particle will move on half of the banana with a drift velocity $v_{D,r}$ and on the second half of the banana with a *different* $v_{D,r}$, which gives a different rate of time evolution of the poloidal angle $\theta(t)$. This is retained as the second term in the expansion of $\sin[\theta(t)]$. It is *this* difference, which is produced by the time variation of the rate of change of the angle $\theta(t)$, that gives in the end a non-zero radial *average* velocity $\langle v_r \rangle$. For a purely periodic banana this should not exist. The motion on a banana is no more periodic.

We have

$$\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} dt\ \cos\theta_0 \frac{1}{r}\frac{1}{2}\left(\frac{\partial v_E}{\partial t}\right) t^2 = \frac{1}{24}\cos\theta_0 \frac{1}{r}\left(\frac{\partial v_E}{\partial t}\right)\tau^2$$

The radial velocity averaged over a period of the motion on banana is at this moment

$$\begin{aligned} \langle v_r \rangle &= \frac{1}{\Omega R}\frac{\mu B}{m}\frac{1}{2}\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} dt\ \sin[\theta(t)] \\ &= \frac{1}{\Omega R}\frac{\mu B}{m}\frac{1}{48}\cos\theta_0 \frac{1}{r}\left(\frac{\partial v_E}{\partial t}\right)\tau^2 \end{aligned}$$

Now we take

$$\begin{aligned} \tau &\equiv \text{time of a period} \\ &\sim \frac{\text{connection length } 2\pi q R}{\text{parallel velocity } v_{\parallel}} \\ &= \frac{2\pi q R}{\left|v_{\parallel}\right|} \end{aligned}$$

It is necessary to define the regime by few parameters.

$$\begin{aligned} \widehat{\nu} &= def = \frac{r\ \nu_{ii}}{\Theta v_{th,i}} \\ &= \frac{\text{freq. of ion collisions}}{\text{freq. of ion transit with poloidal velocity } \Theta v_{th,i} \text{ on poloidal circle}} \end{aligned}$$

the same formula is written

$$\begin{aligned} \widehat{\nu} &= \frac{r\ \nu_{ii}}{\Theta v_{th,i}} \\ &= \frac{r\ \nu_{ii}}{\frac{\varepsilon}{q} v_{th,i}} \\ &= \frac{\nu_{ii}}{v_{th,i}/(qR)} \end{aligned}$$

This parameter compares the collisions with the motion on poloidal direction.

[**note** in **Hazeltine Ware electrostatic trapping** the parameter is $\widehat{\nu} = r\frac{\nu_{e,i}}{\frac{B_\theta}{B} v_{th,e,i}}$, identical].

The parallel velocity of the ions is taken at the limit where the effective ion collision frequency is equal with the parallel transit frequency

$$\nu_{eff} = \frac{v_{\parallel}}{qR}$$

where by definition

$$\nu_{eff} \overset{def}{=} \nu_{ii} \frac{v_{th,i}^2}{v_\parallel^2}$$

We combine the two expression of ν_{eff} and further use the expression for $\widehat{\nu}$

$$\begin{aligned}
\nu_{ii} \frac{v_{th,i}^2}{v_\parallel^2} &= \frac{v_\parallel}{qR} \quad \text{or} \\
\frac{\nu_{ii}}{v_{th,i}/(qR)} &= \frac{v_\parallel^3}{v_{th,i}^3} \\
\widehat{\nu} &= \frac{v_\parallel^3}{v_{th,i}^3}
\end{aligned}$$

from where we derive

$$\frac{v_\parallel}{v_{th,i}} = \widehat{\nu}^{1/3}$$

and we replace $v_\parallel$ with the expression in terms of thermal ion velocity and the effective collision parameter $\widehat{\nu}$

$$v_\parallel = v_{i,th} \widehat{\nu}^{1/3}$$

This is an important connection. It is not universal, it actually is a *choice*.

The connection is between the parallel velocity $v_\parallel$ and the thermal ion velocity, mediated by the parameter which measures the collisions relative to poloidal bounce.

Then the square of the period time τ is

$$\begin{aligned}
\tau^2 &= \frac{(2\pi)^2 q^2 R^2}{v_\parallel^2} \\
&= \frac{(2\pi)^2 q^2 R^2}{v_{i,th}^2} \widehat{\nu}^{-2/3}
\end{aligned}$$

and

$$\begin{aligned}
\langle v_r \rangle &= \frac{1}{\Omega R} \frac{\mu B}{m} \frac{1}{48} \cos\theta_0 \frac{1}{r} \left(\frac{\partial v_E}{\partial t} \right) \tau^2 \\
&= \frac{1}{\Omega R} \frac{\mu B}{m} \frac{1}{48} \cos\theta_0 \frac{1}{r} \left(\frac{\partial v_E}{\partial t} \right) \frac{(2\pi)^2 q^2 R^2}{v_{i,th}^2} \widehat{\nu}^{-2/3} \\
&= \frac{(2\pi)^2}{48} \frac{q^2}{\varepsilon \Omega} \frac{\mu B}{m} \cos\theta_0 \left(\frac{\partial v_E}{\partial t} \right) \widehat{\nu}^{-2/3}
\end{aligned}$$

To calculate the radial current two steps are necessary:

- take the fraction of the particles that are in this regime

- integrate over the positions θ_0. Actually, the parameter θ_0 appears in the magnitude of the magnetic field $B = B_0 (1 - \varepsilon \cos\theta_0)$ and this, in turn, appears in the expression of the magnetic momentum $\mu = v_\perp^2 / (2B)$. Then to integrate over the Maxwell distribution of the variable $v_\perp$ we can equivalently integrate over the variable θ_0 for fixed μ.

The fraction of particles is

$$\sim \frac{v_\parallel}{v_{i,th}}$$

and this is

$$\text{fraction of particles} \sim \widehat{\nu}^{1/3}$$

When we multiply the average radial velocity by this factor

$$\begin{aligned} & \widehat{\nu}^{1/3} \times \widehat{\nu}^{-2/3} \\ = & \widehat{\nu}^{-1/3} \end{aligned}$$

we get a dependence of the effective collisional parameter as $\widehat{\nu}^{-1/3}$ which will be found in the final expression.

The Maxwellian in velocity space for *deep trapped* is

$$\begin{aligned} f_M &= N \exp\left(-\frac{mv^2}{2T}\right) \\ &= N \exp\left(-\frac{m\left(v_\parallel^2 + v_\perp^2\right)}{2T}\right) \sim N \exp\left(-\frac{mv_\perp^2}{2T}\right) \\ &= N \exp\left(-\frac{\mu B}{T}\right) = N \exp\left(-\frac{\mu B_0}{T(1+\varepsilon\cos\theta)}\right) \\ &\approx N \exp\left[-\frac{\mu B_0}{T}(1-\varepsilon\cos\theta)\right] \end{aligned}$$

We use this velocity integration to suppress the indeterminancy given by the presence of θ_0 in the radial current.

$$\frac{\partial v_E}{\partial t} = \frac{1}{B}\left(\frac{\partial E}{\partial t}\right)$$

Then the radial electric current (surface-averaged) induced by the time variation of the radial electric field is

$$\langle j_r \rangle \approx \left(1 + q^2 + \widehat{\nu}^{-1/3} q^2\right) \frac{m}{B^2}\left(\frac{\partial E}{\partial t}\right)$$

In this formula, 1 is the standard polarization term. The second term is the neoclassical polarization term due to ions with comparable parallel and perpendicular velocities, $v_\parallel \approx v_\perp$.

The neoclassical polarization radial current due to radial excursions of the banana (*trapped* particles) when there is fast time variation of the radial electric field, is

$$j_r^{bananas} \approx \varepsilon^{3/2} \frac{c^2}{v_{A\theta}^2} \left(\frac{\partial E_r}{\partial t} \right)$$

We **note** the substantial difference relative to the classical polarization

$$\frac{c^2}{v_A^2} \rightarrow \frac{c^2}{v_{A\theta}^2}$$

and a factor that reflects the fraction of population

$$\varepsilon^{3/2}$$

See **Hinton Robertson**.

The equations are

$$\frac{d\mathbf{r}}{dt} = v_{\parallel}\widehat{\mathbf{n}} + \mathbf{v}_E + \mathbf{v}_D$$

$$\begin{aligned} \frac{dv_{\parallel}}{dt} &= \left(-\frac{v_{\perp}^2}{2}\widehat{\mathbf{n}} + v_{\parallel}\mathbf{v}_E \right) \cdot \boldsymbol{\nabla} \ln B \\ \frac{d}{dt}\left(\frac{v_{\perp}^2}{2} \right) &= \frac{v_{\perp}^2}{2} \left(v_{\parallel}\widehat{\mathbf{n}} + \mathbf{v}_E \right) \cdot \boldsymbol{\nabla} \ln B \end{aligned}$$

The *drift velocity* is

$$\mathbf{v}_D = \frac{1}{\Omega_{ci}} \left(\frac{v_{\perp}^2}{2} + v_{\parallel}^2 \right) \widehat{\mathbf{n}} \times \boldsymbol{\nabla} \ln B + \frac{1}{\Omega_{ci}} \widehat{\mathbf{n}} \times \frac{\partial \mathbf{v}_E}{\partial t}$$

Comparing with previous expressions of the drift velocity v_D we note that there is an additional term, which gives the effect of the fast time variation of the radial electric field $\frac{\partial \mathbf{v}_E}{\partial t}$, like in transitions with rapid change of toroidal and/or poloidal rotation velocity.

We note however that the time variation of the electric drift velocity has the following effect on the drift:

we suppose that

$$\frac{\partial E_r}{\partial t} \sim \widehat{\mathbf{e}}_r$$

exists due to the polarization effect related to the forced increase of the poloidal velocity

$$\frac{\partial v_\theta}{\partial t} \rightarrow \frac{\partial E_r}{\partial t}$$

Then $\mathbf{v}_E$ will increase in two possible directions

$$\frac{\partial \mathbf{v}_E}{\partial t} \sim \frac{1}{B} \left(\frac{\partial E_r}{\partial t} \widehat{\mathbf{e}}_r \times \mathbf{B}_\theta + \frac{\partial E_r}{\partial t} \widehat{\mathbf{e}}_r \times \mathbf{B}_T \right)$$

Then the terms mentioned by **Novakovskii** is

$$\begin{aligned}
\frac{1}{\Omega_{ci}}\widehat{\mathbf{n}}\times\frac{\partial\mathbf{v}_E}{\partial t} \quad &\sim \quad \frac{1}{\Omega_{ci}}\widehat{\mathbf{n}}\times\left(\frac{\partial E_r}{\partial t}\widehat{\mathbf{e}}_r\times\mathbf{B}_\theta\right) \quad \text{almost zero} \\
&+\frac{1}{\Omega_{ci}}\widehat{\mathbf{n}}\times\left(\frac{\partial E_r}{\partial t}\widehat{\mathbf{e}}_r\times\mathbf{B}_T\right) \quad \text{radial}
\end{aligned}$$

Therefore *none* of these contributions is aligned along the toroidal direction, giving a *drift* of the particle population in the toroidal direction.

It seems that a treatment based on the equations of motion of the particles governed by the invariants

$$E,\mu$$

cannot give us a drift of the bananas in the toroidal direction.

Here again the drift-kinetic equation is

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\boldsymbol{\nabla}f+\frac{dv_\parallel}{dt}\frac{\partial f}{\partial v_\parallel}+\frac{d\left(v_\perp^2/2\right)}{dt}\frac{\partial f}{\partial\left(v_\perp^2/2\right)}=St\left(f\right)$$

The paper of **Novakovskii** wants to solve the problem of decay of poloidal rotation in the *plateau* regime.

Then the Drift-Kinetic equation is solved by perturbations.

Zero + order 1 + order 2 are necessary.

In the zeroth order

$$\left(v_\parallel\frac{B_\theta}{B_T}+v_E\right)\frac{\partial f_0}{r\partial\theta}-St\left(f_0\right)=0$$

which gives a Maxwellian function *possibly* shifted in the parallel direction by a velocity U_0.

$$f_0=\left(1-\frac{mv_\parallel U_0}{T}\right)f_M$$

for

$$f_M=\frac{n}{\left(2\pi T/m\right)^{3/2}}\exp\left[-\frac{m\left(v_\parallel^2+v_\perp^2\right)}{2T}\right]$$

NOTE

We remark the combination

$$\begin{aligned}
& v_\parallel\frac{B_\theta}{B_T}+v_E \\
\sim \quad & \Theta\left(v_\parallel+\frac{v_E}{\Theta}\right) \\
\approx \quad & 0 \qquad \text{(since the paranthesis is }\sim 0\text{)}
\end{aligned}$$

The combination $v_\parallel\frac{B_\theta}{B_T}+v_E$ is the *poloidal velocity.*

It is composed of the *projection* of the parallel velocity on θ, using Θ, plus the poloidal velocity due to the radial electric field.

The first term $\left(v_{\parallel}\frac{B_{\theta}}{B_T}+v_E\right)\frac{\partial f_0}{r\partial\theta}$ is the convection of the distribution function f_0 in the poloidal direction.

It is either veru small or zero.

If it exists, it is balanced by collisions.

END

Nothing at this moment suggests there can be a velocity U in the *parallel* direction, *i.e.* along the magnetic field lines. But the equation for f_0 allows it and since we know it can exist, it is introduced at this point.

Note that the velocity along the magnetic field lines comes from a shift in the parallel *particle* velocity, as

$$\begin{aligned} & -\frac{\left(v_{\parallel}-U_0\right)^2}{2T/m} \\ = & -\frac{v_{\parallel}^2}{2T/m}-\frac{2v_{\parallel}U_0}{2T/m}-\frac{U_0^2}{2T/m} \end{aligned}$$

and the last term is much less than 1 since the flow with velocity U_0 is slower than the *thermal* velocity.

Then a substitution is made for the distribution function to extract a rigid body rotation

$$\begin{aligned} f &= f_0+\varepsilon\left(\frac{mv_{\parallel}U_0}{T}\right)f_M \\ &\quad +\widetilde{f} \end{aligned}$$

Then the Drift-kinetic equation to order ε^2 gives

$$\begin{aligned} & \frac{\partial\widetilde{f}}{\partial t}+\left(v_E+v_{\parallel}\frac{B_{\theta}}{B_T}\right)\frac{\partial\widetilde{f}}{r\partial\theta}-St\left(\widetilde{f}\right) \\ = & \frac{\sin\theta}{R}\frac{m\left(v_{\parallel}^2+\frac{v_{\perp}^2}{2}\right)}{T}Wf_M \end{aligned}$$

where W is a velocity in the poloidal direction

$$\begin{aligned} W &\equiv v_E+v_{*n}+U_0\frac{B_{\theta}}{B_T} \\ &\quad +v_{*T}\left[\frac{m\left(v_{\parallel}^2+v_{\perp}^2\right)}{2T}-\frac{3}{2}\right] \end{aligned}$$

$$\begin{aligned} v_{*n} &\equiv \frac{T}{eB}\frac{d}{dr}\ln n \\ v_{*T} &\equiv \frac{T}{eB}\frac{d}{dr}\ln T \end{aligned}$$

COMMENT

Then W is

$$\begin{aligned} W &= \text{poloidal velocity due to the radial electric field} \\ &\quad +\text{diamagnetic-density velocity (poloidal)} \\ &\quad +\text{diamagnetic-temperature velocity (poloidal)} \\ &\quad +\text{poloidal projection of the parallel flow velocity } U_0 \end{aligned}$$

The term $\frac{\sin\theta}{R}\frac{m\left(v_\parallel^2+\frac{v_\perp^2}{2}\right)}{T}Wf_M$ comes from

$$\mathbf{v}\cdot\boldsymbol{\nabla}f_M$$

and reflects the convective variation of the Maxwellian with the flow velocity that exists in the plasma. The space variation of the Maxwellian f_M is $RADIAL$ and the operator $\boldsymbol{\nabla}$ will be reduced to radial derivative

$$\boldsymbol{\nabla}\rightarrow\frac{\partial}{\partial r}=\frac{d}{dr}$$

Then *who is the plasma velocity taht will take advantage of this radial variation of the equilibrium distribution function*? It is the particle's drift velocty $\mathbf{v}_D$ which will act along the minor radius v_{Dr}.

The term is actually

$$v_{Dr}\frac{df_M}{dr}$$

This will become the inhomogeneous term that drives a variation of the distribution function, asking therefore for the existence of a f_1.

But what can the correction do to balance this *radial convective variation of the equilibrium distribution Maxwellian function* ?

The correction f_1 actually has variation in the magnetic surface.

It will again be question of a *convective variation*, which means that there is a poloidal velocity that will advect the function f_1 along its variation.

This poloidal advection of the correction $f_1(\theta)$ will compensate for the *radial* variation of the equilibrium distribution function.

To **comment** further, we note in **Rosenbluth Hazeltine Hinton 1972** the equation

$$v_{Dr}\frac{\partial f_0}{\partial r}+v_\parallel\frac{B_\theta}{B}f_0\frac{\partial\widehat{f}}{r\partial\theta}+|e|\,E_\parallel v_\parallel\frac{\partial f_0}{\partial\epsilon}=C(f)$$

where

$$f=f_0\left(1+\widehat{f}\right)$$

We recognize the same picture:

- the zero-order distribution function f_0

- the perturbation of the distribution function is advected by the parallel velocity, which is the maximum it can get.
- the perturbation to the distribution function $\widehat{f}$ has only variation on the poloidal angle θ.
- the balance is ensured by the advection of the equilibrium distribution function f_0 by the neoclassical drift of the particles. This drift acts only in the radial direction because the equilibrium f_0 only depends on r.
- there is another term, which is energetic. It is the work done by a parallel electric field

  $$v_{\parallel} \ \times \ |e| \, E_{\parallel}$$

  and is affecting the space of velocities. It shows that our interest here is on *instabilities*.

END COMMENT

The collision operator is adopted as

$$St\,(f) = -\nu_{eff} f$$

$$\nu_{eff} = \frac{v_{th}^2}{v_{\parallel}^2} \nu_c$$

Regarding the application of this analysis to the case of a **fast time variation** of the radial electric field (for fast transients of the poloidal or toroidal rotation) the range of validity is established by **Novakovskii et al** by choosing

$$\frac{v_{Th}}{qR} \approx \nu_{eff} \gg \frac{\partial}{\partial t}$$

which means: the frequency of the *bounce* of the trapped particle, $v_{Th}/\,(qR)$, comparable with the frequency of collisions ν_{eff} is much higher than the frequency associated to the variation of the radial electric field, $\partial/\partial t$. Then during the variation of the radial electric field, which is slow, the trapped particle makes many bounces.

Then a new small parameter has been identified and the distribution function can be expanded in a series. The distribution function is only the *correction* to the shifted Maxwellian, *i.e.* the function $\widetilde{f}$ and the series is

$$\widetilde{f} = f_1 + f_2 + ...$$

Separately and related this time to the spatial variation of the distribution function, it is *considered* the variation in the magnetic surface, *i.e.* the dependence of the distribution functions f_i of the *poloidal angle* θ:

$$f_i = \sum_{\sigma=\pm 1} f_{i\sigma} \exp\,(i\sigma\theta)$$

Then we get the solution for the first order correction $f_{1\sigma}$ as

$$f_{1\sigma} = -\frac{\varepsilon \frac{\left(v_{\perp}^2/2+v_{\parallel}^2\right)}{2T/m} W}{v_{\parallel}\left(B_{\theta}/B_T\right) + v_E - \iota\sigma\nu_{eff}} f_M$$

where

$$\iota = -\frac{1}{q}$$

note that usually is $-\frac{2\pi}{\iota} = q$.

$$\sigma = \pm 1$$

NOTE that the denominator

$$\frac{1}{v_{\parallel}\left(B_{\theta}/B_T\right) + v_E - \iota\sigma\nu_{eff}}$$

is not singular only due to collisions. The collisions prevent the resonance.

END.

Using the first order in the small parameter

$$\frac{\partial/\partial t}{v_{Th}/\left(qR\right)} \ll 1$$

and the ordering

$$\begin{aligned} v_E &\ll v_{Th}\frac{B_{\theta}}{B_T} \\ v_E &\ll v_{th}\Theta \end{aligned}$$

the second order contribution to the distribution function $\widetilde{f}$ is obtained from the differential equation

$$\frac{\partial f_1}{\partial t} + v_{\parallel}\frac{B_{\theta}}{B_T}\frac{\partial f_2}{r\partial\theta} = -\nu_{eff} f_2$$

(we **note** that $v_{\parallel}\frac{B_{\theta}}{B_T} = v_{\theta}$) from which a solution is obtained

$$f_{2\sigma} = -\iota\frac{\varepsilon\sigma r\frac{\left(v_{\perp}^2/2+v_{\parallel}^2\right)}{2T/m}}{\left[v_{\parallel}\left(B_{\theta}/B_T\right) - \iota\sigma r\nu_{eff}\right]^2} f_M \; \frac{\partial v_E}{\partial t}$$

COMMENT

The second order correction is obtained from the balance with the *time variation* of the first order perturbation, which means

$$f_2 \sim \frac{\partial f_1}{\partial t}$$

This is because the expression for the first order correction f_1 contains the factor W which was derived from the radial variation of the equilibrium distribution function $f_0 \sim f_M$.

The factor W contains the *electric potential* ϕ that has radial variation

$$\phi = \phi(r)$$

BUT it also has a time variation

$$\phi = \phi(r, t)$$

since the decay of poloidal rotation consists of the chage of the radial electric field that produces the torque.

$$\begin{array}{rcl} & & \text{torque to stop poloidal rotation} \\ & \downarrow & \\ E_r & = & E_r(t) \\ & \sim & \text{decay of } v_\theta = \dfrac{E_r}{B_T} \end{array}$$

Here it is substituted

$$\nu_{eff} \approx \frac{v_{Th}^2}{v_\parallel^2} \nu_i$$

and the second order contribution to the distribution function becomes (omiting the term with $\cos\theta$)

$$f_2 = \frac{\left(\frac{v_\parallel}{v_{Th}}\right)^6 - \widehat{\nu}^2}{\left[\left(\frac{v_\parallel}{v_{Th}}\right)^6 + \widehat{\nu}^2\right]^2} \sin\theta \, \frac{\varepsilon r}{\left(v_{Th}\frac{B_\theta}{B_T}\right)^2} \frac{\left(v_\perp^2/2 + v_\parallel^2\right)}{T/m} \left(\frac{v_\parallel}{v_{Th}}\right)^4 f_M \, \frac{\partial v_E}{\partial t}$$

where it is noted later

$$\frac{v_\parallel}{v_{Th}} \equiv x$$

and

$$\widehat{\nu} \equiv \frac{r}{v_{Th}\frac{B_\theta}{B_T}} \nu_i$$

which is related to the standard neoclassical collisional parameter ν_* by

$$\begin{array}{rcl} \widehat{\nu} & = & \varepsilon^{3/2} \nu_* \\ & = & \text{plateau collisionality parameter} \end{array}$$

In order to calculate the magnetic damping of the poloidal rotation it is necessary to start from the radial electric current which on a magnetic surface must have the average equal to zero

$$\langle j_r \rangle = 0$$

The radial fluxes are considered

$$\langle nV_r \rangle = \frac{1}{2\pi} \int_0^{2\pi} d^3v \; d\theta \; v_r \left(1 + \varepsilon \cos\theta\right) \; f$$

where

$$d^3v = 2\pi dv_{\parallel} d\left(\frac{v_{\perp}^2}{2}\right)$$

The radial component of the *particle drift* velocity is

$$v_r = -\frac{1}{\Omega_c}\left(\frac{v_{\perp}^2}{2} + v_{\parallel}^2\right)\frac{\sin\theta}{R} + \frac{1}{\Omega_c}\frac{\partial v_E}{\partial t}$$

In the integral for the radial particle fluxes one substitutes the *ion* distribution function

$$f = f_0 + f_1 + f_2 + ...$$

the expansion in the small parameter representing the ratio between the characteristic frequency of the variation of the radial electric field and the bounce frequency.

$$\begin{aligned} &\int d^3v d\theta \; \left[f_0 \frac{\partial v_E}{\partial t} - f_2\left(\frac{v_{\perp}^2}{2} + v_{\parallel}^2\right)\frac{\sin\theta}{R}\right] \\ = \; &\int d^3v d\theta \; f_1 \left(\frac{v_{\perp}^2}{2} + v_{\parallel}^2\right)\frac{\sin\theta}{R} \end{aligned}$$

If the plateau collision parameter is small

$$\widehat{\nu} \ll 1$$

then the distribution function in order 1 can be approximated

$$f_1 \approx -\pi q \frac{\frac{v_{\perp}^2}{2} + v_{\parallel}^2}{T/m} W \; f_M \; \delta\left(v_{\parallel}\right) \sin\theta$$

NOTE that the most important contribution to the distribution function, *i.e.* f_1 comes from the *barely trapped ions*. This means that $v_{\parallel} \approx 0$. **End**.

Then the equation for the poloidal velocity becomes

$$\left(1 + q^2\Lambda\right)\frac{\partial v_E}{\partial t} = -\nu_{MP}\left(v_E + v_{*n} + U_0 \frac{B_\theta}{B_T} + \frac{3}{2} v_{*T}\right)$$

where the rate of magnetic pumping damping is

$$\nu_{MP} = \sqrt{\frac{\pi}{2}} \frac{q v_{Th}}{R}$$

and

$$\Lambda \equiv \frac{3}{2} + \Xi \widehat{\nu}^{-1/3}$$

$$\Xi \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \; x^4 \frac{x^6 - 1}{\left(x^6 + 1\right)^2} \exp\left(-\widehat{\nu}^{2/3} \frac{x^2}{2}\right)$$

NOTE compare with **Hassam Drake**.

6 Electric field generation by NBI Cordey Houghton 1973

The paper **1973** on NBI ion effects on plasmas (commented in *NBI*) studies the kinetic equation reduced to the

- time derivative of the distribution function of the *hot (* or *fast) ions*
- collision operator (initially expressed in terms of Rosenbluth potentials and, after adopting $v_i \ll v_{hot} \ll v_e$ reduced at (1) slowing down and (2) pitch angle, with introduction of the critical velocity $v_c = \left(\frac{3\sqrt{\pi}}{4}\right)\left(\frac{m_e}{m_i}\right)^{1/3}$ also mentioned by **Ohkawa**)
- source, strong anisotropy in velocity space

The equation is solved by the usual

- series expansion, and
- separation of variables

which is

$$f^{hot} = \sum_{n=0}^{\infty} A_n(v)\, P_n(\xi)$$

where $\xi \equiv v_{\parallel}/v$.

The paper discusses the creation of large scale electric field, due to the process of charge separation.

The electron from ionization remain close to the magnetic surface, while the ion has to move by the drift velocity.

The *gradient of density* of the beam is important.

Why.

Because the process of charge separation is proportional with the number of new ions generated in the current point, equivalently, with the density of the beam in that point. There is an overall effect of charge separation if there is space variation of the individual "charge separations". [we have used the same idea - and expansion - in *density enhanced confinement*].

Taking

$$\begin{aligned}&\mathbf{B} \text{ along the } z \text{ direction}\\&\text{beam injected along the } y \text{ direction}\end{aligned}$$

The number of hot ions that are generated at the point (x, y) is

$$N^{hot}(x, y)$$

To calculate

$$n\left(x,y\right) \equiv \text{ density of hot ions at the point } \left(x,y\right)$$

one has to integrate over the hot ions that arrive in $\left(x,y\right)$ from various points where they are created by ionization

$$n\left(x,y\right) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} d\theta \ N^{hot}\left[x + 2a\cos^2\theta,\ y + a\sin\left(2\theta\right)\right]$$

The function N^{hot} is Taylor expanded and the integrals are calculated

$$\begin{aligned} n\left(x,y\right) &= N^{hot}\left(x,y\right) \\ &+\rho\frac{\partial N^{hot}}{\partial x} + \frac{3}{4}\rho^2\frac{\partial^2 N^{hot}}{\partial x^2} + \frac{\rho^2}{4}\frac{\partial^2 N^{hot}}{\partial y^2} \\ &+O\left(\rho^3\right) \end{aligned}$$

Here ρ is the distance over which there is separation of charges.

These are written for electrons and for ions since both species are moving after ionization.

The Poisson equation

$$\boldsymbol{\nabla}\cdot\mathbf{D} = \frac{1}{\varepsilon_0}\left(n_i - n_e\right)$$

$$= \frac{1}{\varepsilon_0}e\left[\rho_i\frac{\partial N^{hot}}{\partial x} + \frac{3}{4}\rho_i^2\frac{\partial^2 N^{hot}}{\partial x^2} + \frac{1}{4}\rho_i^2\frac{\partial^2 N^{hot}}{\partial y^2}\right]$$

after neglecting the contribution from electrons.

Now one has to compare

- the distance over which there is charge separation ρ, with
- the length of gradient (spatial variation) of the density of beam

If $\rho \ll L_n^{beam}$ then

$$\frac{\delta D}{\delta r} = \frac{1}{\varepsilon_0}e\rho_i\frac{\delta N^{hot}}{\delta x} + \left[...\left(\frac{\rho_i}{L_n^{beam}}\right)^2\right]$$

$$\delta D = \frac{1}{\varepsilon_0}e\rho_i N^{hot}$$

$$\kappa E_x = \frac{1}{\varepsilon_0}e\rho_i N^{hot}$$

$$\kappa \equiv \frac{\omega_{pi}^2}{\Omega_{ci}^2}$$

where

$$\omega_{pi}^2 = \frac{1}{\varepsilon_0}\frac{n_i\ Z^2e^2}{m_i} \quad \text{(ion plasma frequency)}$$
$$\Omega_{ci}^2 = \left(\frac{ZeB}{m_i}\right)^2 \quad \text{(ion cyclotron frequency)}$$

then

$$\kappa \equiv \frac{\omega_{pi}^2}{\Omega_{ci}^2} = \frac{1}{\varepsilon_0} n_i \frac{1}{B^2} m_i = \frac{1}{\varepsilon_0 \mu_0} \frac{1}{\frac{B^2}{\mu_0 (n_i m_i)}} = \frac{c^2}{v_A^2}$$

By the creation of an electric field there is an effect on the background plasma.

It is $E \times B$ rotation.

This drift is

$$\begin{aligned} V_{Dy} &= \frac{E_x}{B} \\ &= \frac{N^{hot} m^{hot} V^{hot}}{n_i m_i} \end{aligned}$$

where n_i and m_i are of cold background. This results from the conservation of momentum, expressing the fact that the momentum of the beam has been transferred to the cold background plasma.

The electric field will build up perpendicular on the beam, in the direction (beam, y) $\times$ (magnetic field, z) which is $(x) \to E_x$. Then the drift velocity is in the direction $E_x \times B_z = (y)$, which is *parallel* to the beam. The *parallel component* of the beam transfers momentum in the parallel direction.

7 Numerical investigation of *neoclassical* polarization Baek Chang

The velocity of a collisionless ion in a sheared, time-varying, radial electric field is strongly influenced by

$$\frac{\partial E_r}{\partial t} \quad \text{rate of time change of } E_r$$
$$(\Delta r)\frac{\partial E_r}{\partial r} \quad \text{spatial variation of } E_r \text{ assumed of order } \rho_\theta$$

The *classical* polarization drift velocity

$$v_{polariz}^{class} = \frac{m}{(Ze)\, B^2} \frac{\partial E_r}{\partial t}$$

and is due to gyromotion. We recongnize

$$\varepsilon_0 \frac{c^2}{v_A^2} \frac{\partial E_r}{\partial t}$$

(to which one has to add the vacuum polarization $\rightarrow \varepsilon_0 \left(1 + \frac{c^2}{v_A^2}\right) \frac{\partial E_r}{\partial t}$ and this is the radial current, $(Ze)\,\Gamma_{r,i}$.

They use the Hamiltonian formulation of the equations of motion, **White Boozer Littlejohn**.

$$\frac{d\mathbf{x}}{dt} = \frac{1}{D}\left[\frac{e}{m}\widehat{v}_{\|}\mathbf{B} + \frac{e}{m}\widehat{v}_{\|}^2\ \boldsymbol{\nabla}\times\mathbf{B} + \frac{\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\mathcal{H}}{B}\right]$$

$$\frac{d\widehat{v}_{\|}}{dt} = -\frac{1}{D}\frac{1}{B^2}\left[\boldsymbol{\nabla}\mathcal{H}\cdot\mathbf{B} + \widehat{v}_{\|}^2\ \boldsymbol{\nabla}\mathcal{H}\cdot\boldsymbol{\nabla}\times\mathbf{B}\right]$$

where the Hamiltonian is

$$\mathcal{H} = \frac{e}{2m}\widehat{v}_{\|}^2\ B^2 + \frac{1}{e}\mu B + \Phi$$

and the *normalized parallel speed* is

$$\begin{aligned}\widehat{v}_{\|} &\equiv \frac{v_{\|}}{eB/m} \quad \text{(unit: space, like } \rho_i\text{)}\\ &= \frac{v_{\|}}{\Omega_c} \sim \rho_\theta \quad \text{(neoclassic)}\end{aligned}$$

Then the first term in the eq. $d\mathbf{x}/dt$ is

$$\begin{aligned}\frac{e}{m}\widehat{v}_{\|}\mathbf{B} &\rightarrow \frac{e}{m}v_{\|}\frac{1}{\frac{eB}{m}}B\widehat{\mathbf{n}}\\ &\sim v \quad \text{(OK units)}\end{aligned}$$

The denominator

$$D \equiv 1 + \widehat{v}_{\|}\ \frac{\widehat{\mathbf{n}}\cdot(\boldsymbol{\nabla}\times\mathbf{B})}{B}$$

or

$$D = 1 + \frac{v_{\|}}{\frac{eB}{m}}\frac{1}{B}\widehat{\mathbf{n}}\cdot\mu_0\mathbf{j}$$

if we take

$$j_{\|} = env_{\|}$$

$$D = 1 + \frac{v_{\|}^2}{\frac{B^2}{\mu_0 m\ n}} = 1 + \frac{v_{\|}^2}{v_A^2}$$

Note

The hamiltonian

$$\begin{aligned}\frac{e}{2m}\widehat{v}_{\parallel}^{2}\ B^{2}+\frac{1}{e}\mu B+\Phi &= \frac{e}{2m}\frac{v_{\parallel}^{2}}{e^{2}B^{2}/m^{2}}B^{2}+\frac{1}{e}\frac{mv_{\perp}^{2}}{2B}B+\Phi \\ &= \frac{m}{e}\frac{1}{2}v_{\parallel}^{2}+\frac{m}{e}\frac{1}{2}v_{\perp}^{2}+\Phi\end{aligned}$$

if we assume

$$\mu\equiv\frac{mv_{\perp}^{2}}{2B}$$

After rescaling

$$e\mathcal{H}=\frac{1}{2}mv^{2}+e\Phi$$

End

In the numerical experiment it has been adopted a model of *electrostatic potential* with spatial and themporal variation able to reproduce the shear (spatial variation) and time evolution of the radial electric field in different devices.

$$\begin{aligned}\Phi\left(\psi,t\right) &= \left[\frac{C_{1}}{2}\left(\psi-\psi_{0}\right)^{2}+C_{2}\left(\psi-\psi_{0}\right)\right] \\ &\times\frac{t}{\tau}\end{aligned}$$

where $\psi_0 \equiv$ reference surface, $\tau \equiv$ normalization time constant.

