1 Introduction: bootstrap

There is a folder *plasma, models, bootstrap and diamagnetic.*

the bootstrap current is due to:

- the collisional coupling between trapped particles and the passing particles (electrons). The trapped particles, and we can take in particular the case of ions, have a *directed* flow by the unbalance of the local flows of opposite direction on banana orbits, in the presence of the radial gradient of pressure (similar to the case of gyration leading to diamagnetic flow). By collisions this directed flow transfers momentum to the passing electrons and this is a current.
- the process of detrapping of electrons. The electrons that are trapped have a directed flow due to the unbalanced local flows on bananas in the presence of the gradient of pressure. Some of these trapped electrons can be converted collisionally to circulating electrons. Since they had (when trapped) a *directed* flow they will maintain this directed flow while they become untrapped. The generation and sustainment of the bootstrap current from this source is due to a continuous flux of electrons in the velocity space across the boundary trapped / untrapped. The mechanisms for sustainment of this flux are
  - collisions, in a small region in $\mathbf{v}$ space at the boundary trapped/circulating
  - electrostatic fluctuations that modify the conditions under which the electrons are trapped (position along the minor radius, etc.) see **dewitt tang guo turbulence detrapping**

The trapped particles play the essential role.

Therefore the analytical treatment will consist of models that emphasize the banana orbits.

In fluid approach there will be pressure tensor $\boldsymbol{\pi}$ that expresses the anisotropy specific to the motion of the fluid along the line where some of the particles change the sign of their parallel velocity due to magnetic mirror effect and in general the motion is modulated due to the variation of the magnitude of the magnetic field along the line. The pressure stress tensor has a nonzero divergence which reflects the periodic modulation of the parallel/perpendicular velocity along the line. The divergence of the anisotropic pressure tensor $\boldsymbol{\pi}$ is compensated by the friction force. This force balance, projected along $\mathbf{B}$ and averaged on the magnetic surface, provides one of the equations for the bootstrap current.

In kinetic treatment, the Fokker Planck equation has solution expressed in terms of characteristics, and these are the particle's orbits, of which some are trapped and others are circulating. The solution to the kinetic equation will be a function that is perturbed for circulating particles and is simply neoclassic for the trapped region of the velocity space.

Regarding the Neoclassical Tearing Modes.

The current parameter Δ' is essential and is equivalent to the local profile of the current density.

The current contains a component proportional to the gradient of pressure, the *bootstrap* current.

A magnetic island means flattening of all parameters inside the island. In particular of the pressure.

If the pressure inside the island is flat, the gradient is zero and the bootstrap current cannot exist.

The profile of the current is changed because of the absence of the bootstrap current.

Then the conditions for the Tearing Mode are changed.

Now we have Neoclassical Tearing Mode.

The presence of a force $\mathbf{F}$ acting on a particle which is in Larmor gyration leads to modification of gyration with the effect of occurence of a drift of the guiding centre of the particle

$$\frac{\mathbf{F}\times\widehat{\mathbf{n}}}{B}$$

It is explained in **Braginskii, Galeev, etc**. When $\mathbf{F}$ is actually $-\boldsymbol{\nabla}p$ this "force" produces the diamagnetic drift of the particles.

Question

do-we have a similar effect for the bananas ?

Can-we say that it results a *drift of the centers of the bananas* due to modification that the shape of the bananas under the action of the force $\mathbf{F}=-\boldsymbol{\nabla}p$? This would imply that even without collisional transfer of the "unbalanced" local momentum of the trapped particles to the circulating particles, there is however *bootstrap.*

We note however that the bananas are elongated on the magnetic field lines and the modification of the shape of a banana must be examined.

If the result is similar with that of the gyration, the product

$$-\boldsymbol{\nabla}p\times\widehat{\mathbf{n}}$$

is directed $\perp$ $\mathbf{B}$ i.e. mostly in poloidal direction. The bananas cannot drift if the poloidal displacement imposes that the turning point advances deeper in the region of higher B.

NOTE

The formation of a current (*diamagnetic* and in this case "*part of bootstrap*") by the unbalanced fluxes through a point, in the presence of gradient of density - is NOT a valid model.

More clearly this arises in the diamagnetic case.

More explanations are in *research, plasma, boostrap and diamagnetic.* The basic problem is the need for a function δ at the point of consideration, i.e. the independence of the so-called "current" calculated in one point from the "current" produced in another, near-by point. No continuity of the substance that should constitute the current. This means that in this perspective there is no current as real displacement of a continuum of charge. We do not have the presence of a charge in a point $(t, \mathbf{x})$ and precisely the same charge (colored if needed) in the point $(t+\delta t, \mathbf{x}+\delta\mathbf{x})$. Only this is a physical current. Only this can produce a magnetic field.

The diamagnetic current arises from the particle drift which is expressed by

$$\mathbf{v} = \frac{\mathbf{F}\times\widehat{\mathbf{n}}}{B}$$

in the presence of a force. It results from a perturbation of the geometry of the trajectory. In the diamagnetic case, the gyration circle is deformed by the force $\mathbf{F} = -\boldsymbol{\nabla} p$. Using the unbalance of the two opposite fluxes of gyrating particles through the same point does NOT lead to any formula and cannot identify the elements *env*.

Then we must accept that there is a similar situation for the bananas. The banana is the equivalent of the gyration circle. The presence of a force $\mathbf{F}$ modifies the geometry of the banana and from this deformation it arises a drift of the trapped particles $\mathbf{v} = \frac{\mathbf{F}\times\widehat{\mathbf{n}}}{B}$. This part of the bootstrap current does NOT depend on collisions and will survive in the reactor $(\nu \to 0)$ regime.

2 Review Peeters 2000

First considerations are just a repetition of the reasoning for *diamagnetic* unbalance of flows through a point, due to the gradient of density. Instead of gyration circles, one has here the bananas of trapped particles.

It is said that this is *part of the bootstrap current.*

$$w = \frac{v_{\parallel}}{\Omega_{\theta}} \text{ width of an ion banana}$$

$$\begin{aligned} \delta n_{unbalanced}^{trapped} &= w \, \frac{dn^{trapped}}{dr} \\ & \quad \text{the difference between the densities} \\ & \quad \text{of trapped particles} \\ & \quad \text{flowing in opposite direction} \end{aligned}$$

where

$$\begin{aligned} n^{trapped} &\approx \sqrt{\varepsilon}\; n \\ v_{\parallel}^{trapped} &\approx \sqrt{\varepsilon}\; v_{th,i} \end{aligned}$$

Then the flux of trapped ions resulting from uncompensated flows in a point is

$$nu_{i\parallel}^{trapped} = \varepsilon^{3/2} \frac{T_i}{eB_\theta} \frac{dn}{dr}$$

As it is introduced it is independent of collisions.

2.1 Discontinuity trapped/circulating distribution function

In **Galeev Sagdeev**. And in *distribution_function_solution.tex*.

There is a figure from **Threshold Wilson phys plasmas vol3 nr. 1 page 248 january 1996** showing the discontinuities of the distribution function at the boundary trapped/circulating, with smooth transition by collisions.

Fig is discontinuity_.

The distribution function for *circulating* particles

$$\begin{aligned}
& f_{j,untrapped}^{(0)} \\
= & \frac{n_j(r)}{\left(\sqrt{\pi} v_{th,j}\right)^3} \exp\left[-\frac{e\Phi(r)}{T_j} - \frac{v^2}{v_{th,j}^2} - \frac{(v_E/\Theta)^2}{v_{th,j}^2}\right. \\
& \left. -2x_j\, \varepsilon\, \kappa^2 - \sigma \frac{\pi\sqrt{2x_j\varepsilon}}{2} \frac{v_E/\Theta}{v_{th,j}} \int_1^{\kappa^2} \frac{dt}{\sqrt{t} E\left(\frac{1}{\sqrt{t}}\right)}\right] \\
& \times \left\{1 + \sigma \frac{\sqrt{2x_j\varepsilon}}{\Theta} \rho_j \frac{1}{n(r)} \frac{dn(r)}{dr} \left(\sqrt{\kappa^2 - \sin^2\frac{\theta}{2}} - \frac{1}{2}\int_1^{\kappa^2} \frac{dt}{\sqrt{t} E\left(\frac{1}{\sqrt{t}}\right)}\right)\right\}
\end{aligned}$$

The full distribution function for *trapped* particles

$$\begin{aligned}
f_{j,trapped} = & \frac{n(r)}{\left(\sqrt{\pi} v_{th,j}\right)^3} \exp\left(-x_j - c_j^2 - 2x_j\varepsilon\, \kappa^2\right) \\
& \times \left\{1 + \sigma \frac{\sqrt{2x_j\varepsilon}}{\Theta} \rho_j \frac{1}{n(r)} \frac{dn(r)}{dr} \sqrt{\kappa^2 - \sin^2\frac{\theta}{2}}\right. \\
& \left. + \nu_j \frac{2A_j}{v_{th,j}} (r\theta) \frac{1}{\Theta} x_j \varepsilon \left(c_j + \frac{\rho_j}{2\Theta} \frac{1}{n(r)} \frac{dn(r)}{dr}\right)\right\}
\end{aligned}$$

There is a figure in *additional* from *bootstrap*, showing the two discontinuities of the profile of the distribution function in function of parallel velocity, at values of $v_\parallel$ that define the boundaries between trapped (the zone around $\left|v_\parallel\right| \sim 0$) and the circulating particles where $\left|v_\parallel\right|$ can be arbitrarily large.

"The main part of the bootstrap current is carried by the passing particles and is generated

through collisional coupling of trapped and passing particles."

We have

$$\tau_{ei} \gg \tau_{ii}$$

The Coulomb collisions are diffusive.

Then to change the velocity direction by an angle $\delta\theta$, one takes as reference the frequency ν of collisions needed to change the direction with 90^o, and obtain

$$\frac{\nu}{(\delta\theta)^2}$$

[here the quantity that behaves diffusively under the action of scatterings is the angle θ. It is not the angle that occurs in the Rutherford formula for scattering, where at the denominator we have $[\sin(\theta/2)]^4$.]

"trapped ions give momentum to the passing ions on

a typical timescale ε/ν_{ii}. If we assume ε to be small, a passing particle must be scattered over

an angle of roughly 90^o to enter the trapped domain. Only $\sqrt{\varepsilon}$ of its original momentum is

transferred to the trapped particles."

2.2 Balance for transfer of momentum to passing particles

The object of interest is

$$u_{i\|}^{circ}$$

which is the velocity of circulating ions.

The time evolution of the parallel velocity of the circulating particles is given by two effects.

These two effects both involve collisions.

There is a transfer from the population of circulating ions, via ion-ion collisions, to the momentum of the trapped ions

$$\sqrt{\varepsilon}\nu_{ii}m_i n\ u_{i\|}^{circ}$$

where ν_{ii} is the collisionality rate of the ions and $\sqrt{\varepsilon}n$ is the number of trapped ions. The total momentum of the trapped ions will be reduced by these collisions. (?)

The momentum equation is

$$\begin{aligned} nm_i\frac{\partial u_{i\|}^{circ}}{\partial t} &= m_i\frac{\nu_{ii}}{\varepsilon}\varepsilon^{3/2}\frac{T_i}{eB_\theta}\frac{dn}{dr} \\ &\quad -\sqrt{\varepsilon}\nu_{ii}m_i n\ u_{i\|}^{circ} \end{aligned}$$

The first term comes from the "bootstrap" of ions that are on bananas, diamagnetic-like $nu_{i\|}^{trapped} = \varepsilon^{3/2}\frac{T_i}{eB_\theta}\frac{dn}{dr}$. The factor ν_{ii}/ε is the time rate of transfer from the uncompensated trapped flow to the circulating particles.

The second term is a loss of momentum by the circulating ions to the trapped ions $\sqrt{\varepsilon}n$, with usual collision frequency ν_{ii}.

The fluid formulation.

In parallel projection, the full momentum conservation is

$$\begin{aligned} nm_i\mathbf{B}\cdot\frac{d\mathbf{u}_i}{dt} &= \mathbf{B}\cdot\left(-\boldsymbol{\nabla}p\right)+\mathbf{B}\cdot\left(-\boldsymbol{\nabla}\cdot\overline{\boldsymbol{\pi}}\right) \\ &\quad +en\ \mathbf{B}\cdot\mathbf{E}+\mathbf{B}\cdot\mathbf{F} \end{aligned}$$

This is first adapted for *circulating* ions.

It is neglected
- the $\mathbf{B}\cdot\left(-\boldsymbol{\nabla}p\right)$ term,
- the $en\ \mathbf{B}\cdot\mathbf{E}$ term
- the $\mathbf{B}\cdot\mathbf{F}$ term

And what remains is the balance between the pressure anisotropy force with the time-variation of the momentum.

Then it is assumed that the build-up of a parallel velocity of circulating ions by the moment transferred from trapped ions can be represented as anisotropy of the pressure tensor, projected on the magnetic field

$$nm_i\mathbf{B}\cdot\frac{d\mathbf{u}_i}{dt}=\mathbf{B}\cdot\left(-\boldsymbol{\nabla}\cdot\overline{\boldsymbol{\pi}}\right)$$

Here, therefore one replaces the LHS with the formula above

$$nm_i\frac{\partial u_{i\|}^{circ}}{\partial t}=m_i\frac{\nu_{ii}}{\varepsilon}\varepsilon^{3/2}\frac{T_i}{eB_\theta}\frac{dn}{dr}-\sqrt{\varepsilon}\nu_{ii}m_in\ u_{i\|}^{circ}$$

and obtain

$$\begin{aligned} B\ m_i\nu_{ii}\sqrt{\varepsilon}\left(\frac{T_i}{eB_\theta}\frac{dn}{dr}-n\ u_{i\|}^{circ}\right) &= -\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\overline{\boldsymbol{\pi}} \\ B\ m_i\nu_{ii}\sqrt{\varepsilon}\ n\ \left(\frac{T_i}{eB_\theta}\frac{d}{dr}\ln n-u_{i\|}^{circ}\right) &= -\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\overline{\boldsymbol{\pi}} \end{aligned}$$

The first term in the brackets is the diamagnetic velocity of ions

$$\begin{aligned} &\frac{T_i}{eB_\theta}\frac{d}{dr}\ln n-u_{i\|}^{circ} \\ =\ &\frac{B_{tor}}{B_\theta}\left(-v_{i,dia}\right)-u_{i\|}^{circ} \end{aligned}$$

since

$$v_{i,dia}=-\frac{T_i}{eB_{tor}}\frac{d}{dr}\ln n$$

From these two terms it can be formed the *poloidal* velocity of ions. First one multiplies

$$\begin{aligned}
& \left[\frac{T_i}{eB_\theta}\frac{d}{dr}\ln n - u_{i\|}^{circ}\right] \times \frac{B_\theta}{B} \\
= & \frac{B_{tor}}{B} \times \frac{T_i}{eB_{tor}}\frac{d}{dr}\ln n - u_{i\|}^{circ}\frac{B_\theta}{B} \\
= & \frac{B_{tor}}{B} \times (-v_{i,dia}) - u_{i,pol}^{circ} \\
= & -V_{pol}
\end{aligned}$$

this is the poloidal velocity of the circulating ions. The ratio B_{tor}/B close to 1 corrects the diamagnetic velocity which is perpendicular on the magnetic line to be projected on the poloidal direction.

$$V_{pol} = (v_{i,dia})_{pol} + u_{i,pol}^{circ}$$

Now the paranthesis $\left(\frac{T_i}{eB_\theta}\frac{d}{dr}\ln n - u_{i\|}^{circ}\right)$ is replaced by

$$\left(\frac{T_i}{eB_\theta}\frac{d}{dr}\ln n - u_{i\|}^{circ}\right) = \frac{B}{B_\theta}(-V_{pol})$$

and the momentum equation is written

$$\begin{aligned}
B\ m_i\nu_{ii}\sqrt{\varepsilon}\ n\ \left(\frac{T_i}{eB_\theta}\frac{d}{dr}\ln n - u_{i\|}^{circ}\right) &= -\mathbf{B}\cdot\nabla\cdot\overline{\boldsymbol{\pi}} \\
\frac{B^2}{B_\theta}m_i\nu_{ii}\sqrt{\varepsilon}n\ V_{pol} &= \mathbf{B}\cdot\nabla\cdot\overline{\boldsymbol{\pi}}
\end{aligned}$$

The factors $m_i\nu_{ii}\sqrt{\varepsilon}n$ are dynamical viscosity, μ_i

$$\mu_i B^2\frac{V_{pol}}{B_\theta} = \mathbf{B}\cdot\nabla\cdot\overline{\boldsymbol{\pi}}$$

Returning

$$nm_i\mathbf{B}\cdot\frac{d\mathbf{u}_i^{circ}}{dt} = -\mu_i B^2\frac{V_{pol}}{B_\theta}$$

The interpretation given by Peeters

"The density gradient leads to

a diamagnetic velocity in the surface which has a poloidal component. In this direction,

however, the magnetic field strength changes which leads to a viscous force which damps the

poloidal rotation through a build-up of the parallel velocity until the poloidal component of *this velocity cancels the poloidal component of the diamagnetic velocity. The total velocity*

is then in the direction of the symmetry of the system and, therefore, no longer 'feels' the

variation of the field strength. The viscous force that appears in the fluid theory can be traced

back to the friction between trapped and passing. By definition, a trapped particle cannot rotate

in the poloidal direction"

Comment

- there is the diamagnetic flow
- the diamagnetic flow is perpendicular on the magnetic field line
- then there is a poloidal projection of the diamagnetic flow
- this poloidal projection (of the diamagnetic flow) is heavily damped by TTMP
- the process of damping of the poloidal projection of the diamagnetic flow necessarily induce a *parallel* flow
- this parallel flow also has a poloidal projection
- the poloidal projection of the parallel flow and the poloidal projection of the diamagnetic flow *cancel* each other
- there will not be any poloidal rotation.
- all rotation is toroidal

Note that the gradient of temperature is not taken into account.

See explanation of parallel flow in **Stringer**.

2.3 Steady state

Then

$$\mu_i \left(\frac{T_i}{eB_\theta} \frac{d}{dr} \ln n - u_{i\|}^{circ} \right) = 0 \quad \text{for ions}$$

$$\mu_e \left(\frac{T_e}{eB_\theta} \frac{d}{dr} \ln n - u_{e\|}^{circ} \right) = l_{ei} \left(u_{\|,e} - u_{\|,i} \right) \quad \text{for electrons}$$

where

$$
\begin{aligned}
l_{ei} &= m_e n \nu_{ei} \\
& \quad \text{electron friction with ions}
\end{aligned}
$$

Solutions

$$
u_{i\|}^{circ} = \frac{T_i}{eB_\theta}\frac{d}{dr}\ln n
$$

$$
\begin{aligned}
u_{e\|} &= \frac{l_{ei}}{\mu_e + l_{ei}}\frac{T_i}{eB_\theta}\frac{d}{dr}\ln n \\
& \quad -\frac{\mu_e}{\mu_e + l_{ei}}\frac{T_e}{eB_\theta}\frac{d}{dr}\ln n
\end{aligned}
$$

These formulas should be adapted to densities that can be different (?) for electrons and ions

The bootstrap current is the difference

$$
\begin{aligned}
J_{BS} &= \frac{T}{B_\theta}\frac{\mu_e}{\mu_e + n m_e l_{ei}}\left[\frac{dn_e}{dr} + \frac{dn_i}{dr}\right] \\
&= \frac{T}{B_\theta}\sqrt{\varepsilon}\left[\frac{dn_e}{dr} + \frac{dn_i}{dr}\right]
\end{aligned}
$$

3 Bootstrap in island Peeters

Monte Carlo polarization Peeters.

Polarization current Peeters

The formulas for the field

$$
B_{tor} = \frac{B_0}{1 + \varepsilon\cos\theta}
$$

$$
B_{pol} = \frac{B_{\theta 0}}{1 + \varepsilon\cos\theta}
$$

and it is adopted

$$
q(r) = q_0\left(1 + br^2\right)
$$

Then

$$
B_{\theta 0} = \frac{\varepsilon\ B_0}{q(r)\sqrt{1-\varepsilon^2}}
$$

Connect the coordinates

$$
(r, \theta)
$$

with the Boozer coordinates

$$
(\psi, \chi)
$$

as

$$r = R_c \sqrt{1 - \frac{1+b}{b} \tanh^2 \left[\sqrt{b\,(1+b)}\ \ (a_0 - q_0 \psi) \right]}$$

$$\theta = 2 \arctan \left[\sqrt{\frac{1+r}{1-r}}\ \tan \left(\frac{\chi}{2} \right) \right]$$

where

$$a_0 = \frac{\arctan h \sqrt{b\,(b+1)}}{\sqrt{b\,(b+1)}}$$

The bootstrap current is calculated in the presence of the island with

$$j_{bs} = \left\langle en\ v_{i\|}\ B \right\rangle$$

The distribution function is calculated numerically.

Probably the $v_{i\|}$ is averaged. Apparently it is NOT used the gradient of pressure for the bootstrap current.

the boostrap current is just the averaged parallel ion flow.

In other words: if there is parallel flow of ions it can only be produced by the collsional transfer of momentum to circulating ions by the trapped ions which have agradient of pressure.

There are three time scales

- bounce time

$$\tau_B = \frac{1}{\sqrt{\varepsilon}} \frac{qR}{v_{th,i}}$$

- trapped to passing scattering time

$$\tau_s = \frac{1}{\nu_i / \varepsilon}$$

- toroidal drift time of trapped particles

$$\tau_D = 4\pi\varepsilon\Omega_c \frac{R^2}{q v_{th}^2}$$

"the bootstrap current requires a few collisional times to build up (outside the

island). Inside the island, the current oscillates about zero (the poorer statistics is essentially

due to the smaller number of simulation particles in the island)."

4 Notes

"*The bootstrap current is carried by both electron and ion species roughly at an equal amount.*"

from **main ion and impurity rotation Kim**

(also in ERS current hole)

The number of particles that are trapped (including the α's) in the region where there is current hole, i.e. the central region, is very small

$$\begin{aligned} & \sim \sqrt{\varepsilon} \\ \text{for } \varepsilon & = \frac{r}{R} \ll 1 \end{aligned}$$

The "magnetic mirror" seen by the particles is weak. There are little effects of trapping. Then the bootstrap current is not substantial even if the gradient of pressure is strong (due - possibly - to a ITB nearby). Typically the profile of the bootstrap current starts from 0 on the axis, raises slowly to a maximum attained in the confinement region then decays toward the edge.

However in paper **ASDEX current hole EPS** it is said that the bootstrap current can exceed the inductive current and so it reverses the voltage and expells the current, forming a current hole.

4.1 Hinton Kim poloidal rotation 1995

This is interesting only because the flow of ions is supported by the trapped ions, with unbalanced local fluxes arising from density gradients.

