Connection between vorticity and density in fluids and plasmas

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December 12, 2022

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1 Introduction

The objective of these notes is to highlight the connection between the density and vorticity mainly on large spatial scales (background, in contrast to turbulence). The far objective is to prove that, if something happens with the vorticity, then the density will react correspondingly.

Since the field theoretical models suggest that there is an intrinsic organization of the vorticity in the strongly magnetized plasma, including a vorticity pinch toward the axis, the corresponding behavior of the density may partly explain the density pinch in tokamak. In addition, the density will react distinctly according to the sense of rotation of the plasma relative to the direction of the confining magnetic field. However these are *not* discussed in the present notes, focused on the basic connection between ω and n.

In a restricted sense, these notes are meant to assist an elementary examination of the role of the vectorial nonlinearity in various analytical models. It is also called Poisson bracket nonlinearity and arises in all calculations of the nonlinear advection of the fluctuation by its own velocity, but in higher order.

The notes should have to be pedagogical but they are not. Maybe because they are a draft. Everything what is included here can be found, in a better presentation, in many other articles and reviews.

2 The fluid velocity

The first object of interest here is the particle (electrons and ions) velocity.

The main part is the $E \times B$ velocity, the same for both species, where E can be created by the plasma waves or by external factors. A situation where the electric field is imposed from exterior is when there is a stationary, externally sustained, plasma rotation. It requires to be described to introduce the equivalent electrostatic potential

$$\mathbf{v}^{rot} = \frac{-\boldsymbol{\nabla}\phi^{rot} \times \widehat{\mathbf{n}}}{B} \tag{1}$$

Even if the static potential ϕ^{rot} is actually introduced to describe a zonal flow, which is created by the effect of the Reynolds stress of the fluctuating plasma waves, from the point of view of local analysis this potential can be considered external and sustained, and the waves that lead to zonal flow are no more mentioned. However, besides ϕ^{rot} we have to consider potential perturbations $\widetilde{\phi}$, related to the perturbations which make the object of our local analysis. The total potential is denoted ϕ.

The second part of the velocity is the ion polarization drift, to be derived below. Its expression is

$$\begin{aligned} \mathbf{j}_{ip} &= \frac{m_i n_i}{B^2}\frac{d\mathbf{E}_\perp}{dt} \\ &= \frac{m_i n_i}{B^2}\left(\frac{\partial \mathbf{E}_\perp}{\partial t} + \left[\frac{(-\boldsymbol{\nabla}_\perp\phi \times \widehat{\mathbf{n}})}{B}\cdot\boldsymbol{\nabla}\right]\mathbf{E}_\perp\right) \end{aligned} \tag{2}$$

The ion polarization drift is simply a higher order component of the ion fluid velocity (is smaller than $E \times B$ part by a factor $1/B$). However its importance is revealed in some particular cases and inherits its name and physical role from these situations.

For example it may be useful to think of this current as associated with the adiabatic electron response. In drift wave theory the main nonlinear effect is the advection of the perturbed density by the perturbation itself

$$\frac{(-\boldsymbol{\nabla}_\perp\phi \times \widehat{\mathbf{n}})}{B}\cdot\boldsymbol{\nabla}\widetilde{n} \tag{3}$$

Even at low amplitude it is a very effective nonlinearity and much of the renormalization theory was devoted to replace it by something reasonable. This corresponds in particular to the situations where the z direction does not play any role in the dynamics, the problem is purely two-dimensional.

The system can be translated along the z direction and remains invariant. Slab models without magnetic shear are from this category. When magnetic shear is introduced the z-translation invariance is lost and any wave perturbation will aquire a finite variation along the magnetic field lines. The wave perturbation assumed parallel to z direction will intercept the magnetic lines with tilted direction and along any magnetic line there will be maxima and minima of potential. In these cases what matters is the parallel electron motion in these potential spatial oscillations. Since the electron thermal velocity is very high compared with the wave phase velocity (the velocity at which the profile of the potential is changing in time along B lines) the electrons will have enough time to place themselves in the most natural way, *i.e.* in a Boltzmann distribution

$$n_e = n_{e0} \exp\left(\frac{|e|\,\phi}{T_e}\right) \tag{4}$$

which means that a small perturbation of the electron density is

$$\widetilde{n}_e \equiv \frac{n_e}{n_{e0}} - 1 \simeq \frac{|e|\,\phi}{T_e} \tag{5}$$

This is the *adiabatic* response. Due to the quasi-neutrality

$$\widetilde{n}_e \approx \widetilde{n}_i \tag{6}$$

the ion density will do the same thing. In Hasegawa-Mima the parallel wavelength is much greater than the density gradient length

$$k_{\|} L_n \sim O(\varepsilon) \quad \text{where} \quad \varepsilon = \frac{\rho_s}{L_n}$$

and the ion oscillations can be neglected. Two elements have been invoked to arrive here:

1. the loss of z invariance (also called by Hasegawa and Mima *pseudo-three-dimensionality*, for obvious reasons);
2. the Boltzmann response of the electrons in the newly finite potential variations along the magnetic line, due to the very high v_{the}.

In most instabilities the electron response is *not* adiabatic. This is due to all mechanisms that make the electron to not be able to follow exactly the wave: collisions, drifts, trapping. In some cases it is however posible to express these effects in the form: $\widetilde{n} = \frac{|e|\phi}{T_e}(1 + i\delta)$ and this will not change the analysis presented below, when δ is higher order.

When the response is adiabatic, the main nonlinearity becomes ineffective

$$\frac{(-\boldsymbol{\nabla}_{\perp}\phi \times \widehat{\mathbf{n}})}{B} \cdot \boldsymbol{\nabla}\widetilde{n} = \frac{(-\boldsymbol{\nabla}_{\perp}\phi \times \widehat{\mathbf{n}})}{B} \cdot \boldsymbol{\nabla}\left(\frac{|e|\,\phi}{T_e}\right) \equiv 0 \tag{7}$$

(except for strong variations of T_e).

The plasma remains nonlinear but a different nonlinearity becomes dominant, coming from the convection of the potential fluctuations by the next order component of the ion velocity, the ion polarization drift. It can be obtained from the ion momentum conservation equation

$$m_i n_i \left[\frac{\partial \mathbf{v}_i}{\partial t} + (\mathbf{v}_i \cdot \boldsymbol{\nabla})\,\mathbf{v}_i\right] = -\boldsymbol{\nabla} p_i + e_i n_i \mathbf{E} + e_i n_i \mathbf{v}_i \times \mathbf{B} \tag{8}$$

(we do not take the friction yet) by extracting $\mathbf{v}_i$ from the right, via a vectorial product with $\mathbf{B}$. This product will extract, naturally, only the perpendicular on $\mathbf{B}$ component of $\mathbf{v}_i$,

$$\begin{aligned} -\mathbf{v}_{i\perp} B^{\mathbf{2}} = -\mathbf{E} \times \mathbf{B} - & \frac{1}{e_i n_i}(-\boldsymbol{\nabla} p_i \times \mathbf{B}) \\ & + \frac{m_i}{e_i}\left[\frac{\partial \mathbf{v}_i}{\partial t} + (\mathbf{v}_i \cdot \boldsymbol{\nabla})\,\mathbf{v}_i\right] \times \mathbf{B} \end{aligned} \tag{9}$$

This looks like an iteration on $\mathbf{v}_i$ and we must take in the right hand side the principal component of the ion's velocity, the $E \times B$ one. More detailed cases, where $\mathbf{v}_i$ will contain also other contrbutions, already seen in the RHS of the equation, appear to be necessary in different situations. At this moment we take only

$$\mathbf{v}_i \sim \frac{\mathbf{E}_{\perp} \times \mathbf{B}}{B^2} \tag{10}$$

or, for a purely electrostatic origin of $\mathbf{E}_{\perp}$,

$$\mathbf{v}_i = \frac{-\boldsymbol{\nabla}_{\perp}\phi \times \widehat{\mathbf{n}}}{B} \tag{11}$$

and obtain for the component of $\mathbf{v}_{i\perp}$ perpendicular to the magnetic field, after expanding the double vectorial product in the last term of Eq.(9),

$$\begin{aligned} \mathbf{v}_{i\perp} \;\; = \;\; & \frac{(-\boldsymbol{\nabla}_{\perp}\phi \times \widehat{\mathbf{n}})}{B} + \frac{1}{e_i n_i B}(-\boldsymbol{\nabla} p_i \times \widehat{\mathbf{n}}) \\ & - \frac{m_i}{e_i B^2}\left\{\frac{\partial}{\partial t}\boldsymbol{\nabla}_{\perp}\phi + \left[\frac{(-\boldsymbol{\nabla}_{\perp}\phi \times \widehat{\mathbf{n}})}{B} \cdot \boldsymbol{\nabla}_{\perp}\right]\boldsymbol{\nabla}_{\perp}\phi\right\} \end{aligned} \tag{12}$$

(We have neglected the space variation of B when the operator in the square bracket is applied on $\boldsymbol{\nabla}_{\perp}\phi/B$). We recognize the terms in the expression of $\mathbf{v}_{i\perp}$: the first is the $E \times B$ velocity, the second is the diamagnetic velocity and the third is the *polarization drift velocity*.

Several fundamental properties should be reminded, strictly pertaining to the *adiabatic* density case:

1. it is based on the *compressibility* of the plasma. It cannot exist in an incompressible plasma since the density is assumed adiabatic $\widetilde{n} \sim \phi$ and the potential ϕ can change. In incompressible fluids there is a term identical with the second term in the curly brackets. Its origin is however related with the variation of the heigth of a shallow layer of fluid (or atmosphere), this height being equaivalent to the potential ϕ of plasma.

2. it is of *high differential degree*, compared with the convective nonlinearity (now vanishing). While the convection of the fluctuating density by the fluctuating potential itself contains a single space derivative on the potential ϕ, the second term in the ion polarization drift contains three gradient operators. This means that this term will produce a factor $\sim k_{\perp}^{3}$ after the Fourier transformation of ϕ, and this means that the term is more important for high $k_{\perp}$ values. Or, this means small wavelengths and we see that this term dominates the small spatial scales, of the order of the ion sonic Larmor radius, ρ_s.

3. the ion polarization drift is entirely due to the *finite Larmor radius*. It comes from the Lorentz force in the ion momentum equation.

4. it has a *small magnitude* because it contains at the denominator B^2. Only the fast spatial variation of the potential ϕ can make it significant.

5. it has an *inertial* nature (*i.e.* mass matters), since it comes from the Lagrangian (convective) derivative of the momentum, which contains the mass. Therefore the *electron* polarization current will in general be neglected.

To these properties we should add: this term generates vortical motions, on spatial scales of the order of ρ_s. This is typical for the Poisson bracket nonlinearity (**Pavlenko**).

In the general case, where there is *no adiabaticity* of the density, the ion polarization current still can be calculated as a higher order correction to $\mathbf{v}_i$ by iterating the $E \times B$ velocity in the ion momentum equation, exactly as shown above. But the dominant nonlinearity is the $E \times B$ fluctuating convection of the density fluctuations. This acts however on large spatial scales (of the order of the eddies), while at scales of the order ρ_s the ion polarization drift-induced nonlinearity may become dominant. It is no more necessary to invoke the compressibility of plasma when we calculate $\mathbf{v}_{ip}$. The divergence of the ion polarization flow is not necessarly zero.

Concerning the name (ion polarization flow or current). This current is similar to the current that may arise in a semiconductor, for instance. When an electric field is applied on a semiconductor the charges of one sign move in one direction and the charges of opposite sign are fixed or move in the opposite direction. If the charges cannot leave the sample they accumulate at the boundaries where the external field is applied. The accumulation of charges creates an electric field that opposes to the external one. The current that has flown in response to the external electric field and has created the charge polarization of the sample is a *polarization* current. All along the relative motion of the two kinds of charges the local neutrality is preserved, the charges are simply sliding one over the other to produce at the boundaries the charge accumulations that will compensate the external electric field.

Once we adopt this picture, we can easily generalize it to situations where the accumulation of charges are not necessarly only at the boundaries but can also remain in the volume, retained by local balance of forces. But the origin is the one explained.