Note in **Kogan Catto pedestal** (see below) it is assumed for the equilibrium electrostatic potential in the pedestal an expansion around the *drift surface* which is

$$\psi_{*}\approx\psi-I\frac{v_{\parallel}}{\Omega}$$

up to quadratic term

$$\phi_{0}\left(\psi\right)\approx\phi_{0}\left(\psi_{*}\right)+\left(\psi-\psi_{*}\right)\left.\frac{d\phi_{0}}{d\psi}\right|_{\psi_{*}}+\left(\psi-\psi_{*}\right)^{2}\left.\frac{d^{2}\phi_{0}}{d\psi^{2}}\right|_{\psi_{*}}$$

End note

For JET

$$\begin{aligned}C_{1} &= \frac{1}{3}\times10^{7}\ \frac{1}{V\ s^{2}} \\ C_{2} &= 10^{5}\ \frac{1}{s}\end{aligned}$$

With this potential it is calculated

$$\frac{\partial E_{r}}{\partial t}\text{and}\quad\left(\Delta r\right)\frac{\partial E_{r}}{\partial r}$$

where

$$\Delta r \equiv \text{radial excursion of an ions launched at the equatorial plane}$$

during its bounce motion the particle traverses regions with different electri potential $\Phi(\psi)$,

$$\begin{aligned} \frac{\partial \Phi}{\partial \psi} &= \left[C_1(\psi - \psi_0) + C_2\right] \times \frac{t}{\tau} \quad \text{the electric field} \\ \frac{\partial^2 \Phi}{\partial \psi^2} &= C_1 \times \frac{t}{\tau} \quad \text{shear of the electric field} \end{aligned}$$

$$\frac{\partial}{\partial \psi} = \frac{1}{RB_\theta}\frac{\partial}{\partial r}$$

Defining a local shear factor

$$S = 1 + \frac{e}{m}\frac{I^2}{\Omega_{ci}^2}\frac{\partial^2 \Phi}{\partial \psi^2}$$

approx 1... 2.

NOTE in **Novakovskii Liu Sagdeev Galeev** the definition of the *squeezing factor* is

$$\begin{aligned} S &= 1 + \frac{1}{\Omega}\frac{1}{\Theta^2}\frac{\partial V_E}{\partial r} \quad \text{with } V_E = \frac{1}{B_0}\frac{\partial \phi}{\partial r} \\ S &= 1 + \frac{1}{\Omega_i}\frac{B_\varphi^2}{B_\theta^2}\frac{1}{B_0}\frac{\partial^2 \phi}{\partial r^2} \\ &= 1 + \frac{1}{\Omega_i}\frac{1}{\Omega_i}\frac{e}{m}\frac{R^2 B_\varphi^2}{R^2 B_\theta^2}\frac{\partial^2 \phi}{\partial \psi^2}\left|\boldsymbol{\nabla}\psi\right|^2 \\ &= 1 + \frac{e}{m}\frac{I^2}{\Omega_i^2}\frac{\partial^2 \phi}{\partial \psi^2} \end{aligned}$$

End.

The *neoclassical polarization speed* from calculations is

$$V_{NP} \sim 10...20 \quad \left(\frac{m}{s}\right)$$

Follow a particle launched at

$$R_a \ \text{ initial point, or } \psi_a$$

with

$$v \equiv \text{initial speed}$$

$$v_{\parallel} \equiv \text{initial parallel speed}$$

There will be another point b where the orbit will cross the equatorial plane
The invariances
- energy

$$\frac{1}{2}mv_{\parallel a}^2 + \mu B_a + e\Phi_a = \frac{1}{2}mv_{\parallel b}^2 + \mu B_b + e\Phi_b$$

and
- longitudinal invariant

$$R_a m v_{\parallel a} - e\psi_a = R_b m v_{\parallel b} - e\psi_b$$

The electrostatic potential is expanded for small spatial deviations (typical, **Galeev Sagdeev**, expansion in $(r-r_0)^2$ of the second invariant, followed by solution of the second order algebraic equation)

$$\Phi = \Phi_a + (\psi - \psi_a)\frac{\partial \Phi_a}{\partial \psi} + \frac{1}{2}(\psi - \psi_a)^2 \frac{\partial^2 \Phi_a}{\partial \psi^2}$$

The magnetic field between the two points is

$$B_a - B_b = B_a\left(1 - \frac{R_a}{R_b}\right)$$

Define

$$\Delta\psi = \psi_b - \psi_a$$

From the two conservation equations

$$\alpha(\Delta\psi)^2 + \beta(\Delta\psi) + \gamma = 0$$

where

$$\alpha \equiv \frac{e^2}{m} + e\frac{\partial^2 \Phi}{\partial \psi^2}R_b^2$$

$$\begin{aligned} \beta &\equiv eR_a v_{\parallel a} + e\frac{\partial \Phi_a}{\partial \psi}R_b^2 \\ \gamma &\equiv mv^2(R_a - R_b)\left(\xi_a^2 R_a + R_b\right) \end{aligned}$$

An approx relationship

$$\Delta\psi = R_a B_{\theta a}\ \Delta r$$

(**note** the relationship $|\boldsymbol{\nabla}\psi| = 2\pi\ RB_\theta$ according to **Yushmanov**, the factor 2π). Then $\frac{\Delta\psi}{\Delta r} = R_a B_{\theta a}$. This is replaced in the second-order equation for $\Delta\psi$. This is similar to the work **Galeev Berk** where the expansion is in Δr.

It results

$$\Delta r = -2\frac{u_\parallel}{\Omega_\theta}\frac{1}{1 + \frac{e}{m}\frac{I^2}{\Omega_{ci}^2}\frac{\partial^2\Phi}{\partial\psi^2}}$$

where

$$u_{\parallel} \equiv v_{\parallel} + R\frac{\partial \Phi}{\partial \psi}$$

NOTE

We see that the variation of the electrostatic potential on the radial direction, when takes place over the width of the banana, modifies the width, with a *squeezing* factor

$$\frac{1}{S} = \frac{1}{1 + \frac{e}{m}\frac{I^2}{\Omega_{ci}^2}\frac{\partial^2 \Phi}{\partial \psi^2}}$$

This means that the displacement of a new ion, produced by charge exchange or ionization from a NBI neutral particle, in the radial direction, is only partly influenced by the *squeezing factor*. The new fast ion must move radially to take the final position of the "center" of the banana orbit, after which it makes periodic motion. This means that the new ion must move against a radial electric field generated by the displacements of the previous ions, which produced a separation of charge (polarization). Moving against this electric field is NOT described by the effect of banana *squeezing*. Moving against the radial electric field of *polarization* implies a work to be done. The energy of the new ion is reduced after doing this work. At the end of this motion the new ion becomes effectively a new contribution to the separation of charge therefore increasing the polarization electric field and so creating even more difficulty to the other new ions. But: what is the destination of the energy that has been spent by the ion during its motion against E_r? This is what **Honda** associates with a dissipation of collisional origin. But how the collisions intervene in this picture ?

END

NOTE

8 The bounce-averaged kinetic equation (Hahm Fong)

This is also in *general, equations of particle motion*.

The objective is to derive a kinetic equation based on bounce-averaged motion of trapped particles.

A bounce averaged kinetic equation also exists in **Hinton Rosenbluth**.

Three expansion parameters

$$\varepsilon_B \sim \frac{\Lambda_B}{L_B} = \frac{\text{width of banana}}{B/\left|\nabla B\right| \text{ (length of variation of magnetic field)}} \ll 1$$

$$\varepsilon_\phi = \frac{e\phi}{T_i} \sim \frac{1}{k_\perp L_p} \ll 1$$

$$\begin{aligned}
\varepsilon_k &\sim \frac{\omega}{\omega_{bounce}} = \frac{\text{freq. of fluctuations}}{\text{freq. of bounce}} \\
&= k_\perp \Lambda_B \\
&\ll 1
\end{aligned}$$

The velocity of the drift of the particle

$$\begin{aligned}
\mathbf{v} &= v_\parallel \widehat{\mathbf{n}} \\
&\quad + \frac{1}{eB/m}\left(\frac{v_\perp^2}{2} + v_\parallel^2\right)\widehat{\mathbf{n}} \times \boldsymbol{\nabla} \ln B \\
&\quad \left(\text{this is simply } \frac{1}{\Omega}\widehat{\mathbf{n}} \times \left[\mu \boldsymbol{\nabla} B + v_\parallel^2 \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\right)
\end{aligned}$$

Actually the first term is

$$\begin{aligned}
\mu \boldsymbol{\nabla} B &= \frac{v_\perp^2}{2}\frac{1}{B}\boldsymbol{\nabla} B \\
&= \frac{v_\perp^2}{2}\boldsymbol{\nabla} \ln B
\end{aligned}$$

In *particle equation of motion.tex*,

$$\boldsymbol{\nabla} \ln B \approx \frac{1}{R}\left(-\widehat{\mathbf{e}}_R\right) = \boldsymbol{\kappa}$$

The magnetic field is slightly different in definition

$$\mathbf{B} = \frac{\varepsilon_a B_0}{q\left(1 + \varepsilon_a \cos\theta\right)}\widehat{\mathbf{e}}_\theta + \frac{B_0}{1 + \varepsilon_a \cos\theta}\widehat{\mathbf{e}}_\varphi$$

where

$$\begin{aligned}
\varepsilon_a &= \frac{r}{R_0} = \text{local inverse aspect ratio} \\
\text{defined with the } R_0 &\equiv \text{center of bounce} \\
&\quad \left(\text{no connection with } R^{mag-axis}\right)
\end{aligned}$$

then

$$\begin{aligned}
\frac{B_0}{\frac{rB_T}{RB_\theta}\left(1 + \varepsilon_a \cos\theta\right)}\frac{r}{R_0} &= B_\theta \frac{1}{B_T}\frac{B_0}{1 + \varepsilon \cos\theta}\frac{1 + \varepsilon \cos\theta}{1 + \varepsilon_a \cos\theta}\frac{R}{R_0} \\
&= B_\theta \frac{1 + \varepsilon \cos\theta}{1 + \varepsilon_a \cos\theta}\frac{R^{mag-axis}}{1 + \varepsilon \cos\theta}\frac{1}{\frac{R^{mag-axis}}{1 + \varepsilon_a \cos\theta}} \\
&= B_\theta
\end{aligned}$$

The equations of motion

$$\frac{dr}{dt} \approx -\frac{1}{\Omega}\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{R_0}\sin\theta$$

$$\frac{d\theta}{dt} \approx \frac{v_\parallel}{qR_0} - \frac{1}{r}\frac{1}{\Omega}\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{R_0}\cos\theta$$

$$\frac{d\varphi}{dt} \approx \frac{v_\parallel}{R_0}$$

NOTE If we want to integrate these equations we need equations also for the *velocities* $v_\parallel$ and $v_\perp$. See **Berk Galeev**, etc. *particle equations of motion.tex*. **END.**

NOTE that the first equation is the projection of the drift velocity on the radial direction

$$\begin{aligned}
\frac{dr}{dt} &= -\frac{1}{\Omega_c}\frac{1}{R_0}\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)\sin\theta = (\mathbf{v}_D)_r \\
&= \left[\frac{1}{\Omega_c}\widehat{\mathbf{n}}\times\left(\mu\boldsymbol{\nabla}B + v_\parallel^2\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right)\right]_r
\end{aligned}$$

For the second equation we recognize the first term as the poloidal projection of the parallel velocity

$$\begin{aligned}
\left(\frac{rd\theta}{dt}\right)^{(1)} &= \left(v_\parallel\right)_\theta = v_\parallel\frac{B_\theta}{B} \approx v_\parallel\frac{r}{R}\left(\frac{RB_\theta}{rB_T}\right) \approx r\frac{v_\parallel}{qR_0} \\
\text{or}\quad \left(\frac{d\theta}{dt}\right)^{(1)} &= \frac{v_\parallel}{qR_0} \quad \text{the poloidal projection of the parallel velocity}
\end{aligned}$$

The second term in the θ motion equation is due to the poloidal projection of the *drift velocity*

$$\begin{aligned}
(\mathbf{v}_D)_\theta &= \left[\frac{1}{\Omega_c}\widehat{\mathbf{n}}\times\left(\mu\boldsymbol{\nabla}B + v_\parallel^2\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right)\right]_\theta \\
&\approx \frac{1}{\Omega_c}\frac{1}{R_0}\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)\left(\widehat{\mathbf{e}}_{vertical}\right)_\theta \\
&= \frac{1}{\Omega_c}\frac{1}{R_0}\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)\cos\theta
\end{aligned}$$

END NOTE.

NOTE

that for trapped particles the θ function of time is periodic.

What is supplementary in the equation will produce a systematic toroidal drift.

END

The equations of motion, mentioned by **Fong Hahm** by quoting **Kadomtsev Pogutse**

$$\frac{dr}{dt} \approx -\frac{1}{\Omega}\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)\frac{\sin\theta}{R_0}$$

$$\frac{d\theta}{dt} \approx \frac{v_\parallel}{qR_0}-\frac{1}{r}\frac{1}{\Omega}\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)\frac{\cos\theta}{R_0}$$

$$\frac{d\varphi}{dt} \approx \frac{v_\parallel}{R_0}$$

These equations are integrated as functions of θ.

See **Galeev Sagdeev Wong**. In this text.

$$r(\theta)-r_0 = \pm\frac{vq_0}{\Omega\sqrt{\varepsilon}}\left[\frac{v_{\parallel 0}^2}{\varepsilon v_{\perp 0}^2}-2\sin^2\left(\frac{\theta}{2}\right)\right]^{1/2}$$

where

$$\begin{aligned} r_0 &\equiv \text{radius of the surface to which the bounce points belong} \\ q_0 &= q(r_0) \\ v_{\parallel 0} &\equiv \text{velocity at the outer mid-plane} \\ v_{\perp 0} &\equiv \text{velocity at the outer mid-plane} \end{aligned}$$

Characteristic parameters of the motion

$$\Lambda_b = \frac{vq_0}{\Omega\sqrt{\varepsilon}} \equiv \text{banana radius}$$

We note that this width of the banana trajectory is obtained from the *coefficient* of the expression $r(\theta)-r_0$ written above and has the following meaning:

- the departure of the particle from the magnetic surface where its center is located is
$$+\frac{v}{\Omega}\frac{q_0}{\sqrt{\varepsilon}}$$
when the particle advances in the positive toroidal direction: this is the *first half of the banana.*

- the departure of the particle trajectory from the magnetic surface is
$$-\frac{v}{\Omega}\frac{q_0}{\sqrt{\varepsilon}}$$
when the particle moves back: this is the *second half of the banana.*

The other parameter is

$$\kappa^2 \equiv \frac{1}{2\varepsilon}\frac{v_{\parallel 0}^2}{v_{\perp 0}^2} \equiv \text{pitch angle parameter}$$

Note In **Galeev Sagdeev** the parameter is

$$2\kappa^2 = \frac{\left[\Delta v\left(r_0,0\right)\right]^2}{\varepsilon\left(v^2+\frac{v_E^2}{\Theta^2}\right)}$$

End

We can understand how this parameter arises and how we are led to introduce it. The expression under the radical in the equation of the trajectory $r(\theta) - r_0$ must be transformed suce as to make possible the integration

$$\begin{aligned}
& \left[\frac{v_{\|0}^2}{v_{\perp 0}^2}\frac{1}{\varepsilon} - 2\sin^2\left(\frac{\theta}{2}\right)\right]^{1/2} \\
= & \sqrt{2}\left[\frac{v_{\|0}^2}{v_{\perp 0}^2}\frac{1}{2\varepsilon} - \sin^2\left(\frac{\theta}{2}\right)\right]^{1/2} \\
= & \sqrt{2}\left[\kappa^2 - \sin^2\left(\frac{\theta}{2}\right)\right]^{1/2} \\
= & \pm\sqrt{2}\kappa\left[1 - \frac{\sin^2\left(\frac{\theta}{2}\right)}{\kappa^2}\right]^{1/2}
\end{aligned}$$

and a substitution of the variable of integration is made

$$\sin^2\left[\varphi\left(\theta\right)\right] \equiv \frac{\sin^2\left(\frac{\theta}{2}\right)}{\kappa^2}$$

which defines the function $\varphi(\theta)$ by the equation

$$\varphi\left(\theta\right) = \arcsin\left[\frac{\sin\left(\frac{\theta}{2}\right)}{\kappa}\right]$$

We can separate the motion, taking the picture of the meridional plane

$$r_0 \equiv \text{radius of the surface of the "center" of the banana}$$

$$r\left(t\right) = r_0 + \delta r\left(\theta\right)$$

The structure of the trajectory (**Fong Hahm**):

1. bounce center on the surface $r = r_0$,
2. oscillatory motion $r = r_0 \pm \delta r$
   (a) $+\delta r$ when the motion is in positive sense along the line with length $l_{\|}$; this is the *first* half of the banana.

(b) $-\delta r$ when the motion is in negative sense along the line with length $l_{\|}$; this is the *second* half of the banana.

These equations are integrated as functions of θ

$$r(\theta) - r_0 = \pm \frac{v q_0}{\Omega \sqrt{\varepsilon}} \left[\frac{v_{\|0}^2}{\varepsilon v_{\perp 0}^2} - 2 \sin^2 \left(\frac{\theta}{2} \right) \right]^{1/2}$$

where

$$\begin{aligned} r_0 &\equiv \text{radius of the surface to which the bounce points belong} \\ q_0 &= q(r_0) \\ v_{\|0} &\equiv \text{velocity at the outer mid-plane} \\ v_{\perp 0} &\equiv \text{velocity at the outer mid-plane} \end{aligned}$$

Characteristic parameters of the motion

$$\Lambda_b = \frac{v q_0}{\Omega \sqrt{\varepsilon}} \equiv \text{banana radius}$$

$$\begin{aligned} \kappa^2 &= \frac{1}{2\varepsilon} \frac{v_{\|0}^2}{v_{\perp 0}^2} \equiv \text{sin of pitch angle} \\ &< 1 \quad \text{(for trapped, for } \exists \text{ of real solution } \theta) \end{aligned}$$

We can separate the motion, taking the picture of the meridional plane

$$r_0 \equiv \text{radius of the surface of the "center" of the banana}$$

$$r(t) = r_0 + \delta r(\theta)$$

In **Fong Hahm** the polarization velocity

$$\mathbf{v}_{pol} = \frac{1}{\Omega_c} \frac{1}{B} \frac{\partial \mathbf{E}_{\perp}}{\partial t}$$

the polarization current (electrons are neglected)

$$\begin{aligned} \mathbf{j}_{pol} &= e n_{i0} \, \mathbf{v}_{i,pol} \\ &= n_{i0} \frac{m_i}{B^2} \frac{\partial \mathbf{E}_{\perp}}{\partial t} \end{aligned}$$

$$\nabla_{\perp} \cdot \mathbf{E}_{\perp} = \frac{q_{pol}}{\varepsilon_0} = e \frac{n_{pol}}{\varepsilon_0}$$

From the continuity equation

$$\frac{\partial q_{pol}}{\partial t} + \nabla_{\perp} \cdot \mathbf{j}_{pol} = 0$$

$$\frac{\partial\left(e\ n_{pol}\right)}{\partial t}+\mathbf{\nabla}_{\perp}\cdot\mathbf{j}_{pol}=0$$

or

$$\begin{aligned}\frac{\partial n_{pol}}{\partial t}+\mathbf{\nabla}_{\perp}\cdot\left(n_{i0}\frac{m_i}{eB^2}\frac{\partial\mathbf{E}_{\perp}}{\partial t}\right) &= 0\\ n_{pol} &= -\frac{m_i}{eB^2}\mathbf{\nabla}_{\perp}\cdot\left[n_{i0}\left(-\mathbf{\nabla}_{\perp}\phi\right)\right]\end{aligned}$$

Check the units

$$1=\frac{1}{[\Omega]}\frac{1}{[B]}\frac{1}{L}[E]$$

and

$$\frac{E}{B}\sim v\sim\frac{L}{T}$$

$$\begin{aligned}1 &= T\frac{L}{T}\frac{1}{L}=1\\ &\text{OK}\end{aligned}$$

no need of ε_0.

From the first equation $\mathbf{\nabla}_{\perp}\cdot\mathbf{E}_{\perp}=e\frac{n_{pol}}{\varepsilon_0}$ we have

$$n_{pol}=\varepsilon_0\frac{1}{e}\mathbf{\nabla}_{\perp}\cdot\left(-\mathbf{\nabla}_{\perp}\phi\right)$$

?

Result of **Callen**

$$n_{pol}\approx\frac{16}{3\pi\sqrt{2}}\varepsilon^{3/2}\frac{1}{\Omega_{\theta}}\frac{1}{B_{\theta}}n_{i0}\frac{\partial^2\phi}{\partial r^2}$$

The polarization density is (naturally) proportional with the laplacian of the potential (divergence of the electric field). Normally this is NOT a static relationship. The electric field has time variation, the charge separation is time-varying, there is separation and concentration of charges of polarization.

The *polarization* is the response to a process of charge separation.

Static, is just a Gauss relation between the charge density and the divergence of the electric field.

The structure of the trajectory (**Fong Hahm**):

1. bounce center on the surface $r=r_0$,
2. oscillatory motion $r=r_0\pm\delta r$
   (a) $+\delta r$ when the motion is in positive sense along the line with length $l_{\parallel}$; this is the *first* half of the banana.

(b) $-\delta r$ when the motion is in negative sense along the line with length $l_{\|}$; this is the *second* half of the banana.

The motion in toroidal direction (**Fong Hahm**) results from integration of the equation $d\zeta/dt$.

The two expressions correspond to the two halves which means the two expression for $v_{\|}(\theta)$.

$$\begin{aligned}\zeta(\theta) &= q_0\theta \\ &\quad +\sqrt{2}\Lambda_b\left[\left(2q_0'+\frac{q_0}{r_0}\right)C_1(\varphi(\theta),\kappa)\right. \\ &\quad \left.-\left(2q_0'\left(1-\kappa^2\right)+\frac{q_0}{2r_0}\right)C_2(\varphi(\theta),\kappa)\right]\end{aligned}$$

where

$$C_1(\varphi(\theta),\kappa)=\begin{cases} E(\varphi,\kappa)+\mathbf{E}(\kappa) & \text{first half of orbit} \\ 3\mathbf{E}(\kappa)-E(\varphi,\kappa) & \text{second half of orbit}\end{cases}$$

$$C_2(\varphi(\theta),\kappa)=\begin{cases} F(\varphi,\kappa)+\mathbf{K}(\kappa) & \text{first half of orbit} \\ 3\mathbf{K}(\kappa)-F(\varphi,\kappa) & \text{second half of orbit}\end{cases}$$

$$\varphi(\theta)\equiv\arcsin\left(\frac{\sin(\theta/2)}{\kappa}\right)$$

The motion of particle is along the magnetic line with the deviations

$$\begin{aligned}\beta &\equiv \delta\psi_b(r) \quad \text{deviation in the radial direction} \\ \alpha &\equiv \zeta-q(r)\theta \quad \text{deviation non-radial perpendicular}\end{aligned}$$

We know that a magnetic line is given by a fixed value of

$$\zeta-q(r)\theta=\alpha_0$$

and a change of this variable means to quit one line for another line.

In the definition of $\alpha(\theta,t)$ we have ζ (the toroidal angle) and θ which can increase then decrease in the case of trapped particles. In the lowest order, in which we neglect the differences between the two halfs of the bananas (inhomogeneous magnetic field, shear) the trapped particle must return precisely a the same position, with no advancement in ζ. Then if there is a difference

$$\alpha(\theta_{fin})\neq\alpha(\theta_{ini})$$

this is because the two halves of the banana are different. At the end of a time interval

$$\tau_{bounce}$$

the difference between $\alpha' s$ is the *precession* in ζ.

The average *precessional* motion can be characterized by the change in α, given by a precessional frequency

$$\omega_{pr} = \frac{\alpha(\theta_f) - \alpha(\theta_i)}{\tau_{bounce}}$$

The center of bounce has a coordinate α which varies in time according to

$$\alpha_{pr}(t) = \omega_{pr} t(\theta)$$

(this means a linear increase of the coordinate that labels the successive magnetic field lines that the center of the banana changes when it moves along its *precessional* path) where the time has been expressed as function of the poloidal angle by the integrating

$$\frac{dt}{d\theta} \rightarrow t(\theta)$$

This linear increase is extracted from the full function $\alpha(\theta)$. What remains is the the deviation from the average

$$\delta\alpha(\theta) \equiv \alpha(\theta) - \omega_{pr} t(\theta)$$

$$\begin{aligned} \delta\alpha(\theta) &= \mp\sqrt{2}\Lambda_b \left[\left(2q_0' + \frac{q_0}{r_0} \right) \left(\frac{\mathbf{E}(\kappa)}{\mathbf{K}(\kappa)} F(\varphi,\kappa) - E(\varphi,\kappa) \right) \right. \\ &\quad \left. + q_0' \theta \sqrt{\kappa^2 - \sin^2(\theta/2)} \right] \end{aligned}$$

The motion of the bounce center is

$$\begin{aligned} & r_0 \\ & \alpha_{pr}(\theta) \end{aligned}$$

and the motion of the particle is

$$\begin{aligned} r(\theta) &= r_0 + \delta r(\theta) \\ \alpha(\theta) &= \alpha_{pr}(\theta) + \delta\alpha(\theta) \end{aligned}$$

The particularities of the motion of the trapped particles in the toroidal direction

the orbit does not return to the same initial position

The reason is the difference that exists between the travel in the first half of the banana relative to the travel in the second half of the banana

- the intensity of the magnetic field is different

- the curvature of the magnetic lines is different
- the magnetic shear is different

on the two halves.
The result is that the orbit does not close and it is generated a *precession*.

The motion is essentially along the magnetic field line, with deviations

- in the radial direction
$$\beta \equiv \psi_p (r)$$
- nonradial direction (in surface)
$$\begin{aligned} \alpha &= \zeta - q(r)\theta \\ & \quad \text{nonradial perpendicular} \end{aligned}$$
- It is defined an average precessional frequency
$$\omega_{pr} = \frac{\alpha(\theta_{fin}) - \alpha(\theta_{ini})}{\tau_{bounce}}$$

The average position of the bounce motion in α cooerdinate
$$\alpha_{pr}(\theta) = \omega_{pr} \times t(\theta)$$
where $t(\theta)$ is obtained from the equations, but solving for
$$\frac{dt}{d\theta}$$

The deviation is
$$\begin{aligned} \Delta\alpha(\theta) &= \alpha(\theta) - \omega_{pr}\ t(\theta) \\ &= \mp\sqrt{2}\Lambda_B \left[\left(2\frac{dq_0}{dr} + \frac{q_0}{r_0} \right) \left(\frac{\mathbf{E}(\kappa)}{\mathbf{K}(\kappa)} F(\varphi,\kappa) - E(\varphi,\kappa) \right) \right. \\ & \qquad \left. + \frac{dq_0}{dr}\ \theta\ \sqrt{\kappa^2 - \sin^2\left(\frac{\theta}{2}\right)} \right] \end{aligned}$$

These new variables have been introduced to separate the two motions

- the motion of the center of bounce
$$(r_0, \alpha_{pr}(\theta))$$
- the motion of the guiding center on banana
$$\begin{aligned} r(\theta) &= r_0 + \Delta r(\theta) \\ \alpha(\theta) &= \alpha_{pr}(\theta) + \Delta\alpha(\theta) \end{aligned}$$

In view of calculation of the density of ions $n_i(\mathbf{x})$ taking into account the trapping and the precession, one defines the averaging operator

$$\begin{aligned}
\langle g(\theta)\rangle_\theta &= \frac{\oint d\theta \ \frac{dt}{d\theta} \ g(\theta)}{\oint d\theta \ \frac{dt}{d\theta}} \\
&= \frac{\oint d\theta \ \frac{qR}{v_\parallel} \ g(\theta)}{\tau_{bounce}}
\end{aligned}$$

The operator will be used to calculate the *bounce average* of the density of ions.

First the deviations

$$\Delta r(\theta) \quad \text{and} \quad \Delta\alpha(\theta)$$

The first bounce average

$$\begin{aligned}
\langle \Delta r(\theta)\rangle_\theta &= 0 \\
\langle \Delta\alpha(\theta)\rangle_\theta &= 0
\end{aligned}$$

Now the second moment

$$\left\langle [\Delta r(\theta)]^2 \right\rangle_\theta = \Lambda_B^2 \ \kappa^2$$

$$\left\langle [\Delta\alpha(\theta)]^2 \right\rangle = \frac{1}{16}\Lambda_B^2 \ \left(2\frac{dq_0}{dr} - \frac{q_0}{r_0}\right)^2 \ \kappa^4$$

$$\langle \Delta r \ \Delta\alpha\rangle_\theta = 0$$

The ion density is expanded about the position of the *bounce-center* in terms of deviations

$$\begin{aligned}
n_i(\mathbf{X}+\mathbf{\Lambda}) &= n_i(\mathbf{X}) \\
&+\Delta r(\theta) \ \left.\frac{\partial n_i}{\partial r}\right|_{\mathbf{X}} + \Delta\alpha(\theta) \ \left.\frac{\partial n_i}{\partial \alpha}\right|_{\mathbf{X}} \\
&+\frac{1}{2}(\Delta r)^2 \ \left.\frac{\partial^2 n_i}{\partial r^2}\right|_{\mathbf{X}} + \frac{1}{2}(\Delta\alpha)^2 \ \left.\frac{\partial^2 n_i}{\partial \alpha^2}\right|_{\theta} \\
&+(\Delta r \ \Delta\alpha) \ \left.\frac{\partial^2 n_i}{\partial r\partial\alpha}\right|_{\mathbf{X}}
\end{aligned}$$

This expression is bounce-averaged. Then integrated over the space of velocities for trapped particles

$$\begin{aligned}
& n_i(\mathbf{X}) - \langle\langle n_i(\mathbf{X}+\mathbf{\Lambda})\rangle_\theta\rangle_v \\
\approx \ & \frac{1}{4}\Lambda_{B,th}^2 \ \sqrt{2\varepsilon_a} \ n_{i0}\frac{e}{T_i}\frac{\partial^2\phi}{\partial r^2} \\
& +\frac{3}{320}\Lambda_{B,th}^2 \ \left(2\frac{dq_0}{dr} - \frac{q_0}{r_0}\right)^2 \sqrt{2\varepsilon_a} n_{i0}\frac{e\phi}{T_i} \ \frac{\partial^2\phi}{\partial\alpha^2}
\end{aligned}$$

This difference in density is the *polarization*.

The difference in density is expressed in terms of B_θ, as expected for a *neoclassical polarization*

$$\begin{aligned}
& n_i(\mathbf{X}) - \left\langle \left\langle n_i(\mathbf{X}+\mathbf{\Lambda})\right\rangle_\theta \right\rangle_v \\
= \quad & \frac{1}{4}\frac{m_i n_{i0} c^2}{eB_\theta^2}(2\varepsilon_a)^{3/2}\ n_{i0}\frac{e}{T_i} \qquad \times \quad \frac{\partial^2\phi}{\partial r^2} \\
& +\frac{3}{320}\frac{m_i n_{i0} c^2}{eB_\theta^2}\left(2\frac{dq_0}{dr}-\frac{q_0}{r_0}\right)^2(2\varepsilon_a)^{3/2}\, n_{i0}\frac{e\phi}{T_i} \quad \times \quad \frac{\partial^2\phi}{\partial\alpha^2}
\end{aligned}$$

NOTE that the difference in density (local density relative to the averaged density) has the form of a *charge density*, since it is proportional with "Laplacian" of the potential, $\Delta\phi \rightarrow \frac{\partial^2\phi}{\partial r^2}$. This derivative is actually the *vorticity*. **END.**

The density of trapped ions has been approximated

$$\delta n_{i,trapped} \sim \sqrt{2\varepsilon_a} n_{i0}\frac{e\phi}{T_i}$$

In the formula above the first term is due to the radial excursions of the trapped ion.

The second term is due to the toroidal precession.

From the density *of charge* ($\delta n \rightarrow \rho$) induced by polarization one can obtain the density of current $\mathbf{j}$, via the equation of continuity. Here comes the *time derivative of the density,* which is the time derivative of the potential or of the electric field.

The results of previous calculations

$$\mathbf{j}_{pol} \approx \frac{16}{3\pi\sqrt{2}}\ \varepsilon_a^{3/2}\ \frac{n_i m_i c^2}{B_\theta^2}\ \frac{\partial \mathbf{E}_r}{\partial t}$$

For this (approximative) polarization current (which does not include α motion) one can derive again via the equation of continuity the density of charge

$$n_{pol} \approx \frac{16}{3\pi\sqrt{2}}\ \varepsilon_a^{3/2}\ \frac{m_i n_i c^2}{eB_\theta^2}\ \times\ \frac{\partial^2\phi}{\partial r^2}$$

This *is* the neoclassical polarization.

It is in the spirit of physical picture discussed by **Hinton Robertson** and shows how B_θ replaces B in the Alfven speed.

This is different (*i.e.* more) than **Honda** where the non-periodicity of the motion on bananas due to the different conditions on the two halves of the orbit is not included.

NOTE

We compare the result of **Fong Hahm** or **Callen**

$$\mathbf{j}_{pol} \approx \frac{16}{3\pi\sqrt{2}}\ \varepsilon_a^{3/2}\ \frac{n_i m_i c^2}{B_\theta^2}\ \frac{\partial \mathbf{E}_r}{\partial t}$$

or

$$\mathbf{j}_{pol} \approx \frac{16}{3\pi\sqrt{2}}\ \varepsilon_a^{3/2}\ \frac{c^2}{v_{A\theta}^2}\frac{\partial \mathbf{E}_r}{\partial t}$$

and **Novakovskii Liu Sagdeev Galeev** and **Novakovskii Kiu Sagdeev Rosenbluth**. This is for the time evolution of the poloidal rotation generated by the non-ambipolarity of the radial diffusion fluxes of ions (large) and electrons (small)

$$1.52\ nmq^2\sqrt{\frac{S}{\varepsilon}}\frac{\partial V_E}{\partial t} = -\frac{F_\parallel}{\Theta}$$

We remind that $\Theta = \frac{B_\theta}{B_T} = \tan$ (small angle made by the line with toroidal direction). Then $1/\Theta$ is tan (large $\sim \pi/2$ angle between the line and the poloidal direction)

In the LHS we have the time variation of the momentum carried by the electric velocity. This is a force. [we also find the *inertia* q^2 - see **Hassam Drake**, and the squeezing factor].

In RHS we have a force which is parallel $F_\parallel$ and is projected on the poloidal direction $\times\frac{B_T}{B_\theta}$.