[this is the usual reasoning as in the diamagnetic case, a local un-balance of flows of trapped (there: gyrating) particles on bananas. The unbalanceed fluxes are NOT currents but reveal the existence of available momenta that can be transferred collisionally to the circulating particles. This is the same mechanism as for bootstrap]

In the paper **Hinton Kim Kim Brizard Burrell poloidal rotation 1995**

The objective is rotation.

But the reasonning is good for bootstrap too.

The poloidal plasma rotation is the poloidal rotation of the circulating (untrapped) ions.

"*Consider the detailed balance of momentum transfers in collisional trapping and detrapping. A steady state requires that momentum be gained by the trapped ions in the process of collisional trapping at the same rate that it is removed by* *collisional detrapping:"*

$$n_{untr} u_{\| untr} = \left(\frac{\nu_{tr}}{\nu_{untr}} \right) n_{tr} \ u_{\| tr}$$

The ratio of collision frequencies is reduced (after simplifying factors that are common) to

$$\frac{\nu_{tr}}{\nu_{untr}} = \left(\frac{v_{th,i}}{\delta u_{\|}} \right)^2$$

$$\begin{aligned} \delta u_{\|} &= (\varepsilon S)^{1/2} \ v_{th,i} \\ &= \text{range of parallel velocities} \\ &\quad \text{for the trapped particles} \end{aligned}$$

NOTE

This is an essential point in the derivation of the flow of circulating ions sustained by collisions with trapped ions. [this is the bootstrap mechanism]

The ratio of the frequencies of collisions is estimated in terms of concrete parameters, the thermal velocity of ions $v_{th,i}$ and the "range of parallel velocities of trapped ions" $\delta u_{\| tr}$.

But if the frequencis of collisions are zero this argument does NOT work.

END

These considerations are good for equilibrium state, not for any transient process.

For example, a fast process like the ELM modify the number of trapped particles.

Then there is NO balance in *collisional transfer* into trapped and respectively out of trapped.

The RHS term,

$$n_{tr} \ u_{\| tr} \ \nu_{tr}$$

remains the same.

NOTE

that it is taken into account the number

$$\sqrt{\varepsilon}$$

of traped particles and the squeezing factor

$$\sqrt{S}$$

if there is a second derivative of the electrostatic potential, $\partial^2 \phi / \partial r^2$.

The presence of $\sqrt{\varepsilon}$ is the usual **Rosenbluth Hazeltine Hinton** evaluation of the velocity of trapped particles.

END.

NOTE
In **Connor Cordey 1974**
This paper is NBI current drive.

The momentum that is obtained by the background ions i is the momentum lost by the hot ions *hot*

$$m_i n_i u_i + m_{hot} n_{hot} u_{hot} = 0$$

The current carried by the ions

$$\begin{aligned} j_i &= eZ_{hot}\ n_{hot} u_{hot} - eZ_i\ n_i u_i \\ &= eZ_{hot} n_{hot} u_{hot} \left(1 - \frac{m_{hot} Z_i}{m_i Z_{hot}}\right) \end{aligned}$$

There is a visible distinction between the two.

- In **Connor Cordey** the density of momentum (mnu) is fully exchanged between the populations of hot (NBI) ions and of background ions, which are - in principle, distinct types of ions.
- in **Kim Hinton** the *particles* are the same, *ions*, but some are trapped and some are circulating. There are actually *two* transitions
  - transition from circulating ions to trapped ions (collisional trapping with frequency ν_{untr}) involving for the trapped ion population of a gain of momentum per unit time $\nu_{untr} \times n_{untr} u_{\parallel untr}\ m_i$.
  - transition from trapped ions to untrapped condition, collisional with frequency ν_{tr}, with a loss for the trapped population of a momentum per unit time $\nu_{tr} \times n_{tr}\ u_{\parallel tr}\ m_i$.

The balance of the gain/loss of momentum per unit time, at equilibrium, means

$$n_{untr} u_{\parallel untr}\ m_i\ \nu_{untr} = n_{tr}\ u_{\parallel tr}\ m_i\ \nu_{tr}$$

from where the mass m_i disappears.

This balance involves the frequencies of collisions *in* respectively *out* the trapped populations.
END

From simply this equality is hard to give a unique physical picture
- the calculation refers to *conversion* via collisions
(1) of trapped to circulating and
(2) of circulating to trapped

or

, the model without conversion of ions from trapped to circulating and reversed process

- the calculation simply refers to collisional momentum transfer from trapped to circulating, so trapped ions support the electric current (flow) of circulating particles

(1) but, one needs to put a limit to the amount of momentum received by the circulating ions by collisions from the trapped ions

(2) and finds this limit in the return of momentum from circulating to trapped

Returning to poloidal rotation (**Hinton Kim Kim**)

The following considerations are similar to the diamagnetic current calculation.

Counting the flux of trapped particles in a point, from the two directions

$$\begin{aligned} n_{tr}\ u_{\parallel tr} &= \frac{1}{2}\left[n\left(r-\frac{\delta r}{2}\right)-n\left(r+\frac{\delta r}{2}\right)\right] \\ &\quad \times\delta u_{\parallel} \\ &= -\frac{1}{2}\ \delta r\ \delta u_{\parallel}\ \frac{\partial n}{\partial r} \end{aligned}$$

Here δr is taken the banana width

$$\delta r=\sqrt{\frac{\varepsilon}{S}}\ \frac{v_{th,i}}{\Omega_{\theta}}$$

(**note** the usual Rosenbluth Hazeltine Hinton estimation for the width of the banana, modified by the squeezing factor).

In the following resides the essence of the idea of this identification of *bootstrap.*

"identifying the mean ion flow
with the untrapped ion flow,", i.e. circulating.

Here is the mean ion flow, calculated from the old diamagnetic-like reasoning

$$\begin{aligned} n_{tr}\ u_{\parallel tr} &= -\frac{1}{2}\ \delta r\ \delta u_{\parallel}\ \frac{\partial n}{\partial r} \\ \text{where}\ \ \delta r &= \sqrt{\frac{\varepsilon}{S}}\ \frac{v_{th,i}}{\Omega_{\theta}}\quad \text{(width of banana)} \\ \delta u_{\parallel} &= (\varepsilon S)^{1/2}\ \ v_{th,i}\quad \text{(range of parallel velocites)} \\ & \qquad\qquad\qquad\qquad \text{(for the trapped particles)} \end{aligned}$$

then

$$\begin{aligned} n_{tr}\ u_{\parallel tr} &= -\frac{1}{2}\ \delta r\ \delta u_{\parallel}\ \frac{\partial n}{\partial r}=-\frac{1}{2}\sqrt{\frac{\varepsilon}{S}}\ \frac{v_{th,i}}{\Omega_{\theta}}\ (\varepsilon S)^{1/2}\ \ v_{th,i}\frac{\partial n}{\partial r} \\ &= -\frac{1}{2}\varepsilon\frac{v_{th,i}^2}{\Omega_{\theta}}\frac{\partial n}{\partial r} \end{aligned}$$

NOTE that this is just the unbalance of flows of trapped particles in a point, calulated on the basis of *diamagnetic* reasoning. It will be used below in connection with collisional detrapping to calculated how much of *momentum* is lost by the trapped ion population by collisional conversion to circulating particles.
END.

NOTE
From this flux of *trapped particles* the bootstrap current is calculated by multiplying with the fraction of trapped particles

$$\begin{aligned} j_B &= \sqrt{\varepsilon} \times \left[-\frac{1}{2} \varepsilon \frac{v_{th,i}^2}{\Omega_\theta} \frac{\partial n}{\partial r} \right] \\ &= -\varepsilon^{3/2} \frac{T_i}{B_\theta} \frac{\partial n}{\partial r} \end{aligned}$$

END

It will be used below a general estimation for the frequency of collision

$$\nu \sim v^2$$

which will be adapted both for circulating *untr* and for trapped *tr* particles.

We now connect this "artificial flow = imbalance of fluxes of bananas in a point" (*diamagnetic-like*) with the circulating (*untr*) ion flow via the total momentum lost/gained in collisional conversion trapped-untrapped

$$n_{untr} u_{\| untr} \ m_i \ \nu_{untr} = n_{tr} \ u_{\| tr} \ m_i \ \nu_{tr}$$

from where the current (flow) of circulating ions results

$$\begin{aligned} n_{untr} u_{\| untr} &= n_{tr} u_{\| tr} \ \left(\frac{\nu_{tr}}{\nu_{untr}} \right) = n_{tr} u_{\| tr} \ \left(\frac{v_{th,i}}{\delta u_\|} \right)^2 \\ &= n_{tr} u_{\| tr} \ \frac{v_{th,i}^2}{\left[(\varepsilon S)^{1/2} \ \ v_{th,i} \right]^2} = n_{tr} u_{\| tr} \ \frac{1}{\varepsilon S} \\ &= \left[-\frac{1}{2} \varepsilon \frac{v_{th,i}^2}{\Omega_\theta} \frac{\partial n}{\partial r} \right] \frac{1}{\varepsilon S} \end{aligned}$$

$$n_{untr} u_{\| untr} = -\frac{v_{th,i}^2}{2 S \Omega_\theta} \frac{\partial n}{\partial r}$$

Note the mean ion flow is what really can move, the other being fixed on bananas. They are exclusively the untrapped = circulating ions. **End.**

The calculation was done in a referential that move with $\frac{E_r}{B_\theta}$ velocity in toroidal direction, reasonable choice for discussing banana motions.

Now we return to the laboratory frame

$$\begin{aligned}
nu_\parallel &= n\frac{E_r}{B_\theta} - \frac{1}{S}\frac{T}{eB_\theta}\frac{\partial n}{\partial r} \\
&= \text{(electric, parallel)} \\
&\quad + \\
&\quad \text{(dia, projected parallel)}
\end{aligned}$$

Further, one uses

$$u_\theta = u_\perp + \frac{B_\theta}{B}u_\parallel$$

and

$$u_\perp = -\frac{E_r}{B} + \frac{1}{n_i}\frac{1}{e_i B}\frac{\partial p_i}{\partial r}$$

it results

$$u_\theta = \frac{1}{n_i}\,\frac{T_i}{e_i B}\left(1 - \frac{1}{S}\right)\frac{\partial n_i}{\partial r}$$

No contribution from the radial electric field.

"In DIII-D H-mode plasmas, the resulting ion poloidal flow velocity is predicted to be roughly half the ion diamagnetic velocity near the separatrix,"

NOTE

In **Petviashvili Pogutse, Pokhotelov** the flute modes should be unstable in this regime. But they are limited by magnetic shear. On the contrary, if the electric velocity u from $y - ut + \alpha z$ is higher than $v_{dia,i}$ then flute modes are linearly stable but actually they have negative energy due to coupling to the drift waves.

END

NOTE

This work does NOT assume the presence of a mechanism that supports E_r.

The parallel motion of ions is strictly due to the collisional transfer of momentum from the trapped ions to the circulating ions.

This transfer is such that the unbalanced flux of trapped ions (as results from the gradient of density of trapped ions) is fully transferred collisionally to the circulating ions.

This provides an expression for $u_{\parallel i}$ in terms of $\frac{\partial n_i}{\partial r}$.

The radial electric field contributes (however) to the parallel flow (first the calculations are done in a referential moving with the parallel electric velocity, then return to laboratory).

But when one considers the projection of $u_{\parallel i}$ such as to obtain u_θ, the electric field disappears. This is because the perp component of flow contains already E_r/B but with opposite sign.

In conclusion, here the poloidal flow u_θ does NOT depend on the radial electric field.

What is the meaning of that cancellation of the electric field ?

This means that as much E_r offers a poloidal flow, the same amount is lost by the projection of the E_r contribution to parallel flow. The usual constraint to exclude effective poloidal rotation, since it is damped. What however remains on θ is the diamagnetic flow.

This is what **Hinton Kim** find at the end.

The poloidal rotation is like diamagnetic, but contains a factor S and is suppressed if $S = 1$.

There is NO poloidal rotation in this treatment.

END

5 Main ion and impurity rotation Kim 1994

The bootstrap current is carried by both electron and ion species roughly at an equal amount.

6 Kagan Catto enhancement bootstrap current in pedestal

For the pedestal ion

- the variation of the electrostatic energy across a neoclassical orbit
is comparable with that of
- the kinetic energy of the ion.
Then the orbit is strongly modified.

The strong electric field in the pedstal modifies the boundary between trapped and circulating ions in velocity space. It is no longer a cone centered at the origin in $\left(v_\parallel, v_\perp\right)$.

"The maxwellian remains stationary because the $E \times B$ drift cancels the ion diamagnetic field in the pedestal to lowest order"

there are two modifications

1. the parallel velocity must be shifted by a quantity

$$u = I \ \frac{1}{B}\frac{d\phi}{d\psi}$$

(this is the parallel velocity $E_r \times B_\theta$). The deeply trapped particles are NOT at $v_\parallel \approx 0$ but at

$$v_\parallel + u \approx 0$$

2. one must take into account the modification of the ion orbits which influence the momentum conservation in ion-ion collisions. Factors must be introduced.

Then

$$\begin{aligned} f_{i1} &= -I\,\frac{v_\parallel}{\Omega_i} f_{i0}\,\left[\frac{d\ln p_i}{d\psi} + \frac{Ze}{T_i}\frac{d\phi}{d\psi} + \left(\frac{v^2}{2T_i/M} - \frac{5}{2}\right)\frac{d\ln T_i}{d\psi}\right] \\ &\quad + g \end{aligned}$$

$$g \equiv 0 \text{ for trapped particles}$$

This function g differs little ($\sqrt{\varepsilon}$) from

$$h_\sigma = I\,\frac{v_\parallel + u}{\Omega_i}\left[\frac{v^2 + u^2}{2T_i/M} - \sigma\right]\,f_{i0}\,\frac{d\ln T_i}{d\psi}$$

where

$$I = RB_{tor}$$

$$\mathbf{B} = I\boldsymbol{\nabla}\varphi + \boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi$$

There is a factor, σ.

This is introduced to ensure that there is *conservation of momentum* in the ion-ion collisions

$$\sigma = \frac{\int_0^\infty dy\ e^{-y}\left(y + \frac{u^2}{2T_i/M}\right)^{3/2}\left[\nu_\perp\ y + \nu_\parallel\ \frac{u^2}{T_i/M}\right]}{\int_0^\infty dy\ e^{-y}\left(y + \frac{u^2}{2T_i/M}\right)^{1/2}\left[\nu_\perp\ y + \nu_\parallel\ \frac{u^2}{T_i/M}\right]}$$

The collisional factors

$$\nu_\perp = \nu_{ii}\ \frac{3\sqrt{2\pi}}{2}\frac{\mathrm{erf}\,(x) - \Psi\,(x)}{x^3}$$

$$\nu_\parallel = \nu_{ii}\ \frac{3\sqrt{2\pi}}{2}\frac{\Psi\,(x)}{x^3}$$

where

$$x = \sqrt{\frac{v^2}{2T_i/M}}$$

$$\nu_{ii} = \frac{4\sqrt{\pi}}{3}\ \frac{Z^4 e^4}{\sqrt{M}}\ln\Lambda\ \frac{n_i}{T_i^{3/2}}$$

$$\mathrm{erf}\,(x) = \frac{2}{\sqrt{\pi}}\int_0^\infty dy\ \exp\left(-y^2\right)$$

$$\Psi(x) = \frac{1}{2} \frac{\operatorname{erf}(x) - x \frac{d \operatorname{erf}(x)}{dx}}{x^2}$$

With all these, one can calculate the distribution function of the ions.

Then one can calculate the parallel ion velocity

$$\begin{aligned} V_{i\|} &= -I \frac{1}{B} \left(\frac{d\phi}{d\psi} + \frac{1}{Zen_i} \frac{dp_i}{d\psi} \right) \\ &\quad + \frac{7}{6} I \frac{B}{Ze \langle B^2 \rangle} J(U) \frac{dT_i}{d\psi} \end{aligned}$$

for

$$U \equiv \frac{u}{\sqrt{2T_i/M}}$$

$$\langle\rangle \equiv \text{surface average}$$

$$\begin{aligned} J(U) &= \frac{6}{7} \left[\left(\frac{5}{2} - \sigma \right) + U^2 \right] \\ &\quad \text{shape factor} \end{aligned}$$

the perpendicular velocity is

$$V_{i\perp} = \frac{\widehat{\mathbf{n}} \times \boldsymbol{\nabla}\psi}{B} \left[\frac{d\phi}{d\psi} + \frac{1}{Zen_i} \frac{dp_i}{d\psi} \right]$$

Combining these two velocities

$$V_i^{pol} = \frac{7}{6} I \frac{B_p}{Ze \langle B^2 \rangle} J(U) \frac{dT_i}{d\psi}$$

Now, the presence of the impurities.

They are strongly collisional (boron).

The parallel mean free path is $\ll$ than the qR.

Due to high collisionality the ions and the impurity ions have the same *flow*.

Then

$$\begin{aligned} V_z^{pol} &= V_i^{pol} \\ &\quad -I \frac{B_p}{eB^2} \left(\frac{1}{Zn_i} \frac{dp_i}{d\psi} - \frac{1}{Z_z n_z} \frac{dp_z}{d\psi} \right) \end{aligned}$$

is for Pfirsch Schluter regime.

When it is assumed that the "shaping factor" $J = 1$ then

$$V_i^{pol} \text{ and the diamagnetic flow cancel}$$

The function $J(U)$ become negative for $U > 1.2$.

7 Hirschman PF 21 (1978) 1295

(from *models, bootstrap diamagnetic*)

7.1 General expressions of the current density

The toroidal current composed of

- current induced by the electric field of the transformer $\mathbf{E}$ $\left(\text{or } E^{(A)}\right)$.
- Pfirsch Schluter, to keep zero divergence with the diamagnetic current (or, charge neutrality)
- neoclassical current (includes bootstrap)

Physical explanation by **Shaing**

The poloidal flow of the species are non-uniform due to the difference between neoclassical drifts.

Then there is a tentative of compensation consisting of a flow parallel with the line.

This compensation is uni-directional.

This is the bootstrap current.

(this argument seems close to the image of **Stringer**)

The current

$$\mathbf{j} = \frac{1}{B}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p + j_{\|}\widehat{\mathbf{n}}$$

where

$$\mathbf{j}_{\perp} = \frac{1}{B}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p$$

is the diamagnetic current.

The parallel current $j_{\|}\widehat{\mathbf{n}}$ is called "force-free". This is because the vector product with $\mathbf{B}$ is zero *i.e.* $\mathbf{j} \sim \boldsymbol{\nabla} \times \mathbf{B} \sim \mathbf{B}$, they are parallel.

The magnetic field

$$\mathbf{B} = \mathbf{B}_T + \mathbf{B}_p$$

$$\begin{aligned}
\mathbf{B}_T &= F(\psi)\, \boldsymbol{\nabla}\varphi \quad \text{toroidal} \\
\mathbf{B}_p &= \boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi \quad \text{poloidal}
\end{aligned}$$

$$2\pi\psi \equiv \text{poloidal flux}$$

NOTE

The sense of introducing $F(\psi)$: the real expression for the toroidal magnetic field is

$$\mathbf{B}_T = \frac{B_0}{h}\widehat{\mathbf{e}}_\varphi = B_0\frac{R_0}{R_0 h}\widehat{\mathbf{e}}_\varphi = B_0 R_0 \boldsymbol{\nabla}\varphi \sim F\boldsymbol{\nabla}\varphi$$

The function F depends only on the surface label, ψ.

Here there is NO other dependence of F, it is constant in the circular surfaces.

F is I.

In **Hazeltine Hinton**

$$I(\psi,\theta,\varphi) = \sqrt{g}\left(\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta\right)\cdot \mathbf{B}$$

where

$$\begin{aligned} g(\psi,\theta,\varphi) &= \frac{1}{\left|\boldsymbol{\nabla}\psi\cdot\left(\boldsymbol{\nabla}\theta\times\boldsymbol{\nabla}\varphi\right)\right|^2} \\ \text{and } \frac{\partial g}{\partial\varphi} &= 0 \ \text{(axisymmetry)} \end{aligned}$$

In Hamada coordinates

$$\begin{aligned} \text{and more, } g &= \text{const} \\ & \quad \text{(Hamada)} \end{aligned}$$

In general

$$\begin{gathered} \boldsymbol{\nabla}\psi \ \text{ and } \ \boldsymbol{\nabla}\theta \ \text{ are NOT orthogonal} \\ \text{(for example in } D\text{-shape plasma)} \end{gathered}$$

Using usual definitions for *circular plasma* we have

$$g = \frac{1}{\left(RB_\theta \frac{1}{r}\frac{1}{R}\right)^2} = \left(\frac{r}{B_\theta}\right)^2$$

If g =const such a $B_\theta \sim r$ would correspond to a radially uniform profile of $j(r)$, since in this case

$$\begin{aligned} 2\pi r B_\theta(r) &= \mu_0 \int_0^r 2\pi r dr\ j(r) \sim \mu_0 2\pi \frac{r^2}{2} j_{ct} \\ B_\theta(r) &\sim r \times \left(\frac{\mu_0 j_{ct}}{2}\right) \\ \text{and } \frac{r}{B_\theta(r)} &= \text{ct} \end{aligned}$$

But in general (for general current profile) g is not constant.

Then, for *circular plasma*

$$\begin{aligned} I &= \sqrt{g}\left(\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\theta\right)\cdot\mathbf{B} \\ &= \left(\frac{r}{B_\theta}\right) RB_\theta \frac{1}{r}\ B_T = RB_T \end{aligned}$$

But, the configuration is not so simple

$$\boldsymbol{\nabla}\theta \neq \frac{1}{r}\widehat{\mathbf{e}}_{\theta}$$

because for a D-shaped plasma r is not relevant.

Also $|\boldsymbol{\nabla}\psi| \neq RB_{\theta}$ in a D-shaped plasma.

This explains why

$$I \neq \text{const}$$

and $F \neq$const in **Hirshman1978**.

