1 Impurities. Introduction

The other text is accumulation of impurities from *models.*

Basic milestones

- ambipolarity and inward flow of impurities, **Connor**
- **Burrell Wong** temperature screening

Basic result from **Connor 1973**:
the heavy impurities must diffuse inwardly since this is the only way to preserve the AMBIPOLARITY

$$\sum_{ion-species} Z_j \Gamma_j = 0$$

The reason for the fact that only the impurities are involved in the preservation of the ambipolarity is that the electrons have very small diffusion rate, due to the small neoclassical deviation of the electron orbit relative to the magnetic surface. Then the electron radial diffusion cannot compensate the background-ion radial diffusion. If there are heavy impurities.

See below **Longmire Rosenbluth** about the inward diffusion of impurities.
And **Rutherford 1974** about the derived ambipolarity of diffusive fluxes as simple consequence of equality between friction $i-I$ with minus $I-i$ friction. The basis is the force balance (grad-pressure, E and vxB, and friction).

See below **Burrell** for the impurity asymmetries on surface, at injection of gas at the bottom (**Ohkawa**).

Diamond Ware ionization instability, see below.

2 Time scales

In **Helander 3999**.
The collisions frequencies

- ion-ion
- impurity-ion

are comparable. Then

$$Z_{eff} - 1 = \frac{n_Z Z^2}{n_i} \sim O(1)$$

The frequency of collisions

$$\frac{1}{\tau_{ab}} = \frac{e_a^2 \; e_b^2}{(4\pi\varepsilon_0)^2} \frac{1}{m_a^2} \ln \Lambda \; \frac{n_b}{v_{th}^{>} \; v_{th,a}^2}$$

where

$$v_{th}^{>} \equiv \text{the larger of thermal velocities } v_{th,a} \text{ and } v_{th,b}$$

then

$$\frac{n_b}{v_{th}^{>} \; v_{th,a}^2} = \begin{cases} \frac{n_b}{v_{th,a}^3} & \text{if } a \text{ particles are faster} \quad v_{th,a} > v_{th,b} \\ \frac{n_b}{v_{th,b} v_{th,a}^2} & \text{if } b \text{ particles are faster} \quad v_{th,a} < v_{th,b} \end{cases}$$

The time scale of establishing a *parallel* equilibrium

$$\tau_{\parallel} \sim \frac{L_{\parallel}^2}{v_{th,Z}^2 \; \tau_{ZZ}}$$

It looks that the parallel equilibrium is reached by $Z - Z$ collisions (between impurities Z ions).

The time scale associated to perpendicular (cross-field) particle transport

$$\tau_{\perp} \sim \frac{L_{\perp}^2}{\rho_Z^2 / \tau_{Z-i}}$$

where

$$\rho_Z = \frac{v_{th,Z}}{\Omega_Z}$$

The ratio of the time scales

$$\begin{aligned} \frac{\tau_{\parallel}}{\tau_{\perp}} &\sim \frac{Z_{eff} - 1}{Z^{3/2}} \Delta^2 \\ &\ll 1 \end{aligned}$$

where

$$\begin{aligned} \Delta &= \frac{\widetilde{n}_Z}{n_Z} \qquad\qquad [\text{order } \Delta \sim O(1)] \\ &= \text{poloidal modulation of the impurity density} \\ &\sim \delta \; \widehat{\nu}_{i-i} \; Z^2 \left(\text{where } \delta = \frac{\rho_\theta}{L_\perp} \ll 1 \right) \\ \text{and } \widehat{\nu}_{i-i} &= \frac{L_\parallel}{\lambda_{i-i}} \quad (\lambda_{i-i} \equiv \text{mean free path of bulk ions}) \end{aligned}$$

3 General remarks on impurities' dynamics

Helander 3999.

The collisions of impurities with background ions are as frequent as ion-ion.

The perpendicular velocity of high Z ions is due to the *radial* electric field

$$\begin{aligned}\mathbf{V}_{Z\perp} &= \frac{-\boldsymbol{\nabla}\Phi_0 \times \widehat{\mathbf{n}}}{B_0} \\ &= \frac{d\Phi_0}{d\psi}\left(\frac{I}{B}\widehat{\mathbf{n}} - R^2\boldsymbol{\nabla}\varphi\right)\end{aligned}$$

This is similar to the diamagnetic flow and has non-zero divergence.

[remember the *drift velocity* has non-zero divergence and the subject is discussed by **Wong Burrell**].

Then there is a *return* flow of impurities, on the parallel direction, like *Pfirsch Schluter*

$$\boldsymbol{\nabla}\cdot(n_Z\mathbf{V}_Z) = 0$$

$$V_{Z\parallel} = -\frac{I}{B}\frac{d\Phi_0}{d\psi} + \frac{1}{n_Z}K_Z(\psi)B$$

the last term is due to the *poloidal* flow.

The compressional heating

$$\begin{aligned}& p_Z\ \boldsymbol{\nabla}\cdot\mathbf{V}_Z \\ \sim\ & \delta\ p_Z\frac{v_{th,i}}{L_\parallel}\end{aligned}$$

is inefficient in creating a surface non-uniformity of the temperature T_Z since the collisions are able to produce an energy equilibration.

The same for the divergence of the diamagnetic heat flux.

4 Impurity fluxes and plasma rotation

The paper on **Observation of central toroidal rotation in ICRF**, Rice,..., NF 38 (1998) 75. From **Observationtoroidal rotation Alcator-C Rice**:

In the L-mode the impurities rotates toroidally in direction *opposite* to the current. (Counter-current seems a state favorable for confinement of particles, see counter-NBI).

The fact that at the H-mode we have a much better particle confinement extends also over the impurity confinement. In the H-mode there is accumulation of impurities in the center of the plasma.

Regarding the relative direction of impurity radial motion, the text **Density and Omega.tex** presents a hypothesis, in which it is involved the *swirl*.

But the problem is not only *accumulation*, seen as a result of better confinement.

It is also the *inward flow of impurities* in the radial direction.

See **Connor 1973** who shows that the impurities must go inward since the background ions diffuse outward and the electrons cannot compensate the currents.

And **Burrell Wong** about temperature screening.

From the paper **Varenna ICRHTorque**: there is a pinch of particles during the RF heating, because a toroidal acceleration displaces the trapped ion turning points accross the magnetic surfaces. The turning points are displaced inwardly for *heated* ions. And viceversa.

The origin of displacement of the tips of banana after heating is the conservation of J. See **White Chang ICRH**.

Usually the impurities have *inward* flux in a tokamak. This is partly related to radial gradient being created by high impurity-density at the edge and low impurity-density in the plasma core.

The explanation based on *diffusion* and *ambipolarity* is given by **Connor 1973.** Another explanation, *non-diffusive* is given by **Hayashi JT60** the two sides of a banana traverse different Temperature regions (this is also important in the *precession* of the bananas, **Fong Hahm**).

A small amount of Carbon impurities are found in the center of TEXT after a blow up at the edge, much faster than the diffusion.

The paper **NBI influence on impurity Isler** ISX-B tokamak. It is mentioned that theoretical works in neoclassics have revealed that:

1. NBI and electric current in the same direction (*co-injection*) can reduce the inward diffusive flow of impurities.

2. NBI and electric current in opposite direction (*counter-injection*) has the effect of accumulation and *persistence* of impurities in the centre.

In general, in Ohmically heated discharges: classical and neoclassical processes lead to flux of impurities toward the centre. Possibly the *ambipolarity* explanation of **Connor 1973.**

However in reality the accumulation of impurities is slow or even inexistent. One has to use empirical anomalous diffusion coefficients to explain this.

In the paper of **Isler**. The Argon is concentrated in the centre but only in Deuterium and *not* in Hydrogen.

The mass of the background ions is therefore important.

This possibly becomes important through *collisions*.

Conclusion for Argon:

1. the Ar accumulation in the centre is *inhibited* by **co-injection** (injection NBI parallel with the current), and
2. the Ar accumulation in the centre is *enhanced* by **counter-injection**.

[see for comparison the two conclusions for NBI + current (co- and counter-), above]

The same conclusions for the *iron*.

the influx of Fe toward the centre is very fast in *counter-injection* NBI is opposite to the current (strangely: this is H-mode; the outflux of background ions should be reduced and the need for compensation by inward flow of impurities should be correspondingly reduced too).

Plasma begins to cool in the centre rapidly. (**Note** looks like the H-mode; normally the impurities are better confined. However the effect is not due to new atoms absorbed and confined, but to redistribution of the Fe ions. Simply said: Fe ions travel rapidly to the center.)

Redistribution of HEAVY ions with inward flux.

Heavy ions are collected in a cylindrical (toroidal) region around the magnetic axis, with a small radius.

By contrast, the *co-injection* of NBI does not lead to accumulation of Fe in the plasma centre. (**Note** looks like the L-mode).

The evolution of Fe atoms *is not due to supplementary input of Fe in the discharge but to redistribution*.

The change in transport, the only acceptable explanation for this behavior of impurities, cannot be attributed to only the known transport mechanisms (classical and neoclassical) since this would imply a flatter profile of Fe while the experiment shows a peaked profile.

NOTE the possibility to associate this peaking of the impurity (Fe) density in the center with the density pinch induced by the vorticity pinch.

From the article: **BootstrapNeo Hirshman** 1996 :

1. in quiescent H-mode the absolute value of impurity velocity is neoclassic
2. hot-ion mode in JET: neoclassical effect of temperature gradient screening explains the expulsion of the Carbon from the core
3. in PEP (Pellet Enhanced Performance) the same effect (temperature gradient screening) drives impurities inward

When the impurities are introduced in plasma (seeded) as in the experiments on **DIII-D** with Krypton, Argon, Neon, the collisionality for ions increases. As a result there is a higher diffusion of banana in the radial direction. This should be seen in the suplementary *bootstrap* current.

Impurity and rotation? **Fulop Helander**.

5 Impurity inward transport Longmire Rosenbluth

The paper of 1956.

Magnetic field in direction z.

There is a possible gradient of density along x.

$$\mathbf{v}\cdot\boldsymbol{\nabla}f+\frac{e}{m}\left(\mathbf{v}\times\mathbf{B}\right)\cdot\frac{\partial f}{\partial\mathbf{v}}=0$$

$$\begin{aligned} & v_x\frac{\partial f}{\partial x}\quad\text{radial convection}\\ & +\Omega\left(v_y\frac{\partial f}{\partial v_x}-v_x\frac{\partial f}{\partial v_y}\right)\quad\text{gyration}\\ = & \ 0\end{aligned}$$

The solution

$$\begin{aligned} f &= f\left(x+\frac{v_y}{\Omega},\ v_z,\ v_x^2+v_y^2\right)\\ & f\quad\text{is arbitrary collection of gyrations}\end{aligned}$$

The coordinate of the guiding center

$$X=x+\frac{v_y}{\Omega}$$

More restricted

$$f=n\left(x+\frac{v_y}{\Omega}\right)\times g\left(v_z,\ v^2\right)$$

$$\begin{aligned} f &= n\left(x+\frac{v_y}{\Omega}\right)\\ &\times\left(\frac{1}{\pi\left(2kT/m\right)}\right)^{3/2}\exp\left(-\frac{v^2}{2kT/m}\right)\end{aligned}$$

Assume the same temperature for both types of particles.

The flux

It is considered the collision of two *test particles* one of type 1 and one of type 2 not necessarily different.

Calculate the flux of particles of type 1 due to collisions with the particles of type 2.

The flux, from **Chandrasekhar 1943**

$$\begin{aligned} F_1 &= n_1\left(X\right)\ \left\langle\Delta X_1\right\rangle\\ &-\frac{1}{2}\frac{\partial}{\partial X}\left[n_1\left(X\right)\ \left\langle\left(\Delta X_1\right)^2\right\rangle\right]\end{aligned}$$

There is an equation of continuity

$$\frac{\partial n_1(X)}{\partial t} + \frac{\partial}{\partial X} F_1(X) = 0$$

Due to collision the position X_1 of particle 1 suffers a change, a displacement ΔX_1.

The velocity also is modified

$$\Delta v_{1y} = \Omega \times \Delta X_1$$

(**note** that this comes from

$$\begin{aligned} x &= X - \frac{v_y}{\Omega} \\ &= X' - \frac{v_y'}{\Omega} \end{aligned}$$

or

$$\begin{aligned} \Delta X &= X' - X = \frac{1}{\Omega}\left(v_y' - v_y\right) \\ \Delta X &= \frac{1}{\Omega}\Delta v_y \end{aligned}$$

end).

The average

over velocities of particle 1
over velocities of particle 2
over the angle of scattering

The physical position where the collision of particle 1 takes place is

$$x = X - \frac{v_{1y}}{\Omega_1}$$

The density of particles 2 at this physical position is the density of the guiding centers at

$$x + \frac{v_{2y}}{\Omega_2} = X + \left(\frac{v_{2y}}{\Omega_2} - \frac{v_{1y}}{\Omega_1}\right)$$

(note that this is the above equation to which we add v_{2y}/Ω_2 since we want to find the guiding center position of particles 2).

The probability per unit time that a particle 1 with guiding center at X is involved in collision and scattered under solid angle $d\Omega$ is

$$\begin{aligned}
& P\left(v_1, v_2, \Omega\right)\ d^3 v_1\ d^3 v_2\ d\Omega \\
= \quad & d^3 v_1 d^3 v_2\ \left(\frac{m_1}{\pi 2 k T_1}\right)^{3/2} \left(\frac{m_2}{\pi 2 k T_2}\right)^{3/2} \\
& \times \exp\left(-\frac{v_1^2}{2kT/m_1}\right) \\
& \times n_2\left(X + \frac{v_{2y}}{\Omega_2} - \frac{v_{1y}}{\Omega_1}\right) \\
& \times \exp\left(-\frac{v_2^2}{2kT/m_2}\right) \\
& \times \left|\mathbf{v}_1 - \mathbf{v}_2\right|\ \sigma\left(\Omega\right)\ d\Omega
\end{aligned}$$

Regarding the collision: one has to change to *center-of-mass* coordinates

$$\begin{aligned}
M &= m_1 + m_2 \\
m &= \frac{m_1 m_2}{M} \\
\mathbf{V} &= \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{M} \\
\mathbf{v} &= \mathbf{v}_1 - \mathbf{v}_2
\end{aligned}$$

which changes the form of the probability

$$\begin{aligned}
P\left(v, V, \Omega\right) = \quad & d^3 v d^3 V\ \left(\frac{m}{\pi 2 k T}\right)^{3/2} \left(\frac{M}{\pi 2 k T}\right)^{3/2} \\
& \exp\left(-\frac{v^2}{2kT/m}\right) \exp\left(-\frac{V^2}{2kT/M}\right) \\
& \times n_2\left(X + \delta\right) \\
& \times v\ \sigma\left(\Omega\right)\ d\Omega
\end{aligned}$$

where

$$\delta \equiv \frac{1}{B}\left[\left(\frac{m_2}{e_2} - \frac{m_1}{e_1}\right) V_y - m\left(\frac{1}{e_2} + \frac{1}{e_1}\right) v_y\right]$$

The density $n_2\left(X + \delta\right)$ can be expanded

$$\begin{aligned}
n_2\left(X + \delta\right) = \quad & n_2\left(X\right) + \left(\frac{d n_2}{dX}\right)\delta + \frac{1}{2}\left(\frac{d^2 n_2}{dX^2}\right)\delta^2 \\
& + \frac{1}{6}\left(\frac{d^3 n_2}{dX^3}\right)\delta^3 + \ldots
\end{aligned}$$

The displacement of the *guiding center* ΔX_1 is

$$\Delta X_1 = \frac{\Delta v_{1y}}{\Omega_1}$$

where Δv_{1y} is obtained from the conditions of collision in the center-of-mass system. The result is in terms of

$$v, V, \text{ scattering angles } \theta, \varphi$$

$$\Delta X_1 = \frac{m}{m_1} \frac{v}{\Omega_1} \left[\sin\theta \cos\varphi \sin\chi - (1-\cos\theta)\cos\chi \right]$$

$$\begin{aligned} \chi &\equiv \text{angle between } \mathbf{v} \text{ and } \widehat{\mathbf{e}}_y \\ v_y &= v\cos\chi \end{aligned}$$

The averages of the displacement

$$\begin{aligned} \langle \Delta X_1 \rangle &= \int d^3v \; d^3V \; d\Omega \; P(v, V, \Omega) \; \Delta X_1 \\ \left\langle (\Delta X_1)^2 \right\rangle &= \int d^3v \; d^3V \; d\Omega \; P(v, V, \Omega) \; (\Delta X_1)^2 \end{aligned}$$

Keep still unknown $\sigma(v,\theta)$ and perform the integrations over the other variables

$$\begin{aligned} \langle \Delta X_1 \rangle &= \frac{8\pi^2}{3} \left(\frac{m}{\pi 2kT} \right)^{3/2} \left(\frac{m}{e_1 B} \right)^2 \\ &\times \left[\left(1 + \frac{e_1}{e_2} \right) \; I_1 \; \frac{\partial N_2}{\partial X} + \frac{1}{10} \left(\frac{m}{e_1 B} \right)^2 \left(1 + \frac{e_1}{e_2} \right)^3 \; I_2 \frac{\partial^3 N_2}{\partial X^3} \right] \end{aligned}$$

$$\begin{aligned} \left\langle (\Delta X_1)^2 \right\rangle &= \frac{8\pi^2}{3} \left(\frac{m}{\pi 2kT} \right)^{3/2} \left(\frac{m}{e_1 B} \right)^2 \\ &\times \left[I_1 \; N_2 + \frac{1}{20} \left(\frac{m}{e_1 B} \right)^2 \left(1 + \frac{e_1}{e_2} \right)^2 \; I_3 \frac{\partial^2 N_2}{\partial X^2} \right] \end{aligned}$$

The integrals are

$$I_1 = \int_0^\infty dv \int_0^\pi d\theta \; \sigma(v,\theta) \; v^5 \exp\left(-\frac{v^2}{2kT/m} \right) (1-\cos\theta)\sin\theta$$

$$I_2 = \int_0^\infty dv \int_0^\pi d\theta \; \sigma(v,\theta) \; v^7 \exp\left(-\frac{v^2}{2kT/m} \right) (1-\cos\theta)\sin\theta$$

$$\begin{aligned} I_3 &= \int_0^\infty dv \int_0^\pi d\theta \; \sigma(v,\theta) \; v^7 \exp\left(-\frac{v^2}{2kT/m} \right) \\ &\times \left[\sin^2\theta + 3(1-\cos\theta)^2 \right] \sin\theta \end{aligned}$$

Later one will specify σ as Rutherford cross section.

The flux

$$
\begin{aligned}
F_1 &= \frac{8\pi^2}{3}\left(\frac{m}{\pi 2kT}\right)^{3/2}\left(\frac{m}{e_1 B}\right)^2 \\
&\quad \times I_1\left[\left(1+\frac{e_1}{e_2}\right) n_1\frac{\partial n_2}{\partial X} - \frac{\partial}{\partial X}\left(n_1 n_2\right)\right]
\end{aligned}
$$

The flux is ZERO if $n_1 = n_2$, same type.

6 Impurity inward transport Rutherford 1974

The paper by **Rutherford 1974 pfirsch schluter.**
Hydrogen plasma + a single species of impurities.
Mass of impurity $\gg$ mass of ions

$$
\begin{aligned}
m_I &\gg m_i \\
&\rightarrow \text{Lorentz model for collisions}
\end{aligned}
$$

The transport is similar with the case "electrons plus ions".

$$
\begin{aligned}
&\text{ions+impurities} \\
= \\
&\text{electrons+ions}
\end{aligned}
$$

The prototype: "electrons and ions"
The collision operator

- first part: electron-electron
- second part: electron-ion

In the case "ions and impurities"

- first part: proton-proton
- second part proton-impurities

The eq.

$$0 = -\boldsymbol{\nabla} p_i + n_i e\left(\mathbf{E} + \mathbf{v}_i \times \mathbf{B}\right) + \mathbf{R}$$

$$0 = -\boldsymbol{\nabla} p_I + n_I Z_I e\left(\mathbf{E} + \mathbf{v}_I \times \mathbf{B}\right) - \mathbf{R}$$

where $\mathbf{R}$ is the friction on protons due to impurities.

For Pfirsch Schluter only the parallel component is necessary.
In **Burrell1980** it is the **Hirshman** expression for the friction

$$R_{\| a} = \sum_b \left(l_{11}^{ab}\ u_{\| b} + l_{12}^{ab}\ \frac{2}{5}\frac{q_{\| b}}{p_b} \right)$$

$$h_{\| a} = \sum_b \left(l_{21}^{ab}\ u_{\| b} + l_{22}^{ab}\ \frac{2}{5}\frac{q_{\| b}}{p_b} \right)$$

Here it is simplified

$$R_{\|} = -C_1 \frac{m_i n_i}{\tau_{iI}} u_{\|} - C_2\ n_i \nabla_{\|} T_i$$

$$\mathbf{u} \equiv \mathbf{v}_i - \mathbf{v}_I$$

$$\tau_{iI} = \frac{3}{4\sqrt{2\pi}} \left(4\pi\varepsilon_0\right)^2 \frac{\sqrt{m_i}}{Z_I^2 e^2 Z_i^2 e^2} \frac{1}{\ln\Lambda} \times \frac{T_i^{3/2}}{n_I}$$

Note that these are simplified forms of the usual expression of $R_{\|}$ and $H_{\|}$ in terms of *flows* $u_{\|}$ and $q_{\|}/p$ and **Hirshman** coefficients l_{ij}^{ab} (a, b =species, $i, j = 1, 2$ flows). **End.**

The two equations of force balance are used after *radial projection.*
The field E_r is eliminated

$$u_\theta = \frac{B_\theta}{B_T} u_T + \frac{1}{eB_T} \left(\frac{1}{n_i}\frac{\partial p_i}{\partial r} - \frac{1}{n_I Z_I}\frac{\partial p_I}{\partial r} \right)$$

From this equation it is retained only the part that is lowest order in $\varepsilon = r/R$,

$$u_{\theta 0} = \frac{1}{eB_{T0}} \left(\frac{1}{n_i}\frac{\partial p_i}{\partial r} - \frac{1}{n_I Z_I}\frac{\partial p_I}{\partial r} \right)$$

where

$$\begin{aligned} B_T &= \frac{B_{T0}}{1 + \varepsilon\cos\theta} \\ B_\theta &= \frac{B_{\theta 0}}{1 + \varepsilon\cos\theta} \end{aligned}$$

It is neglected the variation on θ.

$$n_i, p_i, n_I, p_I$$
functions on only r

The equation of continuity

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0$$

$$u_\theta = \frac{u_{\theta 0}}{1 + \varepsilon\cos\theta}$$

Using the above equation for u_θ one obtains

$$u_T = -2\frac{r}{R}\cos\theta \; \frac{1}{eB_{\theta 0}}\left(\frac{1}{n_i}\frac{\partial p_i}{\partial r} - \frac{1}{n_I Z_I}\frac{\partial p_I}{\partial r}\right)$$

Projecting the two force balance equation on the *parallel* direction

$$\begin{aligned}\frac{\partial p_i}{r\partial\theta} - n_i e E_\theta &= -\frac{\partial p_I}{r\partial\theta} + n_I Z_I e E_\theta \\ &= \frac{B_T}{B_\theta} R_\parallel\end{aligned}$$

Projecting the force balance for *ions* on the θ direction

$$n_i v_{ir} = -\frac{1}{eB_T}\left(\frac{\partial p_i}{r\partial\theta} - n_i e E_\theta\right)$$

after taking

$$R_\theta \approx 0$$

Using the two equations above **Burrell 1976** finds

$$n_i v_{i,r} = -n_I Z_I v_{I,r} = R_\parallel \frac{1}{B_T}\frac{B}{B_\theta}$$

this equation shows that "*inward flux of impurity ions and the outward flux of protons are always coupled*".

From this expression one obtains the *radial proton flux* averaged on surface

$$\begin{aligned}\Gamma_i &= \left\langle (1+\varepsilon\cos\theta)\, n_i v_{ir}\right\rangle_{av.on\ \theta} \\ &= -\frac{1}{eB_{T0}}\left\langle (1+\varepsilon\cos\theta)^2 \left(\frac{\partial p_i}{r\partial\theta} - n_i e E_\theta\right)\right\rangle_{av.on\ \theta} \\ &= -\frac{2r}{R}\frac{1}{eB_{T0}}\left\langle \cos\theta \left(\frac{\partial p_i}{r\partial\theta} - n_i e E_\theta\right)\right\rangle_{av.on\ \theta}\end{aligned}$$

projecting the force balance for *impurities* on θ direction, the calculations are repeated.

Then

$$\Gamma_I = -\frac{1}{Z_I}\Gamma_i$$

This is simply *ambipolarity* and in case where $I \equiv i$ the flux is zero.

7 Plasma with multiple ion species Connor 1973

7.1 The physical origin of inward impurity flow

In **Hirshman Sigmar review** it is shown how the radial diffusion of ions and electrons is related to *parallel friction* forces (see also **Shaing Houlberg Hirshman**).

The stress tensor and the friction forces are *momentum exchange* processes and arise from collisions.

The friction acting on ions is mainly due to collisions with the *trapped electrons*. [e.g. the NBI fast ions are first slowed down by *electrons* until the fast ion velocity reaches the critical threshold v_c after which the fast ions are slowed down by ions, see **Rosenbluth mirrors** and **Rosenbluth Hinton NBI** and in general *collisions* **Gaffey**].

Normally the radial diffusion of the electrons is smaller than that of the ions (a factor of square root of the ratio of electron mass to the ion mass), since the neoclassical radial departure of an electron from the magnetic surface is smaller than that of an ion. Then the diffusion of the ions would be higher.

The ambipolarity would not be verified.

Actually, due to collisions, the ion rate of diffusion is reduced to that at of the electrons.

Simply, ions make all effort to reduce their diffusion such that to become equal to the electron diffusion. Flows of ions modify the gradient of ion density, to obtain finally $\Gamma_i \approx \Gamma_e$.

In **Connor** and in **Helander** the fact that ions diffuse and the electrons cannot follow at the same pace makes the *impurities* to move inward.

When there are several species of ions.

The collisions between ions produce *friction* that is higher than that due to ion-electrons, by a factor which is

$$\sqrt{\frac{m_i}{m_e}}$$

Then the rate of diffusion of the ions is higher than that of the electrons.

This will be important in the problem of *ambipolarity.*

Ambipolarity will not be obtaind from electrons and ions.

Ambipolarity will be obtained from the mutual behavior of only ions.

The magnetic field is Knorr-type

$$\mathbf{B} = \left(0, \frac{B_0}{1+\varepsilon\cos\theta}\Theta\left(r\right),\ \frac{B_0}{1+\varepsilon\cos\theta}\right)$$

or

$$\Theta\left(r\right) = \frac{B_\theta\left(r\right)}{B_T\left(r\right)} \ll 1$$

The drift velocity

$$v_{D,r} = \frac{m_j}{Z_j e} v_\parallel \frac{\partial}{r\partial\theta}\left(\frac{v_\parallel}{B}\right)$$

The variables

$$w = \frac{v^2}{2}$$

$$\mu = \frac{v_\perp^2}{2B}$$

$$\epsilon = w - \frac{Z_j e \Phi}{m_j}$$

$$v_\parallel = \pm\sqrt{\epsilon - \mu B}$$

The distribution function

$$f \equiv f_j\left(\epsilon, \mu, \sigma = \pm, r, \theta\right)$$

$$f_j = F_{M,j} + \widehat{f}_j$$

$$F_{M,j} = \frac{N_j\left(r\right)}{\left[\pi\ \frac{2T_j(r)}{m_j}\right]^{3/2}} \exp\left(-\frac{\epsilon}{T_j\left(r\right)}\right)$$

The guiding centre equation

$$v_\parallel \frac{\partial \widehat{f}_j}{\partial l_\parallel} + v_{Dj,r}\frac{\partial F_{Mj}}{\partial r} + \frac{Z_j e\ E}{m_j} v_\parallel \frac{\partial F_{M,j}}{\partial \epsilon} = C\left(\widehat{f}_j\right)$$

Clearly, since the radial projection of the *neoclassical drift* velocity $v_{D,r}$ has θ variation, the perturbation to the distribution function, $\widehat{f}_j$ will have a variation on θ, equivalently, a variation along the magnetic field line, $\sim l_\parallel$.

Note in **Shaing Dominguez resonance viscosity** more details on terms in the equation

$$\left(v_\parallel \widehat{\mathbf{n}} + \mathbf{v}_D + \mathbf{V}\right) \cdot \boldsymbol{\nabla} f + \dot{w}\frac{\partial f}{\partial w} = C$$

End.

Here

$$\frac{\partial}{\partial l_\parallel} = \Theta \frac{\partial}{r\partial\theta} = \frac{B_\theta}{B}\frac{\partial}{r\partial\theta} = \frac{1}{qR}\frac{\partial}{\partial\theta}$$

The electric field is

$$E = \frac{E_0}{1 + \varepsilon\cos\theta}$$

The collision operator is a simplification and modification of the Fokker Planck operator

$$C\left(\widehat{f}_j\right) = \sum_k C_{jk}\left(\widehat{f}_j, \widehat{f}_k\right)$$

$$C_{jk} = \nu_{jk}\left(x_j\right)\left[v_\parallel \frac{\partial}{\partial \mu} v_\parallel \frac{\mu}{B}\frac{\partial}{\partial \mu}\widehat{f}_j \ + \ v_\parallel F_{M,j}\frac{\int d^3v \ v_\parallel \nu_{kj} \ \widehat{f}_k}{\int d^3v \ v_\parallel^2 \ \nu_{kj} \ F_{M,k}}\right]$$

Under the constraint on the functions ν_{jk},

$$m_j \int d^3v \ \frac{v_\parallel^2}{v_{th,j}^2} \ \nu_{jk} \ F_{M,j} = m_k \int d^3v \ \frac{v_\parallel^2}{v_{th,k}^2} \ \nu_{kj} \ F_{M,k}$$

then this operator conserves both *momentum* and the *energy* and the *number*.

DIGRESSION

For the two terms in the collisional operator, see first **Rosenbluth Hazeltine Hinton 1972**.

The operator is taken from the Rosenbluth potentials to the final form.

In **Hazeltine Ware electrostatic trapping** the collisions are
electron - ions (including impurity ions)

$$C_{ei} + C_{eZ} = \left(\frac{3}{8}\sqrt{\pi}\right)\frac{1}{\tau_e}\frac{n_e}{\overline{n}_e}Z_{eff}\frac{v_{th,e}^3}{v^3}\left[\frac{\partial}{\partial \xi}\left(1-\xi^2\right)\frac{\partial f_e}{\partial \xi} + \frac{4u_d \ \xi v}{v_{th,e}^2}F_{M,e}\right]$$

and
ion - impurity ion

$$C_{iZ} = \sqrt{2}\left(\frac{3}{8}\sqrt{\pi}\right)\frac{1}{\tau_i}\frac{n_Z}{\overline{n}_i}Z^2\frac{v_{th,i}^3}{v^3}\left[\frac{\partial}{\partial \xi}\left(1-\xi^2\right)\frac{\partial f_i}{\partial \xi} + \frac{4u \ \xi v}{v_{th,i}^2}F_{M,i}\right]$$

and
ion - electrons

$$C_{ie} = -\frac{\overline{F}_e}{p_i}F_{M,i} \ v\xi$$

(where

$$\xi v = v_\parallel$$

)

Here

$$\tau_e = \frac{3}{16\sqrt{\pi}}\frac{m_e^2}{e^4}\frac{1}{\ln \Lambda}\frac{v_{th,e}^3}{\overline{n}_e}$$

$$\tau_i = \sqrt{2}\left(\frac{m_i}{m_e}\right)^{1/2}\left(\frac{T_i}{T_e}\right)^{3/2}\frac{\overline{n}_e}{\overline{n}_i} \ \times \tau_e$$

$$Z_{eff} = \frac{n_i + Z^2 n_Z}{n_e}$$

$$\begin{aligned} u_d &= \frac{n_i V_{\|i} + Z^2 n_Z V_{\|Z}}{n_i + Z^2 n_Z} \\ & \text{velocity of the center of mass} \\ & \text{mass is (ions + impurity ions)} \\ & \text{parallel to } \mathbf{B} \end{aligned}$$

The parallel friction force is

$$\begin{aligned} \overline{F}_{ei} &= \int d^3v \; m_e \; v\xi \; C_{ei} \\ \text{collisional parallel } v\xi &= v_{\|} \text{ momentum friction} \end{aligned}$$

$$\begin{aligned} \overline{F}_e &= \frac{1 + Z^2 \overline{n}_Z}{\overline{n}_i} \overline{F}_{ei} \\ &\approx e \overline{n}_e E_0 \end{aligned}$$

NOTE
the definition in **Helander 3999** is
Parallel friction force between *ions* and *impurity ions*

$$R_{Z\|} = -\int d^3v \; m_i v_{\|} C_{iZ} \left(f_i^{(1)} \right)$$

and can be expressed

$$\begin{aligned} R_{Z\|} &= -\frac{1}{\tau_{iZ}} \frac{I}{\Omega_i} \left(p_i \frac{d \ln p_i}{d\psi} - p_i \frac{3}{2} \frac{d \ln T_i}{d\psi} \right) \\ &+ \frac{1}{\tau_{iZ}} n_i m_i \left(u - \frac{K_Z}{n_Z} \right) B \end{aligned}$$

We **note** that the first term is

$$\sim -\frac{1}{\tau_{iZ}} \frac{1}{\Omega_{i\theta}} \frac{dp_i}{dr}$$

it is like a *diamagnetic flow* (which is poloidal) projected on the parallel direction $\times \frac{B}{B_\theta}$, that is why it occurs Ω_θ.
The definition of the "velocity" u

$$\begin{aligned} u &= \tau_{iZ} \frac{1}{n_i B} \int d^3v \; \nu_{iZ} \; v_{\|} g_i \\ &= \frac{1}{B} \frac{\int d^3v \; \nu_{iZ} \; v_{\|} g_i}{\nu_{iZ} \; n_i} \end{aligned}$$

where the distribution function g is part of the expansion of f_i,

$$\begin{aligned} f_i^{(1)} &= -\frac{I}{\Omega_i} v_\parallel \frac{\partial f_i^{(0)}}{\partial \psi} \ \text{(neoclassic drift)} \\ & -\frac{e\Phi}{T_i} f_i^{(0)} \ \text{(Boltzmann)} \\ & + g_i \left(\epsilon_0, \mu, \psi, \sigma\right) \end{aligned}$$

The function K_Z has been previously introduced as part of the general expression for the velocity

$$V_{Z\parallel} = -\frac{I}{B} \frac{d\Phi_0}{d\psi} + \frac{K_Z(\psi) B}{n_Z}$$

(the first term is

$$\sim \frac{E_r}{B_\theta} \ \text{in toroidal direction}$$

and the second term is in poloidal direction but projected on the parallel direction).