Consider

$$\begin{aligned}
\mathbf{j}_{pol}^{FongHahm} &\approx \frac{16}{3\pi\sqrt{2}}\ \varepsilon_a^{3/2}\ \frac{n_i m_i c^2}{B_\theta^2}\ \frac{\partial \mathbf{E}_r}{\partial t} \\
&= \frac{16}{3\pi\sqrt{2}}\left(\frac{r}{R}\right)^2\frac{1}{\sqrt{\varepsilon}}\frac{B_T^2}{B_\theta^2}\frac{1}{B_T^2}n_i m_i\frac{\partial V_E}{\partial t}B \\
&\approx \frac{16}{3\pi\sqrt{2}}\ q^2\frac{1}{\sqrt{\varepsilon}}\frac{n_i m_i}{B}\frac{\partial V_E}{\partial t} \\
&\approx \frac{1}{B}\times 1.2\ n_i m_i q^2\frac{1}{\sqrt{\varepsilon}}\frac{\partial V_E}{\partial t}
\end{aligned}$$

This will be compared with the expression above from NLSG, of $-\frac{F_\parallel}{\Theta}$.
If we take

$$S \sim 1$$

then the main parameter content of the two formulas is identical, with only a factor $1/B$. We have

$$\begin{aligned}
J_{pol}^{FH} &= \frac{1}{B}\times\left(-\frac{F_\parallel}{\Theta}\right)^{NLSG} \\
&\sim \frac{F_\parallel}{B_\theta}
\end{aligned}$$

this means, if the type of the formula is

$$\frac{\widehat{\mathbf{n}}_\theta \times \mathbf{F}_\parallel}{B_\theta} = \text{flux}$$

that the *polarization current* is *RADIAL*.

The formula for the parallel *VISCOSITY* force is

$$\mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}$$

$$\begin{aligned}
F_\parallel &= (\boldsymbol{\nabla}\pi)_\parallel \\
&= 0.37 \; n_i m_i \nu \sqrt{\frac{S}{\varepsilon}} \left[\frac{V_E}{\Theta} + U_\parallel + \frac{1}{\Theta S}\left(U_p - 1.17 \; U_T\right)\right] \\
&= \frac{1}{\Theta} 0.37 \; n_i m_i \nu \sqrt{\frac{S}{\varepsilon}} \left[V_E + \Theta U_\parallel + \frac{1}{S}\left(U_p - 1.17 \; U_T\right)\right]
\end{aligned}$$

where

$$U_p = \frac{T/m}{(eB/m)} \frac{d \ln (nT)}{dr}$$

This is **Hazeltine**, the neoclassical poloidal rotation when there is a gradient of Temperature ∇T.

And

$$U_T = \frac{T/m}{(eB/m)} \frac{d \ln T}{dr}$$

The velocity V_E is perpendicular on B and the two *diamagnetic* velocities U_p and U_T are also perpendicular. Then $U_\parallel$ must be projected using a factor

$$\frac{1}{\Theta} = \frac{B_T}{B_\theta}$$

The full equation contains now poloidal rotation velocities and is then projected again on the parallel direction.

END.

9 Notes on the physics of radial current, polarization response and return current

This is also in *ITB*.

From the **thesis on NBI**.

"*In essence the creation of fast ions by the ionization of injected neutrals leads to a radial current and therefore produces a buildup of charge. Since a plasma is a polarizable media, a polarization current, J_r^{pol} which results from the changing radial electric field $\partial E_r / \partial t$, cancels the fast ion creation current,* *and the ensuing force due to the polarization current transfers part of the injected momentum to the plasma, $J_r^{pol} \times B$. Since the prompt transfer of injected momentum is proportional to the rate of fast ion creation, then this transfer mechanism occurs immediately after the momentum injection source is turned on whereas the direct collisional interaction between the beam particles and background plasma occurs on a slowing down time scale. When steady state is achieved, the time variation of the radial electric field vanishes $\partial E_r/\partial t = 0$, and therefore the polarization current goes to zero J_r^{pol}.* [this is because the polarization current is always a RESPONSE to a current which is a dynamical separation of charges caused by an external force, as is the case of ionization. The polarization current exists as long as $\partial E/\partial t$ is not zero]. [Note however that plasma is NOT composed of molecules whose dynamic re-orientation creates a response to an external electric field, as is the case of some solid state materials]. [But, the current produced by the ionization continues]. *As a result, the fast ion creation current must now be balanced by other currents* [return current ?]. *It is then these forces, which result from the plasma currents necessary to balance the ion creation current, which cancel the momentum losses in the steady state."*

This explanation does not provide any idea on which is the nature of the NEW response (compensating) currents, when $\partial E/\partial t$ is zero and - in consequence - the polarization current is zero.

Looks similar to the problem of Honda.

In **Wong Burrell** there is an explanation of the dynamics around non-ambipolarity.

Non-ambipolar transport leads to an electric field *with time variation*.

Then $\partial E_r/\partial t$ produces a polarization response, a current that flows to compensate.

The polarization current

$$j^{polarization} = \sum m_i n_i \frac{1}{B_0^2} \frac{\partial}{\partial t} \frac{d\phi_0(r)}{dr}$$

must compensate the non-ambipolar flow, which is estimated as

$$\sum e\Gamma \sim nev_{drift}\ \varepsilon$$

the equality (compensation) exhibits the time scale specific to the process

$$\tau^{-1} \sim \omega^{ion-transit}$$

very fast.

Stix 1973 is mentioned.

NOTE

The paper by **Poli Peeters** on island polarization

$$j_{pol}^{classical} = \frac{1}{\Omega_c} en \frac{dv_E}{dt} = \frac{1}{\Omega_c} en \left[\left(\mathbf{v}_E \cdot \boldsymbol{\nabla} \right) v_E \right]$$

(remark the static part = convection, i.e. we do not have the usual fast decay $\partial/\partial t$ of the poloidal rotation due to TTMP).

Due to the *damping* of the poloidal rotation, in order the poloidal rotation to NOT exist, a *parallel* flow is developed in plasma. This parallel flow velocity is determined by the electric field that has determined the polarization, such that

$$u_{\parallel} = \frac{E}{B_{\theta}}$$

will have a *poloidal projection* which equals and vanishes the *poloidal electric* $E \times B$ rotation.

The neoclassical polarization current is

$$j_{pol}^{neoclass} = \frac{en}{\Omega_{\theta}} \left[\, (\mathbf{v}_E \cdot \boldsymbol{\nabla}) \, u_{\parallel} \right]$$

The ratio

$$\frac{j_{pol}^{class}}{j_{pol}^{neoclass}} \sim q^2 \varepsilon^{-1/2}$$

The charge conservation

$$\boldsymbol{\nabla} \cdot \mathbf{j}_{\perp} = -\boldsymbol{\nabla}_{\parallel} \cdot \mathbf{j}_{\parallel}$$

where

$$\nabla_{\parallel} \sim k_{\parallel} \sim W$$

and

$$\nabla_{\perp} \sim \frac{1}{W}$$

Now take

$$W \equiv \text{ island half-width}$$

then

$$j_{pol}^{\parallel} \sim \frac{1}{W^2}$$

10 Plasma response to the charge separation induced by the displacement of the new ions

This is in *density enhanced confinement.*

After ionization the new ions move to take their neoclassical periodic orbit. Between the point of creation and the "center" of the periodic motion they carry a transitory, short, finite current. As in the main text we consider the ionization to take place in a volume limited between the radii r_1 (left side, closer to the center) r_2 (right side, closer to the last closed magnetic surface). The plasma is considered homogeneous and the ionization generates ions that move to the right a distance Δ^t with velocity v_{Di} while the new electrons can be considered imobile. Then most of the volume between r_1 and r_2 is neutral but at the right end $\sim r_2$ of the ionization interval it results a layer (denoted I) of positive charge, of width Δ^t. This is the source of electric field, resulted from ionization, E^I, directed from r_2 towards r_1. The charge layer I and its field E^I are built up on a time scale $\delta t = \Delta^t / v_{Di}$ in which $\partial E^I / \partial t > 0$. The background plasma responds by modifying the Larmor gyration orbit from the usual circle to a cycloid (actually the new orbit is a *prolate trochoid*). The deformation of the gyration directly indicates the expected $\mathbf{v}_E = \mathbf{E} \times \mathbf{B}/B^2$ motion and also the asymmetry of the charge distribution along the new orbit. The asymmetry creates a new layer (denoted L) of positive (ion) charge, at the left end, $\sim r_1$ and an electric field E^L opposite to E^I. This field is sufficient to almost cancel E^I inside plasma, leaving in the interior a small $E^{int} = E^I - E^L$, directed to the left, like E^I. The asymmetry of the modified Larmor orbit is a manifestation of the polarization drift induced by the variation in time of the electric field E^I (implicitely E^{ind}), a displacement of the background ions in the direction of E^I (*i.e.* to the left). Since the ionization continues to accumulate new ions in the layer I, hence $\partial E^I / \partial t > 0$, the ion's drift of polarization $v_{Di}^{(pol)}$ fills the layer L at the left end, whose electric field E^L (from r_1 toward r_2) continues to quasi-compensate E^I inside plasma.

Qualitatively, this picture conforms to the concept of *return current*, which is universally invoked as the plasma response to any mechanism that is able to produce rotation of only some component of the density: NBI, ICRH, *alpha* particle, etc., to which we add: ionization. In the following we examine the density of charge and respectively the current density arising from ionization, deformation of the Larmor orbit and finally the polarization drift. The ionization-induced charge separation and current are regarded as "external" factors since they are requested by the geometry of the field.

We include a justification of the neglect of the volume-charge accumulation that can be associated with the strong vorticity.

10.1 Charge and current related to the vorticity

At the edge of the tokamak in the H-mode regime there is a layer of strong poloidal rotation, with radial extension of about a banana width calculated for the poloidal magnetic field. The variation of the velocity magnitude is very fast in this layer, or, equivalently, the layer is a concentration of vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v}$ or $\omega \sim \frac{\partial v_\theta}{\partial r}$. Taking as usual $\mathbf{v} = \mathbf{v}_E = \frac{-\boldsymbol{\nabla}\phi \times \widehat{\mathbf{e}}_z}{B}$ we have $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v} = -\frac{\boldsymbol{\nabla}^2 \phi}{B} \widehat{\mathbf{e}}_z$ or

$$\omega = -\boldsymbol{\nabla}^2 \phi / B \tag{1}$$

directed along the magnetic field line. The Laplacian of the electric potential $\Delta\phi$ is the electric charge density and we have the well known situation that a vorticity is formally equivalent to a density of electric charge. If this charge is quantitatively important, it must be taken it into account together with the currents

$$\frac{\partial \rho^V}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J} = 0 \tag{2}$$

where the "charge" $\boldsymbol{\nabla}^2 \phi = -\rho^V / \varepsilon_0$ is the *vorticity*

$$\rho^V = -\varepsilon_0 \omega B \tag{3}$$

We can estimate the magnitude of the charge density ρ^V. If the poloidal velocity has a spatial variation from $v_\theta = 0$ at the edge of the rotation layer and reaches amplitude of $\sim 10\ \ (km/s)$ on a radial extension of $10^{-2}\ (m)$ then $\omega \sim 10^7\ \ \left(s^{-1}\right)$ and this means

$$\rho^V \approx 3 \times 10^{-4}\ \ \left(C/m^3\right) \tag{4}$$

If the formation of this vorticity layer takes place on an interval controlled by the drift of the ions then

$$\delta t \sim \frac{\delta r}{v_{Di}} = \frac{10^{-2}\,(m)}{30\,(m/s)} = 3 \times 10^{-4}\,(s) \tag{5}$$

and the time variation of the charge is

$$\frac{\partial \rho^V}{\partial t} \sim \frac{\rho^V}{\delta t} = \frac{3 \times 10^{-4}\left(C/m^3\right)}{3 \times 10^{-4}\,(s)} = 1\left(\frac{A}{m^3}\right) \tag{6}$$

This must be compared with $\boldsymbol{\nabla} \cdot \mathbf{J}^I$. Taking the value estimated in the text $J^I \sim 10\ \ \left(A/m^2\right)$ and a spatial variation on the same radial extension $\delta r \sim 10^{-2}\,(m)$ we have $\left\| \boldsymbol{\nabla} \cdot \mathbf{J}^I \right\| \sim 10^3\left(A/m^3\right)$. This is much higher than the time derivative of the vorticity-charge, so we can neglect this latter component of the physical picture. We must remember however that $J^I \sim 10\ \ \left(A/m^2\right)$ is obtained for pellets while in other cases (*e.g.* neutrals penetrating from the edge) can be orders of magnitude smaller. Then we have to check the possibility to neglect the vorticity-charge.

10.2 The charge accumulation and the current induced by ionization

The rate of increase of the density of charge ρ^I by influx of the new ions in the region of unbalanced charge at the right end $\sim r_2$ of the segment of ionization (the charge layer I) is $d\rho^I/dt \sim |e|\,\dot{n}_{ioniz}\left[\Theta\left(r - r_2\right)\Theta\left(r_2 + \Delta^t - r\right)\right]\ \ \left(C/m^3/s\right)$, where Θ is the Heaviside function. The width of the layer I is Δ^t and the time scale to fill with newly born ions is $\delta t \sim \Delta^t / v_{Di}$. Using the Gauss law

$$\frac{\partial E^I}{\partial x} = \frac{1}{\varepsilon_0}\rho^I \tag{7}$$

we take the time derivation and integrate over x (actually $x \equiv r$ and we use x instead of r to underline the $1D$ geometry assumed here),

$$\frac{\partial E^I}{\partial t} = \Delta^t \frac{1}{\varepsilon_0} |e| \dot{n}_{ioniz} \left(\frac{V}{ms}\right) \tag{8}$$

According to the source of ionization (neutrals penetrating from the edge, pellets, etc.) $\left\| \partial E^I / \partial t \right\|$ can vary over an interval of three orders of magnitude $\sim 10^9 ... 10^{12} \left(\frac{V}{ms}\right)$. The static magnitude of E^I can be obtained from the surface charge density $\sigma^I = \rho^I \Delta^t$, as $E^I = \sigma^I / \varepsilon_0$.

The current density induced by ionization is calculated in the main text

$$J^I(r) \approx -\frac{1}{2} |e| \dot{n}_0^{ioniz} \left(\frac{\partial S}{\partial r}\right) \rho_i^2 q^2 \varepsilon^{-1/2} \tag{9}$$

However this calculation was adapted to a particular class of cases and it must be reconsidered for other cases.We just mention the order of magnitude $J^I(x_C, t) \approx 10 \ \left[A/m^2\right]$.

10.3 The charge accumulation and the current produced by the ion's drift of polarization

An ion in a constant magnetic field $\mathbf{B} = \widehat{\mathbf{e}}_z B$ performs the Larmor gyration in the transversal plane xOy, on a circle of radius ρ_i with frequency $\Omega_{ci} = |e| B/m_i$. When a constant electric field $\mathbf{E} = \widehat{\mathbf{e}}_x E$ is added, the circle is deformed into a curve of the cycloid type. Integrating the equation

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{|e| E}{m_i} \widehat{\mathbf{e}}_x + \frac{|e| B}{m_i} \frac{d\mathbf{r}}{dt} \times \widehat{\mathbf{e}}_z \tag{10}$$

we obtain

$$x(t) = \frac{1}{\Omega_{ci}^2} \frac{|e| E}{m_i} + \frac{v_{y0}}{\Omega_{ci}} + x_0 - \frac{1}{\Omega_{ci}} \left(\frac{1}{\Omega_{ci}} \frac{|e| E}{m_i} + v_{y0}\right) \cos(\Omega_{ci} t) + \frac{v_{x0}}{\Omega_{ci}} \sin(\Omega_{ci} t) \tag{11}$$

$$y(t) = y_0 - \frac{v_{x0}}{\Omega_{ci}} - \frac{1}{\Omega_{ci}} \frac{|e| E}{m_i} t + \frac{1}{\Omega_{ci}} \left(\frac{1}{\Omega_{ci}} \frac{|e| E}{m_i} + v_{y0}\right) \sin(\Omega_{ci} t) + \frac{v_{x0}}{\Omega_{ci}} \cos(\Omega_{ci} t) \tag{12}$$

10.4 The *cycloid-like* orbits of particles in the $\mathbf{E} \times \mathbf{B}$ field

The equations are

$$m_i \frac{d^2\mathbf{r}}{dt^2} = |e| \mathbf{E} + |e| \mathbf{v} \times \mathbf{B}$$

and

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

The fields are chosen as follows

$$
\begin{aligned}
\mathbf{B} &= \widehat{\mathbf{e}}_z B \\
\mathbf{E} &= \widehat{\mathbf{e}}_x E
\end{aligned}
$$

and we can write

$$
\frac{d^2\mathbf{r}}{dt^2} = \frac{|e|\,E}{m_i}\widehat{\mathbf{e}}_x + \frac{|e|\,B}{m_i}\frac{d\mathbf{r}}{dt} \times \widehat{\mathbf{e}}_z
$$

we introduce the notation

$$
\Omega_{ci} = \frac{|e|\,B}{m_i}
$$

and write the vector product

$$
\begin{aligned}
\frac{d\mathbf{r}}{dt} \times \widehat{\mathbf{e}}_z &= \left(\frac{dx}{dt}\widehat{\mathbf{e}}_x + \frac{dy}{dt}\widehat{\mathbf{e}}_y\right) \times \widehat{\mathbf{e}}_z \\
&= \frac{dx}{dt}\left(-\widehat{\mathbf{e}}_y\right) + \frac{dy}{dt}\widehat{\mathbf{e}}_x
\end{aligned}
$$

Then the equation becomes

$$
\begin{aligned}
\frac{d^2x}{dt^2} &= \frac{|e|\,E}{m_i} + \Omega_{ci}\frac{dy}{dt} \\
\frac{d^2y}{dt^2} &= -\Omega_{ci}\frac{dx}{dt}
\end{aligned}
$$

We can integrate the second

$$
\frac{dy}{dt} = -\Omega_{ci}\ x + \alpha
$$

where

$$
\alpha \equiv \text{const}
$$

and replace in the first equation

$$
\frac{d^2x}{dt^2} = \frac{|e|\,E}{m_i} - \Omega_{ci}^2 x + \Omega_{ci}\alpha
$$

or

$$
\frac{d^2x}{dt^2} + \Omega_{ci}^2 x = \frac{|e|\,E}{m_i} + \Omega_{ci}\alpha \equiv \gamma
$$

10.4.1 The representation of the solutions by periodic functions

Let us consider the equation for $x(t)$,

$$\frac{d^2x}{dt^2} + \Omega_{ci}^2 x = \frac{|e| E}{m_i} + \Omega_{ci}\alpha \equiv \gamma$$

The fundamental system of solutions is determined from the solution of the characteristic equation

$$\begin{aligned} \lambda^2 + \Omega_{ci}^2 &= 0 \\ \lambda_{1,2} &= \pm i\Omega_{ci} \end{aligned}$$

We adopt the form

$$x(t) = C(t)\exp(i\Omega_{ci}t) + D(t)\exp(-i\Omega_{ci}t)$$

then

$$\begin{aligned} &\frac{dx}{dt} \\ = \; &\frac{dC}{dt}\exp(i\Omega_{ci}t) + C(i\Omega_{ci})\exp(i\Omega_{ci}t) \\ &+\frac{dD}{dt}\exp(-i\Omega_{ci}t) + D(-i\Omega_{ci})\exp(-i\Omega_{ci}t) \end{aligned}$$

and according to standard method it is assumed the condition

$$\frac{dC}{dt}\exp(i\Omega_{ci}t) + \frac{dD}{dt}\exp(-i\Omega_{ci}t) = 0$$

Then

$$\frac{dx}{dt} = C(i\Omega_{ci})\exp(i\Omega_{ci}t) + D(-i\Omega_{ci})\exp(-i\Omega_{ci}t)$$

$$\begin{aligned} &\frac{d^2x}{dt^2} \\ = \; &\frac{dC}{dt}(i\Omega_{ci})\exp(i\Omega_{ci}t) + C(i\Omega_{ci})^2\exp(i\Omega_{ci}t) \\ &+\frac{dD}{dt}(-i\Omega_{ci})\exp(-i\Omega_{ci}t) + D(-i\Omega_{ci})^2\exp(-i\Omega_{ci}t) \end{aligned}$$

and it results

$$\begin{aligned} &\frac{dC}{dt}(i\Omega_{ci})\exp(i\Omega_{ci}t) + C(i\Omega_{ci})^2\exp(i\Omega_{ci}t) \\ &+\frac{dD}{dt}(-i\Omega_{ci})\exp(-i\Omega_{ci}t) + D(-i\Omega_{ci})^2\exp(-i\Omega_{ci}t) \\ &+\Omega_{ci}^2 C\exp(i\Omega_{ci}t) + \Omega_{ci}^2 D\exp(-i\Omega_{ci}t) \\ = \; &\gamma \end{aligned}$$

after cancellations

$$
\begin{aligned}
& \frac{dC}{dt}\left(i\Omega_{ci}\right)\exp\left(i\Omega_{ci}t\right)+\frac{dD}{dt}\left(-i\Omega_{ci}\right)\exp\left(-i\Omega_{ci}t\right) \\
= \quad & \gamma
\end{aligned}
$$

We now use the *condition* (constraint) assumed before

$$\frac{dD}{dt}\exp\left(-i\Omega_{ci}t\right)=-\frac{dC}{dt}\exp\left(i\Omega_{ci}t\right)$$

then

$$
\begin{aligned}
2\frac{dC}{dt}\left(i\Omega_{ci}\right)\exp\left(i\Omega_{ci}t\right) &= \gamma \\
\frac{dC}{dt} &= \frac{\gamma}{2i\Omega_{ci}}\exp\left(-i\Omega_{ci}t\right)
\end{aligned}
$$

This is integrated

$$
\begin{aligned}
C\left(t\right) &= \frac{\gamma}{2\left(i\Omega_{ci}\right)\left(-i\Omega_{ci}\right)}\exp\left(-i\Omega_{ci}t\right)+\beta \\
&= \frac{\gamma}{2\Omega_{ci}^{2}}\exp\left(-i\Omega_{ci}t\right)+\beta
\end{aligned}
$$

Now we use the constraint

$$\frac{dC}{dt}\exp\left(i\Omega_{ci}t\right)+\frac{dD}{dt}\exp\left(-i\Omega_{ci}t\right)=0$$

$$
\begin{aligned}
\frac{dD}{dt} &= -\frac{dC}{dt}\exp\left(2i\Omega_{ci}t\right) \\
&= -\left[\frac{\gamma}{2i\Omega_{ci}}\exp\left(-i\Omega_{ci}t\right)\right]\exp\left(2i\Omega_{ci}t\right) \\
&= -\frac{\gamma}{2i\Omega_{ci}}\exp\left(i\Omega_{ci}t\right)
\end{aligned}
$$

which is integrated

$$
\begin{aligned}
D\left(t\right) &= -\frac{\gamma}{2i\Omega_{ci}}\frac{1}{i\Omega_{ci}}\exp\left(i\Omega_{ci}t\right)+\delta \\
&= \frac{\gamma}{2\Omega_{ci}^{2}}\exp\left(i\Omega_{ci}t\right)+\delta
\end{aligned}
$$

With the two expressions for C and D we return to the solution

$$x\left(t\right)=C\left(t\right)\exp\left(i\Omega_{ci}t\right)+D\left(t\right)\exp\left(-i\Omega_{ci}t\right)$$

$$
\begin{aligned}
x\left(t\right) = & \left[\frac{\gamma}{2\Omega_{ci}^{2}}\exp\left(-i\Omega_{ci}t\right)+\beta\right]\exp\left(i\Omega_{ci}t\right) \\
& +\left[\frac{\gamma}{2\Omega_{ci}^{2}}\exp\left(i\Omega_{ci}t\right)+\delta\right]\exp\left(-i\Omega_{ci}t\right)
\end{aligned}
$$

$$\begin{aligned} x(t) &= \frac{\gamma}{\Omega_{ci}^2} \\ &+\beta \exp(i\Omega_{ci}t) + \delta \exp(-i\Omega_{ci}t) \end{aligned}$$

The time derivative is

$$\frac{dx}{dt} = \beta (i\Omega_{ci}) \exp(i\Omega_{ci}t) + \delta (-i\Omega_{ci}) \exp(-i\Omega_{ci}t)$$

Going to the other variable, we use

$$\frac{dy}{dt} = -\Omega_{ci}\ x + \alpha$$

and integrate

$$\begin{aligned} y(t) &= \int [(-\Omega_{ci}) x(t) + \alpha] dt + \sigma \\ &= \int dt \left[-\frac{\gamma}{\Omega_{ci}} - \Omega_{ci}\beta \exp(i\Omega_{ci}t) - \Omega_{ci}\delta \exp(-i\Omega_{ci}t) + \alpha\right] + \sigma \\ &= -\frac{\gamma}{\Omega_{ci}}t - \frac{\beta}{i} \exp(i\Omega_{ci}t) + \frac{\delta}{i} \exp(-i\Omega_{ci}t) + \alpha t + \sigma \end{aligned}$$

or

$$\begin{aligned} y(t) &= \sigma \\ &+ \left(-\frac{\gamma}{\Omega_{ci}} + \alpha\right) t \\ &-\frac{\beta}{i} \exp(i\Omega_{ci}t) + \frac{\delta}{i} \exp(-i\Omega_{ci}t) \end{aligned}$$

Due to the relationship

$$\gamma = \frac{|e| E}{m_i} + \Omega_{ci}\alpha$$

we see that

$$\begin{aligned} & -\frac{\gamma}{\Omega_{ci}} + \alpha \\ = & -\frac{1}{\Omega_{ci}} \frac{|e| E}{m_i} \end{aligned}$$

and α is cancelled.

The initial conditions The initial conditions are

$$\begin{aligned} x(t=0) &= x_0 \\ y(t=0) &= y_0 \end{aligned}$$

$$\begin{aligned} v_x(t=0) &= v_{x0} \\ v_y(t=0) &= v_{y0} \end{aligned}$$

Applying for x, we have

$$\begin{aligned} x(t=0) &= \frac{\gamma}{\Omega_{ci}^2} + \beta + \delta \\ &= x_0 \end{aligned}$$

Now we take the initial condition

$$y(t=0) = y_0$$

$$\sigma - \frac{\beta}{i} + \frac{\delta}{i} = y_0$$

And for the velocity

$$\begin{aligned} v_x(t=0) &= \frac{dx}{dt}(t=0) \\ &= \beta(i\Omega_{ci}) + \delta(-i\Omega_{ci}) \\ &= v_{x0} \end{aligned}$$

with the final form

$$\beta - \delta = \frac{v_{x0}}{i\Omega_{ci}}$$

For y velocity the condition is

$$\frac{dy}{dt}(t=0) = v_{y0}$$

Therefore

$$-\Omega_{ci}x_0 + \alpha = v_{y0}$$

We write again the system of four equations of initial conditions

$$\begin{aligned} \frac{\gamma}{\Omega_{ci}^2} + \beta + \delta &= x_0 \\ \sigma - \frac{\beta}{i} + \frac{\delta}{i} &= y_0 \\ \beta - \delta &= \frac{v_{x0}}{i\Omega_{ci}} \\ -\Omega_{ci}x_0 + \alpha &= v_{y0} \end{aligned}$$

The last equation leads to

$$\alpha = v_{y0} + \Omega_{ci}x_0$$

and we can now find also γ as

$$\gamma = \frac{|e|\, E}{m_i} + \Omega_{ci}\alpha = \frac{|e|\, E}{m_i} + \Omega_{ci}\left(v_{y0} + \Omega_{ci} x_0\right)$$

The second and the third equation lead to

$$\begin{aligned} \beta - \delta &= \frac{v_{x0}}{i\Omega_{ci}} \\ \beta - \delta &= i\sigma - iy_0 \end{aligned}$$

from where it results $i\sigma - iy_0 = -iv_{x0}/\Omega_{ci}$ or

$$\sigma = y_0 - \frac{v_{x0}}{\Omega_{ci}}$$

We note that we must expect real solutions $x(t)$ and $y(t)$. Then we can have β and γ complex numbers, but their sum must be real and their difference must be imaginary

$$\begin{aligned} \beta - \delta &\in \mathrm{Im}\,(C) \\ \beta + \delta &\in \mathrm{Re}\,(C) \end{aligned}$$

We assume a form of the two constants

$$\begin{aligned} \beta &\equiv \beta_1 + i\beta_2 \\ \delta &\equiv \delta_1 + i\delta_2 \end{aligned}$$

and we have

$$\begin{aligned} \beta_1 - \delta_1 &= 0 \\ &\quad \text{and} \\ \beta_2 + \delta_2 &= 0 \end{aligned}$$

This simplifies their form as

$$\begin{aligned} \beta &= \beta_1 + i\beta_2 \\ \delta &= \beta_1 - i\beta_2 \end{aligned}$$

$$\begin{aligned} \beta - \delta &= -i\frac{v_{x0}}{\Omega_{ci}} \\ (\beta_1 - \delta_1) + i\,(\beta_2 - \delta_2) &= -i\frac{v_{x0}}{\Omega_{ci}} \\ i2\beta_2 &= -i\frac{v_{x0}}{\Omega_{ci}} \end{aligned}$$

then

$$\beta_2 = -\frac{v_{x0}}{2\Omega_{ci}} = -\delta_2$$

and

$$\begin{aligned} \beta &= \beta_1 + i\left(-\frac{v_{x0}}{2\Omega_{ci}}\right) \\ \delta &= \beta_1 - i\left(-\frac{v_{x0}}{2\Omega_{ci}}\right) \end{aligned}$$

The first initial condition is

$$\frac{1}{\Omega_{ci}^2}\left(\frac{|e|\,E}{m_i} + \Omega_{ci}\alpha\right) + \beta + \delta = x_0$$

which can now be written

$$\frac{1}{\Omega_{ci}^2}\frac{|e|\,E}{m_i} + \frac{1}{\Omega_{ci}}\left(v_{y0} + \Omega_{ci}x_0\right) + 2\beta_1 = x_0$$

leading to

$$\beta_1 = -\frac{1}{2}\frac{1}{\Omega_{ci}}\left(\frac{1}{\Omega_{ci}}\frac{|e|\,E}{m_i} + v_{y0}\right)$$

Now we can reconstitute

$$\beta = -\frac{1}{2}\frac{1}{\Omega_{ci}}\left(\frac{1}{\Omega_{ci}}\frac{|e|\,E}{m_i} + v_{y0}\right) + i\left(-\frac{v_{x0}}{2\Omega_{ci}}\right)$$

and

$$\delta = -\frac{1}{2}\frac{1}{\Omega_{ci}}\left(\frac{1}{\Omega_{ci}}\frac{|e|\,E}{m_i} + v_{y0}\right) - i\left(-\frac{v_{x0}}{2\Omega_{ci}}\right)$$

Summary of results

$$\begin{aligned} \alpha &= v_{y0} + \Omega_{ci}x_0 \\ \beta &= -\frac{1}{2}\frac{1}{\Omega_{ci}}\left(\frac{1}{\Omega_{ci}}\frac{|e|\,E}{m_i} + v_{y0}\right) + i\left(-\frac{v_{x0}}{2\Omega_{ci}}\right) \\ \delta &= -\frac{1}{2}\frac{1}{\Omega_{ci}}\left(\frac{1}{\Omega_{ci}}\frac{|e|\,E}{m_i} + v_{y0}\right) - i\left(-\frac{v_{x0}}{2\Omega_{ci}}\right) \\ \sigma &= y_0 - \frac{v_{x0}}{\Omega_{ci}} \end{aligned}$$

In addition we have the notation

$$\gamma = \frac{|e|\,E}{m_i} + \Omega_{ci}\alpha = \frac{|e|\,E}{m_i} + \Omega_{ci}\left(v_{y0} + \Omega_{ci}x_0\right)$$

Using the four constants just calculated we have

$$\begin{aligned}
x(t) &= \frac{1}{\Omega_{ci}^2}\frac{|e|E}{m_i} + \frac{v_{y0}}{\Omega_{ci}} + x_0 \\
&\quad -\frac{1}{\Omega_{ci}}\left(\frac{1}{\Omega_{ci}}\frac{|e|E}{m_i} + v_{y0}\right)\cos(\Omega_{ci}t) \\
&\quad +\frac{v_{x0}}{\Omega_{ci}}\sin(\Omega_{ci}t)
\end{aligned}$$

and for $y(t)$ we have

$$\begin{aligned}
y(t) &= -\frac{\gamma}{\Omega_{ci}}t \\
&\quad -\frac{\beta}{i}\exp(i\Omega_{ci}t) + \frac{\delta}{i}\exp(-i\Omega_{ci}t) \\
&\quad +\alpha t + \sigma
\end{aligned}$$

$$\begin{aligned}
y(t) &= -\frac{1}{\Omega_{ci}}\left[\frac{|e|E}{m_i} + \Omega_{ci}(v_{y0} + \Omega_{ci}x_0)\right]t \\
&\quad +(v_{y0} + \Omega_{ci}x_0)t \\
&\quad +y_0 - \frac{v_{x0}}{\Omega_{ci}} \\
&\quad -\frac{1}{i}\left[-\frac{1}{2}\frac{1}{\Omega_{ci}}\left(\frac{1}{\Omega_{ci}}\frac{|e|E}{m_i} + v_{y0}\right) + i\left(-\frac{v_{x0}}{2\Omega_{ci}}\right)\right]\exp(i\Omega_{ci}t) \\
&\quad +\frac{1}{i}\left[-\frac{1}{2}\frac{1}{\Omega_{ci}}\left(\frac{1}{\Omega_{ci}}\frac{|e|E}{m_i} + v_{y0}\right) - i\left(-\frac{v_{x0}}{2\Omega_{ci}}\right)\right]\exp(-i\Omega_{ci}t)
\end{aligned}$$

Simplifications can be made

$$\begin{aligned}
y(t) &= y_0 - \frac{v_{x0}}{\Omega_{ci}} \\
&\quad -\frac{1}{\Omega_{ci}}\frac{|e|E}{m_i}t \\
&\quad +\frac{1}{\Omega_{ci}}\left(\frac{1}{\Omega_{ci}}\frac{|e|E}{m_i} + v_{y0}\right)\sin(\Omega_{ci}t) \\
&\quad +\frac{v_{x0}}{\Omega_{ci}}\cos(\Omega_{ci}t)
\end{aligned}$$

10.5 The time evolution of the average positions

Part 1. The electric field is constant in time Now we have the exact trajectories.