In **Hirshman Sigmar**

$$I = R^2 \mathbf{B} \cdot \boldsymbol{\nabla}\varphi$$

and

$$|\boldsymbol{\nabla}\varphi| = \frac{1}{R}$$

and

$$\begin{aligned} \mathbf{B} &= \frac{1}{2\pi}\frac{\partial\chi}{\partial\psi}\boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi \quad \text{(poloidal)} \\ &\quad +I\boldsymbol{\nabla}\varphi \qquad\qquad\qquad\quad \text{(toroidal)} \end{aligned}$$

here

$$\begin{aligned} \chi' &= \frac{\partial\chi}{\partial\psi} = 2\pi\sqrt{g}\ \mathbf{B}\cdot\boldsymbol{\nabla}\theta \\ &= \text{poloidal magnetic flux density} \end{aligned}$$

In **Hazeltine Hinton**

$$I(\psi, \theta, \varphi) = \sqrt{g}\,(\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta) \cdot \mathbf{B}$$

and the definitions

$$\begin{aligned} \psi(\mathbf{x}) &= \frac{1}{2\pi}\int d^3x\ \boldsymbol{\nabla}\theta \cdot \mathbf{B} \\ &\quad \text{poloidal flux function} \end{aligned}$$

$$\begin{aligned} \phi(\mathbf{x}) &= \frac{1}{2\pi}\int d^3x\ \boldsymbol{\nabla}\theta \cdot \mathbf{B} \\ &\quad \text{toroidal flux function} \end{aligned}$$

$$\begin{aligned} V(\mathbf{x}) &= \int d^3x \\ &\quad \text{volume inside a magnetic surface} \end{aligned}$$

END

Returning to the current $\mathbf{j} = \frac{1}{B}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p + j_{\|}\widehat{\mathbf{n}}$ and to the magnetic field $\mathbf{B} = \mathbf{B}_T + \mathbf{B}_p = F(\psi)\boldsymbol{\nabla}\varphi + \boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi$ expressions in **Hirshman1978**.

The expression of the current is multiplied by the poloidal magnetic field $\mathbf{B}_p$,

$$\begin{aligned}
\mathbf{j} \cdot \mathbf{B}_p &= \left(\frac{1}{B}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p\right) \cdot \mathbf{B}_p + j_{\|}\widehat{\mathbf{n}} \cdot \mathbf{B}_p \\
&= \frac{1}{B}\left(\mathbf{B}_p \times \widehat{\mathbf{n}}\right) \cdot \boldsymbol{\nabla} p + j_{\|}\left(\frac{\mathbf{B}}{B}\right) \cdot \mathbf{B}_p \\
&= \frac{1}{B}\left[(\boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi) \times \frac{\mathbf{B}}{B}\right] \cdot \boldsymbol{\nabla} p + j_{\|}\frac{\mathbf{B}_T + \mathbf{B}_p}{B} \cdot \mathbf{B}_p \\
&= \frac{1}{B}\left[(\boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi) \times \frac{\mathbf{B}_T + \mathbf{B}_p}{B}\right] \cdot \boldsymbol{\nabla} p + j_{\|}\frac{B_p^2}{B} \\
&= \frac{1}{B}\left[(\boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi) \times \frac{F(\psi)\boldsymbol{\nabla}\varphi}{B}\right] \cdot \boldsymbol{\nabla} p + j_{\|}\frac{B_p^2}{B}
\end{aligned}$$

We assume that the pressure only has radial (ψ) variation $\boldsymbol{\nabla} p$, we have

$$\begin{aligned}
(\boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi) \times \boldsymbol{\nabla}\varphi &= -\boldsymbol{\nabla}\varphi\,(\boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla}\psi) + \boldsymbol{\nabla}\psi\,(\boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla}\varphi) \\
&\rightarrow |\boldsymbol{\nabla}\varphi|^2\,\boldsymbol{\nabla}\psi
\end{aligned}$$

and

$$\boldsymbol{\nabla} p = \frac{dp}{d\psi}\boldsymbol{\nabla}\psi$$

The first term is

$$\begin{aligned}
&\frac{1}{B}\left[(\boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi) \times \frac{F(\psi)\boldsymbol{\nabla}\varphi}{B}\right] \cdot \boldsymbol{\nabla} p \\
= &\frac{1}{B}\,|\boldsymbol{\nabla}\varphi|^2\,|\boldsymbol{\nabla}\psi|^2\,\frac{dp}{d\psi}F
\end{aligned}$$

where, for *circular surfaces*

$$\begin{aligned}
|\boldsymbol{\nabla}\varphi| &= \frac{1}{R} \\
|\boldsymbol{\nabla}\psi| &= RB_p
\end{aligned}$$

Then

$$\frac{1}{B}\,|\boldsymbol{\nabla}\varphi|^2\,|\boldsymbol{\nabla}\psi|^2\,\frac{dp}{d\psi}F = \frac{1}{B}\frac{1}{R^2}R^2B_p^2\,\frac{dp}{d\psi}F = \frac{B_p^2}{B^2}\frac{dp}{d\psi}F$$

We have the poloidal projection of the electric current

$$\mathbf{j} \cdot \mathbf{B}_p = \frac{B_p^2}{B^2}\frac{dp}{d\psi}F + j_{\|}\frac{B_p^2}{B}$$

Now one divides to B_p^2 and introduces in the left hand side the notation

$$\begin{aligned} K &\equiv \frac{\mathbf{j}\cdot\mathbf{B}_p}{B_p^2} \quad \text{(definition of } K\text{)} \\ &= \frac{j_p}{B_p} \text{ proportional with the poloidal velocity } (v_p \sim j_p) \end{aligned}$$

and then multiply the equation $K = \mathbf{j}\cdot\mathbf{B}_p/B_p^2 = \frac{1}{B^2}\frac{dp}{d\psi}F + j_{\parallel}\frac{1}{B}$ with B. We have

$$j_{\parallel} = -\frac{Fp'}{B} + KB$$

with $p' = dp/d\psi$

$$j_{\parallel} = -\frac{F}{B}\frac{dp}{d\psi} + KB$$

To prove that $K = \frac{j_p}{B_p}$ is a function of only the surface function ψ.

The poloidal component of the Ampere law

$$\begin{aligned} \nabla\times\mathbf{B} &= \mu_0\mathbf{j} \\ (\nabla\times\mathbf{B})\cdot\mathbf{B}_p &= \mu_0\mathbf{j}\cdot\mathbf{B}_p \\ &= \mu_0 K B_p^2 \end{aligned}$$

The rotational of $\mathbf{B}$ is

$$\begin{aligned} \nabla\times(\mathbf{B}_T+\mathbf{B}_p) &= \nabla\times[F(\psi)\nabla\varphi + \nabla\varphi\times\nabla\psi] \\ &= \nabla\times[F(\psi)\nabla\varphi] \;\; (\rightarrow \text{poloidal current}) \\ &\quad +\nabla\times(\nabla\varphi\times\nabla\psi) \;\; (\rightarrow \text{toroidal current}) \end{aligned}$$

Since this will be scalar-multiplied by $\mathbf{B}_p$, we have only the first term because the second term is a sum of vectors oriented along $\nabla\varphi$ and respectively $\nabla\psi$. Then

$$\begin{aligned} (\nabla\times\mathbf{B})\cdot\mathbf{B}_p &= \{\nabla\times[F(\psi)\nabla\varphi]\}\cdot\mathbf{B}_p \\ &= \left[\frac{dF}{d\psi}\nabla\psi\times\nabla\varphi + F\nabla\times\nabla\varphi\right]\cdot\mathbf{B}_p \\ &= -F'\left(RB_p\frac{1}{R}\widehat{\mathbf{e}}_p\right)\cdot B_p\widehat{\mathbf{e}}_p \\ &= -F'B_p^2 \end{aligned}$$

and the Ampere's law is

$$-F'B_p^2 = \mu_0 K B_p^2$$

or

$$K = -\frac{F'}{\mu_0}$$
$$\text{function of } \psi$$

NOTE
that we have two expressions for K,

$$\begin{aligned} K &= -\frac{1}{\mu_0}\frac{dF}{d\psi} \quad \text{and} \\ K &= \frac{j_p}{B_p} \end{aligned}$$

Since there is an approximation where

$$F \approx B_0 R_0 = \text{constant}$$

it would result

$$j_p \approx 0$$

and all current is toroidal. We know that for the Pfirsch Schluter current and for the Spitzer current, supported by $\mathbf{E}\cdot\mathbf{B}$. But there is the diamagnetic current and now we expect there is a *parallel* current and this means that there is a poloidal projection j_p.
END

Return to the equation for the parallel current

$$j_\parallel = -\frac{Fp'}{B} + KB$$

We multiply this equation by B

$$j_\parallel B = -Fp' + KB^2$$

and note that F, p', K are functions of only the surface ψ. Then we perform surface averaging

$$\left\langle j_\parallel B \right\rangle = -Fp' + K\left\langle B^2 \right\rangle$$

from where

$$K = \frac{Fp'}{\langle B^2\rangle} + \frac{\langle \mathbf{j}\cdot\mathbf{B}\rangle}{\langle B^2\rangle}$$

and this is replaced in the equation for $j_\parallel$,

$$\begin{aligned} j_\parallel &= -\frac{Fp'}{B} + KB \\ &= -\frac{Fp'}{B} + \left(\frac{Fp'}{\langle B^2\rangle} + \frac{\langle \mathbf{j}\cdot\mathbf{B}\rangle}{\langle B^2\rangle}\right)B \\ &= -\frac{Fp'}{B}\left(1 - \frac{B^2}{\langle B^2\rangle}\right) + \frac{\langle \mathbf{j}\cdot\mathbf{B}\rangle}{\langle B^2\rangle}B \end{aligned}$$

The first term is the Pfirsch Schluter current

$$j_{PS} = -\frac{F}{B}\frac{dp}{d\psi}\left(1 - \frac{B^2}{\langle B^2\rangle}\right)$$

Pfirsch Schluter

This current exists in all collisional regimes.
The other term has zero divergence.

Let us make an estimation. Consider circular surfaces and small ε

$$\begin{aligned}
B &= \frac{B_0}{1+\varepsilon\cos\theta} \approx B_0\left(1-\varepsilon\cos\theta\right) \\
B^2 &\approx B_0^2\left(1-2\varepsilon\cos\theta\right) \\
\left\langle B^2\right\rangle &= \frac{\oint \frac{dl_\theta}{B_\theta} B_0^2\left(1-2\varepsilon\cos\theta\right)}{\oint \frac{dl_\theta}{B_\theta}} = \frac{\oint \frac{r\,d\theta}{\frac{b(r)}{1+\varepsilon\cos\theta}} B_0^2\left(1-2\varepsilon\cos\theta\right)}{\oint \frac{r\,d\theta}{\frac{b(r)}{1+\varepsilon\cos\theta}}}
\end{aligned}$$

$$\begin{aligned}
\left\langle B^2\right\rangle &= B_0^2 \frac{\oint d\theta\left(1-2\varepsilon\cos\theta\right)\left(1+\varepsilon\cos\theta\right)}{\oint d\theta\left(1+\varepsilon\cos\theta\right)} \approx B_0^2 \frac{\oint d\theta\left(1-\varepsilon\cos\theta\right)}{\oint d\theta\left(1+\varepsilon\cos\theta\right)} \\
&\approx B_0^2
\end{aligned}$$

then

$$\frac{B^2}{\langle B^2\rangle} \approx 1-2\varepsilon\cos\theta$$

and

$$\begin{aligned}
j_{PS} &= -F\frac{dp}{d\psi}\frac{1}{B}2\varepsilon\cos\theta \\
&\approx -2\frac{1}{B_\theta}\frac{dp}{dr}\varepsilon\cos\theta
\end{aligned}$$

which is the same as

$$J_\parallel \equiv J_\parallel^{PS} = -\varepsilon\frac{2}{B_\theta}\frac{dp}{dr}\cos\theta$$

derived previously from the divergence of the diamagnetic current (*variation on surface.tex*).

The operator of surface averaging was

$$\langle A\rangle = \frac{\oint \frac{dl_\theta}{B_\theta} A}{\oint \frac{dl_\theta}{B_\theta}}$$

Digression
The same problem is explained by **Hazeltine Hinton review**

The main remark is that these flows involve an approximative value of the plasma parameters: they are taken constants on magnetic surface

$$n_a \rightarrow \overline{n}_a$$

where the bar denotes average over the surface. In this approximation it is neglected the variation of the plasma parameters on surface.

The further approximation is first order in δ, the ratio ρ/L.

Then, multiplying vectorially by $\mathbf{B}$ the momentum balance equation, one obtains the *perpendicular* flow

$$(n\mathbf{u}_\perp)_1 = \frac{1}{m\Omega}\widehat{\mathbf{n}} \times \left(\boldsymbol{\nabla}\overline{p} + e\overline{n}\boldsymbol{\nabla}\overline{\Phi}\right)$$

which is the *diamagnetic* plus electric flow. Similarly

$$(\mathbf{q}_\perp)_1 = \frac{5}{2}\frac{1}{m\Omega}\overline{p}\ \widehat{\mathbf{n}} \times \boldsymbol{\nabla}\overline{T}$$

At this level of approximations we have flows that result from Larmor gyration. The flows remain in the surface. There is no effect of *neoclassical drift* yet, no bananas.

Now we impose the zero-divergence constraint on the *total* flows

$$\begin{aligned} \boldsymbol{\nabla}\cdot (n\mathbf{u})_1 &= 0 \\ \boldsymbol{\nabla}\cdot (\mathbf{q})_1 &= 0 \end{aligned}$$

Then there are parallel flows.

The *total* flows must be composed of the two parts, parallel and perpendicular

$$(n\mathbf{u})_1 = \widehat{K}\ \mathbf{B} + \widetilde{K}\ \boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta$$

Formal functions have been introduced as coefficients $\widehat{K}$ and $\widetilde{K}$.

The zero-divergence constraint (of this flux, $\boldsymbol{\nabla}\cdot (n\mathbf{u})_1 = 0$) means

$$\frac{1}{\sqrt{g}}\frac{\partial \widetilde{K}}{\partial \varphi} = \mathbf{B}\cdot\boldsymbol{\nabla}\widehat{K}$$

We have above an expression for the perpendicular flow

$$\widetilde{K} = -\frac{1}{e}\left(\frac{d\overline{p}}{d\psi} + e\overline{n}\frac{d\overline{\Phi}}{d\psi}\right)\sqrt{g}$$

In this expression the paranthesis can only depend on ψ (radial coordinate) and the only factor that can depend on the azimuthal angle φ is $\sqrt{g}$. But g does not depend on φ, then

$$\begin{aligned} \frac{\partial \widetilde{K}}{\partial \varphi} &= 0 \\ &\rightarrow\ \mathbf{B}\cdot\boldsymbol{\nabla}\widehat{K} = 0 \end{aligned}$$

Then the parallel component of the flow $(n\mathbf{u})_1 = \widehat{K}\ \mathbf{B} + \widetilde{K}\ \boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta$, in which we replace the expression of $\widetilde{K}$, is

$$\begin{aligned} \left(nu_{\|}\right)_1 &= \widehat{\mathbf{n}} \cdot (n\mathbf{u})_1 \\ &= -\frac{1}{m\Omega} I \left(\frac{d\overline{p}}{d\psi} + e\overline{n}\frac{d\overline{\Phi}}{d\psi}\right) \\ &\quad + \widehat{K}(\psi)\ B \end{aligned}$$

where the definition is adopted

$$I = \sqrt{g}\ \left(\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta\right) \cdot \mathbf{B}$$

For the heat

$$\begin{aligned} \left(q_{\|}\right)_1 &= -\frac{5}{2}\frac{1}{m\Omega}\ I\ \overline{p}\frac{d\overline{T}}{d\psi} \\ &\quad + \widehat{L}(\psi)\ B \end{aligned}$$

In the calculations two functions have been introduced

$$\widehat{K}(\psi) \quad \text{and} \quad \widehat{L}(\psi)$$

They must be determined.

From the parallel flow one can obtain the parallel current. First we take the expression of the total current, $e\,(n\mathbf{u})_1$,

$$\mathbf{j}_1 = e\,(n\mathbf{u})_1 = e\widehat{K}\ \mathbf{B} + e\widetilde{K}\ \boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta$$

and insert here the formal expression for $\widetilde{K}$,

$$\widetilde{K} = -\frac{1}{e}\left(\frac{d\overline{p}}{d\psi} + e\overline{n}\frac{d\overline{\Phi}}{d\psi}\right)\sqrt{g}$$

Neglecting the electric potential $\overline{\Phi}$,

$$\mathbf{j}_1 = e\widehat{K}\ \mathbf{B} - \frac{d\overline{p}}{d\psi}\sqrt{g}\ \boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta$$

Now we multiply with $\widehat{\mathbf{n}} = \mathbf{B}/B$ to obtain the projection of the total current $\mathbf{j}_1$ on the parallel direction

$$\left(j_{\|}\right)_1 = e\widehat{K} - I\ \frac{1}{B}\frac{d\overline{p}}{d\psi}$$

This is the expression from **Hirshman 1978** where $j_{\|} = KB - F\frac{1}{B}p' = j_{PS} + \frac{\langle \mathbf{j}\cdot\mathbf{B}\rangle}{\langle B^2\rangle}B$, for the known expression $j_{PS} = -\frac{F}{B}\frac{dp}{d\psi}\left(1 - \frac{B^2}{\langle B^2\rangle}\right) \rightarrow -\frac{1}{B_\theta}\frac{dp}{dr}2\varepsilon\cos\theta$.

7.2 A single ion species

In this study (**Hirshman1978**) the first calculations are made for electrons plus only one species of ions.

This is *fluid* description.

This starts with the steady-state momentum equation

$$0 = -\boldsymbol{\nabla} \cdot \boldsymbol{\pi}_e - |e| \, n_e \mathbf{E} + \mathbf{R}_e$$

This equation is projected along the parallel direction $\mathbf{B}\cdot$

$$0 = -\mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_e - |e| \, n_e \mathbf{B} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{R}_e$$

and averaged over the surface

$$0 = -\left\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_e \right\rangle - |e| \, n_e \left\langle \mathbf{B} \cdot \mathbf{E} \right\rangle + \left\langle \mathbf{B} \cdot \mathbf{R}_e \right\rangle$$

The last term is *parallel* momentum exchange by collisional friction

$$\mathbf{R}_e = \int d^3v \; m_e \mathbf{v} \; C_e \left(f_e \right)$$

This term can be expressed in terms of the current density $\mathbf{j} = |e| \, \mathbf{v}_e n_e$ and the *resistivity* due to collisions

$$\mathbf{B} \cdot \mathbf{R}_e = \int d^3v \; \mathbf{B} \cdot \mathbf{v} m_e \; C_e \left(f_e \right) = |e| \, n_e \; \frac{1}{\sigma_S} \, \mathbf{j} \cdot \mathbf{B}$$

$$\left\langle \mathbf{B} \cdot \mathbf{R}_e \right\rangle = |e| \, n_e \; \frac{1}{\sigma_S} \left\langle \, \mathbf{j} \cdot \mathbf{B} \right\rangle$$

where

$$\sigma_S \equiv \text{Spitzer conductivity}$$

With this replacement of the friction term in the momentum equation we have

$$0 = -\left\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_e \right\rangle - |e| \, n_e \left\langle \mathbf{B} \cdot \mathbf{E} \right\rangle + |e| \, n_e \; \frac{1}{\sigma_S} \left\langle \, \mathbf{j} \cdot \mathbf{B} \right\rangle$$

This equation establishes the balance of forces along the magnetic field line:

(1) there is pressure anisotropy which involves the variation of the pressure stress tensor $\boldsymbol{\pi}_e$ along the magnetic line; this ocurs due to neoclassic effect (banana orbits giving variation of the density and temperature on surfaces);

(2) there is electric field with component along the line;

(3) there is collisional force (can accelerate or decelerate the flow of electrons).

The advantage of treating this way the friction force is that we now have the *current* in the equation. We can now attribute this current to the pressure anisotropy and to the electric field.

The simpler problem where an electric field produces in the plasma a flow of electrons (a current) which is saturated by the collisions (electrons with ions) is the Spitzer problem

$$j_S = \sigma_S \frac{\langle \mathbf{E} \cdot \mathbf{B} \rangle}{\langle B^2 \rangle} B$$

and it introduces the classical electrical conductivity

$$|e|\, E_{\|} v_{\|} \frac{1}{T_e}\ f_{e0} = C_{ei}^{lin} \left[f_{a1}, f_{i1} \right]$$

NOTE

The linearized collision operator is

$$C_{ab}^{lin} \left[f_{a1}, f_{b1} \right] \equiv C_{ab} \left[f_{a1}, f_{b0} \right] + C \left[f_{a0}, f_{b1} \right]$$

as in **Hirshman Sigmar 1976**.

END

This equation has as unknown the perturbation f_{e1} to the distribution function (relative to the Maxwellian).

The solution is, formally (with the introduction of τ_{ei} and of F)

$$f_{e1} = \ \tau_{ei}\ \ |e|\, E_{\|} v_{\|} \frac{1}{T_e}\ f_{e0} F_e^{Spitzer}$$

(**Note** the structure of this solution:

$$|e|\, E_{\|} v_{\|} \rightarrow (\text{force}) \times (\text{velocity}) = \frac{(\text{energy})}{(\text{time})}$$

$$\tau_{ei}\ \ |e|\, E_{\|} v_{\|} \rightarrow (\text{energy change in a collision time})$$

and

$$\frac{1}{T_e} f_{e0} F_e^{Spitzer} \rightarrow \frac{\partial}{\partial \epsilon} f$$

with the distribution function modulated in fF.

END

The parallel current (averaged over surface) is composed of two parts

$$\langle \mathbf{j} \cdot \mathbf{B} \rangle = \langle j_S B \rangle + \langle j_{nc} B \rangle$$

where the neoclassical part is

$$j_{nc} = \sigma_S \ \frac{\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_e \rangle\, B}{n_e\, |e|\ \ \langle B^2 \rangle}$$

The part that is driven by radial gradients of pressure and of temperature is the *bootstrap* current.

Consider the steady-state *momentum balance* for electrons where we assume anisotropy of the pressure

$$0 = -\boldsymbol{\nabla} p - \boldsymbol{\nabla} \cdot \boldsymbol{\pi} - |e|\, n_e \mathbf{E} + \mathbf{R}_e$$

where

$$\mathbf{R}_e = \underbrace{\int d^3v \; m_e \mathbf{v} \; C_e\left(f_e\right)}_{\text{friction force}}$$

is the collisional transfer of momentum.

Note this is the usual force equilibrium along a magnetic line. Used also in *drift wave* theory. **End.**

7.3 The presence of several ion species

One starts with the equation defining the *classical* Spitzer problem, electric field which determines a flow of charges whose magnitude is saturated by collisional friction.

Consider this CLASSICAL "**E**-drive/collisional saturation" (Spitzer problem) for the species a,

$$-e_a E_\parallel v_\parallel \frac{1}{T_a} \; f_{a0} = \sum_b C_{ab}^{lin}\left(f_{a1}^c, f_{b1}^c\right)$$

The solution is

$$f_{a1}^c = \tau_{aa} \times e_a E_\parallel v_\parallel \; \times \frac{1}{T_a} f_{a0} F_a^{Spitzer}$$

where

$$\tau_{aa} = \underbrace{\frac{3\sqrt{\pi}}{4} \frac{m_a^2}{4\pi e^4} \frac{1}{\ln\Lambda} \frac{v_{th,a}^3}{n_a}}_{\substack{\text{like-particle collisional} \\ \text{exchange of momentum}}}$$

We must use this *classical* structure, but for *neoclassical* distribution function.