The (ion)-(impurity ion) friction force is part of the *fluid* force balance

$$\begin{aligned} m_Z n_Z \widehat{\mathbf{n}} \cdot \left[\left(\mathbf{V}_Z \cdot \boldsymbol{\nabla}\right) \mathbf{V}_Z\right] &= -\nabla_\parallel p_Z - \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_Z \\ & -n_Z \ Ze \nabla_\parallel \Phi^{(1)} \\ & + R_{Z\parallel} \end{aligned}$$

In this *static* force balance after evaluation of the terms it remains

$$0 = -T_i \nabla_\parallel n_Z - n_Z \ (Ze) \nabla_\parallel \Phi^{(1)} + R_{Z\parallel}$$

(similar to the parallel balance for drift waves, in other range of variation).

Since in this equation occurs two terms with $\nabla_\parallel = \frac{1}{qR} \frac{\partial}{\partial \theta}$ it is suggested to exploit the *periodicity* on θ. This means to multiply by B and take the surface average

$$\left\langle B R_{Z\parallel} \right\rangle = 0$$

where

$$\left\langle B R_{Z\parallel} \right\rangle = \frac{\oint \frac{dl_\theta}{B_\theta} \ B R_{Z\parallel}}{\oint \frac{dl_\theta}{B_\theta}}$$

In this equation it is possible to determine K_Z the second part (poloidal) of the general expression of the velocity.

After finding K_Z one returns to the static *parallel force balance* for impurity ions and replaces the full expression of $R_{Z\parallel}$ (with now K_Z known) and the equation for n_Z is obtained.

Note regarding a paranthesis occuring in the equation for n_Z.

It is

$$\left(1-\frac{\langle n_Z\rangle}{n_Z}\frac{B^2}{\langle B^2\rangle}\right)$$

This in the case $\langle n_Z\rangle = n_Z$ or that the impurities do not matter, is the paranthesis

$$\left(1-\frac{B^2}{\langle B^2\rangle}\right)$$

The term $\left\langle B^2\right\rangle$ appears from two factors. One is the second part (poloidal) in the general expression of the parallel velocity

$$V_{Z\|} = -\frac{I}{B}\frac{d\Phi_0}{d\psi} + \frac{K_Z(\psi)\,B}{n_Z}$$

which is further inserted in the expression of the *parallel friction force* $R_{Z\|}$. The second is from the constraint of periodicity which leads to the condition of surface average

$$\left\langle BR_{Z\|}\right\rangle = 0$$

END digression.

Note

See below **Helander 3999** the collisional operator contains, as here, the pitch angle plus a term of friction ion-impurity due to relative motion. This second part also contains a velocity integration of momentum weighted by collision frequency.

End

NOTE

regarding the form of the collision operator.

In **Connor1973** the form is

$$C\left(\widehat{f}_j\right) = \sum_k C_{jk}\left(\widehat{f}_j, \widehat{f}_k\right)$$

$$C_{jk} = \nu_{jk}(x_j)\left[v_\|\frac{\partial}{\partial\mu}v_\|\frac{\mu}{B}\frac{\partial}{\partial\mu}\widehat{f}_j \;+\; v_\| F_{M,j}\frac{\int d^3v\ v_\|\nu_{kj}\ \widehat{f}_k}{\int d^3v\ v_\|^2\ \nu_{kj}\ F_{M,k}}\right]$$

The second term has a rather strange construction, with parallel momentum $\left(mv_\|\right)$ weighted by collision frequency ν_{kj} and integrated over velocity space.

Here there is no special component, with particular properties, like the *fast particles.*

There are only ions, background and impurities.

The first part is pitch angle scattering, important for the distribution in the *trapped/circulating* space.

The second should be connected with *friction,* transfer of momentum between ion species, k and j.

For comparison we mention here the equation for the distribution function for the *fast ions* from **mirrors Hinton Rosenbluth** (see also **NBI**)

$$\begin{aligned}
& \frac{\partial f}{\partial t}+\left(v_{\|}\widehat{\mathbf{n}}+\mathbf{v}_D\right)\cdot\boldsymbol{\nabla}f \\
& -\nu_s\frac{2m_i}{m_{fast}}\ \frac{v_c^3}{v^3}\xi\frac{1}{B}\frac{\partial}{\partial\lambda}\left(\lambda\xi\frac{\partial f}{\partial\lambda}\right) \qquad \text{(pitch angle scattering)} \\
& -\nu_s\frac{1}{v^2}\frac{\partial}{\partial v}\left[\left(v^3+v_c^3\right)f\right] \qquad \text{(slowing down)} \\
= \ & S\frac{\delta\left(v-v_0\right)}{v_0^2} \qquad \text{(source)}
\end{aligned}$$

The slowing down due to the electron drag is

$$\begin{aligned}
\nu_s &= \frac{m_e}{m_{fast}}Z_{fast}^2\ \frac{1}{\tau_e} \\
\frac{1}{\tau_e} &= \frac{16\sqrt{\pi}}{3}\frac{e^4}{m_e^2}\ln\Lambda\ \frac{n_e}{v_{th,e}^3}
\end{aligned}$$

and the critical velocity

$$v_c=\left(\frac{3\sqrt{\pi}}{4}\frac{m_e}{m_i}\right)^{1/3}\ v_{th,e}$$

Technical advancements.

In the case of *mirrors* the equation can ignore the drift $\mathbf{v}_D$ term

$$\begin{aligned}
& v_{\|}\frac{\partial f}{\partial l}-\nu_s\frac{2m_i}{m_{fast}}\frac{v_c^3}{v^3}\frac{\xi}{B}\frac{\partial}{\partial\lambda}\left(\xi\lambda\frac{\partial f}{\partial\lambda}\right) \\
& -\nu_s\frac{1}{v^2}\frac{\partial}{\partial v}\left[\left(v^3+v_c^3\right)f\right] \\
= \ & 0
\end{aligned}$$

Then it is *bounce averaged*: it is divided by ξ (and in the first term we have $v\xi=v_{\|}$) and then it is integrated over the line. The remaining factor in the first term v is invariant, being the energy

$$\oint dl_{\|}\times$$

and the first term is zero by periodicity.

$$\begin{aligned}
& \frac{2m_i}{m_{fast}}\frac{v_c^3}{v^3}\frac{\partial}{\partial\lambda}\left[\left(\oint dl_{\|}\frac{\xi}{B}\right)\ \lambda\frac{\partial f}{\partial\lambda}\right] \\
& +\left(\oint dl_{\|}\frac{1}{\xi}\right)\ \frac{1}{v^2}\frac{\partial}{\partial v}\left[\left(v^3+v_c^3\right)f\right] \\
= \ & 0
\end{aligned}$$

(This is the mirror equivalent of the periodicity constraint $\left\langle BR_{Z\|}\right\rangle = 0$ determined for impurity friction force in tokamak).

The boundary condition at the limit where the fast particles are generated, v_0, is

$$f \rightarrow \frac{1}{\nu_s\left(v_0^3+v_c^3\right)} \frac{\oint dl_{\|} \frac{1}{\xi} S}{\oint dl_{\|} \frac{1}{\xi}}$$

For the solution.

Substitutions

$$u \equiv \frac{v}{v_c}$$

$$f = \frac{G(u, \lambda)}{v_c^3\left(u^3+1\right)}$$

Results

$$\begin{aligned} & \frac{2m_i}{m_{fast}} \frac{1}{u^3\left(u^3+1\right)} \frac{\partial}{\partial \lambda}\left[I_1(\lambda) \lambda \frac{\partial G}{\partial \lambda}\right] \\ & +I_2(\lambda) \frac{1}{u^2} \frac{\partial G}{\partial u} \\ = \; & 0 \end{aligned}$$

with the definitions

$$I_1(\lambda) = \int_0^{s_t} ds \; \frac{\xi}{B}$$

$$I_2(\lambda) = \int_0^{s_t} ds \frac{1}{\xi}$$

where s_t is the *turning point*

$$\lambda B\left(s_t\right) = 1$$

Another transformation: a new *time-like* variable (but, of course, NOT time)

$$\begin{aligned} \tau & = \frac{2}{3} \frac{m_i}{m_{fast}} \ln \left(\frac{1+u^{-3}}{1+u_0^{-3}}\right) \\ \text{where } u_0 & \equiv \frac{v_0}{v_c} = \frac{\text{velocity at creation}}{\text{critical velocity}} \end{aligned}$$

The equation

$$\frac{\partial G}{\partial \tau} - \frac{1}{I_2(\lambda)} \frac{\partial}{\partial \lambda}\left[I_1(\lambda) \; \lambda \; \frac{\partial G}{\partial \lambda}\right] = 0$$

Note that the first term is what remains from the *slowing down* part of the collision operator. The second term is the *pitch angle* part.

The general form

$$\frac{\partial G}{\partial \tau} = -A(\lambda)\frac{\partial G}{\partial \lambda} + D(\lambda)\frac{\partial^2 G}{\partial \lambda^2}$$

$$A(\lambda) \equiv -\frac{\frac{d}{d\lambda}(\lambda I_1)}{I_2}$$

$$D(\lambda) \equiv \frac{\lambda I_1}{I_2}$$

In the case of mirrors

$$I_2(\lambda) = -2\frac{dI_1}{d\lambda}$$

Substitution

$$G_1(\lambda,\tau) = \exp[\phi(\lambda,\tau)]$$

with

$$\frac{\partial \phi}{\partial \tau} = -A(\lambda)\frac{\partial \phi}{\partial \lambda} + D(\lambda)\left[\frac{\partial^2 \phi}{\partial \lambda^2} + \left(\frac{\partial \phi}{\partial \lambda}\right)^2\right]$$

Further

$$\begin{aligned}\phi(\lambda,\tau) &= -\frac{w(\lambda)}{\lambda} - \frac{1}{2}\ln\tau \\ &\quad +y(\lambda,\tau)\end{aligned}$$

and

$$\begin{aligned}y(\lambda,\tau) &= y_0(\lambda) + y_1(\lambda)\times\tau \\ &\quad +...\end{aligned}$$

After equating terms of the same power in τ it results

$$w(\lambda) = \frac{1}{4}\left[\int_{\lambda_0}^{\lambda}\frac{d\lambda}{\sqrt{D(\lambda)}} + c\right]^2$$

$$y_0(\lambda) = -\frac{1}{4}\ln[\lambda I_1(\lambda)\ I_2(\lambda)] + k$$

The solution

$$\begin{aligned}G_1(\lambda,\tau) &= \frac{Q}{4\pi^{3/2}}\frac{1}{\sqrt{D(\lambda_0)\ \times\tau}} \\ &\quad\times\left[\frac{\lambda_0 I_1(\lambda_0)\ I_2(\lambda_0)}{\lambda I_1(\lambda)\ I_2(\lambda)}\right]^{1/4} \\ &\quad\times\exp\left[-\frac{w_1(\lambda)}{\tau}\right]\end{aligned}$$

where w_1 is w with $c = 0$.

END

Continue with **Connor1973**.
The *frequency of collisions for diffusion in pitch-angle* is

$$\nu_{jk}(x_j) = \sqrt{2}\pi \frac{e^2 Z_j^2 e^2 Z_k^2}{\sqrt{m_j}} \ln\Lambda \; \frac{n_k}{T_j^{3/2}} \; x_j^{-3/2} \; h(x_k)$$

with the function h being (**Trubnikov**)

$$h(x) = \left(1 - \frac{1}{2x}\right)\eta(x) + \frac{d\eta}{dx}$$

$$\eta(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt \; \sqrt{t} \; \exp(-t)$$

where

$$x_j \equiv \frac{v^2}{v_{th,j}^2}$$

(**Note** that frequently the definition is for x_j^2)

$$v_{th,j}^2 = \frac{2T_j}{m_j}$$

Note and comparison. See *collisions*.

In **Bolton Ware** the collision operator is the linearized form of Fokker Planck

Collisions with electrons are neglected since the terms that they contribute are of order $(m_e/m_i)^{1/2}$.

$$C(f) = C(f_1, f_0) + C(f_0, f_1)$$

where

$$\begin{aligned} C(f_1, f_0) \quad = \quad & \frac{3\sqrt{2\pi}}{4} \left\{ [\Phi(u) - G(u)] \frac{1}{2}\frac{1}{u^3}\frac{\partial}{\partial\xi}\left(1-\xi^2\right)\frac{\partial f}{\partial\xi} \right. \\ & \left. + \frac{1}{u^2}\frac{\partial}{\partial u} u^2 \left(\frac{2G(u)}{u}\right)\left(\frac{1}{2}\frac{\partial f}{\partial u} + uf\right) \right\} \end{aligned}$$

compare with the Chandraskhar function

$$\begin{aligned} G(u) &= \frac{\Phi(u) - u\frac{d\Phi}{du}}{2u^2} \\ \Phi(u) &= \text{error function} \\ u &= \frac{v}{v_{th,i}} \end{aligned}$$

we note also that

$$
= \begin{array}{l} \Phi(u) - G(u) \ \text{(Bolton Ware)} \\ \\ h(x) \ \text{(Trubnikov, in Connor1973)} \end{array}
$$

END note and comparison

This operator will have to describe

- electron-ion collisions (including electron-impurity ions)
- ion-ion collisions (including impurity-ion)

For electron $j \equiv e$ colliding with ion k,

$$
x_k = \frac{v^2 \ \text{(electrons)}}{v_{th,k}^2 \ \text{(ions)}} \gg 1
$$

when introduced in the function $h(x_k)$. This is because

$$
m_e \ll m_k
$$

and one obtains

$$
\nu_{ek} = \sqrt{2}\pi \ \frac{e^4 Z_k^2}{\sqrt{m_e}} \ln \Lambda \frac{n_k}{T_e^{1/2}} x_e^{-3/2}
$$

For *ions* k colliding with *electrons*

$$
\nu_{ke} = \sqrt{2} \left(\sqrt{\pi} \frac{4}{3} \frac{m_e}{m_k} \right) \frac{e^4 Z_k^2}{\sqrt{m_e}} \ln \Lambda \frac{n_e}{T_e^{3/2}}
$$

For collisions between ions.
The case where the masses of the two species of ions are very different, say

$$
\begin{array}{rcl} m_k & \gg & m_l \\ \text{(oxygen)} & \gg & \text{(hydrogen)} \end{array}
$$

the collisions are similar with those between the *ions* k with *electrons* (here the notation is l from *light*)

$$
\nu_{k,l-light} = \sqrt{2} \left(\sqrt{\pi} \frac{4}{3} \frac{m_l}{m_k} \right) \frac{e^4 Z_k^2 Z_l^2}{\sqrt{m_l}} \ln \Lambda \frac{n_l}{T_l^{3/2}}
$$

The collisions of light ion species l with the heavy ion species k is

$$
\nu_{l-light,k} = \sqrt{2} \, (\pi) \, \frac{e^4 Z_k^2 Z_l^2}{\sqrt{m_l}} \ln \Lambda \frac{n_k}{T_l^{3/2}} \ x_l^{-3/2}
$$

The case where the ion species have comparable masses: D, T and α particles.

In this case the exchange of momentum is very efficient and the temperatures are equalized rapidly.

The energy dependence of the frquency ν_{kl} is corrected with a factor

$$\nu_{kl} = \alpha_{kl}\ x_k^{-3/2}\ h\left(x_l\right)$$

(h is the function of x_l defined above).

The correction is used to fit the momentum exchange between two drifting Maxwellian ion distributions with velocities u_k and respectively u_l,

$$P_{kl} = -\sqrt{2}\left(\sqrt{\pi}\frac{4}{3}\right)\frac{e^4 Z_k^2 Z_l^2}{\sqrt{m_{kl}}}\ln\Lambda\frac{1}{T}\left(u_k - u_l\right)$$

where

$$T = T_k = T_l$$

(due to the comparable masses)

$$m_{kl} = \frac{m_k m_l}{m_k + m_l} \quad \text{reduced mass}$$

Solution of the drift kinetic equation in the *banana* regime

$$\widehat{f}_j = -\frac{v_\parallel}{\frac{Z_j e B_\theta}{m_j}}\frac{\partial F_{M,j}}{\partial r} + g_j$$

$$\frac{B_\theta}{B}v_\parallel\frac{\partial g_j}{r\partial\theta} = C\left(\widehat{f}_j\right) + Z_j e E\ v_\parallel\frac{1}{T_j}F_{M,j}$$

(last term comes from $\frac{\partial F_{M,j}}{\partial\epsilon}$).

Note

that the perturbation distribution function $\widehat{f}_j$ is composed of

- neoclassical correction term $\left(\sim v_\parallel/\Omega_\theta \times \frac{\partial}{\partial r}\right)$
- Spitzer-like term $\sim E$
- perturbation for ONLY *circulating*, g_j, governed by collision operator, like **Rutherford 1970**.

End.

In the banana regime the collisions are rare (relative to the bounce frequency) and the *energy* effect of the electric field is also small.

Then it is adopted an expansion in the second small neoclassical parameter

$$\frac{\nu_{eff}}{\omega_{bounce}} \to \text{ small}$$

$$g_j = g_j^{(0)} + g_j^{(1)} + ...$$

Replacing in the equation, one obtains

$$\frac{\partial g_j^{(0)}}{\partial \theta} = 0$$

in zero order the correction to the distribution function is uniform on surface

$$\frac{B_\theta}{B} v_\parallel \frac{\partial g_j^{(1)}}{r \partial \theta} = C\left(g_j^{(0)}\right) - \frac{1}{\frac{Z_j e B_\theta}{m_j}} C\left(v_\parallel \frac{\partial F_{M,j}}{\partial r}\right) + Z_j e E \ v_\parallel \ \frac{1}{T_j} F_{M,j}$$

NOTE

The factors in the LHS are B_θ / B and these are

$$\frac{B_\theta}{B} = \frac{\frac{B_{\theta 0}}{h}}{\frac{B_0}{h}} \sim \text{independent of } \theta \text{ dependent only on } r$$

We see that to isolate $\partial g_j^{(1)} / \partial \theta$ in view of exploitation of periodicity on θ, we must divide the equation by $v_\parallel$. This is the way the factor

$$\frac{1}{v_\parallel}$$

appears in the RHS. Further it will appear the difference $\frac{1}{\langle v_\parallel \rangle} - \frac{1}{v_\parallel}$ whose integration is $-1.46\sqrt{\varepsilon}$.

END

The presence of the derivative to θ and the *periodicity* suggest to integrate the equation over θ to eliminate the LHS term and obtain a constraint equation for $g_j^{(0)}$.

Procedure of **Rutherford**.

For circulating particles one integrates the equation

$$\int_0^{2\pi} d\theta \frac{1}{|v_\parallel|} \times (Equation)$$

For trapped particles one uses the annihilator

$$\sum_{\sigma = \pm} \int_{\theta_1}^{\theta_2} d\theta \frac{1}{|v_\parallel|} \times (Equation)$$

$$\begin{aligned} |v_\parallel|_{\theta=\theta_1} &= 0 \\ |v_\parallel|_{\theta=\theta_2} &= 0 \end{aligned}$$

Then for circulating particles one obtains an equation for $g_j^{(0)}$ in the variable μ.

The operator of derivation with respect to μ is of second order. The equation is integrated once to μ and the condition is applied to the *first* derivative to μ, at the limit of *strong circulating*, as follows

$$\left.\frac{\partial g_j^{(0)}}{\partial \mu}\right|_{\mu=0} = \text{finite}$$

We note that $\mu = 0$ is $v_\perp = 0$ and all velocity is "almost" entirely *parallel*. Then

$$\frac{\partial g_j^{(0)}}{\partial \mu} = -\frac{B}{\overline{v}_\parallel}\left\{\frac{1}{\frac{Z_j e B_\theta}{m_j}}\frac{\partial F_{M,j}}{\partial r} + m_j \frac{F_{M,j}}{T_j}\sum_k \frac{\nu_{jk}}{\nu_j}\overline{u}_k^j + Z_j e E\ \frac{1}{\nu_j}\frac{F_{M,j}}{T_j}\right\}$$

where

$$\nu_j = \sum_k \nu_{jk}$$

$$\overline{A} \equiv \frac{1}{2\pi}\int_0^{2\pi} d\theta\ A$$

poloidal averaging

(in other papers also noted $\langle A \rangle)$. Since $v_\parallel$ is function of θ, one will introduce the average parallel flow velocity, $\overline{v}_\parallel$.

$$u_k^j = \frac{\int d^3v\ v_\parallel\ \nu_{kj}\ \widehat{f}_k}{n_k \langle \nu_{kj} \rangle}$$

This is a generalization of the mean longitudinal velocity u_k of the species k. The parallel velocity is averaged over the velocity space weighted by the collision frequency of species k against species j.

$$n_k = \int d^3v\ F_{M,k}$$

$$\langle A \rangle_\epsilon = \frac{4}{3\sqrt{\pi}}\int_0^\infty dx\ x^{3/2}\exp(-x)\ \ A(x)$$

This is an energy-weighted average of A, over the Maxwellian distribution function.

NOTE

In **Hazeltine Ware electrostatic trapping** the solution of the drift kinetic equation is obtained in a *different* way (see *variation on surfaces*).

The distribution function for the *electrons* is composed of

$$f_e^{(1)} = f_{disp,e} + f_{Spitzer,e} + g$$

where

$$\begin{aligned} f_{disp,e} &= \text{ displacement of a Maxwellian} \\ &= \frac{2u\ v\xi}{v_{th,e}} F_{M,e} \\ & \quad (u \equiv \text{ common velocity}) \end{aligned}$$

This function verifies

$$\begin{aligned} C_0\left(f_{disp,e}\right) &= -C_* \\ &= -\left(\frac{3\sqrt{\pi}}{2}\right)\frac{1}{\tau_e}\frac{n_e Z}{\overline{n}_e}\frac{v_{th,e}\ u\xi}{v^2}\ F_{Me} \end{aligned}$$

and C_0 is what remains from the full collision operator after separating the displacement part, *i.e.* for $u_{disp} = 0$.

Next,

$$\begin{aligned} f_{Spitzer,e} &= -\tau_e \\ & \quad \times v_{th,e}\xi\ eE\ \frac{1}{T_e} F_{M,e} \\ & \quad \times S_e\left(\frac{v}{v_{th,e}}\right) \end{aligned}$$

where

$$\begin{aligned} S_e &= A\left(Z_{eff}\right)\left(\frac{v}{v_{th,e}}\right)^2\left[1 + a\left(Z_{eff}\right)\frac{v}{v_{th,e}}\right] \\ & \quad \text{a fitted formula} \end{aligned}$$

(**Note** that the first factor is as if $C \sim \frac{1}{\tau_e}$ and is applied on the RHS, Spitzer inhomogeneous equation; also **Note** that there is no common velocity u as in the case of the *displacement* of the Maxwellian).

Finally

$$\frac{g_e}{F_{M,e}} = \pi\delta\left(\xi\right)G_e + \mathbf{P}\left(\frac{1}{\xi}\right)\int^{\theta} d\theta'\ G_e\left(\theta'\right)$$

The expression for G_e is found in **Hazeltine Ware.**

END

NOTE

From **Hirshman 1977**.

The *first order* distribution function is defined as a series of *flow velocities* u_{ak},

$$f_{a1} = \frac{2v_{\|}}{v_{th,a}^2} f_{Ma} \sum_{k=0}^{\infty}\ u_{ak}\ L_k^{(3/2)}\left(x_a^2\right)$$

The similar definition the *flows* corresponding to higher $k \geq 2$ Sonine polynomials in the expansion are defined as

$$u_{ak} = \frac{1}{n_a} \frac{\int d^3v \ L_k^{(3/2)} \left(x_a^2\right) \ v_{\parallel} f_{a1}}{\left\{ \left[L_k^{(3/2)} \left(x_a^2\right) \right]^2 \right\}}$$

with the notation of a new operator

$$\{A(v)\} = \frac{8}{3\sqrt{\pi}} \int d^3v \ x^4 \exp\left(-x^2\right) \ \times A\left(x v_{th,a}\right)$$

for a function of velocity v since $x v_{th,a} = v$.

There are differences relative to **Connor 1973** where

$$u_k^j = \frac{\int d^3v \ v_{\parallel} \ \nu_{kj} \ \widehat{f}_k}{n_k \left\langle \nu_{kj} \right\rangle}$$

In **Connor 1973** the velocity space integration for u_k^j is parallel velocity but weighted by collision frequency between the two species, k and j.

However

$$\{A(v)\} \ \ (\text{Hirshman1977}) = \langle A \rangle_{\epsilon} \quad (\text{Connor1973})$$

END

NOTE

In **Galeev diffusion 1971**.

The full distribution function starts with the Maxwellian

$$\begin{aligned}
f_j^{(0)} &= \{\text{shifted Maxwellian}\} \times \{\text{series in Sonine polynomials}\} \\
&= \frac{n}{\left(\sqrt{\pi} v_{th,j}\right)^3} \exp\left[-\frac{\left(v_{\parallel} - u_{0j}\right)^2}{2T_j/m_j} - \frac{\mu B_0}{T_j} \right] \times \\
&\quad \times \left\{ 1 - \frac{2 v_{\parallel} u_{0j}}{v_{th,j}^2} \sum_{k=1}^{\infty} a_k \ L_k^{(3/2)} \left(\frac{v^2}{v_{th,j}^2} \right) \right\}
\end{aligned}$$

where

$$\begin{aligned}
u_{0i} &= 0 \\
u_{0e} &= u_0
\end{aligned}$$

The *correction* to this distribution function is due to toroidality

$$\begin{aligned}
f_j^{(1)} &= -\frac{\mu B_0/m_j + v_{\parallel}^2}{R} \left[\frac{B_\theta}{B} \frac{\partial}{\partial v_{\parallel}} + \frac{1}{\Omega_j} \frac{\partial}{\partial r} \right] \\
&\quad \times f_j^{(0)} \left\{ \mathbf{P} \frac{r \cos\theta}{\frac{B_\theta}{B} v_{\parallel} + v_E} - \pi r \ \delta \left(\frac{B_\theta}{B} v_{\parallel} + v_E \right) \sin\theta \right\}
\end{aligned}$$

(to check)
END

NOTE that **Hirshman Sigmar Clarke** find the equation in the form

$$\left\langle \frac{B}{v_\parallel} \left[v_\parallel \frac{e_a E^{(A)}}{T_a} f_{a0} + \sum_b C_{ab}\left(f_{a1}, f_{b1}\right) \right] \right\rangle_{surf-average} = 0$$

END

The equation for $g_j^{(0)}$ in the case of *trapped* particles is

$$\frac{\partial g_j^{(0)}}{\partial \mu} = 0$$

In the RHS of the equation $\partial g_j^{(0)}/\partial \mu$ the functions $\overline{u}_k^j$ are still unknown and must be calculated.

The definition of u_k^j

$$u_k^j = \frac{\int d^3v \; v_\parallel \; \nu_{kj} \; \widehat{f}_k}{n_k \left\langle \nu_{kj} \right\rangle_\epsilon}$$

contains the integration of the distribution function $\widehat{f}_k$.

It will be made an integration by part in the variable μ

Use the velocity-space variables

$$\mu \quad \text{and} \quad w = \frac{v^2}{2}$$

Then, replace $\widehat{f}_j$ in u_k^j by the expression that has introduced g_j,

$$\widehat{f}_j = -\frac{1}{\frac{Z_j e B_\theta}{m_j}} v_\parallel \frac{\partial F_{M,j}}{\partial r} + g_j$$

The result is a set of equations for the functions $\overline{u}_k^j$,

$$\begin{aligned} & \sum_l \left\langle \frac{\nu_{kj}\nu_{kl}}{\nu_k} \right\rangle_\epsilon \left(\overline{u}_k^j - \overline{u}_l^k \right) - \frac{Z_k e E}{m_k} \left\langle \frac{\nu_{kj}}{\nu_k} \right\rangle_\epsilon \\ = \quad & -1.46 \; \varepsilon^{3/2} \left\{ \frac{1}{Z_j e B_\theta} T_k \left[\frac{N_k'}{N_k} - \left(\frac{3}{2} - \frac{\left\langle x_k \; \nu_{kj} \right\rangle}{\left\langle \nu_{kj} \right\rangle} \frac{T_k'}{T_k} \right) \right] \left\langle \nu_{kj} \right\rangle_\epsilon \right. \\ & \left. + \frac{Z_k e E}{m_k} \left\langle \frac{\nu_{kj}}{\nu_k} \right\rangle_\epsilon + \sum_l \left\langle \frac{\nu_{kj} \; \nu_{kl}}{\nu_k} \right\rangle_\epsilon \overline{u}_l^k \right\} \end{aligned}$$

Here $F_{M,k}'$ is the radial derivative of $F_{M,k}$ performed at constant $w = v^2/2$,

$$F_{M,k}' = \left[\frac{N_k'}{N_k} - \left(\frac{3}{2} - x_k \right) \frac{T_k'}{T_k} \right] F_{M,k}$$

It has been used the result of **Hinton Oberman 1969**

$$
\begin{aligned}
& \frac{1}{n_k} \sum_{\sigma=\pm} \iint 2\pi B \; d\mu \; dw \; \frac{\mu B}{\frac{T_k}{m_k}} \; F_{M,k} \left(\frac{1}{\overline{|v_\parallel|}} - \frac{1}{|v_\parallel|} \right) \\
= \quad & -1.46 \; \varepsilon^{1/2} \\
& + O(\varepsilon)
\end{aligned}
$$

In the integral, the term $1/\overline{|v_\parallel|}$ is zero in the trapped region.

We have here an integration over the velocity space with the measure

$$
\int d^3 v = \sum_{\sigma=\pm 1} \int 2\pi \; d\mu \; dw \; \frac{B}{|v_\parallel|}
$$

the integrand being

$$
\frac{v_\perp^2}{v_{th}^2}
$$

Therefore we have the average of the perpendicular energy.

This is the system of equations that must be solved to obtain $\overline{u}_k^j$.

Using this, one can calculate $g_j^{(0)}$ and in this way $\widehat{f}_j$.

Further, using $\widehat{f}_j$ one can calculate the radial fluxes Γ_k and Q_k and the toroidal current J.

From the equation above a result can be obtained for the averaged flows $\overline{u}_k^j$ by summing over j

$$
\begin{aligned}
& \sum_j \langle \nu_{kj} \rangle_\epsilon \left(\overline{u}_k^j - \overline{u}_j^k \right) - \frac{Z_k e E}{m_k} \quad \text{(acceleration)} \\
= \quad & -1.46 \; \varepsilon^{1/2} \left\{ \frac{\frac{T_k}{m_k}}{\frac{Z_k e B_\theta}{m_k}} \left[\frac{N_k'}{N_k} - \left(\frac{3}{2} - \frac{\langle x_k \; \nu_k \rangle}{\langle \nu_k \rangle} \right) \frac{T_k'}{T_k} \right] \langle \nu_k \rangle_\epsilon \right. \\
& \left. + \frac{Z_k e E}{m_k} + \sum_j \langle \nu_{kj} \rangle_\epsilon \overline{u}_j^k \right\}
\end{aligned}
$$

It is *collisional time-change of the relative velocity of the two species.*

The radial fluxes

The particle flux

$$
\Gamma_j = \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{1}{1 + \varepsilon \cos\theta} \int d^3 v \; v_{D,j} \; \widehat{f}_j
$$

surface average

$$
\int d^3 v = \sum_{\sigma=\pm 1} \int 2\pi \; d\mu \; dw \; \frac{B}{|v_\parallel|}
$$

Now we use the drift velocity projected on radial direction and take a part integration on θ,

$$\Gamma_j = -\sum_k \frac{\overline{P}_{jk}}{Z_j e B_\theta} - n_j \frac{E}{B_\theta}$$

It means that the radial flux of the species j (ions, impurities, electrons) is due to

- the $\mathbf{F} \times \mathbf{B}$ drift, where $\mathbf{F}$ is the friction in the parallel direction, between species, and $\mathbf{B}$ is the *poloidal* magnetic field B_θ such as the vector product to lead to radial flux
- the effect of a toroidal electric field E combined with the poloidal magnetic field B_θ.

To find the flux Γ_j one must calculate
the exchange of momentum between the species, in the parallel direction
other words
the momentum exchange P_{kl} between drifting Maxwellian ion distributions with velocities u_k and u_l.

This is

$$P_{kl} = -\sqrt{2}\left(\frac{4}{3}\sqrt{\pi}\right)\frac{e^4 Z_k^2 Z_l^2}{\sqrt{m_{kl}}} \ln\Lambda \; \frac{1}{T^{3/2}}\left(u_k - u_l\right)$$

at equal temperatures $T_k = T_l = T$.