In some cases it is necessary to take in $\mathbf{E}_\perp$ in Eq.(10) both the electrostatic and the magnetic contributions, $\mathbf{E}_\perp = -\boldsymbol{\nabla}_\perp \phi - \partial \mathbf{A}/\partial t$. Then it is convenient for now to keep the general form of the ion polarization current

$$\mathbf{v}_{ip} = \frac{m_i}{e_i B^2} \frac{d\mathbf{E}_\perp}{dt} \tag{13}$$

3 The current

Since when examining the perturbations our instruments are density conservation (electrons and ions) and momentum conservation (parallel and perpendicular, electrons and ions) the flow of particles, or, equivalently, their currents, are needed explicitely. The total (electrons + ions) current is

$$\mathbf{j} = \frac{m_i n_i}{B^2} \frac{d\mathbf{E}_\perp}{dt} + \widehat{\mathbf{n}} \sum_j \frac{n_j e_j^2}{m_j} \int_0^t E_\parallel dt' \tag{14}$$

and the conservation of the charge is

$$\frac{\partial \rho_Q}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0 \tag{15}$$

When we refer to perturbations the charge density has no time variation at quasi-neutrality

$$\frac{\partial \rho_Q}{\partial t} \simeq 0 \tag{16}$$

This may seem strange: on one hand we accept that at stationarity there is a potential ϕ whose space profile leads to a finite Laplacian, therefore to a charge density in every point of the volume

$$\boldsymbol{\nabla}_{\perp}^{2} \phi = -\frac{\rho_Q}{\varepsilon_0} \tag{17}$$

and on the other hand we simultaneously assume quasi-neutrality. There is no contradiction however: the problem of non-neutrality during time variations is rised on the space scale of the Debye length, usually very small. Beyond r_{Debye} the charge neutrality of perturbations is ensured. The finite Laplacean of ϕ, hence a finite charge density ρ_Q exists on large spatial scales (**Petviashvili, Miura**). For example a stationary rotation implicitely means a stationary charge separation. If we can represent the onset of rotation in a plasma layer as being due to some external factor then the increase of the velocity is accompanied by a charge separation. This charge separation will provide the potential from which the velocity in the layer can be calculated as $\mathbf{v} = -\boldsymbol{\nabla}_{\perp} \phi^{ext} \times \widehat{\mathbf{n}}/B$ (*ext* means that is supported from external action, although it is created by internal mechanism, *i.e.* the polarization which follows after externally imposing a rotation). It is the *ion polarization current* that has moved the charges and hence has polarized electrically the layer.

For fast, turbulent-like perturbation, we have therefore the equation

$$\boldsymbol{\nabla} \cdot \mathbf{j} \simeq 0 \tag{18}$$

or

$$\boldsymbol{\nabla} \cdot \left(\frac{m_i n_i}{B^2} \frac{d\mathbf{E}_{\perp}}{dt} \right) + \boldsymbol{\nabla}_{\|} j_{\|} = 0 \tag{19}$$

The meaning is clear: if there is a finite (non-zero) divergence of the ion polarization current then it is only by a corresponding non-zero parallel divergence of the parallel current that this will be compensated, not by the accumulation of any charge. The build-up of the volume charge distribution $-\varepsilon_0 \boldsymbol{\nabla}_{\perp}^{2} \phi$ takes place on much slower time scales and much larger space scales.

4 The balance of density

Now, since we have something about the flows or, equivalently, on electric currents, we can write the equation of continuity. For the ions

$$\frac{\partial n_i}{\partial t} + (\mathbf{v}_E \cdot \boldsymbol{\nabla}) \, n_i + \boldsymbol{\nabla} \cdot (n_i \mathbf{v}_{ip}) = 0 \tag{20}$$

In the second term we could take out the $E \times B$ velocity ($\mathbf{v}_E$) assuming that B has no signifiant spatial variation. Similarly, the divergence of the diamagnetic flow, $n_i \mathbf{v}_{dia}$, is zero when the only contribution to it (except the density, to be discussed below), the space variation of B, is neglected. We recall that in the opposite case the divergence of the diamagnetic flow is the essential component of the neoclassical theory in toroidal systems, leading to the Pfirsch-Schluter currents. Another situation where the space variation of B is essential is when the curvature of lines generates terms leading to a large spatial scale organization of the flow (**Pavlenko**).

In Eq.(20) n_i is the full density (not $\widetilde{n}$) and the second term leads to the usual drive for the drift waves, the term containing $\omega_* = k_\perp \rho_s c_s / L_n$.

We will have to examine two cases. The first assumes a straight, shearless, $\mathbf{B} = \widehat{\mathbf{n}} B$, in which the parallel divergence of the parallel current is zero. In the second case, we retain a finite $\nabla_\| j_\|$.

4.1 The no z-dependence

We take

$$\mathbf{B} = \widehat{\mathbf{n}} B \tag{21}$$

and then there is no parallel variation. We have

$$\nabla_\| j_\| = 0 \tag{22}$$

and from the equation of continuity with no charge variation

$$\boldsymbol{\nabla} \cdot \mathbf{j}_p = 0 \tag{23}$$

or

$$\boldsymbol{\nabla} \cdot \left\{ n_i \frac{\partial}{\partial t} \boldsymbol{\nabla}_\perp \phi + n_i \frac{1}{B} \left[\left(-\boldsymbol{\nabla}_\perp \phi \times \widehat{\mathbf{n}} \right) \cdot \boldsymbol{\nabla}_\perp \right] \boldsymbol{\nabla}_\perp \phi \right\} = 0 \tag{24}$$

to which we add the conservation of the ion density (20), taking into account that $\boldsymbol{\nabla} \cdot \mathbf{j}_p \equiv \boldsymbol{\nabla} \cdot \mathbf{j}_{ip} = 0$,

$$\frac{\partial n_i}{\partial t} + \frac{-\boldsymbol{\nabla}_\perp \phi \times \widehat{\mathbf{n}}}{B} \cdot \boldsymbol{\nabla}_\perp n_i = 0 \tag{25}$$

The problem is how the second equation (of n_i) can help to find a solution of the first one, (24). If the density n_i is almost space-uniform, the Eq.(25) is reduced to an equation for the high order perturbation of the density, since the second term combines the fluctuations of ϕ and $\widetilde{n}_i$. If these fluctuations are small and can be neglected, the density is simply factorized from the first equation and we remain with a single equation,

$$\frac{\partial}{\partial t}\mathbf{\nabla}_{\perp}^{2}\phi+\frac{1}{B}\left[\left(-\mathbf{\nabla}_{\perp}\phi\times\widehat{\mathbf{n}}\right)\cdot\mathbf{\nabla}_{\perp}\right]\mathbf{\nabla}_{\perp}^{2}\phi=0 \tag{26}$$

We can divide the equation by the constant B, and introduce the notation of *streamfunction* ψ

$$\psi\equiv\frac{\phi}{B} \tag{27}$$

and we obtain

$$\frac{\partial}{\partial t}\mathbf{\nabla}_{\perp}^{2}\psi+\left[\left(-\mathbf{\nabla}_{\perp}\psi\times\widehat{\mathbf{n}}\right)\cdot\mathbf{\nabla}_{\perp}\right]\mathbf{\nabla}_{\perp}^{2}\psi=0 \tag{28}$$

which is simply the *Euler* equation,

$$\frac{d\omega}{dt}=0 \tag{29}$$

for the vorticity

$$\omega\equiv\mathbf{\nabla}_{\perp}^{2}\psi \tag{30}$$

and velocity

$$\mathbf{v}=-\mathbf{\nabla}\psi\times\widehat{\mathbf{n}} \tag{31}$$

There is no contradiction between the known characteristic of the *ion polarization drift* of being due to the compressibility of plasma at adiabaticity and the assumption that the density is constant. This simply means that we have retained the dominant term in Eq.(24), if we would have replaced $n_i = n_{i0}(1+\widetilde{n}_i)$. However we note that here the divergence of the ion polarization flow is zero.

4.2 The finite $k_{\parallel}$ case. The case where the magnetic field is tilted, there is parallel dynamics

In the case with shear, there is non-zero B_y and there is a parallel electric field

$$\begin{aligned} E_{\parallel} &= \frac{B_y}{B_0}E_y \\ &= -\frac{B_y}{B_0}\frac{\partial\phi}{\partial y} \end{aligned} \tag{32}$$

There are two competing velocities:

$$\begin{aligned}\frac{\omega}{k_{\|}} &\equiv \text{phase velocity of the perturbation along the magnetic line}\\ v_{the} &\equiv \text{thermal electron velocity}\end{aligned}$$

When the thermal electron velocity is higher than the phase velocity the electron distribution becomes adiabatic and the density perturbation is proportional with the potential perturbation.

$$\begin{aligned}\frac{\omega}{k_{\|}} &< v_{the} \qquad (33)\\ \frac{\omega}{k_y \frac{B_y}{B_0}} &< v_{the}\end{aligned}$$

If the forces in the *parallel* equation of motion of the electrons (parallel gradient of the pressure, parallel electric field, collisions) are small then we turn to the electron equation of continuity to find the parallel current. The parallel divergence of the parallel current is balanced by the time variation of the potential (we should normally write the latter: $-i\omega\phi$).

NOTE
The equation

$$\nabla \cdot \mathbf{B} = 0$$

or

$$\begin{aligned}\mathbf{k}\cdot\mathbf{B} &= 0\\ k_{\|}B_0 + k_\theta B_\theta &= 0\\ k_{\|} &= -k_\theta \frac{B_\theta}{B_0}\end{aligned}$$

However the relation between $\left(k_\theta, k_{\|}\right)$ and (B_θ, B_0) is here a simple geometrical projection.

This is because $\mathbf{k}$ is NOT a Fourier component of the magnetic field and it is NOT introduced here from $\mathbf{k}\cdot\mathbf{B} = 0$. The variable $\mathbf{k}$ is introduced by Fourier representation of the electric potential perturbation, extended along the θ and $\|$ directions.
END of NOTE.

This is a regime specific for drift waves.
The following is the explanation of the term

$$\frac{1}{\rho^2}\frac{\partial \phi}{\partial t}$$

which makes the difference between the Hasegawa-Mima equation and the Euler equation.

In this case the following elements: the adiabatic density and the equation of conservation of the electron density, after expressing the parallel velocity as the parallel current, - show that there is a finite parallel divergence of the parallel current. In this equation, one has to neglect the advection of the density by the two velocities: $\mathbf{v}_E$ and $\mathbf{v}_{dia}$. The result is

$$\nabla_{\|} j_{\|} \simeq \frac{|e|^2 n_0}{T_e} \frac{\partial \phi}{\partial t} \tag{34}$$

In **Horton 1990** it is given the more general form of this equation of continuity, expressed as parallel divergence of the parallel current of the electrons

$$\nabla_{\|} j_{\|} \simeq \frac{|e|^2 n_0}{T_e} \left(\frac{\partial \phi}{\partial t} + v_{dia,e} \frac{\partial \phi}{\partial y} \right) \tag{35}$$

where

$$v_{dia,e} \widehat{\mathbf{e}}_y = -\frac{T_e}{|e| B} \widehat{\mathbf{n}} \times \boldsymbol{\nabla} \ln n_{0e} \tag{36}$$

(with the choices of signs of **Horton Tajima Kamimura**).

Also in Studies.

NOTE regarding the very important role played by the term

$$\frac{1}{\rho^2} \frac{\partial \phi}{\partial t}$$

which makes the diference between: Euler and Hasegawa-Mima.

From **notes density omega**.

The equation

$$\boldsymbol{\nabla} \cdot \mathbf{j} = \boldsymbol{\nabla}_{\perp} \cdot (n_i \mathbf{v}_p) + \nabla_{\|} j_{\|} = 0 \tag{37}$$

and we must provide the expression for parallel current.

One is

$$\nabla_{\|} j_{\|} \simeq \frac{|e|^2 n_0}{T_e} \frac{\partial \phi}{\partial t} \tag{38}$$

and the other: In **Horton 1990** it is given the more general form of this equation of continuity, expressed as parallel divergence of the parallel current of the electrons

$$\nabla_{\|} j_{\|} \simeq \frac{|e|^2 n_0}{T_e} \left(\frac{\partial \phi}{\partial t} + v_{dia,e} \frac{\partial \phi}{\partial y} \right) \tag{39}$$

We NOTE that the second form (Horton1990) provides exactly the structure that we have expected. If we take

$$\frac{\partial}{\partial y} = -\frac{1}{u}\frac{\partial}{\partial t}$$

then

$$\begin{aligned}
\nabla_{\|} j_{\|} &\simeq \frac{|e|^2 n_0}{T_e}\left(\frac{\partial\phi}{\partial t} - \frac{v_{dia,e}}{u}\frac{\partial\phi}{\partial t}\right) \\
&\sim \frac{1}{\rho^2}\left(1 - \frac{v_{dia,e}}{u}\right)\frac{\partial\phi}{\partial t} \\
&\sim \frac{1}{\rho_{eff}^2}\frac{\partial\phi}{\partial t}
\end{aligned}$$

and this confirms that the term that makes the difference between Euler and Hasegawa-Mima is actually dependent of the *effective Larmor radius*.