To involve now the *neoclassical* part of the distribution function, we make the following operations

$$T_a \frac{f_a^{nc}}{f_{a0}} \times \left[-e_a E_\parallel v_\parallel \frac{1}{T_a} \; f_{a0} = \sum_b C_{ab}^{lin}\left(f_{a1}^c, f_{b1}^c\right) \right]$$

next $\displaystyle\int d^3v \; \times$

next $\displaystyle\sum_b$

next Apply self-adjointness of C^{lin}

The last step will formally place the *neoclassical* functions f_{a1}^{nc} in the positions of the **classical** f_{a1}^{c} while the classical functions f_{a1}^{c} will be combined with the other factors and will reproduce the *classical* problem of Spitzer, which means that the function F_S^a will emerge again.

By the operations from the LHS one obtains the neoclassical parallel current

$$j_{\|} = \sum_a e_a \int d^3v \; v_{\|} \; f_{a1}^{nc}$$

$$j_{\|} = -\sum_a \tau_{aa} \; e_a \; \int d^3v \; v_{\|} \; F_a^{Spitzer} \; \sum_b C_{ab}^{lin} \left(f_{a1}^{nc} \; , \; f_{b1}^{nc}\right)$$

One must now determine the Spitzer function for the species a, $F_a^{Spitzer}$.

Remember that the symbol $C_{ab}^{lin} \left(f_{a1}^{nc} \; , \; f_{b1}^{nc}\right)$ means that there are two terms, one is perturbed function of the species a collide with passive background b without disturbing its distribution function and the second takes into account the perturbation of the background b induced by the collisions of a.

7.4 Comments on this determination of the neoclassical current

In the steady state momentum balance equation for electrons one replaces the *friction force* by introducing the *resistivity*, also defined by the collision operator. Doing this we make visible the *current* $\mathbf{j}$.

Then the *electron parallel (i.e.* projected on $\mathbf{B}$) *momentum balance*

$$0 = \langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi} \rangle - |e| \, n_e \left(\langle \mathbf{E} \cdot \mathbf{B} \rangle - \frac{1}{\sigma_{Spitzer}} \langle \mathbf{j} \cdot \mathbf{B} \rangle \right)$$

In this formula it has been replaced the *friction* term using the Spitzer resistivity

$$\int d^3v \; \mathbf{B} \cdot m_e \mathbf{v} \; C_e \left(f_e\right) = |e| \, n_e \frac{1}{\sigma_{Spitzer}} \mathbf{j} \cdot \mathbf{B}$$

From the $\mathbf{B}$-projected momentum, the parallel current $\langle \mathbf{j} \cdot \mathbf{B} \rangle$ has two components: one is produced by the *electric field* $\mathbf{E}$ and this $j_{Spitzer}$ will be called "Spitzer". And the other is produced by the *anisotropy* of the pressure tensor $\boldsymbol{\pi}$.

$$\langle \mathbf{j} \cdot \mathbf{B} \rangle = \langle j_{Spitzer} B \rangle + \langle j_{nc} B \rangle$$

where the Spitzer current is the part due to $\mathbf{E} \cdot \mathbf{B}$

$$j_{Spitzer} = \sigma_{Spitzer} \frac{\langle \mathbf{E} \cdot \mathbf{B} \rangle}{\langle B^2 \rangle} B$$

The neoclassical part is driven by the *electron pressure anisotropy*

$$j_{nc} = \sigma_{Spitzer} \frac{\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi} \rangle \, B}{|e| \, n_e \, \langle B^2 \rangle}$$

This is *bootstrap current*, driven by gradients of pressure and temperature and strictly dependent on

- collisions $\sigma_{Spitzer}$ and
- pressure anisotropy $\boldsymbol{\pi}$. This introduces the *trapping* of particles which is essential in generation of the bootstrap. The anisotropy is connected with the modulation of the parallel (and perpendicular) velocities along the line.

The surface average has removed the Pfirsch Schluter current $\sim \cos\theta$.

When the collisionality is very high, the pressure anisotropy is reduced, since the bananas are less visible. Then the current j_{nc} is reduced due to decrease of the conductivity and of the electron parallel viscosity coefficient

$$\sigma_{Spitzer} \times \eta_{\parallel}^{e} \sim \frac{1}{\nu_e^2}$$

We **NOTE** that the neoclassical current is essentially dependent on collisions, via $\sigma_{Spitzer}$.

The presence of an electric field in plasma produces a force which is balanced by collisional friction.

This is the basic problem of Spitzer resistivity.

Our case is more complex, because we also have *pressure anisotropy* and this contributes to the force balance.

The part of Spitzer problem is taken as starting point.

The equation for the distribution function $f_a^{(1)}$ is

$$-\frac{e_a E_{\parallel}\ v_{\parallel}}{T_a} f_a^{(0)} = \sum_b C_{ab}\left(f_a^{(1)}, f_b^{(1)}\right)$$

This equation expresses the balance between the time-rate of the change of energy $\partial/\partial\epsilon$ (by work done by a charged e_a particle moving with $v_{\parallel}$ in the electric field $E_{\parallel}$, *i.e.* $e_a E_{\parallel} v_{\parallel}$) of the particles of the equilibrium distribution function $f_a^{(0)}$; with the friction force given by collisions with other particles. The collisions create a friction force only if the distribution function is NOT Maxwellian. This balance is possible for a distribution function that is a perturbation from a Maxwellian. The perturbation $f_a^{(1)}$ is obtained by solving the Spitzer problem. The distribution function therefore results from an electric field and a collisional friction.

7.5 Calculation of the Spitzer function and return to friction/flux relationships

This is done by an expansion of $F_a^{Spitzer}$ in series of Sonine polynomials (modified Laguerre polynomials)

$$F_a^{Spitzer} = \sum_{j=0}^{\infty} \Lambda_j^a \; L_j \left(x_a^2 \right)$$

Assuming the coefficients Λ_j^a are known we note that a new function of velocity, $L_j \left(x_a^2 \right)$ appears in the velocity integration

$$\sim \int d^3v \; v_{\parallel} L_j \left(x_a^2 \right) \sum_b C_{ab}^{lin} \left(f_{a1}^{nc}, f_{b1}^{nc} \right)$$

where

$$v_{\parallel} \sim \mathbf{B} \cdot \mathbf{v}$$

Then we are suggested to use the general expressions for the friction forces as they occur in the momentum and heat conservation equations

$$\mathbf{R}_a \;\; \text{and} \;\; \mathbf{H}_a$$

$$\begin{aligned} \mathbf{B} \cdot \mathbf{R}_a &= \int d^3v \; m_a \; \mathbf{B} \cdot \mathbf{v} \; \times L_0 \times \sum_b C_{ab}^{lin} \left(f_{a1}^{nc}, f_{b1}^{nc} \right) \\ \text{where } L_0 &= 1 \end{aligned}$$

$$\begin{aligned} \mathbf{B} \cdot \mathbf{H}_a &= \int d^3v \; m_a \; \mathbf{B} \cdot \mathbf{v} \; \times L_1 \times \sum_b C_{ab}^{lin} \left(f_{a1}^{nc}, f_{b1}^{nc} \right) \\ \text{where } \; L_1 &= x_a^2 - \frac{5}{2} \end{aligned}$$

Comment Here we connect the *fluid* aspect with the *kinetic aspect.* We will need the distribution function $f_a^{(1)}$ to calculate the friction force, by inserting in the expressions of the collision operators. Then we need to *solve* the drift kinetic equation with a adequate collision operator.

The balance of forces *along the line* ($\parallel \sim \mathbf{B}$) in the steady state for the particle species a , surface averaged, is

$$\begin{aligned} 0 &= -\left\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi} \right\rangle + \left\langle e_a \mathbf{E} \cdot \mathbf{B} \right\rangle + \left\langle \mathbf{R}_a \cdot \mathbf{B} \right\rangle \\ & \quad \text{(momentum)} \end{aligned}$$

$$\begin{aligned} 0 &= -\left\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Theta} \right\rangle + \left\langle \mathbf{H}_a \cdot \mathbf{B} \right\rangle \\ & \quad \text{(heat)} \end{aligned}$$

The tensors of pressure anisotropy

$$\boldsymbol{\pi}_a \text{ and } \boldsymbol{\Theta}_a$$

are defined by

$$\boldsymbol{\pi}_a = \int d^3v \; m_a \; [L_0] \; \left(\mathbf{v}\;\mathbf{v} - \frac{1}{3}v^2\;\mathbf{I}\right) \; f_a$$

$$\boldsymbol{\Theta}_a = \int d^3v \; [-L_1] \; \left(\mathbf{v}\;\mathbf{v} - \frac{1}{3}v^2\;\mathbf{I}\right) \; f_a$$

It resulted before that the *neoclassical* parallel current, averaged over surface, is driven by the *parallel anisotropic* pressure $\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}$.

Using the expression for $j_{\parallel,nc}$ in terms of collision operators, (as results from the solution of the Spitzer problem for $f_a^{(1)}$) one can express it further in terms of anisotropic tensors (since the friction forces that are sustained by the collisional operators are balanced by anisotropic tensors $\boldsymbol{\pi}$, $\boldsymbol{\Theta}$).

$$\begin{aligned}
\left\langle j_{\parallel,nc}\;B\right\rangle &= \left\langle \sigma_{Spitzer}\frac{\langle\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}\rangle}{|e|\,n_e}\frac{B^2}{\langle B^2\rangle}\right\rangle \\
&= -\sum_a \sigma_a \frac{1}{e_a n_a}\left[\Lambda_0^a\left\langle\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_a\right\rangle - \Lambda_1^a\left\langle\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Theta}_a\right\rangle\right]
\end{aligned}$$

where

$$\begin{aligned}
\sigma_a &= \frac{n_a e_a^2}{m_a}\tau_{aa} \\
&\quad \text{conductivity}
\end{aligned}$$

Now we should look to *viscosity TTMP*.

There is a connection between the *parallel stress* $\langle\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_a\rangle$ and the *radial flux* Γ_a and Q_a.

This suggests to define

$$I_1^a \equiv T_a\Gamma_a^{nc} = -F\frac{T_a}{e_a}\frac{\langle\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_a\rangle}{\langle B^2\rangle}$$

$$I_2^a \equiv Q_a^{nc} = -F\frac{T_a}{e_a}\frac{\langle\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Theta}_a\rangle}{\langle B^2\rangle}$$

(remember that F is introduced in $j_\parallel = -F\frac{1}{B}\frac{dp}{d\psi} + KB$. It is I).

then in the expression of the (nc) parallel current we replace the stress tensors ($\langle\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_a\rangle$ and $\langle\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Theta}_a\rangle$) by the radial fluxes ($\Gamma_a^{nc}$ and Q_a^{nc}) respectively $I_{1,2}$.

$$\begin{aligned}
\left\langle j_{\parallel,nc}\;B\right\rangle &= -\sum_a \sigma_a \frac{1}{e_a n_a}\left[\Lambda_0^a\left\langle\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_a\right\rangle - \Lambda_1^a\left\langle\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Theta}_a\right\rangle\right] \\
\left\langle j_{\parallel,nc}B\right\rangle &= \frac{\left\langle B^2\right\rangle}{F}\sum_a\frac{\sigma_a}{p_a}\left[\lambda_0^a\;I_1^a + (-\lambda_1^a)\;I_2^a\right]
\end{aligned}$$

where

$$\lambda_{0,1}^{a} = \Lambda_{0,1}^{a}$$

The radial fluxes are supported by radial gradients of backround plasma parameters. These are the *forces* while Γ and Q (equiv. $I_{1,2}$) are the fluxes.

One can now express the radial fluxes $I_{1,2}^{a}$ in terms of

$$\left(\text{coefficients } L_{jk}^{ab}\right) \times (\text{forces} = \text{gradients } A_k)$$

as

$$\begin{aligned}
I_j^a &= \sum_b \sum_{k=1,2} L_{jk}^{ab}\ A_k^b + L_{j3}^{a}\ A_3 \quad \text{for} \quad j = 1,2 \\
I_3 &= \sum_b \sum_{k=1,2} L_{3k}^{b}\ A_k^b\ + L_{33}\ A_3
\end{aligned}$$

These expressions are to be used in the *universal Ohm's law*

$$\langle j^{nc} B \rangle = \frac{\langle B^2 \rangle}{F} \sum_a \frac{\sigma_a}{p_a} \left[\lambda_0^a\ I_1^a + (-\lambda_1^a)\, I_2^a\right]$$

7.6 Comments on the use of friction/flux relationships

Hirshman defines the neoclassic current by the integral of $v_{\parallel}$ over the distribution function in first order $f^{(1)}$

$$j_{\parallel} = \sum_a e_a \int d^3v\ v_{\parallel} f_a^{(1)}$$

NOTE the definition in **Connor 1973**

$$\begin{aligned}
J &= \sum Z_k e n_k \overline{u}_k \\
\overline{u}_k &= \frac{1}{n_k} \int d^3v\ v_{\parallel}\ \widehat{f}_k \\
&= -1.46\ \varepsilon^{1/2} \left\{ \frac{\frac{T_k}{m_k}}{\frac{Z_k e B_\theta}{m_k}} \left[\frac{N_k'}{N_k} + \frac{T_k'}{T_k}\right] + \frac{Z_k e E}{m_k} \left\langle \frac{1}{\nu_k} \right\rangle + \sum_l \left\langle \frac{\nu_{kl}}{\nu_k} \right\rangle \overline{u}_l^k \right\} \\
&\quad + \frac{Z_k e E}{m_k} \left\langle \frac{1}{\nu_k} \right\rangle + \sum_l \left\langle \frac{\nu_{kl}}{\nu_k} \right\rangle \overline{u}_l^k
\end{aligned}$$

END

Here we can insert the *solution* the distribution function $f_a^{(1)}$ that verifies the balance: force due to $E_\parallel$ with friction. The distribution function is solution of an equation with a collision operator

$$\begin{aligned} -\frac{e_a E_\parallel v_\parallel}{T_a} f_a^{(0)} &= \sum_{species-b} C\left(f_a^{(1)}, f_b^{(1)}\right) \\ & \text{equation for } f_a^{(1)} \end{aligned}$$

This equation expresses a balance: acceleration in the electric field $\sim a\frac{\partial}{\partial v}$ is an energetic effect, balanced by collisions.

This equation looks like **Spitzer** equation and is solved using Spitzer functions

$$f_a^{(1)} = \tau_{aa} \times e_a E_\parallel \; v_\parallel \; \frac{1}{T_a} f_a^{(0)} \; \times F_a^{Spitzer}$$

The Spitzer function $F_a^{Spitzer}$ is known, it is determined numerically.

Here it is expressed as

$$F_a^{Spitzer} = \sum_k \Lambda_{a,k} L_k\left(x_a^2\right)$$

where

$$\begin{aligned} L &\equiv \text{Sonine polynomials} \\ L_0 &= 1 \\ L_1\left(x^2\right) &= \frac{5}{2} - x^2 \\ L_2\left(x^2\right) &= \frac{x^4}{2} - \frac{7}{2}x^2 + \frac{35}{8} \end{aligned}$$

and

$$x^2 \equiv \frac{v^2}{v_{th,a}^2}$$

The solution is expressed in terms of Spitzer problem,

$$\begin{aligned} j_\parallel &= \sum_a e_a \int d^3v \; v_\parallel f_a^{(1)} \\ j_\parallel &= -\sum_a \tau_{aa} \; e_a \int d^3v \; F_a^{Spitzer} \sum_{species-b} C_{ab}\left(f_a^{(1)}, f_b^{(1)}\right) \end{aligned}$$

The expression of the distribution function is assumed in terms of the flows: *flow of particles* and *flow of energy.*

The connection between the flows and the friction allows to remove them from the expression of the distribution function and remain with only "forces", *i.e.* gradients.

The surface average of the parallel projection of the non-diagonal part, *i.e.* the pressure tensor

$$\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi} \rangle$$

intervenes in the expression of the surface average of the neoclassical current, as shown above.

The parallel current is the integral over the velocity space of $v_{\parallel}$ with the distribution function that verifes a Spitzer-like equation corrected with pressure anisotropy.

The solution for the distribution function $f_a^{(1)}$ is like Spitzer (*i.e.* it contains $F_a^{Spitzer}$).

We use the expression of the parallel current, in terms of collision operators (since we cannot simply take the Spitzer solution, now we must include $\boldsymbol{\pi}$.

We *average* over surface. In this way the Pfirsch Schluter current is eliminated. The result is the average of the *neoclassical* = *bootstrap* current.

Then we replace the part of the expression of $\left\langle j_{\parallel} \right\rangle$ that consists of collision operator with the friction of momentum and heat $\mathbf{R}_a$ and $\mathbf{H}_a$.

But the friction and heat $\mathbf{R}_a$ and $\mathbf{H}_a$ are connected with the "viscous" tensors of pressure and of heat $\boldsymbol{\pi}_a$ and $\boldsymbol{\Theta}_a$.

In this way the average of the parallel current becomes an expression in terms of the "viscous" tensors

$$\left\langle j_{\parallel} \right\rangle \sim \ \boldsymbol{\pi}_a \quad , \quad \boldsymbol{\Theta}_a$$

General comment

This is the meaning when we say that the bootstrap current is connected with the parallel viscous stresses.

The pressure anisotropy is essential (implicitely the banana orbits).

The collisionality is essential.

It is hard to see here the explanation offered by **Cordey**, **Hazeltine Hinton.**

Or, the role of the region at the boundary *trapped/passing.*

To provide more clarity to this subject one should look for

- an example of calculation, within this framework, of the flux sustained by collision, in velocity space, between regions with different components of stress tensor $\boldsymbol{\pi}$.

- an application, of the above calculation, to the flux in velocity space, across the boundary *trapped-circulating.*

[all in the presence of the gradient of pressure]

[and taking into account the saturation of the flow created by the velocity-space flux, (calculated above) due to collisions with the background, electrons, ions, impurities, trapped and circulating]

7.7 Calculation of the Spitzer functions $F_a^{Spitzer}$

The solution of the first order distribution functions is expressed in terms of *flows*

$$\begin{array}{l} u_{\|a} \ , \\ \dfrac{q_{\|a}}{p_a} \ , \\ u_{a2} \ , \\ \ldots \end{array}$$

$$f_a^{(1)} = \frac{2v_\|}{v_{th,a}^2} f_a^{(0)} \left[u_{\|a} - \frac{2}{5}\frac{q_{\|a}}{p_a} L_1\left(x_a^2\right) + u_{a2} L_2\left(x_a^2\right) \right]$$

Note this is also in **Jhang Chang** below. **End.**

This expression is of the form of definition of Spitzer function and is inserted in the equation that defines the Spitzer problem $-\frac{e_a E_\| v_\|}{T_a} f_a^{(0)} = \sum_{species-b} C\left(f_a^{(1)}, f_b^{(1)}\right)$.

This allows to calculate the Spitzer function $F_a^{Spitzer}$.

Using the result for $F_a^{Spitzer}$ one has

$$f_{a1}^c = \ \tau_{aa} \times e_a E_\| v_\| \frac{1}{T_a} \times f_{a0} F_a^{Spitzer}$$

and can formally calculate the flows $u_{\|a}$ and $q_{\|a}$ as functions of $E_\|$.

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The expression is derived on the basis of the force balance (averaged over magnetic surface) which involves the gradient of the pressure (anisotropic: parallel and perpendicular) and friction.

The balance is for momentum and heat.

The equations are projected along the $\mathbf{B}$ direction and averaged

$$0 = -\left\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_\alpha \right\rangle + \left\langle \mathbf{B} \cdot \mathbf{R}_\alpha \right\rangle \qquad \text{(momentum)}$$

$$0 = \left\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Theta}_\alpha \right\rangle + \left\langle \mathbf{B} \cdot \mathbf{H}_\alpha \right\rangle \quad \text{(heat)}$$

Assume the plasma is of electrons and ions.

The friction forces come from the collisional interaction in the *relative* flows of electrons and ions.

$$\left\langle \mathbf{B} \cdot \mathbf{R}_e \right\rangle = -\left[l_{11} \ \left\langle \mathbf{B} \cdot \left(\mathbf{u}_e - \mathbf{u}_i \right) \right\rangle + \frac{2}{5} \ l_{12} \ \frac{\left\langle \mathbf{B} \cdot \mathbf{q}_e \right\rangle}{p_e} \right]$$

$$\langle \mathbf{B} \cdot \mathbf{H}_e \rangle = -\left[l_{12} \ \langle \mathbf{B} \cdot (\mathbf{u}_e - \mathbf{u}_i) \rangle + \frac{2}{5} \ l_{22} \ \frac{\langle \mathbf{B} \cdot \mathbf{q}_e \rangle}{p_e} \right]$$

The entries of the matrix of coefficients are

$$l_{11} = \frac{n_e m_e}{\tau_{ei}}$$

$$l_{12} = -\frac{3}{2} l_{11}$$

$$l_{22} = \left(\frac{13}{4} + \frac{\sqrt{2}}{Z_i} \right) l_{11}$$

The parallel viscous forces (i.e. the gradient of the pressure tensor projected along $\mathbf{B}$) are expressed in terms of the *flow velocities*

$$u_{pol} = \frac{\mathbf{u} \cdot \boldsymbol{\nabla} \theta}{\mathbf{B} \cdot \boldsymbol{\nabla} \theta}$$

$$q_{pol} = \frac{\mathbf{q} \cdot \boldsymbol{\nabla} \theta}{\mathbf{B} \cdot \boldsymbol{\nabla} \theta}$$

(note the use of the $1/B$ modulation of the physical flows u and q). These are defined for electrons and for ions.

$$-\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_\alpha \rangle = -3 \left\langle \left(\nabla_\parallel B \right)^2 \right\rangle \ \left(\mu_{11}^\alpha \ u_{pol}^\alpha + \frac{2}{5} \mu_{12}^\alpha \ \frac{q_{pol}^\alpha}{p^\alpha} \right)$$

$$-\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Theta}_\alpha \rangle = -3 \left\langle \left(\nabla_\parallel B \right)^2 \right\rangle \ \left(\mu_{12}^\alpha \ u_{pol}^\alpha + \frac{2}{5} \mu_{22}^\alpha \ \frac{q_{pol}^\alpha}{p^\alpha} \right)$$

In the banana regime

$$\mu_{ij} \sim \frac{f_{trapped}}{f_{circulating}} \equiv \frac{f_t}{f_c}$$

Hirshman makes the observation that, when the aspect ratio decreases, the fraction of trapped particles approaches 1. [no circulating particles remain in the limit]

$$f_c = \frac{3}{4} \left\langle B^2 \right\rangle \int_0^{1/B_{\max}} \frac{\lambda \ d\lambda}{\left\langle \sqrt{1 - \lambda B} \right\rangle}$$

Now it is introduced an essential new information.