We want to calculate

$$\begin{aligned}
\overline{x}(t) &= \frac{1}{t}\int_0^t x(t')\,dt' \\
\overline{y}(t) &= \frac{1}{t}\int_0^t y(t')\,dt'
\end{aligned}$$

We start with $x(t)$.

$$\begin{aligned}
x(t) &= \frac{1}{\Omega_{ci}^2}\frac{|e|\,E}{m_i} + \frac{v_{y0}}{\Omega_{ci}} + x_0 \\
&\quad -\frac{1}{\Omega_{ci}}\left(\frac{1}{\Omega_{ci}}\frac{|e|\,E}{m_i} + v_{y0}\right)\cos(\Omega_{ci}t) \\
&\quad +\frac{v_{x0}}{\Omega_{ci}}\sin(\Omega_{ci}t)
\end{aligned}$$

$$x(t) = A + B\cos(\Omega_{ci}t) + C\sin(\Omega_{ci}t)$$

where

$$\begin{aligned}
A &\equiv \frac{1}{\Omega_{ci}^2}\frac{|e|\,E}{m_i} + \frac{v_{y0}}{\Omega_{ci}} + x_0 \\
B &\equiv -\frac{1}{\Omega_{ci}}\left(\frac{1}{\Omega_{ci}}\frac{|e|\,E}{m_i} + v_{y0}\right) \\
C &\equiv \frac{v_{x0}}{\Omega_{ci}}
\end{aligned}$$

Then the integral is carried out

$$\begin{aligned}
\overline{x}(t) &= \frac{1}{t}\int_0^t \left[A + B\cos(\Omega_{ci}t') + C\sin(\Omega_{ci}t')\right]dt' \\
&= \frac{1}{t}\left[At + \frac{B}{\Omega_{ci}}\,\sin(\Omega_{ci}t)\big|_0^t - \frac{C}{\Omega_{ci}}\,\cos(\omega_{ci}t)\big|_0^t\right]
\end{aligned}$$

This leads to

$$\begin{aligned}
\overline{x}(t) &= \frac{1}{\Omega_{ci}^2}\frac{|e|\,E}{m_i} + \frac{v_{y0}}{\Omega_{ci}} + x_0 \\
&\quad -\frac{1}{\Omega_{ci}}\left(\frac{1}{\Omega_{ci}}\frac{|e|\,E}{m_i} + v_{y0}\right)\frac{\sin(\Omega_{ci}t)}{\Omega_{ci}t} \\
&\quad -\frac{v_{x0}}{\Omega_{ci}}\frac{\cos(\Omega_{ci}t) - 1}{\Omega_{ci}t}
\end{aligned}$$

We note that the limit $t \to 0$ is

$$\begin{aligned}
\overline{x}(t \to 0) &= \frac{1}{\Omega_{ci}^2}\frac{|e|\,E}{m_i} + \frac{v_{y0}}{\Omega_{ci}} + x_0 \\
&\quad -\frac{1}{\Omega_{ci}}\left(\frac{1}{\Omega_{ci}}\frac{|e|\,E}{m_i} + v_{y0}\right) \\
&= x_0
\end{aligned}$$

as expected.

At the large-t limit we find

$$\overline{x}\left(t\rightarrow\infty\right)=\frac{1}{\Omega_{ci}^{2}}\frac{\left|e\right|E}{m_{i}}+\frac{v_{y0}}{\Omega_{ci}}+x_{0}$$

Going to $y\left(t\right)$ we note

$$\begin{aligned}
y\left(t\right) &= y_{0}-\frac{v_{x0}}{\Omega_{ci}}\\
&-\frac{1}{\Omega_{ci}}\frac{\left|e\right|E}{m_{i}}t\\
&+\frac{1}{\Omega_{ci}}\left(\frac{1}{\Omega_{ci}}\frac{\left|e\right|E}{m_{i}}+v_{y0}\right)\sin\left(\Omega_{ci}t\right)\\
&+\frac{v_{x0}}{\Omega_{ci}}\cos\left(\Omega_{ci}t\right)
\end{aligned}$$

which we rewrite as

$$y\left(t\right)=U+Vt+W\sin\left(\Omega_{ci}t\right)+Z\cos\left(\Omega_{ci}t\right)$$

with the notations

$$\begin{aligned}
U &\equiv y_{0}-\frac{v_{x0}}{\Omega_{ci}}\\
V &\equiv -\frac{1}{\Omega_{ci}}\frac{\left|e\right|E}{m_{i}}\\
W &\equiv \frac{1}{\Omega_{ci}}\left(\frac{1}{\Omega_{ci}}\frac{\left|e\right|E}{m_{i}}+v_{y0}\right)\\
Z &\equiv \frac{v_{x0}}{\Omega_{ci}}
\end{aligned}$$

And

$$\begin{aligned}
\overline{y}\left(t\right) &= \frac{1}{t}\int_{0}^{t}dt'\left[U+Vt'+W\sin\left(\Omega_{ci}t'\right)+Z\cos\left(\Omega_{ci}t'\right)\right]\\
&= \frac{1}{t}\left[Ut+\frac{1}{2}Vt^{2}-\frac{W}{\Omega_{ci}}\left.\cos\left(\Omega_{ci}t\right)\right|_{0}^{t}+\frac{Z}{\Omega_{ci}}\left.\sin\left(\Omega_{ci}t\right)\right|_{0}^{t}\right]\\
&= U+\frac{V}{2}t\\
&\quad -W\frac{\cos\left(\Omega_{ci}t\right)-1}{\Omega_{ci}t}+Z\frac{\sin\left(\Omega_{ci}t\right)}{\Omega_{ci}t}
\end{aligned}$$

In explicit form it is

$$\begin{aligned}
\overline{y}(t) &= y_0 - \frac{v_{x0}}{\Omega_{ci}} \\
&+\frac{1}{2}\left(-\frac{1}{\Omega_{ci}}\frac{|e|E}{m_i}\right)t \\
&-\frac{1}{\Omega_{ci}}\left(\frac{1}{\Omega_{ci}}\frac{|e|E}{m_i}+v_{y0}\right)\frac{\cos(\Omega_{ci}t)-1}{\Omega_{ci}t} \\
&+\frac{v_{x0}}{\Omega_{ci}}\frac{\sin(\Omega_{ci}t)}{\Omega_{ci}t}
\end{aligned}$$

We note that the limit $t \to 0$ we get

$$\begin{aligned}
\overline{y}(t\to 0) &= y_0 - \frac{v_{x0}}{\Omega_{ci}} + \frac{v_{x0}}{\Omega_{ci}} \\
&= y_0
\end{aligned}$$

as expected.

Also, we see that for small t the velocity of the average position is equal to the velocity of the position itself, since the third line is

$$\begin{aligned}
&-\frac{1}{\Omega_{ci}}\left(\frac{1}{\Omega_{ci}}\frac{|e|E}{m_i}+v_{y0}\right)\frac{\cos(\Omega_{ci}t)-1}{\Omega_{ci}t} \\
\approx &-\frac{1}{\Omega_{ci}}\left(\frac{1}{\Omega_{ci}}\frac{|e|E}{m_i}+v_{y0}\right)\frac{1}{2}(\Omega_{ci}t) \\
= &-\left(\frac{1}{\Omega_{ci}}\frac{|e|E}{m_i}+v_{y0}\right)\frac{1}{2}t
\end{aligned}$$

Combining with the rest

$$\begin{aligned}
\overline{y}(t) &= y_0 - \frac{v_{x0}}{\Omega_{ci}} \\
&+\frac{1}{2}\left(-\frac{1}{\Omega_{ci}}\frac{|e|E}{m_i}\right)t \\
&-\left(\frac{1}{\Omega_{ci}}\frac{|e|E}{m_i}+v_{y0}\right)t \\
&+\frac{v_{x0}}{\Omega_{ci}}
\end{aligned}$$

$$\overline{y}(t) = y_0 - v_{y0}t + \left(-\frac{1}{\Omega_{ci}}\frac{|e|E}{m_i}\right)t$$

The large t domain is

$$\overline{y}(t\to\infty) = y_0 - \frac{v_{x0}}{\Omega_{ci}} + \frac{1}{2}\left(-\frac{1}{\Omega_{ci}}\frac{|e|E}{m_i}\right)t$$

Important NOTE

The velocity of the average position on y is two times slower than the velocity of the variable $y(t)$ itself.

Part 2. The electric field is time-dependent Now consider

$$\mathbf{E} = \mathbf{E}(t)$$

We calculate the average of the successive positions of the particle

$$\begin{aligned} x(t) &= \frac{1}{2i\Omega_{ci}} \frac{|e|}{m_i} \int_0^t dt' E(t') \{\exp[i\Omega_{ci}(t-t')] - \exp[-i\Omega_{ci}(t-t')]\} \\ &+ \frac{\alpha}{\Omega_{ci}} \\ &+ \beta \exp(i\Omega_{ci} t) + \delta \exp(-i\Omega_{ci} t) \end{aligned}$$

and we must calculate

$$\overline{x}(t) = \frac{1}{t} \int_0^t dt' x(t')$$

We note that the limit $t \to 0$ is

$$\begin{aligned} \overline{x}(t \to 0) &= \frac{1}{\Omega_{ci}^2} \frac{|e| E}{m_i} + \frac{v_{y0}}{\Omega_{ci}} + x_0 \\ &- \frac{1}{\Omega_{ci}} \left(\frac{1}{\Omega_{ci}} \frac{|e| E}{m_i} + v_{y0} \right) \\ &= x_0 \end{aligned}$$

as expected.

At the large-t limit we find

$$\overline{x}(t \to \infty) = \frac{1}{\Omega_{ci}^2} \frac{|e| E}{m_i} + \frac{v_{y0}}{\Omega_{ci}} + x_0$$

10.6 The study of displacements

Now one of the methods to see the displacement is

find the first $x(t)$ where the velocity on x (i.e. vertical) is zero, which means that it is parallel with y. Then find the y at this point.

Then

$$\begin{aligned} \frac{dx}{dt} &= \left(\frac{1}{\Omega_{ci}} \frac{|e| E}{m_i} + v_{y0} \right) \sin(\Omega_{ci} t) + v_{x0} \cos(\Omega_{ci} t) \\ &= 0 \end{aligned}$$

for

$$\tan(\Omega_{ci}t_0) = -\frac{v_{x0}}{\frac{1}{\Omega_{ci}}\frac{|e|E}{m_i} + v_{y0}}$$

or

$$t_0 = -\frac{1}{\Omega_{ci}}\arctan\left(\frac{v_{x0}}{\frac{1}{\Omega_{ci}}\frac{|e|E}{m_i} + v_{y0}}\right)$$

from where we find t. We now introduce this t in the expression of $y(t)$ and obtain

$$\begin{aligned} y(t_0) &= -\frac{v_{x0}}{\Omega_{ci}} \\ &\quad -\frac{1}{\Omega_{ci}}\frac{|e|E}{m_i}t_0 \\ &\quad +\frac{1}{\Omega_{ci}}\left(\frac{1}{\Omega_{ci}}\frac{|e|E}{m_i} + v_{y0}\right)\sin(\Omega_{ci}t_0) \\ &\quad +\frac{v_{x0}}{\Omega_{ci}}\cos(\Omega_{ci}t_0) \end{aligned}$$

We can use the tan function to calculate the two other functions

$$\cos(\Omega_{ci}t_0) = \frac{1}{\sqrt{1+\left(-\frac{v_{x0}}{\frac{1}{\Omega_{ci}}\frac{|e|E}{m_i}+v_{y0}}\right)^2}} = \frac{\frac{1}{\Omega_{ci}}\frac{|e|E}{m_i} + v_{y0}}{\sqrt{\left(\frac{1}{\Omega_{ci}}\frac{|e|E}{m_i} + v_{y0}\right)^2 + v_{x0}^2}}$$

$$\begin{aligned} \sin(\Omega_{ci}t_0) &= \sqrt{1-[\cos(\Omega_{ci}t_0)]^2} \\ &= \sqrt{1-\frac{\left(\frac{1}{\Omega_{ci}}\frac{|e|E}{m_i} + v_{y0}\right)^2}{\left(\frac{1}{\Omega_{ci}}\frac{|e|E}{m_i} + v_{y0}\right)^2 + v_{x0}^2}} \\ &= \frac{v_{x0}}{\sqrt{\left(\frac{1}{\Omega_{ci}}\frac{|e|E}{m_i} + v_{y0}\right)^2 + v_{x0}^2}} \end{aligned}$$

Inserting

$$\begin{aligned} y(t_0) &= -\frac{v_{x0}}{\Omega_{ci}} \\ &\quad +\frac{1}{\Omega_{ci}^2}\frac{|e|E}{m_i}\arctan\left(\frac{v_{x0}}{\frac{1}{\Omega_{ci}}\frac{|e|E}{m_i} + v_{y0}}\right) \\ &\quad +\frac{1}{\Omega_{ci}}\left(\frac{1}{\Omega_{ci}}\frac{|e|E}{m_i} + v_{y0}\right)\frac{v_{x0}}{\sqrt{\left(\frac{1}{\Omega_{ci}}\frac{|e|E}{m_i} + v_{y0}\right)^2 + v_{x0}^2}} \\ &\quad +\frac{v_{x0}}{\Omega_{ci}}\frac{\frac{1}{\Omega_{ci}}\frac{|e|E}{m_i} + v_{y0}}{\sqrt{\left(\frac{1}{\Omega_{ci}}\frac{|e|E}{m_i} + v_{y0}\right)^2 + v_{x0}^2}} \end{aligned}$$

$$\mathbf{r}_0(t) = (x, y)$$

$$\begin{aligned} x &= a\phi - b\sin\phi \\ y &= a - b\cos\phi \end{aligned}$$

However in the **book Motion of charged plasma particles** at page 254 it is said that this is a *trochoid*.

In the **book A catalog of special polane curves Lawrence** at page 154 it is shown a *epitrochoid*.

[height=10cm]Figure-6.eps

Figure 1: The deformation of the pure gyration orbit (red) into a trochoid (blue) under the effect of an electric field. An very high value of the electric field (350 kV/m) was used in order to make more visible the deformation. The plasma center is at right.

This curve (see Figure 1) is a *prolate trochoid*. The electric $E \times B$ motion is along the negative y axis. We adopt initial conditions ($x_0 = -\rho_i$, $y_0 = 0$, $v_{x0} = 0$, $v_{y0} = v_{th,i}$) that are identical for the static $E \neq 0$ as well as for pure gyration ($E \equiv 0$). In this way we can see how the *trochoid* is different of the Larmor circle. The orbit has, broadly, two unequal lobes. This asymmetry makes that the "center" of the positions of the particle to be shifted relative to the one of the pure Larmor gyration. For ions the shift is in the direction of the electric field $\mathbf{E}^I$. The ions are now more frequently present to the left of the symmetry axis of the previously symmetric (circle) orbit. There is an effective concentration of ions at the end of the interval on r to which points the electric field produced by the "external" $\mathbf{E}^I$ (generated by ionization). For electrons there is a shift to the opposite direction but much smaller and will be neglected.

Therefore for a given $\mathbf{E}^I$ there is an excess of ion charge at the left ($\sim r_1$) end of the ionization domain. A new layer (called L) of positive charge is generated at the left end, opposite to the ionization-induced layer I. The width δx^L is the amount of deformation relative to the pure Larmor gyration orbit, *i.e.* the distance between the center of the *prolate trochoid* and the center of the Larmor circle, when both trajectories start from the same initial conditions, but with $E \neq 0$ respectively $E = 0$. To find it, we calculate the time evolution of the averages

$$\overline{x}(t) = \frac{1}{t}\int_0^t x(t')\,dt' \quad , \quad \overline{y}(t) = \frac{1}{t}\int_0^t y(t')\,dt' \tag{13}$$

For large t

$$\overline{x}(t \to \infty) = \frac{1}{\Omega_{ci}^2}\frac{|e|\,E}{m_i} + \frac{v_{y0}}{\Omega_{ci}} + x_0 = \frac{1}{\Omega_{ci}B}E \tag{14}$$

and

$$\overline{y}(t \to \infty) = y_0 - \frac{v_{x0}}{\Omega_{ci}} + \frac{1}{2}\left(-\frac{1}{\Omega_{ci}}\frac{|e|\,E}{m_i}\right)t = -\frac{E}{2B}t \tag{15}$$

The *center* of the new orbit has a shift $\delta x^L = \overline{x}(t \to \infty) = \frac{1}{\Omega_{ci} B} E$. The electric field that occurs in the above equation is the *internal* field E^{int}, *i.e.* the ionization-induced field E^I from which we substract the field generated by the new layer L, $E^{int} = E^I - E^L$. Using the shift $\delta x^L = \frac{1}{\Omega_{ci} B} E^{int}$ the surface charge density in the layer L is

$$\sigma^L = |e| \, n^{bg} \frac{1}{\Omega_{ci} B} E^{int} \quad \left(\frac{C}{m^2} \right) \tag{16}$$

The electric field produced by the deformation of the Larmor gyration is

$$E^L = \frac{\sigma^L}{\varepsilon_0} = \frac{1}{\varepsilon_0} |e| \, n^{bg} \frac{1}{\Omega_{ci} B} E^{int} = \frac{c^2}{v_A^2} E^{int} \tag{17}$$

from where we find $E^{int} = E^I - E^L = E^I - \frac{c^2}{v_A^2} E^{int}$, or

$$E^{int} = \frac{E^I}{1 + c^2/v_A^2} \tag{18}$$

If the "external", ionization-induced, electric field E^I continues to increase, there is *increase* in time of the deformation of the trochoid

$$\frac{d}{dt} \overline{x}(t \to \infty) = \frac{1}{\Omega_{ci} B} \frac{dE^{int}}{dt} \tag{19}$$

which is precisely the *polarization drift* of the ions

$$v_{Di}^{(pol)} = \frac{1}{\Omega_{ci} B} \frac{dE^{int}}{dt} \tag{20}$$

i.e. the drift of polarization simply consists of the time variation of the deformation $v_{Di}^{(pol)} = \frac{d}{dt} \overline{x}(t \to \infty)$.

The velocity of the polarization drift of the background ions (of density n^{bg})

$$v_{Di}^{(pol)} = \frac{1}{\Omega_{ci} B} \frac{1}{1 + c^2/v_A^2} \frac{dE^I}{dt} \approx \frac{\varepsilon_0}{|e| \, n^{bg}} \frac{dE^I}{dt} \tag{21}$$

is in general much smaller than the first order drift v_E and than the neoclassical drift v_{Di}. The estimated magnitude varies between $v_{Di}^{(pol)} \sim 0.02...10 \;\; (m/s)$ according to $\dot{n}_{ioniz}$ is determined by slow gas input or pellets. We would be tempted to expect a slower response of the background ions. The build-up of the charge layer induced by ionization is $\delta t = \Delta^t / v_{Di}$ while the build up of the charge layer induced by polarization drift is $\delta t^{(pol)} = \delta x^L / v_{Di}^{(pol)}$. However these two time scales are identical. Using Eqs.(19-20) and (18)

$$\delta t^{(pol)} = \frac{\delta x^L}{v_{Di}^{(pol)}} = \left(\frac{d}{dt} \ln E^I \right)^{-1} \tag{22}$$

and inserting Eq.(8) and $\delta t = \Delta^t / v_{Di}$ we find

$$\delta t^{(pol)} = \delta t \tag{23}$$

Using again Eq.(8) we obtain

$$v_{Di}^{(pol)} = \frac{1}{\Omega_{ci} B} \frac{1}{1 + c^2/v_A^2} \Delta^t \frac{1}{\varepsilon_0} |e| \dot{n}_{ioniz} = \Delta^t \frac{\dot{n}_{ioniz}}{n^{bg}} \tag{24}$$

This can be translated in the language of currents. By definition

$$J^{(pol)} = |e| \, n^{bg} v_{Di}^{(pol)} \tag{25}$$

and

$$J^I = |e| \, \Delta^t \dot{n}_{ioniz} \tag{26}$$

and Eq.(24) shows that the polarization current $J^{(pol)}$ is equal and opposite to the "externally" imposed current J^I.

$$J^I = J^{(pol)} \tag{27}$$

The plasma response $J^{(pol)}$ is the return current, involving the background ions.

11 The polarization response through inertial drift of ions

This is also in *density enhanced confinement*.

In the paper **Particle motion.pdf** (from a Russian book) it is derived the *polarization drift* for the particles generated by the time-variation of a force that acts on the particles

$$\mathbf{v}_d^{(1)} = \frac{1}{m\Omega_c^2} \frac{d\mathbf{F}}{dt}$$

where the force can be

$$\mathbf{F}_\perp = Ze\mathbf{E}_\perp$$

then

$$\mathbf{v}_d^{(1)} = \frac{1}{\Omega_c B} \frac{d\mathbf{E}}{dt}$$

This is called by them *inertial drift*.

The direction of drift is *opposite* for electrons and ions (it can produce current).

The magnitude of the drift for the ions is much larger than for the electrons (due to Ω_c in the denominator).

$$\mathbf{v}_{d,i}^{(1)} = \frac{m_i}{|e|\, B^2} \frac{d\mathbf{E}}{dt}$$

Note that the derivative is Lagrangian.

Then at a variation of the electric field the first to react are the ions.

The current is

$$\begin{aligned}
\mathbf{j}_d^{(1)} &= n\,|e| \left(\mathbf{v}_{di}^{(1)} - \mathbf{v}_{de}^{(1)} \right) \\
&\approx n\,|e|\,\mathbf{v}_{di}^{(1)} \\
&= \frac{nm_i}{B^2} \frac{\partial \mathbf{E}}{\partial t} \quad \left(\text{after introducing the expression for } v_{di}^{(1)} \right)
\end{aligned}$$

To this current one has to add the *vacuum* current

$$\mu_0 \mathbf{j}_{vac}^{ind} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

and results

$$\begin{aligned}
\mathbf{j} &= \mathbf{j}_d^{(1)} + \mathbf{j}_{vac}^{ind} \\
&= \frac{nm_i}{B^2} \frac{\partial \mathbf{E}}{\partial t} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \varepsilon_0 \left(1 + \frac{1}{\varepsilon_0} \frac{1}{\mu_0} \frac{\mu_0 n m_i}{B^2} \right) \frac{\partial \mathbf{E}}{\partial t} \\
&= \varepsilon_0 \left(1 + \frac{c^2}{v_A^2} \right) \frac{\partial \mathbf{E}}{\partial t}
\end{aligned}$$

since

$$\frac{1}{\varepsilon_0 \mu_0} = c^2 \quad \text{and} \quad \frac{B^2}{\mu_0 n m_i} = v_A^2$$

The Alfven speed is

$$\begin{aligned}
v_A &= \frac{B}{\sqrt{\mu_0 \rho}} \\
\text{where } \rho &= \text{density, } \frac{kg}{m^3}
\end{aligned}$$

Then

$$\begin{aligned}
\frac{m}{s} &= \frac{T}{\sqrt{\frac{H}{m} \frac{kg}{m^3}}} \\
\frac{m^2}{s^2} &= \frac{T^2}{\frac{H}{m} \frac{kg}{m^3}}
\end{aligned}$$

The main result of this analysis is rather strange:

a variation in time of the electric field $\partial \mathbf{E}/\partial t$ leads to a drift motion of electrons and ions.

This drift is slow since it is higher order, it actually is a *polarization* drift, it has B^2 at the denominator.

In addition this drift of polarization contains as factor the *mass* therefore the electrons have a drift of polarization that is much smaller than the drift of polarization of the ions and usually is neglected.

At a time variation of the electric field the first to respond are the *ions* and the response is a slow motion in the same direction as the derivative of the electric field

$$\mathbf{v}_d^{(1)} \text{ has the same sign as the time-variation } \frac{\partial \mathbf{E}}{\partial t}$$

For example, during the ionization process new ions are moving radially outward (say) and produce an accumulation of ions at the *right* end of the ionization interval, which we can suppose it is close to the last magnetic surface. The electric field is radial and directed from the last surface inward, *i.e.* toward the plasma center. This electric field increases steadly due to the steady accumulation of new ions. The direction of $\partial \mathbf{E}/\partial t$ is inward. So must be the inertial-polarization drift of the ions $\mathbf{v}_{di}^{(1)}$, *i.e.* the drift is inward.

Then the inertial drift is *opposite* to the current of the new ions.

This is a polarization current.

It involves *all* ions, including the background ones.

But the velocity is small.

It is a polarization response.

Therefore the current that acts to reduce the electric field $\mathbf{E}$ that has the tendency to increase (by accumulation of newly created ions) is due to the ion polarization drift and reduces this electric field.

What remains from the electric field is

$$\begin{aligned} E_p &= \frac{1}{1 + c^2/v_A^2} E \\ &= \frac{E}{\varepsilon/\varepsilon_0} \quad \text{very small} \end{aligned}$$

See also **Honda**.

12 Neoclassical radial E_r Wang 2001

The flux is the sum of

- neoclassical diffusion [scattering with space step $\sim \rho_{i\theta}$], and

- polarization current

$$\begin{aligned}\langle(\mathbf{j}_{diff}+\mathbf{j}_{pol})\cdot\boldsymbol{\nabla}\psi\rangle &= e\Gamma_i \\ &\equiv e\int d^3v\ (\mathbf{v}_D\cdot\boldsymbol{\nabla}\psi)\ f_i\end{aligned}$$

In the limit of small banana orbit the transport is ambipolar in second order in

$$\frac{\rho_{i\theta}}{L_n}$$

The electric field is determined from the *toroidal angular momentum flux*

$$\Pi_i=\left\langle\int d^3v\ I\frac{v_\parallel}{\Omega_i}\ (\mathbf{v}_D\cdot\boldsymbol{\nabla}\psi)\ f\right\rangle$$

Saturation is the vanishing of the toroidal angular momentum transport.

$$\ln\frac{\widetilde{n}}{n_i}-\frac{e\phi}{T_i}=\text{const}\times\psi$$

(due to the $\partial^2/\partial\psi^2$ operator in $\Pi_i=0$).

13 Neoclassical polarization Robertson Hinton

Neoclassical dielectric Robertson Hinton.

13.1 Basic physics of neoclassical polarization

Consider
- loss of ions from plasma
- loss of runaways

Then there is

$$E_r(t)$$

and one must assume that there is time variation of this field.

The field produces a toroidal velocity

$$u_\varphi=\frac{E_r}{B_\theta}$$

and the toroidal velocity must also have time variation $u_\varphi(t)$.

Angular momentum input with the same direction as the current (*co-injection*, it refers to NBI) results in increase of the radial electric field (because of the increase of the toroidal velocity u_φ and the direction of B_θ: $E_r\widehat{\mathbf{e}}_r=-\mathbf{u}\times\mathbf{B}_\theta$, where B_θ is directed from up to down (at the equatorial plane) when the current flows from the reader away, with $u\widehat{\mathbf{e}}_\varphi$ directed in the same direction, which produces a radial E_r outward relative to the magnetic surface.

$$\text{co-injection} \rightarrow \frac{\partial E_r}{\partial t} > 0$$

and for *counter-injection*, decrease of E_r,

$$\text{counter-injection} \rightarrow \frac{\partial E_r}{\partial t} < 0$$

Therefore the NBI co-injection means modification of the angular momentum content of the plasma.

This requires a torque acting on plasma.

The torque can be generated by a radial current $j_r\widehat{\mathbf{e}}_r$ flowing outward through the magnetic surfaces.

From the momentum balance

$$\rho\frac{\partial u_\varphi}{\partial t} = j_r B_\theta$$

Then

$$j_r = \frac{c^2}{B_\theta^2/\rho}\frac{\partial E_r}{\partial t}$$

Introducing

$$v_{A\theta}^2 = \frac{B_\theta^2}{\mu_0\rho}$$

j_r is the *neoclassical* polarization current.

The Maxwell equation (Ampere's law) must involve the *total* current, including

- an external current j^{ext} and
- the *induction* one, j_r

$$0 = \mu_0 j_r^{tot} + \frac{1}{c^2}\frac{\partial E_r}{\partial t}$$

since $(\boldsymbol{\nabla}\times\mathbf{B})_r$ averaged over surface must be zero.

The external radial current j^{ext} is due to "external" causes (like NBI) and is at the origin of the change of the angular momentum.

$$\mu_0\left(j_r^{ext} + j_r\right) = -\frac{1}{c^2}\frac{\partial E_r}{\partial t}$$

where $c^2 = \frac{1}{\varepsilon_0\mu_0}$ and $j_r = \frac{c^2}{B_\theta^2/\rho}\frac{\partial E_r}{\partial t}$. It results

$$\frac{\partial E_r}{\partial t} = -\frac{1}{\varepsilon_0\left(1+\frac{c^2}{v_{A\theta}^2}\right)}j_r^{ext}$$

This defines the neoclassical dielectric constant

$$\varepsilon^{neo} = \varepsilon_0 \left(1 + \frac{c^2}{v_{A\theta}^2}\right)$$

and

$$\varepsilon_0 \left(1 + \frac{c^2}{v_{A\theta}^2}\right) \frac{\partial E_r}{\partial t} = -j_r^{ext}$$

What are the elements of this derivation

- the plasma velocity is only *toroidal*. This is the reason that only the *poloidal* magnetic field is involved;

- the radial current produced by an external source (NBI fast ion motion), j_r^{ext}, is the cause for the change of the plasma *angular momentum*;

- the condition $\langle (\mathbf{\nabla} \times \mathbf{B})_r \rangle = 0$ is imposed

Objective

- show that the trapped ions have a polarization current

- show that the untrapped ions, due to collisions with trapped ones, will also produce radial polarization current

Note that electrons, due to their small deviation from the surface (produced by the neoclassical drift) can be neglected.

The polarization current of ions is closely related with the *banana toroidal precession*.

The trapped ion precession (neoclassical effect) is caused by the work done by E_r during the radial motion of an ion along the banana orbit.

The change of $E_r(t)$ is an acceleration of the precession, in the follwoing way: the work done along the first half of banana is different from the work done along the second half.

For untrapped ions the effect exists but

- mainly for those that are close to the boundary with trapped, and

- the deviation relative to the surface is small for circulating ions

Trapped ions

$$\begin{aligned} v_\theta &= u \frac{B_\theta}{B} \quad \text{(poloidal projection of parallel } \mathbf{u}\text{)} \\ &\quad -\frac{E_r}{B} \quad \left(\frac{E_r}{B_\theta} \times \frac{B_\theta}{B} = \text{poloidal projection}\right) \\ &\quad -v_D \cos\theta \quad \text{(neo drift)} \end{aligned}$$

where

$$v_D = \frac{1}{\Omega}\left(\frac{v_\perp^2}{2} + u^2\right)$$

Now, for trapped ions, one can take the average over the bounce. Then

$$\overline{v}_\theta = 0$$

from where

$$\overline{u} = \frac{E_r}{B_\theta} + \frac{B}{B_\theta}\overline{v_D \cos\theta}$$

First note: the parallel velocity of the banana (i.e. averaged over bounce, like a "guiding centre") is *not* 0.

It is the toroidal precession (with the approximation that u is actually parallel).

(no magnetic shear, E_r is non-zero)

The definition of the toroidal φ component of the magnetic potential is modified by the time variation of the electric field

$$\begin{aligned} RA_\varphi(r,t) &= -R_0 \int^r dr' \ B_\theta(r') \\ &\quad -R_0 \int^t dt' \ E_\varphi(t') \end{aligned}$$

The last term results from $E_\varphi = -\frac{\partial A_\varphi}{\partial t}$, integrated on time. Axisymmetry exclude charge accumulation on φ therefore $-\nabla_\varphi \phi = 0$.

The φ component of the magnetic potential contributes to the generalized momentum, conserved

$$e\ R\ A_\varphi + m\ R\ u = \text{const}$$

The first term is replaced in the expression of A_φ,

$$\left(-\frac{m}{e}Ru + \text{const}\right) = -R_0 \int^r dr' \ B_\theta(r') - R_0 \int^t dt' \ E_\varphi(t')$$

where

$$\begin{aligned} R &= R_0 h \\ h &= 1 + \frac{r}{R_0}\cos\theta \end{aligned}$$

and this equation is differentiated to t,

$$\frac{d}{dt}\left(-\frac{m}{e}hu\right) = -B_\theta(r)\frac{dr}{dt} - E_\varphi(t)$$

$$\frac{dr}{dt} = \frac{1}{(eB_\theta)/m}\frac{d}{dt}(hu) - \frac{E_\varphi}{B_\theta}$$

Now one calculates the time average over the bounce of trapped ions

$$\begin{aligned}
\overline{v}_r &= \overline{\frac{dr}{dt}} \\
&= \frac{1}{(eB_\theta)/m} \overline{\frac{d}{dt}(u)} - \frac{E_\varphi}{B_\theta} \\
&\quad \text{(after taking } h \approx 1\text{)}
\end{aligned}$$

For $\overline{u}$ we have a formula derived above

$$\overline{u} = \frac{E_r}{B_\theta} + \frac{B}{B_\theta}\overline{v_D \cos\theta}$$

When this is time-differentiated only the first term remains

$$\frac{d\overline{u}}{dt} = \frac{1}{B_\theta}\frac{\partial E_r}{\partial t}$$

then

$$\begin{aligned}
\overline{v}_r &= \frac{1}{(eB_\theta)/m}\frac{1}{B_\theta}\frac{\partial E_r}{\partial t} \quad \text{neoclassical polarization} \\
&\quad - \frac{E_\varphi}{B_\theta} \quad \text{Ware pinch of bananas}
\end{aligned}$$

To extend the considerations on neoclassical polarization one studies now the difference between the two halves of the banana.

During the time spent by the particle to travel on first half, the conditions on the second half have been modified.

13.1.1 The first half of the banana

This is the part above the equatorial plane.

