1 Physics of ITG

1.1 General properties of the observed fluctuations

In the paper **CowleyKulsrudSudan** it is done an analysis of the transport produced by drift waves. Principal characteristics:

frequency spectrum at k is broad, centered on ω_*	$\omega_* = k_\perp \frac{\rho_i v_{thi}}{L_n} \frac{T_e}{T_i}$ (electrons)
wavenumber spectrum is decreasing	$k^{-\beta}$
level of fluctuations	$\frac{\delta n}{n_0} \sim \frac{1}{k L_n}$
typical wavelength	$k_\perp \rho_i \sim 1$

Other characteristics

the range of phase velocity	$k_\parallel v_{thi} < \omega \ll k_\parallel v_{the}$
the electron density perturbation is adiabatic	$\frac{\delta n}{n_0} = \frac{e\phi}{T_e}$
it is an ion sound mode, combined with $\boldsymbol{\nabla} p_0$	
the rate of pressure relaxation along the field line:	$k_z c_s$

ITG can be studied even without a density gradient, $\boldsymbol{\nabla} n_0 = 0$.

The process is, shortly: parallel compression of ions with generation of an electric potential, radial advection of cold ions by $E \times B$ to the zone of compressed density with the effect of local reduction of ion pressure, continuation of the process of ion suction into the perturbed zone and repetition of this scenario.

The radial extension of the mode is determined by the effect of the Finite Larmor Radius. It is *not* determined by the coupling of the poloidal harmonics belonging to different resonant magnetic surfaces. This is because the mode exists even if the ion drift ω_D is not considered.

Two physical reasons for which the ITG mode has a radially extended structure, both physical reasons being related to the Finite Larmor Radius effect:

1. the existence of a finite k_x makes even more effective the averaging which is produced by the Larmor rotation; the wave potential seen by the particle is averaged on the gyration.

2. the existence of a finite k_x means that the wave potential has a variation on the radial direction. This generates a finite gradient of the potential on the radius (an electric field) which combines like $E \times B$ with the equilibrium magnetic field, producing a drift in the poloidal, θ direction. This motion simply expands the energy without contributing to the *radial* advection of the temperature, which sustains the instability.

The representation of the eddies in the form of *twisted eddies* is suggested by the necessity of minimising everywhere $k_\parallel$. This simply means that locally the mode tries to align with the field.

There are two branches of the drift mode in the presence of the gradient of the ion temperature:

1. the *slab* branch, which is a drift wave coupled with the ion acoustic wave, destabilised by the gradient of the ion temperature;
2. the *interchange* branch, which is destabilised by the bad curvature of the magnetic field lines in the presence of the ion temperature gradient;

When the **magnetic shear** effect is stronger than the **magnetic curvature** effect, the interchange mode goes into the slab branch.

Later: magnetic shear is important to confine radially the eigenmodes, as for any drift mode. The magnetic curvature is the main part of the particle's drift which enters in the propagator of the mode. This is a shift of frequency which affects the resonance of the propagator and induces a response of the particle orbit to the electric perturbation such that exchange of energy becomes possible. The fact that magnetic curvature is important means that the main source of growth-damping of the perturbation is the drift of the ions. It is more important than the bounding brought about by the radial magnetic shear.

The compressibility of the perpendicular ion velocity due to the polarization drift does not play any role in the *linear* stability of the mode. This is because the two nonlinearities: (1) divergence of the ion polarization drift velocity, and (2) advection of the fluctuation of the density in the poloidal direction by the $E \times B$ velocity of the wave in the radial direction, are not contributing to the *linear* destabilization, but are considered via *renormalization*.

Also in Plasma.tex.

The ITG is **an electrostatic sound wave driven unstable by the ion-pressure gradient.**

$$\eta_i = \frac{L_n}{L_{T_i}}$$

and the condition for stability is:

$$\eta_i \gtrsim \eta_{ic} \sim 1.5$$

In the *slab* geometry with straight magnetic field it is **a sound wave destabilized by ion temperature gradients sufficiently steep with respect to the density gradients (**i.e. L_{T_i} **small** and/or L_n **large**). It is driven by the ion dynamics parallel to the magnetic field.

Except near the marginal stability where the kinetic effects are important, the mode is **fluid-like**, with

$$\begin{aligned} \gamma_k &> \omega_k > \omega_{Di} \ \ , \ \ k_\parallel v_i \\ k_\perp \rho_i &< 1 \end{aligned}$$

The fast growing mode is **ballooning significantly to the outside of the torus** and is driven by the **unfavorable magnetic curvature**.