It is the balance of forces on the direction which is *normal* to the surfaces

[this is actually the same as the condition that there is NO radial electric current after averaging on surfaces]

$$\left\langle B^2 \right\rangle \ u_{pol}^\alpha = -V_0^\alpha \ \left(A_1^\alpha + \frac{e_\alpha}{T_\alpha} \frac{d\Phi}{d\psi} \right) + \langle \mathbf{u}_\alpha \cdot \mathbf{B} \rangle$$

$$\left\langle B^2 \right\rangle \frac{2}{5} q_{pol}^{\alpha} = -V_0^{\alpha} \; A_2^{\alpha} + \frac{2}{5} \left\langle \mathbf{q}_{\alpha} \cdot \mathbf{B} \right\rangle$$

where

$$V_0^{\alpha} = -\frac{2\pi I}{\left(\frac{d\chi}{d\psi}\right)} \frac{T_{\alpha}}{e_{\alpha}}$$

and the "forces" are

$$A_1^{\alpha} = \frac{1}{p_{\alpha}} \frac{dp_{\alpha}}{d\psi} = \frac{d}{d\psi} \ln p_{\alpha}$$

$$A_2^{\alpha} = \frac{d}{d\psi} \ln T_{\alpha}$$

It is now possible to introduce and calculate the parallel current (averaged over surface)

$$\left\langle \mathbf{j} \cdot \mathbf{B} \right\rangle = - \left| e \right| \; n \; \left\langle \left(\mathbf{u}_e - \mathbf{u}_i \right) \cdot \mathbf{B} \right\rangle$$

This is obtained after replacing the velocities, as determined from balance equations

$$\begin{aligned} \left\langle \mathbf{j} \cdot \mathbf{B} \right\rangle \quad &= \quad \frac{d_1}{D} \left[j_0 \; \left(A_1^e - \frac{|e|}{T_e} \frac{d\Phi}{d\psi} \right) + |e| \, n \; \left\langle \mathbf{u}_i \cdot \mathbf{B} \right\rangle \right] \\ &\quad + \frac{d_2}{D} \; j_0 \; A_2^e \end{aligned}$$

where

$$\begin{aligned} j_0 \quad &\equiv \quad - \left| e \right| \; n \; V_0^e \\ &= \quad -\frac{2\pi \; I}{\left(\frac{d\chi}{d\psi}\right)} p_e \end{aligned}$$

$$d_1 = \widehat{\mu}_{11}^e \left(\widehat{\mu}_{22}^e + l_{22} \right) - \widehat{\mu}_{12}^e \left(\widehat{\mu}_{12}^e + l_{12} \right)$$

$$d_2 = \widehat{\mu}_{12}^e \left(\widehat{\mu}_{22}^e + l_{22} \right) - \widehat{\mu}_{22}^e \left(\widehat{\mu}_{12}^e + l_{12} \right)$$

$$D = \left(\widehat{\mu}_{11}^e + l_{11} \right) \left(\widehat{\mu}_{22}^e + l_{22} \right) - \left(\widehat{\mu}_{12}^e + l_{12} \right)^2$$

New notations have been introduced

$$\widehat{\mu}_{ij}^e = 3 \; \mu_{ij}^e \; \frac{\left\langle \left(\nabla_{\parallel} B \right)^2 \right\rangle}{\left\langle B^2 \right\rangle}$$

It still remains to calculate the *ion* flow, which appears in the equation as $\left\langle \mathbf{u}_i \cdot \mathbf{B} \right\rangle$.

$$n_e \left| e \right| \left\langle \mathbf{u}_i \cdot \mathbf{B} \right\rangle = j_0 \frac{1}{Z_i} \frac{T_i}{T_e} \left[\left(A_1^i + \frac{Z_i \left| e \right|}{T_i} \frac{d\Phi}{d\psi} \right) + \alpha_i \; A_2^i \right]$$

where

$$\alpha_i = \frac{\widehat{\mu}_{12}^{ion}}{\widehat{\mu}_{11}^{ion}} \frac{l_{22}^{ion}}{\widehat{\mu}_{22}^{ion} + l_{22}^{ion} - \frac{\left(\widehat{\mu}_{12}^{ion}\right)^2}{\widehat{\mu}_{11}^{ion}}}$$

This expression (of the ion velocity projected along the magnetic field and averaged over surface) allows us to obtain the parallel current (i.e. the bootstrap).

It is only necessary to provide explict form of the two matrices $\widehat{\mu}_{ij}$ and l_{ij}, for electrons and for ions, as follows.

For electrons

$$\widehat{\mu}_{11}^{e} = \frac{n_e m_e}{\tau_{ee}} \; x \; \left(0.533 + Z_i\right)$$

$$\widehat{\mu}_{12}^{e} = -\frac{n_e m_e}{\tau_{ee}} \; x \; \left(0.625 + 1.5 \; Z_i\right)$$

$$\widehat{\mu}_{22}^{e} = \frac{n_e m_e}{\tau_{ee}} \; x \; \left(1.386 + 3.25 \; Z_i\right)$$

For ions

$$\widehat{\mu}_{11}^{ion} = \frac{n_i m_i}{\tau_{ii}} \sqrt{2} \; x \; 0.377$$

$$\widehat{\mu}_{12}^{ion} = -\frac{n_i m_i}{\tau_{ii}} \sqrt{2} \; x \; 0.442$$

$$\widehat{\mu}_{22}^{ion} = \frac{n_i m_i}{\tau_{ii}} \sqrt{2} \; x \; 0.980$$

It is used the notation

$$x \equiv \frac{f_t}{f_c}$$

Then

$$\begin{aligned} \langle \mathbf{j} \cdot \mathbf{B} \rangle &= L_{31} \left[A_1^e + \frac{1}{Z_i} \frac{T_i}{T_e} \left(A_1^i + \alpha_i A_2^i \right) \right] \\ &\quad + L_{32} \; [\; A_2^e \;] \end{aligned}$$

where

$$L_{31} = j_0 \frac{1}{D(x)} \; x \; \left[0.754 + 2.21 \; Z_i + Z_i^2 + x \left(0.348 + 1.243 \; Z_i + Z_i^2 \right) \right]$$

$$L_{32} = j_0 \frac{1}{D(x)} \; x \; \left[0.884 + 2.074 \; Z_i \right]$$

and

$$\alpha_i = -\frac{1.172}{1 + 0.462 \; x}$$

$$
\begin{aligned}
D(x) \quad = & \quad\quad 1.414\ Z_i + Z_i^2 \\
& +x\left(0754 + 2.657\ Z_i + 2Z_i^2\right) \\
& +x^2\left(0.384 + 1.243\ Z_i + Z_i^2\right)
\end{aligned}
$$

In a concrete example, take

$$\varepsilon = \frac{r}{R_0}$$

and

$$B = \frac{B_0}{1 + \varepsilon \cos\theta}$$

then

$$x \approx \frac{\left(1.46\ \sqrt{\varepsilon} + 2.40\ \varepsilon\right)}{\left(1-\varepsilon\right)^{3/2}}$$

9 Bootstrap current in the limit of aspect ratio close to 1

It is the paper **Shaing1995 boostrap current aspect ration eq** 1.

It is found that

$$
\begin{aligned}
A &= \frac{R}{a} \rightarrow 1 \\
\varepsilon &= \frac{1}{A}
\end{aligned}
$$

still maintain the boostrap. The explanation is that the viscous forces

$$
\begin{aligned}
\left\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi} \right\rangle &\rightarrow \infty \\
\left\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Theta} \right\rangle &\rightarrow \infty
\end{aligned}
$$

at the limit $A \rightarrow 1$.

In this limit

$$f_{circ} \rightarrow 0$$

there are no more circulating particles (all are trapped).

For concentric surfaces

$$\left\langle \left(\nabla_{\parallel} B\right)^2 \right\rangle = \frac{1}{2} \frac{\varepsilon^2}{\left(1-\varepsilon^2\right)^{3/2}} \left(\frac{B_0}{r} \frac{B_\theta}{B_{tor}}\right)$$

This becomes singular as $\varepsilon \rightarrow 1$. Also $B = \frac{B_0}{1+\varepsilon\cos\theta}$ becomes singular for $\varepsilon \rightarrow 1$ at $\theta = 0$.

10 Boostrap current for arbitrary aspect ratio and arbitrary collisionality

It is the paper **bootstrap_arbitrary_aspect_ratio_collisions_houlberg1997**.

It is discussed the squeezing of banana.

The experimental data are from NTSX and from TFTR in RS and ERS.

It seems that there are two transport barriers.

Impurities are introduced by the effective charge method.

11 Bootstrap ECRH Helander Hastie Connor

This text is also in *plasma models bootstrap diamagnetic*.

This paper discusses the *bootstrap*, it is a good reference.

The physical picture:

the bootstrap results from the trapped particles. Due to the gradient of density the trapped particles have a finite directed velocity (like the un-balanced fluxes in the diamagnetic case).

The directed velocity of the *trapped particles* is transferred by collisions to the *passing* electrons.

The passing electrons now carry a current: bootstrap.

The equation *for electrons* is

$$\left(\mathbf{v}_{\parallel}+\mathbf{v}_D\right)\cdot\boldsymbol{\nabla}f=C\left(f\right)+Q^{ECRH}\left(f\right)$$

where $Q\left(f\right)$ is the quasilinear operator that describe the ECRH in the space of velocities

$$Q^{ECRH}\left(f\right)=\frac{1}{v_{\perp}}\frac{\partial}{\partial v_{\perp}}\left(v_{\perp}D\frac{\partial f\left(v_{\parallel},v_{\perp}\right)}{\partial v_{\perp}}\right)$$

It is a *diffusion in the space of perpendicular velocity*, with

$$D=\sum_{N,k_{\parallel}}D_{N,k_{\parallel}}\delta\left(\omega-N\Omega-k_{\parallel}v_{\parallel}\right)$$

$$\Omega\equiv\Omega_e=\frac{-\left|e\right|B}{m_e}$$

The lowest order in the expansion

$$f=f_0+f_1+...$$

obeys the equation

$$v_{\parallel}\nabla_{\parallel}f_0=C\left(f_0\right)+Q\left(f_0\right)$$

To exploit periodicity of the toroidal geometry we write

$$\left\langle \frac{B}{v_{\parallel}} \left[C\left(f_0\right) + Q\left(f_0\right) \right] \right\rangle = 0$$

for

$$\left\langle ... \right\rangle = \frac{\oint \frac{rd\theta}{B_\theta} \left(... \right)}{\oint \frac{rd\theta}{B_\theta}}$$

and

$$B = I\left(\psi\right) \boldsymbol{\nabla}\varphi + \boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi$$

$$f_0 = f_0\left(w, \lambda, \sigma, \psi\right)$$

where the *invariants* of the particle motion are

$$\begin{aligned} w &= \frac{v^2}{2} \\ \lambda &= \frac{v_\perp^2}{v^2} \frac{1}{B} \end{aligned}$$

NOTE

Reiterate the procedure that consists in producing a factor that turns the surface average into an annihilator. We have

$$v_{\parallel} \nabla_{\parallel} f = A$$

The LHS is

$$v_{\parallel} \frac{d}{dl_{\parallel}} f$$

with

$$\begin{aligned} \frac{dl_\theta}{dl_{\parallel}} &= \frac{B_\theta}{B} \rightarrow \frac{1}{dl_{\parallel}} = \frac{B_\theta}{B} \frac{1}{dl_\theta} \\ \frac{df}{dl_{\parallel}} &= \frac{B_\theta}{B} \frac{df}{dl_\theta} = \frac{B_\theta}{B} \frac{df}{rd\theta} \end{aligned}$$

Then the LHS operator is

$$v_{\parallel} \nabla_{\parallel} f = v_{\parallel} \frac{B_\theta}{B} \frac{df}{rd\theta}$$

Knowing the structure of the operator $\oint \frac{rd\theta}{B_\theta} \left(... \right)$ of surface average, that will be applied on the two sides, we note that the factor $\frac{v_{\parallel}}{B}$ "perturbs" the immediate exploitation of the periodicity in θ in the RHS. Then we divide by $v_{\parallel}$ and multiply by B

$$\left\langle \frac{B}{v_{\parallel}} \ \times \ v_{\parallel} \nabla_{\parallel} f \right\rangle = \left\langle B_\theta \frac{\partial f}{r \partial \theta} \right\rangle$$

The averaging operation becomes

$$\left\langle B_\theta \frac{\partial f}{r\partial\theta} \right\rangle = \frac{\oint \frac{rd\theta}{B_\theta} B_\theta \frac{\partial f}{r\partial\theta}}{\oint \frac{rd\theta}{B_\theta}} = \frac{\oint \frac{\partial f}{\partial\theta} d\theta}{\oint \frac{rd\theta}{B_\theta}}$$
$$= 0 \text{ by periodicity}$$

This teaches us that the RHS multiplied by B and divided by $v_\parallel$ has zero surface average

$$0 = \left\langle \frac{B}{v_\parallel} (C + Q) \right\rangle$$

END

In preparation of the next order one remarks the presence of the convection by the drift velocity

$$\mathbf{v}_D \cdot \boldsymbol{\nabla} f$$

and this can be represented as

$$\mathbf{v}_D \cdot \boldsymbol{\nabla} \psi = I v_\parallel \nabla_\parallel \left(\frac{v_\parallel}{\Omega} \right)$$

which is to be applied to the *zero* order function, f_0 that will be derived

$$\frac{\partial f_0}{\partial \psi}$$

The term has the same *operator* form,

$$v_\parallel \nabla_\parallel$$

as the parallel convection of the first order distribution.

Then they both can be placed together (as if the operator $v_\parallel \nabla_\parallel$ is factored out).

Next order

$$v_\parallel \nabla_\parallel \left(f_1 + \frac{I v_\parallel}{\Omega} \frac{df_0}{d\psi} \right) = C(f_1) + Q(f_1)$$

This equation suggests to introduce a function

$$g = \frac{I v_\parallel}{\Omega} \frac{df_0}{d\psi} + f_1$$

Again, the surface averaging of $\frac{B}{v_\parallel} \times (LHS)$ is zero, as explained above. Then

$$\left\langle \frac{B}{v_\parallel} [C(f_1) + Q(f_1)] \right\rangle = 0$$

The equation above contains the orbit-width corrections to f_0.

This means that the equation above contains the *bootstrap* current.

The correction to the Maxwellian distribution, f_1, is localized in the region of the velocity space where there are *trapped* particles

$$1 - \varepsilon < \lambda B_0 < 1 + \varepsilon$$

and around it

$$|1 - \lambda B_0| \sim O(\varepsilon)$$

In the determination of f_1 the collision operator is the *pitch angle scattering*.

$$\begin{aligned} \nu_e \mathcal{L} &= \nu_e \frac{1}{2} \frac{\partial}{\partial \xi} \left(1 - \xi^2\right) \frac{\partial}{\partial \xi} \\ &= \nu_e \frac{v_\parallel}{wB} \frac{\partial}{\partial \lambda} \lambda v_\parallel \frac{\partial}{\partial \lambda} \end{aligned}$$

where

$$\xi = \frac{v_\parallel}{v}$$

$$\nu_e = \nu_{ee} + \nu_{ei}$$

Obs.

$$\nu_{ee} \equiv \text{the rate at which the collisions drive the distribution towards a Maxwellian}$$

(remember that this is ECRH situation and the heated component is *electrons*. Their distribution function is no more Maxwellian but collisions drive it toward Maxwellian).

The main component in the equation that determines f_1 is

$$\begin{aligned} C(f_1) &\sim \nu_e \mathcal{L}(f_1) \\ &\sim \nu_e \frac{f_1}{\varepsilon} \end{aligned}$$

The paper examines the ECRH and this is represented in the equation for f_1 by

$$Q(f_1) \sim D \frac{f_1}{v^2}$$

which may be neglected since

$$\nu_{ee} \sim \frac{D}{v^2}$$

These estimations simplify the equation for f_1 to

$$\left\langle \frac{B}{v_\parallel} C(f_1) \right\rangle = 0$$

The first information about f_1 comes from the composition

$$f_1 = -I \frac{v_\parallel}{\Omega} \frac{\partial f_0}{\partial \psi} + g$$

where

$$\begin{aligned} g &\equiv g\left(w, \lambda, \sigma, \psi\right) \text{ for circulating particles} \\ &= 0 \text{ for trapped} \end{aligned}$$

This is a standard neoclassical *general* form of the electron distribution function and we expect to give the final form by fixing on physical basis the expression for the collision operator. Then the equation for g can be solved.

11.0.1 The Lorentzian limit of the collisionality

This means a large difference in the masses of the species, and

$$Z >> 1$$

and only the *electron-ion* collisions must be retained.

This means that the ECR heated electron component will loose the excess of energy to ions.

$$\begin{aligned} \nu_{ei} &= \nu_0 \frac{n_i Z^2}{n_e} \left(\frac{v_{th,e}}{v}\right)^3 \\ & \left(\text{note as usual } \sim \frac{n}{T^{3/2}}\right) \end{aligned}$$

and

$$\nu_0 = \frac{e^4}{4\pi\epsilon_0^2} \frac{1}{m^2} \ln \Lambda \frac{1}{v_{th,e}^3}$$

the bootstrap current is determined only by the distribution of trapped particles

After replacing in the collision operator f_1 by its expression in terms of g the equation for g is

$$\left\langle \frac{\partial}{\partial \lambda} \lambda v_\parallel \frac{\partial}{\partial \lambda} \left(g - I \frac{v_\parallel}{\Omega} \frac{\partial f_0}{\partial \psi} \right) \right\rangle = 0$$

The solution

$$\begin{aligned} \frac{\partial g}{\partial \lambda} &= \frac{1}{\left\langle v_\parallel \right\rangle} \left\langle v_\parallel \frac{\partial}{\partial \lambda} I \frac{v_\parallel}{\Omega} \frac{\partial f_0}{\partial \psi} \right\rangle \\ \text{for } 0 &< \lambda < 1 - \varepsilon \\ & \text{circulating} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial g}{\partial \lambda} &= 0 \\ \text{for } 1-\varepsilon &< \lambda < 1+\varepsilon \\ & \text{trapped} \end{aligned}$$

Now we know f_1 since we know g.
We calculate the current

$$\begin{aligned} \langle j_{elect} \rangle &= -e \left\langle \int d^3v \; f_1 v_\parallel \right\rangle \\ &= 2\pi e \sum_{\sigma=\pm 1} \int_0^\infty w dw \; \int_0^{1-\varepsilon} d\lambda \; B_0 \; \left(\lambda \frac{\partial g}{\partial \lambda} + I \frac{\langle |v_\parallel| \rangle}{\Omega} \frac{\partial f_0}{\partial \psi} \right) \end{aligned}$$

here we have used the measure

$$d^3v = \sum_{\sigma=\pm 1} \frac{2\pi B}{|v_\parallel|} w \; dw \; d\lambda$$

The integration over the velocity space must be restricted to the *passing* particles.

There is a contribution from the trapped particles

$$j_{trapped} = Ie \int_{trapped} d^3v \; v_\parallel^2 \frac{\partial f_0}{\partial \psi}$$

but this is smaller as

$$\sqrt{\varepsilon}$$

compared to the contribution to the current of the passing particles.

In the expression of the current one introduces the distribution function, solution obtained above

$$\begin{aligned} \left\langle j^{elect} \right\rangle &= -2^{3/2}\pi \; mR \sum_\sigma \int_0^\infty dw \; w^{3/2} \left(\frac{\langle \xi^2 \rangle}{\langle |\xi| \rangle} \left. \frac{\partial f_0}{\partial \psi} \right|_{\lambda B_0 = 1-\varepsilon} \right. \\ & \left. + \int_0^{1-\varepsilon} d\lambda B_0 \; \left| \langle \xi \rangle - \left\langle \xi \frac{\partial}{\partial \lambda} \frac{\lambda \xi}{\langle \xi \rangle} \right\rangle \right| \; \frac{\partial f_0}{\partial \psi} \right) \end{aligned}$$

The authors make the observation that in the *passing region* of the velocity space the terms

$$\langle \xi \rangle \quad \text{and} \quad \left\langle \xi \frac{\partial}{\partial \lambda} \frac{\lambda \xi}{\langle \xi \rangle} \right\rangle$$

almost cancel each other. Or, the interval $0 < \lambda < 1-\varepsilon$ is the region of passing. And this is the interval of integration of the last integral.

Take

$$B = B_0 \left(1 - \varepsilon \cos\theta\right)$$

and calculate the averages for the *passing* particles

$$\langle \xi \rangle = \frac{2E\left(k\right)}{\pi} \sqrt{\frac{2\varepsilon}{k^2 + 2\varepsilon}}$$

and

$$\left\langle \xi^2 \right\rangle = \frac{2\varepsilon}{k^2 + 2\varepsilon}$$

where the *trapping parameter* is

$$k^2 = \frac{2\varepsilon\lambda B_0}{1 - \lambda B_0 \left(1 - \varepsilon\right)}$$

replacing these formulas in the expression of the current

$$\begin{aligned} \left\langle j^{elect} \right\rangle &= -2\pi^2 \varepsilon^{1/2}\ mR \sum_{\sigma} \int_0^{\infty} dw\ w^{3/2}\ \left\{ \left. \frac{\partial f_0}{\partial \psi} \right|_{k=1} \right. \\ &\quad \left. + \int_0^1 \frac{dk^2}{E\left(k\right)\left(k^2 + 2\varepsilon\right)^{3/2}} \left[\frac{2\varepsilon}{k^2 + 2\varepsilon} \left(\frac{4E^2\left(k\right)}{\pi^2} - 1 \right) + \frac{d \ln E\left(k\right)}{d \ln k^2} \right] \frac{\partial f_0}{\partial \psi} \right\} \end{aligned}$$

At the limit of very large aspect ratio,

$$\varepsilon \to 0$$

we have

$$\lambda B_0 = 1$$

and the zero distribution function is factored out from the integral and is calculated for *deep trapped* particles

$$\begin{aligned} & f_0\left(v_\perp, v_\parallel = 0\right) \\ = & \text{ distribution function of DEEP trapped particles} \end{aligned}$$

It is neglected the term with

$$\frac{4E^2\left(k\right)}{\pi^2} - 1 \sim \sqrt{\varepsilon}$$

Then

$$\left\langle j^{elect} \right\rangle = -1.46 \frac{\sqrt{\varepsilon}}{B_\theta}\ \frac{d}{dr} \int_0^{\infty} dv_\perp\ \frac{4\pi m\ v_\perp^4}{3}\ f_0\left(v_\perp, v_\parallel = 0\right)$$

For comparison the pressure of the trapped electrons, after averaging over the surface

$$\begin{aligned}
& \text{pressure of trapped} \\
p^{trapped} &= \left\langle \int_{trapped} d^3v \; \frac{mv^2}{3} \; f_0\left(v_\perp, v_\parallel\right) \right\rangle \\
&= \frac{2\sqrt{2\varepsilon}}{\pi} \int_0^\infty dv_\perp \; \frac{4\pi m \; v_\perp^4}{3} f_0\left(v_\perp, v_\parallel = 0\right)
\end{aligned}$$

this is an approximation, the trapping is very *deep* since we take $v_\parallel = 0$.