The energy variation is due to the work done by the electric field.

$$\frac{d}{dt}\left(\frac{mu^2}{2} + \mu B\right) = e\left(u\widehat{\mathbf{n}} + \mathbf{v}_D\right)\cdot\mathbf{E}$$

The two limit points are

$$\frac{mu_2^2}{2} + \mu B_2 - \frac{mu_1^2}{2} - \mu B_1 = e\int_{t_1}^{t_2} dt\ v_r\ E_r(t)$$

where

$$v_r = \frac{dr}{dt}$$

The points 1 and 2 are chosen to be on the equatorial plane, $\theta = 0$, 1 is exterior, 2 is interior.

$$B_2 = B_1 + \frac{B_0}{R_0}(r_1 - r_2)$$

since

$$B = \frac{B_0}{R/R_0}$$

Now we calculate the work

$$\begin{aligned}
e\int_{t_1}^{t_2} dt\ v_r\ E_r(t) &= e\int_{t_1}^{t_2} dt\ \frac{dr}{dt}\ E_r(t) \\
&= e\left[r\ E_r(t)\right]_{t=t_1}^{t=t_2} - e\int_{t_1}^{t_2} dt'\ r\ \frac{\partial E_r(t')}{\partial t'} \\
&\approx er_2 E_r(t_2) - er_1 E_r(t_1) \\
&\quad -e\left(\frac{\partial E_r}{\partial t}\right)\int_{t_1}^{t_2} dt'\ r(t')
\end{aligned}$$

We will have to use the time average of the position of the particle on the banana. It is approx

$$\frac{r_2 + r_1}{2}$$

then

$$\int_{t_1}^{t_2} dt'\ r(t') \approx \frac{r_2 + r_1}{2}(t_2 - t_1)$$

The two values of the radial electric field in t_2 respectively in t_1 are connected using a linear approximation

$$E_r(t_1) = E_r(t_2) - (t_2 - t_1)\frac{\partial E_r}{\partial t}$$

Further, the conservation of the generalized momentum gives approximative connection between thetwo spatial positions $r_{1,2}$ in terms of the velocities there $u_{1,2}$,

$$r_1 - r_2 \approx \frac{u_1 - u_2}{\Omega_\theta}$$

Introducing this in the equation that connects the energy in moments t_2 and t_1,

$$\begin{aligned}
&\frac{m\left(u_2^2 - u_1^2\right)}{2} \\
&+\mu\left[B_1 + \frac{B_0}{R_0}(r_1 - r_2)\right] - \mu B_1 \\
\approx\ &er_2 E_r(t_2) - er_1\left[E_r(t_2) - (t_2 - t_1)\frac{\partial E_r}{\partial t}\right] \\
&-e\left(\frac{\partial E_r}{\partial t}\right)\frac{r_2 + r_1}{2}(t_2 - t_1)
\end{aligned}$$

or

$$\begin{aligned}
& m\frac{\left(u_2^2-u_1^2\right)}{2}+\mu\frac{B_0}{R_0}\frac{h}{h}\left[\frac{u_1-u_2}{\Omega_\theta}\right] \\
\approx\quad & r_2E_r\left(t_2\right)-r_1E_r\left(t_2\right) \\
& +r_1\left(t_2-t_1\right)\left(\frac{\partial E_r}{\partial t}\right)-\left(\frac{\partial E_r}{\partial t}\right)\frac{r_2+r_1}{2}\left(t_2-t_1\right)
\end{aligned}$$

$$\begin{aligned}
& m\frac{1}{2}\left(u_2+u_1\right)\left(u_2-u_1\right)-\mu\frac{B_0h}{R}\frac{u_2-u_1}{\Omega_\theta} \\
\approx\quad & eE_r\left(t_2\right)\left(r_2-r_1\right) \\
& +e\left(\frac{\partial E_r}{\partial t}\right)\left(t_2-t_1\right)\left(r_1-\frac{r_2+r_1}{2}\right)
\end{aligned}$$

In the RHS two times occurs the factor

$$\left(r_1-r_2\right)\approx\frac{u_1-u_2}{\Omega_\theta}$$

$$\begin{aligned}
& m\frac{1}{2}\left(u_2+u_1\right)\left(u_2-u_1\right)-\mu\frac{B_0h}{R}\frac{u_2-u_1}{\Omega_\theta} \\
\approx\quad & \left[eE_r\left(t_2\right)-e\left(\frac{\partial E_r}{\partial t}\right)\left(t_2-t_1\right)\right]\left(r_2-r_1\right)
\end{aligned}$$

$$\begin{aligned}
& m\frac{1}{2}\left(u_2+u_1\right)\left(u_2-u_1\right)-\mu\frac{B_0h}{R}\frac{u_2-u_1}{\Omega_\theta} \\
\approx\quad & eE_r\left(t_2\right)\frac{u_2-u_1}{\Omega_\theta} \\
& -e\left(\frac{\partial E_r}{\partial t}\right)\left(t_2-t_1\right)\frac{u_2-u_1}{\Omega_\theta}
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2}\left(u_2+u_1\right)-\mu\frac{B_0h}{R}\frac{1}{m\Omega_\theta} \\
=\quad & \frac{1}{m\Omega_\theta}\left[eE_r\left(t_2\right)-e\left(\frac{\partial E_r}{\partial t}\right)\left(t_2-t_1\right)\right]
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2}\left(u_2+u_1\right) \\
=\quad & \mu\frac{B_0h}{R}\frac{1}{m\Omega_\theta}+\frac{1}{m\Omega_\theta}eE_r\left(t_2\right)-\frac{1}{m\Omega_\theta}e\left(\frac{\partial E_r}{\partial t}\right)\left(t_2-t_1\right)
\end{aligned}$$

$$\begin{aligned}
& \frac{u_2+u_1}{2} \\
=\quad & \frac{1}{m\Omega_\theta}\left[\frac{\mu B}{R}+eE_r\left(t_2\right)-e\left(\frac{\partial E_r}{\partial t}\right)\left(t_2-t_1\right)\right]
\end{aligned}$$

NOTE
At this point there is a single problem with $B_0 h \rightarrow B$.
END

13.1.2 The second half of the banana

Starts from the point 2 of the equatorial plane and goes down and returns intersecting this plane at 3.

We have

$$r_2 < r_3$$

The result

$$\frac{u_3 + u_2}{2} = \frac{1}{m\Omega_\theta}\left[\frac{\mu B}{R} + eE_r(t_2) + \left(\frac{\partial E_r}{\partial t}\right)(t_3 - t_2)\right]$$

Note the change in sign of the last term compared with the first half.

13.1.3 Both halves of banana

The two expressions are subtracted

$$u_3 - u_1 = \frac{1}{B_\theta}\left(\frac{\partial E_r}{\partial t}\right)(t_3 - t_1)$$

The conservation of the canonical momentum

$$r_3 - r_1 = \frac{1}{\Omega_\theta}(u_3 - u_1)$$

The time-average radial drift

$$\begin{aligned}\overline{v}_r &= \frac{r_3 - r_1}{t_3 - t_1} \\ &= \frac{1}{\Omega_\theta B_\theta}\left(\frac{\partial E_r}{\partial t}\right)\end{aligned}$$

13.2 Lagrangian approach

The phase space Lagrangian.

The variable u is a parallel velocity.

$$L\left(u, \mathbf{x}, \dot{\mathbf{x}}\right) = \left[mu\widehat{\mathbf{n}} + e\mathbf{A}\right]\cdot\frac{d\mathbf{x}}{dt} - H$$

where

$$H = \frac{mu^2}{2} + \mu B + e\Phi$$

with

$$\frac{d\mathbf{x}}{dt} = u\widehat{\mathbf{n}} + \mathbf{v}_D$$

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{\partial \mathbf{A}}{\partial t}$$

Coordinates

$$(r, \theta, \varphi)$$

$$B_r = 0$$

gauge

$$A_r = 0$$

Neglect of

$$mub_\theta$$

compared with

$$eA_\theta$$

Then

$$\begin{aligned} L &= eA_\theta \, \frac{rd\theta}{dt} \\ &\quad + [mub_\varphi + eA_\varphi] \; R\frac{d\varphi}{dt} \\ &\quad -H \end{aligned}$$

Approximation

$$b_\varphi \approx 1$$

The Euler-Lagrange equation for u is

$$R\frac{d\varphi}{dt} - u = 0$$

Canonical angular momentum conservation

$$\begin{aligned} p_\varphi &= (mRu + eRA_\varphi) \\ &= \text{const} \end{aligned}$$

Denote

$$p_\theta \equiv r \; eA_\theta$$

with the approximation

$$rA_\theta \approx B_0 \frac{r^2}{2}$$

and

$$\frac{dp_\theta}{dt} = -\frac{\partial H}{\partial \theta}$$

The Euler-Lagrange equation for r is

$$\frac{d\theta}{dt} = \frac{\partial H}{\partial p_\theta}$$

It is

$$r = \sqrt{\frac{2}{eB_0}}\ \sqrt{p_\theta}$$

which can be replaced in the Hamiltonian.

Also

$$\begin{aligned} RA_\varphi &= (RA_\varphi)_0 \\ &\quad - \int_{r_0}^{r} dr'\ R_0 B_\theta\,(r') \\ &\quad - \int_0^t dt'\ R_0 E_\varphi\,(t') \end{aligned}$$

Small banana width approximation

$$r = r_0 + \delta r$$

with the equations

$$\frac{d}{dt}\delta r = -v_D\ \sin\theta$$

$$r_0 \frac{d\theta}{dt} = \left(\frac{B_\theta}{B_0}\right) u - \frac{E_r}{B_0}$$

with

$$v_D = \frac{1}{\Omega_0 R_0}\left(u^2 + \frac{\mu B_0}{m}\right)$$

We use

$$\begin{aligned} uh &= \Omega_\theta\ \delta r \\ &\quad + \frac{1}{mR_0}\left[p_\varphi - (eRA_\varphi)_0\right] \\ &\quad + \frac{e}{m}\int_0^t dt'\ E_\varphi\,(t') \end{aligned}$$

After derivation to time

$$\begin{aligned} m\frac{du}{dt} &= -\frac{B_\theta}{R_0 B_0}\left[\mu B_0 + \frac{mu\ E_r}{B_\theta}\right]\sin\theta \\ &\quad + eE_\varphi \end{aligned}$$

Simplifying by ignoring the gravitation-like term, with E_r, the equation is reduced to

$$\frac{d^2\theta}{dt^2} + \alpha\ \sin\theta = \varepsilon$$

where

$$\alpha = \frac{\mu B_\theta^2}{m B_0 R_0} \frac{1}{r_0}$$

$$\begin{aligned}\frac{d\varepsilon}{dt} &= \frac{1}{r_0 B_0} \frac{\partial E_r}{\partial t} \\ &\quad + \frac{1}{r_0 B_0} \Omega_\theta E_\varphi\end{aligned}$$

It is integrated once

$$\frac{d\theta}{dt} = \sqrt{2\left[\alpha\left(2k-1\right) + \alpha\cos\theta + \varepsilon\theta\right]}$$

$$\begin{aligned}\delta r &= -\frac{1}{\Omega_\theta} \frac{1}{m R_0} \left[p_\varphi - \left(eRA_\varphi\right)_0\right] \\ &\quad + \frac{B_0 r_0}{\Omega_\theta B_\theta} \frac{d\theta}{dt} \\ &\quad + \frac{1}{\Omega_\theta} \frac{E_r}{B_\theta} \\ &\quad - \frac{1}{B_\theta} \int_0^t dt' \ E_\varphi\end{aligned}$$

It is approximated

$$E_r\left(T\right) - E_r\left(0\right) \approx \frac{\partial E_r}{\partial t} T$$

$$\begin{aligned}\overline{v}_r &= \frac{\delta r\left(T\right) - \delta r\left(0\right)}{T} \\ &= \frac{1}{\Omega_\theta} \frac{1}{B_\theta} \frac{\partial E_r}{\partial r} \quad \text{(neoclassical polarization)} \\ &\quad - \frac{E_\varphi}{B_\theta} \quad \text{(Ware)}\end{aligned}$$

14 Radial electric field and radial current Honda

This is the paper of **Honda** on fast ion current and return current.

The magnetic field

$$\mathbf{B} = \boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi + I\boldsymbol{\nabla}\varphi$$

and Ampere's law

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}$$

the density of current is *total*, includes imposed current and plasma response (polarization).

Surface averged

$$\langle \mathbf{j} \cdot \boldsymbol{\nabla}\psi \rangle = \varepsilon_0 \frac{\partial}{\partial t} \left(\left\langle |\boldsymbol{\nabla}\psi|^2 \right\rangle \frac{\partial \phi}{\partial \psi} \right)$$

The radial current (surface-averaged) is balanced by the time derivative of the radial electric field.

The determination of the radial current will be done in *fluid* formulation.
The current crossing the surface (radial) is determined by friction forces acting in toroidal momentum balance.

$$\begin{aligned} m_a n_a \frac{d\mathbf{u}_a}{dt} &= -\boldsymbol{\nabla} p_a - \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a \\ &\quad + e_a n_a \mathbf{E} + \mathbf{e}_a n_a \mathbf{u}_a \times \mathbf{B} \\ &\quad + \mathbf{R}_a \end{aligned}$$

There will be two distinct direction of study of the balance of forces

parallel, $\mathbf{B}\cdot$
toroidal, $R^2 \boldsymbol{\nabla}\varphi\cdot$

The difference between them is very important

- the toroidal projection of the momentum equation still contains the term $j \times B$ which is *radial* and allows to calculate the radial flux
- the parallel projection does NOT contain the radial flux

The momentum equation is

- projected on the toroidal direction, $R^2 \boldsymbol{\nabla}\varphi\cdot$, plus $\langle\rangle$
- [multiplied by R^2
- surface averaged]

The two first steps are obtained by multiplying with

$$R^2 \boldsymbol{\nabla}\varphi\cdot$$

Then (toroidal)

$$\begin{aligned} \left\langle m_a n_a \, R^2 \boldsymbol{\nabla}\varphi \cdot \frac{d\mathbf{u}_a}{dt} \right\rangle &= -\left\langle R^2 \boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a \right\rangle \\ &\quad + \left\langle e_a n_a \, R^2 \boldsymbol{\nabla}\varphi \cdot \mathbf{E} \right\rangle \\ &\quad + \left\langle e_a n_a \, R^2 \boldsymbol{\nabla}\varphi \cdot (\mathbf{u}_a \times \mathbf{B}) \right\rangle \text{ this will give radial current } j_r \\ &\quad + \left\langle R^2 \boldsymbol{\nabla}\varphi \cdot \mathbf{R}_a \right\rangle \end{aligned}$$

The *toroidal* projection of the momentum equation contains the term $\mathbf{j} \times \mathbf{B}$ from which the radial current will be extracted.

The *parallel* projection of the momentum equation, that will be used below, does not keep the product $\mathbf{B} \cdot (\mathbf{u}_a \times \mathbf{B})$ which is zero. It is

$$\sum_{a=e,i} \left\langle \mathbf{B} \cdot m_a n_a \frac{\partial \mathbf{u}_a}{\partial t} \right\rangle = - \sum_{a=e,i} \langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a \rangle + \sum_{a=e,i} \left\langle \mathbf{B} \cdot \mathbf{R}_a^{non-C} \right\rangle$$

(where $\mathbf{E}$ and the Coulombian collisional friction $\mathbf{R}^C$ disappear by summation over species).

The density is variable on surface

$$n_a = \langle n_a \rangle + \delta n_a (\theta)$$

but this effect is neglected in **Honda**.

We note that

$$\begin{aligned}
\mathbf{B} \times \boldsymbol{\nabla}\varphi \quad &\rightarrow \quad \mathbf{B} \times \widehat{\mathbf{e}}_\varphi \frac{1}{R} \\
&= \quad B \widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_\varphi \frac{1}{R} = B \ \frac{1}{R} \sin(\alpha) \quad \widehat{\mathbf{e}}_{\boldsymbol{\nabla}\psi} \sim \text{radial direction} \\
&= \quad B \ \times \frac{1}{R} \sin(\text{small angle between the magnetic line and } \widehat{\mathbf{e}}_\varphi) \\
&\qquad \times \widehat{\mathbf{e}}_{\boldsymbol{\nabla}\psi}
\end{aligned}$$

and

$$\begin{aligned}
&\sin(\text{small angle between magnetic line and } \widehat{\mathbf{e}}_\varphi) \\
= \quad &\frac{B_\theta}{B}
\end{aligned}$$

$$\begin{aligned}
\mathbf{B} \times \boldsymbol{\nabla}\varphi \quad &= \quad B \ \frac{1}{R} \frac{B_\theta}{B} \widehat{\mathbf{e}}_{\boldsymbol{\nabla}\psi} \\
&= \quad \frac{1}{R^2} R B_\theta \widehat{\mathbf{e}}_{\boldsymbol{\nabla}\psi}
\end{aligned}$$

and the last three factors are

$$R B_\theta \widehat{\mathbf{e}}_{\boldsymbol{\nabla}\psi} = \boldsymbol{\nabla}\psi$$

Then the third term in the RHS (the Lorentz force term)

$$\begin{aligned}
\left\langle e_a n_a \ R^2 \boldsymbol{\nabla}\varphi \cdot (\mathbf{u}_a \times \mathbf{B}) \right\rangle \quad &= \quad \left\langle e_a n_a \ R^2 \ \mathbf{u}_a \cdot (\mathbf{B} \times \boldsymbol{\nabla}\varphi) \right\rangle \\
&= \quad \left\langle e_a n_a R^2 \ \frac{1}{R^2} \mathbf{u}_a \cdot \boldsymbol{\nabla}\psi \right\rangle \\
&= \quad \langle \mathbf{j}_a \cdot \boldsymbol{\nabla}\psi \rangle
\end{aligned}$$

This term from the *toroidal* $\left(R^2\boldsymbol{\nabla}\varphi\cdot\right)$ projection of the momentum for species a is the surface average of the radial current.

In addition

$$\left\langle R^2\boldsymbol{\nabla}\varphi\cdot\left(\mathbf{u}\cdot\boldsymbol{\nabla}\right)\mathbf{u}\right\rangle = 0$$

The demonstration is given by **Hirshman** and mentioned by **Hinton Rosenbluth** also detailed by **Honda**; it involves the periodicity. This will reduce the total time derivative to the explicit derivative to time.

We separate the main object of interest, the surface-averaged radial current of the species a. The radial current is connected (by its combination with $\mathbf{B}$) with every force (change of momentum in time) which acts in toroidal direction

$$\begin{aligned}
\left\langle \mathbf{j}_a\cdot\boldsymbol{\nabla}\psi\right\rangle &= -\left\langle e_a n_a\ R^2\boldsymbol{\nabla}\varphi\cdot\mathbf{E}\right\rangle \text{ toroidal electric field} \\
&\quad -\left\langle R^2\boldsymbol{\nabla}\varphi\cdot\mathbf{R}_a\right\rangle \quad \text{friction, toroidal} \\
&\quad +\left\langle m_a n_a\ R^2\boldsymbol{\nabla}\varphi\cdot\frac{\partial\mathbf{u}_a}{\partial t}\right\rangle \quad \text{polarization} \\
&\quad +\left\langle R^2\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_a\right\rangle \text{ viscosity}
\end{aligned}$$

Later we will also use the *parallel* balance, but this cannot involve $\mathbf{j}_r$.

We **note** the third term, $\left\langle m_a n_a\ R^2\boldsymbol{\nabla}\varphi\cdot\frac{\partial\mathbf{u}_a}{\partial t}\right\rangle$.

This term shows that a time variation of the toroidal flow $\boldsymbol{\nabla}\varphi\cdot\frac{\partial\mathbf{u}_a}{\partial t}$ is related to a *radial current*.

This is normal, in the direction $\mathbf{j}_r$ →torque→toroidal acceleration, since the radial current $\times$ the poloidal magnetic field $\mathbf{j}_r\times\mathbf{B}_\theta$ is a force which injects momentum in every unit of time in the heavy component, ions, of plasma, in the toroidal direction. The toroidal plasma flow is *accelerated* $(\partial/\partial t\neq 0)$ when there is a radial current, even if the radial current is constant in time.

The reversed relation is less usual: if a factor generates *acceleration* of the toroidal flow, then a radial current should arise.

The resistive friction $\mathbf{R}$ is separated in two parts: Coulombian and non-Coulombian (turbulent)

$$\mathbf{R}_a = \mathbf{R}_a^C + \mathbf{R}_a^{non-C}$$

The coulombian part is coupled with the electric field.

We write, for species a,

$$\begin{aligned}
\left\langle \mathbf{j}_a\cdot\boldsymbol{\nabla}\psi\right\rangle &= -\left\langle R^2\boldsymbol{\nabla}\varphi\cdot\left(\mathbf{R}_a^C + e_a n_a\mathbf{E}\right)\right\rangle \text{ resistive flux} \\
&\quad +\left\langle m_a n_a\ R^2\boldsymbol{\nabla}\varphi\cdot\frac{\partial\mathbf{u}_a}{\partial t}\right\rangle \text{ polarization flux} \\
&\quad +\left\langle R^2\boldsymbol{\nabla}\varphi\cdot\left(\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_a - \mathbf{R}_a^{non-C}\right)\right\rangle \text{ orthogonal conduction flux}
\end{aligned}$$

Regarding the first term:

the coulombian collisions conserve momentum

then the sum over species will give zero.

In addition we assume neutrality of plasma and this makes the terms with toroidal electric E field to disappear when summing over species.

$\sum_{e,i} \left\langle R^2 \boldsymbol{\nabla}\varphi \cdot \left(\mathbf{R}_a^C + e_a n_a \mathbf{E}\right)\right\rangle \rightarrow 0.$

The term that comes from the change of momentum orthogonal on surfaces contains the *pressure stress.*

$$\boldsymbol{\pi}_a = \boldsymbol{\pi}_a^{(1)} + \boldsymbol{\pi}_a^{(2)}$$

where Chow Goldberger Low form

$$\boldsymbol{\pi}_a^{(1)} = \left(p_{a\|} - p_{a\perp}\right)\left(\widehat{\mathbf{n}}\,\widehat{\mathbf{n}} - \frac{1}{3}\mathbf{I}\right)$$

For this part the operations ($R^2\boldsymbol{\nabla}\varphi\cdot$ followed by surface averaging) will give *zero*

$$\left\langle R^2 \boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a^{(1)}\right\rangle = 0$$

The demonstration is in **Hirshman Sigmar**.

See also demonstration in *viscosity.tex.*

Only the higher orders can contribute to exchange of toroidal momentum in the perpendicular direction on surface (what remains after surface averaging).

Then one can sum over species

$$\begin{aligned}
\left\langle \mathbf{j}\cdot\boldsymbol{\nabla}\psi\right\rangle &= \sum_{a=e,i}\left[\left\langle m_a n_a\, R\frac{\partial u_{a\varphi}}{\partial t}\right\rangle \quad \text{polarization flux}\right.\\
&\quad + \left\langle R^2\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_a^{(2)}\right\rangle \text{ perpendicular viscosity}\\
&\quad \left. - \left\langle R^2\boldsymbol{\nabla}\varphi\cdot\mathbf{R}_a^{non-C}\right\rangle\right] \quad \text{cross-field resistive flux (small)}
\end{aligned}$$

It is now introduced a new object of interest

$$\begin{aligned}
& \text{polarization current}\\
\equiv\ & \text{(def) } \mathbf{j}_a^{pol}
\end{aligned}$$

Honda assumes that the radial component of the *polarization* current is connected with the time variation (acceleration) of the toroidal rotation

$$\begin{aligned}
& \left\langle \mathbf{j}_a^{pol}\cdot\boldsymbol{\nabla}\psi\right\rangle\\
=\ & \sum_{a=e,i}\left\langle m_a n_a\, R\frac{\partial u_{a\varphi}}{\partial t}\right\rangle
\end{aligned}$$

This is only a part of the momentum equation.

Therefore we are entitled to name it *polarization* current.

NOTA

This is an interesting suggestion for the connection between a *radial* flow (of charges) when there is a change in the *toroidal* flow.

It is like a pinch.

It is like the Bernoulli law for pressure (static, dynamic).

Much further, it is like an axial anomaly. See **Boyanovsky**.

END

The general form of the velocity has componenents that are

- toroidal $\sim \nabla\varphi$,
- parallel, $\sim \mathbf{B}$,

$$\begin{aligned}\mathbf{u}_a &= \omega_a \; R^2\nabla\varphi + \widehat{u}_{a\theta} \; \mathbf{B} \\ & \quad \text{(toroidal)} + \text{(parallel)}\end{aligned}$$

$$\begin{aligned}\omega_a &= -\frac{1}{e_a n_a}\frac{\partial p_a}{\partial\psi} - \frac{\partial\phi}{\partial\psi} \\ & \quad \text{diamagnetic and electric} \\ & \quad \text{only depend on } \psi\end{aligned}$$

We must **note** that the *diamagnetic* and the *electric* rotations are perpendicular on the magnetic field since they come from

$$\widehat{\mathbf{n}} \times \nabla\psi \rightarrow \text{perp. on } \mathbf{B}$$

Then ω_a refers to a rotation which is perpendicular on $\mathbf{B}$, which however contains the *poloidal magnetic field* B_θ since

$$\frac{\partial}{\partial\psi} = \frac{1}{|\nabla\psi|}\frac{\partial}{\partial r} = \frac{1}{RB_\theta}\frac{\partial}{\partial r}$$

To obtain the *toroidal* rotation one has to project it on the toroidal direction $\nabla\varphi$.

Projecting $\mathbf{u}_a$ on toroidal direction

$$u_{a\varphi} = \omega_a R + \frac{I}{R}\widehat{u}_{a\theta}$$

Projecting $\mathbf{u}_a$ on parallel direction

$$u_{a\|} = \omega_a \frac{I}{B} + \widehat{u}_{a\theta} \ B$$

NOTE that a generic form of the parallel velocity is adopted, with the *poloidal* term left unknown, as KB and determined by a constraint equation. **END.**

We will need the surface average of $Ru_{a\varphi}$.

Then we multiply the first by R (equivalent to multiplying by $R^2\boldsymbol{\nabla}\varphi\cdot$) and average

$$\langle Ru_{a\varphi}\rangle = \omega_a \left\langle R^2\right\rangle + I \left\langle \widehat{u}_{a\theta}\right\rangle$$

To eliminate $\widehat{u}_{a\theta}$ we first multiply the parallel equation to B and average

$$\left\langle Bu_{a\|}\right\rangle = \omega_a I + \left\langle \widehat{u}_{a\theta} B^2\right\rangle$$

Now we have to take into account that

$$\left\langle \widehat{u}_{a\theta}\right\rangle = \widehat{u}_{a\theta}$$

then

$$\frac{\left\langle Bu_{a\|}\right\rangle}{\left\langle B^2\right\rangle} = \omega_a \frac{I}{\left\langle B^2\right\rangle} + \left\langle \widehat{u}_{a\theta}\right\rangle$$

Multiply this equation by $-I$ and add the two equations

$$\langle Ru_{a\varphi}\rangle - I\frac{\left\langle Bu_{a\|}\right\rangle}{\left\langle B^2\right\rangle} = \omega_a \left[\left\langle R^2\right\rangle - I^2 \frac{1}{\left\langle B^2\right\rangle}\right]$$

Since we expect the *neoclassical polarization* to be derived we separate

$$B^2 = \frac{I^2}{R^2} + B_\theta^2$$

and take surface average

$$\left\langle B^2\right\rangle = I^2 \left\langle \frac{1}{R^2}\right\rangle + \left\langle B_\theta^2\right\rangle$$

next we rewrite to find the ratio

$$\frac{\left\langle B_\theta^2\right\rangle}{\left\langle B^2\right\rangle} = 1 - I^2 \frac{1}{\left\langle B^2\right\rangle}\left\langle \frac{1}{R^2}\right\rangle$$

In the expression that multiplies ω_a we separate a quantity that is known

$$\begin{aligned} & \left\langle R^2\right\rangle - I^2 \frac{1}{\left\langle B^2\right\rangle} \\ = \quad & \left\langle R^2\right\rangle - I^2 \frac{\left\langle B_\theta^2\right\rangle}{\left\langle B^2\right\rangle}\frac{1}{\left\langle B_\theta^2\right\rangle} \end{aligned}$$

Let us add and subtract a term

$$\left\langle R^2\right\rangle I^2 \frac{1}{\langle B^2\rangle}\left\langle\frac{1}{R^2}\right\rangle$$

$$\begin{aligned}
& \left\langle R^2\right\rangle - \left\langle R^2\right\rangle I^2 \frac{1}{\langle B^2\rangle}\left\langle\frac{1}{R^2}\right\rangle + \left\langle R^2\right\rangle I^2 \frac{1}{\langle B^2\rangle}\left\langle\frac{1}{R^2}\right\rangle - I^2\frac{\left\langle B_\theta^2\right\rangle}{\langle B^2\rangle}\frac{1}{\left\langle B_\theta^2\right\rangle} \\
= & \left\langle R^2\right\rangle - \left\langle R^2\right\rangle I^2 \frac{1}{\langle B^2\rangle}\left\langle\frac{1}{R^2}\right\rangle + I^2\left\langle R^2\right\rangle \frac{1}{\left\langle B_\theta^2\right\rangle}\left[\frac{\left\langle B_\theta^2\right\rangle}{\langle B^2\rangle}\left\langle\frac{1}{R^2}\right\rangle - \frac{1}{\langle R^2\rangle}\frac{\left\langle B_\theta^2\right\rangle}{\langle B^2\rangle}\right] \\
= & \left\langle R^2\right\rangle - \left\langle R^2\right\rangle I^2 \frac{1}{\langle B^2\rangle}\left\langle\frac{1}{R^2}\right\rangle + \left\langle R^2\right\rangle \frac{\left\langle B_\theta^2\right\rangle}{\langle B^2\rangle}\frac{I^2}{\left\langle B_\theta^2\right\rangle}\left[\left\langle\frac{1}{R^2}\right\rangle - \frac{1}{\langle R^2\rangle}\right]
\end{aligned}$$

For the last factor in the last line we introduce the notation

$$\frac{I^2}{\left\langle B_\theta^2\right\rangle}\left[\left\langle\frac{1}{R^2}\right\rangle - \frac{1}{\langle R^2\rangle}\right] \equiv 2\widehat{q}^2$$

and the last line takes the form

$$\left\langle R^2\right\rangle - \left\langle R^2\right\rangle I^2 \frac{1}{\langle B^2\rangle}\left\langle\frac{1}{R^2}\right\rangle + \left\langle R^2\right\rangle \frac{\left\langle B_\theta^2\right\rangle}{\langle B^2\rangle} 2\widehat{q}^2$$

One can prove that the first two terms can be combined as

$$\left\langle R^2\right\rangle - \left\langle R^2\right\rangle I^2 \frac{1}{\langle B^2\rangle}\left\langle\frac{1}{R^2}\right\rangle = \left\langle R^2\right\rangle \frac{\left\langle B_\theta^2\right\rangle}{\langle B^2\rangle}$$

To prove that this is correct, one starts by using the relation derived before

$$\frac{\left\langle B_\theta^2\right\rangle}{\langle B^2\rangle} = 1 - I^2 \frac{1}{\langle B^2\rangle}\left\langle\frac{1}{R^2}\right\rangle$$

Here we isolate

$$I^2 \frac{1}{\langle B^2\rangle}\left\langle\frac{1}{R^2}\right\rangle = 1 - \frac{\left\langle B_\theta^2\right\rangle}{\langle B^2\rangle}$$

and replace in the LHS

$$\begin{aligned}
& \left\langle R^2\right\rangle - \left\langle R^2\right\rangle I^2 \frac{1}{\langle B^2\rangle}\left\langle\frac{1}{R^2}\right\rangle \\
= & \left\langle R^2\right\rangle - \left\langle R^2\right\rangle\left(1 - \frac{\left\langle B_\theta^2\right\rangle}{\langle B^2\rangle}\right) \\
= & \left\langle R^2\right\rangle \frac{\left\langle B_\theta^2\right\rangle}{\langle B^2\rangle} \quad \text{OK}
\end{aligned}$$

Then indeed the term in paranthesis multiplying ω_a is

$$\begin{aligned}\left\langle R^2\right\rangle - I^2\frac{1}{\left\langle B^2\right\rangle} &= \left\langle R^2\right\rangle \frac{\left\langle B_\theta^2\right\rangle}{\left\langle B^2\right\rangle} + \left\langle R^2\right\rangle \frac{\left\langle B_\theta^2\right\rangle}{\left\langle B^2\right\rangle} 2\widehat{q}^2 \\ &= \left\langle R^2\right\rangle \frac{\left\langle B_\theta^2\right\rangle}{\left\langle B^2\right\rangle}\left(1+2\widehat{q}^2\right)\end{aligned}$$

Replacing

$$\begin{aligned}\left\langle R u_{a\varphi}\right\rangle &= I\frac{\left\langle B u_{a\|}\right\rangle}{\left\langle B^2\right\rangle} + \omega_a \left[\left\langle R^2\right\rangle - I^2\frac{1}{\left\langle B^2\right\rangle}\right] \\ &= I\frac{\left\langle B u_{a\|}\right\rangle}{\left\langle B^2\right\rangle} + \omega_a \left\langle R^2\right\rangle \frac{\left\langle B_\theta^2\right\rangle}{\left\langle B^2\right\rangle}\left(1+2\widehat{q}^2\right)\end{aligned}$$

We return to the expression of the polarization current. The polarization current is the part with acceleration of the toroidal flow, $\partial/\partial t$

$$\begin{aligned}&\left\langle \mathbf{j}_a^{pol}\cdot\boldsymbol{\nabla}\psi\right\rangle = (\text{def}) \\ &= \sum_{a=e,i}\left\langle m_a n_a\, R\frac{\partial u_{a\varphi}}{\partial t}\right\rangle\end{aligned}$$

Now we express the *toroidal velocity* $u_{a\varphi}$ by *parallel velocity* $u_{a\|}$, using the equation $\left\langle R u_{a\varphi}\right\rangle - I\frac{\left\langle B u_{a\|}\right\rangle}{\left\langle B^2\right\rangle} = \omega_a\left[\left\langle R^2\right\rangle - I^2\frac{1}{\left\langle B^2\right\rangle}\right]$ which we have commented above

$$\left\langle R u_{a\varphi}\right\rangle (\text{toroidal}) - I\frac{\left\langle B u_{a\|}\right\rangle (\text{parallel})}{\left\langle B^2\right\rangle} = \omega_a\left[\left\langle R^2\right\rangle - I^2\frac{1}{\left\langle B^2\right\rangle}\right]$$

Then from the definition of the polarization current

$$\begin{aligned}&\left\langle \mathbf{j}_a^{pol}\cdot\boldsymbol{\nabla}\psi\right\rangle \overset{\text{def}}{=} \sum_{a=e,i}\left\langle m_a n_a\, R\frac{\partial u_{a\varphi}}{\partial t}\right\rangle \\ &= \sum_{a=e,i} m_a n_a \left[I\frac{1}{\left\langle B^2\right\rangle}\frac{\partial\left\langle B u_{a\|}\right\rangle}{\partial t}\right. \\ &\quad \left. + \left\langle R^2\right\rangle \frac{\left\langle B_\theta^2\right\rangle}{\left\langle B^2\right\rangle}\left(1+2\widehat{q}^2\right)\ \frac{\partial\omega_a}{\partial t}\right]\end{aligned}$$

The first term $I\frac{1}{\left\langle B^2\right\rangle}\frac{\partial\left\langle B u_{a\|}\right\rangle}{\partial t}$ introduces the time variation of the *parallel* velocity $u_{a\|}$, the *acceleration* of the parallel velocity.

This is because **Honda** associates the polarization *current* with the effect of the torque = injection of moment per unit time in the toroidal direction i.e. acceleration of the *toroidal rotation.*

This is in contrast to **Novakovskii Liu Sagdeev Rosenbluth** who consider the *poloidal rotation damping* and coupling with Geodesic Acoustic Modes.