From the conservation of momentum

$$\sum_{jk} \overline{P}_{jk} = 0$$

we find the ambipolarity

$$\sum_j Z_j \Gamma_j = 0$$

The definition

$$P_{jk} = n_j m_j \int d^3v \; v_\parallel C_{jk}$$

$$\begin{aligned}\Gamma_j &= \frac{1}{\frac{Z_j e B_\theta}{m_j}} \sum_k n_j \; \left\langle \nu_{jk} \right\rangle_\epsilon \left(\overline{u}_j^k - \overline{u}_k^j\right) \\ &\quad - n_j \frac{E}{B_\theta}\end{aligned}$$

where we replace the difference between the two velocities

$$\begin{aligned}\Gamma_j &= -1.46\varepsilon^{1/2} \, n_j \frac{1}{\frac{Z_j e B_\theta}{m_j}} \left\{ \frac{\frac{T_j}{m_j}}{\frac{Z_j e B_\theta}{m_j}} \left[\frac{N_j'}{N_j} - \left(\frac{3}{2} - \frac{\left\langle x_j \nu_j \right\rangle_\epsilon}{\left\langle \nu_j \right\rangle_\epsilon}\right)\frac{T_j'}{T_j}\right] \left\langle \nu_j \right\rangle_\epsilon \right. \\ &\qquad \left. + Z_j e E \frac{1}{m_j} + \sum_k \left\langle \nu_{jk} \right\rangle_\epsilon \overline{u}_k^j \right\}\end{aligned}$$

The transport of heat

$$Q_j = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{1+\varepsilon \cos\theta} \int d^3v \; m_j w \; v_{D,j} \; \widehat{f}_j$$

7.2 Application for two ion components, one heavy

The case

$$m_2 \gg m_1$$

The frequency of collisions

$$\nu_j = \sum_k \nu_{jk}$$

with ν_{kl} defined above.

In this case the frequency for the *heavy impurity* is

$$\nu_2 = \nu_{22} + O\left(\sqrt{\frac{m_1}{m_2}}\right)$$

and is dominated by collisions within the heavy species. Also

$$m_2 n_2 \langle \nu_2 \rangle \gg m_1 n_1 \langle \nu_1 \rangle$$

For the neutrality, the background ions and the electrons ensure the balance

$$\begin{aligned} n_a Z_1 &\approx n_e \\ &\gg n_2 Z_2 \end{aligned}$$

The heavy impurity ions do not contribute.

It is however admitted that for the Z_{eff} the heavy ions are important

$$n_1 Z_1^2 \sim n_2 Z_2^2$$

The flux of the background ions

$$\begin{aligned} \Gamma_1 &= -0.73 \sqrt{\varepsilon} \; \nu \; \rho_{e\theta}^2 \; n_1 \sqrt{\frac{m_1 T_e^3}{m_e T_1^3}} \; Z_1^2 \; (0.53 + \alpha) \\ &\quad \times \left\{ \frac{T_1}{Z_1 T_e} \left[\frac{n_1'}{n_1} - \left(\frac{0.09 + 0.5\;\alpha}{0.5 + \alpha} \right) \frac{T_1'}{T_1} \right] \right. \\ &\quad \left. - \frac{T_2}{Z_2 T_e} \left[\frac{n_2'}{n_2} - 0.17 \frac{T_2'}{T_2} \right] \right\} \end{aligned}$$

where

$$\rho_{e\theta}^2 = \frac{\frac{2T_e}{m_e}}{\left(\frac{eB_\theta}{m_e}\right)^2}$$

$$\nu = \sqrt{2}\left(\frac{4}{3}\sqrt{\pi}\right)\frac{e^4}{\sqrt{m_e}}\ln\Lambda\;\frac{n_e}{T_e^{3/2}}$$

$$\alpha = \frac{n_2 Z_2^2}{n_1 Z_1^2}$$

7.3 Discussion on Connor1973

The impurities that enter the plasma at the edge are almost imobile, they are cold and their parallel velocity is small.

The background ions are flowing rapidly along the lines.

Then we have a strong collisional coupling in the parallel direction, with substantial transfer of momentum between ions and impurity. This collisional force, combined with $\mathbf{B}_\theta$ (the first term in the equation for Γ_j) gives a strong *inflow* of the impurity ions.

For the same reason, but from the point of view of the ions, the radial drift due to parallel friction with impurities $\times$ B_θ is in the opposite direction and ions are going radially outward.

The argument that the background ions have significat parallel flow and so they collisionally exchange momentum with the much slower (almost imobile) impurities requires an effective parallel flow for the basic ions.

Equivalently: for the *inward* diffusion of the impurity ions to take place it is necessary that the collisional friction force exerced by background ions on impurity ions to keep the same direction (*i.e.* to not be symmetric along the magnetic field line, but singly oriented along the line). This means that we need a *flow* of background ions.

This toroidal flow *exists* as combination between the radial gradient of the plasma parameters n' and T' with the poloidal magnetic field B_θ.

Then, after taking the average over θ (on surface) of the momentum exchanged collisionally between background ions and impurities, it still remains something important, a directed parallel flow if the basic ions exists.

This is the reason of the presence of the second factor B_θ at the denominator in the expression Γ_j. Formally it comes from the neoclassical correction ρ_θ/L in the expression of the distribution function $g^{(0)}$ inserted in the formula for the *flow* u.

8 Temperature screening Helander 2017

The diffusion flux of impurities

$$\Gamma_z = \frac{1}{Ze}\left\langle uBR_{zi\parallel} + \left(p_{z\parallel} - p_{z\perp}\right)\frac{\nabla_\parallel\left(uB^2\right)}{2B}\right\rangle$$

The first term,

$$uBR_{zi\|} \equiv \text{friction with background ions}$$

the second term

$$\left(p_{z\|} - p_{z\perp}\right) \frac{\nabla_{\|}\left(uB^2\right)}{2B} \equiv \text{due to pressure anisotropy}$$

Here

$$u = \frac{j_{\|}}{Bp'(r)}$$

$$\begin{aligned} R_{zi\|} &= \int d^3v \; m_z v_{\|} \; C_{zi}\left[f_z, f_i\right] \\ &= \int d^3v \; m_i v_{\|} \; \nu_D^{iz} \; f_i - \frac{m_i n_i V_{z\|}}{\tau_{iz}} \end{aligned}$$

$$\begin{aligned} \nu_D^{iz} &\equiv \text{deflection frequency} \\ &= \frac{3\sqrt{\pi}}{4} \frac{1}{\tau_{iz}} \; \left(\frac{v_{th,i}}{v}\right)^3 \end{aligned}$$

$$\tau_{iz} = \frac{3}{4\sqrt{2\pi}} \; \sqrt{m_i} \; \frac{\left(4\pi\varepsilon_0\right)^2}{Z^2 e^4} \frac{1}{\ln\Lambda} \frac{T_i^{3/2}}{n_z}$$

9 Collisions, friction, viscosity and transport fluxes

See also *collisions*.
And *viscosity*.

9.1 Collisions multiple-ion species (Hirschman 1977)

9.1.1 Classical transport since the regime is Pfirsch Schluter

This starts with the parallel balance of forces

$$\begin{aligned} & v_{\|} \left[\left(\nabla_{\|} \ln p_a - \frac{e_a}{T_a} E_{\|}\right) \right. \\ & \qquad \left. - L_1^{(3/2)} \left(x_a^2\right) \left(\nabla_{\|} \ln T_a\right)\right] f_{Ma} \\ = & \sum_b \left[C_{ab}\left(f_{a1}, f_{Mb}\right) + C\left(f_{Ma}, f_{b1}\right)\right] \end{aligned}$$

The two *generalized forces* are

$$\begin{aligned} A_{1a} &\equiv \nabla_{\|} \ln p_a - \frac{e_a}{T_a} E_{\|} \\ A_{2a} &\equiv \nabla_{\|} \ln T_a \end{aligned}$$

It is clear that the equation is not trivial for only *variation of the plasma variables in the magnetic surface* (see **Stringer**).

Then the equation shows balance at equilibium of forces, only permitted by collisions.

This is a typical situation of Pfirsch Schluter regime since the latter is exclusively determined by variation of the basic plasma variables in the magnetic surface.

Also, the Pfirsch Schluter regime is strongly collisional and the bananas are not visible. Then the perturbation to the distribution function that results from the necessary variation of plasma variables in the magnetic surface f_{a1} is only governed by the collisional friction in response to the parallel gradients of pressure, temperature, transported (advected) by the parallel velocity. We have no reason to introduce in the balance the term of advection by the drift velocity of the radially-varying Maxwellian $v_D \partial f_{Ma}/\partial r$. The drift is not visible.

$$\begin{aligned}
x_a^2 &= \left(\frac{v}{v_{th,a}}\right)^2 \\
v_{th,a} &= \left(\frac{2T_a}{m_a}\right)^{1/2} \\
L_k^{(3/2)}\left(x_a^2\right) &\equiv \text{Sonin polynomials of order } l+\frac{1}{2}=3/2 \text{ and index } k
\end{aligned}$$

See **Galeev** for the expansion in Sonine polynomials.

We write the expressions of the Laguerre polynomials

$$L_0^{(3/2)}\left(x_a^2\right) = 1$$

$$L_1^{(3/2)}\left(x_a^2\right) = \frac{5}{2} - x_a^2$$

$$L_2^{(3/2)}\left(x_a^2\right) = \frac{35}{8} - \frac{7}{2}x_a^2 + \frac{1}{2}x_a^4$$

NOTE

Let us compare **Hirschman Sigmar Clarke** (see above)

$$v_\parallel \nabla_\parallel \left(f_{a1} + I\frac{v_\parallel}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial \psi}\right) = v_\parallel \frac{e_a E^{(A)}}{T_a} f_{a0} + \sum_b C_{ab}\left(f_{a1}, f_{b1}\right)$$

with **Hirschman 1977** (multiple-ion species)

$$\begin{aligned}
& v_\parallel \left[\left(\nabla_\parallel \ln p_a - \frac{e_a}{T_a}E_\parallel\right)\right. \\
& \qquad \left. - L_1^{(3/2)}\left(x_a^2\right)\left(\nabla_\parallel \ln T_a\right)\right] f_{Ma} \\
= \; & \sum_b \left[C_{ab}\left(f_{a1}, f_{Mb}\right) + C\left(f_{Ma}, f_{b1}\right)\right]
\end{aligned}$$

We must note that in the second equation the Left Hand Side is

$$\sim v_{\parallel} \nabla_{\parallel} f_{Ma}$$

which is possible since the plasma parameters inside f_{Ma}, the density and the temperature, have spatial variations along the magnetic field lines, $n(r,\theta)$ and $T(r,\theta)$.

This is NOT the first equation, where the neoclassical aspect $\rho_\theta \partial f_{Ma}/\partial r$ is present.

Then we say that in this approach the variation *of plasma variables* n, T_a , p_a *in the magnetic surface, along the magnetic lines leads to parallel advection* $v_{\parallel} \nabla_{\parallel}$ *which is only compensated (balanced) by collisions.*

The present approach is related to Pfirsch Schluter regime, where there is high collisionality and NO trace of banana. Therefore the perturbation to the distribution function f_{a1} does not contain a neoclassical part $\rho_\theta \partial f_{Ma}/\partial r$.

The theory developed by **Hirschman Sigmar Clarke** retains both terms

$$\begin{array}{c} v_{\parallel} \nabla_{\parallel} f^{(1)} \quad \text{and} \\ \mathbf{v}_{drift} \cdot \boldsymbol{\nabla} f^{(0)} \end{array}$$

the second brings the basic neoclassical perturbation of drift-velocity-advection of the Maxwellian distribution which has radial variation. This perturbation has the factor ρ_θ .

$$\begin{aligned} & \mathbf{v}_{Drift} \cdot \boldsymbol{\nabla} f_{Ma} \\ = \; & \mathbf{v}_{drift} \cdot \boldsymbol{\nabla} \psi \times \frac{\partial f_{Ma}}{\partial \psi} \\ = \; & v_{\parallel} \nabla_{\parallel} \left(I \frac{v_{\parallel}}{\Omega_{ca}} \right) \times \frac{\partial f_{Ma}}{\partial \psi} \end{aligned}$$

In **Hirschman 1977** the Maxwellian is considered as parametric function of the *(1) density* and *(2) temperature*. Then there is a parallel variation of the Maxwellian distribution, through these paramaters. Then the operator

$$v_{\parallel} \nabla_{\parallel} f_M$$

provides the parallel advection (by the very high parallel velocity) of the small perturbation of the parameters *in the surface*.

$$\begin{aligned} n(r,\theta) &= n_0(r) + n_1(r,\theta) \\ T(r,\theta) &= T_0(r) + T_1(r,\theta) \end{aligned}$$

This advection is halted (balanced) by collisions.

This is in the deep collisional regime Pfirsch Schluter.

END

9.1.2 Method of moments

The solution of the equation for the perturbation to the distribution function f_{a1} is obtained by the method of *moments*.

The equation is multiplied by

$$m_a v_\parallel \; L_k^{(3/2)}\left(x_a^2\right)$$

and integrated over the velocity space.

$$\begin{aligned} & p_a A_{1a} \delta_{0k} \\ & -\frac{5}{2} p_a A_{2a} \delta_{1k} \\ = & \int d^3 v \; m_a v_\parallel L_k^{(3/2)} \; C_a \end{aligned}$$

for

$$C_a = \sum_b C_{ab}$$

The first moment, with $L_0^{(3/2)}\left(x_a^2\right)$,

$$\nabla_\parallel p_a - e_a n_a E_\parallel = R_a$$

The second moment, with $L_1^{(3/2)}\left(x_a^2\right)$,

$$\frac{5}{2} n_a \nabla_\parallel T_a = H_a$$

The superior moments $k \geq 2$ produce *constraining equations*

$$\begin{aligned} \int d^3 v \; m_a v_\parallel \; L_k^{(3/2)}\left(x_a^2\right) \; C_a &= 0 \\ k &\geq 2 \end{aligned}$$

because these terms $L_{k\geq 2}^{(3/2)}\left(x_a^2\right)$ are missing in the left hand side of the balance of parallel forces.

Two *friction forces* R_a and H_a can be defined directly

$$R_a = \int d^3 v \; m_a v_\parallel \; C_a$$

$$H_a = \int d^3 v \; m_a v_\parallel \; \left[\left(\frac{v}{v_{th,a}}\right)^2 - \frac{5}{2}\right] \; C_a$$

NOTE the first moment, selects

$$k = 0$$

in the term

$$p_a A_{1a} \delta_{0k}$$

and this leaves

$$p_a \nabla_{\|} \ln p_a = \nabla_{\|} p_a$$

This is the variation of the pressure in the surface. There will be also the term with the electric field $E_{\|}$. But what is important is the balance

$$\nabla_{\|} p_a \sim R_a$$

which expresses exactly what we have defined as present regime: balance of variation of parameters in the surface by collisions.

END

The perturbation to the distribution function is expanded in a series of flows with coefficients Sonin polynomials

$$f_{a1} = \frac{2v_{\|}}{v_{th,a}^2} f_{Ma} \sum_{k=0}^{\infty} u_{ak} \ L_k^{(3/2)} \left(x_a^2\right)$$

We **NOTE** the following. In the equation expressing the balance of forces along the parallel direction, the perturbation to the distribution function f_{a1} is only contained in the collisional operator, in the right hand side. The left hand side consists of parallel forces: the parallel gradient of the pressure; they come from assuming that the parameters of a Maxwellian distribution function (local) have small variation along the parallel direction and the operator

$$v_{\|} \nabla_{\|}$$

acting on the full distribution function

$$f_{Ma}$$

will extract these forces that must further be balanced by collisional friction (right hand side).

Now, the collision operator will be linearized, $f_{Ma} + f_{a1}$.

It is natural to expect that the perturbation f_{a1} will have as coefficients

$$\begin{array}{l} v_{\|} \\ \text{and} \ \ f_{Ma} \end{array}$$

then

$$f_{a1} \sim v_{\|} f_{Ma} \times ...$$

END

Only the first two terms in the Sonine-polynomials expansion have a name: flow velocity $u_{a\|}$ and heat flow $\frac{q_{a\|}}{p_a}$.

The expansion in Sonin polynomials is

$$\begin{aligned} f_{a1} &= \frac{2v_{\|}}{v_{th,a}^2} f_{Ma} \left[L_0^{(3/2)}\left(x_a^2\right) \boxed{\mathrm{u}_{a\|}} \qquad \left(\text{flow } u_{a\|}\right) \right. \\ & \qquad -\frac{2}{5} L_1^{(3/2)}\left(x_a^2\right) \boxed{\frac{q_{a\|}}{p_a}} \qquad \left(\text{flow } \frac{q_{a\|}}{p_a}\right) \\ & \qquad \left. + \sum_{k=2}^{\infty} L_k^{(3/2)}\left(x_a^2\right) \boxed{\mathrm{u}_{ak}} \right] \qquad \text{(higher order flows } u_{ak}\text{)} \end{aligned}$$

where the *flows* corresponding to higher $k \geq 2$ Sonine polynomials in the expansion are defined as

$$u_{ak} = \frac{1}{n_a} \frac{\int d^3v \; L_k^{(3/2)}\left(x_a^2\right) \; v_{\|} f_{a1}}{\left\{ \left[L_k^{(3/2)}\left(x_a^2\right) \right]^2 \right\}}$$

with the notation

$$\{A(v)\} = \frac{8}{3\sqrt{\pi}} \int d^3v \; x^4 \exp\left(-x^2\right) \; \times A\left(x v_{th,a}\right)$$

for a function of velocity v. This symbol occurs from the condition of normalization of the Sonine polynomials, where there is a *weight function* of the set of functions.

We NOTE that here we have already assumed that the equilibrium distribution functions are MAXWELLIAN.

This is what makes interesting the Sonine polynomials: the wight function for orthonormality is an exponential like a Maxwell distribution.

NOTE in **viscosity Shaing Zarnstroff** the *friction* forces and the *viscous stress* are expanded in series of flows

- the momentum flow u
- the heat flow q/p

in principle there can also be higher "moments" but they are ignored.

The coefficients of the $k = 0$ Sonine polynomial is

$$u_{\|a} = \frac{1}{n_a} \int d^3v \; v_{\|} f_{a1}$$

parallel flow velocity

The coefficient of the $k = 1$ Sonine polinomial is

$$q_{\|a} = \int d^3v \left(\frac{mv^2}{2} - \frac{5}{2} T_a \right) v_{\|} f_{a1}$$

parallel random heat flux

Here "random" means "statistically determined".

We now have an expression for the perturbation to the distribution function, as a series of Sonine polynomials.

This is NOT yet a solution.

It is not a solution because the terms of the expansion consist of integrals over the velocity space of product of Laguerre (Sonine) polynomials and the perturbation function f_{a1}, exactly what we want to find.

Then it is a symbolical solution.

We can replace this "solution" in the collision operator, after linearization $f_{Ma} + f_{a1}$.

After doing this, we can *purely formally* integrate in the expressions defining the friction flow and the friction of heat.

This will make the two flows $u_{a\|}$ and $q_{a\|}/p_a$ to be exhibited in the expressions of R_a and H_a. The coefficients are numerical, a matrix denoted l_{ij}^{ab}.

$$R_a = \sum_b \left[l_{11}^{ab} \; u_{b\|} + \frac{2}{5} l_{12}^{ab} \; \left(\frac{q_{b\|}}{p_b} \right) \right]$$

$$H_a = \sum_b \left[l_{21}^{ab} \; u_{b\|} + \frac{2}{5} l_{22}^{ab} \; \left(\frac{q_{b\|}}{p_b} \right) \right]$$

The new coefficients

$$l_{1j}^{ab} \equiv \text{classical friction}$$

And

$$\begin{aligned} \left(l_{ij}^{ab} \right) &\equiv \text{(classical friction coeff.)} \\ &= \text{(classical trapsort coeff)}^{-1} \end{aligned}$$

The series expansion in $L_k^{(3/2)} \left(x_a^2 \right)$ is *infinite*.

It must be cut at a finite number of terms.

The $N = 0$ (no higher terms, just $u_{a\|}$ and $q_{a\|}/p_a$ flows, the first two "moments") is incorect.

Take $N = 1$, which means to include just one (higher) flow, u_{a2} in addition

$$\begin{aligned} u_{a0} &= u_{a\|} \\ u_{a1} &= \frac{q_{a\|}}{p_a} \\ u_{a2} &= \text{the only higher order flow retained} \\ & \text{it proves to be sufficient} \end{aligned}$$

Then

$$\begin{aligned} f_{a1}^{(N=1)} &= \frac{2v_\parallel}{v_{th,a}^2} f_{Ma} \left[L_0^{(3/2)}\left(x_a^2\right)\ u_{a\parallel} \right. \\ &\qquad -\frac{2}{5} L_1^{(3/2)}\left(x_a^2\right)\ \frac{q_{a\parallel}}{p_a} \\ &\qquad \left. +L_2^{(3/2)}\left(x_a^2\right)\ u_{a2} \right] \end{aligned}$$

The flow u_{a2} must be calculated from the constraints, in terms of the first two flows.

$$\int d^3v\ v_\parallel L_2^{(3/2)}\left(x_a^2\right)\ C_a\left[f_{a1}^{(N=1)}\ ,\ f_{b1}^{(N=1)}\right] = 0$$

After separation of the unknown functions (flows) u_{a2},

$$\begin{aligned} u_{a2} &= -\sum_b \alpha_{ab}\ u_{b\parallel} \\ &\quad +\frac{2}{5}\sum_b \beta_{ab}\ \left(\frac{q_{b\parallel}}{p_b}\right) \\ &\quad +\sum_b \Delta_{ab}\ u_{b2} \end{aligned}$$

where

$$\alpha_{ab} = \frac{1}{M_a^{22}+N_{aa}^{22}}\left[M_a^{20}\ \delta_{ab} + \left(\frac{v_{th,a}}{v_{th,b}}\right)\ N_{ab}^{20}\right]$$

$$\beta_{ab} = \frac{1}{M_a^{22}+N_{aa}^{22}}\left[M_a^{21}\ \delta_{ab} + \left(\frac{v_{th,a}}{v_{th,b}}\right)\ N_{ab}^{21}\right]$$

$$\Delta_{ab} = \frac{1}{M_a^{22}+N_{aa}^{22}}\left[-\left(\frac{v_{th,a}}{v_{th,b}}\right)\ N_{ab}^{22}\ \left(1-\delta_{ab}\right)\right]$$

The matrices are obtained by integration over the velocity space

$$M_{ab}^{jk} = \int d^3v\ \left(\frac{v_\parallel}{v_{th,a}}\right) L_j^{(3/2)}\left(x_a^2\right)\ C_{ab}\left[\frac{2v_\parallel}{v_{th,a}} f_{Ma} L_k^{(3/2)}\left(x_a^2\right)\quad ,\quad f_{Mb}\right]$$

$$N_{ab}^{jk} = \int d^3v\ \left(\frac{v_\parallel}{v_{th,a}}\right) L_j^{(3/2)}\left(x_a^2\right)\ C_{ab}\left[\ f_{Ma}\quad ,\quad \frac{2v_\parallel}{v_{th,b}} f_{Mb} L_k^{(3/2)}\left(x_b^2\right)\right]$$

The symbol

$$M_a^{jk} = \sum_b M_{ab}^{jk}$$

For the elements of the matrices we have

$$\begin{aligned} M_{ab}^{jk} &= M_{ab}^{kj} \\ T_a N_{ab}^{jk} &= T_b N_{ba}^{kj} \end{aligned}$$

These elements can be calculated explicitely in the case of Maxwellian f_{Ma}, if the collision operator is Coulombian linearized.

Then we have an algebraic equation for the flows

$$u_{a2}$$

which can be solved by inverting a matrix.

Two coefficients appear in this algebraic inversion

$$\widehat{\alpha}_{ab} = \alpha_{ab} + \sum_c \Delta_{ac}\alpha_{cb} + ...$$

$$\widehat{\beta}_{ab} = \beta_{ab} + \sum_c \Delta_{ac}\beta_{cb} + ...$$

Using these coefficients we find the first higher flow u_{a2} in terms of the lower flows $u_{b\|}$ and $\frac{q_{b\|}}{p_b}$ of *all the other species*.

$$u_{a2} = -\sum_b \widehat{\alpha}_{ab}\ u_{b\|} + \frac{2}{5}\sum_b \widehat{\beta}_{ab}\left(\frac{q_{b\|}}{p_b}\right)$$

Returning to the relations between the friction forces and the flows

$$l_{11}^{ab} = m_a\left[M_a^{00}\ \delta_{ab} - M_a^{02}\ \widehat{\alpha}_{ab} + \left(\frac{v_{th,a}}{v_{th,b}}\right)\ N_{ab}^{00} - \sum_s\left(\frac{v_{th,a}}{v_{th,s}}\right)\ N_{as}^{02}\ \widehat{\alpha}_{sb}\right]$$

$$l_{12}^{ab} = m_a\left[-M_a^{01}\ \delta_{ab} + M_a^{02}\ \widehat{\beta}_{ab} - \left(\frac{v_{th,a}}{v_{th,b}}\right)\ N_{ab}^{01} + \sum_s\left(\frac{v_{th,a}}{v_{th,s}}\right)\ N_{as}^{02}\ \widehat{\beta}_{sb}\right]$$

$$l_{22}^{ab} = m_a\left[M_a^{11}\ \delta_{ab} - M_a^{12}\ \widehat{\beta}_{ab} + \left(\frac{v_{th,a}}{v_{th,b}}\right)\ N_{ab}^{11} - \sum_s\left(\frac{v_{th,a}}{v_{th,s}}\right)\ N_{as}^{12}\ \widehat{\beta}_{sb}\right]$$

The balance of the friction forces

$$R_{ab} + R_{ba} = 0$$

leads to

$$m_a v_{th,a} M_{ab}^{0j} + m_b v_{th,b} N_{ba}^{0j} = 0$$

9.1.3 Calculation of the matrix elements M_{ab}^{ik} and N_{ab}^{ij}

The method is to use the generating function of the Sonine polynomials $L_k^{(3/2)}\left(x_a^2\right)$.

$$\sum_{j,k=0} \xi^j \eta^k\ M_{ab}^{jk} = -\frac{n_a}{\tau_{ab}}\overline{m}_{ab}\left(\xi,\eta\right)$$

$$\sum_{j,k=0} \xi^j \eta^k\ N_{ab}^{jk} = \frac{T_a}{T_b}\frac{v_{th,a}}{v_{th,b}}\frac{n_a}{\tau_{ab}}\overline{n}_{ab}\left(\xi,\eta\right)$$

9.1.4 The transport in the Pfirsch Schluter regime

$$\Gamma_a^{\psi} = \left\langle \int d^3v \ \mathbf{v} \cdot \boldsymbol{\nabla}\psi \ f_a \right\rangle$$

$$\frac{q_a^{\psi}}{T_a} = \left\langle \int d^3v \ \mathbf{v} \cdot \boldsymbol{\nabla}\psi \ \left[x_a^2 - \frac{5}{2} \right] \ f_a \right\rangle$$

10 Electrostatic trapping Hazeltine Ware

The text is in *plasma general, variation on surface.*

11 Reversal of impurity flow (Ohkawa Burrell Wong)

This is also in *collisions.tex*.

The paper is **NF20, 1021 (1980) Burrell**.

It is about the action that should be taken to *reverse the transport of impurity from inward to outward.*

Ohkawa proposal: gas puff localized at bottom, to create a poloidally asymmetric particle source.

Explanation by **Burrell**.

The vertical drifts of the ions combines with the radial gradients of density and pressure to create *variation of plasma variables in the magnetic surface.*

This is now well known and described and includes the Pfirsch Schluter current.

An impurity ion is acted upon by two frictional forces:

- a slippage friction force, which results from the different velocities of the backgroud ions and of the impurity ions along the magnetic lines

$$\begin{array}{rl} & \text{slippage friction} \\ \sim & v_{\parallel i} - v_{\parallel Z} \end{array}$$

  To fix our image, we consider that the ion neoclassical drift $\mathbf{v}_D$ is vertical. This means that the main magnetic field points into the page and the current also points in the same direction. This will create the helicity of the magnetic field toward right-down. The flow of the impurities results from the motion of the ions along this direction (as the current) and from the fact that this collisional pressure dominates a collisional resistance from Z ions. Then the impurity ions will have an induced flow that in projection is seen as downward.

- a friction force resulting from the dependence of the collisional force on the temperature along the magnetic field line. This second frictional force involves, therefore, the variation of plasma variables in the magnetic surface. The variables of plasma that have significant neoclassical (intrinsic) variation in the magnetic surface are density $n_1(r,\theta)$, and the electrostatic potential $\phi_1(r,\theta)$. But the temperature is higher at the low-field side and this means that the collisional force exerted from the low field side toward regions at smaller $R = R_0 + r\cos\theta$, is smaller than the collisional force exerted from smaller R toward low field side $R = R_0 + r$. This is therefore a force towards the low field side, opposed to the slippage force.

The second is explained as follows:
the collisional cross section depends on the velocity as

$$\frac{n_Z}{v_{th,Z}^3}$$

and this makes a difference between regions where the temperature (*i.e.* the thermal velocity $v_{th,Z}$) is different. In the expression of Burrell, the hot regions *hit less* than the cold regions. This creates a directed frictional force, from the cold toward the hot regions, along the magnetic line.

This *thermal frictional force* opposes the *slippage frictional force.*

Since usually the slippage force is larger than the thermal frictional force, there is *dominant flow of impurities downward.*

There is a slight accumulation of impurities at the bottom of the tokamak.

This is a poloidal nonuniformity of the density of impurities.

Combined with the equilibrium gradients leads to *inward transport of Z ions.*

See **Connor 1973** for an explanation based on ambipolarity.

11.1 Impurity influx, sources in Pfirsch Schluter Burrell 1976 and 1980

Like in **Rutherford 1974** (see above)

simplified

$$R_{\|} = -C_1 \frac{m_i n_i}{\tau_{iI}} u_{\|} - C_2 \, n_i \nabla_{\|} T_i$$

$$\mathbf{u} \equiv \mathbf{v}_i - \mathbf{v}_I$$

$$\tau_{iI} = \frac{3}{4\sqrt{2\pi}} (4\pi\varepsilon_0)^2 \frac{\sqrt{m_i}}{Z_I^2 e^2 Z_i^2 e^2} \frac{1}{\ln\Lambda} \times \frac{T_i^{3/2}}{n_I}$$

Note that these are simplified forms of the usual expression of $R_{\|}$ and $H_{\|}$ in terms of *flows* $u_{\|}$ and $q_{\|}/p$ and **Hirshman** coefficients l_{ij}^{ab} (a, b =species, $i, j = 1, 2$ flows). **End.**

The two equations of force balance are used after *radial projection.*
The field E_r is eliminated

$$u_\theta = \frac{B_\theta}{B_T} u_T + \frac{1}{eB_T} \left(\frac{1}{n_i} \frac{\partial p_i}{\partial r} - \frac{1}{n_I Z_I} \frac{\partial p_I}{\partial r} \right)$$

It is neglected the variation on θ for

$$\begin{array}{l} n_i, p_i, n_I, p_I \\ \text{functions on only } r \end{array}$$

Projecting the two force balance equation on the *parallel* direction

$$\begin{aligned} \frac{\partial p_i}{r\partial\theta} - n_i e E_\theta &= -\frac{\partial p_I}{r\partial\theta} + n_I Z_I e E_\theta \\ &= \frac{B_T}{B_\theta} R_\parallel \end{aligned}$$

NOTE
HERE is the origin of the *ambipolarity.* Both species are force-balanced by mutual collision, *i.e.* the unicity of $R_\parallel$.
END

Projecting the force balance for *ions* on the θ direction

$$n_i v_{ir} = -\frac{1}{eB_T} \left(\frac{\partial p_i}{r\partial\theta} - n_i e E_\theta \right)$$

after taking

$$R_\theta \approx 0$$

Using the two equations above **Burrell 1976** finds

$$n_i v_{i,r} = -n_I Z_I v_{I,r} = R_\parallel \frac{1}{B_T} \frac{B}{B_\theta}$$

this equation shows that "*inward flux of impurity ions and the outward flux of protons are always coupled*".

the equation with *sources*

$$\nabla \cdot (n_a \mathbf{v}_a) = S_a (r, \theta)$$

it is assumed that the sources have zero average

$$\begin{aligned} \langle S_a (r, \theta) \rangle &= \int \frac{d\theta}{2\pi} (1 + \varepsilon \cos\theta) S_a (r, \theta) \\ &= 0 \end{aligned}$$

The parallel velocity

$$u_{\parallel} = \frac{B}{B_{\theta}} u_{\theta} + \frac{B_T}{B} \frac{1}{eB_{\theta}} \left(\frac{1}{n_i Z_i} \frac{\partial p_i}{\partial r} - \frac{1}{n_I Z_I} \frac{\partial p_I}{\partial r} \right)$$

The average of $u_{\parallel}$ on θ is zero

$$\overline{u}_{\parallel} = \int \frac{d\theta}{2\pi} u_{\parallel} = 0$$

Using this equation one finds

$$\begin{aligned} u_{\parallel} &= \frac{2r}{R} \cos\theta \frac{1}{B_{\theta 0}} \frac{B}{B_T} \left(\frac{1}{n_i Z_i} \frac{\partial p_i}{\partial r} - \frac{1}{n_I Z_I} \frac{\partial p_I}{\partial r} \right) \\ &+ \frac{B}{B_{\theta}} \frac{r}{h} \frac{1}{n_i} \left[I_i\left(r,\theta\right) - \sqrt{1-\varepsilon^2 \left(\frac{I_i\left(r,\theta\right)}{h\left(\theta\right)} \right)} \right] \\ &+ O\left(\varepsilon^2\right) \end{aligned}$$

where

$$I_a\left(r,\theta\right) = \int_{\pi/2}^{\theta} d\theta' \ h\left(\theta'\right) \ S_a\left(r,\theta'\right)$$

Equations

$$\frac{\partial n_a}{\partial t} + \nabla \cdot \mathbf{\Gamma}_a = S_a$$

$$\begin{aligned} m_a \frac{\partial \mathbf{\Gamma}_a}{\partial t} &= -\nabla \cdot \mathbf{P}_a + Z_a e \ \left(n_a \mathbf{E} + \mathbf{\Gamma}_a \times \mathbf{B} \right) + \mathbf{R}_a \\ &-b_a \mathbf{\Gamma}_a + \mathbf{L}_a \end{aligned}$$

$$\frac{\partial}{\partial t} \left(\frac{3}{2} p_a \right) = -\nabla \cdot \mathbf{Q}_a + Z_a e \ \mathbf{\Gamma}_a \cdot \mathbf{E} + \mathbf{Q}_{\Delta,a} + \mathbf{G}_a$$

$$\begin{aligned} \frac{\partial \mathbf{Q}_a}{\partial t} &= -\nabla \cdot \mathbf{\Theta}_a + \frac{Z_a e}{m_a} \mathbf{E} \cdot \left(\frac{3}{2} p_a \mathbf{I} + \mathbf{P}_a \right) + \frac{Z_a e}{m_a} \mathbf{Q}_a \times \mathbf{B} + \mathbf{H}_a \\ &- \frac{b_a}{m_a} \mathbf{Q}_a + \mathbf{M}_a \end{aligned}$$

where (**Hirschman Sigmar review**)

$$\begin{aligned} \mathbf{\Theta}_a &\equiv \text{energy-weighted stress tensor} \\ \mathbf{Q}_a &\equiv \text{energy flux} \\ \mathbf{Q}_{\Delta,a} &\equiv \text{collisional energy change} \\ S &\equiv \text{particle source} \\ \mathbf{H}_a &\equiv \text{collisional change in the energy flux} \\ \mathbf{L}_a &\equiv \text{momentum source} \\ G_a &\equiv \text{energy source} \\ \mathbf{M}_a &\equiv \text{energy-weighted momentum source} \end{aligned}$$

$$
\begin{aligned}
b_a \mathbf{\Gamma}_a &\equiv \text{drag term, friction with neutral atoms} \\
\frac{b_a}{m_a}\mathbf{Q}_a &\equiv \text{drag term friction with neutral atoms}
\end{aligned}
$$

the dominant interaction of ions with neutral atoms is *charge exchange*.