The domain of integration of the first integral is the *trapped particles in the velocity space.*

The fraction is

$$f_t = \sqrt{\varepsilon\left(1 + \cos\theta\right)}$$

with

$$\langle f_t \rangle = \frac{2\sqrt{2\varepsilon}}{\pi}$$

Then the surface average of the electron bootstrap current

$$\left\langle j^{elect} \right\rangle = -1.62 \frac{\sqrt{\varepsilon}}{B_\theta} \frac{d}{dr} \left(\frac{p^{trapped}}{\sqrt{\varepsilon}} \right)$$

Conclusion of the authors

For the Lorentz collision operator, the bootstrap current depends exclusively on the distribution function of the trapped electrons.

However

they say that for Lorentz operator, the *ion-electron* collisions dominate the collision operator.

This means that the equation for the *electron* distribution function f_1^{elect} that will produce the bootstrap current

$$\left\langle \frac{B}{v_\parallel} C\left(f_1^{elect}\right) \right\rangle = 0$$

involves the collisions electrons-ions.

The physical picture makes a connection between *electrons* and *ions* in two ways

- the electrons are circulating and they are collided by trapped ions
- the electrons that have acquired momentum (*i.e.* current) from trapped particles lose momentum until saturation by collision with background ions.

It seems that the second situation is represented here.

Comment.

Reference to the explanation that the bootstrap current is produced by the *detrapping* of the trapped electrons, while the latter have a directed flow (a current) due to the gradient of pressure.

[the idea that there is a stationary flux in *velocity space* across the boundary between the trapped and circulating particles, a flux that continuosly convert trapped particles into circulating ones, through the discontinuity of the derivative of the distribution function at that boundary, **Galeev Berk**]

The pitch angle scattering is a transfer in the space of velocity.

In the space of velocity the regions of *trapped* and *circulating* particles are essential. The pitch angle collision can produce fluxes of particles between the two regions. We recall **Galeev Sagdeev JETP** who have determined the distribution function in the regions of *trapped* and of *untrapped* particles and have found a discontinuity (also **Berk Galeev**). Then at the region around the transition there can be a flow of particles sustained by *pitch angle* collisions.

In this collisions there is momentum transfer but there is no substantial energetic transfer. The main effect of the pitch angle scattering seems to be the conversion of trapped to circulating and of circulating to trapped.

[What is the effect of acceleration and transfer of momentum to another species?]

The distribution function is of electrons.

There is a finite momentum of $v_{\|}$ which is the *bootstrap* current. Which is the function that contributes to the current? The two terms in the integrand are

- the derivative of g to the pitch variable λ in the region of passing electrons
- and and the neoclassical correction to the Maxwellian

The next step in the work of **Helander ECRH** is to include the *electron-electron* collisions.

They are the agent that progressively turns the distribution function of heated electron population into the Maxwellian one.

$$\begin{aligned}
\nu_{ee} &= \nu_0 \frac{\Phi(x) - G(x)}{x^3} \\
\Phi(x) &\equiv \text{error function} \\
G(x) &\equiv \text{Chandrasekhar function} \\
&= \frac{\Phi(x) - x\frac{d\Phi(x)}{dx}}{2x^2} \\
\text{where } x &\equiv \frac{v}{v_{th,e}}
\end{aligned}$$

The equation is the same and we already know the solution g.

We return with g to the solution f_1 and write

$$\begin{aligned}
\frac{\partial f_1}{\partial \lambda} &= \frac{I}{\Omega}\left[\mathbf{H}(\lambda)\frac{\left\langle v_{\parallel}^2\right\rangle}{\left\langle v_{\parallel}\right\rangle} - v_{\parallel}\right]\frac{\partial f_0}{\partial\lambda\partial\psi} \\
&\quad - I\frac{wB}{\Omega}\left[\mathbf{H}(\lambda)\frac{1}{\left\langle v_{\parallel}\right\rangle} - \frac{1}{v_{\parallel}}\right]\frac{\partial f_0}{\partial\psi}
\end{aligned}$$

where **H** is the heaviside function.

The last term is close to the **Hinton Oberman** formula, from **Connor 1973**.

Something similar is calculated by **Rosenbluth Hazeltine Hinton 1972** in appendix.

The difference relative to the previous use of the solution for f_1 in the calculation of the current is that, - this time, the region of velocity space which is NOT close to the boundary trapped/circulating has a significant contribution.

This is due to the *resistivity* of the plasma, an effect that occurs through the electron-electron scattering and is known from Spitzer calculations. There is a function of distribution that verifies

$$C\left(v_{\parallel}E_{\parallel}f^{Spitzer}\right) = -\frac{e}{T_e}E_{\parallel}v_{\parallel}\ f_M$$

Then

$$\begin{aligned}
j^{elect} &= \int d^3v\ T_e\frac{f_1}{f_M}C\left(v_{\parallel}f^{Spitzer}\right) \\
&= \int d^3v\ T_e\frac{f^{Spitzer}}{f_M}\ v_{\parallel}C\left(f_1\right)
\end{aligned}$$

and this is separated into two contributions.

Self-adjointness of the collision operator has been applied.

In the space of velocity there are two regions.

Each region has a separate contribution to the integral for j^{elect}.

- The contribution of the region around the *trapped/circulating* boundary

  $$\left|1-\lambda B\right| = O\left(\varepsilon\right)$$

  is j_1;

- the contribution of the region of *deep passing* electrons

  $$\left|1-\lambda B\right| = O\left(1\right)$$

  is j_2; In this region the contribution comes from the *heating source* $Q\left(f_1\right)$.

The first part is

$$j_1^{elect} = \int d^3v \; T_e \; \frac{f^{Spitzer}}{f_M} \left(\nu_{ee} + \nu_{ei}\right) \; \frac{v_\parallel}{B} \frac{\partial f_1}{\partial \lambda}$$

We dispose of the expression for $\frac{\partial f_1}{\partial \lambda}$.
replacing in the current

$$\begin{aligned} \left\langle j^{elect} \right\rangle &= -\sqrt{2}\;\pi m R \sum_\sigma \int_0^\infty dw \; w^{3/2} \; \frac{T_e}{e} \frac{f^{Spitzer}}{f_M} \left(\nu_{ee} + \nu_{ei}\right) \\ &\times \left\langle \int_0^{1+\varepsilon\cos\theta} d\lambda \; B_0 \; \left(\left| \mathbf{H} \frac{\left\langle \xi^2 \right\rangle}{\langle \xi \rangle} - \xi \right| \frac{\partial^2 f_0}{\partial \lambda B_0 \; \partial \psi} - \left| \mathbf{H} \frac{1}{\langle \xi \rangle} - \frac{1}{\xi} \right| \frac{\partial f_0}{\partial \psi} \right) \right\rangle \end{aligned}$$

H Heaviside that excludes trapped

For evaluation

$$\frac{\partial^2 f_0}{\partial \lambda B_0 \; \partial \psi} \quad \sim \quad \sqrt{\varepsilon}$$

is neglected

In the term

$$\int_0^{1+\varepsilon\cos\theta} d\lambda \; B_0 \; \left(\left| -\mathbf{H} \frac{1}{\langle \xi \rangle} + \frac{1}{\xi} \right| \frac{\partial f_0}{\partial \psi} \right)$$

the region with significant contribution is *trapped/passing boundary*

$$\left|1 - \lambda B\right| \sim O\left(\varepsilon\right)$$

Note that **Connor 1973** says that this integral is calculated by **Hinton Oberman** and is $-1.46\varepsilon^{1/2} + ...$

The rest of the integral has been calculated by **Rosenbluth Hazeltine Hinton.**

The result for the *collision-driven* bootstrap current is

$$\left\langle j_1^{elect} \right\rangle = -1.46 \frac{\sqrt{\varepsilon}}{B_\theta} \int_0^\infty dv_\perp \frac{4\pi m v_\perp^4}{3} \; \left(\nu_{ee} + \nu_{ei}\right) \frac{T_e}{e} \frac{f^{Spitzer}}{f_M} \; \frac{\partial f_0 \left(v_\perp, v_\parallel = 0\right)}{\partial r}$$

For Spitzer functions see **Hirshman 1978 neoclassical current**.

12 Peeling mode ELM Gimblett 2006

The peeling ballooning mode.

$$\mu_0 j^{bootstrap} = -2\sqrt{\varepsilon} \frac{1}{B_\theta} \mu_0 \frac{dp}{dr}$$

$$\mu_0 j^{bootstrap} = \frac{1}{q R_0} B_0 \frac{\alpha}{\sqrt{\varepsilon}}$$

13 The bootstrap of the ALPHA particles Hsu Shaing Gormley Sigmar

The text is also in *bootstrap diamagnetic*.

The paper is **PF B 4 (1992) 4023 bootstrap of ALPHA particles.**

There is a parallel flow of electrons and ions, "of diamagnetic nature".

The flows of electrons and ions are different and there is a friction between species (Stringer ?)

The fact that the friction tries to equilibrate the parallel flows has a consequence: it is generated a relative poloidal flow (*i.e.* poloidal flows with distinct poloidal veocities).

These poloidal flows arise because they must compensate for the un-usual equilibration of the parallel flows. This is the ideal situation.

Although there is parallel friction (that should equilibrate the most important flow in tokamak, the parallel flow of different species), the balance is not fully realized and a finite *relative* parallel flow still remain. This is due to the trapped orbits, since they cannot contribute to the poloidal flows.

The remaining unbalanced parallel flow of the species is the *bootstrap* current.

See above more detailed comment.

[in other words: the friction along the parallel direction attempts to equalize the velocities of the species in parallel direction.

This is a force acting along the magnetic field line. If there is, for example, a decrease of the velocity of parallel flow of some species, due to this friction, then this is automatically followed by a poloidal flow of that species. The velocity of this poloidal flow will produce a *projection* along the line equal to the decrease due to friction.

This necessarily will produce poloidal flows, for each species.

The fact that the parallel flows become equal (by friction) means that the poloidal flows must be different for different species.

But not any imposed poloidal flow is allowed.

Poloidal flow is subject to a geometrical restriction, for any species. This is the fact that the trapped particles *cannot* flow poloidally.

So the intention to establish some poloidal flows for the species, in order to obtain equilibration of the parallel flows of species (by friction) is a project that *cannot* be realized.

Therefore the intention to equilibrate the parallel flows CANNOT be realized.

It still remains a parallel flow, against all attempts of friction to equalize the velocities.

This parallel flow is the boostrap.]

Comment

This explanation is strange since it does not make obvious the role of the gradient of pressure of the trapped particles.

The *alpha* particles are created isotropically.

The other kind of fast particles, the NBI ions, are created anisotropically.

The physical explanation of the bootstrap current, according to this paper.

The different species have different parallel flows and the collisional friction attempts to suppress the relative motion, *i.e.* to reduce it to a single common *parallel* velocity. This is only possible if *poloidal* flows are generated. The poloidal flows are necessary because by their projection on the parallel direction, they can compensate for the differences of the parallel flow velocities. Bringing all parallel velocities to one single value (as collisional friction intends to do) means to generate poloidal flows. Of course these are different for different species.

However, the *trapped* particles cannot move poloidally, so the poloidal flows cannot be created to the magnitude of their velocity as requested by the frictional suppression of the differences in the parallel flows. Then the frictional suppression of the differences between the parallel velocities will remain *incomplete*.

There will still be a relative motion between species, in the parallel direction. **Hsu Gormley Shaing Sigmar** consider that this is the source of the *bootstrap* current.

In general the current in tokamak

$$\begin{aligned}
\mathbf{J} &= |e|\, n_e \left(\mathbf{V}_i - \mathbf{V}_e\right) \quad \text{electron-ions} \\
&\quad + |e|\, n_\alpha Z_\alpha \left(\mathbf{V}_\alpha - \mathbf{V}_i\right) \quad \text{alpha particles} \\
&\quad + |e|\, n_I Z_I \left(\mathbf{V}_I - \mathbf{V}_i\right) \quad \text{impurities}
\end{aligned}$$

These are currents due to relative motions between charges of different species.

A **note** arising from the presence of the force $\mathbf{F}$ of bulk ions in the drift kinetic equation for the ALPHA particles, **Hsu Shaing Gormley Sigmar**.

The distinction is very important and substantial: the two components

- the ALPHA particles, which are the object of interest
- the background plasma, in particular the bulk ions

are only in interaction and must be treated separately:

the neoclassical drift $\mathbf{v}_D$ of the ALPHA particles must work against the *force* $\mathbf{F}$ opposed by the background plasma. This is an energy effect that modifies f_α in velocity space.

It is interesting that the neoclassical drift $\mathbf{v}_D$ must work against the poloidal electric field $E_\theta = -\partial\widetilde{\Phi}/\partial\left(r\theta\right)$ and this is an energetic term that must be included in the drift kinetic equation **Hazeltine Ware electrostatic trapping**.

Derivation of the equation for the *alpha* particles.

Choose the referential where the *background ions* are static.

Fokker Planck equation, change of variables

$$\mathbf{v} \rightarrow \mathbf{v} - \mathbf{V}_i$$

and perform gyroaverage.

This part is discussed also later, below.

To the first order in gyroradius

$$\begin{aligned}
& v_{\parallel} \nabla_{\parallel} \overline{f}_{\alpha} + \mathbf{v}_D \cdot \left(\boldsymbol{\nabla} f_{\alpha}^{(0)} + \mathbf{F}^{(0)} \frac{\partial}{\partial w} f_{\alpha}^{(0)} \right) \\
& + v_{\parallel} F_{\parallel}^{(1)} \frac{\partial}{\partial w} f_{\alpha}^{(0)} \\
& - \left[\left(\widehat{\mathbf{n}} \cdot \mathbf{W}_i \cdot \widehat{\mathbf{n}} \right) \frac{3v_{\parallel}^2 - v^2}{2} + \frac{2}{3} v^2 \ \boldsymbol{\nabla} \cdot \mathbf{V}_i \right] \frac{\partial}{\partial w} f_{\alpha}^{(0)} \\
= \ & C\left(\overline{f}_{\alpha} \right) \\
& + \frac{S}{4\pi v^2} \delta \left(v - v_0 \right)
\end{aligned}$$

NOTE

The term

$$\mathbf{v}_D \cdot \mathbf{F}^{(0)} \frac{\partial}{\partial w} f_{\alpha}^{(0)}$$

is the energy spent (the work done) by particles when they move in neoclassical drift $\mathbf{v}_D$ against the radial gradient of pressure, since $\mathbf{F}^{(0)} \sim \boldsymbol{\nabla} p$.

END.

NOTE

The term

$$v_{\parallel} F_{\parallel}^{(1)} \frac{\partial}{\partial w} f_{\alpha}^{(0)}$$

is the energetic effort made by particles against the force $F_{\parallel}^{(1)}$ which is due to the parallel electric field, since $F_{\parallel}^{(1)} \sim E_{\parallel}$.

END.

The variables

$$f_{\alpha} = f_{\alpha} \left(\psi, \theta; w, \lambda \right)$$

The forces that are present in the Fokker Planck equation are acting on the particle.

However the force is obtained from the fluid equilibrium

$$\begin{aligned}
\mathbf{F} \ = \ & \left(\frac{m_i Z_{\alpha}}{m_{\alpha} Z_i} - 1 \right) \left(\frac{\partial \mathbf{V}_i}{\partial t} + \left(\mathbf{V}_i \cdot \boldsymbol{\nabla} \right) \mathbf{V}_i \right) \\
& + \frac{Z_{\alpha}}{m_{\alpha} Z_i n_i} \left(\boldsymbol{\nabla} p_i + \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_i - \mathbf{R}_i \right)
\end{aligned}$$

$$\mathbf{v}_D = v_\parallel \widehat{\mathbf{n}} \times \boldsymbol{\nabla} \left(\frac{v_\parallel}{\Omega} \right)$$

and

$$\begin{aligned} \mathbf{W}_i &= \frac{1}{2}\left[(\boldsymbol{\nabla}\ \mathbf{V}_i) + (\boldsymbol{\nabla}\ \mathbf{V}_i)^T \right] - \frac{1}{3} (\boldsymbol{\nabla} \cdot \mathbf{V}_i)\ \mathbf{I} \\ & \quad \text{ion velocity strain tensor} \end{aligned}$$

NOTE

the rather unusual mixture of particle and fluid quantities.

The first occurence of the *force* is from the energy that a particle with the usual neoclassical drift $\mathbf{v}_D$ must spend to work against a force with which the *fluid* plasma acts against it. This is because we have a kinetic equation for the distribution function of α particles, while the *force* of the plasma acting against the α particles comes from the bulk plasma, of ions.

The effects of the ions on the α particles are complex, even before considering the friction (collisions).

The effects are energetic.

END

The ion velocity is

$$\begin{aligned} \mathbf{V}_i &= K_i \mathbf{B} + \omega_i R^2 \boldsymbol{\nabla} \varphi \\ & \quad (\text{parallel}) + (\text{toroidal}) \end{aligned}$$

and the *force* is separated into parts of different orders.

The first term will contribute to the *poloidal* flow

$$K_i B \sim \text{poloidal velocity}$$

Zero order *force*, it is almost radial like the gradient of pressure

$$\mathbf{F}^{(0)} = \frac{Z_\alpha}{m_\alpha Z_i n_i} \boldsymbol{\nabla} p_i$$

(this is the force that creates the *diamagnetic drift* of particles, as in the universal case **Galeev Sagdeev**)

And the first order *force*, it is along the magnetic line, due to a parallel electric field (contains the variation with θ)

$$F_\parallel^{(1)} = \frac{Z_\alpha e}{m_\alpha} E_\parallel$$

The flow is divergenceless

$$\boldsymbol{\nabla} \cdot \mathbf{V}_i = 0$$

which simplifies the *ion velocity strain* to

$$\widehat{\mathbf{n}} \cdot \mathbf{W} \cdot \widehat{\mathbf{n}} = K_i \nabla_\parallel B$$

and this is the *mirror force* due to the ondulations of the magnitude of the magnetic field B along the line.

The equation that results

$$
\begin{aligned}
& v_{\|} \nabla_{\|}\left(\overline{f}_{\alpha}+I \frac{v_{\|}}{\Omega_{\alpha}} \frac{\partial}{\partial \psi} f_{\alpha}^{(0)}-v_{\|} V_{\| i}^{*} \frac{\partial}{\partial w} f_{\alpha}^{(0)}\right) \\
& +\frac{Z_{\alpha} e}{m_{\alpha}} v_{\|} E_{\|} \frac{\partial}{\partial w} f_{\alpha}^{(0)} \\
= \; & C_{\alpha}\left(\overline{f}_{\alpha}\right) \\
& +\frac{S}{4 \pi v^{2}} \delta\left(v-v_{0}\right) \text{ (source of } \alpha \text{)}
\end{aligned}
$$

where **Hsu Shaing Gormley Sigmar PFB4 1992**,

$$
\begin{aligned}
V_{\| i}^{*} \;=\; & -\frac{I}{m_{i} \Omega_{c i}} \frac{1}{n_{i}} \frac{\partial p_{i}}{\partial \psi} \left(\begin{array}{c}\text{parallel "diamagnetic" } \sim \frac{1}{e B_{\theta}} \frac{1}{n_{i}} \frac{\partial p_{i}}{\partial r} \\ \text{(radial gradient) } / \; B_{\theta}\end{array}\right) \\
& +K_{i} B \quad \text{(poloidal)} \\
& \text{both projected on parallel direction (?)}
\end{aligned}
$$

(see also **Hirshman1978**).

The collision operator between α and electrons

$$
\begin{aligned}
C_{\alpha e} \;=\; & \frac{1}{\tau_{s}} \frac{\partial}{\partial \mathbf{v}} \cdot\left(\mathbf{v} f_{\alpha}+\frac{T_{e}}{m_{\alpha}} \frac{\partial}{\partial \mathbf{v}} f_{\alpha}\right) \\
& +\frac{\tau_{e}}{\tau_{s}} \frac{1}{m_{e} n_{e}} \mathbf{F}_{e i} \cdot \frac{\partial}{\partial \mathbf{v}} f_{\alpha}^{(0)}
\end{aligned}
$$

with $\mathbf{F}_{ei}$ $\equiv$electron ion friction force.

The first term is the *divergence of a flux in velocity space.*

The flux is composed of two terms.

The first is a flow of the α's.

The second is due to the gradient in the space of velocity. The coefficient is the thermal velocity of α's with T_e. The divergence of this gradient is determined by collisions τ_s.

The second term is a time-change of the distribution function, $\partial f/\partial t$, due to a coolisional friction acting as a force $\mathbf{F}_{ei}/n_e$. This force produces an acceleration $\mathbf{F}_{ei}/m_e$ which, acting in the time τ_e produces a change of velocity $\Delta v \sim \tau_e \times \mathbf{F}_{ei}/(m_e n_e)$ that exploits the gradient of the distribution function in the velocity space. Then we obtain $\Delta v \times \frac{\partial f_\alpha^{(0)}}{\partial \mathbf{v}}$ $\equiv$ a change of the distribution function due to this acceleration originating in $\mathbf{F}_{ei}$. This change is then distributed over τ_s, the time of collisions.

And the collisions of the α's with the ions

$$
C_{\alpha i}=\frac{1}{2} \frac{1}{\tau_{s}} v_{b}^{3} \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{U} \cdot \frac{\partial}{\partial \mathbf{v}} f_{\alpha}+\frac{1}{\tau_{s}} v_{c}^{3} \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{v} \frac{1}{v^{3}} f_{\alpha}
$$

Then the equation for the α function becomes

$$\begin{aligned}
& v_\parallel \nabla_\parallel \left(\overline{f}_\alpha + I \frac{v_\parallel}{\Omega_\alpha} \frac{\partial}{\partial \psi} f_\alpha^{(0)} - v_\parallel V_{\parallel i}^* \frac{\partial}{\partial w} f_\alpha^{(0)} \right) \\
& + \frac{Z_\alpha e}{m_\alpha} v_\parallel E_\parallel^* \frac{\partial}{\partial w} f_\alpha^{(0)} \\
= & \frac{1}{\tau_s} \left[\frac{1}{2} v_b^3 \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{U} \cdot \frac{\partial}{\partial \mathbf{v}} \overline{f}_\alpha + \frac{\partial}{\partial \mathbf{v}} \cdot \left(1 + \frac{v_c^3}{v^3} \right) \mathbf{v} \overline{f}_\alpha \right] \text{(coll)} \\
& \qquad + \frac{S}{4\pi v^2} \delta \left(v - v_0 \right) \quad \text{(source)}
\end{aligned}$$

From here one obtains equations for V_{nk}, $k = 1, 2, 3$. See below.