The part of the polarization current that comes from the parallel motion $u_{a\|}$ can be calculated on the basis of *viscous* exchange of momentum.

$$\sum_{a=e,i}\left\langle \mathbf{B}\cdot m_a n_a \frac{\partial \mathbf{u}_a}{\partial t}\right\rangle = -\sum_{a=e,i}\left\langle \mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_a\right\rangle + \sum_{a=e,i}\left\langle \mathbf{B}\cdot\mathbf{R}_a^{non-C}\right\rangle$$

In the content of $\boldsymbol{\pi}_a$ the *neoclassical* part is more important

$$\boldsymbol{\Pi}_a = \boldsymbol{\pi}_a^{neo}$$

$$\left\langle \mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_a\right\rangle$$

which is determined from

$$\left\langle \mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_a\right\rangle = 3\left\langle \left(\nabla_{\|}B\right)^2\right\rangle \left[\widehat{\mu}_{a1}\ \widehat{u}_{a\theta} + \widehat{\mu}_{a2}\ \frac{2}{5}\frac{\widehat{q}_{a\theta}}{p_a}\right]$$

(**Hirshman Sigmar**) where

$$\widehat{\mu}_{a\ 1,2} \equiv \text{neoclassical viscosity coefficients}$$

Honda comments

$$\widehat{\mu}_{a1}\ \widehat{u}_{a\theta}\ \ \text{damps the flow}$$
$$\widehat{\mu}_{a2}\ \frac{2}{5}\frac{\widehat{q}_{a\theta}}{p_a}\ \ \text{drives the flow through } \boldsymbol{\nabla}T \text{ (Hazeltine)}$$

Returning to the equation for the parallel velocity

$$\begin{aligned}\sum_{a=e,i} m_a n_a \frac{\partial}{\partial t}\left\langle B u_{a\|}\right\rangle &= -\sum_{a=e,i}\left\langle \mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_a\right\rangle \ \text{(with the expression from above)} \\ &\quad + \sum_{a=e,i}\left\langle \mathbf{B}\cdot\mathbf{R}_a^{non-C}\right\rangle \ \ \text{(turbulent)}\end{aligned}$$

Returning to the polarization current, it becomes

$$\begin{aligned}\left\langle \mathbf{j}^{pol}\cdot\boldsymbol{\nabla}\psi\right\rangle &= -\frac{I}{\left\langle B^2\right\rangle}\sum_{a=e,i}\left\langle \mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_a\right\rangle \ \ \text{neoclassical viscosity} \\ &\quad + \frac{I}{\left\langle B^2\right\rangle}\sum_{a=e,i}\left\langle \mathbf{B}\cdot\mathbf{R}^{non-C}\right\rangle \ \ \text{non-Coul friction} \\ &\quad - \frac{1}{\mu_0}\frac{\left\langle R^2\right\rangle\left\langle B_\theta^2\right\rangle}{v_A^2}\left(1+2\widehat{q}^2\right)\ \frac{\partial}{\partial t}\frac{\partial \phi}{\partial \psi} \ \ \text{time variation of } E_r \text{ from } \frac{\partial \omega_a}{\partial t}\end{aligned}$$

NOTE.

Just for understanding the content of this equation, we will remove ψ

$$\begin{aligned} \nabla\psi &\rightarrow RB_\theta \text{ in the LHS} \\ \partial\psi &\rightarrow RB_\theta \text{ at denominator in the last term} \end{aligned}$$

This makes $\sim R^2 B_\theta^2$ and it will simplify with the same factor in the last term. What remains in the last term is

$$\frac{1}{\mu_0}\frac{1}{v_A^2}\left(1+2\widehat{q}\right)\frac{\partial E_r}{\partial t}$$

which is the usual polarization current. We do not have $v_{A\theta}$ instead of v_A.

NOTHING neoclassical is taken into account (except for the factor $1+2\widehat{q}^2$ of order units).

In the *fluid* formulation as of **Honda** one cannot introduce the difference between the motion on the two halves of the banana.

END

Return to the *total radial current*, after we have derived the polarization current. The total radial current includes, besides the polarization current, the effect of the higher order toroidal viscosity plus the effect of the non-Coulombian friction force

$$\begin{aligned} \langle \mathbf{j}\cdot\nabla\psi\rangle &= -\frac{1}{\mu_0}\frac{\left\langle R^2\right\rangle\left\langle B_\theta^2\right\rangle}{v_A^2}\left(1+2\widehat{q}^2\right)\frac{\partial}{\partial t}\frac{\partial\phi}{\partial\psi} \\ &\quad+\sum_{a=e,i}\left[-\frac{I}{\langle B^2\rangle}\langle \mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_a\rangle+\left\langle R^2\nabla\varphi\cdot\nabla\cdot\boldsymbol{\pi}_a^{(2)}\right\rangle\right. \\ &\quad\left.+\frac{I}{\langle B^2\rangle}\left\langle \mathbf{B}\cdot\mathbf{R}_a^{non-C}\right\rangle-\left\langle R^2\nabla\varphi\cdot\mathbf{R}_a^{non-C}\right\rangle\right] \end{aligned}$$

From the identity

$$\frac{\mathbf{B}\times\nabla\psi}{B^2}=\frac{I}{B^2}\mathbf{B}-R^2\nabla\varphi$$

it is suggested to define the vector

$$\mathbf{r}_\wedge\equiv\frac{I}{\langle B^2\rangle}\mathbf{B}-R^2\nabla\varphi$$

This vector is composed of a *parallel* and of a *toroidal* components.

We take from here the *toroidal* component

$$R^2\nabla\varphi=\frac{I}{\langle B^2\rangle}\mathbf{B}-\mathbf{r}_\wedge$$

$$
\begin{aligned}
& \frac{I}{\langle B^2\rangle}\left\langle \mathbf{B}\cdot\mathbf{R}_a^{non-C}\right\rangle - \left\langle R^2\boldsymbol{\nabla}\varphi\cdot\mathbf{R}_a^{non-C}\right\rangle \\
= & \frac{I}{\langle B^2\rangle}\left\langle \mathbf{B}\cdot\mathbf{R}_a^{non-C}\right\rangle - \left\langle\left(\frac{I}{\langle B^2\rangle}\mathbf{B}-\mathbf{r}_\wedge\right)\cdot\mathbf{R}_a^{non-C}\right\rangle \\
= & \left\langle \mathbf{r}_\wedge\cdot\mathbf{R}_a^{non-C}\right\rangle
\end{aligned}
$$

then

$$
\begin{aligned}
\langle \mathbf{j}\cdot\boldsymbol{\nabla}\psi\rangle = & -\frac{1}{\mu_0}\frac{\langle R^2\rangle\langle B_\theta^2\rangle}{v_A^2}\left(1+2\widehat{q}^2\right)\frac{\partial}{\partial t}\frac{\partial\phi}{\partial\psi} \\
& +\sum_{a=e,i}\left[-\frac{I}{\langle B^2\rangle}\langle\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_a\rangle+\left\langle R^2\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_a^{(2)}\right\rangle\right. \\
& \left.+\left\langle\mathbf{r}_\wedge\cdot\mathbf{R}_a^{non-C}\right\rangle\right]
\end{aligned}
$$

With this we can return to the *vacuum* equation of Maxwell, to include the vacuum induction term

$$
\begin{aligned}
& \varepsilon_0\left[\left\langle|\boldsymbol{\nabla}\psi|^2\right\rangle+\frac{c^2}{v_A^2}\langle R^2\rangle\langle B_\theta^2\rangle\left(1+2\widehat{q}^2\right)\right]\frac{\partial}{\partial t}\frac{\partial\phi}{\partial\psi} \\
= & \sum_{a=e,i}\left[-\frac{I}{\langle B^2\rangle}\langle\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_a\rangle+\left\langle R^2\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_a^{(2)}\right\rangle+\left\langle\mathbf{r}_\wedge\cdot\mathbf{R}_a^{non-C}\right\rangle\right] \\
& +\left\langle\mathbf{j}^{fast}\cdot\boldsymbol{\nabla}\psi\right\rangle
\end{aligned}
$$

Define

$$
\begin{aligned}
\varepsilon_\perp & \equiv 1+\frac{\langle R^2\rangle\langle B_\theta^2\rangle}{\left\langle|\boldsymbol{\nabla}\psi|^2\right\rangle}\frac{c^2}{v_A^2}\left(1+2\widehat{q}^2\right) \\
& \equiv 1+\kappa
\end{aligned}
$$

where

$$
\begin{aligned}
\kappa & \equiv \frac{\langle R^2\rangle\langle B_\theta^2\rangle}{\left\langle|\boldsymbol{\nabla}\psi|^2\right\rangle}\frac{c^2}{v_A^2}\left(1+2\widehat{q}^2\right) \\
& \gg 1
\end{aligned}
$$

The equation for the evolution of the *electric field*

$$
\begin{aligned}
& \varepsilon_0\varepsilon_\perp\left\langle|\boldsymbol{\nabla}\psi|^2\right\rangle\frac{\partial}{\partial t}\frac{\partial\phi}{\partial\psi} \\
= & \sum_{a=e,i}\left[-\frac{I}{\langle B^2\rangle}\langle\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_a\rangle+\left\langle R^2\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_a^{(2)}\right\rangle+\left\langle\mathbf{r}_\wedge\cdot\mathbf{R}_a^{non-C}\right\rangle\right] \\
& +\left\langle\mathbf{j}^{fast}\cdot\boldsymbol{\nabla}\psi\right\rangle
\end{aligned}
$$

the radial plasma current

$$\begin{aligned}
& \left\langle \mathbf{j}\cdot\boldsymbol{\nabla}\psi\right\rangle \\
= & -\frac{\kappa}{1+\kappa}\left\langle \mathbf{j}^{fast}\cdot\boldsymbol{\nabla}\psi\right\rangle \\
& +\frac{1}{1+\kappa}\sum_{a=e,i}\left[-\frac{I}{\left\langle B^{2}\right\rangle}\left\langle \mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_{a}\right\rangle +\left\langle R^{2}\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_{a}^{(2)}\right\rangle +\left\langle \mathbf{r}_{\wedge}\cdot\mathbf{R}_{a}^{non-C}\right\rangle \right] \\
\approx & -\left\langle \mathbf{j}^{fast}\cdot\boldsymbol{\nabla}\psi\right\rangle
\end{aligned}$$

and

$$\left\langle \mathbf{j}\cdot\boldsymbol{\nabla}\psi\right\rangle \approx -\left\langle \mathbf{j}^{fast}\cdot\boldsymbol{\nabla}\psi\right\rangle$$

approximately cancels current of the fast particles, since

$$\kappa \gg 1$$

At stationarity, during NBI injection, the charge separation due to the fast ions, = the radial current, is still continuing.

But the electric field E_r will reach the asymptotic constant value

$$\begin{aligned}
-\sum_{a=e,i}\left\langle R^{2}\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_{a}^{(2)}\right\rangle = & \frac{I}{\left\langle B^{2}\right\rangle}\sum_{a=e,i}\left\langle \mathbf{B}\cdot\mathbf{S}_{a}^{m}\right\rangle \\
& +\left\langle \mathbf{j}^{fast}\cdot\boldsymbol{\nabla}\psi\right\rangle
\end{aligned}$$

(turbulence, collisions, charge separation).

The set of equations that must be solved to obtain the time evolution of the radial electric field $\partial E_r/\partial t$.

$$\frac{\partial n_s}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\ u_{sr}\ n_s\right)=S_s$$

$$\begin{aligned}
& \frac{\partial}{\partial t}\left(m_s n_s u_{sr}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\ u_{sr}\ m_s n_s u_{sr}\right)-\frac{1}{r}m_s n_s u_{sr}^{2} \\
= & -\frac{\partial}{\partial r}\left(n_s T_s\right) \\
& +e_s n_s E_s+e_s n_s u_{s\theta}B_{\varphi}-e_s n_s u_{s\varphi}B_{\theta}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial}{\partial t}\left(m_s n_s u_{s\theta}\right)+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\ u_{sr}\ m_s n_s u_{s\theta}\right)-\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^3\ m_s n_s\ \mu_s\ \frac{\partial}{\partial r}\left(\frac{u_{s\theta}}{r}\right)\right] \\
= & e_s n_s E_{\theta}-e_s n_s u_{sr}B_{\varphi} \\
& +F_{s\theta}^{NC} \\
& +F_{s\theta}
\end{aligned}$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(m_s n_s u_{s\varphi}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\ u_{sr}\ m_s n_s u_{s\varphi}\right)-\frac{1}{r}\frac{\partial}{\partial r}\left(r\ m_s n_s\ \mu_s \frac{\partial u_{s\varphi}}{\partial r}\right)\\
= \quad & e_s n_s E_\varphi + e_s n_s u_{sr}\ B_\theta\\
&+F_{s\varphi}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{3}{2}n_s T_s\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\ \frac{5}{2}u_{sr}\ n_s T_s\right)-u_{sr}\frac{\partial}{\partial r}\left(n_s T_s\right)\\
= \quad & \frac{1}{r}\frac{\partial}{\partial r}\left(r\ n_s \chi_s \frac{\partial T_s}{\partial r}\right)\\
&+e_s n_s E_\theta u_{s\theta}+e_s n_s E_\varphi u_{s\varphi}\\
&+P_s
\end{aligned}
$$

The perpendicular viscosity

$$\mu_s$$

and the thermal conductivity

$$\chi_s$$

are produced by the turbulence.

The Maxwell equations

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rE_r\right)=\frac{1}{\varepsilon_0}\sum_s e_s n_s$$

$$\frac{1}{c^2}\frac{\partial E_\theta}{\partial t}=-\frac{\partial B_\varphi}{\partial r}-\mu_0\sum_s e_s n_s u_{s\theta}$$

$$\frac{1}{c^2}\frac{\partial E_\varphi}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left(rB_\theta\right)-\mu_0\sum_s e_s n_s u_{s\varphi}$$

$$\frac{\partial B_\theta}{\partial t}=\frac{\partial E_\varphi}{\partial r}$$

$$\frac{\partial B_\varphi}{\partial t}=-\frac{1}{r}\frac{\partial}{\partial r}\left(rE_\theta\right)$$

The connections between the equations.
Take the Gauss equation $\frac{1}{r}\frac{\partial}{\partial r}\left(rE_r\right)=\frac{1}{\varepsilon_0}\sum_s e_s n_s$ where we replace

$$E_r=-\frac{\partial\phi}{\partial r}$$

The Gauss equation can be integrated

$$\varepsilon_0\frac{\partial\phi}{\partial r}=-\frac{1}{r}\int_0^r\sum_s e_s n_s\ r'\ dr'$$

This equation is differentiated with respect to time

$$\varepsilon_0 \frac{\partial}{\partial t} \frac{\partial \phi}{\partial r} = -\frac{1}{r} \int_0^r \sum_s \frac{\partial n_s}{\partial t} \; r' dr'$$

Now consider the equation of continuity $\frac{\partial n_s}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \; u_{sr} \; n_s) = 0$ (ignore the source)

$$\begin{aligned} \varepsilon_0 \frac{\partial}{\partial t} \frac{\partial \phi}{\partial r} &= \sum_s e_s n_s \; u_{sr} \\ &= j_r^{tot} \end{aligned}$$

Honda then retains the *zeroth order* in the momentum balance equation.

This is the equilibrium balance of forces, and is the most usual equation that connects the Radial electric field with the diamagnetic flow and the $\mathbf{v} \times \mathbf{B}$ terms.

This means that we have a saturated E_r, the very special parameter that we were looking for above.

Above we used to say that E_r has time variation due to the continuous charge separation consisting of the fast ion finite displacements after being born.

The only reason for saturation of E_r and suppression of its time variation was that the new ion must work against the *polarization* electric field in order to produce the plasma rotation. This work is saturated by the *friction* that is opposed to the rotation. This reasoning does NOT calculate the electric field E_r but just gives a reason for its saturation.

A discussion on the time variation of the components of the physical balance.

At NBI, the expansion of the new ions toward their banana orbits is a radial current.

There is a torque $\mathbf{j}_r \times \mathbf{B}_\theta$.

There is a acceleration of the rotation.

It results a radial electric field.

The E_r from polarization (time variation of the parallel velocity) grows immediately. Its derivative to time equals j^{fast}.

But there is also a return current of the bulk.

At a certain limit $\partial E_r / \partial t \rightarrow 0$ stops increasing and reaches a stationary value.

The current J^{fast} still exist however.

There are collisions, poloidal.

This is the mechanism that saturates the increase of the radial elecrtric field: *the collisionality balances the parallel velocity acceleration.*

The time derivative of the parallel velocity tends to zero balanced by collisions.

All momentum injection along parallel direction, which would have produced acceleration of the flow, is now lost through collisions.

Therefore, if the toroidal acceleration of the toroidal flow velocities is SATURATED TO ZERO then this is due to the collisional loss of momentum along parallel direction.

Discussion on numerical results.

Due to the very large $\varepsilon_0 \varepsilon_\perp$, a slight change in time of E_r, leads to a *polarization* current that offsets the fast ion radial current. [we conclude from this proposition that for Honda the polarization current is like the return current].

15 Pedestal poloidal flow Kagan Catto

15.1 Invariants and orbits

The distinction between

$$\begin{aligned} \psi &\equiv \text{flux surface} \\ \psi_* &\equiv \text{drift surface} \end{aligned}$$

with

$$\psi_* \approx \psi - I \frac{v_\parallel}{\Omega}$$

The drift surface departs from the flux surface with an amount given by the poloidal Larmor radius

$$\approx \rho_\theta$$

The canonical momentum

$$p_\varphi \equiv ZeA_\varphi R + m_i v_\varphi R$$

Define the poloidal flux function

$$\psi \equiv -A_\varphi R$$

and define

$$\begin{aligned} \psi_* &\equiv \text{constant of motion} \\ \psi_* &\equiv \frac{1}{Ze} p_\varphi \\ &= \psi + \frac{m_i}{Ze} v_\varphi R \end{aligned}$$

for large ρ_θ this is

$$\psi_* \approx \psi - I \frac{v_\parallel}{\Omega}$$

The motion of an ion is made at fixed ψ_*.

The quantity

$$\frac{v_{\parallel}}{\Omega}$$

has variation with the poloidal variable θ.

Then the orbit of the ion crosses several magnetic surfaces, does not stay on the surface ψ.

The distance of this drift is

$$\sim \rho_{\theta}$$

In the pedestal this distance is large and plasma parameters have substantial change over this distance.

It is necessary to make the difference between the electrostatic potential on the surface

$$\phi\left(\psi\right)$$

and the electrostatic potential effectively seen by the ion

$$\phi\left(\psi_{*}=\text{const}\right)$$

The equation of motion on poloidal direction

$$\begin{aligned}
\frac{d\theta}{dt} &= v_{\parallel}\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta\right) \\
&\quad +I\frac{1}{B}\frac{d\phi}{d\psi}\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta\right)
\end{aligned}$$

It is assumed for the equilibrium electrostatic potential in the pedestal an expansion around the *drift surface* which is

$$\psi_{*}\approx\psi-I\frac{v_{\parallel}}{\Omega}$$

up to quadratic term

$$\phi_{0}\left(\psi\right)\approx\phi_{0}\left(\psi_{*}\right)+\left(\psi-\psi_{*}\right)\left.\frac{d\phi_{0}}{d\psi}\right|_{\psi_{*}}+\left(\psi-\psi_{*}\right)^{2}\left.\frac{d^{2}\phi_{0}}{d\psi^{2}}\right|_{\psi_{*}}$$

This should correspond to the *parabolic* shape of the electrostatic potential in the pedestal.

The equation becomes, after taking $B\left(\psi,\theta\right)\approx B\left(\psi_{*},\theta\right)$, which means weak spatial variation of the modulus of B,

$$\begin{aligned}
qR_{0}\frac{d\theta}{dt} &= \left(1+I^{2}\frac{1}{B\Omega}\left.\frac{d^{2}\phi_{0}}{d\psi^{2}}\right|_{\psi_{*}}\right)v_{\parallel} \\
&\quad +I\frac{1}{B}\left.\frac{d\phi_{0}}{d\psi}\right|_{\psi_{*}}
\end{aligned}$$

where

$$R_0 \equiv R_{\max} \quad \text{i.e. } \theta = 0$$

Notations adopted

$$\left.\frac{d\phi_0}{d\psi}\right|_{\psi_*} \equiv \phi'_*$$

$$\left.\frac{d^2\phi_0}{d\psi^2}\right|_{\psi_*} \equiv \phi''_0$$

It is suggested

$$S \equiv 1 + I^2 \frac{\phi''_*}{B\Omega}$$

the sqeezing factor

and the notation

$$u_* \equiv I \frac{\phi'_*}{B} \frac{1}{S}$$

then

$$qR_0 \frac{d\theta}{dt} = S\left(v_\parallel + u_*\right)$$

Note we recognize here the usual structure of the equation of motion on θ, with the projected-parallel velocity and the electric velocity. Small differences are taken into account here, for pedestal condition.

End.

A comparison is necessary to reveal the distinction between

$$u = I \frac{1}{B} \frac{d\phi}{d\psi}$$

is the $E \times B$ velocity of an ion in the magnetic surface ψ. And

$$u_* = \frac{1}{S} I \frac{1}{B} \left.\frac{d\phi}{d\psi}\right|_{\psi_*}$$

is the velocity of a *trapped* or *barely circulating* ion preserving ψ_*.

The magnetic field is written as

$$B = B_0 \frac{1+\varepsilon}{1+\varepsilon\cos\theta}$$

The subscript 0 means that the quantity is on the equatorial plane.

B_0 $\equiv$magnetic field at the outer point on the equatorial plane. As in **Cordey**.

Def

$$v_{\parallel 0} \equiv v_\parallel (\theta = 0)$$

$$\begin{aligned} u_{*0} &\equiv u_*(\theta=0) \\ &= \frac{1}{S_0} I \frac{1}{B_0} \left.\frac{d\phi}{d\psi}\right|_{\psi_*} \end{aligned}$$

The equation of motion in the point on the equatorial plane

$$qR_0 \left.\frac{d\theta}{dt}\right|_{\theta=0} = S_0 \left(v_{\|0} + u_0\right)$$

The enrgy conservation

$$\begin{aligned} E &= \frac{v_{\|}^2}{2} + \mu B + \frac{Ze}{M}\phi(\psi) \\ &= \text{const} \end{aligned}$$

Here $\phi(\psi)$ is expanded such as to exhibit the quadratic form, as was adopted above. And

$$\psi - \psi_* = I \frac{v_{\|}}{\Omega}$$

then the new expression of the energy

$$\frac{S\left(v_{\|} + u_*\right)^2}{2} + \mu B - \frac{S u_*^2}{2} = \text{const}$$

which is an invariant

$$\frac{S\left(v_{\|} + u_*\right)^2}{2} + \mu B - \frac{S u_*^2}{2} = \frac{S_0\left(v_{\|0} + u_{*0}\right)^2}{2} + \mu B_0 - \frac{S_0 u_{*0}^2}{2}$$

This equation is only in θ and $\frac{d\theta}{dt}$.

We have to replace here u and S with expressions of their dependence on θ. This dependence on θ comes from the magnetic field $B(\theta)$.

The solution

$$\left(v_{\|} + u_*\right) = \pm\left(v_{\|0} + u_{*0}\right)\sqrt{1 - \kappa^2 \sin^2\left(\frac{\theta}{2}\right)}$$

for

$$\kappa^2 = \frac{4\varepsilon}{S_0} \frac{u_{*0}^2 + \mu B_0}{\left(u_{*0} + v_{\|0}\right)^2}$$

in the condition

$$4\varepsilon \frac{S_0 - 1}{S_0} \ll 1$$

which means that S_0 is close to 1.

For

$$4\varepsilon\frac{1}{S_0} \ll 1$$

the ions are located near the boundary

$$\kappa = 1$$

between trapped and circulating.

New variables

$$\begin{aligned} W &= \frac{\left(v_{\parallel 0} + u_0\right)^2}{2S_0} + \left(\mu B_0 + u_0^2\right) \\ \lambda &\equiv \frac{\mu B_0 + u_0^2}{W} \end{aligned}$$

They are *constants of motion* to $\sqrt{\varepsilon S}$.

$$\kappa^2 = \frac{2\varepsilon\lambda}{1-\lambda}$$

$$\lambda = \frac{\kappa^2}{\kappa^2 + 2\varepsilon}$$

The vector

$$\frac{\partial}{\partial \mathbf{v}}\lambda$$

is perpendicular on the trapped/passing boundary $\kappa = 1$.

Near the boundary

$$\begin{aligned} v_{\parallel} + u_0 &\sim v_{th,i}\sqrt{\varepsilon} \\ \text{when } S &\approx 1 \end{aligned}$$

The transition to other variables

$$\frac{\left(v_{\parallel} + u\right)^2}{2S} = W\left(1 - \frac{\lambda B}{B_0}\right)$$

Evaluation of gradients in the space of the new variables

$$\frac{\partial W}{\partial \mathbf{v}} = \widehat{\mathbf{n}}\frac{v_{\parallel} + u}{S} + v_{\perp}\widehat{\mathbf{e}}_{\perp}$$

$$\frac{B}{B_0}W\,\frac{\partial \lambda}{\partial \mathbf{v}} = \left(1 - \frac{\lambda B}{B_0}\right)\frac{\partial W}{\partial \mathbf{v}} - \widehat{\mathbf{n}}\frac{v_{\parallel} + u}{S}$$

With these expressions one calculates the scalar product (the projection) of the vector fields $\left(\frac{\partial \lambda}{\partial \mathbf{v}} \text{ and } \frac{\partial E}{\partial \mathbf{v}}\right)$

$$\begin{aligned} &\frac{B}{B_0}W\,\frac{\partial \lambda}{\partial \mathbf{v}} \cdot \frac{\partial E}{\partial \mathbf{v}} \\ &= \mu B\frac{\left(v_{\parallel} + u\right)^2}{SW} - \lambda\frac{B}{B_0}\frac{\left(v_{\parallel} + u\right)^2}{S^2} \end{aligned}$$

In the vecinity of the trapped/passing boundary the parallel velocity is

$$v_{\|} + u \sim v_i \sqrt{\varepsilon S}$$

and the scalar product is close to zero, which means that

$$\begin{array}{l} \dfrac{\partial \lambda}{\partial \mathbf{v}} \text{ and } \dfrac{\partial E}{\partial \mathbf{v}} \\ \text{are almost perpendicular} \end{array}$$

or

$$\begin{array}{l} \text{the lines of } W \text{ and of } \lambda \\ \text{are almost perpendicular} \\ \text{in velocity space} \end{array}$$

15.2 The collision operator

The collision operator (Rosenbluth) is expressed as the divergence of a flux in velocity space

$$C[\delta f] = \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{\Gamma}[\delta f]$$

with the flux in velocity space defined as

$$\mathbf{\Gamma}[\delta f] = \gamma \; f_0 \; \frac{\partial}{\partial \mathbf{v}} \frac{\partial}{\partial \mathbf{v}} G_M \; \cdot \; \frac{\partial}{\partial \mathbf{v}} \left(\frac{\delta f}{f_M} \right)$$

The function G_M is the solution of

$$\gamma \frac{\partial}{\partial \mathbf{v}} \frac{\partial}{\partial \mathbf{v}} \; G_M = \frac{\nu_\perp}{4} \left(v^2 \mathbf{I} - \mathbf{v} \; \mathbf{v} \right) + \frac{\nu_\|}{2} \mathbf{v} \; \mathbf{v}$$

The collision frequencies

$$\nu_\perp = \nu_B \frac{3\sqrt{2\pi}}{2} \frac{1}{x^3} \left[\operatorname{erf}(x) - \Psi(x) \right]$$

$$\nu_\| = \nu_B \; \frac{3\sqrt{2\pi}}{2} \frac{1}{x^3} \Psi(x)$$

with

$$\begin{aligned} \nu_B \quad &= \quad \frac{4\sqrt{\pi}}{3} \frac{Z^4 e^4}{\sqrt{m_i}} \ln \Lambda \; \frac{n_i}{T^{3/2}} \\ & \qquad \text{Braginskii} \end{aligned}$$

$$\Psi(x) = \frac{\operatorname{erf}(x) - x \; \frac{d \operatorname{erf}(x)}{dx}}{2x^2}$$

$$\mathrm{erf}\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt\ \exp\left(-t^2\right)$$
$$x \equiv \frac{v}{v_{th,i}}$$

From the velocity space variables $\mathbf{v}$ one changes to

$$W\ ,\ \lambda\ ,\ \zeta\ \text{(gyrophase)}$$

and the operator of collision becomes

$$\begin{aligned} C\left[\delta f\right] &= \frac{1}{J}\frac{\partial}{\partial\lambda}\left(J\ \mathbf{\Gamma}\cdot\frac{\partial\lambda}{\partial\mathbf{v}}\right) \\ &+\frac{1}{J}\frac{\partial}{\partial W}\left(J\ \mathbf{\Gamma}\cdot\frac{\partial W}{\partial\mathbf{v}}\right) \\ &+\frac{1}{J}\frac{\partial}{\partial\zeta}\left(J\ \mathbf{\Gamma}\cdot\frac{\partial\zeta}{\partial\mathbf{v}}\right) \end{aligned}$$

Since everything that results from the finite gyroradius is neglected - being classical effect, not neoclassical, it is adopted $\frac{\partial}{\partial\zeta}=0$.

To determine the Jacobian it is used the previous results for

$$\frac{\partial W}{\partial\mathbf{v}}$$

and

$$\frac{\partial\lambda}{\partial\mathbf{v}}$$

and results

$$\begin{aligned} J &= \frac{d^3v}{d\zeta dW d\lambda} \\ &= \frac{2BWS}{B_0\left(v_\parallel+u\right)} \end{aligned}$$

Now define

$$\begin{aligned} h_\sigma &\equiv \text{neoclassical collisional drive term} \\ & \quad\ \text{in the kinetic equation} \end{aligned}$$

$$g \equiv \text{kinetic function, response to } h_\sigma$$

The kinetic equation with collisions

$$\begin{aligned} & \mathbf{\Gamma}\left[g-h_\sigma\right]\cdot\frac{\partial\lambda}{\partial\mathbf{v}} \\ \approx\ & f_M\frac{\partial\lambda}{\partial\mathbf{v}}\cdot\left[\frac{1}{4}\nu_\perp\left(v^2\mathbf{I}-\mathbf{v}\ \mathbf{v}\right)+\frac{1}{2}\nu_\parallel\mathbf{v}\ \mathbf{v}\right]\cdot\frac{\partial\lambda}{\partial\mathbf{v}}\ \frac{\partial}{\partial\lambda}\left(\frac{g-h_\sigma}{f_M}\right) \end{aligned}$$

The term with

$$\frac{\partial}{\partial W}$$

is neglected.

For particles that are close to the boundary *trapped/circulating* the follwoing approximation van be made

$$\lambda \approx \frac{B_0}{B} \approx 1$$

and this leads to

$$W^2 \left(\frac{\partial \lambda}{\partial \mathbf{v}}\right)^2 \approx \frac{\left(v_\parallel + u\right)^2}{S^2}$$

$$W^2 \left(\mathbf{v} \cdot \frac{\partial \lambda}{\partial \mathbf{v}}\right)^2 \approx u^2 \frac{\left(v_\parallel + u\right)^2}{S^2}$$

This modifies the equation

$$\begin{aligned} & \mathbf{\Gamma}\left[g - h_\sigma\right] \cdot \frac{\partial \lambda}{\partial \mathbf{v}} \\ \approx \quad & f_M \frac{\left(v_\parallel + u\right)^2}{S^2 W^2} \left[\frac{1}{4} \nu_\perp \ v^2 + \left(\frac{\nu_\parallel}{2} - \frac{\nu_\perp}{4}\right) u^2\right] \ \frac{\partial}{\partial \lambda} \left(\frac{g - h_\sigma}{f_M}\right) \end{aligned}$$

This equation allows to identify the expression of the collision operator

$$\begin{aligned} & C\left[g - h_\sigma\right] \\ = \quad & \frac{B_0}{B} \left(v_\parallel + u\right) \ \frac{\partial}{\partial \lambda} \left[\frac{B}{B_0} \frac{1}{v_\parallel + u} \mathbf{\Gamma} \cdot \frac{\partial \lambda}{\partial \mathbf{v}}\right] \end{aligned}$$

Now it is proposed a procedure intended to ensure the conservation of momentum through the collisions represented by this operator.

The procedure is particular.

It is introduced a new parameter in the expression of the collision operator, *i.e.* in the expression of the projection of the flux $\mathbf{\Gamma}$ onto the derivative $\partial \lambda / \partial \mathbf{v}$.

This parameter works by modifying the distribution function as

$$h \rightarrow h_\sigma$$

$$h_\sigma = h - f_M \sigma \ I \frac{1}{\Omega} \left(v_\parallel + u\right) \frac{\partial \ln T}{\partial \psi}$$

The new equation for the projection is

$$\begin{aligned} & \mathbf{\Gamma}\left[g - h_\sigma\right] \cdot \frac{\partial \lambda}{\partial \mathbf{v}} \\ = \quad & f_M \frac{\left(v_\parallel + u\right)^2}{S^2 W^2} \left[\frac{1}{4} \nu_\perp \ v^2 + \left(\frac{\nu_\parallel}{2} - \frac{\nu_\perp}{4}\right) u^2\right] \ \frac{\partial}{\partial \lambda} \left[\frac{g - h}{f_M} + \sigma \ I \frac{1}{\Omega} \left(v_\parallel + u\right) \frac{\partial \ln T}{\partial \psi}\right] \end{aligned}$$

15.3 Solution of the kinetic equation

Looking for the kinetic equation for the perturbed distribution function that produces the neoclassical transport

At stationarity the *constraint* on the kinetic function is

$$\overline{C\left[g-h\right]}=0$$

with the operation of transit averaging

$$g\equiv \text{ non-diamagnetic perturbation}$$

$$\begin{aligned} h &\equiv \text{ neoclassical drive} \\ &= f_M\ I\ \frac{v_\parallel}{\Omega}\frac{v^2}{v_{th,i}^2}\ \frac{\partial\ln T}{\partial\psi} \end{aligned}$$

The operation

$$\overline{Q}=\frac{\oint_* d\theta\frac{Q}{v_\parallel+u}}{\oint_* d\theta\frac{1}{v_\parallel+u}}$$

Note this is to be compared with the bounce averaging usually defined (in the core). **End.**

Note the operation of "*bounce" averaging* is equivalent with dividing by $v_\parallel$ and multiplying by B in the usual framework of the first order expanded kinetic equation, when it is used the *periodicity* of the LHS, *i.e.* one suppresses the LHS by the θ integration while the RHS is modified by the factor $B/v_\parallel$ that cleans the LHS and makes it explicitely a θ derivative, good for periodicity exploitation.

End.

The *neoclassical drive* is modified to reflect the presence of a strong electric field and rotation u,

$$h=f_M\ I\ \frac{\left(v_\parallel+u\right)}{\Omega}\frac{v^2+u^2}{v_{th,i}^2}\ \frac{\partial\ln T}{\partial\psi}$$

Consider the region

$$\lambda\approx 1$$

then

$$\frac{1}{2}\left(v^2+u^2\right)\approx W$$

and

$$h\approx f_M\ I\ \frac{v_\parallel+u}{\Omega}\frac{1}{2}\frac{W}{v_{th,i}^2}\ \frac{\partial\ln T}{\partial\psi}$$

Remember that for *trapped* particles we have

$$g \equiv 0$$

whicle for circulating particles, in the banana regime, the dependence of g on θ is weak and can be neglected. Here is how this is used.