In the Pfirsch Schluter regime, the components of *friction* terms that are parallel - are important.

$$
R_{\|a} = \sum_b \left(l_{11}^{ab} \; u_{\|b} + l_{12}^{ab} \; \frac{2}{5} \frac{q_{\|b}}{p_b} \right)
$$

$$
h_{\|a} = \sum_b \left(l_{21}^{ab} \; u_{\|b} + l_{22}^{ab} \; \frac{2}{5} \frac{q_{\|b}}{p_b} \right)
$$

where

$$
\begin{aligned}
\mathbf{u}_a &= \text{flow flux} \\
\mathbf{q}_a &= \mathbf{Q}_a - \frac{5}{2} T_a \mathbf{\Gamma}_a \quad \text{heat flux} \\
\mathbf{h}_a &= \frac{m_a}{T_a}\mathbf{H}_a - \frac{5}{2}\mathbf{R}_a \quad \text{heat friction}
\end{aligned}
$$

First the perpendicular components (*diamagnetic*)

$$
\mathbf{\Gamma}_{\perp a} = \frac{1}{Z_a e B} \widehat{\mathbf{n}} \times \left(\boldsymbol{\nabla} p_a + n_a Z_a e \; \boldsymbol{\nabla} \Phi \right)
$$

$$
\frac{\mathbf{q}_{\perp a}}{p_a} = \frac{1}{Z_a e B} \; \frac{5}{2} \widehat{\mathbf{n}} \times \boldsymbol{\nabla} T_a
$$

Similar to the calculation for the *diamagnetic* - *Pfirsch Schluter*, one finds

$$
\begin{aligned}
\Gamma_{\|a} = & -\frac{I}{Z_a e} \frac{1}{B} \left(1 - \frac{B^2}{\langle B^2 \rangle} \right) \left(\frac{\partial p_a}{\partial \psi} + n_a Z_a e \frac{\partial \Phi}{\partial \psi} \right) \\
& + B \int^{\theta} \frac{r d\theta}{B_\theta} \; S_a - B \left\langle \frac{B^2}{\langle B^2 \rangle} \int^{\theta} \frac{r d\theta}{B_\theta} \; S_a \right\rangle \\
& + B \frac{\langle \Gamma_{\|a} B \rangle}{\langle B^2 \rangle}
\end{aligned}
$$

Ohkawa proposal:

inject hydrogen ions at the bottom, since in their motion they will reverse the sense of the slippage force.

In addition, since there is local cooling, the thermal force will be enhanced, since it points from cold to hot along the line. This time we enhance the collisional friction by raising the force due to decrease of the temperature at the bottom.

As result, there will be a *flow of impurities upward.* (reversed)

The reversal of the impurity flow

The influence of poloidally asymmetric sources of particles upon the flow of impurities and reversal of the inward flow.

The paper **Burrell PfirschSchluter 1980**.

Essentially

$$\Gamma_{I,r} = -\frac{Z_i}{Z_I}\Gamma_{i,r}$$

from ambipolarity.

The paper by **Wong**.

It is about inward flux of impurities in the regime of Pfirsch Schluter.

It can be corrected (reversed flow) by poloidally asymmetric sources.

The perpendicular fluxes

$$\mathbf{\Gamma}_{\perp}^{(1)} = \frac{1}{m\Omega}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p$$

and continuity

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{\Gamma} = S_n$$

In first order

$$\frac{\partial n}{\partial t_1} = \langle S_n \rangle$$

The parallel flow is

$$\begin{aligned} \mathbf{B} \cdot \boldsymbol{\nabla} \left(\frac{\mathbf{\Gamma}^{(1)}}{B} \right) &= -\boldsymbol{\nabla} \cdot \frac{1}{m\Omega}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p \\ &\quad + \widetilde{S}_n \end{aligned}$$

where

$$\widetilde{S}_n = S_n - \langle S_n \rangle$$

The equation for the parallel flow is integrated

$$\begin{aligned} \Gamma_{\parallel}^{(1)} &= -\frac{I}{e}\frac{1}{B}\frac{\partial p}{\partial \psi}\left(1 - \frac{B^2}{\langle B^2 \rangle}\right) \\ &\quad + B\left(\int_0^{l_\theta} \frac{dl_\theta}{B_\theta}\widetilde{S}_n - \frac{B^2 \int_0^{l_\theta} \frac{dl_\theta}{B_\theta}\widetilde{S}_n}{\langle B^2 \rangle}\right) \\ &\quad + \frac{\left\langle \Gamma_{\parallel}^{(1)} B \right\rangle}{\langle B^2 \rangle} B \end{aligned}$$

NOTE
regarding the Pfirsch Schluter term.
In *variation on surfaces.tex* it is started from $j_{\|} = -\frac{Fp'}{B} + KB$ and found

$$\begin{aligned} j_{\|} &= -\frac{F}{B}\frac{\partial p}{\partial \psi}\left(1 - \frac{B^2}{\langle B^2 \rangle}\right) \\ &\quad + \frac{\langle \mathbf{j} \cdot \mathbf{B} \rangle}{\langle B^2 \rangle} B \end{aligned}$$

where

$$F = \frac{I}{e}$$

END

12 Inward flow of impurities

12.1 Impurities in a rotating plasma (Wong Burrell 1982)

Impurities and toroidal rotation **Wong Burrell 1982**
The equations are in *derivation drift kinetic equation.tex.*
Some comments are also in *solutions drift kinetic.*

The intention is to derive a drift kinetic equation.

12.2 Equations of motion of the particles

The derivation of the equations of motion is in the text *particles equations of motion.*

For a circular geometry, when the potential can be separated into a main part (constant on magnetic surface) and a part that depends on (r, θ),

$$\phi = \phi_0(r) + \phi_1(r, \theta)$$

$$\frac{dx_\theta}{dt} = v_{\|}\frac{B_\theta}{B_T} + \frac{1}{B_0}\frac{d\phi_0}{dr}$$

$$\frac{dx_r}{dt} = -\frac{1}{B_0}\frac{\partial \phi_1}{r\partial \theta} - \frac{1}{\Omega_{ci}}\frac{v_\perp^2/2 + v_{\|}^2}{R}\sin\theta$$

$$\frac{d}{dt}\left(\frac{v_\perp^2}{2}\right) = \left(\frac{v_\perp^2}{2}\right) v_{\|}\frac{B_\theta}{B_T}\frac{\sin\theta}{R} + \left(\frac{v_\perp^2}{2}\right)\frac{1}{B_0}\left(\frac{d\phi_0}{dr}\right)\frac{\sin\theta}{R}$$

$$\frac{dv_{\|}}{dt} = -\left(\frac{v_\perp^2}{2}\right)\frac{B_\theta}{B_T}\frac{\sin\theta}{R} + v_{\|}\frac{1}{B_0}\left(\frac{d\phi_0}{dr}\right)\frac{\sin\theta}{R} - \frac{e}{m}\frac{B_\theta}{B_T}\frac{\partial\phi_1}{r\partial\theta}$$

This is **Wong Burrell 1982**.

12.3 Drift kinetic equation

The drift-kinetic equation is written as

$$\frac{\partial f}{\partial t}+\boldsymbol{\nabla}\cdot\left(\frac{d\mathbf{x}}{dt}f\right)+\frac{\partial}{\partial v_{\parallel}}\left(\frac{dv_{\parallel}}{dt}f\right)+\frac{\partial}{\partial\left(v_{\perp}^{2}/2\right)}\left(\frac{d\left(v_{\perp}^{2}/2\right)}{dt}f\right)=C\left(f,f\right)$$

It is remarked that the volume in phase space is NOT preserved by the equations of motion

$$\begin{aligned}&\boldsymbol{\nabla}\cdot\frac{d\mathbf{x}}{dt}+\frac{\partial}{\partial v_{\parallel}}\left(\frac{dv_{\parallel}}{dt}\right)+\frac{\partial}{\partial\left(v_{\perp}^{2}/2\right)}\left(\frac{d\left(v_{\perp}^{2}/2\right)}{dt}\right)\\ =\;&\frac{1}{B^{2}}\left(\boldsymbol{\nabla}\times\mathbf{B}\right)\cdot\boldsymbol{\nabla}\phi+\boldsymbol{\nabla}\cdot\mathbf{v}_{D}\end{aligned}$$

This is not so large and later it will be neglected.

From **Wong Burrell** we have

$$\mathbf{v}_{D}=\frac{1}{\Omega}\left(v_{\parallel}^{2}+\frac{v_{\perp}^{2}}{2}\right)\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\ln B$$

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{v}_{D}&=\frac{m}{e}\left(v_{\parallel}^{2}+\frac{v_{\perp}^{2}}{2}\right)\frac{\mathbf{J}\cdot\boldsymbol{\nabla}B}{B^{3}}\\&\approx\frac{\varepsilon^{2}v_{D}}{r}\end{aligned}$$

and

$$\frac{J_{\theta}}{B}\sim\varepsilon^{2}\frac{1}{r}$$

This divergence of the drift velocity is neglected.

It is also mentioned by **Hinton Waltz** with the intention to derive exact equation for the heating of plasma by instabilities.

Then the drift kinetic equation is

$$\frac{d\mathbf{x}}{dt}\cdot\boldsymbol{\nabla}f+\frac{dv_{\parallel}}{dt}\frac{\partial f}{\partial v_{\parallel}}+\left[\frac{d}{dt}\left(\frac{v_{\perp}^{2}}{2}\right)\right]\frac{\partial f}{\partial\left(v_{\perp}^{2}/2\right)}=C\left(f,f\right)$$

and an expansion is made

$$f=f_{0}+f_{1}+...$$

(multiple time-scale expansion)

$$\frac{dx_{\theta}}{dt}\frac{\partial f_{0}}{r\partial\theta}=C\left(f_{0},f_{0}\right)$$

or

$$\frac{dx_{\theta}}{dt}=\frac{d\left(r\theta\right)}{dt}$$

$$f_0 = \frac{1}{\left[\pi\left(\frac{2T}{m}\right)\right]^{3/2}} n \ \exp\left[-\frac{\left(v_\parallel - U\right)^2}{(2T/m)} - \frac{v_\perp^2}{(2T/m)}\right]$$

where $U \equiv$parallel flow velocity

$$\begin{aligned}
&\left(v_\parallel + \frac{1}{B_\theta}\frac{d\phi_0}{dr}\right)\frac{B_\theta}{B_T}\frac{\partial f_1}{r\partial\theta} \quad \text{(poloidal advection of } f_1\text{)} \\
&+\left(v_\parallel - U\right)\frac{e}{T}\frac{B_\theta}{B_T}\frac{\partial\phi_1}{r\partial\theta} f_0 - C^{lin}\left(f_1\right) \\
= \quad & \varepsilon\frac{B_\theta}{B_T}\frac{1}{r}\left\{\left[\frac{v_\perp^2}{(2T/m)}\left(U + \frac{1}{B_\theta}\frac{d\phi_0}{dr}\right) + 2\frac{v_\parallel\left(v_\parallel - U\right)}{(2T/m)}\frac{1}{B_\theta}\frac{d\phi_0}{dr}\right]\sin\theta\right. \\
&+\frac{T}{eB_\theta}\left[\frac{v_\parallel^2 + \frac{v_\perp^2}{2}}{(T/m)}\sin\theta + \frac{1}{\varepsilon}\frac{e}{T}\frac{\partial\phi_1}{\partial\theta}\right]\times \\
&\qquad \left.\times\left[\frac{d}{dr}\ln n + \left(\frac{\left(v_\parallel - U\right)^2}{(2T/m)} + \frac{v_\perp^2}{(2T/m)} - \frac{3}{2}\right)\frac{d}{dr}\ln T + \frac{\left(v_\parallel - U\right)\frac{dU}{dr}}{(T/m)}\right]\right\}
\end{aligned}$$

A term like

$$\begin{aligned}
\left(v_\parallel - U\right)\frac{e}{T}\frac{B_\theta}{B_T}\frac{\partial\phi_1}{r\partial\theta} f_0 \quad &\sim \quad V_\parallel e\frac{1}{qR}\frac{\partial\phi_1}{\partial\theta}\times\frac{\partial f_0}{\partial\epsilon} \sim V_\parallel \ e\widetilde{E}_\parallel \times \frac{\partial f_0}{\partial\epsilon} \\
&= \quad \text{work done by parallel flow against } \widetilde{E}_\parallel
\end{aligned}$$

is energetic. It is added in the LHS to the poloidal advection of the order-1 function f_1 .

Wong Burrell consider that the first term in the RHS

$$\varepsilon\frac{B_\theta}{B_T}\frac{1}{r}\left[\frac{v_\perp^2}{(2T/m)}\left(U + \frac{1}{B_\theta}\frac{d\phi_0}{dr}\right) + 2\frac{v_\parallel\left(v_\parallel - U\right)}{(2T/m)}\frac{1}{B_\theta}\frac{d\phi_0}{dr}\right]\sin\theta$$

is related with the parallel velocity and with $E \times B$ velocity, in the inhomogeneous magnetic field.

We note

$$\varepsilon\frac{B_\theta}{B_T}\frac{1}{r}\sin\theta = \frac{B_\theta}{B_T}\frac{1}{R}\sin\theta$$

and we have

$$U + \frac{1}{B_\theta}\frac{d\phi_0}{dr} \sim \text{parallel velocity incl. } E_r \times B_\theta$$

$$2\frac{v_\parallel\left(v_\parallel - U\right)}{(2T/m)}\frac{1}{B_\theta}\frac{d\phi_0}{dr} \sim \text{parallel velocity incl. } E_r \times B_\theta$$

These terms act to produce poloidal advection of the perturbed distribution function f_1.

They are called *magnetic pumping* terms.

The other terms in the RHS are due to the drift: curvature and gradient, and the self-consistent perturbation of the potential on the surface ϕ_1,

$$\begin{aligned}
\frac{T}{eB_\theta}\left[\frac{v_\parallel^2 + \frac{v_\perp^2}{2}}{(T/m)}\sin\theta + \frac{1}{\varepsilon}\frac{e}{T}\frac{\partial\phi_1}{\partial\theta}\right] &\sim \text{drift} + \frac{1}{B_\theta}\frac{\partial\phi_1}{r\partial\theta} \\
&\sim \text{drift} + \frac{E_\theta^{(1)}}{B_\theta}
\end{aligned}$$

See below the comparison.

12.4 First order distribution function and electrostatic potential

The terms due to the parallel velocity in the RHS (*magnetic pumping*)

are larger by a factor

$$\frac{L_n}{\rho_\theta}$$

than the drift + ϕ_1 terms in the RHS.

Then the drift $+\phi_1$ terms are neglected.

The first order distribution function for the species α is

$$\begin{aligned}
f_{\alpha 1} &= h_{\alpha 1}\cos\theta \\
\frac{e\phi_1}{T} &= \varepsilon W\cos\theta
\end{aligned}$$

$$W \equiv \frac{\sum_\alpha Z_\alpha n_\alpha m_\alpha U^2}{\sum_\alpha Z_\alpha^2 n_\alpha T}$$

or

$$W \equiv \frac{\sum_\alpha \left[Z_\alpha n_\alpha m_\alpha\right] U^2}{\sum_\alpha Z_\alpha \ \left[Z_\alpha n_\alpha m_\alpha\right](T/m_\alpha)}$$

is a ratio of the parallel energy U^2 weighted by the charge Z_α to the thermal energy (T/m_α) weighted by the charge.

This W should be small if U is much smaller than the thermal speed of the species-α particles.

Where

$$\begin{aligned} h_{\alpha 1} &= \varepsilon \left(\frac{v_{\parallel} U}{T/m_{\alpha}} - Z_{\alpha} \; W \right) f_{\alpha 0} \\ \alpha &\equiv \text{species, electrons, ions,...} \end{aligned}$$

Interesting.

The first-order f_1 correction to the zeroth distribution function (a $U-$shifted Maxwellian) is

- variable on θ like $\cos\theta$;
- is of the order $\sim \varepsilon$ (toroidality) $\times$ (parallel flow).

Note

the velocity U is here *imposed* from outside (for ex. NBI).

The first order f_1 would be zero if we can assume that the imposed velocity U is zero.

But this is not possible, U cannot be zero.

Constraint in this order

$$\frac{1}{B_{\theta}} \frac{d\phi_0}{dr} + U = 0$$

well known resonance condition of *parallel* flow.

The electrostatic potential ϕ_0 results from ambipolarity condition.

The presence of an imposed parallel velocity U means that there is an electric potential ϕ_0.

See **Kagan Catto enhanced bootstrap**.

End.

From Wang Burrell 1982

"The magnetic pumping terms are typically larger than the magnetic drift terms by a factor

$$\frac{L_n}{\rho_{\theta}}$$

for each species"

The *magnetic pumping* is the variation of the magnitude of the magnetic field along the line, due to the toroidality.

The *magnetic drift* is

$$\mathbf{v}_D^{mag} = \frac{1}{\Omega} \widehat{\mathbf{n}} \times (\mu \boldsymbol{\nabla} B)$$

with neglect of the curvature drift.

12.4.1 The magnetic pumping terms

The group of terms in the equation for the correction to the distribution function

$$\sin\theta \times \varepsilon \frac{B_\theta}{B_T}\frac{1}{r}\left[\frac{v_\perp^2}{v_{th}^2}\left(U + \frac{1}{B_\theta}\frac{d\phi_0}{dr}\right) + \frac{mv_\parallel\left(v_\parallel - U\right)}{T}\frac{1}{B_\theta}\frac{d\phi_0}{dr}\right] f_0$$

arises from *parallel* and $E \times B$ *flow* and are called magnetic pumping terms.

The second term in the square paranthesis

$$\begin{aligned}
& \sin\theta \times \varepsilon \frac{B_\theta}{B_T}\frac{1}{r} \times \frac{mv_\parallel\left(v_\parallel - U\right)}{T}\frac{1}{B_\theta}\frac{d\phi_0}{dr} f_0 \\
\rightarrow \quad & \sin\theta \times \varepsilon \frac{B_\theta}{B_T}\frac{1}{r}\frac{1}{B_\theta}\frac{d\phi_0}{dr} \times v_\parallel \times \frac{\partial}{\partial v_\parallel}\exp\left[-\frac{\left(v_\parallel - U\right)^2}{(2T/m)}\right] \\
\rightarrow \quad & \sin\theta \frac{1}{R}\left(\frac{1}{B_T}\frac{d\phi_0}{dr}\right) \times v_\parallel \frac{\partial f_0}{\partial v_\parallel}
\end{aligned}$$

we recall that $d\phi_0/dr$ is the radial electric field, and with B_T gives a poloidal velocity, $\sim v_E$.

Now, from the equation of motion

$$\frac{dv_\parallel}{dt} = -\left(\frac{v_\perp^2}{2}\right)\frac{B_\theta}{B_T}\frac{\sin\theta}{R} + v_\parallel \frac{1}{B_0}\left(\frac{d\phi_0}{dr}\right)\frac{\sin\theta}{R} - \frac{e}{m}\frac{B_\theta}{B_T}\frac{\partial\phi_1}{r\partial\theta}$$

we retain the second term

$$\left(\frac{dv_\parallel}{dt}\right)^{electric} = v_\parallel \frac{1}{B_0}\left(\frac{d\phi_0}{dr}\right)\frac{\sin\theta}{R}$$

and the two expressions (*i.e.* second term in the square paranthesis above) are coincident since they represent the *energetic* effect along the parallel motion

$$\left(\frac{dv_\parallel}{dt}\right)^{electric}\frac{\partial f_0}{\partial v_\parallel}$$

where the particle parallel motion $v_\parallel$ is against the *parallel* electric field (originated from ϕ_0 and B_θ).

It is the energetic *magnetic mirror* effect.

12.4.2 The drift terms

The second group of terms is

$$\begin{aligned}
& \varepsilon \frac{B_\theta}{B_T}\frac{1}{r}\frac{T}{eB_\theta}\left[\frac{m}{T}\left(v_\parallel^2 + \frac{v_\perp^2}{2}\right)\sin\theta + \frac{1}{\varepsilon}\frac{e}{T}\frac{\partial\phi_1}{\partial\theta}\right] \\
\times & \left[\frac{d\ln n}{dr} + \left(\frac{m\left(v_\parallel - U\right)^2}{2T} + \frac{mv_\perp^2}{2T} - \frac{3}{2}\right)\frac{d\ln T}{dr} + \frac{m\left(v_\parallel - U\right)}{T}\frac{dU}{dr}\right]
\end{aligned}$$

This comes from *curvature* and *gradient drifts*. It contains ϕ_1 which is the variation of the electrostatic potential in the magnetic surface.

$$\begin{aligned}
& \varepsilon \frac{B_\theta}{B_T}\frac{1}{r}\frac{T}{eB_\theta}\left[\frac{m}{T}\left(v_\parallel^2+\frac{v_\perp^2}{2}\right)\sin\theta+\frac{1}{\varepsilon}\frac{e}{T}\frac{\partial\phi_1}{\partial\theta}\right] \\
= & \frac{1}{R}\frac{1}{B_T}\frac{T}{e}\times\frac{m}{T}R\left[\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{R}\sin\theta\times\frac{\Omega}{\Omega}+\frac{e}{m}\frac{\partial\phi_1}{r\partial\theta}\right] \\
= & \frac{1}{B_T}\frac{m}{e}\left[\frac{eB}{m}\ v_r^{drift}+\frac{e}{m}\frac{\partial\phi_1}{r\partial\theta}\right]\approx v_r^{drift}+\frac{1}{B}\frac{\partial\phi_1}{r\partial\theta} \\
= & v_r^{drift}+v_r^{\widetilde{E}} \qquad \text{(radial velocity)}
\end{aligned}$$

This factor is the radial advection (by drift and poloidal electric field) of the other factor, which consists of radial gradients of n, of T and of the flow velocity U

$$\left(v_r^{drift}+v_r^{\widetilde{E}}\right)\frac{\partial f_0}{\partial r}$$

12.4.3 Comparison *mirror* to *drift* terms

Now we want to make a comparison between the two terms.

For this we simplify $U=0$, and take $\phi_1\equiv 0$, and write

$$\begin{aligned}
& \sin\theta\times\left[\frac{v_\perp^2}{v_{th}^2}\frac{1}{B_\theta}\frac{d\phi_0}{dr}+\frac{mv_\parallel^2}{T}\frac{1}{B_\theta}\frac{d\phi_0}{dr}\right] \\
= & \sin\theta\times\frac{1}{B_\theta}\frac{d\phi_0}{dr}\frac{m}{T}\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)
\end{aligned}$$

and the second group of terms

$$\sin\theta\times\frac{T}{eB_\theta}\frac{m}{T}\left(v_\parallel^2+\frac{v_\perp^2}{2}\right)\frac{1}{L_n}\left[1+\eta\left(\frac{v^2}{v_{th}^2}-\frac{3}{2}\right)\right]$$

and we recall that

$$\frac{T}{eB_\theta}\frac{1}{L_n}\left[1+\eta\left(\frac{v^2}{v_{th}^2}-\frac{3}{2}\right)\right]=\frac{B}{B_\theta}v_T^{dia}$$

then

$$\sin\theta\times\frac{m}{T}\left(v_\parallel^2+\frac{v_\perp^2}{2}\right)\frac{B}{B_\theta}v_T^{dia}$$

Now we have to compare

$$\begin{aligned}
& \frac{\text{(mirror, energetic)}}{\text{(drift, convection)}}\sim\frac{\sin\theta\times\frac{1}{B_\theta}\frac{d\phi_0}{dr}\frac{m}{T}\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)}{\sin\theta\times\frac{m}{T}\left(v_\parallel^2+\frac{v_\perp^2}{2}\right)\frac{B}{B_\theta}v_T^{dia}} \\
= & \frac{\frac{1}{B_\theta}\frac{d\phi_0}{dr}}{\frac{B}{B_\theta}v_T^{dia}}=\frac{\frac{1}{B}\frac{d\phi_0}{dr}}{v_T^{dia}}
\end{aligned}$$

then we have to compare the poloidal electric $E \times B$ velocity due to radial variation of the electric potential, with the diamagnetic velocity.

Note For us the rotation velocity V_E is greater than the diamagnetic velocity

$$V_E \gg v_{dia}$$

(see *rho-effective*) but the diamagnetic velocity is steadly increasing $L_n \rightarrow \rho_\theta$.

$$\frac{\frac{1}{B}\frac{d\phi_0}{dr}}{v_T^{dia}} \sim \frac{V_E}{v_{dia}} \sim \frac{L_n}{\rho_\theta}$$

When

$$\frac{1}{B}\frac{d\phi_0}{dr} \sim v_T^{dia}$$

$$\begin{aligned} \frac{\rho_\theta}{L_n} &\rightarrow 1 \\ \text{then } 1 - \frac{v_{dia}}{V_E} &\rightarrow 0 \ \text{ and } \ \frac{1}{\rho_{eff}^2} \rightarrow 0 \end{aligned}$$

Reaching comparative magnitudes $L_n \searrow \rho_\theta$.

For the distribution function

a the time variation due to energetic effect of mirror (magnetic modulation)

is equal to

b the time variation due to the convection by drift velocity

End

Wang Burrell asserts that the first group of terms ("pumping", energetic, mirror, $\left(\frac{dv_\parallel}{dt}\right)^{electric} \frac{\partial f_0}{\partial v_\parallel}$) is larger than the second group of terms ("drift", advection of equilibrium gradients, $\left(v_r^{drift} + v_r^{\widetilde{E}}\right) \frac{\partial f_0}{\partial r}$) as

$$\frac{(\text{mirror})}{(\text{drift})} \sim \frac{\left(\frac{dv_\parallel}{dt}\right)^{electric} \frac{\partial f_0}{\partial v_\parallel}}{\left(v_r^{drift} + v_r^{\widetilde{E}}\right) \frac{\partial f_0}{\partial r}} \sim \frac{L_n}{\rho_\theta}$$

It is difficult to understand why.

For trapped part of the velocity space is OK. The modulations due to the magnetic mirror is substantial, it turns back $v_\parallel$.

NOTE
This is however *favorable* for the idea of *rho-effective*.
Here seems to involve the equality between L_n and ρ_θ.
END

12.5 The two constraints: neutrality and ambipolarity

The neutrality

$$\sum e \int d^3v \; f_1 = 0$$

This equation will determine ϕ_1, the variation of the electrostatic potential on the surface.

The ambipolarity.

We have

$$\Gamma = \int \frac{d\theta}{2\pi} \; h \; \int d^3v \; (\mathbf{v}_D + \mathbf{v}_E) \cdot \widehat{\mathbf{e}}_r \; f$$

Ambipolarity

$$\sum Z_\alpha e \Gamma_\alpha = 0$$

is an equation for ϕ.

"To maintain neutrality the radial electric field will acquire time dependence

$$\frac{\partial E_r}{\partial t}$$

producing a *polarization* current which will completely cancel the radial current $\sum e\Gamma$.

The polarization current is

$$\begin{aligned} & \text{polarization current} \\ = \quad & -\sum_\alpha m_\alpha n_\alpha \frac{1}{B_0^2} \frac{\partial}{\partial t} \frac{d\phi}{dr} \end{aligned}$$

Assume

$$\sum e\Gamma \sim nev_{drift} \; \varepsilon$$

taking equality it results

$$\frac{\partial}{\partial t} \sim \omega^{ion-transit}$$

This time scale cannot be present in a *transport* calculation."

This implies that the ambipolarity is not automatic and must be imposed and the equation will provide a value for $\phi_0(r)$.

The existence of a potential on surfaces $\phi_0(r)$ implicitely makes U non-zero.

Then both $h_{\alpha 1}$ and ϕ_1 cannot be zero since U and W cannot be zero.

12.6 Extend the first order

When the expressions like

$$\begin{aligned} f_{\alpha 1} &= h_{\alpha 1}\cos\theta \\ &\ldots \end{aligned}$$

have been assumed we actually neglected the effect of the drift $+\phi_1$ terms (the second set of terms in RHS).

Now we advance beyond this level of approximation and add corrections

$$f_{\alpha 1} = h_{\alpha 1}\cos\theta + g_{\alpha 1}$$

$$\frac{e\phi_1}{T} = e\ W\ \ \cos\theta + \widetilde{\phi}_1$$

$$\frac{1}{B_\theta}\frac{d\phi_0}{dr} + U = u$$

Note that u is a small corection to the basic equilibrium which consists of cancelling the parallel flow (the *resonance* $\frac{1}{B_\theta}\frac{d\phi_0}{dr} + U = 0$), **Kagan Catto**.

Note that $\widetilde{\phi}_1$, the supplement, actually is normalized to e/T.

the next step is the change to the *moving referential*

$$\begin{aligned} v_\parallel &\rightarrow v'_\parallel = v_\parallel - U \\ w' &= \frac{1}{2}v'^2_\parallel + \frac{1}{2}v^2_\perp \quad \text{(new energy)} \end{aligned}$$

The equation becomes

$$\begin{aligned} &\left(v'_\parallel + u\right)\frac{B_\theta}{B_\varphi}\frac{\partial g_{\alpha 1}}{r\partial\theta} \\ &+\left[v'_\parallel - \frac{T/m_\alpha}{\left(\frac{Z_\alpha e B_\theta}{m_\alpha}\right)}\left(\frac{d}{dr}\ln n_\alpha + \overline{\frac{w'}{T/m_\alpha} - \frac{3}{2}}\frac{d}{dr}\ln T + \frac{v'_\parallel}{T/m_\alpha}\frac{dU}{dr}\right)\right]\frac{B_\theta}{B_\varphi}Z_\alpha\frac{\partial\widetilde{\phi}_1}{r\partial\theta}f_{\alpha 0} \\ &-C^{lin}\left[g_{\alpha 1}\right] \\ = \ &\varepsilon\frac{1}{r}\frac{B_\theta}{B_\varphi}H_\alpha\left[u + \frac{T/m_\alpha}{\left(\frac{Z_\alpha e B_\theta}{m_\alpha}\right)}\left(\frac{d}{dr}\ln n_\alpha + \overline{\frac{w'}{T/m_\alpha} - \frac{3}{2}}\frac{d}{dr}\ln T + \frac{v'_\parallel}{T/m_\alpha}\frac{dU}{dr}\right)\right]f_{\alpha 0}\ \sin\theta \end{aligned}$$

with the notation

$$H_\alpha \equiv \frac{1}{T/m_\alpha}\left(v'^2_\parallel + \frac{v^2_\perp}{2}\right) + \frac{U^2}{T/m_\alpha} - Z\ W + \frac{v'_\parallel}{T/m_\alpha}\frac{dU}{dr}$$

The velocity $v'_\parallel$ is measured in the referential moving with the flow velocity U. Here the first term $\frac{1}{T/m_\alpha}\left(v'^2_\parallel + \frac{v^2_\perp}{2}\right)$ in H_α compares the energy in the moving frame with the thermal energy.

The next term $\frac{U^2}{T/m_\alpha}$ in H_α is a ratio between the energy of the flow to the thermal energy.