When the effective Larmor radius is very large, because the plasma rotation is very close to the diamagnetic velocity, this term is practically suppressed and the equation becomes the Euler equation.

We still should note that the equation that results when

$$\rho_{eff}^2 \to \infty$$

is NOT the Euler equation. According to Gruzinov it is the degenerate Hasegawa Mima equation: it keeps the memory of the existence of a spatial scale, which is in the vectorial nonlinearity and it is

$$\rho_s$$

END note.

The equation of current conservation will be changed by the new term, from Eq.(38)

$$\boldsymbol{\nabla}\cdot\mathbf{j} = \boldsymbol{\nabla}_{\perp}\cdot(n_i\mathbf{v}_p) + \nabla_{\|}j_{\|} = 0 \tag{40}$$

Taking their expression we have

$$\begin{aligned}
\boldsymbol{\nabla}_{\perp}\cdot\left\{-\frac{c^2 m_i n_0}{B^2}\left[\frac{\partial(\boldsymbol{\nabla}_{\perp}\phi)}{\partial t} + \frac{c}{B}\left[(-\boldsymbol{\nabla}_{\perp}\phi\times\widehat{\mathbf{n}})\cdot\boldsymbol{\nabla}_{\perp}\right]\boldsymbol{\nabla}_{\perp}\phi\right]\right\} + \nabla_{\|}j_{\|} &= 0 \quad (41) \\
-\frac{c^2 m_i n_0}{B^2}\left\{\frac{\partial}{\partial t}\boldsymbol{\nabla}_{\perp}^2\phi + \frac{c}{B}\left[(-\boldsymbol{\nabla}_{\perp}\phi\times\widehat{\mathbf{n}})\cdot\boldsymbol{\nabla}_{\perp}\right]\boldsymbol{\nabla}_{\perp}^2\phi\right\} + \frac{|e|^2 n_0}{T_e}\frac{\partial\phi}{\partial t} &= 0
\end{aligned}$$

We divide the equation by n_0 and transform the terms. These transformations are actually intended to normalize the terms, for example to place a

ρ_s factor to each $\boldsymbol{\nabla}_\perp$ operator, etc. They are given in detail just for future comparison with other normalizations.

The first term

$$\begin{aligned}
-\frac{c^2 m_i}{B^2}\frac{\partial}{\partial t}\boldsymbol{\nabla}_\perp^2\phi &= -B\left|e\right|^2 c^2 \frac{T_e/m_i}{\left|e\right|^2 B^2/\left(c^2 m_i^2\right)}\frac{1}{c^2}\frac{1}{T_e}\frac{\partial}{\partial t}\boldsymbol{\nabla}_\perp^2\frac{\phi}{B} \\
&= -B\left|e\right|^2 \frac{1}{T_e}\frac{1}{c}\frac{c_s^2}{\Omega_{ci}^2}\frac{\partial}{\partial t}\boldsymbol{\nabla}_\perp^2\frac{c\phi}{B} \\
&= -B\left|e\right|^2 \frac{1}{T_e}\frac{1}{c}\frac{\partial}{\partial t}\left[\rho_s^2\boldsymbol{\nabla}_\perp^2\varphi\right]
\end{aligned} \tag{42}$$

where

$$\varphi \equiv \frac{c\phi}{B} \tag{43}$$

and

$$\rho_s = \frac{\left(T_e/m_i\right)^{1/2}}{\left|e\right| B/\left(c m_i\right)} \tag{44}$$

The second term

$$\begin{aligned}
&-\frac{c^2 m_i}{B^2}\frac{c}{B}\left[\widehat{\mathbf{n}}\cdot\left(\boldsymbol{\nabla}_\perp\phi\times\boldsymbol{\nabla}_\perp\right)\right]\boldsymbol{\nabla}_\perp^2\phi \qquad (45) \\
= & -\left|e\right|^4\frac{1}{c^4}B\frac{T_e^2/m_i^2}{\left|e\right|^4 B^4/\left(c m_i\right)^4}c^3\frac{1}{m_i}\frac{1}{T_e^2}B^2\left[\left(-\boldsymbol{\nabla}_\perp\frac{\phi}{B}\times\widehat{\mathbf{n}}\right)\cdot\boldsymbol{\nabla}_\perp\right]\boldsymbol{\nabla}_\perp^2\frac{\phi}{B} \\
= & -\left|e\right|^4\frac{1}{c^4}B\frac{c_s^4}{\Omega_{ci}^4}c^3\frac{1}{m_i}\frac{1}{T_e^2}B^2\left[\left(-\boldsymbol{\nabla}_\perp\frac{\phi}{B}\times\widehat{\mathbf{n}}\right)\cdot\boldsymbol{\nabla}_\perp\right]\boldsymbol{\nabla}_\perp^2\frac{\phi}{B} \\
= & -\left|e\right|^4\frac{1}{c^4}B\frac{c_s^4}{\Omega_{ci}^4}c\frac{1}{m_i}\frac{1}{T_e^2}B^2\left[\left(-\boldsymbol{\nabla}_\perp\frac{c\phi}{B}\times\widehat{\mathbf{n}}\right)\cdot\boldsymbol{\nabla}_\perp\right]\boldsymbol{\nabla}_\perp^2\frac{c\phi}{B} \\
= & -\left|e\right|^4\frac{1}{c^3}B^3\frac{1}{m_i T_e^2}\left[\left(-\rho_s\boldsymbol{\nabla}_\perp\varphi\times\widehat{\mathbf{n}}\right)\cdot\rho_s\boldsymbol{\nabla}_\perp\right]\rho_s^2\boldsymbol{\nabla}_\perp^2\varphi
\end{aligned}$$

The third term

$$\begin{aligned}
\frac{\left|e\right|^2}{T_e}\frac{\partial\phi}{\partial t} &= B\left|e\right|^2\frac{1}{c}\frac{1}{T_e}\frac{\partial}{\partial t}\frac{c\phi}{B} \\
&= \left|e\right|^2 B\frac{1}{c}\frac{1}{T_e}\frac{\partial\varphi}{\partial t}
\end{aligned}$$

Putting together

$$
\begin{aligned}
& -B\left|e\right|^2 \frac{1}{T_e}\frac{1}{c}\frac{\partial}{\partial t}\left[\rho_s^2\boldsymbol{\nabla}_\perp^2\varphi\right] \\
& -\left|e\right|^4 \frac{1}{c^3}B^3\frac{1}{m_i T_e^2}\left[\left(-\rho_s\boldsymbol{\nabla}_\perp\varphi\times\widehat{\mathbf{n}}\right)\cdot\rho_s\boldsymbol{\nabla}_\perp\right]\rho_s^2\boldsymbol{\nabla}_\perp^2\varphi \\
& +\left|e\right|^2 B\frac{1}{c}\frac{1}{T_e}\frac{\partial\varphi}{\partial t} \\
= & \;\; 0
\end{aligned} \tag{46}
$$

We divide by $B^2\left|e\right|^3/\left(cT_e\right)$ and multiply by cm_i,

$$
\begin{aligned}
& -\frac{cm_i}{\left|e\right|B}\frac{\partial}{\partial t}\left[\rho_s^2\boldsymbol{\nabla}_\perp^2\varphi\right] \\
& -\left|e\right|B\left(cm_i\right)\frac{1}{c^2}\frac{1}{m_i}\frac{1}{T_e}\left[\left(-\rho_s\boldsymbol{\nabla}_\perp\varphi\times\widehat{\mathbf{n}}\right)\cdot\rho_s\boldsymbol{\nabla}_\perp\right]\rho_s^2\boldsymbol{\nabla}_\perp^2\varphi \\
& +\frac{cm_i}{\left|e\right|B}\frac{\partial\varphi}{\partial t} \\
= & \;\; 0
\end{aligned} \tag{47}
$$

The coefficient of the second term is

$$
\begin{aligned}
-\left|e\right|B\left(cm_i\right)\frac{1}{c^2}\frac{1}{m_i}\frac{1}{T_e} &= \frac{\left|e\right|B}{cm_i}\frac{m_i}{T_e} \\
&= \frac{\Omega_{ci}}{c_s^2}
\end{aligned} \tag{48}
$$

$$
\begin{aligned}
& -\frac{1}{\Omega_{ci}}\left[\frac{\partial}{\partial t}\left(\rho_s^2\boldsymbol{\nabla}_\perp^2\varphi\right)-\frac{\partial\varphi}{\partial t}\right] \\
& -\frac{\Omega_{ci}}{c_s^2}\left[\left(-\rho_s\boldsymbol{\nabla}_\perp\varphi\times\widehat{\mathbf{n}}\right)\cdot\rho_s\boldsymbol{\nabla}_\perp\right]\rho_s^2\boldsymbol{\nabla}_\perp^2\varphi \\
= & \;\; 0
\end{aligned} \tag{49}
$$

The unit of φ (which is the streamfunction) is

$$
\frac{c_s^2}{\Omega_{ci}} = \frac{L^2}{T} = \text{(diffusion-like)} \tag{50}
$$

It is obtained the *Charney-Hasegawa-Mima* equation

$$
\left(1-\rho_s^2\boldsymbol{\nabla}_\perp^2\right)\frac{\partial\varphi}{\partial\left(\Omega_{ci}t\right)}-\frac{\Omega_{ci}}{c_s^2}\left[\left(-\rho_s\boldsymbol{\nabla}_\perp\varphi\times\widehat{\mathbf{n}}\right)\cdot\rho_s\boldsymbol{\nabla}_\perp\right]\rho_s^2\boldsymbol{\nabla}_\perp^2\varphi = 0 \tag{51}
$$

Conditions for Hasegawa-Mima model:

1. cold ions, $T_i \approx 0$;
2. no *parallel* dynamics, $k_\parallel \to 0$.
3. electrons are adiabatic, $\widetilde{n}_e/n_0 = |e|\,\varphi/T_e$.
4. Transversal wavelength smaller but comparable to ρ_s:

$$\begin{aligned}\left|\frac{\rho_s^2\Delta\varphi}{\varphi}\right| &\gtrsim 1\\ \text{or,}\ \ \rho_s^2 k_\perp^2 &\gtrsim 1\\ \lambda_\perp &\lesssim \rho_s\end{aligned}$$

5. time variation much slower than ion cyclotron frequency

$$\left(\frac{\partial}{\partial t}\right)^{-1} \ll \Omega_{ci}$$

This is the regime for *Rossby waves* in atmosphere. The planet rotates much faster than the speed of propagation of the Rossby wave perturbations and $\lambda_{Rossby} \lesssim \frac{\sqrt{gH_0}}{f}$ (the Rossby radius).

NOTE

In the review **1990 Physics Reports Horton** a different normalization is adopted

$$\begin{aligned}\mathbf{x}_\perp &\to \rho_s\mathbf{x}_\perp\\ z &\to L_n z\\ \varphi &\to \frac{\rho_s}{L_n}\varphi\\ \frac{\mathbf{v}}{c_s} &\to \frac{\rho_s}{L_n}\mathbf{v}\\ t &\to \frac{L_n}{c_s}t\end{aligned} \tag{52}$$

and the equation that is obtained (Eq.2.12 in that reference)

$$\left(\frac{\partial\varphi}{\partial t} - \frac{\partial}{\partial t}\boldsymbol{\nabla}_\perp^2\varphi\right) + v_{dia}\frac{\partial\varphi}{\partial y} - \left[\varphi, \boldsymbol{\nabla}_\perp^2\varphi\right] = 0 \tag{53}$$

where

$$
\begin{aligned}
\left[\phi, \boldsymbol{\nabla}_\perp^2 \phi\right] &= \frac{\partial \phi}{\partial x}\frac{\partial \boldsymbol{\nabla}_\perp^2 \phi}{\partial y} - \frac{\partial \phi}{\partial y}\frac{\partial \boldsymbol{\nabla}_\perp^2 \phi}{\partial x} \\
&= \left(-\frac{\partial \phi}{\partial y}\widehat{\mathbf{e}}_x + \frac{\partial \phi}{\partial x}\widehat{\mathbf{e}}_y\right) \cdot \left(\widehat{\mathbf{e}}_x \frac{\partial \boldsymbol{\nabla}_\perp^2 \phi}{\partial x} + \widehat{\mathbf{e}}_y \frac{\partial \boldsymbol{\nabla}_\perp^2 \phi}{\partial y}\right) \\
&= \left[\left(-\frac{\partial \phi}{\partial y}\widehat{\mathbf{e}}_x + \frac{\partial \phi}{\partial x}\widehat{\mathbf{e}}_y\right) \cdot \left(\widehat{\mathbf{e}}_x \frac{\partial}{\partial x} + \widehat{\mathbf{e}}_y \frac{\partial}{\partial y}\right)\right] \boldsymbol{\nabla}_\perp^2 \phi \\
&= \left[\left(-\boldsymbol{\nabla}_\perp \phi \times \widehat{\mathbf{n}}\right) \cdot \boldsymbol{\nabla}_\perp\right] \boldsymbol{\nabla}_\perp^2 \phi
\end{aligned}
\tag{54}
$$

and

$$
v_{dia} = \frac{c_s \rho_s}{L_n} \rightarrow 1 \tag{55}
$$

in these units.