Consider the constraint on the collision operator

$$\oint_{*} d\theta \ \left(v_{\parallel} + u\right) \frac{\partial}{\partial \lambda} \left[\frac{g}{f_M} - I \frac{v_{\parallel} + u}{\Omega} \frac{W - \sigma \frac{1}{2} v_{th}^2}{\frac{1}{2} v_{th}^2} \frac{\partial \ln T}{\partial \psi} \right] = 0$$

after taking into account the weak dependence on θ,

$$\begin{aligned} & \frac{\partial}{\partial \lambda} \left(\frac{g}{f_M} \right) \oint_{*} d\theta \ \left(v_{\parallel} + u\right) \\ = & \oint_{*} d\theta \ \left(v_{\parallel} + u\right) \frac{\partial}{\partial \lambda} \left[I \frac{v_{\parallel} + u}{\Omega} \frac{W - \sigma \frac{1}{2} v_{th}^2}{\frac{1}{2} v_{th}^2} \frac{\partial \ln T}{\partial \psi} \right] \end{aligned}$$

here one uses

$$\left(v_{\parallel} + u\right) \frac{\partial \left(v_{\parallel} + u\right)}{\partial \lambda} = -SW \frac{B}{B_0}$$

Setting

$$\frac{B}{B_0} \approx 1$$

$$\frac{\partial}{\partial \lambda} \left(\frac{g}{f_M} \right) \approx -SI \frac{W}{\Omega_0} \frac{1}{\left\langle v_{\parallel} + u \right\rangle} \frac{W - \sigma \frac{1}{2} v_{th}^2}{\frac{1}{2} v_{th}^2} \frac{\partial \ln T}{\partial \psi}$$

Here $\langle\rangle$ is the simple surface averaging

$$\left\langle v_{\parallel} + u \right\rangle = \frac{1}{2\pi} \oint_{*} d\theta \ \left(v_{\parallel} + u\right)$$

For the operator

$$\frac{\partial}{\partial \lambda} \left(\frac{g - h_{\sigma}}{f_M} \right) = SI \frac{W}{\Omega_0} \frac{W - \sigma \frac{1}{2} v_{th}^2}{\frac{1}{2} v_{th}^2} \frac{\partial \ln T}{\partial \psi} \left(\frac{1}{v_{\parallel} + u} - \frac{1}{\left\langle v_{\parallel} + u \right\rangle} \right)$$

after flux-surface averaging

$$\begin{aligned} & \left\langle \frac{1}{v_{\parallel} + u} \right\rangle - \frac{1}{\left\langle v_{\parallel} + u \right\rangle} \\ = & \left\langle \frac{1}{\sqrt{1 - \kappa^2 \sin^2 \frac{\theta}{2}}} \right\rangle - \frac{1}{\left\langle \sqrt{1 - \kappa^2 \sin^2 \frac{\theta}{2}} \right\rangle} \end{aligned}$$

To estimate it, one remarks that close to the boundary

$$\begin{aligned} \lambda &= \frac{1}{1+2\varepsilon} \\ \text{for } \kappa^2 &= 1 \end{aligned}$$

For very small departure from the boundary,

$$\kappa^2 \ll 1$$

Then

$$\left\langle \frac{1}{\sqrt{1-\kappa^2 \sin^2 \frac{\theta}{2}}} \right\rangle - \frac{1}{\left\langle \sqrt{1-\kappa^2 \sin^2 \frac{\theta}{2}} \right\rangle} \sim O\left(\kappa^4\right) \ll 1$$

and this means that one can restrict the range of λ to near the boundary.

It results that

$$\frac{\partial}{\partial \lambda}\left(\frac{g-h_\sigma}{f_M}\right) \approx O\left(\frac{1}{\varepsilon}\right)$$

15.4 Imposing momentum conservation

It is determined from this constraint the parameter σ

16 The time variation of the radial electric field and of the poloidal velocity Novakovskii, Liu Sagdeev Rosenbluth

It is in *rotation*.

Very good for magnetic damping and for Geodesic Acoustic Modes. In general for very rapid time variation of the electric field.

17 Drift-kinetic calculation of the radial current for *alpha* particles

See the texts: *Rotation.tex* and *Neoclassical.tex*.

This is also in *density enhanced confinement*.

Before discussing **Hinton Rosenbluth** one has to refer to **Cordey, Connor Cordey, Hsu Catto,** for NBI and for *alphas*, where the collisional operator is written in detail.

The problem is treated in

- the paper **Rosenbluth Hinton 1996** (for alpha particles) and
- the Ref. **Hinton Rosenbluth 1999** (for NBI)

The following equation is used to describe the *new, fast* ions:

$$\frac{\partial f}{\partial t} + \left(v_{\parallel}\widehat{\mathbf{b}} + \mathbf{v}_d\right) \cdot \boldsymbol{\nabla} f = C_{ie}\, f + S_i$$

where the drift velocity of the guiding centre is

$$\mathbf{v}_d = -v_{\parallel}\widehat{\mathbf{b}} \times \boldsymbol{\nabla}\left(\frac{v_{\parallel}}{\Omega}\right)$$

and the notations are introduced

$$\begin{aligned} \xi &= \frac{v_{\parallel}}{v} \\ &= (1-\lambda B)^{1/2} \\ E &= \frac{v^2}{2} \\ \mu &= \lambda E \end{aligned}$$

We **note** that the variable λ is

$$\lambda \equiv \frac{\mu}{E} = \frac{\frac{v_{\perp}^2}{2B}}{\frac{v^2}{2}} = \frac{1}{B}\frac{v_{\perp}^2}{v^2}$$

The **spatial variables** are

$$\psi, \theta, \phi$$

and the **velocity space** variables are

$$v, \lambda, \sigma \; (= \text{sign of } v_{\parallel}\,)$$

The **new ion collision term** is

$$C_{ie} f = \nu_s \frac{1}{v^2}\frac{\partial}{\partial v}\left(v^3 f\right)$$

NOTE
this is a simplified form of the *slowing down* part of the collision operator.
The *alpha* particles are slowed down by collisions with electrons mainly, for $v > v_{crit}$. **Cordey, Gaffey**.
The pitch angle scattering (alphas on ions with similar energies) is not included at this point.
END.

where the **collision frequency** is

$$\nu_s = \frac{m_e}{m_f} Z_f^2 \, \frac{1}{\tau_e}$$

and the **electron collision time** is

$$\tau_e = \frac{16\sqrt{\pi}}{3}\frac{e^4}{m_e^2}\ln\Lambda\ \frac{n_e}{v_{th,e}^3}$$

with the thermal electron velocity

$$v_{th,e} = \sqrt{\frac{2T_e}{m_e}}$$

In this treatment it is assumed that:

- Slowing down to thermal ions can be neglected. This means that the first effect on the new ions comes from electrons.
- The pitch angle scattering is kept.

The following parameters are considered small and this permits the perturbative solution of the drift-kinetic equation: **the ratio of the slowing down rate of the fast ions to the transit time frequency**: the fast ion performs many bounces or transits in the typical time of slowing down.

$$\begin{aligned}
\frac{\nu_s}{\left(\frac{v_{th}}{qR}\right)} &\ll 1 \\
\frac{\text{time of transit}}{\text{time of slowing down}} &\ll 1
\end{aligned}$$

And **the ratio of the guiding centre drift frequency to the bounce frequency**

$$\begin{aligned}
\frac{\frac{v_d}{L}}{\frac{v_{th}}{qR}} &\ll 1 \\
\frac{\text{time of transit}}{\text{time of drift motion}} &\ll 1
\end{aligned}$$

The two parameters are considered of the same order of magnitude δ and both very small, $\delta \ll 1$. **Note**: there is no reference to a space-type small parameter like the ratio of the banana width to the minor radius.

The perturbative expansion of f for the solution of the drift-kinetic equation for the fast ions.

$$f = f_{-1} + f_0 + f_1 + ...$$

The reason of the existence of the order -1 , f_{-1} is the *source* of particles.

The lowest order

$$\begin{aligned}
\widehat{\mathbf{b}} \cdot \boldsymbol{\nabla} f_{-1} &= 0 \\
\nabla_{\|} f_{-1} &= 0
\end{aligned}$$

This means that the lowest order function f_{-1} is **constant** along the magnetic lines and then on the **magnetic surfaces**; it still can be dependent on the radial coordinate ψ.

Actually the function f_{-1} has time variation that is determined by the *source* and by the *collisions*. This dependence will be obtained after *bounce average*.

It says nothing about the velocity space dependence.

The **zeroth order equation** is

$$\begin{aligned} v_{\parallel}\widehat{\mathbf{b}}\cdot\boldsymbol{\nabla}f_0 &= -\mathbf{v}_d\cdot\boldsymbol{\nabla}\psi\frac{\partial f_{-1}}{\partial\psi}-\frac{\partial f_{-1}}{\partial t}+ \\ &\quad +C_{ei}f_{-1}+S_i \end{aligned}$$

(it is identical for the NBI particles).

We note that the scalar product with $\mathbf{v}_d$ retains only the **radial** derivative in the gradient of f_{-1} since it does not vary on the surface. The function f_0 should be sensitive to the spatial modifications which are introduced by:

- the radial dependence of the distribution function f_{-1} (due to static plasma gradients, like in density) and
- the space changes due to the *source* of fast ions and due to *collisions*

[By *bounce averaging* the function f_0 is eliminated. This is because f_0 is spatially periodic. The spatial variation of f_0 is due to the neoclassical drift $\mathbf{v}_D$. The parallel gradient of f_0 must balance the radial advection of the previous order distribution, f_{-1}, induced by the neoclassical drift velocity. This is periodic since no difference is assumed between the motion on the two halves of the banana (as it is in neoclassical polarization).]

This gives the equation

$$\frac{\partial f_{-1}}{\partial t}=\overline{C}_{ei}f_{-1}+\overline{S}_i-\overline{\mathbf{v}_d\cdot\boldsymbol{\nabla}\psi}\frac{\partial f_{-1}}{\partial\psi}$$

The operator of *bounce averaging* is

$$\begin{aligned} \overline{A} &= \frac{1}{T}\oint\frac{d\theta}{v_{\parallel}\widehat{\mathbf{b}}\cdot\boldsymbol{\nabla}\theta}A \\ &= \frac{1}{T}\oint\frac{d\theta}{\left(\frac{v_{\parallel}}{qR}\right)}A \end{aligned}$$

where the **bounce time** is

$$\begin{aligned} T &= \oint\frac{d\theta}{v_{\parallel}\widehat{\mathbf{b}}\cdot\boldsymbol{\nabla}\theta} \\ &= \oint\frac{d\theta}{\left(\frac{v_{\parallel}}{qR}\right)} \qquad (s) \end{aligned}$$

We **note** that the bounce average should actually involve a time integration

$$\oint dt\ A$$

and it is made a change of variables

$$dt \rightarrow \frac{dt}{d\theta}\ d\theta$$

Now, the equation of motion along the banana or along the circulation orbit, is

$$\begin{aligned} r d\theta &= \Theta v_{\parallel}\ dt \\ &= \frac{B_\theta}{B_T} v_{\parallel} dt \end{aligned}$$

and we have

$$\frac{dt}{d\theta} = \frac{r}{\Theta v_{\parallel}} = \frac{r}{\frac{B_\theta}{B_T} v_{\parallel}} = \frac{qR}{v_{\parallel}}$$

and the operator of *bounce average* becomes

$$\oint dt\ A = \oint \frac{d\theta}{v_{\parallel} / (qR)}\ A$$

END.

The *limits of integrations* are

$$-\pi\ ,\ \pi \text{ for untrapped ions}$$

and for trapped ions the integral is defined

$$\oint d\theta = \sum_{\sigma} \sigma \int_{\theta_1}^{\theta_2} d\theta$$

where θ_1 and θ_2 are the *turning points*.

We write the **radial part of the guiding centre drift velocity**

$$\mathbf{v}_d \cdot \boldsymbol{\nabla} \psi = I v_{\parallel} \widehat{\mathbf{b}} \cdot \boldsymbol{\nabla} \theta \frac{\partial}{\partial \theta} \left(\frac{v_{\parallel}}{\Omega} \right)$$

Nota: this formula, using

$$\begin{aligned} \psi &= \int 2\pi R dr B_\theta (r) \\ d\psi &= 2\pi R B_\theta dr \\ \text{or} \quad |\nabla \psi| &= 2\pi R B_\theta \end{aligned}$$

and

$$\widehat{\mathbf{b}} \cdot \boldsymbol{\nabla} \theta = \nabla_{\parallel} \theta = \frac{d\theta}{dl_{\parallel}} = d\theta \frac{1}{r d\theta \frac{B_T}{B_\theta}} = \frac{1}{r} \frac{B_\theta}{B_T} = \frac{1}{qR}$$

$$\mathbf{v}_d \cdot \boldsymbol{\nabla}\psi = v_{d,r} \times 2\pi R B_\theta$$

$$v_{d,r} \times 2\pi R B_\theta = I v_{\|} \left(\frac{1}{r}\frac{B_\theta}{B_T}\right) \frac{\partial}{\partial\theta}\left(\frac{v_{\|}}{\Omega_{ci}}\right)$$

can be written ??

$$v_{d,r}\left(...\right) = I \frac{v_{\|}}{qR}\frac{\partial}{\partial\theta}\left(\frac{v_{\|}}{\Omega_{ci}}\right)$$

By performing the *bounce averaging* the radial displacements average to zero (because there is no radial electric field and no ripple)

$$\overline{(\mathbf{v}_d \cdot \boldsymbol{\nabla}\psi)} = 0$$

[This refers to the possibility that the two halves of the trapped particle's orbit (banana) to traverse regions with different plasma parameters which makes that the orbit is NOT periodic and the banana does not close peridically. The radial shift due to a radial electric field or different temperatures on the two halves leads to non-periodicity of the banana.

A consequence of this difference between the two halves of the banana is the *neoclassical polarization* **Hinton Robertson** and the hypothesis that for a high-Z impurity ion the two halves are at different temperatures which means different levels of ionization, Z is different]

Then the equation for f_{-1} which is the *bounce averaged zeroth-order* equation, becomes

$$\frac{\partial f_{-1}}{\partial t} = \overline{C}_{ei} f_{-1} + \overline{S}_i$$

(It is identical for NBI)

Note. It results that the lowest order function f_{-1} which is constant on the magnetic surfaces has a variation in the *velocity* space given by the *source* of fast ions and *the collisions*. It will result that the **source** is essential for a stationary **radial current** of new ions, even if the collisions and the limitator do not remove ions from plasma. **End.**

The source of *alpha particles* in that paper is

$$S_\alpha = \frac{\dot{n}\left(\psi,t\right)}{4\pi v_0^2}\delta\left(v - v_0\right)$$

where

$$\begin{aligned} \dot{n}\left(\psi,t\right) &\equiv \text{birth rate} \\ v_0 &\equiv \text{birth velocity} \end{aligned}$$

The solution of the *bounce averaged zero order EQUATION* for f_{-1} is

$$f_{-1} = \frac{\dot{n}\left[\psi, t - \tau\left(v\right)\right]}{4\pi v^3}\frac{1}{\nu_s}\left[\Theta\left(v - v_0 e^{-\nu_s t}\right) - \Theta\left(v - v_0\right)\right]$$

where

$$\tau(v) = \frac{1}{\nu_s} \ln \frac{v_0}{v}$$

The equation for the function f_0 can now be written since we have the solution for f_{-1}. The equation is

$$v_{\parallel} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} f_0 = -v_{\parallel} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \left(\frac{I v_{\parallel}}{\Omega_{ci}} \right) \frac{\partial f_{-1}}{\partial \psi}$$

and the solution for α particles is adopted in the form

$$\begin{aligned} f_0 &= -I \left(\frac{v_{\parallel}}{\Omega_{ci}} \right) \frac{\partial f_{-1}}{\partial \psi} \quad \text{(usual neoclassical term)} \\ &\quad +g \quad \text{(which only exists for circulating ions)} \end{aligned}$$

where

$$\begin{aligned} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} g &= 0 \\ &\rightarrow \ g \text{ has no } \theta \text{ dependence} \end{aligned}$$

meaning that g is constant on the magnetic lines, *i.e.* surfaces.

A distribution function for bananas can never be constant on the magnetic surfaces because the trajectory stops somewhere.

This function g verifies the equation

$$\begin{aligned} \frac{\partial g}{\partial t} &= \nu_s \frac{1}{v^2} \frac{\partial}{\partial v} \left(v^3 g \right) \\ &\quad + I \overline{\left(\frac{v_{\parallel}}{\Omega_{ci}} \right)} \left[\frac{\partial}{\partial \psi} \frac{\partial f_{-1}}{\partial t} - \nu_s \frac{1}{v^3} \frac{\partial}{\partial v} \left(v^4 \frac{\partial f_{-1}}{\partial \psi} \right) \right] \end{aligned}$$

The solution of this equation is

$$g = I \overline{\left(\frac{v_{\parallel}}{\Omega_{ci}} \right) \frac{\partial f_{-1}}{\partial \psi}}$$

This function does not exist for the *trapped* ions, but only for *circulating* ones.

Note this is interesting: the *parallel* variation of f_0 (equivalently the θ variation of f_0) is given by the perpendicular variation of f_{-1} transported by $v_D \sim \sin\theta$, just as the relationship between $f_1^{neoclassic}$ with f_M. We have

$$I = R^2 \mathbf{B} \cdot \boldsymbol{\nabla} \varphi = R B_{\varphi}$$

and

$$\begin{aligned} d\psi &= dr \frac{d}{dr} \int_0^r 2\pi R dr B_{\theta}(r) \\ &= 2\pi R B_{\theta} dr \end{aligned}$$

$$RB_\varphi \left(\frac{v_\parallel}{|e| \, B_T / m_i} \right) \frac{1}{RB_\theta} \frac{\partial f_{-1}}{\partial r} \sim \frac{v_\parallel}{\frac{|e| B_\theta}{m_i}} \frac{\partial f_{-1}}{\partial r} \sim \rho_\theta \frac{\partial f_{-1}}{\partial r}$$

This formula looks like

$$\rho_\theta \frac{\partial f_M}{\partial r}$$

which is the neoclassical correction to Maxwell function.

End.

The next step is the equation for f_1 the *first order*, which actually comes from the expansion in the equation written at *zero*-order

$$\frac{\partial f_0}{\partial t} + v_\parallel \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} f_1 + \mathbf{v}_d \cdot \boldsymbol{\nabla} f_0 = C_{ie} f_0$$

We **note** that everything relies on the variation of the first order correction function f_1 along the magnetic lines

$$v_\parallel \nabla_\parallel f_1 + ...$$

This variation of the order-one correction of the distribution function (f_1) for trapped particles actually takes place on the magnetic surface. It must be found as a variation with the poloidal angle θ:

$$f_1 \equiv f_1 \left(\theta, ... \right)$$

Therefore we look for the contribution of this kind from the other terms

$$\mathbf{v}_d \cdot \boldsymbol{\nabla} \rightarrow \mathbf{v}_d \cdot \boldsymbol{\nabla} \theta \frac{\partial}{\partial \theta} + ...$$

we now can take separately this factor

$$\mathbf{v}_d \cdot \boldsymbol{\nabla} \theta = -I v_\parallel \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \theta \frac{\partial}{\partial \psi} \left(\frac{v_\parallel}{\Omega_{ci}} \right)$$

Now we should remember that

$$\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \theta = \frac{1}{qR}$$

Again it is found that the *bounce average* of the drift term acting on the 0-order distribution function is zero

$$\overline{(\mathbf{v}_d \cdot \boldsymbol{\nabla} f_0)} = 0$$

It is time to calculate the *radial current*, on the basis of the distribution function in orders -1, 0, 1.

This is obtained from tha radial projection of the drift velocity

$$\mathbf{v}_d \cdot \boldsymbol{\nabla}\psi = I v_{\|} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta \frac{\partial}{\partial \theta}\left(\frac{v_{\|}}{\Omega}\right)$$

The current is projected on the radial direction and the result is averaged over the magnetic surface.

$$\langle \mathbf{j} \cdot \boldsymbol{\nabla}\psi \rangle = |e| \left\langle \int d^3v \left(\mathbf{v}_d \cdot \boldsymbol{\nabla}\psi\right) f \right\rangle$$

The expression of the drift velocity projected on radial direction is replaced in the integral and an integration by parts over θ is made. Then the $\partial/\partial\theta$ operator now goes to the distribution function as $\nabla_{\|} f$.

$$\langle \mathbf{j} \cdot \boldsymbol{\nabla}\psi \rangle = -|e|\, I \left\langle \int d^3v \left(\frac{v_{\|}}{\Omega_{ci}}\right) v_{\|} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} f \right\rangle$$

Two terms are absent

- the order -1 distribution function f_{-1} does not contribute
- the order 0 does not contribute

The first order to have a contribution to this current (averaged over surface) is f_1.

From the equation written in order 1 :

$$\frac{\partial f_0}{\partial t} + \underline{v_{\|} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} f_1} + \mathbf{v}_d \cdot \boldsymbol{\nabla} f_0 = C_{ie} f_0$$

we take separately the underlined term

$$v_{\|} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} f_1 = C_{ie} f_0 - \frac{\partial f_0}{\partial t} - \mathbf{v}_d \cdot \boldsymbol{\nabla} f_0$$

and the averaged current is

$$\begin{aligned} \langle \mathbf{j} \cdot \boldsymbol{\nabla}\psi \rangle &= -|e|\, I \left\langle \int d^3v \left(\frac{v_{\|}}{\Omega_{ci}}\right) v_{\|} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} f \right\rangle \text{ here } f = f_1 \\ &= -|e|\, I \left\langle \int d^3v \left(\frac{v_{\|}}{\Omega_{ci}}\right) \left(C_{ie} f_0 - \frac{\partial f_0}{\partial t} - \mathbf{v}_d \cdot \boldsymbol{\nabla} f_0\right) \right\rangle \end{aligned}$$

From this expression the drift term does not contribute

$$\left\langle \int d^3v \left(\frac{v_{\|}}{\Omega_{ci}}\right) \left(\mathbf{v}_d \cdot \boldsymbol{\nabla} f_0\right) \right\rangle = 0$$

Then the part of this current that does not come from *collisions* , called transitory, is

$$\left\langle \mathbf{j}^{transit} \cdot \boldsymbol{\nabla}\psi \right\rangle = |e|\, I \left\langle \int d^3v \left(\frac{v_{\|}}{\Omega_{ci}}\right) \frac{\partial f_0}{\partial t} \right\rangle$$

This is the polarization current.

This is the answer. What remains is to calculate the function in 0 order, f_0, by solving its equation derived above from the drift-kinetic equation.

Now we replace the zero-order function

$$
\begin{aligned}
f_0 &= -I\left(\frac{v_\parallel}{\Omega_{ci}}\right)\frac{\partial f_{-1}}{\partial\psi}+g \\
&= -I\left(\frac{v_\parallel}{\Omega_{ci}}\right)\frac{\partial f_{-1}}{\partial\psi}+I\overline{\left(\frac{v_\parallel}{\Omega_{ci}}\right)}\frac{\partial f_{-1}}{\partial\psi}
\end{aligned}
$$

and take the time derivative, then return to the current

$$
\frac{\partial f_0}{\partial t}=-I\left[\left(\frac{v_\parallel}{\Omega_{ci}}\right)-\overline{\left(\frac{v_\parallel}{\Omega_{ci}}\right)}\right]\frac{\partial^2 f_{-1}}{\partial\psi\partial t}
$$

This equation gives us the response: we now have the time derivative of f_0 which allows us to calculate the (surface averaged radial component of the) current.

The time variation of the distribution of the order -1 can be obtained since f_{-1} has explicit expression.

We need

$$
\begin{aligned}
\frac{\partial f_{-1}}{\partial t} &= \frac{\partial}{\partial t}\left\{\frac{\dot{n}\left[\psi,t-\tau\left(v\right)\right]}{4\pi v^3}\frac{1}{\nu_s}\left[\Theta\left(v-v_0e^{-\nu_s t}\right)-\Theta\left(v-v_0\right)\right]\right\} \\
&= \dot{n}\frac{1}{4\pi v^2}\delta\left(v-v_0e^{-\nu_s t}\right) \\
&\quad +\frac{\partial\dot{n}}{\partial t}\frac{1}{4\pi v^3}\frac{1}{\nu_s}\left[\Theta\left(v-v_0e^{-\nu_s t}\right)-\Theta\left(v-v_0\right)\right]
\end{aligned}
$$

The result is

$$
\left\langle\mathbf{j}^{transit}\cdot\boldsymbol{\nabla}\psi\right\rangle=-\left|e\right|I^2\frac{v_0^2}{2}\mathcal{I}\frac{\partial}{\partial\psi}\left[\dot{n}\left(\psi,0\right)e^{-2\nu_s t}+\int_0^t d\tau e^{-2\nu_s t}\frac{\partial}{\partial\tau}\dot{n}\left(\psi,t-\tau\right)\right]
$$

If the source is switched on at $t=0$, and then remains constant

$$
\left\langle\mathbf{j}^{transit}\cdot\boldsymbol{\nabla}\psi\right\rangle=-\left|e\right|I^2\frac{v_0^2}{2}\mathcal{I}\frac{\partial}{\partial\psi}\left[\dot{n}\left(\psi,0\right)e^{-2\nu_s t}\right]
$$

The quantity that remains to be calculated is

$$
\mathcal{I}=\sum_\sigma\left\langle\frac{1}{2\Omega_{ci}}\int Bd\lambda\left[\frac{\xi}{\Omega_{ci}}-\overline{\left(\frac{\xi}{\Omega_{ci}}\right)}\right]\right\rangle
$$

where

$$\begin{aligned}
\xi &= \left|(1-\lambda B)^{1/2}\right| \\
&= \frac{\left|v_{\|}\right|}{v}
\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{I} &= \frac{(2\varepsilon)^{3/2}}{\Omega_{ci}^2}\left\{\frac{8}{9\pi}+\int_0^1 \frac{dk}{k^{5/2}}\left[\frac{2}{\pi}E\left(k^{1/2}\right)-\frac{\pi}{2K\left(k^{1/2}\right)}\right]\right\} \\
&\approx \frac{(2\varepsilon)^{3/2}}{\Omega_{ci}^2}\times 0.38
\end{aligned}$$

The transient part of the current is

$$\left\langle \mathbf{j}^{transit}\cdot\boldsymbol{\nabla}\psi\right\rangle = -0.54\times\varepsilon^{3/2}\left|e\right| I^2\frac{v_0^2}{\Omega_{ci}^2}\frac{\partial}{\partial\psi}\left[\dot{n}\left(\psi,0\right)e^{-2\nu_s t}\right]$$

NOTE

On the calculation by **Rosenbluth Hinton** of the radial current from *alpha* particles.

The essential result is for the (surface averaged) radial current

$$\left\langle \mathbf{j}^{transit}\cdot\boldsymbol{\nabla}\psi\right\rangle = \left|e\right| I\left\langle \int d^3v\left(\frac{v_{\|}}{\Omega_{ci}}\right)\frac{\partial f_0}{\partial t}\right\rangle$$

can be schematically simplified to (combining $I=RB_T$ with Ω_{ci} and with $\left|\boldsymbol{\nabla}\psi\right|$)

$$j_r \sim \int d^3v \quad \rho_\theta\frac{\partial f_0}{\partial t}$$

and shows that the current is produced by the *time variation of the zero-order distribution function.*

The meaning is as follows.

The *neoclassical* correction is $f^{(1)}\sim\rho_\theta\frac{\partial f_0}{\partial r}$ and reflects the departure of the particles relative to the magnetic surface, due to the neoclassical *drift* $v_{D,r}$.

But ρ_θ is also a dimension of displacement of a new (fast, at NBI) ion created by ionization or charge exchange. It is the length of a *current* since this is a charge moving on a finite distance.

The current results however only after we take into account the amount of such ions that are created, the *source.*

Whatever is the physical process (ionization, charge exchange) the amount of new "charge carriers" is measured by the *time variation* of the basic distribution function. In the kinetic equation for f_0 there should be somewhere a *source* that changes in time f_0 and provides moving charges. It is of course ionization or charge exchange. This is the current. All velocities are good, that is why there is d^3v integration.

Since f_0 is calculated from the *source function* f_{-1},

$$\frac{\partial f_0}{\partial t} = -I\left[\left(\frac{v_\parallel}{\Omega_{ci}}\right) - \overline{\left(\frac{v_\parallel}{\Omega_{ci}}\right)}\right]\frac{\partial^2 f_{-1}}{\partial\psi\partial t}$$

or

$$\begin{aligned}\frac{\partial f_0}{\partial t} &\sim \frac{\partial}{\partial t}\left(\left[\rho_\theta - \overline{\rho_\theta}\right]\frac{\partial f_{-1}}{\partial r}\right)\\ &= \left[\rho_\theta - \overline{\rho_\theta}\right]\frac{\partial}{\partial t}\left[\text{space variation of the source of }\alpha\text{'s}\right]\end{aligned}$$

Since ρ_θ is only function of $v_\parallel$, the velocity space integration that will be used to calculate finally the *current*, j_r, will include all possible velocities.

The space variation of the source and the time variation of the source are both essential for the current.

But it is hard to see where the finite radial displacement is included.

Possibly in the fact that the radial displacement is exactly ρ_θ.

See **Fong Hahm**.
END

18 Similar (Rosenbluth Hinton) treatment of the NBI ions

This is **PhysLett259, 1999,267**.

Before solving the equation, we calculate the **radial current of fast ions**.
The *magnetic surface-averaged radial fast ion current density* is

$$\langle \mathbf{j}_{fast}\cdot\boldsymbol{\nabla}\psi\rangle = e_{fast}\left\langle\int d^3v\,\mathbf{v}_d\cdot\boldsymbol{\nabla}\psi\, f\right\rangle$$

where we have to replace f by the solution of the *drift-kinetic equation for the fast ions*. Then f here should be seen as $f_{-1} + f_0 + \ldots$.

The following **surface average operator** is used

$$\langle A\rangle = \frac{\oint \frac{d\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}A}{\oint\frac{d\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}}$$

We remind

$$\frac{d\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta} = \frac{rd\theta}{B_\theta}$$

and this integrand contains

$$B_\theta = \frac{b(r)}{h}$$

and the integral is over

$$\frac{r d\theta}{b(r)} h \times$$

The part that depends on θ is h.

Just for a connection, the *bounce average* is of the type

$$\oint \frac{dt}{d\theta} \, d\theta \ \times ...$$

and the equation for $dt/d\theta$ is obtained from the equation of motion along the line.

The equation (**??**) gives

$$\begin{aligned} \left\langle \mathbf{j}_{fast} \cdot \boldsymbol{\nabla}\psi \right\rangle &= e_{fast} \left\langle \int d^3v \, \mathbf{v}_d \cdot \boldsymbol{\nabla}\psi \, f \right\rangle \\ &= e_{fast} \left\langle \int d^3v \ I v_\parallel \widehat{\mathbf{b}} \cdot \boldsymbol{\nabla}\theta \frac{\partial}{\partial\theta} \left(\frac{v_\parallel}{\Omega} \right) f \right\rangle \end{aligned}$$

We **NOTE**

that the calculation is based on the expression of the *fast ion current* composed of the product of their charge, the neoclassical drift and their distribution function.

In *fluid* treatments like **Honda** one has a connection between radial current (surface-averaged) and the *toroidal acceleration* of the flow.

END.

We perform an integration by parts in θ:

$$e_{fast} \left\langle \int d^3v \, \mathbf{v}_d \cdot \boldsymbol{\nabla}\psi \, f \right\rangle = -e_{fast} I \left\langle \int d^3v \ \left(\frac{v_\parallel}{\Omega} \right) v_\parallel \widehat{\mathbf{b}} \cdot \boldsymbol{\nabla} f \right\rangle$$

Now we must replace f by its expansion (**??**). The lowest order f_{-1} does not depend on the parallel coordinate (in the surface) so it will not contribute. The first contribution is from the term f_0. This is obtained in equation (**??**) where we have to insert the equation (**??**)

$$\frac{\partial f_{-1}}{\partial t} = \overline{C}_f \ f_{-1} + \overline{S}_f$$

which shows that the *time variation* of the lowest order distribution f_{-1} is only due to *collisions* and *source*. There is no rotation damping, polarization, transient rise of a radial electric field due toa non-ambipolar loss of ions or similar.

The result is

$$
\begin{aligned}
v_\parallel \widehat{\mathbf{b}} \cdot \boldsymbol{\nabla} f_0 &= -\mathbf{v}_d \cdot \boldsymbol{\nabla}\psi \frac{\partial f_{-1}}{\partial \psi} \\
&+ C_{fast} f_{-1} - \overline{C}_{fast} f_{-1} \\
&+ S_{fast} - \overline{S}_{fast}
\end{aligned}
$$

Again, a **Comment**.

This equation has a structure which is neoclassic: the parallel variation of the f_0 is due to the radial advection by the *drift velocity* of the lower order distribution f_{-1}.

Certainly, the collisions and the source must occur and modify this elementary neoclassical balance.

However we note the way they appear:

as difference between the real, actual, value and the bounce-averaged value.

This is what **Fong Hahm** will also do.

End

The first term on the right will not contribute after **surface-averageing**

$$
\begin{aligned}
e_{fast} I \left\langle \int d^3v \, \left(\frac{v_\parallel}{\Omega}\right) \mathbf{v}_d \cdot \boldsymbol{\nabla}\psi \frac{\partial f_{-1}}{\partial \psi} \right\rangle &= e_{fast} I \left\langle \int d^3v \, \left(\frac{v_\parallel}{\Omega}\right) I v_\parallel \widehat{\mathbf{b}} \cdot \boldsymbol{\nabla}\theta \frac{\partial}{\partial \theta} \left(\frac{v_\parallel}{\Omega}\right) \frac{\partial f_{-1}}{\partial \psi} \right\rangle \\
&= \frac{1}{2} e_{fast} I^2 \left\langle \int d^3v \, v_\parallel \widehat{\mathbf{b}} \cdot \boldsymbol{\nabla}\theta \frac{\partial}{\partial \theta} \left[\left(\frac{v_\parallel}{\Omega}\right)^2\right] \frac{\partial f_{-1}}{\partial \psi} \right\rangle
\end{aligned}
$$

The integrand is the product of a *odd function* of $v_\parallel$ ($v_\parallel$) with an *even function* of $v_\parallel$ (f_{-1}). The integral on the velocity space is **zero**. This result is great importance: it shows that the surface averaged radial current of the ions is zero. If we expected to have an effective current simply due to the difference in radial drift motions between the ions and the electrons, this result shows clearly that it cannot exist. (A particularity of this calculation is that we assume that the current is entirely due to the ions, *i.e.* the electrons are considered tied to the magnetic lines. This changes nothing since actually the electrons really have small radial drifts compared to ions. The total radial current is practically the ions radial current.)

The radial current due to the difference between iond and electron drifts exists, but only locally. When it is integrated over the magnetic surface, it gives zero.

In conclusion we have to find somewhere else to produce an effective radial current (*i.e.* whose surface average is not zero). We find:

- the collisions, and
- the source of fast ions [*note that this is exactly our idea: it is the initial, unique, finite motion of the new ion*]

NOTE We have to compare this with the work of Stix 73 where the surface averaged radial current seems to depend on the plasma rotation. If there is no plasma rotation (in that paper's notation, $U = W = 0$ then it still remains in the expression of the "flow" the *diamagnetic velocity*.

The **surface-averaged** radial current of fast ions is

$$\langle \mathbf{j}_{fast} \cdot \boldsymbol{\nabla}\psi \rangle = -e_{fast} \left\langle \int d^3v \, \left(\frac{v_\parallel}{\Omega}\right) \left[C_{fast} f_{-1} - \overline{C}_{fast} f_{-1} + S_{fast} - \overline{S}_{fast} \right] \right\rangle$$

Note

This explains very clearly the current: it is due to the difference between the actual value and the bounce-averaged value.

End.