If we will have this H_α in combination with other factors, we would see

$$H_\alpha \to \frac{1}{v_{th}^2} R\Omega \times \frac{1}{\Omega} \frac{v_\parallel'^2 + \frac{v_\perp^2}{2}}{R}$$

which is

$$\frac{R}{\rho_s} \frac{1}{v_{th}} \times (v_{drift})$$

Let us see

$$\left[v_\parallel' - \frac{T/m_\alpha}{\left(\frac{Z_\alpha e B_\theta}{m_\alpha}\right)} \left(\frac{d}{dr} \ln n_\alpha + \overline{\frac{w'}{T/m_\alpha} - \frac{3}{2}} \frac{d}{dr} \ln T + \frac{v_\parallel'}{T/m_\alpha} \frac{dU}{dr} \right) \right] \frac{B_\theta}{B_\varphi} Z_\alpha \frac{\partial \widetilde{\phi}_1}{r \partial \theta} f_{\alpha 0}$$

The gradients are

$$\left(\frac{d}{dr} \ln n_\alpha + \overline{\frac{w'}{T/m_\alpha} - \frac{3}{2}} \frac{d}{dr} \ln T + \frac{v_\parallel'}{T/m_\alpha} \frac{dU}{dr} \right) \sim \frac{\partial}{\partial r} \ln f_0$$

if we ignore the shear of U. Also

$$\frac{T/m_\alpha}{\left(\frac{Z_\alpha e B_\theta}{m_\alpha}\right)} = \frac{v_{th}^2}{\Omega_{Z,\theta}} \sim v_{th}\ \rho_\theta$$

$$\begin{aligned} & v_\parallel' - v_{th} \left[\rho_\theta \frac{\partial}{\partial r} \ln f_0 \right] \\ \sim\ & V_\parallel \end{aligned}$$

then

$$\begin{aligned} & V_\parallel \times \frac{B_\theta}{B_\varphi} \times Z_\alpha \frac{\partial \widetilde{\phi}_1}{r \partial \theta} \\ =\ & V_\theta\ \widetilde{E}_\theta \\ \to\ & V_\theta\ \widetilde{E}_\theta \times f_0 \end{aligned}$$

We should have $\frac{df_0}{d\epsilon}$. This is an energetic term.

12.7 Approximations in the equation for $g_{\alpha 1}$

It is made the substitution

$$g'_{\alpha 1} = g_{\alpha 1} + Z_\alpha \widetilde{\phi}_1 \ f_{\alpha 0}$$

(remember $\widetilde{\phi}_1$ contains e/T). After this, the neutrality condition is

$$\sum Z_\alpha^2 n_\alpha \widetilde{\phi}_1 = \sum Z_\alpha \int d^3v \ g'_{\alpha 1}$$

and the ambipolarity condition

$$\sum Z_\alpha \Gamma_\alpha = 0$$

which determine the velocity u, *i.e.* the perturbation that prevents, in higher order, the resonance $\frac{1}{B_\theta}\frac{d\phi_0}{dr} + U = 0$.

The explanation of the definition of the new $g'_{\alpha 1}$ results from an approximation that can be done on the equation for $g_{\alpha 1}$.

12.7.1 First term

First term is

$$\left(v'_\parallel + u\right) \frac{B_\theta}{B_\varphi} \frac{\partial g_{\alpha 1}}{r \partial \theta}$$

and in it the velocity u is neglected. Then we have

$$v'_\parallel \frac{B_\theta}{B_\varphi} \frac{\partial g_{\alpha 1}}{r \partial \theta}$$

12.7.2 Second term

In the square paranthesis of the second term

$$v'_\parallel - \frac{T/m_\alpha}{\left(\frac{Z_\alpha e B_\theta}{m_\alpha}\right)} \left(\frac{d}{dr} \ln n_\alpha + \overline{\frac{w'}{T/m_\alpha} - \frac{3}{2}} \ \frac{d}{dr} \ln T + \frac{v'_\parallel}{T/m_\alpha} \frac{dU}{dr} \right)$$

the second part contains at the denominator $\Omega_{\alpha\theta}$ which means that it will contain ρ_θ. It is neglected.

The second term will be

$$v'_\parallel \times \frac{B_\theta}{B_\varphi} Z_\alpha \frac{\partial \widetilde{\phi}_1}{r \partial \theta} f_{\alpha 0}$$

12.8 Collision term

Now adopt a strong simplification of the collision term.

At the end, one takes

$$\nu \to 0+$$

12.9 After approximations

We have for the LHS

$$v_{\|}' \frac{B_\theta}{B_\varphi} \frac{\partial g_{\alpha 1}}{r\partial\theta} + v_{\|}' \times \frac{B_\theta}{B_\varphi} Z_\alpha \frac{\partial \widetilde{\phi}_1}{r\partial\theta} f_{\alpha 0} - C$$

or

$$\begin{aligned} & v_{\|}' \frac{B_\theta}{B_\varphi} \frac{\partial}{r\partial\theta} \left(g_{\alpha 1} + Z_\alpha \widetilde{\phi}_1 \ f_{\alpha 0} \right) \\ \to \quad & v_{\|} \frac{B_\theta}{B_\varphi} \frac{\partial}{r\partial\theta} g_{\alpha 1}' \end{aligned}$$

The drift kinetic equation becomes

$$\begin{aligned} & v_{\|}' \frac{B_\theta}{B_\varphi} \frac{\partial}{r\partial\theta} g_{\alpha 1}' - C^{lin}\left[g_{\alpha 1}'\right] \\ = \quad & \varepsilon \frac{B_\theta}{B_\varphi} \frac{1}{r} H_\alpha \left[u + \frac{T/m_\alpha}{\left(\frac{Z_\alpha e B_\theta}{m_\alpha}\right)} \left(\frac{d}{dr} \ln n_\alpha + \overline{\frac{w'}{T/m_\alpha} - \frac{3}{2}} \frac{d}{dr} \ln T + \frac{v_{\|}'}{T/m_\alpha} \frac{dU}{dr} \right) \right] f_{\alpha 0} \ \sin\theta \end{aligned}$$

To see what is this

$$\begin{aligned} & \varepsilon \frac{B_\theta}{B_\varphi} \frac{1}{r} \times \frac{R}{\rho_s} \frac{1}{v_{th}} \times (v_{drift}) \times \left[u + v_{th} \left(\rho_\theta \frac{d}{dr} \ln f_0 \right) \right] \sin\theta \\ \to \quad & \frac{B_\theta}{B_\varphi} \frac{1}{\rho_s v_{th}} \times \left[v_{drift} \sin\theta \right] \times \left[V_{\|} \right] \\ \to \quad & \frac{1}{\rho_s v_{th}} \times \left[v_{drift}^{radial} \right] \times \left[V_\theta \right] f_0 \end{aligned}$$

and

$$\begin{aligned} & v_{\|}' \frac{B_\theta}{B_\varphi} \frac{\partial}{r\partial\theta} g_{\alpha 1}' \\ \to \quad & v_{\|}' \nabla_{\|} g_{\alpha 1}' \end{aligned}$$

then

$$\begin{aligned} v_{\|}' \nabla_{\|} g_{\alpha 1}' - C^{lin} \quad & \sim \quad \frac{1}{\rho_s v_{th}} \times \left[v_{drift}^{radial} \right] \times \left[V_\theta \right] f_0 \\ & \sim \quad (\text{velocity}) \times \frac{f_0}{\rho_s} \end{aligned}$$

The order of magnitude

$$\frac{g'_{\alpha 1}}{f_{\alpha 0}} \sim \varepsilon \frac{\rho_\theta}{L_n}$$

The transport fluxes

$$\Gamma = \int \frac{d\theta}{2\pi}\ h\ \int d^3v\ \left(\mathbf{v}_D + \mathbf{v}_E\right) \cdot \widehat{\mathbf{e}}_r\ f$$

$$\Gamma_\alpha = -\frac{T/m_\alpha}{\Omega_{0\alpha}} \frac{1}{R} \int \frac{d\theta}{2\pi} \sin\theta \int d^3v\ \left(H_\alpha\ g'_{\alpha 1} + \frac{Z_\alpha}{\varepsilon \sin\theta} \frac{\partial \widetilde{\phi}_1}{\partial\theta}\ g_{\alpha 1} \right)$$

Here will be replaced the solution $g'_{\alpha 1}$ obtained with P and δ below.

12.10 Comments on the contributions

The density gradients enter in the equations through the combination

$$u + \frac{T}{Z_\alpha e B_\theta} \frac{d}{dr} \ln n_\alpha$$

The velocity-deffect u will be determined from the *condition of ambipolarity*. Then in the expression of the flux there will be a combination like

$$\frac{1}{Z_\alpha} \frac{d}{dr} \ln n_\alpha - \frac{1}{Z_\beta} \frac{d}{dr} \ln n_\beta$$

In other treatments this combination occurs due to conservation of momentum in the Coulomb collisions.

12.11 Solution

Solution of the equation

$$v'_\parallel \frac{B_\theta}{B_T} \frac{\partial g'_{\alpha 1}}{r\partial\theta} + \nu g'_{\alpha 1} = A\left(v'_\parallel, v_\perp^2\right) f_0 \sin\theta$$

is

$$g'_{\alpha 1} = \frac{r B_\varphi}{B_\theta} A\ f_0 \left[-P\left(\frac{1}{v'_\parallel}\right) \cos\theta + \pi\delta\left(v'_\parallel\right) \sin\theta \right]$$

The principal part to be taken at the integration over the variable $v_\parallel$.

Similar to **Galeev** and with **Rozhansky Tendler**.

Note that this is *drift kinetic* and is NOT an instability kinetic equation.

See **Rutherford1970**.

What shows the formal solution.

It contains a $\delta\left(v_{\|}'\right)$.

Therefore the transport comes exclusively from the particles that correspond to the resonance condition

$$v_{\|}' = v_{\|} - U = 0$$

which is selected by the δ function.

Using this expression for the distribution function $g_{\alpha 1}'$ one can calculate the fluxes of transport.

First for every species one introduces

$$z_{\alpha} \equiv \frac{U^2}{2T/m_{\alpha}} - \frac{1}{2} Z_{\alpha}\ W$$

This variable is a measure of the magnitude of the energy in the parallel flow U compared with the *thermal* energy.

It is the two middle terms in H_{α} (the last in H_{α} is the shear of the parallel flow, and the first is the relative magnitude of the energy in the moving frame to the thermal energy).

Then, with

$$a_{\alpha} = 1 + 2z_{\alpha} + 2z_{\alpha}^2$$
$$b_{\alpha} = \frac{3}{2} + z_{\alpha} - z_{\alpha}^2$$
$$c_{\alpha} = 3 + 4z_{\alpha} + 2z_{\alpha}^2$$

and

$$v_{th,\alpha} = \sqrt{\frac{2T}{m_{\alpha}}}$$

The transport fluxes are

$$\Gamma_{\alpha} = -n_{\alpha} \frac{T/m_{\alpha}}{\Omega_{0\alpha}} \frac{1}{R} \varepsilon \frac{1}{v_{th,\alpha}} \sqrt{\pi} \left[a_{\alpha} \left(u + \frac{T/m_{\alpha}}{\Omega_{\theta\alpha}} \frac{d}{dr} \ln n_{\alpha} \right) + b_{\alpha} \frac{T/m_{\alpha}}{\Omega_{\theta\alpha}} \frac{d}{dr} \ln T \right]$$

$$\frac{\widetilde{Q}}{T} =$$

$$\Pi = \sum_{\alpha} m_{\alpha} \Gamma_{\alpha} U$$

We note that in these expressions occurs u which is the departure from the resonance and which is NOT yet determined. Only ambipolarity will determine it.

Particular case, only ions Z_i , m_i.

$$\Gamma_i = 0 \quad \text{the ambipolarity}$$

(neglect of electron fluxes).

From Γ_i

$$\left[a_\alpha \left(u + \frac{T/m_\alpha}{\Omega_{\theta\alpha}} \frac{d}{dr} \ln n_\alpha \right) + b_\alpha \frac{T/m_\alpha}{\Omega_{\theta\alpha}} \frac{d}{dr} \ln T \right] = 0$$

$$U + \frac{1}{B_\theta} \frac{d\phi}{dr} + \frac{T/m_i}{\Omega_{\theta i}} \left(\frac{d}{dr} \ln n_i + K \frac{d}{dr} \ln T \right) = 0$$

where

$$K \equiv \frac{b_i}{a_i}$$

For no parallel flow $U = 0$

$$K \to \frac{3}{2}$$

then the ambipolarity which is simply $\Gamma_{ion} = 0$ gives for the toroidal flow velocity of the ions

$$U_{ion} = -\frac{1}{B_\theta} \frac{d\phi}{dr} - \frac{T/m_i}{\Omega_{\theta i}} \left(\frac{d}{dr} \ln n_i + \frac{3}{2} \frac{d}{dr} \ln T \right)$$

When however U is known (and it fulfills the ambipolarity constraint) it can be taken as input in the previous formula to calculate the electrostatic potential ϕ_0.

Then with $\frac{d\phi_0}{dr}$ replaced in the expression of $\Gamma_{electrons}$ one has

$$\begin{aligned} & \Gamma^{ambipolar} \\ = & -\frac{\sqrt{\pi}}{4} \left(1 + \frac{1}{Z_i} \right) q^2 \omega_{transit}^{elect} \rho_e^2 \\ & \times \left[\left(1 + \frac{2 z_{ion}}{Z_i} \right) \frac{d}{dr} \ln n_e + \left(\frac{3}{2} + \frac{3 - Z_i}{1 + Z_i} \frac{z_{ion}}{Z_i} \right) n_e \frac{d}{dr} \ln T \right] \end{aligned}$$

$$\begin{aligned} z_{ion} &= \frac{1}{1 + Z_i} \frac{V^2}{2T/m_i} \\ \omega_{transit}^{elect} &= \frac{B_\theta}{B_T} \frac{\overline{v}_e}{r} \end{aligned}$$

It is also obtained the perturbation of the potential, with harmonic components $\sin\theta$ and $\cos\theta$,

$$\begin{aligned} \widetilde{\phi}_1 = & \frac{\pi\varepsilon}{1 + Z_i} \frac{T/m_i}{\Omega_{\theta i}} \left[\frac{1}{\sqrt{\pi}} \left(\frac{1}{2} - z_i - (1 + 2z_i) K \right) \frac{d}{dr} \ln T \, \frac{1}{v_{th,i}} \times \sin\theta \right. \\ & \left. + \frac{U}{T/m_i} \left(3 \frac{d}{dr} \ln T - 2 \frac{d}{dr} \ln U \right) \times \cos\theta \right] \end{aligned}$$

The $\sin\theta$ part is most important.

Then $\widetilde{\phi}_1 \sim \sin\theta$.

Important case.

The presence of a single impurity species (I) beside the background ions and electrons.

Negligible terms

$$\frac{U^2}{2T/m_e} \ll 1$$

$$\frac{U^2}{2T/m_i} \ll 1$$

Define

$$\gamma \equiv \sqrt{\frac{m_I}{m_i}}\ \frac{n_I}{n_i}$$

$$\alpha_I \equiv \frac{Z_I^2 n_I}{n_e + Z_i^2 n_i}$$

$$y \equiv \frac{U^2}{2T/m_I}$$

The variables become

$$z_i = -\frac{Z_i}{Z_I}\frac{\alpha_I}{1+\alpha_I} y$$

$$z_I = \frac{1}{1+\alpha_I} y$$

The ambipolar condition

$$Z_i \Gamma_i + Z_I \Gamma_I = 0$$

becomes

$$\begin{aligned}
& U + \frac{1}{B_\theta}\frac{d\phi}{dr} \qquad \text{(should be 0 at resonance)} \\
& + \frac{T/m_i}{\Omega_{\theta i}}\ \frac{1}{a_i + \gamma a_I}\left[\frac{a_i}{Z_i}\frac{d}{dr}\ln n_i + \gamma\frac{a_I}{Z_I}\frac{d}{dr}\ln n_I + \left(\frac{b_i}{Z_i} + \gamma\frac{b_I}{Z_I}\right)\frac{d}{dr}\ln T\right] \\
= \ & 0
\end{aligned}$$

This constraint allows to determine the velocity u.

The flux is then

$$\begin{aligned}
Z_I \Gamma_I &= -\frac{\sqrt{\pi}}{4} Z_i^2 n_i q^2\ \omega_{ti}\ \rho_i^2 \\
& \times \left[F_1\left(\frac{1}{Z_I}\frac{1}{n_I}\frac{dn_i}{dr} - \frac{1}{Z_i}\frac{d}{dr}\ln n_i\right) + F_2\frac{d}{dr}\ln T\right]
\end{aligned}$$

$$F_1 = \gamma\frac{a_I a_i}{a_i + \gamma a_I}$$

$$
\begin{aligned}
F_2 &= \gamma \frac{a_I a_i}{a_i + \gamma a_I} \\
&\quad \times \left(\frac{1}{Z_I} \frac{b_I}{a_I} - \frac{1}{Z_i} \frac{b_i}{a_I} \right)
\end{aligned}
$$

(to be checked).

F_1 is positive, F_2 is negative.

Assume that the impurities are very rare, just a trace.

Then $\frac{n_I}{n_i} \ll 1$ and is neglected.

The radial flux of the impurity atoms is

$$
\begin{aligned}
\Gamma_I &= -\frac{\sqrt{\pi}}{4} q^2 \omega_{tI}\ \rho_I^2\ \times a_I \times \left[\left(\frac{d}{dr} \ln n_I - \frac{Z_I n_I}{Z_i n_i} \frac{d}{dr} \ln n_i \right) \right. \\
&\qquad \left. - \left(\frac{3}{2} \frac{Z_I}{Z_i} - \frac{b_I}{a_I} \right) n_I \frac{d}{dr} \ln T \right]
\end{aligned}
$$

Here one observes the *<u>inward</u> convection of impurities due to the gradient of temperature.*

$$
\Gamma_I \sim \frac{\sqrt{\pi}}{4} q^2 \omega_{tI}\ \rho_I^2\ \times a_I \times \left[\left(\frac{3}{2} \frac{Z_I}{Z_i} - \frac{b_I}{a_I} \right) n_I \frac{d}{dr} \ln T \right]
$$

Both the main ion density n_i gradient and the temperature gradient $\frac{d}{dr} \ln T$ correspond to *inward* convection

$$
\begin{aligned}
F_1 &> 0 \\
F_2 &< 0
\end{aligned}
$$

Possibly this is what is called *temperature screening* for impurities.

We need a physical picture of how the ambipolarity (determination of u) leads to the final signs in Γ_I.

13 Ion - impurity collsional coupling (Hirshman Sigmar Clarke)

This is also in *plasma general drift kinetic solutions.*

See also **Helander3999**.

Very good for qualitative analysis of the regimes of collisionality and relative exchange of momentum between impurities, main ions and electrons.

The calculations are repeated below.

The equation for the species a is

$$\begin{aligned}
& \frac{\partial f_a}{\partial t}+\left(v_{\|}\widehat{\mathbf{n}}+\mathbf{v}_{d,a}\right)\cdot\boldsymbol{\nabla} f_a \\
& +\left[\mu\frac{\partial B}{\partial t}+\frac{e_a}{m_a}\left(v_{\|}E_{\|}^{(A)}+\frac{\partial\phi}{\partial t}\right)\right]\frac{\partial f_a}{\partial\epsilon} \\
= & \sum_b C_{ab}\left(f_a,f_b\right)
\end{aligned}$$

where

$$\epsilon=\frac{1}{2}v^2+\frac{e_a}{m_a}\phi$$

with

$$e_a=Z_a\left|e\right|$$

$$\begin{aligned}
\mathbf{E}^{(A)} &= E_{\varphi 0}\frac{R_0}{R}\widehat{\mathbf{e}}_{\varphi} \\
&\sim \frac{1}{h}
\end{aligned}$$

the toroidal electric field produced by induction by the tokamak transformer.

We **NOTE**

there is a term

$$\mu\frac{\partial B}{\partial t}$$

which is due to the change of the magnetic surfaces. The magnetic surfaces are modified as a *transient* phenomenon, in few situations

- rise of the main plasma current, in the initial phases of the discharge
- damping of the poloidal rotation, by TTMP
- enhanced rotation, produced by external mechanism, like NBI, ICRH
- a radial current exists; this is also transient and is *non-ambipolarity*.

This term is work done by a particle when the magnetic field changes, while μ is conserved. It changes the distribution function in the velocity space.

The drift velocity

$$\begin{aligned}
\mathbf{v}_{d,a} &= -v_{\|}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\left(\frac{v_{\|}}{\Omega_{ca}}\right) \\
&\quad+\frac{v_{\|}^2}{\Omega_{ca}}\frac{1}{B}\left(\boldsymbol{\nabla}\times\mathbf{B}\right)_{\perp}
\end{aligned}$$

The last term is

$$\sim \frac{1}{\Omega_{ca}} v_{\parallel}^{2} \frac{J_{\perp}}{B}$$

and is a drift of particles which exists when there is a current perpendicular to the magnetic field.

A more detailed expression for $\mathbf{v}_D$ is given by **Shaing Domiguez resonant viscosity**.

Note we cannot use $\widehat{\mathbf{n}} \times \boldsymbol{\nabla} f \rightarrow \boldsymbol{\nabla} \times (\widehat{\mathbf{n}} f)$ as in two-dimensions because $\widehat{\mathbf{n}}$ depends on space here. **End.**

NOTE that this can also be

$$\frac{1}{\Omega} \frac{\partial V_E}{\partial t} \quad \text{polarization drift}$$

that will lead to polarization current. **END**

Since the drift velocity will multiply the spatial gradient of the zero-order distribution function,

$$\mathbf{v}_{d,a} \cdot \boldsymbol{\nabla} f_{a0}$$

and since $f_{a0} = f_{Ma}$ has only variation in a direction perpendicular on the magnetic surface

$$\boldsymbol{\nabla} f_{Ma} = \boldsymbol{\nabla} \psi \frac{\partial f_{Ma}}{\partial \psi}$$

we will have to use

$$\begin{aligned} \mathbf{v}_{d,a} \cdot \boldsymbol{\nabla} \psi &= v_{\parallel} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \left(I \frac{v_{\parallel}}{\Omega_{ca}} \right) \\ &= v_{\parallel} \nabla_{\parallel} \left(I \frac{v_{\parallel}}{\Omega_{ca}} \right) \end{aligned}$$

(in this form the term $\mathbf{v}_{d,a} \cdot \boldsymbol{\nabla} f$ which actually is $\mathbf{v}_{d,a} \cdot \boldsymbol{\nabla} \psi \frac{\partial f_{Ma}}{\partial \psi}$ can be coupled to $v_{\parallel} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} f_{a1}$ they have the same *operator* $v_{\parallel} \nabla_{\parallel}$ which is factored).

The expansion in neoclassical small parameter ρ_{θ}/L. No collisions here

$$f_a = f_{a0} + f_{a1} + ...$$

$$f_{a0} = f_{Ma} = n_a(\psi) \frac{1}{\left[\pi \frac{2T_a(\psi)}{m_a}\right]^{3/2}} \exp\left[-\frac{\epsilon}{\frac{T_a(\psi)}{m_a}} \right]$$

$$n_a(\psi) = n_{a0} \exp\left[\frac{e_a \phi}{T_{a0}(\psi)} \right]$$

The equation for the first order

$$
\begin{aligned}
& v_{\|} \nabla_{\|} f_{a1} + v_{\|} \nabla_{\|} \left(I \frac{v_{\|}}{\Omega_{ca}} \frac{\partial f_{a0}}{\partial \psi} \right) \\
& - v_{\|} \frac{e_a E^{(A)}}{T_a} f_{a0} \\
= & \sum_b C_{ab} \left(f_{a1}, f_{b1} \right)
\end{aligned}
$$

Note that here we have taken $\partial f_{a0}/\partial\psi$ inside the paranthesis that comes from the drift velocity. Then it is convenient to separate from f_{a1} the part resulting from the drift motion advection of the equilibrium f_{a0} distribution function. This part is called by **Hirshman1976** *diamagnetic*.

$$
\begin{aligned}
f_{a1} &= -I \frac{v_{\|}}{\Omega_{ca}} \frac{\partial f_{a0}}{\partial \psi} \\
& \quad + g_a \left(\epsilon, \mu, \psi \right)
\end{aligned}
$$

where (**Hirshman Sigmar Clarke**)

$$
\begin{aligned}
-I \frac{v_{\|}}{\Omega_{ca}} \frac{\partial f_{a0}}{\partial \psi} &\equiv \text{diamagnetic response of the species } a, \ \sim \frac{\rho_\theta}{L_n} \\
g_a &\equiv \text{collisional response of the species } a, \text{ only in passing}
\end{aligned}
$$

Again the separation of **Rutherford**,

- a part which comes from neoclassical effects and is a series of functions f in orders of the neoclassical spatial small parameter ρ_θ/L and
- a part that comes from collisions and is a series of functions g in orders of collisionality parameter

Consider the surface average

$$
\langle A(\mathbf{x}) \rangle = \frac{2\pi}{\frac{\partial V}{\partial \psi}} \oint d\chi \frac{A}{\boldsymbol{\nabla}\chi \cdot \mathbf{B}}
$$

$$
\frac{\partial V}{\partial \psi} = 2\pi \oint d\chi \frac{1}{\boldsymbol{\nabla}\chi \cdot \mathbf{B}}
$$

Returning to the drift-kinetic equation, we have

$$
v_{\|} \nabla_{\|} \left(f_{a1} + I \frac{v_{\|}}{\Omega_{ca}} \frac{\partial f_{a0}}{\partial \psi} \right) = v_{\|} \frac{e_a E^{(A)}}{T_a} f_{a0} + \sum_b C_{ab} \left(f_{a1}, f_{b1} \right)
$$

We divide by $v_\parallel$ and note that the surface average is applied to

$$\begin{aligned}
& \oint d\chi \frac{1}{\mathbf{B}\cdot\boldsymbol{\nabla}\chi}\nabla_\parallel () \\
= & \oint d\theta \frac{1}{B_\theta \frac{1}{r}} \frac{d}{dl_\parallel} ()
\end{aligned}$$

but

$$dl_\parallel = \frac{B}{B_\theta} dl_\theta = r d\theta \frac{B}{B_\theta}$$

Then

$$\frac{d}{dl_\parallel} = \frac{B_\theta}{B}\frac{d}{rd\theta} = \frac{1}{qR}\frac{\partial}{\partial\theta}$$

and

$$\begin{aligned}
\oint d\theta \frac{1}{B_\theta \frac{1}{r}} \frac{d}{dl_\parallel} () &= \oint d\theta \frac{1}{B_\theta \frac{1}{r}} \frac{B_\theta}{B}\frac{d}{rd\theta} () \\
&= \oint d\theta \frac{1}{B}\frac{d}{d\theta} ()
\end{aligned}$$

The magnitude of B is

$$B \approx \frac{B_0}{h} = B_0 \frac{1}{1+\varepsilon\cos\theta}$$

and B is a function of θ. But it is of order ε and multiples a quantity which is also of order ε. Then $\frac{1}{B}$ can be inserted in the paranthesis

$$\begin{aligned}
\nabla_\parallel \left(f_{a1} + I\frac{v_\parallel}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial\psi}\right) &= \frac{1}{v_\parallel}\left[v_\parallel \frac{e_a E^{(A)}}{T_a} f_{a0} + \sum_b C_{ab}\left(f_{a1}, f_{b1}\right)\right] \\
\frac{B_\theta}{B}\frac{d}{rd\theta} \left(f_{a1} + I\frac{v_\parallel}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial\psi}\right) &= \frac{1}{v_\parallel}\left[v_\parallel \frac{e_a E^{(A)}}{T_a} f_{a0} + \sum_b C_{ab}\left(f_{a1}, f_{b1}\right)\right] \\
B_\theta\frac{d}{rd\theta} \left(f_{a1} + I\frac{v_\parallel}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial\psi}\right) &= \frac{B}{v_\parallel}\left[v_\parallel \frac{e_a E^{(A)}}{T_a} f_{a0} + \sum_b C_{ab}\left(f_{a1}, f_{b1}\right)\right]
\end{aligned}$$

Now we take the surface average

$$\left\langle B_\theta\frac{d}{rd\theta} \left(f_{a1} + I\frac{v_\parallel}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial\psi}\right)\right\rangle = \left\langle \frac{B}{v_\parallel}\left[v_\parallel \frac{e_a E^{(A)}}{T_a} f_{a0} + \sum_b C_{ab}\left(f_{a1}, f_{b1}\right)\right]\right\rangle$$

In the left hand side we replace

$$\begin{aligned}
\langle ... \rangle &= \frac{2\pi}{V'}\oint d\theta \frac{1}{B_\theta \frac{1}{r}} (...) \\
\left\langle B_\theta\frac{d}{rd\theta} \left(f_{a1} + I\frac{v_\parallel}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial\psi}\right)\right\rangle &= \frac{2\pi}{V'}\oint d\theta \frac{1}{B_\theta \frac{1}{r}} B_\theta \frac{d}{rd\theta}\left(f_{a1} + I\frac{v_\parallel}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial\psi}\right) \\
&= 0
\end{aligned}$$

due to periodicity.

This is what results

$$\left\langle \frac{B}{v_{\parallel}} \left[v_{\parallel} \frac{e_a E^{(A)}}{T_a} f_{a0} + \sum_b C_{ab}\left(f_{a1}, f_{b1}\right) \right] \right\rangle = 0$$

(remember the calculation of the *bootstrap* current in **Helander ECRH**; the equation is the same).

NOTE that this equation, as equation for f_{a1} looks like the Spitzer problem.

The expression of the collision operator.

HSC write

$$\begin{aligned} C_{ab}\left(f_{a1}, f_{b1}\right) &= \nu_{ab}^{defl} \frac{v_{\parallel}}{B} \frac{\partial}{\partial \mu} \mu v_{\parallel} \frac{\partial}{\partial \mu} f_{a1} \\ &+ \left[\nu_{ab}^{defl} - \nu_{ab}^{slowing} \right] \frac{v_{\parallel} u_{a1}(v)}{v^2} f_{a0} \\ &+ \frac{2 v_{\parallel}}{v_{th,a}^2} r_{ba} \, \nu_{ab}^{slowing} f_{a0} \end{aligned}$$

One can make a different grouping

$$\begin{aligned} & C_{ab}\left(f_{a1}, f_{b1}\right) \\ = \; & \nu_{ab}^{defl} \left[\frac{v_{\parallel}}{B} \frac{\partial}{\partial \mu} \mu v_{\parallel} \frac{\partial}{\partial \mu} f_{a1} + \frac{v_{\parallel} u_{a1}(v)}{v^2} f_{a0} \right] \quad \text{deflection} \\ & + \nu_{ab}^{slowing} \left[\frac{2 v_{\parallel}}{v_{th,a}^2} r_{ba} \; f_{a0} - \frac{v_{\parallel} u_{a1}(v)}{v^2} f_{a0} \right] \quad \text{slowing down} \end{aligned}$$

The definitions

The <u>*deflection*</u> frequency

$$\nu_{ab}^{defl} = \nu_{ab} \frac{\Phi\left(\frac{v}{v_{th,b}}\right) - G\left(\frac{v}{v_{th,b}}\right)}{\left(\frac{v}{v_{th,a}}\right)^3}$$

The <u>*slowing down*</u> frequency

$$\nu_{ab}^{slowing} = 2 \frac{T_{a0}}{T_{b0}} \left(1 + \frac{m_b}{m_a}\right) \nu_{ab} \frac{G\left(\frac{v}{v_{th,b}}\right)}{\left(\frac{v}{v_{th,a}}\right)}$$

These frequencies are functions of the velocity v.

The frequency of collisions

$$\nu_{ab} = \frac{4\pi}{2^{3/2}} \frac{e_a^2 e_b^2}{\sqrt{m_a}} \ln \Lambda \frac{n_{b0}}{T_a^{3/2}}$$

The function

$$\begin{aligned} G(x) &= \frac{\Phi(x) - x\frac{d\Phi(x)}{dx}}{2x^2} \\ &\text{the Chandrasekhar function} \end{aligned}$$

where

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt \exp\left(-t^2\right)$$

In the formulas above it has been introduced

$$\begin{aligned} u_{a1}(v) &= \frac{1}{f_{a0}} \frac{3}{4\pi} \int v_{\parallel} f_{a1} d\Omega \\ d\Omega &\equiv \text{solid angle in velocity space} \end{aligned}$$

$$d\Omega = \pi \sum_{\sigma=\pm1} \frac{Bd\mu}{\epsilon} \frac{1}{\left|v_{\parallel}\right|/v}$$

$$d\Omega = \pi \sum_{\sigma=\pm1} \frac{d\left(\frac{v_{\perp}^2}{2}\right)}{\frac{v^2}{2}} \frac{1}{\sqrt{1-\frac{\lambda}{h}}} = \pi \sum_{\sigma=\pm1} d\left(\frac{v_{\perp}^2}{v^2}\right) \frac{1}{\sqrt{1-\frac{v_{\perp}^2}{v^2}}}$$

the *momentum restoring* coefficient

$$r_{ab} \equiv \frac{\int d^3v \; m_b \nu_{ba}^{slowing} \; f_{b1}}{m_a n_{a0} \left\{\nu_{ab}^{slowing}\right\}}$$

The following operator is introduced

$$\{F_{ab}(v)\} \equiv 2 \int d^3v \; \left(\frac{v_{\parallel}}{v_{th,a}}\right)^2 \; \frac{f_{a0}}{n_{a0}} \; F_{ab}(v)$$

To solve the drift-kinetic equation

$$\lambda \equiv \mu \frac{\left\langle B^2 \right\rangle^{1/2}}{\epsilon - \frac{e_a \phi(\psi)}{m_a}}$$

The quantity

$$\left\langle B^2 \right\rangle^{1/2}$$

can be related with the known gemetrical parameter

$$h = \frac{\left\langle B^2 \right\rangle^{1/2}}{B}$$

Usually we write $B \approx \frac{B_0}{h}$ so we can think of it as

$$\left\langle B^2 \right\rangle^{1/2} \approx B_0$$

Define

$$f_{a1} = \widehat{f}_{a1} f_{a0}$$

It results

$$\begin{aligned}
\widehat{f}_{a1} &= -I v_{\parallel} \frac{1}{\frac{e_a \langle B^2 \rangle^{1/2}}{m_a}} \frac{\left\langle B^2 \right\rangle^{1/2}}{B} \frac{\partial}{\partial \psi} \ln f_{a0} \\
&+ I \frac{1}{\frac{e_a \langle B^2 \rangle^{1/2}}{m_a}} \mathbf{H} \left[V_{\parallel} \frac{\partial f_{a0}}{\partial \psi} \right] \quad (\mathbf{H} : \text{does NOT exist for trapped}) \\
&+ \mathbf{H} \left[\frac{2 V_{\parallel}}{v_{th,a}^2} \left(\frac{e_a \overline{E}_{\parallel}^{(A)}}{m_a} \frac{1}{\nu_a^{defl}} + \left(1 - \frac{\nu_a^{slow}}{\nu_a^{defl}} \right) \frac{1}{2 x_a^2} \overline{u}_{a1}(v) + \sum_b \overline{r}_{ab} \frac{\nu_{ab}^{slow}}{\nu_{ab}^{defl}} \right) \right]
\end{aligned}$$

The terms that contain the Heaviside function **H** only exist for circulating particles.