END

We see that until now we have two possible equations describing the fluid flow structure in $2D$:

1. the *Euler* equation, obtained at z-translation invariant system (no magnetic shear) and constant density
2. the *Charney-Hasegawa-Mima* equation, obtained when there is a magnetic shear (tilted magnetic lines) and drift-wave-like parallel motion, where the parallel divergence of the parallel flow is given by the time variation of the electric potential, from the density conservation equation. The density is adiabatic.

We see that the difference between the Euler equation and the Charney-Hasegawa-Mima equation consists of a term of explicit time variation of the potential $\partial\varphi/\partial t$. This comes from parallel divergence of the parallel (electron) current, determined from the electron equation of continuity in the adiabatic-density case. It is then confirmed that the parallel dynamics (pseudo-three-dimensionality) is at the origin of the change from Euler to CHM equation.

The difference is huge.

The Euler equation is conformal invariant and has no intrinsic space scale. This can be seen from the fact that the dimension of ψ (streamfunction, $\psi = \phi/B$) are m^2/s and this eliminates any space unit from the Euler equation. A transformation $\psi \rightarrow \lambda\psi$ will affect equally the streamfunction and the units of the gradient operators, such that the equation is invariant. The conformal transformations are more general than the scaling and will be discussed separately.

The CHM is dominated by ρ_s. This cannot be removed by a scaling.

The practical consequence of this difference can be seen in solving the two equations.

For the Euler equation it is important to fix the extension of the spatial domain where the flow exists. This will introduce a length L in the problem (see **Kraichnan and Montgomery**).

For the CHM equation, fixing L is not sufficient, we need to specify ρ_s. For short time scales the dynamics will be mainly confined to ρ_s scale, with weak interaction between distant regions. For large time scales the full spatial domain will be coherently covered (**Petviashvili**).

These differences are highlighted by the discrete models which are considered to be equivalent to the continuum ones. The Euler fluid is equivalent with a discrete set of point-like vortices interacting in plane by a potential which is the sum over the natural logarithm of the relative distances. The ln potential means Coulombian, long range, and the Euler equation is able to describe large space scales. This is because of the absence of the necessity to compensate in every point a variation of the current in the plane by a variation of current along the third direction. For the CHM equation, the potential is short range (the function is K_0) and CHM is bound to describe ρ_s-scale vortices. From these small scale vortices however a large scale structure can be build.

We need to separate clearly various aspects of the theory where the ion polarization drift is implied.

First we remind that the ion polarization drift is a higher order (in $1/B$) component of the ion fluid velocity. It exists in all cases where the ion fluid velocity is present (with the observation that it will be important especially for small space scales). There are two large classes of situations where the ion fluid velocity is essential: the *turbulence* and the evolution of the *background*.

For the turbulence the space scales are much smaller than the "box", say a in tokamak, and involve the space extension of the turbulent eddies and of the ion sonic Larmor radius ρ_s. The balance of the parallel divergence of the parallel current and the perpendicular divergence of the perpendicular current takes place on these small scales. The perturbations can be taken oscillatory in time, for example Eq.(38)

$$\nabla_{\|} j_{\|} \simeq \frac{|e|^2 n_0}{T_e} \frac{\partial \phi}{\partial t} = \frac{|e|^2 n_0}{T_e} (-i\omega) \phi \tag{56}$$

and this means that $\nabla_{\|} j_{\|}$ has a time oscillation like ϕ. Correspondingly $\boldsymbol{\nabla}_{\perp} \cdot \mathbf{J}_{ip}$ has the same time oscillation, to preserve neutrality. These fast time variations are associated with small spatial scale (ρ_s) vortical motions.

The *background* balance is slower but has the same origin, the equations are the same. Nothing in the derivation of the full expression of the ion fluid velocity, including the polarization flow, has invoked a small spatial scale or a fast time variation (once we are sufficiently far from Ω_{ci}). However the ion polarization drift needs fast spatial variation for the gradients in its expression to become quantitatively significant. In consequence, when the two-dimensional approximation is acceptable and the dynamics of the parallel current is not significant the conservation of charge $\boldsymbol{\nabla}_{\perp} \cdot \mathbf{J}_{ip} = 0$ leads to the Euler equation, which may be applied to study the dynamics on the scale a. When the parallel dynamics is significant the small (ρ_s) scales governed by CHM equation are organized into local vortical motions, out of which, by collisions and coalescence, large scale ($\sim a$) structures are build up.

5 The theorem of Ertel

5.1 Vorticity in fluid physics, in particular in planetary atmosphere

Consider a fluid that undergoes an uniform rotation with angular velocity Ω. We take this as a solid body rotation. This is a good approximation for the case of the planetary atmosphere, of the protoplanetary disks, or the fluid in rotating water tank experiments, etc. The $2D$ plasma case is identical, with only the replacement

$$2\boldsymbol{\Omega} \rightarrow \boldsymbol{\Omega}_{ci} \tag{57}$$

The angular frequency of rotation of the planet is projected onto the local vertical of a point on the planet, giving the frequency Ω.

Assuming that the cartezian system of reference has the z direction along the vector of the angular momentum $\boldsymbol{\Omega}$ we have

$$\boldsymbol{\Omega} \equiv \widehat{\mathbf{e}}_z \Omega \tag{58}$$

In a point $\mathbf{r}$ in the fluid the velocity is

$$\mathbf{v} = \boldsymbol{\Omega} \times \mathbf{r} \tag{59}$$

or

$$\begin{aligned}
\mathbf{v} &= (\widehat{\mathbf{e}}_z\Omega) \times (x\widehat{\mathbf{e}}_x + y\widehat{\mathbf{e}}_y + z\widehat{\mathbf{e}}_z) \\
&= \begin{pmatrix} \widehat{\mathbf{e}}_x & \widehat{\mathbf{e}}_y & \widehat{\mathbf{e}}_z \\ 0 & 0 & \Omega \\ x & y & z \end{pmatrix} = \widehat{\mathbf{e}}_x(-\Omega y) + \widehat{\mathbf{e}}_y(\Omega x)
\end{aligned} \tag{60}$$

$$\begin{aligned}
v_x &= -\Omega y \\
v_y &= \Omega x \\
v_z &= 0
\end{aligned} \tag{61}$$

Then the vorticity is

$$\begin{aligned}
\boldsymbol{\omega} &= \boldsymbol{\nabla} \times \mathbf{v} \\
&= \begin{pmatrix} \widehat{\mathbf{e}}_x & \widehat{\mathbf{e}}_y & \widehat{\mathbf{e}}_z \\ \partial_x & \partial_y & \partial_z \\ -\Omega y & \Omega x & 0 \end{pmatrix} \\
&= \widehat{\mathbf{e}}_z(\Omega + \Omega) \\
&= (2\Omega)\,\widehat{\mathbf{e}}_z
\end{aligned} \tag{62}$$

The vorticity is equal to 2Ω. The coefficient 2 comes from taking a *solid body* rotation, *i.e.* Eq.(59). In plasma the equivalent rotation is the gyration Ω_{ci} and the problem of a solid body rotation of plasma must be treated separately.

It is often used the *circulation* Γ of the fluid on a closed curve C,

$$\oint_C \mathbf{v}\cdot d\mathbf{r} = \iint_A \boldsymbol{\omega}\cdot\widehat{\mathbf{n}}\,dA \tag{63}$$

The circulation is equal to the flux of the vorticity through the surface bounded by the curve C.

In the following two kinds of vorticity will be used:

1. *relative* vorticity; it is what we calculate from the physical velocity of a fluid, taking the rotational, $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v}$;

2. the *absolute* vorticity; it is the relative vorticity plus the vorticity of the solid body rotation: $\boldsymbol{\omega}_a = \boldsymbol{\omega} + 2\boldsymbol{\Omega}$;

The *absolute strength* or *flux* of a vortex tube is

$$\Gamma_a = \iint \boldsymbol{\omega}_a \cdot \widehat{\mathbf{n}} dA \tag{64}$$

The strength is constant along the length of the vortex tube. Then a vortex tube cannot end or appear in the volume.

The following notes are strictly confined to the text of **Pedlosky**.
The flux of the absolute vorticity Γ_a through the surface A is

$$\begin{aligned}\Gamma_a &= \iint_A \boldsymbol{\omega}_a \cdot \widehat{\mathbf{n}} dA \\ &= \iint_A \boldsymbol{\omega} \cdot \widehat{\mathbf{n}} dA + \iint_A 2\boldsymbol{\Omega} \cdot \widehat{\mathbf{n}} dA \\ &= \Gamma + \iint_A 2\boldsymbol{\Omega} \cdot \widehat{\mathbf{n}} dA\end{aligned} \tag{65}$$

The last integral can be done since $\boldsymbol{\Omega}$ is constant

$$\Gamma_a = \Gamma + 2\Omega A_n \tag{66}$$

where A_n is the area of the projection of A on a surface perpendicular to Ω.

The circulation is defined for the *absolute* Γ_a and for the *relative* vorticity, Γ.

Consider the circulation Γ of the *relative* vorticity.

Take a curve C as a material curve, *i.e.* a curve that moves with the fluid.

$$\begin{aligned}\frac{d\Gamma}{dt} &= \frac{d}{dt} \oint_C \mathbf{v} \cdot d\mathbf{r} \\ &= \oint_C \frac{d\mathbf{v}}{dt} \cdot d\mathbf{r} + \oint_C \mathbf{v} \cdot \frac{d}{dt}(d\mathbf{r})\end{aligned} \tag{67}$$

The second term contains the change of an infinitesimal element of length along the contour C. This infinitesimal element will change due to the local velocity of the fluid, that moves the contour

$$\frac{d}{dt} d\mathbf{r} = d\mathbf{v} \tag{68}$$

Then

$$\begin{aligned}\frac{d\Gamma}{dt} &= \oint_C \frac{d\mathbf{v}}{dt}\cdot d\mathbf{r}+\oint_C \mathbf{v}\cdot d\mathbf{v} \\ &= \oint_C \frac{d\mathbf{v}}{dt}\cdot d\mathbf{r}+\frac{1}{2}\oint_C d\left|\mathbf{v}\right|^2 \\ &= \oint_C \frac{d\mathbf{v}}{dt}\cdot d\mathbf{r}\end{aligned} \tag{69}$$

The second term vanishes due to periodicity, since it is the integration over a closed contour of a total differential.

Now we will use for $d\mathbf{v}/dt$ the *equation of motion*

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\boldsymbol{\nabla}p - 2\boldsymbol{\Omega}\times\mathbf{v} + \frac{1}{\rho}\mathcal{F} \tag{70}$$

where $\mathcal{F}$ is the friction.

When introduced in the expression of $d\Gamma/dt$ we obtain

$$\frac{d\Gamma}{dt} = \oint_C \left(-\frac{1}{\rho}\boldsymbol{\nabla}p - 2\boldsymbol{\Omega}\times\mathbf{v} + \frac{1}{\rho}\mathcal{F}\right)\cdot d\mathbf{r} \tag{71}$$

will have to analyse *three* terms which contribute to the change of the *relative circulation.*

1. The effect of the *Coriolis force.*

The Coriolis force per unit mass $-2\boldsymbol{\Omega}\times\mathbf{v}$ acts on a body (element of fluid) that moves with velocity $\mathbf{v}$ in the plane of rotation, say from the center to perifery of the disk bounded by C, and deflects it in plane in a direction perpendicular on $\mathbf{v}$ (the body is in the rotating frame of reference).