NOTE ON THE TORQUE

We can continue along this line and use the above expression of the *surface-averaged radial fast ion current* to calculate the torque $\mathbf{j} \times \mathbf{B}$ exerted on plasma at injection. It must be added the **torque due to the frictional forces** in the lowest order of the distribution function, f_{-1}:

$$\mathbf{F}_{fast-e} + \mathbf{F}_{fast-i} = \int d^3v \; m_{fast} \mathbf{v} \, C_{fast} f_{-1}$$

It results from the equation (**??**)

$$T = -m_{fast} \left\langle \frac{I}{B} \int d^3v \; v_\parallel \left[\overline{C}_{fast} f_{-1} - S_{fast} + \overline{S}_{fast} \right] \right\rangle$$

and using the equation (**??**)

$$T = -m_{fast} \left\langle \frac{I}{B} \int d^3v \; v_\parallel \left[\frac{\partial f_{-1}}{\partial t} - S_{fast} \right] \right\rangle$$

If the injection is made in **trapped particle region** the function f_{-1} will not depend on the **direction** of the parallel velocity so its integral is zero. The result is then

$$\begin{aligned} T &= m_{fast} \left\langle \frac{I}{B} \int d^3v \; v_\parallel S_{fast} \right\rangle \qquad (28) \\ &= \left\langle \dot{M}_\phi \right\rangle \end{aligned}$$

The conclusion is that the torque exists only if there is a source of fast ions.

Also at stationarity, the time-derivative of the distribution function is zero and, for **trapped and untrapped** particles, the torque is given by equation (28) which reflects **the angular momentum conservation in the neutral injection**.

The equation (**??**) must be solved for

trapped ions

untrapped ions

18.0.1 Solution of the equation for the lowest order distribution function of the fast ions injected into untrapped orbits, f_{-1}^{untr}

The equation is (**??**) and gives the nostationar dist. function from the source and the collisions averaged over the bounce. The notation f_{-1}^{untr} will be simplified to f. The bounce-averaged collision operator is

$$\begin{aligned}\overline{C}_{fast}f &= 2\frac{m_i}{m_{fast}}\frac{\nu_s v_c^3}{v^3}\frac{1}{I_2(\lambda)}\frac{\partial}{\partial\lambda}\lambda I_1(\lambda)\frac{\partial f}{\partial\lambda} \qquad \text{(pitch angle on ions)}\\ &+\frac{\nu_s}{v^2}\frac{\partial}{\partial v}\left(v^3 f\right) \qquad \text{(slowing down on electrons)}\end{aligned}$$

where

$$I_1(\lambda) = \int_{-\pi}^{\pi}\frac{d\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}\xi$$

$$I_2(\lambda) = \int_{-\pi}^{\pi}\frac{d\theta}{\widehat{\mathbf{b}}\cdot\boldsymbol{\nabla}\theta}\frac{1}{\xi}$$

Note
that these are the ubiquituous

$$\left\langle\frac{v_\parallel}{v}\right\rangle$$

and

$$\left\langle\frac{v}{v_\parallel}\right\rangle$$

Find the result in **Connor1973**.
END.

The *bounce-averaged source term* for injection into trapped orbits is

$$\overline{S}_{fast} = N(\psi)\,\delta_{\sigma,\sigma_0}\Theta(t)\,\delta(\lambda-\lambda_0)\,\frac{\delta(v-v_0)}{\pi v_0^2}$$

where

$$N(\psi) = \frac{\int_{-\pi}^{\pi}\frac{d\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}\dot{n}_{fast}}{\int_{-\pi}^{\pi}\frac{d\theta}{\widehat{\mathbf{b}}\cdot\boldsymbol{\nabla}\theta}\frac{1}{|\xi_0|}}$$

The function step Θ is introduced since the injection begins at a specific moment.

18.1 Comment on the sources in the *alpha* - particles and respectively in the NBI ions

The source of *alpha particles* is

$$S_\alpha = \frac{\dot{n}(\psi,t)}{4\pi v_0^2}\delta(v-v_0)$$

where

$$\begin{aligned}\dot{n}(\psi,t) &\equiv \text{birth rate}\\ v_0 &\equiv \text{birth velocity}\end{aligned}$$

The source for *NBI* is

$$\begin{aligned}S_{fast} &= \dot{n}_{fast}(\psi,\theta,t)\oint\frac{d\zeta}{2\pi}\delta(\mathbf{v}-\mathbf{v}_0)\\ &= \dot{n}_{fast}(\psi,\theta,t)\frac{\left|(1-\lambda_0 B)^{1/2}\right|}{B}\delta_{\sigma\sigma_0}\delta(\lambda-\lambda_0)\frac{\delta(v-v_0)}{\pi v_0^2}\end{aligned}$$

The condition for an ion to be trapped is

$$\lambda_0 B_{\max} > 1$$

because this means that somewhere the factor

$$\lambda B$$

will become 1 and the parallel velocity becomes zero in that point.

After averaging over the bounce

$$\overline{S}_{fast} = N(\psi)\,\delta_{\sigma,\sigma_0}\Theta(t)\,\delta(\lambda-\lambda_0)\frac{\delta(v-v_0)}{\pi v_0^2}$$

Other quantities

$$\begin{aligned}\xi &= (1-\lambda B)^{1/2}\\ &= \frac{v_\parallel}{v}\end{aligned}$$

In the absence of collisions the *torque* that is introduced in plasma by fast ions NBI is

$$\mathcal{T} = m_{fast}\sigma_0 v_0 I\left[\left\langle\frac{\dot{n}_{fast}|\xi_0|}{B}\right\rangle - \frac{\left\langle\dot{n}_{fast}\right\rangle}{\left\langle\frac{B}{|\xi_0|}\right\rangle}\right]$$

Now we can see how these formulas are similar

$$|\xi_0| \sim \frac{v_\parallel}{v}$$

and

$$I = RB_\varphi \sim RB$$

then

$$\mathcal{T} \sim \dot{n}_{fast}\,|e|\,I\left[\left\langle\frac{v_\parallel}{\Omega}\right\rangle - \frac{1}{\left\langle\frac{\Omega}{v_\parallel}\right\rangle}\right]$$

19 Reduced, non-collisional model for the drift-kinetic derivation of the current produced by the new ions

The problem is treated in the paper **RosenbluthHinton1996** (for alpha particles) and the Ref. **HintonRosenbluth1999** (for NBI) the following equation is used to describe the *new* ions:

$$\frac{\partial f}{\partial t} + \left(v_{\parallel}\widehat{\mathbf{n}} + \mathbf{v}_{Di}\right) \cdot \boldsymbol{\nabla} f = S^{ioniz}$$

where the drift velocity of the guiding centre is $\mathbf{v}_{Di} = -v_{\parallel}\widehat{\mathbf{n}} \times \boldsymbol{\nabla}\left(\frac{v_{\parallel}}{\Omega_{ci}}\right)$ and the neoclassical notations will be used: $\xi = v_{\parallel}/v = (1-\lambda/h)^{1/2}$, $\lambda = hv_{\perp}^2/v^2 = h\sin^2\zeta$, $h = 1+\varepsilon\cos\theta$. The *limits* of the trapped particle region in the variable λ are $1-\frac{r}{R} < \lambda < 1+\frac{r}{R}$. The velocity space variables are v, λ and σ (=sign of $v_{\parallel}$).

Since in this simple treatment we neglect the effect of collisions the neoclassical small parameter is the ratio of the banana width to the minor radius, $\delta = \Delta^{t\pm}/a \ll 1$. In usual neoclassical perturbative solution of the drift kinetic equation the zero order distribution function is the Maxwellian. In the present case, the perturbative expansion of f , the solution of the drift-kinetic equation for the new ions, must contain a term which is directly related to the source and, since it is determined by external factors, cannot be ordered as powers of δ. The first term is formally of order -1, f_{-1}.

$$f = f_{-1} + f_0 + f_1 + ... \tag{29}$$

The lowest order, we consider that the source does not have a spatial variation which would be comparable with the dominant term, formally δ^{-1}.

$$\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} f_{-1} = 0 \tag{30}$$

shows that f_{-1} is constant along the magnetic lines, or $f_{-1} = f_{-1}(\psi)$. This result is connected with an assumption about the distribution of ionization processes in space: they have a rate which is constant on a magnetic surface.

However the function in this lowest order f_{-1} has variation due to :

- collisions
- source of NBI ions

these effects will be introduced through bounce averaging, below.

The zeroth order equation is

$$v_{\parallel}\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} f_0 = -\mathbf{v}_{Di} \cdot \boldsymbol{\nabla}\psi \frac{\partial f_{-1}}{\partial \psi} - \frac{\partial f_{-1}}{\partial t} + S^{ioniz} \tag{31}$$

As in any multiple space-time scale analysis we average at this level (0) to obtain a solution on the level (-1). We apply the operator of *bounce averaging* to eliminate the function f_0. This gives the equation

$$\frac{\partial f_{-1}}{\partial t} = \overline{S}^{ioniz} - \overline{\mathbf{v}_{Di} \cdot \boldsymbol{\nabla}\psi \frac{\partial f_{-1}}{\partial \psi}} \tag{32}$$

The operator of *bounce averaging* is

$$\begin{aligned} \overline{A} &= \frac{1}{T} \oint \frac{d\theta}{v_{\parallel} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta} A \\ &= \frac{1}{T} \oint \frac{d\theta}{v_{\parallel} / (qR)} A \end{aligned}$$

The bounce time is

$$T = \oint \frac{d\theta}{v_{\parallel} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta} = \oint \frac{d\theta}{v_{\parallel} / (qR)}$$

The *limits of integrations* for untrapped ions are $[-\pi \ , \ \pi]$ and for trapped ions the integral is defined

$$\oint d\theta = \sum_{\sigma} \sigma \int_{-\theta_0}^{+\theta_0} d\theta$$

where $-\theta_0$ and $+\theta_0$ are the *turning points* of the banana. The radial projection of the guiding centre drift velocity can be written

$$\mathbf{v}_{Di} \cdot \boldsymbol{\nabla}\psi = I v_{\parallel} \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \right) \left(\frac{v_{\parallel}}{\Omega_{ci}} \right) = I v_{\parallel} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta \frac{\partial}{\partial \theta} \left(\frac{v_{\parallel}}{\Omega_{ci}} \right) \tag{33}$$

where $I = R^2 \mathbf{B} \cdot \boldsymbol{\nabla}\varphi = RB_T \equiv I(\psi)$ is a function of only the magnetic surface variable (ψ) and $\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta = 1/(qR)$. At this point we assume that the new ion has reached the asymptotic periodic motion on the banana. Then the radial displacements during the periodic banana orbit average to zero

$$\overline{(\mathbf{v}_{Di} \cdot \boldsymbol{\nabla}\psi)} = \frac{1}{T} \sum_{\sigma} \int_{-\theta_0}^{+\theta_0} \frac{d\theta}{v_{\parallel} / (qR)} I(\psi) \frac{v_{\parallel}}{qR} \frac{\partial}{\partial \theta} \left(\frac{v_{\parallel}}{\Omega_{ci}} \right) = 0$$

From Eq.(32) the equation for f_{-1} which is the bounce averaged zeroth-order equation, becomes

$$\frac{\partial f_{-1}}{\partial t} = \overline{S}^{ioniz} \tag{34}$$

and indeed f_{-1} appears as a direct result of the "external" source.

NOTE the absence of the collision operator, compared with *alpha*. **END.**

The source of new ions of velocity v_0, with direction σ_0 and trapping parameter λ_0 is (**RosenbluthHinton alpha**)

$$\overline{S}^{ioniz} = \overset{.}{n}^{ioniz} (\psi, t) \, \delta_{\sigma,\sigma_0} \Theta(t) \, \delta(\lambda - \lambda_0) \, \frac{\delta(v - v_0)}{\pi v_0^2}$$

The Eq.(34) simply describes the accumulation of new ions with $(v_0, \sigma_0, \lambda_0)$ on the surface ψ. The motion of these ions toward the banana trajectories and the periodic motion that follows must be found at higher orders.

Returning to Eq.(31) we can find f_0 in terms of f_{-1}, using (**??**)

$$v_\parallel \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} f_0 \equiv v_\parallel \nabla_\parallel f_0 = -v_\parallel \nabla_\parallel \left(\frac{I v_\parallel}{\Omega_{ci}} \right) \frac{\partial f_{-1}(\psi)}{\partial \psi}$$

and the solution is

$$f_0(\psi, \theta, t) = -I \left(\frac{v_\parallel}{\Omega_{ci}} \right) \frac{\partial f_{-1}(\psi)}{\partial \psi} + g(\psi, \theta, t) \tag{35}$$

where a constant of integration of the operator $\nabla_\parallel$ is introduced. We make few remarks. First, the time dependence of f_0 inherrited from f_{-1} will be essential for the presence in the theory of the first, transitory and unique, part of the trajectory. Further, the first term can be approximated, using for circular geometry $\frac{I}{B_T} \frac{\partial}{\partial \psi} \simeq \frac{1}{B_\theta} \frac{\partial}{\partial r}$,

$$-I \left(\frac{v_\parallel}{\Omega_{ci}} \right) \frac{\partial f_{-1}(\psi)}{\partial \psi} \approx -\frac{B}{B_\theta} \frac{v_\parallel}{\Omega_{ci}} \frac{\partial f_{-1}(r)}{\partial r} \sim -\rho_\theta \frac{\partial}{\partial r} f_{-1}(r)$$

and we see that the distribution function f_0 is different from the one of the previous level (f_{-1}) by a radial shift of the space argument, of the order of the poloidal Larmor radius$\rho_\theta \sim v_\parallel / \Omega_{ci}$. This is the same relationship as between the first order neoclassical distribution function relative to the Maxwellian equilibrium distribution **HazeltineHinton RMP**. Finally we note that g is constant on the magnetic lines, *i.e.* on surfaces, $\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} g = 0$. A distribution function for *bananas* can never be constant on the magnetic surfaces because the trajectory stops somewhere. Therefore g must only be added to (35) if we consider circulating particles. For our purpose it is not retained.

The next step is the equation for the *first order* f_1, which is derived from the equation written at *zero*-order

$$\frac{\partial f_0}{\partial t} + v_\parallel \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} f_1 + \mathbf{v}_{Di} \cdot \boldsymbol{\nabla} f_0 = 0 \tag{36}$$

This involves the variation of the first order correction function f_1 along the magnetic lines

$$v_\parallel \nabla_\parallel f_1 + ...$$

The variation of the order-one correction of the distribution function (f_1) for trapped particles takes place in the magnetic surface, *i.e.* $v_\parallel \nabla_\parallel f_1$ is a variation along the poloidal angle θ: $f_1 \equiv f_1(\theta, ...)$. Therefore we look for the contribution of this kind from the other terms

$$\mathbf{v}_{Di} \cdot \boldsymbol{\nabla} f_0 \rightarrow \mathbf{v}_{Di} \cdot \boldsymbol{\nabla} \theta \frac{\partial f_0}{\partial \theta} + \mathbf{v}_{Di} \cdot \boldsymbol{\nabla} \psi \frac{\partial f_0}{\partial \psi}$$

where

$$\begin{aligned}
\mathbf{v}_{Di}\cdot\boldsymbol{\nabla}\theta &= -v_{\parallel}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\left(\frac{v_{\parallel}}{\Omega_{ci}}\right)\cdot\boldsymbol{\nabla}\theta \\
\mathbf{v}_{Di}\cdot\boldsymbol{\nabla}\psi &= -v_{\parallel}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\left(\frac{v_{\parallel}}{\Omega_{ci}}\right)\cdot\boldsymbol{\nabla}\psi
\end{aligned}$$

We calculate the first line

$$\left[\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\left(\frac{v_{\parallel}}{\Omega_{ci}}\right)\right]\cdot\boldsymbol{\nabla}\theta = \left[\boldsymbol{\nabla}\theta\times\widehat{\mathbf{n}}\right]\cdot\left[\boldsymbol{\nabla}\theta\frac{\partial}{\partial\theta}+\boldsymbol{\nabla}\psi\frac{\partial}{\partial\psi}\right]\left(\frac{v_{\parallel}}{\Omega_{ci}}\right)$$

and we have

$$\boldsymbol{\nabla}\theta\times\widehat{\mathbf{n}} = -\frac{1}{r}\widehat{\mathbf{e}}_{\psi}$$

Then

$$\begin{aligned}
\mathbf{v}_{Di}\cdot\boldsymbol{\nabla}\theta &= -v_{\parallel}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\left(\frac{v_{\parallel}}{\Omega_{ci}}\right)\cdot\boldsymbol{\nabla}\theta \\
&= -v_{\parallel}\left(-\frac{1}{r}\widehat{\mathbf{e}}_{\psi}\right)\cdot\left\{\left[\boldsymbol{\nabla}\theta\frac{\partial}{\partial\theta}+\boldsymbol{\nabla}\psi\frac{\partial}{\partial\psi}\right]\left(\frac{v_{\parallel}}{\Omega_{ci}}\right)\right\} \\
&= v_{\parallel}\frac{1}{r}RB_{\theta}\frac{\partial}{\partial\psi}\left(\frac{v_{\parallel}}{\Omega_{ci}}\right)
\end{aligned}$$

and similarly

$$\left[\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\left(\frac{v_{\parallel}}{\Omega_{ci}}\right)\right]\cdot\boldsymbol{\nabla}\psi = \left[\boldsymbol{\nabla}\psi\times\widehat{\mathbf{n}}\right]\cdot\left[\boldsymbol{\nabla}\theta\frac{\partial}{\partial\theta}+\boldsymbol{\nabla}\psi\frac{\partial}{\partial\psi}\right]\left(\frac{v_{\parallel}}{\Omega_{ci}}\right)$$

and we replace

$$\boldsymbol{\nabla}\psi\times\widehat{\mathbf{n}} = \widehat{\mathbf{e}}_{\theta}\left(RB_{\theta}\right)$$

from where we obtain

$$\begin{aligned}
\mathbf{v}_{Di}\cdot\boldsymbol{\nabla}\psi &= -v_{\parallel}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\left(\frac{v_{\parallel}}{\Omega_{ci}}\right)\cdot\boldsymbol{\nabla}\psi \\
&= -v_{\parallel}\left[\widehat{\mathbf{e}}_{\theta}\left(RB_{\theta}\right)\right]\cdot\left\{\left[\boldsymbol{\nabla}\theta\frac{\partial}{\partial\theta}+\boldsymbol{\nabla}\psi\frac{\partial}{\partial\psi}\right]\left(\frac{v_{\parallel}}{\Omega_{ci}}\right)\right\} \\
&= -\frac{v_{\parallel}}{r}RB_{\theta}\frac{\partial}{\partial\theta}\left(\frac{v_{\parallel}}{\Omega_{ci}}\right)
\end{aligned}$$

Returning to the initial expression

$$\begin{aligned}
\mathbf{v}_{Di}\cdot\boldsymbol{\nabla}f_0 &= \mathbf{v}_{Di}\cdot\boldsymbol{\nabla}\theta\frac{\partial f_0}{\partial\theta}+\mathbf{v}_{Di}\cdot\boldsymbol{\nabla}\psi\frac{\partial f_0}{\partial\psi} \\
&= v_{\parallel}\frac{1}{r}RB_{\theta}\frac{\partial}{\partial\psi}\left(\frac{v_{\parallel}}{\Omega_{ci}}\right)\frac{\partial f_0}{\partial\theta}-\frac{v_{\parallel}}{r}RB_{\theta}\frac{\partial}{\partial\theta}\left(\frac{v_{\parallel}}{\Omega_{ci}}\right)\frac{\partial f_0}{\partial\psi} \\
&= I\frac{v_{\parallel}}{qR}\left[\frac{\partial}{\partial\psi}\left(\frac{v_{\parallel}}{\Omega_{ci}}\right)\frac{\partial f_0}{\partial\theta}-\frac{\partial}{\partial\theta}\left(\frac{v_{\parallel}}{\Omega_{ci}}\right)\frac{\partial f_0}{\partial\psi}\right]
\end{aligned}$$

then

$$\begin{aligned}
\mathbf{v}_{Di}\cdot\boldsymbol{\nabla}f_0 &= Iv_\parallel\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta\left[\frac{\partial}{\partial\psi}\left(\frac{v_\parallel}{\Omega_{ci}}\right)\frac{\partial f_0}{\partial\theta}-\frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{\Omega_{ci}}\right)\frac{\partial f_0}{\partial\psi}\right]\\
&= I\frac{v_\parallel}{qR}\boldsymbol{\nabla}f_0\times\boldsymbol{\nabla}\left(\frac{v_\parallel}{\Omega_{ci}}\right)
\end{aligned}$$

The *bounce average* of the drift term acting on the 0-order distribution function is zero

$$\overline{(\mathbf{v}_{Di}\cdot\boldsymbol{\nabla}f_0)}=\frac{1}{T}I\int_{-\theta_0}^{+\theta_0}\frac{d\theta}{v_\parallel/(qR)}\frac{v_\parallel}{qR}\left[\frac{\partial}{\partial\psi}\left(\frac{v_\parallel}{\Omega_{ci}}\right)\frac{\partial f_0}{\partial\theta}-\frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{\Omega_{ci}}\right)\frac{\partial f_0}{\partial\psi}\right]$$

At this moment we replace f_0 with its expression in terms of f_{-1};

$$f_0(\psi,\theta,t)=-I\left(\frac{v_\parallel}{\Omega_{ci}}\right)\frac{\partial f_{-1}(\psi)}{\partial\psi}$$

$$\begin{aligned}
&\overline{(\mathbf{v}_{Di}\cdot\boldsymbol{\nabla}f_0)}\\
=\;&\frac{1}{T}I\int_{-\theta_0}^{+\theta_0}d\theta\left[\frac{\partial}{\partial\psi}\left(\frac{v_\parallel}{\Omega_{ci}}\right)\frac{\partial f_0}{\partial\theta}-\frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{\Omega_{ci}}\right)\frac{\partial f_0}{\partial\psi}\right]\\
=\;&\frac{1}{T}I\int_{-\theta_0}^{+\theta_0}d\theta\left\{\frac{\partial}{\partial\psi}\left(\frac{v_\parallel}{\Omega_{ci}}\right)(-I)\frac{\partial}{\partial\theta}\left[\left(\frac{v_\parallel}{\Omega_{ci}}\right)\frac{\partial f_{-1}(\psi)}{\partial\psi}\right]-(-I)\frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{\Omega_{ci}}\right)\frac{\partial}{\partial\psi}\left[\left(\frac{v_\parallel}{\Omega_{ci}}\right)\frac{\partial f_{-1}(\psi)}{\partial\psi}\right]\right\}\\
=\;&-\frac{I^2}{T}\int_{-\theta_0}^{+\theta_0}d\theta\left\{\frac{\partial}{\partial\psi}\left(\frac{v_\parallel}{\Omega_{ci}}\right)\frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{\Omega_{ci}}\right)\frac{\partial f_{-1}(\psi)}{\partial\psi}+\frac{\partial}{\partial\psi}\left(\frac{v_\parallel}{\Omega_{ci}}\right)\left(\frac{v_\parallel}{\Omega_{ci}}\right)\frac{\partial^2 f_{-1}(\psi)}{\partial\theta\partial\psi}\right.\\
&\left.-\frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{\Omega_{ci}}\right)\frac{\partial}{\partial\psi}\left(\frac{v_\parallel}{\Omega_{ci}}\right)\frac{\partial f_{-1}(\psi)}{\partial\psi}-\frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{\Omega_{ci}}\right)\left(\frac{v_\parallel}{\Omega_{ci}}\right)\frac{\partial^2 f_{-1}}{\partial\psi^2}\right\}
\end{aligned}$$

The first and third terms cancel. In addition we know that f_{-1} is constant on the magnetic surfaces and does not depend on θ. Then the second term is zero. It remains

$$\overline{(\mathbf{v}_{Di}\cdot\boldsymbol{\nabla}f_0)}=\frac{I^2}{T}\frac{\partial^2 f_{-1}}{\partial\psi^2}\int_{-\theta_0}^{+\theta_0}d\theta\frac{\partial}{\partial\theta}\left[\frac{1}{2}\left(\frac{v_\parallel}{\Omega_{ci}}\right)^2\right]$$

which is zero. We conclude

$$\overline{\mathbf{v}_{Di}\cdot\boldsymbol{\nabla}f_0}=0\tag{37}$$

We can now calculate the *radial current*, on the basis of the distribution function in orders -1, 0, 1.

This is obtained from tha radial projection of the drift velocity, Eq.(**??**)

$$\mathbf{v}_{Di}\cdot\boldsymbol{\nabla}\psi=I\frac{v_\parallel}{qR}\frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{\Omega_{ci}}\right)$$

The current is projected on the radial direction and the result is averaged over the magnetic surface.

$$\langle \mathbf{j} \cdot \boldsymbol{\nabla} \psi \rangle = |e| \left\langle \int d^3 v \left(\mathbf{v}_{Di} \cdot \boldsymbol{\nabla} \psi \right) f \right\rangle$$

The expression of the drift velocity projected on radial direction is replaced in the integral and an integration by parts over θ is made.

$$\langle \mathbf{j} \cdot \boldsymbol{\nabla} \psi \rangle = - |e| \, I \left\langle \int d^3 v \left(\frac{v_{\parallel}}{\Omega_{ci}} \right) v_{\parallel} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} f \right\rangle$$

Two terms are absent

- the order -1 distribution function f_{-1} does not contribute due to (30)
- the order 0 does not contribute, due to (37)

The first orde to have a contribution to this current (averaged over surface) is f_1.

From the equation (36) we take the term

$$v_{\parallel} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} f_1 = - \frac{\partial f_0}{\partial t} - \mathbf{v}_{Di} \cdot \boldsymbol{\nabla} f_0$$

and the current, averaged on surface, is

$$\langle \mathbf{j} \cdot \boldsymbol{\nabla} \psi \rangle = - |e| \, I \left\langle \int d^3 v \left(\frac{v_{\parallel}}{\Omega_{ci}} \right) \left(- \frac{\partial f_0}{\partial t} - \mathbf{v}_{Di} \cdot \boldsymbol{\nabla} f_0 \right) \right\rangle$$

From this expression the drift term does not contribute

$$\left\langle \int d^3 v \left(\frac{v_{\parallel}}{\Omega_{ci}} \right) \left(\mathbf{v}_{Di} \cdot \boldsymbol{\nabla} f_0 \right) \right\rangle = 0$$

Then the part of this current called transitory, is

$$\left\langle \mathbf{j}^{transit} \cdot \boldsymbol{\nabla} \psi \right\rangle = |e| \, I \left\langle \int d^3 v \left(\frac{v_{\parallel}}{\Omega_{ci}} \right) \frac{\partial f_0}{\partial t} \right\rangle$$

This is the answer. What remains is to calculate the function in 0 order, f_0, by solving its equation derived above from the drift-kinetic equation.

Now we replace the zero-order function, which is the first neoclassical correction, $\sim \rho_\theta \frac{\partial f}{\partial r}$,

$$f_0 = -I \left(\frac{v_{\parallel}}{\Omega_{ci}} \right) \frac{\partial f_{-1}}{\partial \psi}$$

and take the time derivative, then return to the current

$$\frac{\partial f_0}{\partial t} = -I\left(\frac{v_\parallel}{\Omega_{ci}}\right)\frac{\partial^2 f_{-1}}{\partial\psi\partial t}$$

This equation gives us the respnse: we now have the time derivative of f_0 which allows us to calculate the (surface averaged radial component of the) current.

We need

$$\begin{aligned}\frac{\partial f_{-1}}{\partial t} &= \overline{S}^{ioniz} = \overset{.}{n}^{ioniz}\frac{1}{4\pi v^2}\delta\left(v-v_0\right)\\ &= \overset{.}{n}_0^{ioniz} S\left(r,t\right)\frac{1}{4\pi v^2}\delta\left(v-v_0\right)\end{aligned}$$

where we have used our previous factorization of the rate of ionization.

The result is

$$\begin{aligned}&\langle \mathbf{j}\cdot\widehat{\mathbf{e}}_r RB_\theta\rangle\\ &= \left|e\right| I\left\langle\int d^3v\left(\frac{v_\parallel}{\Omega_{ci}}\right)\frac{\partial f_0}{\partial t}\right\rangle\\ &= -\left|e\right| I^2\left\langle\int d^3v\left(\frac{v_\parallel}{\Omega_{ci}}\right)^2\frac{\partial^2 f_{-1}}{\partial\psi\partial t}\right\rangle\end{aligned}$$

We restrict to circular surfaces

$$\langle \mathbf{j}\cdot\widehat{\mathbf{e}}_r\rangle \approx -\left|e\right|\frac{1}{RB_\theta}\left(RB_T\right)^2\frac{1}{RB_\theta}\left\langle\int d^3v\left(\frac{v_\parallel}{\Omega_{ci}}\right)^2\frac{\partial^2 f_{-1}}{\partial r\partial t}\right\rangle$$

We note that the squared factor is actually

$$\left(\frac{v_\parallel}{\Omega_{\theta ci}}\right)^2 = \rho_\theta^2 = \left(\Delta^{t\pm}\right)^2$$

The spatial average contains

$$\begin{aligned}\frac{B_T^2}{B_\theta^2}\left\langle\frac{1}{\Omega_{ci}^2}\right\rangle &= \frac{B_T^2}{B_\theta^2}\frac{m_i^2}{e^2}\left\langle\frac{1}{B^2}\right\rangle\\ &= \frac{B_T^2}{B_\theta^2}\frac{m_i^2}{e^2}\frac{1}{B_0^2}\left\langle\frac{B_0^2}{B^2}\right\rangle\\ &\approx \frac{1}{\Omega_{\theta ci}^2}\left(1+\frac{3\varepsilon^2}{2}\right)\end{aligned}$$

After replacing the expression of $\partial f_{-1}/\partial t$ we have

$$\begin{aligned}
\langle j_r \rangle &= -|e| \left\langle \int d^3 v \rho_\theta^2 \frac{\partial^2 f_{-1}}{\partial r \partial t} \right\rangle \\
&= -|e| \left\langle \int d^3 v \rho_\theta^2 \frac{\partial}{\partial r} \dot{n}_0^{ioniz} S(r,t) \frac{1}{4\pi v^2} \delta(v - v_0) \right\rangle \\
&\sim -|e| \dot{n}_0^{ioniz} \frac{\partial S(r,t)}{\partial r} \langle \rho_\theta^2 \rangle \\
&= -|e| \dot{n}_0^{ioniz} \frac{\partial S(r,t)}{\partial r} \left(\Delta^{t\pm}\right)^2 \gamma' \\
&= -|e| \dot{n}_0^{ioniz} \frac{\partial S(r,t)}{\partial r} \rho_i^2 q^2 \varepsilon^{-1} \gamma' \\
&= -\gamma |e| \dot{n}_0^{ioniz} \frac{\partial S(r,t)}{\partial r} \rho_i^2 q^2 \varepsilon^{-1/2}
\end{aligned}$$

It is understood that ρ_θ and further $\Delta^{t\pm} = \rho_i q \varepsilon^{-1/2}$ are calculated at the velocity v_0.

The constant γ is calculated, for more general conditions, in [**?**] It results

$$\gamma \sim \text{const} \times \varepsilon^{3/2}$$

If the source is switched on at $t = 0$, and then remains constant

$$\left\langle \mathbf{j}^{transit} \cdot \boldsymbol{\nabla} \psi \right\rangle = -|e| I^2 \frac{v_0^2}{2} \mathcal{I} \frac{\partial}{\partial \psi} \left[\dot{n}(\psi, 0) e^{-2\nu_s t} \right]$$

The quantity that remains to be calculated is

$$\mathcal{I} = \sum_\sigma \left\langle \frac{1}{2\Omega_{ci}} \int B d\lambda \left[\frac{\xi}{\Omega_{ci}} - \overline{\left(\frac{\xi}{\Omega_{ci}}\right)} \right] \right\rangle$$

where

$$\begin{aligned}
\xi &= \left| (1 - \lambda B)^{1/2} \right| \\
&= \frac{|v_\parallel|}{v}
\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{I} &= \frac{(2\varepsilon)^{3/2}}{\Omega_{ci}^2} \left\{ \frac{8}{9\pi} + \int_0^1 \frac{dk}{k^{5/2}} \left[\frac{2}{\pi} E\left(k^{1/2}\right) - \frac{\pi}{2K\left(k^{1/2}\right)} \right] \right\} \\
&\approx \frac{(2\varepsilon)^{3/2}}{\Omega_{ci}^2} \times 0.38
\end{aligned}$$

The transient part of the current is

$$\left\langle \mathbf{j}^{transit} \cdot \boldsymbol{\nabla}\psi \right\rangle = -0.54 \times \varepsilon^{3/2} \left| e \right| I^2 \frac{v_0^2}{\Omega_{ci}^2} \frac{\partial}{\partial \psi} \left[\dot{n} \left(\psi, 0 \right) e^{-2\nu_s t} \right]$$

NOTE

On the calculation by **Rosenbluth Hinton** of the radial current from *alpha* particles.

The essential result

$$\left\langle \mathbf{j}^{transit} \cdot \boldsymbol{\nabla}\psi \right\rangle = \left| e \right| I \left\langle \int d^3 v \left(\frac{v_{\parallel}}{\Omega_{ci}} \right) \frac{\partial f_0}{\partial t} \right\rangle$$

can be schematically simplified to

$$j_r \sim \int d^3 v \quad \rho_\theta \frac{\partial f_0}{\partial t}$$

and shows that the current is produced by the *time variation of the zero-order distribution function.*

Since f_0 is calculated from the *source function* f_{-1},

$$\frac{\partial f_0}{\partial t} = -I \left[\left(\frac{v_{\parallel}}{\Omega_{ci}} \right) - \overline{\left(\frac{v_{\parallel}}{\Omega_{ci}} \right)} \right] \frac{\partial^2 f_{-1}}{\partial \psi \partial t}$$

or

$$\begin{aligned} \frac{\partial f_0}{\partial t} &\sim \frac{\partial}{\partial t} \left(\left[\rho_\theta - \overline{\rho_\theta} \right] \frac{\partial f_{-1}}{\partial r} \right) \\ &= \left[\rho_\theta - \overline{\rho_\theta} \right] \frac{\partial}{\partial t} \left[\text{space variation of the source of } \alpha\text{'s} \right] \end{aligned}$$

Since ρ_θ is only function of $v_{\parallel}$, the velocity space integration that will be used to calculate finally the *current*, j_r, will include all possible velocities.

The space variation of the source and the time variation of the source are both essential for the current. [similar to Cordey Houghton, where the *gradient* of the density of the beam is essential for the generation of the radial electric field $\rightarrow$ torque.

But it is hard to see where the finite radial displacement is included.

Possibly in the fact that the radial displacement is exactly ρ_θ.

END

20 Helicity from polarization

The *classical* polarization drift velocity

$$v_{polariz}^{class} = \frac{m}{\left(Ze \right) B^2} \frac{\partial E_r}{\partial t}$$

and is due to gyromotion. We recognize

$$\varepsilon_0 \frac{c^2}{v_A^2} \frac{\partial E_r}{\partial t}$$

The current of polarization

$$j_r^{pol} \sim v^{pol} \sim \frac{\partial E_r}{\partial t}$$

The electric field that arises at NBI or ionization, is *radial*.
This electric field has a time variation up to the saturation.

In the paper **MonteCarlo polarization Peeters** it is mentioned the helical component of the current.
There is an island.