The notations are

$$\frac{\left\langle B^2 \right\rangle^{1/2}}{B} \equiv h$$

$$x_a = \frac{v}{v_{th,a}}$$

$$V_{\parallel}(\lambda) = \int_{\lambda}^{\lambda_c} d\lambda \, \frac{\frac{v^2}{2}}{\left\langle v_{\parallel} \right\rangle}$$

The following parameter gives the limit (in variable λ) of the *circulating* particles

$$\lambda_c = \frac{\left\langle B^2 \right\rangle^{1/2}}{B_c}$$

It is

$$\lambda = \frac{v_{\perp}^2}{v^2} h$$

$$\begin{aligned}
B_c(\psi) &= \text{maximum of } B \text{ in a surface } \psi \\
B &= \frac{B_0}{h} \quad \text{and} \quad h_{\min} = 1 - \varepsilon \rightarrow B_c(\psi) = \frac{B_0}{1 - \varepsilon} \\
\lambda_c &= \frac{B_0}{B_c} = \frac{B_0}{\frac{B_0}{1-\varepsilon}} = 1 - \varepsilon
\end{aligned}$$

For all λ's smaller than λ_c the particles are circulating.

And

$$\begin{aligned}\nu_a^{defl} &= \sum_b \nu_{ab}^{defl} \\ &\quad \text{sum of deflections from other species } b\end{aligned}$$

$$\begin{aligned}\nu_a^{slow} &= \sum_b \nu_{ab}^{slow} \\ &\quad \text{sum of slowing down from other species } b\end{aligned}$$

a new avareging

$$\overline{A} \equiv \left\langle \frac{A}{h} \right\rangle$$

To obtain an explicit expression for $\widehat{f}_{a1}$ we must eliminate u_{a1}. The present form of $\widehat{f}_{a1}$, which contains $\overline{u}_{a1}$ is introduced in the formula for u_{a1} and the integration over the angular space $d\Omega$ is performed. The result is an equation for u_{a1} whose result is

$$\begin{aligned}\frac{\overline{u}_{a1}(v)}{x_a^2} &= -f_T \; I \frac{v_{th,a}^2}{\widehat{\Omega}_a} \frac{\partial}{\partial \psi} \ln f_{a0} \times \frac{\nu_a^{defl}}{\nu_a} \\ &\quad +2f_c \left(\frac{e_a \overline{E}_\parallel^{(A)}}{m_a} \frac{1}{\nu_a} + \sum_b \overline{r}_{ab} \frac{\nu_{ab}^{slow}}{\nu_a} \right)\end{aligned}$$

where the following integration is done over circulating particles, leading to the definition of the fraction of circulating particles f_c

$$\frac{3}{4\pi} \int d\Omega \; v_\parallel \; \mathbf{H}\left[V_\parallel\right] = f_c \; v^2 \frac{1}{h}$$

The proportions of circulating and trapped particles

$$\begin{aligned}f_c &= \frac{3}{4} \int_0^{\lambda_c} \frac{\lambda d\lambda}{\left\langle \sqrt{1-\frac{\lambda}{h}} \right\rangle} \\ f_T &= 1 - f_c \\ &\approx 1.46 \times \sqrt{\varepsilon}\end{aligned}$$

Note this integral is calculated in **Rosenbluth Hazeltine Hinton**. **End.**

Since

$$\lambda = \frac{v_\perp^2}{v^2} h$$

we have

$$\sqrt{1-\frac{\lambda}{h}} = \sqrt{1-\frac{v_\perp^2}{v^2}} = \frac{\left|v_\parallel\right|}{v}$$

$$\left\langle \sqrt{1-\frac{\lambda}{h}} \right\rangle = \left\langle \sigma \frac{v_{\parallel}}{v} \right\rangle$$

The proportion of trapped particles is indicated in general by the degree of spatial modulation of the magnetic field along the line

$$f_T \sim \sqrt{\frac{\delta B}{B}}$$

Comment by **HSK**

For $f_T \ll 1$ the circulating particles couple along the magnetic field to minimize the interspecies friction.

To zeroth order in the fraction of trapped particles f_T the friction produces a *common* flow, the toroidal diamagnetic flow.

The combination of collisional effects: slowing down on circulating and deflection on trapped.

$$\nu_a = f_c \nu_a^{slow} + f_T \nu_a^{defl}$$

The function $\overline{u}_{a1}(v)$ is replaced in the expression of $\widehat{f}_{a1}$.

$$\begin{aligned}
\widehat{f}_{a1} &= -I \frac{1}{\widehat{\Omega}_a} h v_{\parallel} \frac{\partial}{\partial \psi} \ln f_{a0} \\
&\quad + I \frac{1}{\widehat{\Omega}_a} \mathbf{H} \left[V_{\parallel} \right] \frac{\partial}{\partial \psi} \ln f_{a0} \quad \text{(does not exist for trapped)} \\
&\quad + \mathbf{H} \left[\frac{2V_{\parallel}}{v_{th,a}^2} \left(\frac{e_a \overline{E}_{\parallel}^{(A)}}{m_a} \frac{1}{\nu_a} + \sum_b \frac{\nu_{ab}^{slow}}{\nu_a} \overline{r}_{ba} \right) \right] \quad \text{(does not exist for trapped)}
\end{aligned}$$

$$\mathbf{H} \equiv \text{heaviside function}$$

The *restoring coefficients* must still be determined.

Notations

$$\widehat{\Omega}_a = \frac{e_a \left\langle B^2 \right\rangle^{1/2}}{m_a}$$

13.1 Calculation of cross field fluxes

This is transport for the species a due to the drift and due to the collisions.

$$\Gamma_a = \left\langle \int d^3 v \ \left(\mathbf{v}_{Da} \cdot \boldsymbol{\nabla} \psi \right) \ f_a \right\rangle$$

$$Q_a = \left\langle \int d^3 v \ \left(\mathbf{v}_{Da} \cdot \boldsymbol{\nabla} \psi \right) \left[m \epsilon - e_a \Phi \right] \ f_a \right\rangle$$

The flux across the magnetic surface is obtained from the equation for the distribution function f_{a1} by multiplying with

$$I(\psi)\,\frac{v_{\|}}{\Omega_a}$$

This is equivalent to the procedure in the case of fluid equation for conservation of the momentum $n_a m_a \frac{\partial \mathbf{u}_a}{\partial t}$... where the term

$$e_a m_a n_a \mathbf{v} \times \mathbf{B}$$

after vectorial product with $\mathbf{B}$ gives the *drift velocity* generated by the action of a force $\mathbf{F}$,

$$\frac{1}{\Omega_a}\left(\widehat{\mathbf{n}} \times \mathbf{F}\right)$$

The equation

$$\Gamma_a = -I(\psi)\left\langle \frac{\sum_b R_{ab}}{m_a \Omega_a} \right\rangle - I(\psi)\left\langle \frac{n_{a0} E_{\|}^{(A)}}{B} \right\rangle$$

The first term is of the type

$$\frac{(R/m_a) \sim \text{force/mass}}{\Omega} \sim \text{drift velocity}$$

and is the drift produced by a force acting in gyrating particles transversal on the force and on the magnetic field.

The force here is the *friction*, the result of the collisional changes of momentum

$$R_{ab} = \int d^3v \ m_a v_{\|} \ C_{ab}\left(f_{a1}, f_{b1}\right)$$

here

$$\begin{aligned} m_a v_{\|} &\sim \text{momentum} \\ C_{ab} &\sim \frac{1}{\text{time}} = \text{frequency of collisions} \end{aligned}$$

$$\begin{aligned} R_{ab} &\sim m_a v_{\|}\ C_{ab} \sim \text{change of momentum in unit of time} \\ &\sim \text{force} \end{aligned}$$

and

$$\frac{R_{ab}}{m_a \Omega_a} \sim \frac{\text{force/mass}}{\Omega_a} \sim \text{drift velocity}$$

The problem is that we need here that the magnetic field is *poloidal* because the friction is a force which is parallel.

here it is evident the ambipolarity

$$\sum_a e_a \Gamma_a = \langle \mathbf{j} \cdot \boldsymbol{\nabla} \psi \rangle = 0$$

The collisions cannot produce/destroy the angular momentum.

Using the expression of the distribution function f_{a1} in the formula for R_{ab} one obtains

$$R_{ab} = -n_{a0} m_a \ \left\{ \nu_{slowing}^{ab} \right\} \ (r_{ab} - r_{ba})$$

using also the momentum conservation in collisions

$$n_{a0} m_a \left\{ \nu_{slowing}^{ab} \right\} = n_{b0} m_b \left\{ \nu_{slowing}^{ba} \right\}$$

In the expression of the friction force R_{ab} :

$$r_{ab} = r_{ab}^{(1)} \ h + f_c \ r_{ab}^{(2)} \frac{1}{h}$$

$$\begin{aligned} & \left\{ \nu_{slowing}^{ab} \right\} \ r_{ab}^{(1)} \\ = \quad & (def) = -I \frac{v_{th,a}^2}{2\widehat{\Omega}_a} \left[\left\{ \nu_{slowing}^{ab} \right\} \ \left(A_{1a} + \frac{e_a}{T_{a0}} \frac{\partial \Phi}{\partial \psi} \right) + \left\{ \frac{v^2}{v_{th,a}^2} \nu_{slowing}^{ab} \right\} \ A_{2a} \right] \end{aligned}$$

$$\begin{aligned} & \left\{ \nu_{slowing}^{ab} \right\} \ r_{ab}^{(2)} \\ = \quad & (def) = I \frac{v_{th,a}^2}{2\widehat{\Omega}_a} \left[\left\{ \frac{\nu_{slowing}^{a} \ \nu_{slowing}^{ab}}{\nu_a} \right\} \left(A_{1a} + \frac{e_a}{T_{a0}} \frac{\partial \Phi}{\partial \psi} \right) \right. \\ & \qquad \qquad \qquad \left. + \left\{ \frac{v^2}{v_{th,a}^2} \frac{\nu_{slowing}^{a} \ \nu_{slowing}^{ab}}{\nu_a} \right\} A_2 \right] \\ & + \frac{e_a}{m_a} \left\{ \frac{\nu_{slowing}^{ab}}{\nu_a} \right\} A_3 \\ & + \sum_k \left\{ \frac{\nu_{slowing}^{ab} \ \nu_{slowing}^{ak}}{\nu_a} \right\} \overline{r}_{ka} \end{aligned}$$

The forces

$$A_{1a} = \frac{\partial}{\partial \psi} \ln n_{a0} - \frac{3}{2} \frac{\partial}{\partial \psi} \ln T_{a0}$$

$$A_{2a} = \frac{\partial}{\partial \psi} \ln T_{a0}$$

$$A_3 = \frac{\langle \mathbf{E} \cdot \mathbf{B} \rangle}{\langle B^2 \rangle^{1/2}} = \overline{E}_{\parallel}^{(A)}$$

The following part is also in *impurity accumulation.*

14 Asymmetric distribution of impurities on magnetic surfaces

14.1 Poloidal distribution impurities Wesson.

The toroidal rotation is assumed and attributed to the NBI.

The toroidal velocity v_φ

$$v_\varphi = \omega R$$

produces a centrifugal force. This force is oriented along the major radius and so it has a projection on the surface. On the surface this projection is balanced by the gradient of the pressure that is induced: the densities are variable on the surface

$$\begin{aligned} \nabla p &= F^{centrif} \\ \frac{T_i \delta n}{(\Delta R)} &= n m_i \omega^2 R \end{aligned}$$

where ΔR is the change of R on a surface.

The modification of the uniform profile of the density on the magnetic surface

$$n_0 \rightarrow n_0 + \delta n$$

can reach a level where the variation equals the equilibrium value

$$\delta n \sim n_0$$

Then

$$\omega^2 R^2 \sim \frac{R}{\Delta R} \frac{T_i}{m_i}$$

which means that such very important variation $\delta n \sim n_0$ occurs when

$$v_\varphi \sim v_{th,i}$$

The distribution of impurities is described by the balance of forces along the parallel direction.

The variable on the surface is

$$s$$

and

$$\frac{d}{ds} = \frac{dR}{ds} \frac{d}{dR} = \cos\theta \frac{d}{dR}$$

The equation of balance is

$$\begin{aligned} T_j \frac{dn_j}{ds} &= n_j m_j \omega^2 R \cos\theta + n_j e Z_j E_s \\ E_s &= -\frac{d\phi}{ds} = -\frac{d\phi}{dR} \frac{dR}{ds} = -\cos\theta \frac{d\phi}{dR} \end{aligned}$$

then

$$T_j \frac{dn_j}{ds} = n_j m_j \omega^2 R \cos\theta - n_j e Z_j \cos\theta \frac{d\phi}{dR}$$

it is integrated

$$n_j = n_{j0} \exp\left[\frac{\frac{1}{2} m_j \omega^2 \left(R^2 - R_0^2\right) - e Z_j \phi}{T_j}\right]$$

14.2 The asymmetries of impurity density Helander 3999

General structure of Neoclassical particle transport **Helander 3999.**

The impurities play an essential role, n_Z.
There is rotation Φ_0.
There is electrostatic potential $\widetilde{\Phi} \sim \theta$ on surface, the density of impurities is NOT uniform on θ.
Neutrality is used to connect $\widetilde{\Phi}$ with n_Z, n_i, n_e.
The equation is drift kinetic:

- neoclassic ρ_θ / L_n (from $\mathbf{v}_D \cdot \boldsymbol{\nabla} f_i^{(0)}$) : $-\frac{I}{\Omega_{ci}} v_\parallel \frac{\partial f_{M,i}}{\partial \psi}$;
- plus energetic $v_\parallel \nabla_\parallel \widetilde{\Phi} \frac{\partial f_i^{(0)}}{\partial \psi}$ coming from variation in the surface $\widetilde{\Phi} \ll \Phi_0$;
- the usual function h (or g in other notations).

Collisions are $ions \ - \ Z \ imp. \ ions$:

- (pitch-angle) +
- (friction i- with $V_{Z,\parallel}$, the latter from Φ_0 and from K_Z poloidal, created by friction-drag).

Friction R_{iZ} is STATIC in parallel momentum balance.
Solubility constraint $\langle B R_{iZ} \rangle = 0$ determines the poloidal flow K_Z from $V_{Z\parallel}$.
With flow velocity known and with friction R_{iZ} known, the parallel balance becomes equation for $n_Z / \langle n_Z \rangle$ and various regimes are found, including up-down or inner-outer asymmetries.

The paper **PoP 5 (1998) 3999**.
Bifurcated neoclassical particle transport.
The general scheme is standard neoclassical.
The scales

$$\delta = \frac{\rho_{\theta i}}{L_\perp}$$

and

$$\Delta = \delta \widehat{\nu}_{ii} z^2$$

$$\begin{aligned}
\widehat{\nu}_{ii} &= \frac{L_{\parallel}}{\lambda_{ii}} \\
\lambda_{ii} &\equiv \text{mean free path of BULK ions}
\end{aligned}$$

The study of *parallel dynamics*
First, two time scales are compared.
Frequency of collisions of species a with species b,

$$\begin{aligned}
\frac{1}{\tau_{ab}} &\sim \frac{e_a^2 e_b^2}{(4\pi\varepsilon_0)^2} \frac{1}{m_a^2} \ln\Lambda \frac{n_b}{v_{th}^{>}\ v_{th,a}^2} \\
& \quad \text{note the ratio } \frac{n_b}{v_{th}^{>}\ v_{th,a}^2} \text{ of the type } \frac{n}{T^{3/2}} \text{ as usual} \\
v_{th}^{>} &\equiv \text{the larger of } (v_{th,a}, v_{th,b})
\end{aligned}$$

Def

$$Z_{eff} - 1 = \frac{n_Z Z^2}{n_i} \sim O(1)$$

and

$$\tau_{\perp} = \frac{L_{\perp}^2}{\rho_Z^2/\tau_{Zi}}$$

$$\tau_{\parallel} = \frac{L_{\parallel}^2}{v_{th,Z}^2\ \tau_{ZZ}}$$

The ratio

$$\begin{aligned}
\frac{\tau_{\parallel}}{\tau_{\perp}} &\sim \frac{(Z_{eff}-1)\,\Delta^2}{Z^{3/2}} \\
&\sim \text{small}
\end{aligned}$$

This ratio is *small*.

The conclusion is that one can examine parallel dynamics on each flux surface independently (neglecting influences in the direction perpendicular to the surface = radial).

Other assumption

$$Z \gg 1$$

which leads to

$$n_Z Z \ll n_i$$

and the electrostatic potential is almost constant on the magnetic surface

$$\Phi \approx \Phi_0(\psi)$$

There is however a small variation of the potential on the surface $\widetilde{\Phi}(r,\theta)$.

The drift kinetic equation for the background ions

$$v_{\parallel}\nabla_{\parallel}\left(f_i^{(1)} + I\frac{v_{\parallel}}{\Omega_i}\frac{\partial f_{i0}}{\partial \psi}\right) + e\nabla_{\parallel}\Phi_1 \frac{v_{\parallel}}{T_i} f_{i0} = C_i(f_{i1})$$

We **note**.

The first part is neoclassical, it results from

$$\mathbf{v}_D \cdot \boldsymbol{\nabla} f_0$$

$$v_\parallel \nabla_\parallel \left(I \frac{v_\parallel}{\Omega_{ca}} \frac{\partial f_{a0}}{\partial \psi} \right)$$

and applying

$$\begin{aligned} \mathbf{v}_D \cdot \boldsymbol{\nabla} \psi \frac{\partial f_{a0}}{\partial \psi} &= I v_\parallel \nabla_\parallel \left(\frac{v_\parallel}{\Omega_{ca}} \right) \frac{\partial f_{a0}}{\partial \psi} \\ &= v_\parallel \nabla_\parallel \left(I \frac{v_\parallel}{\Omega_{ca}} \frac{\partial f_{a0}}{\partial \psi} \right) \end{aligned}$$

since the factors are only functions of ψ, the surface parameter.

The second part

$$e \nabla_\parallel \Phi_1 \frac{v_\parallel}{T_i} f_{i0}$$

comes from an energetic effect of the *variation of the electrostatic potential along the line* on the Maxwellian, equilibrium function. The factor $1/T_i$ comes from a derivation to the *energy*, $\partial f_{i0}/\partial \epsilon$.

The two *convective* terms

- the parallel change for the perturbation f_1,
- the neoclassical drift $\mathbf{v}_D$ of the equilibirum (Maxwellian) - which gives ρ_θ / L_n,

are combined with the energetic perturbation produced by the electrostatic potential variable on the surface.

It is a energy effect involved when moving along a magnetic line since now there is an electrostatic potential that is varying along the line.

All is balanced by collisions.

The solution

$$\begin{aligned} f_i^{(1)} &= -I \frac{v_\parallel}{\Omega_i} \frac{\partial f_i^{(0)}}{\partial \psi} \\ &\quad - \frac{e\widetilde{\Phi}}{T_i} f_i^{(0)} \\ &\quad + h_i (\epsilon_0, \mu, \psi, \sigma) \end{aligned}$$

The neutrality

$$\begin{aligned} n_Z &= \frac{n_e - n_i}{Z} \\ &= \frac{2 n_0}{T_0} e \widetilde{\Phi} \end{aligned}$$

with two new parameters introduced

$$\frac{2n_0}{T_0} = \frac{n_e^{(0)}}{T_e} + \frac{n_i^{(0)}}{T_i}$$

The perpendicular velocity of the impurities

$$\begin{aligned}\mathbf{V}_{Z\perp} &= \frac{-\boldsymbol{\nabla}\Phi_0 \times \widehat{\mathbf{n}}}{B} \\ &= \frac{d\Phi_0}{d\psi}\left(\frac{I}{B}\widehat{\mathbf{n}} - R^2\boldsymbol{\nabla}\varphi\right)\end{aligned}$$

The first factor is along $\mathbf{B}$:

$$\begin{aligned}\frac{I}{B}\frac{d\Phi_0}{d\psi}\widehat{\mathbf{n}} &= \frac{RB_T}{B}\frac{d\Phi_0}{dr}\frac{1}{|\boldsymbol{\nabla}\psi|}\widehat{\mathbf{n}} = \frac{d\Phi_0}{dr}\frac{RB_T}{B}\frac{1}{2\pi RB_\theta}\widehat{\mathbf{n}} \\ &\sim \frac{E_r}{B_\theta\left(\frac{B}{B_T}\right)} \text{ along the magnetic line, } \widehat{\mathbf{n}}\end{aligned}$$

The second term

$$\begin{aligned}-\frac{d\Phi_0}{d\psi}R^2\boldsymbol{\nabla}\varphi &= -\frac{d\Phi_0}{dr}\frac{1}{|\boldsymbol{\nabla}\psi|}R^2\frac{1}{R}\widehat{\mathbf{e}}_\varphi \\ &\sim E_r\frac{1}{2\pi RB_\theta}R\widehat{\mathbf{e}}_\varphi \\ &\sim \frac{E_r}{B_\theta} \text{ along toroidal direction } \varphi\end{aligned}$$

The combination

$$\frac{I}{B}\widehat{\mathbf{n}} - R^2\boldsymbol{\nabla}\varphi$$

is directed perpendicular relative to $\mathbf{B}$. From the parallel direction $\widehat{\mathbf{n}}$ it is subtracted the toroidal direction, with adequate coefficients. We must check that

$$\frac{I}{B}\widehat{\mathbf{n}} - R^2\boldsymbol{\nabla}\varphi \sim \boldsymbol{\nabla}\psi \times \widehat{\mathbf{n}}$$

The equation of continuity

$$\boldsymbol{\nabla}\cdot(n_Z\mathbf{V}_Z) = 0$$

This implies that *a parallel impurity RETURN current* must exist.
It is called return current because it is like Pfirsch Schluter current.

$$V_{Z\parallel} = -\frac{I}{B}\frac{d\Phi_0}{d\psi} + \frac{K_Z(\psi)B}{n_Z}$$

adopted as general form

The function K_Z is proportional with the poloidal flow velocity.

The collision operator

$$\begin{aligned} C_{iZ} &= \frac{1}{2}\nu_{iZ}(v)\frac{\partial}{\partial \xi}\left(1-\xi^2\right)\frac{\partial}{\partial \xi} \quad \text{pitch angle} \\ &\quad +\nu_{iZ}\frac{m_i v_{\|} V_{Z\|}}{T_i} f_i^{(0)} \quad \text{friction with parallel flow} \end{aligned}$$

Assuming that we are able to solve the drift kinetic equation and to find the solution for h_i (which means $f_i^{(1)}$ which also means f_i) we can use this distribution function to calculate the *friction* between ions and impurities.

In fluid version

$$\begin{aligned} R_{Z\|} &= -\int d^3v \; m_i v_{\|} \; C_{iZ}\left(f_i^{(1)}\right) \\ &= -\frac{1}{\tau_{iZ}}\frac{I}{\Omega_i}\left(p_i\frac{d\ln p_i}{d\psi} - p_i\frac{3}{2}\frac{d\ln T_i}{d\psi}\right) + \frac{1}{\tau_{iZ}}n_i m_i\left(u - \frac{K_Z}{n_z}\right)B \end{aligned}$$

where

$$u = \tau_{iZ}\frac{1}{n_i B}\int d^3v \; v_{\|}\nu_{iZ} \; h_i$$

In **Connor 1973** it is defined a similar quantity.

The parallel momentum balance, which involves the *ion-impurity friction.*

$$\begin{aligned} m_Z n_Z \; \widehat{\mathbf{n}}\cdot\left[\left(\mathbf{v}_Z\cdot\boldsymbol{\nabla}\right)\mathbf{v}_Z\right] &= -\nabla_{\|}p_Z - \widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_Z \\ &\quad -n_Z \; Ze \; \nabla_{\|}\Phi^{(1)} \\ &\quad +R_{Z\|} \end{aligned}$$

but

$$\frac{m_Z n_Z \; \widehat{\mathbf{n}}\cdot\left[\left(\mathbf{v}_Z\cdot\boldsymbol{\nabla}\right)\mathbf{v}_Z\right]}{R_{Z\|}} \sim \frac{\delta}{Z\widehat{\nu}_{ii}}$$

Most contributions are overwhelmed by the *ion-impurity energy equilibration.*

Then it remains

$$0 = -T_i\nabla_{\|}n_Z - n_Z \; Ze \; \nabla_{\|}\Phi^{(1)} + R_{Z\|}$$

(like in the drift waves).

The solubility constraint, resulting by imposing periodicity

$$\left\langle B \; R_{Z\|}\right\rangle = 0$$

provides an equation for the poloidal rotation K_Z.

This equation is the constraint resulting from *periodicity.*
The part K_z is eliminated and the constraint is applied

$$\begin{aligned}
&\left(T_i + \frac{n_z z^2}{2n_0}\right) \nabla_{\|} n_z \\
= \quad & -\frac{1}{\tau_{iz}} \frac{I}{\Omega_{ci}} \left(p_i \frac{d \ln p_i}{d\psi} - p_i \frac{3}{2} \frac{d \ln T_i}{d\psi}\right) \left(1 - \frac{\langle n_z \rangle}{n_z} \frac{B^2}{\langle B^2 \rangle}\right) \\
& + \frac{1}{\tau_{iz}} \frac{1}{n_z} m_i n_i \, u \left(n_z - \frac{\langle n_z B^2 \rangle}{B^2}\right) B
\end{aligned}$$

This is the derivative of the density of impurity ions with respect to the parallel coordinate or, equivalently, with respect to the poloidal angle θ.
The density of impurity ions has variation on the magnetic surface (along the poloidal angle θ) which is nonlinear but has a θ-variation which is given by *friction* and by *drift of electric field.*

15 Drift kinetic eq. for the ion - impurity collisional coupling (Hirshman Sigmar Clarke)

this is also in *general, drift kinetic solutions.*
As in the preceding Section.
ă**note** the collision operator introduced here is used by **Novakovskii Rosenbluth**î

There is electrostatic variation on magnetic surfaces $\widetilde{\Phi}(r, \theta)$ due to $n_Z(\theta)$.
then, there is $E_{\|}$ which further requires an *energetic* term in the drift kinetic equation.
This work also includes a variation of the magnetic field *in time*

$$\mu \frac{\partial B}{\partial t} \times \frac{\partial}{\partial \epsilon} f \ \text{ energy term}$$

which exists when transiently the magnetic surfaces are changed.

For $f_t \ll 1$ (very few trapped particles) the *circulating particles collisionally couple along the magnetic field to minimize the interspecies friction* and the produces *a common diamagnetic toroidal flow of the plasma.*

For large aspect ratio limit there are two time scales.
The fast one. Collisions between ions and the toroidal diamagnetic flows of the individual plasma species couple to produce a unique common flow.
The slow one is transport. The gradients are slowly modified by transport such that

- the common parallel flow is maintained
- equalizing the individual diamagnetic responses, to minimize the parallel friction

The equation for the species a is

$$\begin{aligned} & \frac{\partial f_a}{\partial t}+\left(v_{\|}\widehat{\mathbf{n}}+\mathbf{v}_{d,a}\right)\cdot\boldsymbol{\nabla}f_a+\left[\mu\frac{\partial B}{\partial t}+\frac{e_a}{m_a}\left(v_{\|}E_{\|}^{(A)}+\frac{\partial\phi}{\partial t}\right)\right]\frac{\partial f_a}{\partial\epsilon} \\ = & \sum_b C_{ab}\left(f_a,f_b\right) \end{aligned}$$

where

$$\epsilon=\frac{1}{2}v^2+\frac{e_a}{m_a}\phi$$

with

$$e_a=Z_a\left|e\right|$$

$$\mathbf{E}^{(A)}=E_{\varphi 0}\frac{R_0}{R}\widehat{\mathbf{e}}_{\varphi}$$

the electric field produced by induction by the tokamak transformer.

The drift velocity

$$\begin{aligned} \mathbf{v}_{d,a} = & -v_{\|}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\left(\frac{v_{\|}}{\Omega_{ca}}\right) \\ & +\frac{v_{\|}^2}{\Omega_{ca}}\frac{1}{B}\left(\boldsymbol{\nabla}\times\mathbf{B}\right)_{\perp} \end{aligned}$$

The last term is

$$\sim\frac{1}{\Omega_{ca}}v_{\|}^2\frac{J_{\perp}}{B}$$

and is a drift of particles which exists when there is a current perpendicular to the magnetic field.

The magnetic field

$$\mathbf{B}=I\left(\psi\right)\boldsymbol{\nabla}\varphi+\boldsymbol{\nabla}\varphi\times\boldsymbol{\nabla}\psi$$

$$I=RB_T$$

$$\mathbf{B}_p=\boldsymbol{\nabla}\varphi\times\boldsymbol{\nabla}\psi$$

Since the drift velocity will multiply the spatial gradient of the zero-order distribution function,

$$\mathbf{v}_{d,a}\cdot\boldsymbol{\nabla}f_{a0}$$

and since $f_{a0} = f_{Ma}$ has only variation in a direction perpendicular on the magnetic surface

$$\nabla f_{Ma} = \nabla \psi \frac{\partial f_{Ma}}{\partial \psi}$$

we will have to use

$$\begin{aligned}
\mathbf{v}_{d,a} \cdot \nabla \psi &= v_{\|} \widehat{\mathbf{n}} \cdot \nabla \left(I \frac{v_{\|}}{\Omega_{ca}} \right) \\
&= v_{\|} \nabla_{\|} \left(I \frac{v_{\|}}{\Omega_{ca}} \right)
\end{aligned}$$

(in this form the term $\mathbf{v}_{d,a} \cdot \nabla f$ which actually is $\mathbf{v}_{d,a} \cdot \nabla \psi \frac{\partial f_{Ma}}{\partial \psi}$ can be coupled to $v_{\|} \widehat{\mathbf{n}} \cdot \nabla f_{a1}$).