Consider the following identity

$$\begin{aligned}-\left(2\boldsymbol{\Omega}\times\mathbf{v}\right)\cdot d\mathbf{r} &= -2\boldsymbol{\Omega}\cdot\left(\mathbf{v}\times d\mathbf{r}\right) \\ &= -2\boldsymbol{\Omega}\cdot\widehat{\mathbf{n}}_A\, v_\perp dr\end{aligned} \tag{72}$$

where: $v_\perp$ is the component of the velocity of the fluid along the normal to $d\mathbf{r}$, $\widehat{\mathbf{n}}_A$ is the versor along the direction of the vector product $\mathbf{v}\times d\mathbf{r}$. The motion will displace the infinitesimal element dr of the contour on an elementary transversal distance $dl = v_\perp dt$, which modifies the area inside the contour,

$$\begin{aligned}\delta A &= dr\ dl \\ &= dr\ v_\perp dt\end{aligned} \tag{73}$$

or

$$\frac{d}{dt}\delta A = v_{\perp} dr \tag{74}$$

Then we get

$$-\left(2\boldsymbol{\Omega}\times\mathbf{v}\right)\cdot d\mathbf{r} = -2\Omega\frac{d}{dt}\delta A_n \tag{75}$$

and the second term in the time-derivative of the circulation, Eq.(71)

$$-\oint_C \left(2\boldsymbol{\Omega}\times\mathbf{v}\right)\cdot d\mathbf{r} = -2\Omega\frac{dA_n}{dt} \tag{76}$$

where A_n is the total area enclosed by C, projected on a plane perpendicular on $\boldsymbol{\Omega}$. (**Note**. It may seem strange that a loop integral $\oint_C \delta A_n$ gives the total area A_n. Actually integrating δA_n along the loop we just get the increase of the area A_n, say ΔA_n but since in Eq.(76) we apply a *time derivative* and the initial area is a constant, we have $d\left(\Delta A_n\right)/dt = d\left(A_n^{initial} + \Delta A_n\right)/dt = d\left(A_n\right)/dt$. **End)**

Thus, in the presence of the planetary vorticity $\boldsymbol{\Omega}$ an increase of the area A_n (*i.e.* $dA_n/dt > 0$) leads to a decrease of the *relative* circulation $\frac{d\Gamma}{dt} < 0$.

The flux of the relative vorticity through the loop C will be decreased in direct proportion as the number of planetary-vorticity filaments captured by the loop is increased.

When the area bounded by C expands in size it will collect more planetary-vorticity filaments (represented by $\boldsymbol{\Omega}$), and these will produce a negative circulation of the *relative* velocity around the loop C. This decreases the *relative* vorticity contained in C.

The phenomenon is identical with the *induction* of an electric field in a loop-wire that moves in an uniform magnetic field.

2. The second mechanism that can lead to a change in the *relative* Γ circulation is the non-vanishing *baroclinic* term.

We have, applying the Stokes theorem on the first term in the right hand side (RHS) of Eq.(71)

$$\begin{aligned} -\oint_C \frac{\boldsymbol{\nabla} p}{\rho} &= -\iint_A \boldsymbol{\nabla}\times\left(\frac{\boldsymbol{\nabla} p}{\rho}\right)\cdot\widehat{\mathbf{n}}\, dA \\ &= \iint_A \frac{\boldsymbol{\nabla}\rho\times\boldsymbol{\nabla} p}{\rho^2}\cdot\widehat{\mathbf{n}}\, dA \end{aligned} \tag{77}$$

If the *surfaces of constant density* and the *surfaces of constant pressure* do not coincide the state of the fluid is called *baroclinic.*

If the surfaces of constant p and the surfaces of constant ρ *coincide* the fluid is called *barotropic.*

Horton notes that the ITG, Ion-Temperature-Gradient mode (the η_i mode) is *not* barotropic (**Horton, Phys.Rep. 1990**, page 54).

3. the third term is the line integral of the external friction force $\mathcal{F}$ along the contour C. This will be discussed separately.

5.2 Kelvin theorem

The connection between the *absolute* and the *relative* circulations is

$$\Gamma_a = \Gamma + 2\Omega A_n \tag{78}$$

Then this quantity can be attached to a differential equation that is obtained from Eqs.(71) and (76)

$$\frac{d\Gamma_a}{dt} = -\oint_C \frac{\nabla p}{\rho} \cdot d\mathbf{r} + \oint_C \frac{\mathcal{F}}{\rho} \cdot d\mathbf{r} \tag{79}$$

Now, assume :

1. the fluid is *barotropic* on C, which means that the pressure and density have the same equilines
2. the frictional force $\mathcal{F}$ is zero.

then, the absolute circulation is conserved on contours that follow the motion

$$\frac{d\Gamma_a}{dt} = 0 \tag{80}$$

This is *Kelvin*'s theorem.

It shows that there is a transfer between the *absolute* and *relative* vorticity. The mechanism that allows to transfer from absolute to relative vorticity is the *vorticity induction.*

This hightlights the meaning of *absolute* vorticity tubes: the absolute vorticity filaments move with the fluid.

The Kelvin equation and the Ertel's equation can be considered the integral and respectively the differential form of the same conservation law.

5.3 The equation of the vorticity

Consider the identity

$$\begin{aligned}(\boldsymbol{\nabla}\times\mathbf{v})\times\mathbf{v} &\equiv \\ \boldsymbol{\omega}\times\mathbf{v} &= (\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}-\boldsymbol{\nabla}\frac{\left(|\mathbf{v}|^2\right)}{2}\end{aligned} \tag{81}$$

using this formula, the momentum equation Eq.(70) can be written

$$\begin{aligned}\frac{\partial\mathbf{v}}{\partial t}+(2\boldsymbol{\Omega}+\boldsymbol{\omega})\times\mathbf{v} &= -\frac{1}{\rho}\boldsymbol{\nabla}p \\ &\quad +\boldsymbol{\nabla}\left(-\frac{|\mathbf{v}|^2}{2}\right)+\frac{\mathcal{F}}{\rho}\end{aligned} \tag{82}$$

We need an equation for the vorticity $\boldsymbol{\omega}=\boldsymbol{\nabla}\times\mathbf{v}$ and we will apply a rotational operator on the equation. The following vectorial identity is used

$$\begin{aligned}\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) &= \mathbf{A}\,(\boldsymbol{\nabla}\cdot\mathbf{B})+(\mathbf{B}\cdot\boldsymbol{\nabla})\,\mathbf{A} \\ &\quad -\mathbf{B}\,(\boldsymbol{\nabla}\cdot\mathbf{A})-(\mathbf{A}\cdot\boldsymbol{\nabla})\,\mathbf{B}\end{aligned} \tag{83}$$

Applied to our vectors it gives

$$\begin{aligned}&\boldsymbol{\nabla}\times[(2\boldsymbol{\Omega}+\boldsymbol{\omega})\times\mathbf{v}] \\ =\ &(2\boldsymbol{\Omega}+\boldsymbol{\omega})\,(\boldsymbol{\nabla}\cdot\mathbf{v})+(\mathbf{v}\cdot\boldsymbol{\nabla})\,(2\boldsymbol{\Omega}+\boldsymbol{\omega}) \\ &-\mathbf{v}\,[\boldsymbol{\nabla}\cdot(2\boldsymbol{\Omega}+\boldsymbol{\omega})]-[(2\boldsymbol{\Omega}+\boldsymbol{\omega})\cdot\boldsymbol{\nabla}]\,\mathbf{v}\end{aligned} \tag{84}$$

The third term of this equation is zero since the absolute vorticity (as well as the relative $\boldsymbol{\omega}$ vorticity) have *zero* divergence, being defined by *curl* operators. Using this identity in the momentum equation on which we have applied the rotational operator (to get the vorticity equation) we obtain

$$\begin{aligned}\frac{d\boldsymbol{\omega}}{dt} &= (\boldsymbol{\omega}_a\cdot\boldsymbol{\nabla})\,\mathbf{v}-\boldsymbol{\omega}_a\,(\boldsymbol{\nabla}\cdot\mathbf{v}) \\ &\quad +\frac{\boldsymbol{\nabla}\rho\times\boldsymbol{\nabla}p}{\rho^2} \\ &\quad +\boldsymbol{\nabla}\times\frac{\mathcal{F}}{\rho}\end{aligned} \tag{85}$$

where the *absolute* vorticity is defined as usual

$$\boldsymbol{\omega}_a=\boldsymbol{\omega}+2\boldsymbol{\Omega} \tag{86}$$

To understand the first two terms we take a particular geometry. The frame has the z axis along the *absolute vortex* filament direction, tangent to it. The other two coordinates are perpendicular on this tangent. Then the absolute vorticity is in this frame $\omega_a \widehat{\mathbf{e}}_z$.

$$
\begin{aligned}
& (\boldsymbol{\omega}_a \cdot \boldsymbol{\nabla})\,\mathbf{v} - \boldsymbol{\omega}_a\,(\boldsymbol{\nabla} \cdot \mathbf{v}) \\
= \; & \left(\omega_a \frac{\partial}{\partial z}\right)(u\widehat{\mathbf{e}}_x + v\widehat{\mathbf{e}}_y + w\widehat{\mathbf{e}}_z) - (\omega_a \widehat{\mathbf{e}}_z)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \\
= \; & \widehat{\mathbf{e}}_x \omega_a \frac{\partial u}{\partial z} + \widehat{\mathbf{e}}_y \omega_a \frac{\partial v}{\partial z} - \omega_a \widehat{\mathbf{e}}_z \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)
\end{aligned}
\tag{87}
$$

We can particularize the analysis considering the *rate of change* of the z-component of the relative vorticity, $\boldsymbol{\omega}$. This is obtained from the equation (85) where we *neglect* for the moment the baroclinic term and the frictional term (second and third lines) and using Eq.(87).

$$
\frac{d\omega_z}{dt} = -\left(\omega_a\right)_z \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{88}
$$

We see that the time change of the z-component vorticity (*i.e.* along the vortex line) is proportional with the z-vorticity multiplied by the *convergence* of the velocity components in the plane perpendicular to the line absolute-vortex. If the convergence of the velocity is positive

$$
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} < 0 \tag{89}
$$

(*i.e.* the fluid comes to the centre) the absolute-vorticity filaments will be gathered closer together, increasing the magnitude of the vorticity vector by decreasing the cross-sectional area of the local vortex tube. The Kelvin theorem is applied.

$$
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{1}{A_\perp}\frac{dA_\perp}{dt} \tag{90}
$$

where $A_\perp$ is a differential area perpendicular to the line vortex. Then

$$
\frac{d\left(\omega_a\right)_z}{dt} = -\frac{\left(\omega_a\right)_z}{A_\perp}\frac{dA_\perp}{dt} \tag{91}
$$

or

$$
\frac{d}{dt}\left[\left(\omega_a\right)_z A_\perp\right] = 0 \tag{92}
$$

and this is another expression of the Kelvin theorem. This mechanism of increasing the absolute vorticity *along* the vortex by reducing the cross-section of the vortex tube is called *vortex stretching*.

5.4 The Ertel's theorem

The equation for the vorticity is obtained from Eq.(85)

$$\begin{aligned} \frac{d\boldsymbol{\omega}_a}{dt} &= (\boldsymbol{\omega}_a \cdot \boldsymbol{\nabla})\,\mathbf{v} - \boldsymbol{\omega}_a\,(\boldsymbol{\nabla} \cdot \mathbf{v}) \\ &\quad + \frac{\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla}p}{\rho^2} \quad \text{(baroclinic term)} \\ &\quad + \boldsymbol{\nabla}\times\frac{\mathcal{F}}{\rho} \end{aligned} \tag{93}$$

To this equation we add the equation of continuity

$$\boldsymbol{\nabla} \cdot \mathbf{v} = -\frac{1}{\rho}\frac{d\rho}{dt} \tag{94}$$

and with this we can eliminate the divergence of the velocity vector, $\boldsymbol{\nabla} \cdot \mathbf{v}$,

$$\begin{aligned} \frac{d}{dt}\left(\frac{\boldsymbol{\omega}_a}{\rho}\right) &= \left(\frac{\boldsymbol{\omega}_a}{\rho} \cdot \boldsymbol{\nabla}\right)\mathbf{v} \\ &\quad + \boldsymbol{\nabla}\rho \times \frac{\boldsymbol{\nabla}p}{\rho^3} \\ &\quad + \left(\boldsymbol{\nabla}\times\frac{\mathcal{F}}{\rho}\right)\frac{1}{\rho} \end{aligned} \tag{95}$$

Separately, consider a quantity λ that has a Lagrangian evolution

$$\frac{d\lambda}{dt} = \Psi \tag{96}$$

where Ψ is a source.