The distribution function for the *alpha* particles is, to first order of neoclassical theory

$$\begin{aligned}
f_{\alpha 1} = & -\frac{I v_\parallel}{\Omega_\alpha} \frac{\partial}{\partial \psi} f_{\alpha 0} \quad (\text{neoclassic } \sim \rho_\theta / L_n) \\
& + v_\parallel V_{\parallel i}^* \frac{\partial}{\partial w} f_{\alpha 0} \\
& + P \left(\lambda, w, \psi \right)
\end{aligned}$$

(First term is

$$\frac{I v_\parallel}{\Omega_\alpha} \frac{\partial}{\partial \psi} f_{\alpha 0} \approx \rho_\theta \left| \boldsymbol{\nabla} f_M \right|$$

the usual first order correction in the neoclassical distribution function, due to the particle's neoclassical drifts that carry the distribution function (zero order) along the radial gradient)

The second term is the usual expansion at the exponent of the energy in the equilibrium distribution

$$-\frac{\left(v_\parallel - V_i^* \right)^2}{2T/m} \sim -\frac{2 v_\parallel V_i^*}{2T/m} \sim \frac{v_\parallel V_i^*}{w}$$

which is obtained after derivation to the energy w of the distribution $f_{\alpha 0}$.

Here is **Rewoldt Tang Frieman integral formulation**

$$\begin{aligned}
f_M = & \frac{n(r)}{\left[2\pi T(r)/m \right]^{3/2}} \exp \left\{ -\frac{\left(v_\parallel - u_0 \right)^2 + v_\perp^2}{2T/m} \right\} \\
f_M^C \approx & \left(1 + \frac{2 u_0 v_\parallel}{v_{th}^2} \right) f_M
\end{aligned}$$

The last term P is like the function g usually considered for *circulating* and zero for *trapped*. This is neoclassic.

NOTE
Remember the substitution

$$\overline{f}_1 = \left(-I \frac{1}{2\Omega_0} v \left. \frac{\partial f_0}{\partial \psi} \right|_{\epsilon = \text{const}} \right) (2h\xi + P)$$

which is done in **Hsu Catto Sigmar** transport of fast *alphas*.
END

The zeroth order for α's is similar to the function for NBI fast ions

$$f_\alpha^{(0)} (\psi, v) = \frac{S}{4\pi (v^2 + v_c^3)} \tau_s \Theta (v_0 - v)$$

$$\begin{aligned} \frac{\partial}{\partial \psi} f_\alpha^{(0)} &= \left. \frac{\partial}{\partial \psi} f_\alpha^{(0)} \right|_v \\ &= \left(\frac{\partial}{\partial \psi} \ln (S \tau_s) - \frac{v_c^3}{v^3 + v_c^3} \frac{\partial}{\partial \psi} \ln \left(v_c^3 \right) \right) f_\alpha^{(0)} \end{aligned}$$

See also **Hazeltine Ware** for *rotation*.
Later used by **Fulop Helander** for large gradients, rotation, impurities.

The part of the distribution function

$$P (\lambda, w, \psi) = \sum_{j=1,2,3} \left(\sum_{n=1}^{\infty} \Lambda_n (\lambda, \psi) V_{nj} (w, \psi) \right) A_j (w, \psi)$$

The function $P (\lambda, w, \psi)$ is the distribution function that is zero in the trapped region.
See **Cordey, Hsu Catto Sigmar**.

NOTE
In **Hsu Catto Sigmar** transport of *alphas*, we have the separation of variables λ and v.

$$P (\psi, v, \lambda) = \sum_{n=1} \Lambda_n (\psi, \lambda) \; V_n (\psi, v)$$

but NO factors coming from the *forces* $A_j \left(\frac{v^2}{2}, \psi \right)$. What is the reason ?

In the case of *bootstrap* we need the perturbation P (to the function of distribution) to exibit explicitely the relation with the gradient of pressure. The first *force* is

$$A_1 \equiv -I \frac{v}{2\Omega_0} \frac{\partial}{\partial \psi} f_{\alpha 0}$$

and this contains the gradient of pressure.

END

The notations

$$w \equiv \frac{v^2}{2}$$

$$\lambda \equiv \frac{v_\perp^2}{v^2} h$$

$$h \equiv \frac{B_0}{B} = 1 + \frac{r}{R}\cos\theta$$

$$\begin{aligned} I &\equiv \mathbf{B} \cdot R^2 \boldsymbol{\nabla}\varphi \\ &\simeq RB_T \end{aligned}$$

The ion parallel flow without the: $E \times B$-induced return flow

$$\begin{aligned} V_{\parallel i}^{*} &= -\frac{I}{n_i m_i \Omega_i}\frac{\partial}{\partial \psi} p_i \quad \left(\text{this is } \frac{\partial p_i}{\partial r}\widehat{\mathbf{e}}_r \;\times\; B_\theta \widehat{\mathbf{e}}_\theta \sim \widehat{\mathbf{e}}_\parallel\right) \\ &\quad +K_i B \end{aligned}$$

The first part is (above) the combination between the radial gradient of ion pressure with the poloidal magnetic field, $\frac{1}{eB_\theta}\frac{1}{n_i}\frac{\partial p_i}{\partial r}$, which is a parallel velocity and

$$K_i \equiv \mathbf{V}_i \cdot \frac{\boldsymbol{\nabla}\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta} \quad \text{the poloidal flow}$$

NOTE
see also **Hirshman neoclassic current.**

$$j_\parallel = -F\frac{1}{B}\frac{dp}{d\psi} + KB$$

where

$$K = -\frac{dF}{d\psi}\frac{1}{4\pi}$$

and

$$F \sim I$$

END

(**Note** that this is an explicit form of the flow within the surface introduced by **Hinton Hazeltine**). **Note** that the diamagnetic part is

$$n\mathbf{v}_{dia} = \frac{1}{m\Omega}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p$$

and we replace

$$\frac{I}{B_0}\frac{\partial}{\partial\psi} \simeq \frac{1}{B_{pol}}\frac{\partial}{\partial r}$$

(because $I = RB_{\varphi}$ and $\nabla\psi\frac{\partial}{\partial\psi} = \frac{\partial}{\partial r}$, and then

$$\begin{aligned} -\frac{I}{m_i n_i \Omega_i}\frac{\partial p}{\partial \psi} &= -\frac{1}{m_i n_i \Omega_i}\frac{B_0}{B_{pol}}\frac{\partial p_i}{\partial r} \\ &= -\left(\mathbf{v}_{dia,i}\right)_{\perp}\frac{B_0}{B_{pol}} \\ &\simeq -\left(\text{parallel projection of the ion-diamagnetic velocity}\right) \\ &\quad \text{(if this is considered poloidal, with } \parallel \text{ component)} \end{aligned}$$

This is the usually considered parallel flow, due to the radial gradients, combined with the poloidal magnetic field B_θ. This is also the basis for the *simple* definition of the bootstrap current, the one where the *seed* toroidal current creates a B_θ ($\equiv$ toroidality, modulation of magnetic field magnitude, *trapped orbits banana*) that further, with the gradients, produce the toroidal current and its poloidal magnetic field, etc.: from here, - the name *bootstrap*).

End.

$$\Lambda_n \equiv \text{eigenfunction of the pitch angle scattering operator}$$

It is defined an *average over the surface*

$$\langle\rangle$$

The driving <u>forces</u> for $P(\lambda, w, \psi)$ are

$$A_1 = -\frac{Iv}{2\Omega_0}\frac{\partial}{\partial\psi}f_{\alpha 0}$$

(gradient of density)

$$A_2 = \left\langle \frac{V_{\parallel i}^*}{h} \right\rangle \frac{1}{2}\frac{\partial}{\partial v}f_{\alpha 0}$$

$$\begin{aligned} A_3 &= \frac{Z_\alpha |e|}{m_\alpha}\left\langle \frac{E_\parallel}{h} \right\rangle \frac{\tau_s}{2}\frac{\partial}{\partial v}f_{\alpha 0} \\ &\quad \text{(note the presence of collisions } \tau_s\text{)} \end{aligned}$$

13.1 Derivation of the function $V_{n1,2,3}$

The following part has been partly discussed above.

To the first order in gyroradius

$$
\begin{aligned}
& v_{\|}\nabla_{\|}\overline{f}_{\alpha}+\mathbf{v}_{D}\cdot\left(\boldsymbol{\nabla} f_{\alpha}^{(0)}+\mathbf{F}^{(0)}\frac{\partial}{\partial w}f_{\alpha}^{(0)}\right) \\
& +v_{\|}F_{\|}^{(1)}\frac{\partial}{\partial w}f_{\alpha}^{(0)} \\
& -\left[\left(\widehat{\mathbf{n}}\cdot\mathbf{W}_{i}\cdot\widehat{\mathbf{n}}\right)\frac{3v_{\|}^{2}-v^{2}}{2}+\frac{2}{3}v^{2}\ \boldsymbol{\nabla}\cdot\mathbf{V}_{i}\right]\frac{\partial}{\partial w}f_{\alpha}^{(0)} \\
= \quad & C\left(\overline{f}_{\alpha}\right) \\
& +\frac{S}{4\pi v^{2}}\delta\left(v-v_{0}\right)
\end{aligned}
$$

NOTE
The term

$$\mathbf{v}_{D}\cdot\mathbf{F}^{(0)}\frac{\partial}{\partial w}f_{\alpha}^{(0)}$$

is the energy spent by particles when they move in neoclassical drift against the radial gradient of pressure, since $\mathbf{F}^{(0)}\sim\boldsymbol{\nabla}p$.
END.

NOTE
The term

$$v_{\|}F_{\|}^{(1)}\frac{\partial}{\partial w}f_{\alpha}^{(0)}$$

is the energetic effort made by particles against the force $F_{\|}^{(1)}$ which is due to the parallel electric field, since $F_{\|}^{(1)}\sim E_{\|}$.
END.

Now it is used the procedure of **Hsu Catto Sigmar neoclassical transport of isotropic fast ions PF-B2 (1990) 280**.

The conjugated *fluxes* are

$$V_{n1}=\sigma_{n}\left[1-\frac{\left(\kappa_{n}-1\right)v_{b}^{3}}{\left(v_{c}^{3}+v^{3}\right)v\frac{\partial f_{\alpha0}}{\partial\psi}}\int_{v}^{v_{0}}du\left(\frac{v^{3}\left(v_{c}^{3}+u^{3}\right)}{u^{3}\left(v_{c}^{3}+v^{3}\right)}\right)^{\frac{Q_{p}\kappa_{n}}{3}}\frac{\partial f_{\alpha0}\left(u\right)}{\partial\psi}\right]$$

$$V_{n2}=\sigma_{n}\left[1-\frac{\left(\kappa_{n}-1\right)v_{b}^{3}}{\left(v_{c}^{3}+v^{3}\right)\frac{\partial f_{\alpha0}}{\partial\psi}}\int_{v}^{v_{0}}du\left(\frac{v^{3}\left(v_{c}^{3}+u^{3}\right)}{u^{3}\left(v_{c}^{3}+v^{3}\right)}\right)^{\frac{Q_{p}\kappa_{n}}{3}}\frac{\partial f_{\alpha0}\left(u\right)}{\partial\psi}\right]$$

$$V_{n3}=\frac{-\sigma_{n}}{\left(v^{3}+v_{c}^{3}\right)\frac{\partial f_{\alpha0}}{\partial v}}\int_{v}^{v_{0}}du\left(\frac{v^{3}\left(v_{c}^{3}+u^{3}\right)}{u^{3}\left(v_{c}^{3}+v^{3}\right)}\right)^{\frac{Q_{p}\kappa_{n}}{3}}u^{2}\frac{\partial f_{\alpha u}\left(u\right)}{\partial u}$$

where

$$Q_p = \frac{v_b^3}{v_c^3} \sim 1$$

etc.

The notation

$$E_\parallel^* = E_\parallel - \frac{Z_\alpha}{n_i e} F_{ei\parallel}$$

Here

$$\begin{aligned} F_{ei\parallel} &\equiv \text{electron-ion parallel friction force} \\ &\sim \text{resistivity } \eta \end{aligned}$$

Note that the last term contains $F_{ei\parallel}$ which is also defined in **Hazeltine Ware electrostatic trapping**, as

$$\begin{aligned} C_{ie} &= -\frac{\overline{F}_{ei}}{p_i} f_{M,i}\ v\xi \\ F_{ei} &= \int d^3v\ m_e v\xi\ C_{ei} \quad \left(\text{collisional parallel momentum } m_e v_\parallel\right) \\ \overline{F}_e &= \left(1 + \frac{Z^2 \overline{n}_Z}{\overline{n}_i}\right) \overline{F}_{ei} \\ &\approx e\overline{n}_e E_0 \quad (\text{then this term will be coupled with } E_0) \end{aligned}$$

and this is the reason for which it is transformed

$$E_\parallel \to E_\parallel^*$$

END

NOTE the solutions in **Cordey** and in **Hsu Catto Sigmar**.
END

NOTE that these are the result of using the full expression of the pitch angle scattering, as in **Fowler**. See **Gaffey**. **END.**

The parallel velocity of α can be calculated from the neoclassical distribution function

$$\begin{aligned} n_\alpha V_{\alpha i} &= \int d\mathbf{v} v_\parallel f_{\alpha 1} \\ &= \frac{\pi}{2} \sum_{\sigma=\pm 1} \sigma \int_0^{v_0} v^3 dv \int_0^h d\lambda f_{\alpha 1} \end{aligned}$$

so we need to solve the drift-kinetic equation for $f_{\alpha 1}$.

NOTE that the integration over $\lambda = \left(v_\perp^2/v^2\right) \times h$ is taken in the region of CIRCULATING particles, *i.e.* for small $v_\perp^2$ or small λ, not trapped.

This is natural since only the circulating particles can carry current.

END.

The term which is important in the *parallel current*

$$\begin{aligned}\left\langle V_{ie\|} B\right\rangle &= \left\langle V_{ie\|}^{(0)} B\right\rangle \left[1 + O\left(\frac{n_\alpha Z_a^2}{n_e Z_{eff}}\right)\right] \\ &\quad - \left(1 - F_\mu^i\right) \frac{n_\alpha Z_a^2}{n_e Z_{eff}} \left\langle V_{\alpha i\|} B\right\rangle\end{aligned}$$

The flux defined above

$$\begin{aligned}&\left\langle n_\alpha \frac{V_{\alpha i\|}}{h}\right\rangle \\ =\ &\left(1 - \left\langle \frac{1}{h^2}\right\rangle \sum_{n=1}^{\infty} \frac{\gamma_n}{\kappa_n}\right) \left(U_1 - n_\alpha \left\langle \frac{V_{\|i}^*}{h}\right\rangle\right) \\ &- \left\langle \frac{1}{h^2}\right\rangle \sum_{n=1}^{\infty} \left[\gamma_n \left(1 - \frac{1}{\kappa_n}\right) \left(U_{2n} - N_{2n} \left\langle \frac{V_{\|i}^*}{h}\right\rangle\right)\right] \\ &+ N_3 \frac{Z_\alpha e}{m_\alpha} \tau_s \left\langle \frac{E_\|^*}{h}\right\rangle\end{aligned}$$

We **note** that the $1/h$-weighted parallel *relative* α-ion FLOW $n_\alpha V_{\alpha i\|}$ is expressed in terms of the *ion* parallel flow $V_{i\|}^*$.

Here

$$\begin{aligned}U_1 &\equiv \int d^3v\ \frac{v^2}{3} \left(-\frac{I}{\Omega_0} \frac{\partial}{\partial \psi} f_{\alpha 0}\right) \\ &= -\frac{I}{m_\alpha \Omega_0} \frac{\partial p_\alpha}{\partial \psi} \\ &= S \tau_s v_0 \sum_{l=1}^{2} (-1)^{l+1} L_l X_l \\ &\quad \text{parallel "diamagnetic" } \alpha \text{ flow}\end{aligned}$$

and

$$U_{2n} \equiv \int d^3v\ \frac{v^2}{3}\ H_n(v) \left(-\frac{I}{\Omega_0} \frac{\partial f_{\alpha 0}}{\partial \psi}\right)$$

$$H_n(v) = \frac{4}{v^4} \int_0^v du\ u^3\ \left(\frac{u^3 \left(v^3 + v_c^3\right)}{v^3 \left(u^3 + v_c^3\right)}\right)^{\frac{Q_p \kappa_n}{3}}$$

Note that the integrand is the same parallel diamagnetic flow velocity but modulated by the factor $\left(\frac{u^3\left(v^3+v_c^3\right)}{v^3\left(u^3+v_c^3\right)}\right)^{\frac{Q_p\kappa_n}{3}}$ which results from collisional operator.

There is a related function, multiplying $\left\langle\frac{V_{\parallel i}^*}{h}\right\rangle$.

$$N_{2n} \equiv \int d^3v \ \left(H_n(v) + \frac{v}{3}\frac{\partial}{\partial v}H_n(v)\right) f_{\alpha 0}$$

Similar formulas

$$N_3 \equiv \sum_{n=1}^{\infty} \gamma_n \int d^3v \ \left(G_n(v) + \frac{v}{3}\frac{\partial}{\partial v}G_n(v)\right) f_{\alpha 0}$$

$$G_n(v) \equiv \frac{1}{v}\int_0^v du \ \left(\frac{u^3\left(v^3+v_c^3\right)}{v^3\left(u^3+v_c^3\right)}\right)^{\frac{Q_p\kappa_n}{3}} \ \frac{u^3}{u^3+v_c^3}$$

$$L_l \equiv \frac{1}{3\chi_0^{3(l-1)}}\int_0^1 dx \frac{x^4}{\left(x^3+\frac{1}{\chi_0^3}\right)^l}$$

$$X_1 \equiv -\frac{I}{\Omega_0}v_0\frac{\partial}{\partial\psi}\left[\ln\left(S\tau_s\right)\right]$$

$$X_2 \equiv -\frac{I}{\Omega_0}v_0\frac{\partial}{\partial\psi}\left[\ln\left(v_c^3\right)\right]$$

$$\chi_0 \equiv \frac{v_0}{v_c}$$

ration between velocity of α birth to critical velocity

13.2 The simplification

The authors eliminate the small effects

$$\begin{aligned} & \left\langle n_\alpha \frac{V_{\alpha i\parallel}}{h}\right\rangle \\ = \ & f_t^p \ U_1 \\ & -\left\langle\frac{1}{h^2}\right\rangle \sum_{n=1}^{\infty}\left[\gamma_n\left(1-\frac{1}{\kappa_n}\right)U_{2n}\right] \end{aligned}$$

$$\begin{aligned} f_t^p & \equiv \text{fraction of trapped particles} \\ p & \equiv \text{pitch-angle-dominated} \end{aligned}$$

$$f_t^p = 1 - f_c^p$$

13.3 The calculation of V_{ei}

In the following the usual force balance for electrons are written from the averaged parallel momentum equation.

But the presence of α will be modify the relative velocity between ions and electrons, in the expression of the forces.

For this one needs the collision operators for

$$\begin{array}{c} \alpha \text{ and } e \\ i \text{ and } e \end{array}$$

$$\begin{aligned} C_{e\alpha} &= \frac{1}{\tau_s}\frac{3}{4}\sqrt{\pi}v_{th,e}^3 \frac{n_\alpha Z_\alpha^2}{n_e Z_{eff}} \\ &\times \left(\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{U} \cdot \frac{\partial}{\partial \mathbf{v}} f_e + \frac{2\mathbf{v}\cdot\mathbf{V}_{\alpha i}}{v_{th,e}^2 \; v^3} f_{Me} \right) \end{aligned}$$

and with the collisions between electrons and ions

$$\begin{aligned} & C_{e\alpha} + C_{ei} \\ = \; & \frac{1}{\tau_e}\frac{3}{4}\sqrt{\pi}v_{th,e}^3 \\ & \times \left[\left(1 + \frac{n_\alpha Z_\alpha^2}{n_e \; Z_{eff}} \right) \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{U} \cdot \frac{\partial}{\partial \mathbf{v}} f_e + \frac{n_\alpha Z_\alpha^2}{n_e \; Z_{eff}} \frac{2\mathbf{v}\cdot\mathbf{V}_{\alpha i}}{v_{th,e}^2 \; v^3} f_{Me} \right] \end{aligned}$$

where

$$\frac{1}{\tau_s} = \frac{m_e}{m_\alpha}\frac{Z_\alpha^2}{Z_{eff}} \times \frac{1}{\tau_e}$$

Use of the momentum equations for electrons.

The divergence of the anisotropic pressure tensor is a force, equilibrated by the electric force and by the collisional friction.