The expansion

$$f_a = f_{a0} + f_{a1} + ...$$

$$f_{a0} = f_{Ma} = n_a(\psi) \frac{1}{\left[\pi \frac{2T_a(\psi)}{m_a}\right]^{3/2}} \exp\left[-\frac{\epsilon}{\frac{T_a(\psi)}{m_a}}\right]$$

$$n_a(\psi) = n_{a0} \exp\left[\frac{e_a \phi}{T_{a0}(\psi)}\right]$$

The equation for the first order

$$\begin{aligned}
& v_{\|} \nabla_{\|} f_{a1} + v_{\|} \nabla_{\|} \left(I \frac{v_{\|}}{\Omega_{ca}} \frac{\partial f_{a0}}{\partial \psi} \right) \\
& -v_{\|} \frac{e_a E^{(A)}}{T_a} f_{a0} \quad \text{(external } E\text{)} \\
= & \sum_b C_{ab}(f_{a1}, f_{b1})
\end{aligned}$$

Note that here we have taken $\partial f_{a0}/\partial \psi$ inside the paranthesis that comes from the drift velocity. Then it is convenient to separate from f_{a1} the part resulting from the drift motion advection of the equilibrium f_{a0} distribution function.

$$\begin{aligned}
f_{a1} &= -I \frac{v_{\|}}{\Omega_{ca}} \frac{\partial f_{a0}}{\partial \psi} \\
&\quad + g_a(\epsilon, \mu, \psi)
\end{aligned}$$

where (**Hirshman Sigmar Clarke**)

$$\begin{aligned}
-I \frac{v_{\|}}{\Omega_{ca}} \frac{\partial f_{a0}}{\partial \psi} &\equiv \text{diamagnetic response of the species } a \\
g_a &\equiv \text{collisional response of the species } a
\end{aligned}$$

For a comment on the name *diamagnetic* in relation with *gyration/bananas* see **the text** *reference plasma.*

Consider the surface average

$$\langle A(\mathbf{x})\rangle = \frac{2\pi}{\frac{\partial V}{\partial \psi}}\oint d\chi \frac{A}{\boldsymbol{\nabla}\chi\cdot\mathbf{B}}$$

$$V' \equiv \frac{\partial V}{\partial \psi} = 2\pi \oint d\chi \frac{1}{\boldsymbol{\nabla}\chi\cdot\mathbf{B}}$$

Returning to the drift-kinetic equation, we have

$$v_\parallel \nabla_\parallel \left(f_{a1} + I\frac{v_\parallel}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial \psi}\right) = v_\parallel \frac{e_a E^{(A)}}{T_a} f_{a0} + \sum_b C_{ab}(f_{a1}, f_{b1})$$

We divide by $v_\parallel$ and note that the surface average is applied to

$$\begin{aligned} & \oint d\chi \frac{1}{\mathbf{B}\cdot\boldsymbol{\nabla}\chi}\nabla_\parallel () \\ = & \oint d\theta \frac{1}{B_\theta \frac{1}{r}}\frac{d}{dl_\parallel}() \end{aligned}$$

but

$$dl_\parallel = \frac{B}{B_\theta} dl_\theta = r d\theta \frac{B}{B_\theta}$$

Then

$$\frac{d}{dl_\parallel} = \frac{B_\theta}{B}\frac{d}{rd\theta}$$

and

$$\begin{aligned} \oint d\theta \frac{1}{B_\theta\frac{1}{r}}\frac{d}{dl_\parallel}() &= \oint d\theta \frac{1}{B_\theta \frac{1}{r}}\frac{B_\theta}{B}\frac{d}{rd\theta}() \\ &= \oint d\theta \frac{1}{B}\frac{d}{d\theta}() \end{aligned}$$

The magnitude of B is

$$B \approx \frac{B_0}{h} = B_0 \frac{1}{1+\varepsilon\cos\theta}$$

and B is a function of θ. But it is of order ε and multiples a quantity which is also of order ε. Then $\frac{1}{B}$ can be inserted in the paranthesis

$$\begin{aligned} \nabla_\parallel \left(f_{a1} + I\frac{v_\parallel}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial \psi}\right) &= \frac{1}{v_\parallel}\left[v_\parallel \frac{e_a E^{(A)}}{T_a} f_{a0} + \sum_b C_{ab}(f_{a1}, f_{b1})\right] \\ \frac{B_\theta}{B}\frac{d}{rd\theta}\left(f_{a1} + I\frac{v_\parallel}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial \psi}\right) &= \frac{1}{v_\parallel}\left[v_\parallel \frac{e_a E^{(A)}}{T_a} f_{a0} + \sum_b C_{ab}(f_{a1}, f_{b1})\right] \\ B_\theta \frac{d}{rd\theta}\left(f_{a1} + I\frac{v_\parallel}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial \psi}\right) &= \frac{B}{v_\parallel}\left[v_\parallel \frac{e_a E^{(A)}}{T_a} f_{a0} + \sum_b C_{ab}(f_{a1}, f_{b1})\right] \end{aligned}$$

Now we take the surface average

$$\left\langle B_\theta \frac{d}{rd\theta}\left(f_{a1} + I\frac{v_\parallel}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial\psi}\right)\right\rangle = \left\langle \frac{B}{v_\parallel}\left[v_\parallel \frac{e_a E^{(A)}}{T_a} f_{a0} + \sum_b C_{ab}\left(f_{a1}, f_{b1}\right)\right]\right\rangle$$

In the left hand side we replace

$$\begin{aligned}
\langle ...\rangle &= \frac{2\pi}{V'}\oint d\theta \frac{1}{B_\theta \frac{1}{r}}(...) \\
\left\langle B_\theta \frac{d}{rd\theta}\left(f_{a1} + I\frac{v_\parallel}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial\psi}\right)\right\rangle &= \frac{2\pi}{V'}\oint d\theta \frac{1}{B_\theta \frac{1}{r}} B_\theta \frac{d}{rd\theta}\left(f_{a1} + I\frac{v_\parallel}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial\psi}\right) \\
&= 0
\end{aligned}$$

due to periodicity.

NOTE

As explained in *particle equations of motion* before averaging one must multiply the equation with B.

This will turn the averaging operation into an integration over the poloidal angle θ of a total derivative to θ. Then the periodicity on θ will vanish one of the terms.

The general form is

$$\left\langle \frac{B}{v_\parallel} C(f)\right\rangle = 0$$

END

This is what results

$$\left\langle \frac{B}{v_\parallel}\left[v_\parallel \frac{e_a E^{(A)}}{T_a} f_{a0} + \sum_b C_{ab}\left(f_{a1}, f_{b1}\right)\right]\right\rangle = 0$$

(remember the calculation of the *bootstrap* current in **Helander**; the equation is the same).

The expression of the collision operator.

Hirshman Sigmar Clarke write

$$\begin{aligned}
C_{ab}\left(f_{a1}, f_{b1}\right) &= \nu_{ab}^{defl}\ \frac{v_\parallel}{B}\frac{\partial}{\partial\mu}\mu v_\parallel \frac{\partial}{\partial\mu} f_{a1} \\
&\quad + \left[\nu_{ab}^{defl} - \nu_{ab}^{slowing}\right]\frac{v_\parallel u_{a1}(v)}{v^2}\ f_{a0} \\
&\quad + \frac{2v_\parallel}{v_{th,a}^2} r_{ba}\ \nu_{ab}^{slowing}\ f_{a0}
\end{aligned}$$

The first is the *pitch-angle scattering*.

The second, with ν_{ab}^{defl} of the paranthesis, implies two relative velocities even inside a species with elements in relative motion.

The second term of the paranthesis is a friction tending to suppress the differences in velocities.

The last term contains r_{ab} ≡momentum restoring coefficient.

One can make a different grouping

$$\begin{aligned}
& C_{ab}\left(f_{a1}, f_{b1}\right) \\
= \quad & \nu_{ab}^{defl}\left[\frac{v_{\parallel}}{B}\frac{\partial}{\partial\mu}\mu v_{\parallel}\frac{\partial}{\partial\mu}f_{a1}+\frac{v_{\parallel}u_{a1}\left(v\right)}{v^{2}}\ f_{a0}\right] \quad \text{deflection} \\
& +\nu_{ab}^{slowing}\left[\frac{2v_{\parallel}}{v_{th,a}^{2}}r_{ba}\quad f_{a0}-\frac{v_{\parallel}u_{a1}\left(v\right)}{v^{2}}\ f_{a0}\right] \quad \text{slowing down}
\end{aligned}$$

The definitions

The *deflection* frequency

$$\nu_{ab}^{defl}=\nu_{ab}\frac{\Phi\left(\frac{v}{v_{th,b}}\right)-G\left(\frac{v}{v_{th,b}}\right)}{\left(\frac{v}{v_{th,a}}\right)^{3}}$$

The *slowing down* frequency

$$\nu_{ab}^{slowing}=2\frac{T_{a0}}{T_{b0}}\left(1+\frac{m_{b}}{m_{a}}\right)\ \nu_{ab}\ \frac{G\left(\frac{v}{v_{th,b}}\right)}{\left(\frac{v}{v_{th,a}}\right)}$$

The frequency of collisions (**Trubnikov**)

$$\nu_{ab}=\frac{4\pi}{2^{3/2}}\frac{e_{a}^{2}e_{b}^{2}}{\sqrt{m_{a}}}\ln\Lambda\frac{n_{b0}}{T_{a}^{3/2}}$$

The function

$$\begin{aligned}
G\left(x\right) \quad &= \quad \frac{\Phi\left(x\right)-x\frac{d\Phi(x)}{dx}}{2x^{2}} \\
& \quad\quad \text{the Chandrasekhar function}
\end{aligned}$$

where

$$\Phi\left(x\right)=\frac{2}{\sqrt{\pi}}\int_{0}^{x}dt\exp\left(-t^{2}\right)$$

In the formulas above it has been introduced

$$\begin{aligned}
u_{a1}\left(v\right) \quad &= \quad \frac{1}{f_{a0}}\frac{3}{4\pi}\int v_{\parallel}f_{a1}d\Omega \\
d\Omega \quad &\equiv \quad \text{solid angle in velocity space}
\end{aligned}$$

$$d\Omega = \pi \sum_{\sigma=\pm 1} \frac{B d\mu}{\epsilon} \frac{1}{\left|v_{\parallel}\right| / v}$$

Note

It is expected that f_{a1} will depend on the direction in the space of velocities, through a direct dependence on $v_{\parallel}$. The integration over the solid angle will filter-out the part of the space of velocities where the function f_{a1} does not have a significant value. Since this function is multiplied in the integrand by $v_{\parallel}$ we get a *flow* in the parallel direction, by the weight of f_{a1}.

This will contain the flow in velocity space across the boundary between trapped/circulating.

End.

NOTE

A similar object is introduced by **Helander 3999**

$$u \equiv \tau_{iZ} \frac{1}{n_i B} \int d^3 v \; m v_{\parallel} \; C_{iZ} \left(f_i^{(1)} \right)$$

END

the *momentum restoring* coefficient

$$r_{ab} \equiv \frac{\int d^3 v \; \nu_{ba}^{slowing} \; m_b v_{\parallel} \;\; f_{b1}}{m_a n_{a0} \left\{ \nu_{ab}^{slowing} \right\}}$$

This quantity is defined on the basis of the momentum exchange between species a and b. It measures the mutual friction between species a and b. $\nu_{ba}^{slowing}$ is the frequency of collisions with transfer of momentum $m_b v_{\parallel}$. the function f_{b1} is a measure of the difference between species and this is the origin of mutual momentum transfer between species.

The following operator is introduced, for any function that depends on two species indices, a and b

$$\left\{ X_{ab}\left(v\right) \right\} \equiv 2 \frac{1}{n_{a0}} \int d^3 v \; \left(\frac{v_{\parallel}}{v_{th,a}} \right)^2 \; f_{a0} \; X_{ab}\left(v\right)$$

The function X_{ab} depends on velocity. The integral is over this quantity with the Maxwellian of the velocities, but weighted by the square of the parallel velocity $\sim$ energy of the parallel motion.

To solve the drift-kinetic equation

$$\lambda \equiv \mu \frac{\left\langle B^2 \right\rangle^{1/2}}{\epsilon - \frac{e_a \phi(\psi)}{m_a}}$$

and extract the zero-distribution f_{a0} as factor from f_{a1}

$$f_{a1} = \widehat{f}_{a1} f_{a0}$$

results

$$\begin{aligned}
\widehat{f}_{a1} &= -Iv_{\parallel} \frac{1}{\frac{e_a \langle B^2 \rangle^{1/2}}{m_a}} \frac{\langle B^2 \rangle^{1/2}}{B} \frac{\partial}{\partial \psi} \ln f_{a0} \\
&\quad + I \frac{1}{\frac{e_a \langle B^2 \rangle^{1/2}}{m_a}} \mathbf{H} \left[V_{\parallel} \frac{\partial \ln f_{a0}}{\partial \psi} \right] \quad \text{(Heaviside-like excludes the Trapped region)} \\
&\quad + \mathbf{H} \left[\frac{2V_{\parallel}}{v_{th,a}^2} \left(\frac{e_a \overline{E}_{\parallel}^{(A)}}{m_a} \frac{1}{\nu_a^{defl}} + \left(1 - \frac{\nu_a^{slow}}{\nu_a^{defl}} \right) \frac{1}{2x_a^2} \overline{u}_{a1}(v) + \sum_b \overline{r}_{ab} \frac{\nu_{ab}^{slow}}{\nu_{ab}^{defl}} \right) \right]
\end{aligned}$$

The terms that contain the Heaviside-like function $\mathbf{H}$ only exist for circulating particles.

For example the first two terms can be

$$\begin{aligned}
&hv_{\parallel} - V_{\parallel} \quad \text{for passing particles} \\
&hv_{\parallel} \quad \text{for trapped particles}
\end{aligned}$$

The notations are

$$\frac{\langle B^2 \rangle^{1/2}}{B} \equiv h$$

$$x_a = \frac{v}{v_{th,a}}$$

$$V_{\parallel}(\lambda) = \int_{\lambda}^{\lambda_c} d\lambda \, \frac{\frac{v^2}{2}}{\langle v_{\parallel} \rangle}$$

integration over circulating particles

$$\lambda_c = \frac{\langle B^2 \rangle^{1/2}}{B_c}$$

As shown below for a different notation (and definition of λ) the integral that defines $V_{\parallel}(\lambda)$ is on the interval

$$\begin{aligned}
& [\lambda, \lambda_c] \quad \text{which is passing-domain} \\
\lambda_c \quad &\equiv \quad \text{limit passing} \rightarrow \text{trapped}
\end{aligned}$$

so that the integration is for *passing* particles, up to the boundary passing/trapped.

$$\begin{aligned}
B_c(\psi) \quad &= \quad \text{maximum of } B \text{ in a surface } \psi \\
&\quad\quad \text{it is } B \text{ at the farthest equatorial point}
\end{aligned}$$

and

$$\nu_a^{defl} = \sum_b \nu_{ab}^{defl}$$

$$\nu_a^{slowing} = \sum_b \nu_{ab}^{slowing}$$

a new averaging

$$\overline{A} \equiv \left\langle \frac{A}{h} \right\rangle$$

Note some physical quantities have variation over the magnetic surface with the factor h. For example the poloidal velocity must be divided to B_θ such as to obtain a function that only depends on ψ.

$$\begin{aligned} \frac{u_p(r,\theta)}{B_\theta} &\rightarrow U_{pol}(r) \\ \frac{1}{b(r)} u_p(r,\theta) h &\rightarrow U_{pol}(r) \end{aligned}$$

In **Su Yushmanov Dong Horton**,

$$V_{pol} \equiv \left\langle v_\theta \left(1 + \frac{r}{R}\cos\theta\right)\right\rangle$$

$$v_\theta = \frac{V_{pol}(r)}{1 + \frac{r}{R}\cos\theta} = \frac{V_{pol}(r)}{h}$$

Therefore some quantity contains as *main* poloidal variation the factor h. Then it is better to start by extracting this poloidal variation, by dividing with $h(\theta)$ and the remaining θ variation, of A/h , must be averaged.

End.

NOTE and Comment on parameters

This is in *particle equations of motion*

parameters

$$w = \frac{v^2}{2} = \epsilon - \frac{e\phi}{m}$$

$$\lambda = \frac{\mu}{w}$$

or

$$\lambda = \frac{v_\perp^2}{v^2} \frac{1}{B(\mathbf{x})}$$

We have

$$B(\mathbf{x}) = \frac{B_0}{h}$$

and

$$\lambda \rightarrow \lambda' = \frac{v_\perp^2}{v^2} h$$

then, since here $\lambda = \frac{v_\perp^2}{v^2}\frac{1}{B(\mathbf{x})}$ we have

$$v_\parallel = \sigma\sqrt{2w\left(1-\lambda B\right)}$$

We define

$$\begin{aligned}
\lambda_m &= \frac{1}{B\left(\mathbf{x}\right)} \\
&= \text{the largest } \lambda \text{ for which the function} \\
&\quad\; f\left(\mathbf{x},\lambda,w\right) \text{ is defined} \\
\text{corresponds to } v_\perp^2 &= v^2 \text{ (no parallel velocity, deep trapped)}
\end{aligned}$$

and

$$\begin{aligned}
\lambda_c &= \frac{1}{B_{\max}} \\
&\quad \text{where} \\
B_{\max} &= \text{the maximum of } \left|\mathbf{B}\right| \text{ along a field line}
\end{aligned}$$

or

$$B_{\max} = \frac{B_0}{h_{\min}} = \frac{B_0}{1-(r/R_0)}$$

then

$$\lambda_c = \frac{1-(r/R_0)}{B_0}$$

and λ_c is the critical λ for trapping.

The *trapped region* is

$$\lambda_c < \lambda < \lambda_m \quad \text{trapped}$$

the *untrapped* (passing, circulating) region

$$0 < \lambda < \lambda_c \quad \text{circulating}$$

All *circulating* particles have the perpendicular velocity sufficiently small (*i.e.* λ small) such that at the given energy the parallel velocity to be high enough for the particle to overcome the magnetic barrier along the line.

Then λ must be *small* for the particle to be *circulating*.

END of Note and Comment

To obtain an explicit expression for $\widehat{f}_{a1}$ we must eliminate u_{a1}. The present form of $\widehat{f}_{a1}$, which contains $\overline{u}_{a1}$ is introduced in the formula for u_{a1} ($u_{a1}(v) = \frac{1}{f_{a0}}\frac{3}{4\pi}\int v_\parallel f_{a1} d\Omega$) and the integration over the angular space $d\Omega$ is performed. The result is an equation for u_{a1}.

After averaging

$$\begin{aligned}
\frac{\overline{u}_{a1}(v)}{x_a^2} &= -f_T \; I \frac{v_{th,a}^2}{\widehat{\Omega}_a} \frac{\partial}{\partial \psi} \ln f_{a0} \times \frac{\nu_a^{defl}}{\nu_a} \qquad \text{(trapped)} \\
&+2 f_c \left(\frac{e_a \overline{E}_{\|}^{(A)}}{m_a} \frac{1}{\nu_a} + \sum_b \overline{r}_{ab} \frac{\nu_{ab}^{slow}}{\nu_a} \right) \qquad \text{(circulating)}
\end{aligned}$$

where

$$\frac{3}{4\pi} \int d\Omega \; v_{\|} \; \mathbf{H} \left[V_{\|} \right] = f_c \; v^2 \frac{1}{h}$$

Note that due to the presence of Heaviside-like $\mathbf{H}$ the integration is made over the circulating particle domain.

The proportions of circulating and trapped particles

$$\begin{aligned}
f_c &= \frac{3}{4} \int_0^{\lambda_c} \frac{\lambda d\lambda}{\left\langle \sqrt{1 - \frac{\lambda}{h}} \right\rangle} \\
f_T &= 1 - f_c \\
&\approx 1.46 \times \sqrt{\varepsilon}
\end{aligned}$$

[**Note** the meaning

$$\int_0^{\lambda_c} \frac{\lambda d\lambda}{\left\langle \sqrt{1 - \frac{\lambda}{h}} \right\rangle} = \int \frac{\lambda d\lambda}{\left\langle \frac{v_{\|}}{v} \right\rangle} = \int \lambda d\lambda \frac{1}{\langle \xi \rangle}$$

This integral is done, in the complete expression, by **Rosenbluth Hazeltine Hinton** in Appendix. **End.**]

NOTE

We remark that the definition of $V_{\|}(\lambda)$ does NOT involve any distribution function. It is just an organization in the space of velocity. The integration is over the portion of the velocity space where there are *circulating* particle.

In consequence $\frac{3}{4\pi} \int d\Omega \; v_{\|} \; \mathbf{H} \left[V_{\|} \right] = f_c \; v^2 \frac{1}{h}$ gives a fraction of circulating particles, again independent of any distribution function.

$$\begin{aligned}
f_c &= \frac{3}{4\pi} \frac{h}{v^2} \int d\Omega \; v_{\|} \; \mathbf{H} \left[V_{\|} \right] \\
&= \frac{3}{4\pi} \frac{h}{v^2} \int d\Omega \; v_{\|} \int_{\lambda}^{\lambda_c} d\lambda \; \frac{\frac{v^2}{2}}{\langle v_{\|} \rangle} \\
&= \frac{3}{4} \int_0^{\lambda_c} \frac{\lambda d\lambda}{\left\langle \sqrt{1 - \frac{\lambda}{h}} \right\rangle}
\end{aligned}$$

END.

Further

$$\nu_a = f_c \nu_a^{slow} + f_T \nu_a^{defl}$$

This frequency of collision combines the slowing-down collisions (friction) due to *circulating* particles and the deflection collisions due to the *trapped* particles.

The function $\overline{u}_{a1}(v)$ is introduced in the expression of $\widehat{f}_{a1}$.

$$\begin{aligned}
\widehat{f}_{a1} &= -I\frac{1}{\widehat{\Omega}_a} h v_\parallel \frac{\partial}{\partial\psi}\ln f_{a0} \\
&\quad + I\frac{1}{\widehat{\Omega}_a}\frac{\nu^a_{slow}}{\nu^a}\mathbf{H}\left[V_\parallel\right]\frac{\partial}{\partial\psi}\ln f_{a0} \\
&\quad + \mathbf{H}\left[\frac{2V_\parallel}{v_{th,a}^2}\left(\frac{e_a\overline{E}_\parallel^{(A)}}{m_a}\frac{1}{\nu_a} + \sum_b \frac{\nu_{ab}^{slow}}{\nu_a}\overline{r}_{ba}\right)\right]
\end{aligned}$$

The *restoring coefficients* $\overline{r}_{ba}$ for all the other species b must still be determined.

Transport

This is

$$\begin{aligned}
\Gamma_a &= \left\langle \int d^3v \ \left(\mathbf{v}_{Da}\cdot\boldsymbol{\nabla}\psi\right) f_a \right\rangle \\
&= -I(\psi)\left\langle \frac{\sum_b R_{ab}}{m_a\Omega_a}\right\rangle - I(\psi)\left\langle \frac{n_{a0}E_\parallel^{(A)}}{B}\right\rangle
\end{aligned}$$

where

$$\begin{aligned}
R_{ab} &= \int d^3v \ v_\parallel C_{ab}\left(f_{a1}, f_{ba}\right) \\
&= \text{parallel collisional friction} \\
&\quad\ \text{between species } a \text{ and } b
\end{aligned}$$

Note that, as usual, the parallel force (here *friction*) produces perpendicular flux.

There is a velocity V parallel with the field, which results from *neoclassical momentum balance.*

It results from the collisional coupling between passing particles among themselves and between passing particles and the trapped particles.

[**Observation**. In the absence of the component of trapped particles, *i.e.* for

$$f_T \to 0$$

the velocity V is *indeterminate*. This is because now there Galilean transformation is arbitrary and does not allow to fix a frame. Only if there is *trapping* which is equivalent to the existence of *modulation of the magnitude of the magnetic field* along the line, $\delta B/B$, which is equivalent to provide a choice for the reference system and suppresses the arbitrary Galilean transformation.]

Pitch angle scattering in the region close to the separatrix *trapped/passing* λ_c produces a continuous flow of particles in velocity space, from trapped to passing.

Next the interspecies collisions will try to eliminate the differences between the velocities of the flows of *passing* particles.

First case

A plasma consisting of electrons and one species of ions.

In this simple plasma there is strong coupling between electrons and ions, since

$$\nu^{ee} \sim \nu^{ei}$$

and there is easily a common flow. It is defined by

$$R_{ii} = 0$$

where R_{ii} is the parallel friction force ion-ion.

Second case

A plasma with electrons and ions and impurity ions.

There is coupling of electrons to ions.

There is also coupling of ions to impurity ions.

For this latter case, assume

$$\begin{array}{ccl} \nu^{ii} & > & \nu^{zi} \\ & & \text{frequent ion-ion colisions} \\ & & \text{impurities are not coupled} \end{array}$$

Now assume

$$\begin{array}{ccl} \nu^{zi} & \geq & \nu^{zz} \\ & & \text{the impurity colides mainly with ions} \\ & & \text{than between them} \end{array}$$

Then

$$R_{iz} \to 0$$

means that a flow will be established such that

$$u_{\parallel z} = u_{\parallel i}$$

ions and impurities flow together.

16 Impurities and rotation in large gradients Fulop Helander

The parameter

$$\begin{aligned}\Delta &= \delta\widehat{\nu}_{ii}Z^2 \\ &\sim \frac{\text{ion-impurity friction force}}{\text{parallel impurity-pressure gradient}}\end{aligned}$$

Strong rotation, *centrifugal force.*

For the order of magnitude.
Consider the usual neoclassical small parameter

$$\delta \equiv \frac{\rho_\theta}{L_n} \ll 1$$

$$Z \gg 1 \quad \text{(the impurity charge)}$$

Define

$$M_i^2 = \frac{\omega^2 R^2}{(2T_i/m_i)} \text{mach number of the ions}$$

very small since the rotation velocity is much smaller than the thermal velocity of the ions.

$$\begin{aligned}M_z^2 &= \frac{\omega^2 R^2}{(2T_i/m_z)} \quad \text{Mach number for impurity} \\ &\sim 1\end{aligned}$$

since the "thermal" velocity of the impurities is much smaller than that of ions and it is comparable with the rotation.

Parameter of collisionality of background ions

$$\widehat{\nu}_{ii} = \frac{L_\parallel}{\lambda_{free-path,i}}$$

and $L_\parallel \sim qR$.

The parameter

$$\Delta = \delta\ \widehat{\nu}_{ii}\ Z^2$$

This is NOT small in the edge region.

When

$$\Delta \sim 1$$

the two effects on impurity dynamics are *comparable*

- the parallel gradient of the pressure of the impurity ions, and

- the force of friction between the impurity ions and the background ions

Conclusion at this point in **FH**: the only presence of the parallel gradient, without any competing force, leads to uniform distribution of density of impurities on surfaces.

It is not so when $\Delta \sim 1$.

Regarding the collisionality.

The concentration of impurities in tokamak makes comparable the

- ion-ion collision frequency, and
- ion-impurity collision frequency

The variation with θ (i.e. on the magnetic surface) of the electrostatic potential is weak.

No electrostatic trapping.

The comparison

$$\frac{1}{z} = O\left(\delta^{1/2}\right)$$

Usually there is accumulation at large R and lectrostatic variation on surface, $\rightarrow$ it results in $\nabla_{\parallel}\widetilde{\Phi}$

The drift kinetic equation with convection $v_{\parallel}$ and energetic $v_{\parallel}\nabla_{\parallel}\widetilde{\Phi}$ and sources (force) $\times$ (flux).

16.1 Drift kinetic equation for any species (ions, impurities, electrons)

The drift-kinetic equation for species a based on **Hinton Wong** (in *rotation*).

The variation on surface $\widetilde{\Phi}$ is essential in **HW**.

Due to *centrifugal* force.

$$\begin{aligned} & v_{\parallel}\nabla_{\parallel}f_a + v_{\parallel}\ e_a\left(-\nabla_{\parallel}\widetilde{\Phi}\right)\frac{\partial f_a}{\partial H} - C\left(f\right) \\ = \quad & v_{\parallel}f_{a0}\sum_{j=1}^{3}A_{aj}\nabla_{\parallel}\alpha_{aj} \qquad \text{(sources)} \end{aligned}$$

(**Hinton Wong**).

We see here the parallel convection (i.e. from the parallel variation of the density f_a), the energy change due to work against the parallel electric field (since $\Phi \sim \theta$) and collisions. The rest are sources.

The equilibrium

$$f_a^{(0)} = \frac{n_a\left(\psi\right)}{\left(\sqrt{\pi}v_{th,a}\right)^3}\exp\left(-\frac{H}{T_a}\right)$$

The *thermodynamic forces* are

$$\begin{aligned}
A_{a1} &= \frac{n_a'}{n_a} + \frac{T_a'}{T_a} \\
A_{a2} &= \frac{T_a'}{T_a} \\
A_{a3} &= \frac{\omega'}{\omega}
\end{aligned}$$

(the last "force" is analogous to the β-plane variation of the Coriolis frequency).

The *fluxes* are

$$\begin{aligned}
\alpha_{a1} &= \frac{m_a}{e_a}\left(\frac{I v_\parallel}{B} + \omega R^2\right) \\
\alpha_{a2} &= \left(\frac{H}{T_a} - \frac{5}{2}\right)\alpha_{a1} \\
\alpha_{a3} &= \frac{m_a^2 \omega}{2 e_a T_a}\left[\left(\frac{I v_\parallel}{B} + \omega R^2\right) + \mu \frac{|\boldsymbol{\nabla}\psi|^2}{m_a B}\right]
\end{aligned}$$

We **note** that for α_{a1} we have the first part

$$\frac{I}{\Omega_a} v_\parallel$$

or, including the other factors, the parallel gradient of this "flux" (which is dual to the "force" $\sim \frac{\partial}{\partial\psi} p$)

$$v_\parallel \nabla_\parallel \left(\frac{m_a}{e_a}\frac{I}{B} v_\parallel\right)\frac{\partial f_M}{\partial \psi} = \frac{m_a}{e} I \; v_\parallel \nabla_\parallel \left(\frac{v_\parallel}{B}\right) \times \frac{\partial f_M}{\partial \psi}$$

and this is $\mathbf{v}_D \cdot \boldsymbol{\nabla} f_M$, since

$$\mathbf{v}_D \cdot \boldsymbol{\nabla}\psi = v_\parallel \nabla_\parallel \left(I \frac{v_\parallel}{\Omega_a}\right)$$

The other notations

$$\mu = \frac{m_a v_\perp^2}{2B} \; \text{(note the mass)}$$

$$H = \frac{m_a v^2}{2} + e_a \widetilde{\Phi} - \frac{m_a \omega^2 R^2}{2}$$

$$\mathbf{B} = I(\psi)\, \boldsymbol{\nabla}\varphi + \boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi$$

Now important assumptions on the rotation are done

$$\omega = -\frac{d\Phi_0}{d\psi}$$

where

$$\Phi_0 \equiv \text{ average potential on surface } \psi$$

this may come from various sorces, basically there is Φ_0 of equilbrium due to diffusion.

Rotation velocity results from $E \times B$ where E is the radial $\sim \psi$ variation and the magnetic field is B_θ.

$$\begin{aligned} \mathbf{v}_{rot} &= \omega R^2 \boldsymbol{\nabla} \varphi \\ &\sim \text{toroidal} \end{aligned}$$

NOTE
This velocity does not refer explicitely to the NBI momentum injection.
The source of Φ_0 is not explicit. If it is only diffusion then E_r is small.
END

The transformation to the rotating frame.
The velocity $\mathbf{v}$ above is in this rotating frame.

NOTE
For comparison, in **Rosenbluth Hazeltine Hinton** see *solutions drift kinetic.tex.*

$$f_M \left\{ v_D|_r \ \left[A_1 + A_2 \left(\epsilon - e\Phi \right) \right] + v_\parallel \left(\frac{B_\theta}{B} \frac{\partial \widehat{f}}{r \partial \theta} - \frac{eE_\parallel}{T} \right) \right\} = C\left(f \right)$$

This can be rewritten

$$v_\parallel \frac{B_\theta}{B} \frac{\partial \widehat{f}}{r\partial\theta} - v_\parallel \left(-e \nabla_\parallel \overline{\Phi} \right) \frac{\partial f_M}{\partial \epsilon} - C\left(f \right) = -v_{D,r} \left[A_1 + A_2 \left(\epsilon - e\Phi \right) \right] f_M$$

where the *forces* are, for species a

$$A_{1a} = \frac{d}{dr} \ln n_a - \frac{3}{2} \frac{d}{dr} \ln T_a + \frac{e_a}{T_a} \frac{d\Phi}{dr}$$

$$A_{2a} = \frac{1}{T} \frac{d}{dr} \ln T$$
$$a \equiv e, i$$

and

$$\begin{aligned} f_M &= \frac{n_a \left(\psi \right)}{\left[\pi \ 2T_a / m_a \right]^{3/2}} \exp \left(-\frac{\epsilon}{T_a} \right) \\ \epsilon &= m_a \frac{v^2}{2} + e_a \overline{\Phi} \end{aligned}$$

In **Rosenbluth Hazeltine Hinton** there is a derivative of $\overline{\Phi}$ to r which occurs in A_{1a} with the other derivations of f_M. Or,

$$\frac{d\overline{\Phi}}{d\psi} = -\omega$$

END comparison.

Now follows two levels of approximations:

$$\delta^{1/2}$$

It is assumed that the electric potential on the surface varies like

$$\frac{e\widetilde{\Phi}}{T} \sim \delta^{1/2}$$

and

$$\delta$$

In the $\delta^{1/2}$ order

$$v_{\|}\nabla_{\|}f_i^{(1/2)} + v_{\|}e\nabla_{\|}\widetilde{\Phi}\frac{1}{T_i}f_i^{(0)} = C_i^{lin}\left(f_i^{1/2}\right)$$

In the order δ,

$$v_{\|}\nabla_{\|}f_i^{(1)} - C\left(f_i^{(1)}\right) = -v_{\|}e\nabla_{\|}\widetilde{\Phi}\frac{1}{T_i}\ f_i^{(1/2)} - v_{\|}f_i^{(0)}\sum A_j\nabla_{\|}\alpha_j$$

From the two equations

$$\begin{aligned} f_i \quad = \quad & f_i^{(0)} - \frac{e\widetilde{\Phi}}{T_i}f_i^{(0)} + \frac{1}{2}\left(\frac{e\widetilde{\Phi}}{T_i}\right)^2 f_i^{(0)} \\ & - \sum_{j=1}^{3} A_j\alpha_j\ f_i^{(0)} \\ & + h_i \end{aligned}$$

the new function-correction h results from the balance of parallel convection and the collisions

The solution of the eq

$$v_{\|}\nabla_{\|}h_i = C_i^{lin}\left(f_i^{(1)}\right)$$

This function depends on

$$h_i = h_i\left(H, \mu, \psi, \sigma\right)$$

and

$$h_i \equiv 0 \ \text{ in the trapping region}$$

It is the neoclassical correction that takes into account the distinction between trapped and circulating particles.

For not an extreme sheared electric field

$$\begin{aligned} f_i &= f_i^{(0)} \exp\left(-\frac{e\widetilde{\Phi}}{T_i} + M_i^2\right) \quad \text{(Boltzmannian + rotation)} \\ & -I\frac{v_\parallel}{\Omega_i}\frac{\partial f_i^{(0)}}{\partial \psi} \qquad \text{(neoclassical drift correction)} \\ & +h_i\left(H,\mu,\psi,\sigma\right) \quad \text{(neoclassical correction for trapped)} \end{aligned}$$

where

$$\begin{aligned} M_i^2 &= \frac{m_i \omega^2 R^2}{2T_i} \\ & \quad \text{Mach number of the background ions} \\ & \quad \text{in a rotation with } \omega \end{aligned}$$

or $M_i = \frac{\omega R}{v_{th,i}}$.

NOTE one could say that h_i is the usual function g of **Rosenbluth**.