We multiply this conservation law with the ratio $\boldsymbol{\omega}_a/\rho$, and use the identity

$$\frac{\boldsymbol{\omega}_a}{\rho} \cdot \frac{d}{dt}\boldsymbol{\nabla}\lambda = \left(\frac{\boldsymbol{\omega}_a}{\rho} \cdot \boldsymbol{\nabla}\right)\frac{d\lambda}{dt} - \left[\left(\frac{\boldsymbol{\omega}_a}{\rho} \cdot \boldsymbol{\nabla}\right)\mathbf{v}\right] \cdot \boldsymbol{\nabla}\lambda \tag{97}$$

This is one of the equations we need. The second one is obtained after multiplying the equation (95) for $\boldsymbol{\omega}_a/\rho$ by the *gradient* of λ,

$$\begin{aligned} \boldsymbol{\nabla}\lambda \cdot \frac{d}{dt}\left(\frac{\boldsymbol{\omega}_a}{\rho}\right) &= \left[\left(\frac{\boldsymbol{\omega}_a}{\rho} \cdot \boldsymbol{\nabla}\right)\mathbf{v}\right] \cdot \boldsymbol{\nabla}\lambda \\ &\quad + \boldsymbol{\nabla}\lambda \cdot \left(\boldsymbol{\nabla}\rho \times \frac{\boldsymbol{\nabla}p}{\rho^3}\right) \\ &\quad + \frac{\boldsymbol{\nabla}\lambda}{\rho} \cdot \left(\boldsymbol{\nabla}\times\frac{\mathcal{F}}{\rho}\right) \end{aligned} \tag{98}$$

The two equations are now summed and we take into account the equation of conservation of the quantity λ

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\boldsymbol{\omega}_a}{\rho}\cdot\boldsymbol{\nabla}\lambda\right) &= \frac{\boldsymbol{\omega}_a}{\rho}\cdot\boldsymbol{\nabla}\Psi \\
&+\boldsymbol{\nabla}\lambda\cdot\left(\frac{\boldsymbol{\nabla}\rho\times\boldsymbol{\nabla}p}{\rho^3}\right) \\
&+\frac{\boldsymbol{\nabla}\lambda}{\rho}\cdot\left(\boldsymbol{\nabla}\times\frac{\mathcal{F}}{\rho}\right)
\end{aligned} \tag{99}$$

Assume now the following conditions

1. there is no source for λ, $\Psi\equiv 0$; (the first term in the RHS vanishes)
2. the friction is absent $\mathcal{F}\equiv 0$; (the last term in the RHS vanishes)
3. the fluid is *barotropic*, which means that the *baroclinic* term vanishes $\boldsymbol{\nabla}\rho\times\boldsymbol{\nabla}p = 0$; (the second line in the RHS vanishes)

Alternatively the third condition is replaced with : λ is a function of p and ρ.

Then it is possible to introduce the *potential vorticity*

$$\Pi \equiv \frac{\boldsymbol{\omega}_a}{\rho}\cdot\boldsymbol{\nabla}\lambda = \frac{\boldsymbol{\omega}+2\boldsymbol{\Omega}}{\rho}\cdot\boldsymbol{\nabla}\lambda \tag{100}$$

and the result of the equation (99) is that the potential vorticity is conserved along the flow, by each fluid element

$$\frac{d\Pi}{dt} = 0 \tag{101}$$

In the expression of Π the *absolute* vorticity is projected along the gradient of λ.

This is the theorem of Ertel.

NOTE the neglect of the parallel dynamics

$$\nabla_{\|}v_{\|} = 0$$

END.

5.5 A derivation of Ertel's theorem for plasma in strong magnetic field

A good reference is **Hasegawa Maclennan Kodama**.

The two equations for the ion fluid are the continuity and the momentum conservation. Several simplifications are done: no variation of the magnetic field $\mathbf{B}$ in space (this ensures that Ω_{ci} is a constant, like the Coriolis frequency), the ion temperature is constant and uniform, the electrons have higher temperature (cold ions), which also means that the ion pressure can be neglected, $p = 0$ (this will eliminate the *baroclinic* term). The process is purely electrostatic and there are no friction forces.

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v} = \frac{|e|}{m_i}(-\boldsymbol{\nabla}\phi) + \mathbf{v}\times\boldsymbol{\Omega}_{ci} \tag{102}$$

The equation for the ion density is

$$\begin{aligned}\frac{\partial n}{\partial t} + \boldsymbol{\nabla}\cdot(n\mathbf{v}) &= \\ \frac{\partial n}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\,n + n\,(\boldsymbol{\nabla}\cdot\mathbf{v}) &= 0\end{aligned} \tag{103}$$

or

$$\boldsymbol{\nabla}\cdot\mathbf{v} = -\frac{d}{dt}\ln n \tag{104}$$

It is assumed a Boltzmann distribution of the electrons

$$n_e = n_0\exp\left(\frac{|e|\,\phi}{T_e}\right) \tag{105}$$

and the neutrality

$$n_i = n_e \tag{106}$$

Then the equation for the ion density becomes

$$\boldsymbol{\nabla}\cdot\mathbf{v} = -\frac{d}{dt}\left(\ln n_0 + \frac{|e|\,\phi}{T_e}\right) \tag{107}$$

To get the *vorticity* we apply the rotational on the ion velocity equation, Eq.(102).

The following two identities Eqs.(81) and (84) are used

$$(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v} = \frac{1}{2}\boldsymbol{\nabla}\left(v^2\right) - \mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{v}) \tag{108}$$

(the last term is $\mathbf{v}\times\boldsymbol{\omega}$).

$$\boldsymbol{\nabla}\times(\mathbf{v}\times\boldsymbol{\omega})=\mathbf{v}(\boldsymbol{\nabla}\cdot\boldsymbol{\omega})-\boldsymbol{\omega}(\boldsymbol{\nabla}\cdot\mathbf{v})+(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{v}-(\mathbf{v}\cdot\boldsymbol{\nabla})\boldsymbol{\omega} \tag{109}$$

The first term is zero ($\boldsymbol{\omega}$ is a curl) and the divergence of the velocity vector $\mathbf{v}$ along the direction of $\boldsymbol{\omega}$ (which is perpendicular on the plane where $\mathbf{v}$ is contained), *i.e.* the third term, is zero. Then the second identity is

$$\boldsymbol{\nabla}\times(\mathbf{v}\times\boldsymbol{\omega})=-\boldsymbol{\omega}(\boldsymbol{\nabla}\cdot\mathbf{v})-(\mathbf{v}\cdot\boldsymbol{\nabla})\boldsymbol{\omega} \tag{110}$$

We also note that the double vectorial product we calculate in Eq.(109) extracts the part of the velocity $\mathbf{v}$ which is perpendicular on $\boldsymbol{\omega}$, and we can write

$$\boldsymbol{\nabla}\times(\mathbf{v}\times\boldsymbol{\omega})=-\boldsymbol{\omega}(\boldsymbol{\nabla}_\perp\cdot\mathbf{v}_\perp)-(\mathbf{v}\cdot\boldsymbol{\nabla})\boldsymbol{\omega} \tag{111}$$

The same will apply below where $\boldsymbol{\omega}$ is replaced by $\boldsymbol{\Omega}_{ci}$. We obtain

$$\begin{aligned}\frac{\partial}{\partial t}(\boldsymbol{\nabla}\times\mathbf{v})+\boldsymbol{\nabla}\times[(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}] &= \boldsymbol{\nabla}\times(\mathbf{v}\times\boldsymbol{\Omega}_{ci}) \qquad (112)\\ \frac{\partial\boldsymbol{\omega}}{\partial t}+\boldsymbol{\nabla}\times\left[\frac{1}{2}\boldsymbol{\nabla}\left(v^2\right)-\mathbf{v}\times\boldsymbol{\omega}\right] &= -\boldsymbol{\Omega}_{ci}(\boldsymbol{\nabla}_\perp\cdot\mathbf{v}_\perp)\end{aligned}$$

The curl of the first term in the square brackets is zero and for the second we apply Eq.(110)

$$\frac{\partial\boldsymbol{\omega}}{\partial t}-[-\boldsymbol{\omega}(\boldsymbol{\nabla}_\perp\cdot\mathbf{v}_\perp)-(\mathbf{v}\cdot\boldsymbol{\nabla})\boldsymbol{\omega}]=-\boldsymbol{\Omega}_{ci}(\boldsymbol{\nabla}_\perp\cdot\mathbf{v}_\perp) \tag{113}$$

or

$$\frac{\partial\boldsymbol{\omega}}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})\boldsymbol{\omega}+(\boldsymbol{\omega}+\boldsymbol{\Omega}_{ci})(\boldsymbol{\nabla}_\perp\cdot\mathbf{v}_\perp)=0 \tag{114}$$

The first two terms are the convective derivative of the vorticity vector, to which we can add the constant vector $\boldsymbol{\Omega}_{ci}$,

$$\frac{d}{dt}(\boldsymbol{\omega}+\boldsymbol{\Omega}_{ci})+(\boldsymbol{\omega}+\boldsymbol{\Omega}_{ci})(\boldsymbol{\nabla}_\perp\cdot\mathbf{v}_\perp)=0 \tag{115}$$

The hypothesis of Hasegawa and Mima is that the variations of the system along the z direction are much smaller than the perpendicular variations

$$\left|\frac{dv_z}{dz}\right|\ll|\boldsymbol{\nabla}_\perp\cdot\mathbf{v}_\perp| \tag{116}$$

This is called *pseudo-three-dimensionality.*

Now, if we simply replace Eq.(104) in Eq.(115) we have

$$\begin{aligned}\frac{d}{dt}(\omega+\Omega_{ci})-(\omega+\Omega_{ci})\frac{d}{dt}\ln n &= 0\\ \frac{d}{dt}\ln\frac{(\omega+\Omega_{ci})}{n} &= 0\end{aligned}$$

Which is the Ertel's theorem.

5.6 Derivation of the Charney-Hasegawa-Mima equation for a plasma in strong magnetic field

Using Eq.(116), instead of Eq.(107) we will have

$$\nabla_\perp \cdot \mathbf{v}_\perp \simeq -\frac{d}{dt}\left(\ln n_0 + \frac{|e|\,\phi}{T_e}\right) \tag{117}$$

Inserting Eq.(117) into Eq.(115) we obtain

$$\frac{d}{dt}\left(\ln\frac{\Omega_{ci}}{n_0} + \frac{\omega}{\Omega_{ci}} - \frac{|e|\,\phi}{T_e}\right) = 0 \tag{118}$$

Taking here

$$\omega = \frac{1}{B}\nabla_\perp^2 \phi$$

and

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \frac{-\nabla_\perp \phi \times \widehat{\mathbf{n}}}{B} \cdot \nabla_\perp$$

we obtaine the CHM equation, in the form

$$\frac{\partial}{\partial t}\left(1 - \nabla_\perp^2\right)\phi - \left[\left(-\nabla_\perp \phi \times \widehat{\mathbf{n}}\right) \cdot \nabla_\perp\right]\left(\nabla_\perp^2 \phi - \ln\frac{n_0}{\Omega_{ci}}\right) = 0$$

and with the units:

$$\begin{array}{rcl} \mathbf{x}/\rho_s & \rightarrow & \mathbf{x} \\ t\Omega_{ci} & \rightarrow & t \\ \dfrac{|e|\,\phi}{T_e} & \rightarrow & \phi \end{array}$$

5.7 The vorticity equation for the plasma in strong magnetic field

This is derived as before. Compared with the Eq.(85) and Eq.(93) we now eliminate the variation of $\mathbf{v}$ in the direction of $\boldsymbol{\omega} + \boldsymbol{\Omega}_{ci}$ and neglect the friction. Compared with Eq.(115), now we keep the ion pressure. The result is (**Hasegawa Mima**)

$$\frac{d\left(\boldsymbol{\omega} + \boldsymbol{\Omega}_{ci}\right)}{dt} = -\left(\boldsymbol{\omega} + \boldsymbol{\Omega}_{ci}\right)\left(\nabla \cdot \mathbf{v}\right) + \frac{1}{m_i n^2}\nabla n \times \nabla p \tag{119}$$

where n is the *full* ion density. Here we recognize the *baroclinic* term, $\frac{1}{m_i n^2}\nabla n \times \nabla p$, active when the density and the pressure are not constant along the same lines.

It results from Eq.(119) that the *baroclinic* term is a source of

1. fluid vorticity $\boldsymbol{\omega}$
2. vertical magnetic field, via $\boldsymbol{\Omega}_{ci} = |e| B/m_i$.