This balance is projected on $\mathbf{B}$ and averaged on surface

$$\left\langle \mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_e \right\rangle = -n_e e \left\langle E_\parallel B \right\rangle + \left\langle F_{e\parallel} \; B \right\rangle$$

$$\left\langle \mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Theta}_e \right\rangle = \left\langle F_{e\parallel}^{(2)} \; B \right\rangle$$

where

$$\boldsymbol{\Theta}_e \equiv \int d^3v \; \frac{m_e}{2} \left(v^2 - \frac{5}{2}v_{th,e}^2 \right) \left(\mathbf{v} \; \mathbf{v} - \frac{v^2}{3}\mathbf{I} \right) \; f_e$$

$$\mathbf{F}_e^{(2)} \equiv \int d^3v \; \frac{m_e \mathbf{v}}{2} \left(v^2 - \frac{5}{2}v_{th,e}^2 \right) \; C_e \left[f_e \right]$$

The friction is due to collisions between *electrons* and α's and background ions. The operator is $C_e = C_{e\alpha} + C_{ei}$,

$$\mathbf{F}_e = m_e n_e \nu_{ei} \left(\mathbf{V}_* + \frac{3}{5} \frac{\mathbf{q}_e}{n_e T_e} \right)$$

$$\mathbf{F}_e^{(2)} = -\frac{3}{2} n_e T_e \left[\nu_{ei} \mathbf{V}_* + \frac{8}{15} \left(\nu_{ee} + \frac{13}{8} \nu_{ei} \right) \frac{\mathbf{q}_e}{n_e T_e} \right]$$

where we have introduced the *revaltive* velocity between electrons and ions+alphas, $\mathbf{V}_*$, as

$$\mathbf{V}_* = \mathbf{V}_{ie} + \frac{n_\alpha Z_\alpha^2}{n_e Z_{eff}} \mathbf{V}_{\alpha i}$$

Returning to the viscous forces

$$\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Pi}_e \rangle = m_e n_e \left\langle B^2 \right\rangle \left(\mu_1 V_p + \frac{2}{5} \mu_2 q_p \right)$$

$$\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Theta}_e \rangle = n_e T_e \left\langle B^2 \right\rangle \left(\mu_2 V_p + \frac{2}{5} \mu_3 q_p \right)$$

$$p \equiv \text{pitch angle dominated}$$

Approx

$$\mu_1 \approx \frac{f_t^p}{f_c^p} \left(\nu_{ei} + 0.754 \ \nu_{ee} \right)$$

$$\mu_2 \approx -\frac{f_t^p}{f_c^p} \left(1.5 \ \nu_{ei} + 0.884 \ \nu_{ee} \right)$$

$$\mu_3 \approx \frac{f_t^p}{f_c^p} \left(3.25 \ \nu_{ei} + 1.94 \ \nu_{ee} \right)$$

The relative parallel *electron-ion* velocity

$$\begin{aligned} \left\langle V_{ie\|} B \right\rangle &= \left\langle V_{ie\|}^{(0)} \ B \right\rangle \left[1 + O\left(\frac{n_\alpha Z_\alpha^2}{n_e Z_{eff}} \right) \right] \\ &\quad - \left(1 - F_\mu^e \right) \ \frac{n_\alpha Z_\alpha^2}{n_e Z_{eff}} \left\langle \mathbf{V}_{\alpha i \|} \ B \right\rangle \end{aligned}$$

where

$$\begin{aligned} F_\mu^e &= \frac{\mu_1 \left[\mu_3 + 2 \left(\nu_{ee} + \frac{13}{8} \nu_{ei} \right) \right] - \mu_2 \left(\mu_2 - \frac{3}{2} \nu_{ei} \right)}{\left(\mu_1 + \nu_{ei} \right) \left[\mu_3 + 2 \left(\nu_{ee} + \frac{13}{8} \nu_{ei} \right) \right] - \left(\mu_2 - \frac{3}{2} \nu_{ei} \right)^2} \\ &\equiv \text{effective fraction of trapped electrons} \end{aligned}$$

The electron-ion collisions must be calculated with the collision operator that includes the effect of impurities

$$\nu_{ei} = \nu_{ei}^{(0)} \left(1 + \frac{n_\alpha Z_\alpha^2}{n_e Z_{eff}} \right)$$

Upper label $(0) \equiv$ the result without α particles.

In the following calculations will occur the effective fraction of trapped particles
- F_μ^e of electrons
- F_μ^i of ions
- F_μ^α of alphas

13.4 Bootstrap current

With $\left\langle V_{ie\|} \, B \right\rangle$ one can now calculate the bootstrap current.

The total current is

$$\begin{aligned} \left\langle j_\| \, B \right\rangle &= \left\langle j_\|^{(0)} B \right\rangle \left[1 + O\left(\frac{n_\alpha Z_\alpha^2}{n_e Z_{eff}} \right) \right] \\ &\quad + n_\alpha Z_\alpha e \left(1 - \frac{Z_\alpha}{Z_{eff}} \left(1 - F_\mu^e \right) \right) \left\langle V_{\alpha i \|} \, B \right\rangle \end{aligned}$$

The part that comes from α particles

$$\begin{aligned} j_{bs}^\alpha &= \frac{\left\langle j_{bs}^\alpha B \right\rangle}{B_0} \\ &= \left(1 - \frac{Z_\alpha}{Z_{eff}} \left(1 - F_\mu^e \right) \right) \, F_\mu^\alpha \, \left(-I \frac{1}{B_0} \frac{\partial p_\alpha}{\partial \psi} \right) \end{aligned}$$

where

$$\begin{aligned} F_\mu^\alpha &\equiv f_t^p \\ &= - \frac{\left\langle \frac{1}{h^2} \right\rangle \sum_{n=1}^{\infty} \gamma_n \left(1 - \frac{1}{\kappa_n} \right) U_{2n}}{U_1} \\ &\quad \text{effective fraction of trapped } \alpha' s \end{aligned}$$

The term

$$\frac{Z_\alpha}{Z_{eff}} \left(1 - F_\mu^e \right) \equiv \text{electron screening effect}$$

The background bootstrap current is

$$
\begin{aligned}
& j_{bs}^{(0)} \\
= \quad & \left(-I\frac{1}{B_0}\right)\left[F_\mu^e \frac{\partial}{\partial\psi}\left(p_e+p_i\right)\right. \\
& \left. -F_\mu^e\ F_{\mu T}^i\ n_i\frac{\partial}{\partial\psi}T_i - F_{\mu T}^e\ n_e\frac{\partial}{\partial\psi}T_e\right]
\end{aligned}
$$

where

$$F_{\mu T}^e = \frac{-2\mu_2\left(\nu_{ee}+\frac{13}{8}\nu_{ei}\right)-\frac{3}{2}\mu_3\nu_{ei}}{\left(\mu_1+\nu_{ei}\right)\left[\mu_3+2\left(\nu_{ee}+\frac{13}{8}\nu_{ei}\right)\right]-\left(\mu_2-\frac{3}{2}\nu_{ei}\right)^2}$$

and,

$$F_{\mu\sigma}^e = \frac{\left[\mu_3+2\left(\nu_{ee}+\frac{13}{8}\nu_{ei}\right)\right]\nu_{ei}}{\left(\mu_1+\nu_{ei}\right)\left[\mu_3+2\left(\nu_{ee}+\frac{13}{8}\nu_{ei}\right)\right]-\left(\mu_2-\frac{3}{2}\nu_{ei}\right)^2}$$

is the correction to the Spitzer resistivity

The neoclassical correction to Spitzer conductivity modifies the Ohms law

$$J_{OH} = F_{\mu\sigma}^e\ \sigma_{Spitzer}\ \left\langle\frac{E_\parallel}{h}\right\rangle$$

By letting

$$
\begin{aligned}
\nu_{ei} &\rightarrow 0 \\
\nu_{ee} &\rightarrow \nu_{ii}
\end{aligned}
$$

and

$$\varepsilon \ll 1$$

it results

$$
\begin{aligned}
F_{\mu T}^e \quad \approx & \quad 1.17 \\
& \quad \text{thermal friction force}
\end{aligned}
$$

The α particles do not modify significantly the radial fluxes of electrons.

One uses the relations between radial fluxes and the parallel moments

$$
\begin{aligned}
\Gamma_e &= \left\langle n_e\mathbf{V}_e\cdot\boldsymbol{\nabla}\psi\right\rangle \\
&\approx \frac{I}{\left\langle B^2\right\rangle}\frac{c}{e}\ \left\langle\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_e\right\rangle
\end{aligned}
$$

14 Bootstrap current from alphas Jhang Chang

The paper intends to make a bridge between two limit states of the theoretical model.

In these limit-states the description is possible in analytical terms.

But not in-between.

The paper proposes a variational approach for the domain between the two analytically treatable extrema states

Where should-we look for the bootstrap ?

The bootstrap current is calculated from the parallel flow of the alpha particles. Then it must involve of

- radial gradient of the pressure
- the poloidal magnetic field B_θ

We must hope that the physical origin of the bootstrap is visible in this treatment.

14.1 Electron Fokker Planck equation

The first objective in the calculation of the bootstrap current sustained by the alphas is the distribution function of the *electrons*. This is because they interact collisionally with the alphas and produce the slowing down and pitch angle scattering.

As usual, two expansions

- the small banana width

$$f_e = f_{e0} + f_{e1} + ...$$

- small collisions relative to bounce frequency

The equation

$$v_\parallel \nabla_\parallel f_{e1} = C_e^{lin}\left[f_{e1}\right] + I v_\parallel \nabla_\parallel \left(\frac{v_\parallel}{\Omega_e}\frac{\partial f_{e0}}{\partial \psi}\right)$$

$$\begin{aligned} C_e^{lin} &= C_{ee}^{lin} + C_{ea}^{lin} \\ a &\equiv \text{ species} \end{aligned}$$

$$\begin{aligned} C_{ea}^{lin}\left[f_{e1}\right] &= \sum_{a=i,\alpha} \nu_{ea}(v)\left\{\mathcal{L}\left[f_{e1}\right] + \frac{2v_\parallel u_{a\parallel}}{v_{th,e}^2} f_{e0}\right\} \\ i &\equiv \text{ all thermalized ion species, } \alpha \equiv \text{ alphas} \\ u_{a\parallel} &\equiv \text{ parallel flow of species } a \text{ relative to electrons} \end{aligned}$$

$$\mathcal{L} \equiv 2\left(\frac{v_\parallel}{v}\right) h \frac{\partial}{\partial\lambda}\lambda\xi\frac{\partial}{\partial\lambda} \text{ pitch angle scattering}$$

$$\nu_{ea} = \frac{4\pi Z_a^2 e^4}{m_e^2} \ln\Lambda \left(\frac{v_{th,e}}{v}\right)^3 \frac{n_a}{v_{th,e}^3}$$

$$\xi \equiv \frac{v_\parallel}{v}$$

$$\lambda \equiv \frac{\mu B_0}{w} = \frac{v_\perp^2}{v^2} h$$

The second term in the collision operator is dependent on the relative velocity $u_{a\parallel}$ between electrons the species a, i.e. ions and α's.

Extract from the expression of the frequency of collisions a factor which does not depend on species, denoted ν_{e0},

$$\nu_{ea} = \frac{n_a Z_a^2}{n_e}\nu_{e0}$$

Then

$$C_{ea}^{lin}\left[f_{e1}\right] = \nu_{e0} Z_{eff}\left\{\mathcal{L}\left[f_{e1}\right] + \frac{2v_\parallel}{v_{th,e}^2}\frac{n_i Z_i^2 u_{i\parallel} + n_\alpha Z_\alpha^2 u_{\alpha\parallel}}{n_e Z_{eff}} f_{e0}\right\}$$

The velocity that combines the contributions from various thermal ions (including impurities) and alphas

$$u_* = \frac{n_i Z_i^2 u_{i\parallel} + n_\alpha Z_\alpha^2 u_{\alpha\parallel}}{n_e Z_{eff}}$$

$$Z_{eff} = \frac{\sum_a n_a Z_a^2}{n_e}$$

This u_* summarizes all about the relative flow of electrons and ions/α.

It is adopted the first substitution

$$\begin{aligned} f_{e1} &= \frac{2v_\parallel u_*}{v_{th,e}^2} f_{e0} \\ &\quad + H_e \end{aligned}$$

Then

$$\begin{aligned} v_\parallel \nabla_\parallel H_e &= \\ &= \nu_{e0} Z_{eff}\ \mathcal{L}\left[H_e\right] \\ &\quad + I v_\parallel \nabla_\parallel\left(\frac{v_\parallel}{|\Omega_e|}\frac{\partial f_{e0}}{\partial\psi}\right) \\ &\quad - v_\parallel \nabla_\parallel\left(\frac{2v_\parallel u_*}{v_{th,e}^2} f_{e0}\right) \end{aligned}$$

Now assume that the collisions can be neglected. Then there is no integration needed, but adding a constant G_e. The full function $H_e = H_e^{non-c} + G_e$,

$$H_e = I\frac{v_\parallel}{|\Omega_e|}\frac{\partial f_{e0}}{\partial \psi} - \frac{2v_\parallel u_*}{v_{th,e}^2} f_{e0} + G_e$$

where the constant of integration for the operator $\nabla_\parallel$ is

$$G_e \equiv \text{constant along the magnetic field line}$$

Returning to the equation for H_e we collect everything under the operator

$$v_\parallel \nabla_\parallel$$

plus the term involving the pitch angle operator $\mathcal{L}$ which is separated.

Then it is made a bounce averaging $\{\}$, which eliminates all terms with the exception of

$$\{\mathcal{L}[H_e]\} = 0$$

This becomes the solubility condition to determine G_e.

It is seen that G_e is a *linear* function of the *driving forces* = gradients plus the flows.

Then the representation of G_e must indicate this. From G_e it is extracted a factor that contains the flows, similar with u_*, whose definition is $u_* = \frac{n_i Z_i^2 u_{i\parallel} + n_\alpha Z_\alpha^2 u_{\alpha\parallel}}{n_e Z_{eff}}$.

$$G_e = \left[\frac{n_\alpha Z_\alpha^2}{Z_{eff}} u_{\alpha\parallel} + \frac{n_i Z_i^2}{Z_{eff}} u_{i\parallel}\right]\widehat{G} + G_\psi$$

We have

$$\begin{aligned} H_e + \frac{2v_\parallel u_*}{v_{th,e}^2} f_{e0} &= I\frac{v_\parallel}{|\Omega_e|}\frac{\partial f_{e0}}{\partial \psi} + G_e \\ &= I\frac{v_\parallel}{|\Omega_e|}\frac{\partial f_{e0}}{\partial \psi} + \left[\frac{n_\alpha Z_\alpha^2}{Z_{eff}} u_{\alpha\parallel} + \frac{n_i Z_i^2}{Z_{eff}} u_{i\parallel}\right]\widehat{G} + G_\psi \end{aligned}$$

The current of all plasma species

$$j = -\left\langle \frac{e\int d^3v\ v_\parallel f_{e1}}{h}\right\rangle + \sum_{a=\alpha,i} Z_a e\ n_a\ u_{a\parallel}$$

$$
\begin{aligned}
= \quad & -\left\langle \frac{e\int d^3v \; v_\parallel \left(H_e + \frac{2v_\parallel u_*}{v_{th,e}^2} f_{e0}\right)}{h} \right\rangle \\
& + \sum_{a=\alpha,i} Z_a e \; n_a \; u_{a\parallel}
\end{aligned}
$$

Here we replace the paranthesis with

$$
I\frac{v_\parallel}{|\Omega_e|}\frac{\partial f_{e0}}{\partial \psi} + \left[\frac{n_\alpha Z_\alpha^2}{Z_{eff}} u_{\alpha\parallel} + \frac{n_i Z_i^2}{Z_{eff}} u_{i\parallel} \right] \widehat{G} + G_\psi
$$

and write

note here a factor f_{e0} must be placed on the terms with G functions

$$
\begin{aligned}
= \quad & -\left\langle \frac{1}{h} e \int d^3v \; v_\parallel \left[I\frac{v_\parallel}{|\Omega_e|}\frac{\partial f_{e0}}{\partial \psi} + \frac{n_i Z_i^2}{Z_{eff}} \widehat{G} \; u_{i\parallel} + \frac{n_\alpha Z_\alpha^2}{Z_{eff}} \widehat{G} \; u_{\alpha\parallel} + G_\psi \right] \right\rangle \\
& + \sum_{a=\alpha,i} Z_a e \; n_a \; u_{a\parallel}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & Z_\alpha e \; n_\alpha \; u_{\alpha\parallel} \left(1 - \frac{Z_\alpha}{Z_{eff}} F_p \right) \\
& + j_{bulk}
\end{aligned}
$$

where

$$
\begin{aligned}
F_p \quad = \quad & \left\langle \frac{e\int_{circulating} \; d^3v \; \widehat{G} \; f_{e0}}{h} \right\rangle \\
= \quad & \text{effective passing fraction}
\end{aligned}
$$

Therefore

in order to obtain the bootstrap current due to alphas, one needs to calculate the flow velocity of alphas

$$
u_{\alpha\parallel}
$$

This is calculated from the *radial gradients of pressure*.

14.2 Alphas Fokker Planck equation

The form is the same

$$
v_\parallel \nabla_\parallel f_{\alpha 1} = C_\alpha^{lin} \left[f_{\alpha 1} \right] - I v_\parallel \nabla_\parallel \left(\frac{v_\parallel}{\Omega_\alpha} \frac{\partial f_{\alpha 0}}{\partial \psi} \right)
$$

There will be another expansion (beside the small banana width relative to L_n), in a small parameter that compares the collisions with the bounce.

$$
\begin{aligned}
f_{\alpha 1}^{(0)} \quad = \quad & -I\frac{v_\parallel}{\Omega_\alpha}\frac{\partial f_{\alpha 0}}{\partial \psi} \\
& + G
\end{aligned}
$$

from which the equation is

$$\left\{ C_{\alpha}^{lin}\left[f_{\alpha 1}^{(0)} \right] \right\} = 0$$

which becomes

$$\left\{ C_{\alpha}^{lin}\left[G \right] \right\} = -\left\{ C_{\alpha}^{lin}\left[-\frac{v_{\parallel}}{\Omega_{\alpha}} \frac{\partial f_{\alpha 0}}{\partial \psi} \right] \right\}$$

The expression of the linearized collision operator

$$\begin{aligned} C_{\alpha}^{lin}\left[f \right] &= C_{v}\left[f \right] \\ &\quad +\frac{1}{\tau_{s}} \frac{v_{b}^{3}}{v^{3}} \mathcal{L}\left[f \right] \end{aligned}$$

$\mathcal{L} \equiv$pitch angle scattering

$$\begin{aligned} C_{v}\left[f \right] &= \text{energy scattering operator} \\ &= \frac{1}{\tau_{s}} \frac{1}{v^{2}} \frac{\partial}{\partial v} \left[\left(v^{3} + v_{c}^{3} \right) f_{\alpha} + \frac{T_{e}}{m_{\alpha}} \frac{v^{2}}{} \left(1 + \frac{v_{c}^{3}}{v^{3}} \beta \right) \frac{\partial f_{\alpha}}{\partial v} \right] \end{aligned}$$

The components of this "energy scattering"

$$\begin{aligned} &\frac{1}{\tau_{s}} \frac{1}{v^{2}} \frac{\partial}{\partial v} \left[\left(v^{3} + v_{c}^{3} \right) f_{\alpha} \right] \\ &= \text{slowing down drag of energetic particles} \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{\tau_{s}} \frac{1}{v^{2}} \frac{\partial}{\partial v} \left[\frac{T_{e}\ v^{2}}{m_{\alpha}} \left(1 + \frac{v_{c}^{3}}{v^{3}} \beta \right) \frac{\partial f_{\alpha}}{\partial v} \right] \\ &= \text{diffusion in energy} \end{aligned}$$

14.2.1 Limiting cases

In the region where the slowing down is dominant

$$G_{v} = -I \frac{v}{2\Omega_{\alpha 0}} \frac{\partial f_{\alpha 0}}{\partial \psi} \times \frac{1}{\frac{\partial \langle \xi \rangle}{\partial \lambda}}$$

In the region where the pitch angle is dominant

$$G_{\lambda} = -I \frac{v}{2\Omega_{\alpha 0}} \frac{\partial f_{\alpha 0}}{\partial \psi} \times \int_{\lambda_{c}}^{\lambda} d\lambda \frac{1}{\langle \xi \rangle}$$

In the space of velocity there is a boundary between trapped and circulating particles

This boundary is a *layer*

$$|\lambda - \lambda_c| < \Delta\lambda$$

In this layer, the *pitch angle* solution is dominant.

In the rest of the velocity space

$$|\lambda - \lambda_c| > \Delta\lambda$$

the *slowing down* solution is dominant.

At the birth energy (highest energy) the slowing down solution is dominant.

The width of the pitch angle scattering layer is

$$(\Delta\lambda)^2 \sim \frac{v_b^3}{v^3}$$

The parallel velocity of the α's is approximated as a sum of three terms

$$u_{\parallel\alpha} = A\sqrt{\varepsilon} + B\ \varepsilon + C\ \varepsilon^{3/2}$$

14.2.2 For large aspect ration $\varepsilon \ll 1$

The quantity

$$\langle|\xi|\rangle = \left(1 + 2\varepsilon - \frac{\lambda}{\lambda_c}\right)^{1/2} \frac{2}{\pi}\ E(k)$$

$$\begin{aligned} \lambda_c &= 1 - \varepsilon \\ &\quad \text{value of } \lambda \text{ at the boundary} \end{aligned}$$

$$k = 2\varepsilon \frac{2 - \frac{\lambda}{\lambda_c}}{1 - \frac{\lambda}{\lambda_c} + 2\varepsilon}$$

The two functions

$$G_v \sim \text{slowing down}$$

$$G_\lambda \sim \text{pitch angle}$$

are non-zero only in the circulating part of the velocity space

$$0 < \lambda < \lambda_c$$

Near the limit

$$\begin{aligned} \lambda &\rightarrow 0 \\ \text{which is } v_\perp &\sim 0 \text{ fast circulating} \end{aligned}$$

$$k \to \frac{4\varepsilon}{1+2\varepsilon}$$

$$\begin{aligned} \frac{2}{\pi}E(k) &\sim \frac{2}{\pi}E\left(\frac{4\varepsilon}{1+2\varepsilon}\right) \\ &\sim 1-\frac{\varepsilon}{1+2\varepsilon} \end{aligned}$$

Near the limit

$$\begin{aligned} \lambda &\to \lambda_c \\ &\text{slow circulating, close to trapping} \end{aligned}$$

$$k \to 1$$
$$\frac{2}{\pi}E \to \frac{2}{\pi}$$

When λ is not close to λ_c,

$$\langle|\xi|\rangle \approx \sqrt{1-\frac{\lambda}{\lambda_c}}$$

and plus a correction

$$\begin{aligned} \langle|\xi|\rangle &\approx \langle|\xi^*|\rangle \\ &= \sqrt{1-\frac{\lambda}{\lambda_c}+2\alpha\varepsilon} \\ \text{for} \quad \varepsilon &\ll 1 \end{aligned}$$

Comments

"Considering the fact that the background bootstrap current is peaked too much toward the plasma periphery, one of the beneficial features of the alpha-driven bootstrap current may be in that the peak of the alpha bootstrap current appears at smaller minor radius than that of the background bootstrap current. Thus, by increasing the alpha bootstrap current fraction in a tokamak reactor, the total bootstrap current profile could be made to be closer to that required for a reversed shear equilibrium"

"the alpha bootstrap fraction in the ITER CDA under similar plasma conditions is 7.2% for $Z_{eff} \sim 1.5$."

"The direction of the alpha-driven bootstrap current follows the sign of $Z_{eff} - 2F_p$, which may yield a negative

current for $Z_{eff} < 2$ near the magnetic axis where $F_p \sim 1$. However, for practical values of $Z_{eff} \geq 1.5$, the alpha-driven bootstrap current is positive over most of the plasma volume"

The effective fraction of circulating electrons

$$F_p = 1 - 1.46\left(1 + \frac{0.67}{Z_{eff}}\right)\sqrt{\varepsilon} + 0.46\left(1 + \frac{2.1}{Z_{eff}}\right)\varepsilon$$

in the low collisional regime.