The known neaoclassical correction is ρ_θ / L_n;

The visible difference is that in order zero, instead of a Maxwellian we have a Maxwellian corrected with

- the Boltzmannian *variation of the electrostatic potential on surface* $\widetilde{\Phi}(\theta)$, and
- the centrifugal force, as a new potential.

END

In view of applying *neutrality* one calculates the densities

$$n_i\left(\psi,\theta\right) = n_i^{(0)}\left(1 - \frac{e\widetilde{\Phi}}{T_i} + M_i^2 \ + O\left(\delta\right)\right)$$

$$n_e\left(\psi,\theta\right) = n_e^{(0)}\left(1 + \frac{e\widetilde{\Phi}}{T_e}\right)$$

16.2 Parallel friction force between ions and impurities

Now, calculate the friction $i-Z$ force using the collision operator

$$\begin{aligned} R_{iZ,\|} &= -\int d^3v\ m_i v_\| \ \nu_{iZ}(v) \\ &\times \left[\mathcal{L}\left(f_i - f_i^{(0)}\right) + \frac{m_i v_\|}{T_i} V_{Z\|}\ f_i^{(0)}\right] \end{aligned}$$

The ion-impurity collision operator is

$$C_{iZ}^{lin} = \nu_{iZ}(v)\left[\mathcal{L} + \frac{m_i v_\|}{T_i}\ V_{Z\|}\ f_i^{(0)}\right]$$

The Lorentz operator is

$$\mathcal{L} = \frac{2}{v^2}\frac{1}{B}v_\| \frac{\partial}{\partial\lambda}\lambda v_\| \frac{\partial}{\partial\lambda} \quad \text{pitch angle}$$

with

$$\lambda = \frac{v_\perp^2}{v^2}\frac{1}{B}$$

(note that $B \approx B_0/h$ normally would leave only h in expression).

$$\nu_{iZ}(v) = \frac{3\sqrt{\pi}}{4}\frac{1}{\tau_{iZ}}\frac{1}{x^3}$$

$$x = \frac{v}{v_{th,i}}$$

$$\begin{aligned} \tau_{iZ} &= \frac{3(2\pi)^{3/2}}{Z^2 e^4}\varepsilon_0^2 \frac{\sqrt{m_i}}{\ln\Lambda}\frac{T_i^{3/2}}{n_Z} \\ &\text{in SI} \end{aligned}$$

Note $T_i^{3/2}/n_z$ as usual. The full formula is however function of v, not yet averaged.

The parallel flow velocity of the *impurities* is (**Helander3999**)

$$\begin{aligned} V_{Z\|} &= -I\frac{1}{B}\frac{d\Phi_0}{d\psi} \left(\text{this will be } \sim \frac{1}{B_\theta}\left(-\frac{d\Phi_0}{dr}\right) = \frac{E_r}{B_\theta}\right) \\ &+\frac{K_Z(\psi)\ B}{n_Z} \end{aligned}$$

(as usual this is an assumed general form which introduces an unknown function $K_Z(\psi)$ which must represent the poloidal flow).

Then one can calculate the collisional friction

$$\begin{aligned} R_{iZ,\parallel} &= -\int d^3v\ m_i v_\parallel\ \nu_{iZ}(v) \\ &\times \left[\mathcal{L}\left(f_i - f_i^{(0)}\right) + \frac{m_i v_\parallel}{T_i} V_{Z\parallel}\ f_i^{(0)}\right] \end{aligned}$$

between background ions and impurities Z.

$$\begin{aligned} R_{iZ,\parallel} &= -\frac{1}{\tau_{iZ}}\ p_i I \frac{1}{\Omega_i}\left(\frac{p_i'}{p_i} - \frac{3}{2}\frac{T_i'}{T_i}\right) \\ &+ \frac{n_i m_i}{\tau_{iZ}}\left(u - \frac{K_Z}{n_Z}\right) B \end{aligned}$$

where

$$u = \tau_{iZ}\frac{1}{n_i B}\int d^3v\ \nu_{iZ}(v)\ \ v_\parallel h_i$$

(similar to **Connor1973**).

The friction associated with the parallel flow of impurities

$$n_Z V_{Z,\parallel} = K_Z(\psi) B$$

The parallel flow of impurities is inversely proportional with the cross section of a tube of flow. The section is small where the magnetic field is large. There, the parallel impurity flow velocity is large

$$V_{Z\parallel} \sim B$$

For comparison the first part (which contains gradients p_i/p diamagnetic) consists of a flow similar to the diamagnetic one.

The friction force is proportional with
the cross section of the guiding center orbit

Then it is

$$\text{first term} \sim \frac{1}{B}$$

This means

$$\text{force} \sim \frac{1}{B} + B$$

expected various regimes.

the correction h to the distribution function, due to the distinction trapped/untrapped.
The equation

$$v_\parallel \nabla_\parallel h_i = C_i^{lin}\left(f_i^{(1)}\right)$$

is multiplied by $\frac{B}{v_\parallel}$ and it is calculated the flux surface average

$$\left\langle \frac{B}{v_\parallel} \, C_i^{lin} \left(h - I \frac{v_\parallel}{\Omega_i} \frac{\partial f_{i0}}{\partial \psi} \right) \right\rangle = 0$$

then

$$\begin{aligned} \frac{\partial h_i}{\partial \lambda} &= x^2 \frac{f_{i0}}{\langle \widehat{n} \; v_\parallel \rangle} \left(-I \frac{T_i}{e} \frac{\partial \ln f_{i0}}{\partial \psi} + I \frac{d\Phi}{d\psi} - K_z \frac{\langle B^2 \rangle}{\langle n \rangle_z} \right) \Theta \left(\lambda_c - \lambda \right) \\ & \quad \text{only for circulating particles} \end{aligned}$$

with $\widehat{n} = n_z / \langle n_z \rangle$, $\lambda_c = \frac{1}{B_{\max}}$.

Returning to the expression of the parameter u that occurs in $R_{Zi,\parallel}$ as a velocity

$$u = \tau_{iZ} \frac{1}{B} \, \frac{1}{n_i} \int d^3 v \; \left(v_\parallel h_i \right) \; \nu_{iZ}$$

where

$$d^3 v = \sum_\sigma \int \frac{2\pi B}{m_i^2 \left| v_\parallel \right|} \; d\epsilon \; d\mu$$

and ν_{iZ} is the frequency of collisions ion-Z. Replacing here h_i

$$u = f_c \left[- \frac{T_i}{e \langle B^2 \rangle} \frac{1}{L_\perp} + \frac{K_Z}{\langle n_Z \rangle} \right]$$

$$\frac{1}{L_\perp} = -I \left(\frac{p_i'}{p_i} - \frac{3}{2} \frac{T_i'}{T_i} \right)$$

$$f_c = \frac{3 \langle B^2 \rangle}{4} \int_0^{\lambda_c} \frac{\lambda \; d\lambda}{\langle \widehat{n} \sqrt{1 - \lambda B} \rangle}$$

16.3 Neutrality and determination of $\widetilde{\Phi}$

The condition of neutrality is

$$Z n_Z = n_e - n_i$$

Equation of *parallel momentum balance*

$$\begin{aligned} m_Z n_Z \widehat{\mathbf{n}} \cdot \left(\mathbf{V}_Z \cdot \boldsymbol{\nabla} \right) \mathbf{V}_Z &= -Z n_Z e \; \nabla_\parallel \widetilde{\Phi} \\ & \quad - T_i \nabla_\parallel n_Z \\ & \quad + R_{Zi,\parallel} \end{aligned}$$

We **note** that it is *static*.

It is a balance of forces, along the line.

As in the case of the balance of the parallel forces for the instability of *drift wave*, we have

- the gradient of pressure, of the impurities,
- the friction and
- the parallel electric field.

Here they are static and - in addition, they cannot exactly compensate.

For *drift waves* $d/dt = 0$ and the parallel convection of the parallel velocity is not included. There, this equation of force balance leads to the occurence of the *collision frequency* at the denominator in the expression of the parallel velocity which is further used in the equation of continuity.

The condition

$$\frac{Ze\nabla_{\|}\widetilde{\Phi}}{T_i} = \frac{T_0}{2T_i n_0}\nabla_{\|}\left(Z^2 n_Z + Zn_{i0}M_i^2\right)$$

(the parallel electric force is balanced by the parallel gradient of the pressure) and

$$2\frac{n_0}{T_0} \equiv \frac{n_{e0}}{T_e} + \frac{n_{i0}}{T_i}$$

One introduces the notations

$$\alpha \equiv \langle n_Z\rangle Z^2 \frac{1}{T_i}\frac{T_0}{2n_0}$$

The impurity modified Mach number

$$M^2 = \underbrace{\frac{m_Z\omega^2 R^2}{2T_i}}_{\text{of order 1}}\left(1 - \frac{Zm_i}{m_Z}\frac{T_e}{T_e - T_i}\right)$$

The normalized magnetic field strength

$$b \equiv \frac{B}{\sqrt{\langle B^2\rangle}}$$

The parameters

$$g \equiv \frac{n_i m_i}{n_Z}\frac{1}{e}\frac{1}{\tau_{iZ}}\frac{1}{L_\perp}\frac{1}{\langle \mathbf{B}\cdot\boldsymbol{\nabla}\theta\rangle}$$

$$\gamma \equiv \frac{e}{T_i}L_\perp\left\langle B^2\right\rangle\ u$$

A new coordinate, defined as

$$\frac{d\vartheta}{d\theta} = \frac{\langle \mathbf{B}\cdot\boldsymbol{\nabla}\theta\rangle}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}$$

The parallel momentum balance is

$$(1+\alpha n)\,\nabla_{\|} n = n\nabla_{\|} M^2 + \frac{R_{Zi,\|}}{\langle n_Z\rangle\, T_i}$$

The flux surface average is replaced by an "averaging" by integration over the "poloidal" new coordinate ϑ.

The equation

$$\begin{aligned}(1+\alpha n)\,\frac{\partial n}{\partial \vartheta} &= g\left[n+\gamma\left(n-\frac{K_Z}{\langle n_Z\rangle\, u}\right)b^2\right] \\ &\quad +\frac{\partial M^2}{\partial\vartheta}\; n\end{aligned}$$

16.4 Poloidal rotation of impurities K_Z fixed by periodicity on ϑ.

We now impose the periodicity on ϑ.

By taking average on ϑ and imposing to be zero.

This will produce a constraint equation.

This constraint will be used to determine K_Z which is the poloidal impurity rotation part in $\mathbf{V}_Z$.

$$K_Z = \langle n_Z\rangle\;\; u\;\; \left[\left\langle nb^2\right\rangle + \frac{1}{\gamma} + \frac{1}{\gamma g}\left\langle n\frac{\partial M^2}{\partial\vartheta}\right\rangle\right]$$

This expression of K_z is inserted in the parallel balance of forces

$$\begin{aligned}(1+\alpha n)\,\frac{\partial n}{\partial \vartheta} &= g\left[n-b^2+\gamma\left(n-\left\langle nb^2\right\rangle\right)\right]b^2 \\ &\quad +\frac{\partial M^2}{\partial\vartheta}\; n \\ &\quad -\left\langle n\frac{\partial M^2}{\partial\vartheta}\right\rangle b^2\end{aligned}$$

The solution will give the profile on

$$\theta \in (0, 2\pi)$$

of the density n.

17 Poloidal asymmetries of impurity density Mollen Fulop

The flux of impurities

The velocity is $v_r \sim \frac{E_\theta}{B} \sim \frac{-\nabla \phi}{B} \sim \frac{k_\theta \phi}{B}$,

$$
\begin{aligned}
\Gamma_z &= \mathrm{Im}\left[\widehat{n}_z \left(-\frac{k_\theta \phi^*}{B}\right)\right] \\
&= \mathrm{Im}\left[\int d^3v \; J_0\left(\frac{k_\perp v_\perp}{\Omega_{cz}}\right) \; g_z \; \left(-\frac{k_\theta \phi^*}{B}\right)\right]
\end{aligned}
$$

where

$$k_\perp = \sqrt{1+\widehat{s}^2\theta^2} \; k_\theta$$

There is an electrostatic potential at equilibrium, ϕ_0 and it is *uniform* on surfaces.

The non-adiabatic part of the distribution function for impurities

$$
\begin{aligned}
& \frac{v_\parallel}{qR}\frac{\partial g_z}{\partial \vartheta} - i\left(\omega - \omega_{Dz} - \omega_E\right) g_z - C\left(g_z\right) \\
=& \quad -\left(i\omega - i\omega_{*z}^T\right)\frac{Ze\phi}{T_z} J_0\left(\frac{k_\perp v_\perp}{\Omega_{cz}}\right) \; f_{z0}
\end{aligned}
$$

where

$$f_{z0} = \frac{n_{z0}}{\left(2\pi \; T_z/m_z\right)^{3/2}} \exp\left(-\frac{\epsilon}{T_z}\right)$$

$$\epsilon = \frac{m_z v^2}{2} + Ze\phi_0$$

$$n_z\left(\mathbf{r}\right) = n_{z0}\exp\left(-\frac{Ze\phi_0\left(\mathbf{r}\right)}{T_z}\right)$$

$$
\begin{aligned}
\omega_{*z} &= -k_\theta \frac{T_z}{ZeB}\frac{1}{L_{nz}} \\
L_{nz}^{-1} &= -\frac{d\ln n_z}{dr} \\
L_{Tz}^{-1} &= -\frac{d\ln T_z}{dr}
\end{aligned}
$$

$$\omega_{*z}^T = \omega_{*z}\left[1 - L_{nz}\frac{Ze}{T_z}\frac{\partial \phi_0}{\partial r} + \left(\frac{\epsilon}{T_z} - \frac{Ze\phi_0}{T_z} - \frac{3}{2}\right)\frac{L_{nz}}{L_{Tz}}\right]$$

$$\mu = \frac{m_z v_\perp^2}{2B}$$

The magnetic drift

$$\omega_{Dz} = k_\theta\left(-2\right)\frac{\left(\epsilon - Ze\phi_0 - \frac{\mu B}{2}\right)}{R}\frac{D\left(\theta\right)}{eB}$$

$$D(\theta) = \cos\vartheta + \widehat{s}\vartheta\sin\vartheta$$

$$\widehat{s} = \frac{rq'}{q}$$

$$\omega_E = k_\theta \left(\frac{1}{B}\frac{\partial \phi_0}{\partial r}\right) - k_\theta \widehat{s}\vartheta \frac{\partial \phi_0}{r\partial \vartheta}$$

The approximation adopted in this work

$$\text{no rotation,} \quad \frac{\partial \phi_0}{\partial r} = 0$$

The $\mathbf{E} \times \mathbf{B}$ drift consists of the combination

$$E_\theta \times B_\varphi$$

and is due to the poloidally varying electric field.

$$J_0\left(\frac{k_\perp v_\perp}{\Omega_{cz}}\right) \sim 1$$

The poloidal variation of the function g_z is neglected since the

- trapping of impurities along the magnetic field line due to $\nabla_\parallel B$ or the electrostatic potential with non-uniform profile on θ, and
- parallel dynamics

are neglected.
It remains

$$-i\left(\omega - \omega_{Dz} - \omega_E\right) g_z - C\left(g_z\right) = -\left(i\omega - i\omega_{*z}^T\right) \frac{Ze\phi}{T_z} f_{z0}$$

It is adopted a collision operator

$$\begin{aligned} C\left(g_z\right) &= \frac{1}{2}\nu_D(x)\ \mathcal{L}\left(g_z\right) \\ &= \frac{1}{2}\nu_D(x)\frac{\partial}{\partial \xi}\left[\left(1-\xi^2\right)\frac{\partial g_z}{\partial \xi}\right] \end{aligned}$$

where, using

$$\begin{aligned} x &\equiv \frac{v}{v_{th,Z}} \\ v_{th,Z} &= \sqrt{\frac{2T_Z}{m_z}} \end{aligned}$$

$$\nu_D(x) = \widehat{\nu}_{ZZ}\frac{\mathrm{erf}(x) - G(x)}{x^3}$$

$$\widehat{\nu}_{ZZ} = \frac{Z^4 e_Z^4}{4\pi\varepsilon_0^2 \sqrt{m_Z} \left(2T_Z\right)^{3/2}} n_Z \ln\Lambda$$

$$G(x) = \frac{\mathrm{erf}(x) - x \frac{d\,\mathrm{erf}(x)}{dx}}{2x^2}$$

Chandrasekhar function

The eigenfunctions of the pitch angle operator

$$\frac{\partial}{\partial\xi}\left[\left(1-\xi^2\right)\frac{\partial}{\partial\xi}\right] \equiv \mathcal{L}$$

$$\mathcal{L}\ P_n(\xi) = -n(n+1)\ P_n(\xi)$$

18 Ware Diamond. Drift instability induced by the poloidally nonuniform ionization rate

This is **Ware Diamond 1993**.

It is also commented upon in *instabilities.tex*.

The equations

$$\frac{\partial n_i}{\partial t} + \boldsymbol{\nabla}\cdot(n_i \mathbf{v}_i) = n_e n_N \langle\sigma v\rangle_I$$

note that the neutrals are ionized by electron collisions. This is the source of new ions and of new electrons.

$$\frac{\partial n_e}{\partial t} + \boldsymbol{\nabla}\cdot(n_e \mathbf{v}_e) = n_e n_N \langle\sigma v\rangle_I$$

$$\begin{aligned} n_i m_i \frac{dv_{\|i}}{dt} &= -\nabla_{\|} p_i + \mu_{\|} \nabla_{\|}^2 v_{\|i} \\ &\quad - n_i m_i\ n_N \left(v_{\|i} - v_{\|N}\right)\left(\langle\sigma v\rangle_{cx} + \langle\sigma v\rangle_I\right) \quad \text{source} \end{aligned}$$

This is the *parallel momentum* conservation for the ions. There is parallel gradient of the pressure and there is viscosity that generically acts like $\mu\Delta v$. It may be collisional or Landau damping. The *source of momentum* comes from the *ionization* and *charge exchange*. For both there is the *relative velocity* that matters

$$\begin{aligned} & v_{\|i} - v_{\|N} \\ = & \text{ relative velocity of ions with reference to neutral's} \end{aligned}$$

At every charge exchange event the ion population receives the momentum of the previous *neutral* which now is an ion

$$m_i v_{\|N} \times n_i n_N \langle\sigma v\rangle_{cx}$$

and loses the momentum of the previous *ion* which has become a neutral

$$m_i v_{\parallel i} \times n_i n_N \left\langle \sigma v \right\rangle_{cx}$$

Finally, the source of momentum from charge exchange

$$\begin{aligned} & m_i v_{\parallel N} \times n_i n_N \left\langle \sigma v \right\rangle_{cx} - m_i v_{\parallel i} \times n_i n_N \left\langle \sigma v \right\rangle_{cx} \\ = \quad & -n_i m_i n_N \left(v_{\parallel i} - v_{\parallel N} \right) \left\langle \sigma v \right\rangle_{cx} \end{aligned}$$

At every ionization event the ion population receives the momentum of the previous *neutral* that has become a new ion

$$m_i v_{\parallel N} \times n_i n_N \left\langle \sigma v \right\rangle_I$$

The net momentum that can be a source for the ion population is that which is due to the *relative* velocity

$$m_i \left(v_{\parallel N} - v_{\parallel i} \right) \times n_i n_N \left\langle \sigma v \right\rangle_I$$

In charge exchange one ion disappears and a new ion appears.
In ionization just a new ion appears. (The neutrals are only a background).

The equation for electron thermal energy, which is essentially connected with the ionization processes

$$\begin{aligned} \frac{3}{2} n \frac{dT_e}{dt} = \quad & -p_e \left(\boldsymbol{\nabla} \cdot \mathbf{v}_e \right) \quad \text{(compressibility)} \\ & +0.71 \frac{T_e}{e} \nabla_{\parallel} j_{\parallel} \quad \text{(Braginsky)} \\ & +\chi_{\parallel} \nabla_{\parallel}^2 T_e \qquad \text{(parallel diffusional conduction)} \\ & -\left(W_I + \frac{3}{2} T_e \right) n_e n_N \left\langle \sigma v \right\rangle_I \quad \text{(loss by ionization)} \end{aligned}$$

NOTE regarding the Braginsky term $0.71 \frac{T_e}{e} \nabla_{\parallel} j_{\parallel}$, there is the density conservation

$$\frac{dn}{dt} = -\frac{1}{e} \nabla_{\parallel} j_{\parallel}$$

END

The ion perpendicular momentum equation is the ion polarization, with the *vorticity* evolution

$$\frac{1}{B^2} n_i m_i \frac{d}{dt} \nabla_{\perp}^2 \phi = \nabla_{\parallel} j_{\parallel} \quad \text{vorticity}$$

Note. Assuming that the density is constant, the only source for creation of *vorticity* is parallel variation of the parallel velocity of the electrons. This has two sources

- neoclassical effect of magnetic mirror, as shown by the equations of motion of the *particle*;
- variation of $v_\parallel$ due to instability and electrostatic potential ϕ, result of balance of forces along the line: gradient of pressure, parallel electric field, collisional friction (or resistivity); in addition, parallel gradient of the temperature Braginsky.

Since along a magnetic field line the velocity of the electrons has variation (for example **Hassam Kulsrud**) there should exist intervals along the line where the vorticity is increased or decreased. This means that the *shear* of the local flow changes. **End**.

Note that the current $j_\parallel$ is carried by the electrons. When we discuss about flow we refer to ions, since they are heavy. **End.**

The conservation of the electron parallel momentum, with electron inertia neglected

$$
\begin{aligned}
0 &= -\frac{1}{en_e}\nabla_\parallel p_e - \nabla_\parallel \phi \\
&\quad -\eta_\parallel j_\parallel \\
&\quad -\frac{0.71}{e}\nabla_\parallel T_e
\end{aligned}
$$

The current is entirely attributed to electrons.

The energy lost per ionization collision

$$W_I \approx 20 \ eV$$

The velocities are for *fluids*

$$
\begin{aligned}
\mathbf{v}_e &= \left(v_{\parallel i} - \frac{j_\parallel}{en_e} \right) \widehat{\mathbf{n}} \qquad (e - i \text{ relative velocity}) \\
&\quad + \frac{-\boldsymbol{\nabla}\phi \times \widehat{\mathbf{n}}}{B} \qquad \text{(electric)} \\
&\quad + \frac{1}{n_e}\frac{1}{eB}\left(-\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p_e \right) \qquad \text{(diamagnetic)}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{v}_i &= v_{\parallel i}\widehat{\mathbf{n}} \\
&\quad + \frac{-\boldsymbol{\nabla}\phi \times \widehat{\mathbf{n}}}{B} \qquad \text{(electric)} \\
&\quad + \frac{1}{B\Omega_i}\frac{d}{dt}\boldsymbol{\nabla}_\perp \phi \qquad \text{(polarization)}
\end{aligned}
$$

(Indeed see **Hirshman bootstrap**).

The divergence of the *electric flux* in the ion continuity equation produces poloidal mode coupling

$$
\begin{aligned}
& \nabla\cdot \left(n\mathbf{V}_E\right) \\
= & \nabla\cdot \left(n\frac{-\nabla\phi\times\widehat{\mathbf{n}}}{B}\right) = n\nabla\phi\cdot\nabla\times\left(\frac{\widehat{\mathbf{n}}}{B}\right) \\
= & n\nabla\phi\cdot\left[\widehat{\mathbf{n}}\times\nabla\ln B+\widehat{\mathbf{n}}\times\left(\widehat{\mathbf{n}}\cdot\nabla\right)\widehat{\mathbf{n}}\right]
\end{aligned}
$$

The two terms in the paranthesis: variation of the magnetic field and curvature. They produce neoclassical drifts of the ions. *But, no velocities as coefficients.*

Later in the same work

$$
\begin{aligned}
\mathbf{v}_{De} &= -\rho_s c_s\ \widehat{\mathbf{n}}\times\nabla\left(\frac{\widehat{\mathbf{n}}}{B}\right) \\
&\approx 2\rho_s c_s\frac{1}{R_0}\left(\widehat{\mathbf{e}}_\theta\ \cos\theta+\widehat{\mathbf{e}}_r\ \sin\theta\right)
\end{aligned}
$$

is the drift velocity of the electrons.

$$
2\rho_s c_s = 2\frac{mc_s^2}{eB}=\frac{2T_e/m_i}{|\Omega_e|}
$$

The thermal velocity $2T_e/m_i$ is approximated with the *velocity* $\frac{v_\perp^2}{2}+v_\parallel^2$ (?)

$$
2\rho_s c_s\frac{1}{R_0}\approx\frac{1}{|\Omega_c|}\frac{v^2}{R_0}
$$

Note

We have the expressions

$$
\begin{aligned}
& \nabla\times\left(\frac{\widehat{\mathbf{n}}}{B}\right) \\
\approx & 2\frac{1}{R_0}\left(\widehat{\mathbf{e}}_\theta\ \cos\theta+\widehat{\mathbf{e}}_r\ \sin\theta\right) \\
= & \widehat{\mathbf{n}}\times\nabla\ln B+\widehat{\mathbf{n}}\times\left(\widehat{\mathbf{n}}\cdot\nabla\right)\widehat{\mathbf{n}}
\end{aligned}
$$

For this expression $\nabla\times\left(\frac{\widehat{\mathbf{n}}}{B}\right)$ to be a part of the drift velocity one has to assume that

$$
\frac{v_\perp^2}{2}\approx v_\parallel^2
$$

but these are velocities of a particle, not guiding center velocities, the first is $\mu=v_\perp^2/(2B)$.

End

This is a source of toroidal coupling.

Another source of toroidal mode coupling is the *variation of the ionization rate*

$$\gamma_I \equiv n_N \langle \sigma v \rangle_I$$

with the poloidal angle

$$\gamma_I = \gamma_{I_0} \left[1 + \delta_I \cos(\theta - \theta_0) \right]$$

For TEXT

$$\begin{aligned} \delta_I &\approx 0.8 \\ \theta_0 &= 0 \end{aligned}$$

We consider perturbations to the equilibrium variables.

The normalizations

$$\widetilde{\phi} = \frac{e\phi}{T_{e0}}$$

$$\widetilde{T} = \frac{\widetilde{T}_e}{T_{e0}}$$

$$\widetilde{n} = \frac{\widetilde{n}}{n_0}$$

$$\widetilde{v}_{\|} = \frac{\widetilde{v}_{\| i}}{c_s}$$

where

$$c_s = \sqrt{\frac{T_{e0}}{m_i}}$$

The neutrality

$$\widetilde{n}_i = \widetilde{n}_e = \widetilde{n}$$

The equations for perturbations

$$\begin{aligned} \frac{d\widetilde{n}}{dt} &= (-\mathbf{v}_{*n} - \mathbf{v}_{de}) \cdot \boldsymbol{\nabla} \widetilde{\phi} && \text{diamagnetic and DRIFT advection of the adiabatic density } e\phi/T \\ & -\frac{d}{dt} \rho_s^2 \nabla_\perp^2 \widetilde{\phi} && \text{polarization} \\ & -c_s \nabla_{\|} \widetilde{v}_{\|} && \text{compressibility} \\ & +\gamma_I \widetilde{n} && \text{ionization} \\ & +\beta \widetilde{T} && \text{variation of } \gamma_I \text{ with temperature} \end{aligned}$$

The diamagnetic velocity due to the gradient of the density L_n is

$$\begin{aligned} \mathbf{v}_{*n} &= \frac{\rho_s c_s}{L_n} \widehat{\mathbf{e}}_\theta \\ &= \rho_s c_s \boldsymbol{\nabla} n_0 \times \widehat{\mathbf{n}} \end{aligned}$$

The *drift velocity* in the *magnetic* toroidal field

$$\begin{aligned}
\mathbf{v}_{de} &= -\rho_s c_s \widehat{\mathbf{n}} \times \boldsymbol{\nabla} \left(\frac{\widehat{\mathbf{n}}}{B} \right) \\
&\approx \frac{2\rho_s c_s}{R_0} \left(\cos\theta \; \widehat{\mathbf{e}}_\theta + \sin\theta \; \widehat{\mathbf{e}}_r \right)
\end{aligned}$$

The compressibility comes from the thermal energy equation and

$$\widetilde{v}_\parallel = \frac{\widetilde{v}_{\parallel i}}{c_s} \text{ is the normalization.}$$

The parameter

$$\beta = T_{e0} \frac{\partial \gamma_I}{\partial T_{e0}}$$

The equation for the velocity: it is for the *ions* since they are the plasma flow.

$$\begin{aligned}
\frac{d\widetilde{v}_\parallel}{dt} &= -c_s \nabla_\parallel \left(\widetilde{n} + \widetilde{T} \right) \quad \text{(parallel gradient of pressure)} \\
&\quad + \mu_\parallel \nabla_\parallel^2 \widetilde{v}_\parallel \quad \text{(viscosity)} \\
&\quad - \gamma_{cx}^{eff} \; \widetilde{v}_\parallel \quad \text{(source)}
\end{aligned}$$

The first term is very familiar: it is the force resulted from the parallel gradient of the pressure. Typical for drift waves. The second is also typical, a dissipation, as usual it is proportional with the Laplacian of the velocity.

The last term is the source of *momentum*, generated by charge exchange.

There is no electric field $\nabla_\parallel \phi$ for the ions. Such a term is acting for electrons.

The equation for the thermal energy of electrons

$$\begin{aligned}
\frac{d\widetilde{T}}{dt} &= -\mathbf{v}_{*T} \cdot \boldsymbol{\nabla} \widetilde{\phi} \qquad \left(\text{electric work, } v_{dia}^T \times eE \right) \\
&\quad + \chi_\parallel \nabla_\parallel^2 \widetilde{T} \quad \text{(parallel conduction)} \\
&\quad - \frac{2}{3} \alpha \chi \nabla_\parallel^2 \left(\widetilde{\phi} - \widetilde{n} - \alpha \widetilde{T} \right) \quad \text{(thermal Braginsky)} \\
&\quad - \frac{2}{3} c_s \nabla_\parallel \widetilde{v}_\parallel \quad \text{(compression)} \\
&\quad - \overline{W}_I \; \gamma_I \widetilde{n} \quad \text{(loss of energy at every ionization)} \\
&\quad - \left(\gamma_I + \widetilde{W}_I \beta_I \right) \widetilde{T} \quad \text{(loss of energy FOR ionization)}
\end{aligned}$$

where

$$\alpha = 1.71$$

The equation for vorticity

$$\frac{d}{dt}\rho_s^2\nabla_\perp^2\widetilde{\phi} = -\mathbf{v}_{de}\cdot\boldsymbol{\nabla}\left(\widetilde{n}+\widetilde{T}\right) \quad \text{(diamagnetic advection of the pressure)}$$
$$-\chi\nabla_\parallel^2\left(\widetilde{\phi}-\widetilde{n}-\alpha\widetilde{T}\right) \quad \text{(dissipation)}$$

Let us consider

$$-\mathbf{v}_{de}\cdot\boldsymbol{\nabla}\left(\widetilde{n}+\widetilde{T}\right) \quad \text{(magnetic drift velocity } \times \text{ force of grad-pressure)}$$

It is a source of vorticity. Then it must have a content like

$$\nabla_\parallel j_\parallel$$

The convection by the electron-drift velocity $\mathbf{v}_{de}$ of the density perturbation. This involves the vertical drift of the particles and the gradient, which is radial, of the pressure. The result is a change that is parallel with the line.

For notations

$$\gamma_{cx}^{eff} = \gamma_{cx}+\gamma_I$$
$$\overline{W}_I = 1+\frac{2}{3}\frac{W_I}{T_{e0}}$$

Ballooning representation for the potential

$$\begin{aligned}\widetilde{\phi} &= \exp(-i\omega t)\\ &\times\sum_n \exp(in\varphi)\sum_m \exp(-im\theta)\\ &\times\int d\eta\ \exp\{i[m-nq(r)]\eta\}\ \widetilde{\phi}_n(\eta)\end{aligned}$$

where

$$\eta \equiv \text{cordinate along the magnetic field line}$$

The resulting equation

$$\begin{aligned}&\left(\frac{\partial^2}{\partial\eta^2}+Q(\Omega,\eta)\right)\widetilde{\phi}_n\\ &\qquad +\widetilde{Q}\widetilde{\phi}_n\\ &= 0\end{aligned}$$

with

$$\begin{aligned}-Q(\Omega,\eta) &= -\eta_s^2\Omega^2\left(1+i\frac{\overline{\gamma}_{cx}}{\Omega}\right)\\ &\times\left[1-\frac{1}{\Omega}+b\left(1+\widehat{s}^2\eta^2\right)\right.\\ &+2\frac{\varepsilon_n}{\Omega}(\cos\eta+\widehat{s}\eta\sin\eta)\\ &\left.-i\frac{\overline{\gamma}_I}{\Omega}[1+\delta_I\cos(\eta-\eta_0)]\right]\end{aligned}$$

with

$$\eta_s = qR\frac{\omega_{*n}}{c_s} = \frac{\omega_{*n}}{c_s/\left(qR\right)} = \frac{\text{dia freq.}}{\text{transit freq.}}$$

$$b = k_\theta^2 \rho_s^2$$

$$\widehat{s} = \frac{rq'}{q}$$

$$\eta_0 = \theta_0$$

$$\Omega = \frac{\omega}{\omega_{*n}}$$

$$\overline{\gamma}_I = \frac{\gamma_{I_0}}{\omega_{*n}}$$

$$\overline{\gamma}_{cx} = \frac{\gamma_{cx}^{eff}}{\omega_{*n}}$$

$$\overline{\beta}_I = \frac{\beta_{I_0}}{\omega_{*n}}$$

There is a source which results from the *non-adiabatic* density response

$$\widetilde{h}_n = \widetilde{n}_n - \widetilde{\phi}_n$$