If the baroclinic term is zero, we scalar-multiply Eq.(119) with $\boldsymbol{\omega} + \boldsymbol{\Omega}_{ci}$ and replace the equation of continuity

$$\boldsymbol{\nabla} \cdot \mathbf{v} = -\frac{d}{dt} \ln n \tag{120}$$

This gives

$$\frac{d}{dt} \ln \left(\frac{\omega + \Omega_{ci}}{n} \right)^2 = 0 \tag{121}$$

or

$$\frac{d}{dt} \left(\frac{\omega + \Omega_{ci}}{n} \right)^2 = 0$$

which is again Ertel's theorem. The Eq.(121) is considered by **HM** as a generalized enstrophy conservation.

5.8 The vorticity and the density. Notes on the Ertel's theorem for plasma in strong magnetic field

Horton (**Phys.Rep. 1990**) applies this theorem to the $2D$ model of the tokamak plasma, since there

$$k_{\|} \simeq 0 \tag{122}$$

and

$$\boldsymbol{\nabla} \lambda = \widehat{\mathbf{n}} \tag{123}$$

from which we have

$$\frac{d}{dt} \left(\frac{\omega_z + \Omega_{ci}}{n} \right) = 0 \tag{124}$$

Horton Su Morrison reminds that the Ertel 's theorem can be derived only when there is NO *parallel compressibility.*

$$\nabla_{\|} v_{\|} = 0$$

As noted by **Horton 1990** this equation contains the Hasegawa-Mima equation. To get the known form of that equation we need to take in the calculation of ω_z the $E \times B$ velocity and the *adiabatic* density, $n = n_0 \exp(\varphi)$ with small perturbation φ compared with the bulk plasma energy density, $\varphi \ll 1$.

This equation also contains the effects of the spatial variation of the magnetic field, ∇B, whose analog in the physics of the atmosphere is the β-plane effect, *i.e.* the variation with altitude of the Coriolis frequency f.

In the Ertel's equation other effect are included: the velocity in the operator d/dt may include the diamagnetic velocity and higher order effects, which in geophysics are called *non-geostrophic* effects. The *geostrophic* case consists of assuming that the velocity of the fluid is strictly given by the equivalent of the $E \times B$ velocity in plasma.

Using the Ertel's theorem we can examine the consequence of moving an element of plasma toward the center.

When the vorticity has the same direction as the confining magnetic field, the approach to the center is an effective increase of the numerator in

$$\frac{\Omega_{ci} + |\omega|}{n}$$

and this means that the density should also increase, to keep this ratio constant.

When the element coming toward the center has a vorticity that is opposite to the direction of the magnetic field then the constancy of

$$\frac{\Omega_{ci} - |\omega|}{n}$$

imposes a reduction of the density with the increase of the physical vorticity.

These two cases correspond to co- and counter- rotation that introduce effective external torque in the plasma.

5.8.1 Estimations for the Ertel's theorem in the H mode

Consider that the vorticity results from a strongly sheared velocity layer,

$$v_\theta \in [0, c_s]$$

We calculate the sound speed

$$\begin{aligned} c_s &= 9.79 \times 10^5 \sqrt{T_e} \ \ (cm/s) \\ &\sim 10^4 \sqrt{T_e} \ \ (m/s) \end{aligned}$$

and for

$$T_e \sim 10^3$$

we have

$$c_s \sim 10^4 \times 30 \quad (m/s) = 3 \times 10^5 \ (m/s)$$

And it varies of a distance of 1 (cm). This means that the vorticity is

$$\begin{aligned} \omega &\sim \frac{\delta v_\theta}{\delta r} = \frac{3 \times 10^5 \ (m/s)}{10^{-2} \ (m)} \\ &= 3 \times 10^7 \ \left(s^{-1}\right) \end{aligned}$$

This should be compared with the ion cyclotron frequency

$$\Omega_{ci} = 9.58 \times 10^3 B \, (Gs) \ \left(s^{-1}\right)$$

and for

$$\begin{aligned} B &= 3 \ (T) \\ &= 3 \times 10^4 \ (Gs) \end{aligned}$$

we have

$$\begin{aligned} \Omega_{ci} &\sim 10^4 \times 3 \times 10^4 \ \left(s^{-1}\right) \\ &= 3 \times 10^8 \ \left(s^{-1}\right) \end{aligned}$$

We have

$$\frac{\Omega_{ci} + |\omega|}{n} = \text{const on lines}$$

$$\begin{aligned} \frac{\Omega_{ci}}{n} &= \frac{\Omega_{ci} + |\omega|}{n'} \\ \frac{3 \times 10^8}{n} &= \frac{3 \times 10^8 + 3 \times 10^7}{n'} \end{aligned}$$

Then

$$n' = n \, (1 + 0.1)$$

It means that the density increases with 10% where there is sheared poloidal velocity.

NOTE

In **Hasegawa Maclennan Kodama** it is estimated the amplitude of the nonlinearity using

$$\begin{aligned} \Omega_{ci} &\sim 10^8 ... 10^9 \ \left(s^{-1}\right) \\ \omega_* &\sim 10^6 ... 10^7 \ \left(s^{-1}\right) \\ \rho_s &\sim 10^{-3} \ (m) \\ \frac{|e| \, \phi}{T_e} &\sim 10^{-2} ... 10^{-1} \\ \frac{\omega}{\Omega_{ci}} &= \frac{1}{\Omega_{ci}} \frac{1}{B_0} \boldsymbol{\nabla}_\perp^2 \phi = k^2 \rho_s^2 \frac{|e| \, \phi}{T_e} = 10^{-2} ... 10^{-3} \end{aligned}$$

but it is question of *turbulence*.

6 The expulsion of the vorticity of one sign toward the periphery of the cross section

There are numerical simulations that suggests that this is the case.

Kai Neffaa.

Hasegawa-Wakatani. (hw_a.eps).

Shecter Dubin.

Marcus.

7 The co- and the counter-injection of NB and density peaking

7.1 Notes on experiments and other studies

In the paper **density peaking Ida** it is mentioned the result of an experiment of NBI in the tokamak JFT-2M. The following situations appear:

1. NB injection in a sense *opposite* to the current in tokamak (counter-injection). Leads to
   (a) density peaking (and density pinch)
   (b) improvement of energy confinement (possibly due to reduction of η_i turbulence)
   (c) more negative radial electric field, E_r;
   (d) impurities accumulate in the center; then T_e starts to decrease;
   (e) the ion temperature increases, T_i;
2. NB injection in a sense *direct* relative to the current (co-injection). Leads to
   (a) density flattening

In the presentation **APS Kamada** about the rotation experiments made on JT60U.

Various results, in particular: fast ion loss due to ripple induces COUNTER toroidal rotation in the edge. At page 29:

drop in v_{tor} is observed when a steep gradient of the ion temperature is observed.

In the paper **Rotation Gurcan** it is discussed the intrinsic toroidal rotation. The observation from experiments is mentioned: the intrinsic toroidal rotation is in the *co-current* (**note**: like H-mode) direction and: the rotation decreases when the poloidal current increases.

7.2 A unique direction for the poloidal rotation

The statement is:

for fixed direction of the confining magnetic field, B_0;

for fixed direction of the toroidal current $\mathbf{J}_T$ created by inductive electric field

then

the poloidal rotation of plasma induced by the NBI is always in the same direction, irrespective of the direction of the NBI.

7.3 A discussion on the relative directions of flows

There is a new file **tex.**

We know that a NB leads to charge exchange and ionizations.

Suppose that there is created a pair electron-ion and that the particles have mainly perpendicular energies. This should be calculated.

According to Rosenbluth-Hinton, the two trapped particles will evolve such that to occupy their bananas. The deviation of the ion from the magnetic surface where it was created is large. The deviation of the electron is small. There is a difference in these deviations, in the radial direction. Then there is a radial current

$$j_r$$

This current combines with the main magnetic field B_0 and generates a torque which is applied on plasma

$$\mathbf{j}_r \times \mathbf{B}_0$$

and this is always in the same poloidal direction, for fixed $\mathbf{B}_0$, irrespective of the sense of the current or of the sense of the NB.

The combination of this poloidal rotation and the toroidal rotation induced by the NB gives a *swirl* motion of plasma under the effect of the NB.

The direction of the *swirl* motion, *i.e.* its helicity, is only dependent on the direction of the injection of the NB.

Suppose we have the *current* and the *magnetic field* in the same direction. Then the helicity of the magnetic lines is *right* (to the right when looking in the direction of $\mathbf{B}_0$).

Suppose the injection of the NB is co-current. Then we have

1. a toroidal rotation that is co-curent and co-magnetic field

2. a poloidal rotation that is *left* (to the left wehn we look in the direction of $\mathbf{B}_0$). This is always so, since $\mathbf{B}_0$ is fixed.

This makes a *swirl* that is *left*, therefore opposite to the helicity of the magnetic lines.

This should impede the evolution of the vorticity, or would expell it to the border even more efficiently.

Suppose the injection of the NB is counter-current. Then we have

1. a toroidal rotation that is counter-curent and counter-magnetic field

2. a poloidal rotation that is *left* (to the left wehn we look in the direction of $\mathbf{B}_0$). This is always so, since $\mathbf{B}_0$ is fixed.

This makes a *swirl* that is *right* when we look in the direction of counter-injection, therefore is the same as the helicity of the magnetic lines.

The compatibility of the *swirl* with the magnetic *helicity* makes the evolution of the vorticity easier.

The NB injects torque and then the poloidal rotation is sustained (against magnetic pumping decay).

The vorticity evolves for one of the reasons:

1. either the maximum of the vorticity must coincide with the maximum of the current density (but they have opposite signs)

2. it is allowed by the Euler or Liouville equation to evolve by pinching to the axis. Here the conformal transformations should be used to show that there are solutions that can be obtained from an initial distribution.

Although the vorticity is driven/dissipative, the structured states should play a role. And they are given by one of:

1. *sinh*-Poisson eq.

2. Liouville eq.

3. our equation.

In both cases, the density will follow. It will be expelled (together with the impurities) at co-injection. And it will be pinched together with the impurities at counter-injection.

8 Appendix

8.1 Stability of swirling jets

This is from **Swirl thesis**.

Consider a jet immersed ina flow and rotating as a solid body with Ω. Define

$$\widetilde{\omega} \equiv \omega - m\Omega - k_z V_z$$

The situation that is examined is of a Rankine vortex in the interior part of the swirling jet and an uniform vertical velocity in the external region, where also the azimuthal rotation velocity has $1/r$ decay. It is noted

$$\psi \equiv \sqrt{k_z^2 \left(\frac{4\Omega^2}{\widetilde{\omega}^2} - 1 \right)}$$

and a *swirl parameter*

$$S \equiv \frac{\Omega R}{\delta v}$$

where δv is the difference in v_z velocity between the background and the jet.

The dispersion relation for helical perturbations ($m \neq 0$) is

$$\left(\widetilde{\omega} + k_z R\right)^2 \left(-2mS + \widetilde{\omega}\psi \frac{J_m'(\psi)}{J_m(\psi)} \right) + \frac{\widetilde{\omega}^3 \psi^2}{|k_z|} \frac{K_m'(|k_z|)}{K_m(|k_z|)} = 0$$

8.2 Notes to Ertel's theorem

The paper **Nonlinear Fluid Holm Abarbanel** presents fluids described by barotropic model and derives the Ertel's theorem. See also Horton, **Drift wave Transport** Phys.Rep.1990.

Two useful equations, more than we need, however.

The time equation of the *vorticity vector* (**Miura NIFS 372**)

$$\begin{aligned}\frac{\partial \boldsymbol{\omega}}{\partial t} &= -\boldsymbol{\nabla}\times\left[\left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{v}\right] \\ &\quad+\frac{\boldsymbol{\nabla}\rho\times\boldsymbol{\nabla}p}{\rho^2} \quad \text{(baroclinic)} \\ &\quad+\frac{1}{\mathrm{Re}}\frac{\boldsymbol{\nabla}^2\boldsymbol{\omega}}{\rho} \\ &\quad-\frac{1}{\mathrm{Re}}\frac{\boldsymbol{\nabla}\rho\times\left[\boldsymbol{\nabla}^2\mathbf{v}+\frac{1}{3}\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{v}\right)\right]}{\rho^2}\end{aligned} \tag{125}$$

NOTE

There is the identity

$$\boldsymbol{\nabla}\times\left[\left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{v}\right] = \left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\boldsymbol{\omega}+\boldsymbol{\omega}\left(\boldsymbol{\nabla}\cdot\mathbf{v}\right)-\left(\boldsymbol{\omega}\cdot\boldsymbol{\nabla}\right)\mathbf{v} \tag{126}$$

The first term in the right of (125) can be simplified assuming incompressibility

$$\boldsymbol{\nabla}\cdot\mathbf{v} = 0$$

and transfering the first term to the left

$$\begin{aligned}\frac{\partial \boldsymbol{\omega}}{\partial t}+\left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\boldsymbol{\omega} &= -\left(\boldsymbol{\omega}\cdot\boldsymbol{\nabla}\right)\mathbf{v} \\ &\quad+\frac{\boldsymbol{\nabla}\rho\times\boldsymbol{\nabla}p}{\rho^2}\end{aligned}$$

for an ideal plasma. The first term in the right is small since the velocity does not change along the z-direction, which is the direction of the vorticity $\boldsymbol{\omega}$. Then we have

$$\frac{d\omega}{dt}\sim\left(\frac{1}{n^2}\frac{dn}{dr}\right)\nabla_{\|}p$$

and it results that the change of vorticity is sustained by the parallel variation of the pressure. A perturbation (cooling the edge with impurities) produces modes that advect perpendicularly the temperature and permit the propagation of the vorticity perturbation, such that the total value remains constant.

END

The equation for the *generation of enstrophy*

$$Q=\frac{1}{2}\left|\boldsymbol{\omega}\right|^2$$

$$
\begin{aligned}
\frac{\partial Q}{\partial t} &= -\boldsymbol{\omega}\cdot\left\{\boldsymbol{\nabla}\times\left[\left(\mathbf{u}\cdot\boldsymbol{\nabla}\right)\mathbf{u}\right]\right\} \\
&+\boldsymbol{\omega}\cdot\frac{\boldsymbol{\nabla}\rho\times\boldsymbol{\nabla}p}{\rho^2} \\
&+\frac{1}{\mathrm{Re}}\frac{\boldsymbol{\omega}\cdot\boldsymbol{\nabla}^2\boldsymbol{\omega}}{\rho} \\
&-\frac{1}{\mathrm{Re}}\frac{\boldsymbol{\omega}\cdot\left\{\boldsymbol{\nabla}\rho\times\left[\boldsymbol{\nabla}^2\mathbf{u}+\frac{1}{3}\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{u}\right)\right]\right\}}{\rho^2}
\end{aligned}
$$

The spatial average is

$$
\begin{array}{rcc}
\left\langle\frac{\partial Q}{\partial t}\right\rangle = & -\left\langle\boldsymbol{\omega}\cdot\left\{\boldsymbol{\nabla}\times\left[\left(\mathbf{u}\cdot\boldsymbol{\nabla}\right)\mathbf{u}\right]\right\}\right\rangle & \text{rotational} \\
& +\left\langle\boldsymbol{\omega}\cdot\frac{\boldsymbol{\nabla}\rho\times\boldsymbol{\nabla}p}{\rho^2}\right\rangle & \text{baroclinic} \\
& +\frac{1}{\mathrm{Re}}\left\langle\frac{\boldsymbol{\omega}\cdot\boldsymbol{\nabla}^2\boldsymbol{\omega}}{\rho}\right\rangle & \text{viscous} \\
& -\frac{1}{\mathrm{Re}}\left\langle\frac{\boldsymbol{\omega}\cdot\left\{\boldsymbol{\nabla}\rho\times\left[\boldsymbol{\nabla}^2\mathbf{u}+\frac{1}{3}\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{u}\right)\right]\right\}}{\rho^2}\right\rangle & \text{viscous-compressional}
\end{array}
$$

Angles that are made by the vectors from these terms. The terms evolve due to alignement and uniformization of the angles.

An extended version of the Ertel's theorem can be found in **Magnetic Penetration Hall Effect** where we have

$$
n\frac{d}{dt}\frac{\omega+\Omega_{ci}}{n}=\frac{1}{m}\left[\frac{1}{n},\boldsymbol{\nabla}p\right]
$$

which is a kind of baroclinic term.

8.3 The Hasegawa-Wakatani equations

The theory of **Wakatani Hasegawa** is about the *turbulence* in $2D$. The equation for the ion vorticity

$$
\boldsymbol{\nabla}\times\mathbf{v}=\widehat{\mathbf{e}}_z\left(\frac{\boldsymbol{\nabla}^2\phi}{B_0}\right) \tag{127}
$$

is

$$
\frac{d}{dt}\left(\frac{\boldsymbol{\nabla}^2\phi}{B_0\Omega_{ci}}\right)=\frac{1}{en_0}\frac{\partial J_z}{\partial z}+\mu\boldsymbol{\nabla}^2\left(\frac{\boldsymbol{\nabla}^2\phi}{B_0\Omega_{ci}}\right) \tag{128}
$$

where μ is the viscosity.

The *fluctuating* parallel current has ionic and electronic components

$$
J_z=J_z^e+J_z^i \tag{129}
$$

It is considered that the most important role of the *ion* current, J_z^i is to dissipate by Landau damping the ion vorticity. This happens when the frequency becomes comparable with the inverse parallel transit time

$$\omega_* \sim \frac{v_{thi}}{qR} \tag{130}$$

Here it is used the "frequency" that is most typical, *i.e.* the diamagnetic frequency, as for the drift waves. The equality Eq.(130) is reminiscent of the drift wave theory, where the radial extension of the eigenmode is limited by the fact that wave reaches the *ion turning point*, where the energy is absorbed by the ions whose thermal speed equals the wave phase velocity.

This means

$$\frac{1}{en_0}\frac{\partial J_z^i}{\partial z} = \begin{cases} \omega_*\left(k_\perp\right)\frac{T_i}{T_e}\frac{e\phi}{T_e} & \text{for} \quad \omega_*\left(k_\perp\right) \sim \frac{v_{thi}}{qR} \\ 0 & \text{for} \quad \omega_*\left(k_\perp\right) > \frac{v_{thi}}{qR} \end{cases} \tag{131}$$

The *electron* parallel current J_z^e is obtained from the electron continuity equation

$$\begin{aligned} \frac{d}{dt}\left(n_0+\widetilde{n}\right) &= \\ &= \left(\frac{\partial}{\partial t} + \frac{-\nabla\phi\times\widehat{\mathbf{n}}}{B_0}\cdot\nabla\right)\left(n_0+\widetilde{n}\right) \\ &= \frac{1}{e}\frac{\partial J_z^e}{\partial z} \end{aligned} \tag{132}$$

We have also to write the *equation of motion* for the electrons, in the parallel direction

$$J_z^e = \frac{T_e}{e\eta}\frac{\partial}{\partial z}\left(\frac{\widetilde{n}}{n_0} - \frac{e\phi}{T_e}\right) \tag{133}$$

This is the equilibrium in the parallel direction, of the pressure force, the electric force and the friction due to collisions. Or: the parallel gradient of the pressure, the parallel electric field lead to a force which is balanced by parallel friction.

The last two equations allow to eliminate the electron parallel current J_z^e.

After normalization

$$\begin{aligned} \frac{e\phi}{T_e} &\rightarrow \phi \\ t\Omega_{ci} &\rightarrow t \\ x/\rho_s &\rightarrow x \\ \frac{\widetilde{n}}{n_0} &\rightarrow n \end{aligned} \tag{134}$$

the following equations result

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}+\frac{-\nabla\phi\times\widehat{\mathbf{n}}}{B_0}\cdot\nabla\right)\nabla^2\phi &= \overline{C}_1\left(\phi-n\right)+C_2\nabla^4\phi+C_3\phi \qquad (135)\\
\left(\frac{\partial}{\partial t}+\frac{-\nabla\phi\times\widehat{\mathbf{n}}}{B_0}\cdot\nabla\right)\left(n+\ln n_0\right) &= \overline{C}_1\left(\phi-n\right)
\end{aligned}
$$

where

$$\overline{C}_1\equiv-\frac{T_e}{e^2n_0\eta\Omega_{ci}}\frac{\partial^2}{\partial z^2} \tag{136}$$

$$C_2\equiv\frac{\mu}{\rho_s^2\Omega_{ci}} \tag{137}$$

$$C_3=\begin{cases}\frac{\omega_*}{\Omega_{ci}}\frac{T_i}{T_e} & \text{for} \quad \omega_*=v_{thi}/\left(qR\right)\\ 0 & \text{otherwise}\end{cases} \tag{138}$$

NOTES on Wakatani-Hasegawa.

The parallel equation of motion of electrons plays an essential role. The parallel forces arising from : the pressure gradient, the electric field of the wave and the friction are balanced. This fixes the amount of electron flow that is needed for this equilibrium. This balance is typical for a wave-like perturbation, for an instability consisting of a perturbed potential ϕ and a perturbed density $\widetilde{n}$. All this stays on a background equilibrium.

9 The connection between the toroidal and poloidal rotation through the baroclinic term

The equation

$$
\begin{aligned}
\frac{\partial\boldsymbol{\omega}}{\partial t}+\left(\mathbf{v}\cdot\nabla\right)\boldsymbol{\omega} &= -\left(\boldsymbol{\omega}\cdot\nabla\right)\mathbf{v}\\
&\quad+\frac{\nabla\rho\times\nabla p}{\rho^2}
\end{aligned}
$$

has components. The vorticity along the toroidal direction ω_φ results from poloidal rotation.

$$\omega_\varphi\sim\frac{\partial v_\theta}{\partial r}$$

$$
\begin{aligned}
\frac{\partial}{\partial t}\omega_\varphi+\left(v_\theta\frac{\partial}{r\partial\theta}+v_\varphi\frac{\partial}{R\partial\varphi}\right)\omega_\varphi &= -\left(\omega_\theta\frac{\partial}{r\partial\theta}+\omega_\varphi\frac{\partial}{R\partial\varphi}\right)v_\varphi\\
&\quad+\left.\frac{\nabla\rho\times\nabla p}{\rho^2}\right|_\varphi
\end{aligned}
$$

and without the $\partial/\partial\varphi$ terms it is

$$\begin{aligned}\frac{\partial}{\partial t}\omega_\varphi + \left(v_\theta \frac{\partial}{r\partial\theta}\right)\omega_\varphi &= -\left(\omega_\theta \frac{\partial}{r\partial\theta}\right) v_\varphi \\ &\quad + \left.\frac{\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla}p}{\rho^2}\right|_\varphi\end{aligned}$$

And

$$\begin{aligned}\frac{\partial}{\partial t}\omega_\theta + \left(v_\theta \frac{\partial}{r\partial\theta} + v_\varphi \frac{\partial}{R\partial\varphi}\right)\omega_\theta &= -\left(\omega_\theta \frac{\partial}{r\partial\theta} + \omega_\varphi \frac{\partial}{R\partial\varphi}\right) v_\theta \\ &\quad + \left.\frac{\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla}p}{\rho^2}\right|_\theta\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial t}\omega_\theta + \left(v_\theta \frac{\partial}{r\partial\theta}\right)\omega_\theta &= -\left(\omega_\theta \frac{\partial}{r\partial\theta}\right) v_\theta \\ &\quad + \left.\frac{\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla}p}{\rho^2}\right|_\theta\end{aligned}$$

Let us repeat the first relationship

$$\begin{aligned}\frac{\partial}{\partial t}\omega_\varphi + \left(v_\theta \frac{\partial}{r\partial\theta}\right)\omega_\varphi &= -\left(\omega_\theta \frac{\partial}{r\partial\theta}\right) v_\varphi \\ &\quad + \left.\frac{\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla}p}{\rho^2}\right|_\varphi\end{aligned}$$

If the plasma has toroidal rotation with different toroidal velocities v_φ on different magnetic surfaces (sheared) $v_\varphi(r)$;

then it has a poloidal component of the vorticity ω_θ

if the toroidal rotation velocity $v_\varphi(r)$ gets a variation on the magnetic surface

$$v_\varphi(r) \rightarrow v_\varphi(r,\theta)$$

then the term

$$-\left(\omega_\theta \frac{\partial}{r\partial\theta}\right) v_\varphi$$

induces an increase of the variation of the poloidal velocities on nearby magnetic surfaces

$$\frac{\partial \omega_\varphi}{\partial t}$$

the poloidal rotation gets accelerated on some surfaces.

The simple fact that the toroidal velocity gets a variation in the magnetic surface leads to generation of poloidal rotation.

Then: why the toroidal velocity would get a variation with θ ?

Pellet?

And

$$\frac{\partial}{\partial t}\ln\omega_\theta = -\frac{\partial}{r\partial\theta}v_\theta$$

$$\omega_\theta\left(t\right) = \omega_\theta^0 - \alpha\exp\left[-\frac{\partial v_\theta}{r\partial\theta}\times t\right]$$

and the toroidal rotation can be reversed.

Is-it possible that the torque is localized?