The *toroidal* ITG mode aligns along the magnetic field and is **driven by the perpendicular drifts of the ions induced by the toroidal geometry** $\mathbf{v}_D$. These drifts are destabilizing when they are going in the same direction as the ion-diamagnetic flow,

$$\mathbf{v}_D \parallel \mathbf{v}_i^{dia}$$

which happens usually on the outer side of the torus. The mode tends to ballon and it is localized in the 'unfavorable curvature' region. It is an interchange-type instability. Since the particles moving here can be captured on banana orbits the mode can be coupled with trapped particle instabilities.

In **Horton** it is introduced the *cuasi-toroidal* approximation, in which the drift of the ions is taken at equatorial plane and considered constant over the surface, to not introduce mode-couplings due to different resonant surfaces traversed by the drifting ions.

A characteristic of the ITG mode:

$$\omega_k \ll \gamma_k$$

i.e. the mode is fast growing and the renormalization based on weak coupling does not work.

1.2 Phenomenology of the Ion-temperature-gradient mode (ITG)

The ITG is **an electrostatic sound wave driven unstable by the ion-pressure gradient.**

$$\eta_i = \frac{L_n}{L_{T_i}}$$

and the condition for stability is:

$$\eta_i \gtrsim \eta_{ic} \sim 1.5$$

In the *slab* geometry with straight magnetic field it is **a sound wave destabilized by ion temperature gradients sufficiently steep with respect to the density gradients** (i.e. L_{T_i} **small** and/or L_n **large**: flat density and steep temperature). It is driven by the ion dynamics parallel to the magnetic field.

Except near the marginal stability where the kinetic effects are important, the mode is **fluid-like**, with

$$\begin{aligned} \gamma_k &> \omega_k > \omega_{Di} \;\;,\;\; k_\parallel v_i \\ k_\perp \rho_i &< 1 \end{aligned}$$

The *growth rate is higher than the REAL part of the frequency of the mode. It is almost PURELY GROWING.*

The fast growing mode is **ballooning significantly to the outside of the torus** and is driven by the **unfavorable magnetic curvature**.

The *toroidal* ITG mode aligns along the magnetic field and is **driven by the perpendicular drifts of the ions induced by the toroidal geometry**, $\boldsymbol{\nabla} B$ and *curvature*, $\mathbf{v}_D$. These drifts are destabilizing when they are going in the same direction as the ion-diamagnetic flow, which happens usually on the outer side of the torus. The mode tends to balloon and it is localized in the 'unfavorable curvature' region. It is an interchange-type instability. Since the particles moving here can be captured on banana orbits the mode can be coupled with trapped particle instabilities.

A characteristic of the ITG mode:

$$\omega_k \ll \gamma_k$$

i.e. the mode is fast growing and the renormalization based on weak coupling does not work.

1.2.1 Estimations

The mixing length estimations.

The **growth rate**:

$$\gamma = \left(\frac{1+\eta_i}{\tau} \right)^{1/2} \frac{k_y \rho_s c_s}{L_s}$$

Radial mode width:

$$\lambda_r = \left(\frac{1+\eta_i}{\tau} \right)^{1/2} \rho_s$$

Pressure fluctuation level:

$$\frac{\widetilde{p}_i}{p_{0i}} \sim \frac{\lambda_r}{L_p}$$

The thermal diffusivity:

$$\chi_i \sim \left(\frac{1+\eta_i}{\tau} \right)^{1/2} \frac{\rho_s^2 c_s}{L_s}$$

Frequently invoked mixing length rules:

$$\frac{\widetilde{p}_i}{p_{0i}} \sim \frac{1}{k_x L_p} \text{ and } D \sim \frac{\gamma}{k_x^2}$$

2 Model I Carreras (renormalisation)

About ions. The following model has been proposed by **Carreras**:

There is a velocity imposed by external means $\mathbf{V}$. It is considered to be of the $E \times B$ type and one introduces a formal potential $\phi_0(\psi)$ whose gradient will generate the radial electric field E_r.

$$\frac{\partial \widetilde{n}_i}{\partial t} + \widetilde{V}_x \frac{dn_0}{dx} + \widetilde{\mathbf{V}} \cdot \boldsymbol{\nabla} \widetilde{n}_i = -n_0 \left(\boldsymbol{\nabla}_\perp \cdot \widetilde{\mathbf{V}}_\perp + \boldsymbol{\nabla}_\parallel \cdot \widetilde{\mathbf{V}}_\parallel \right)$$

where the perpendicular ion velocity is due to the $\mathbf{E} \times \mathbf{B}$ motion in the wave and polarization drift (no diamagnetic)

$$\mathbf{v}_\perp = \mathbf{v}_{ExB} + \mathbf{v}_{pol}$$

In these works the variation of the drift waves along the magnetic field is neglected

$$\boldsymbol{\nabla}_\parallel = 0$$

and the model becomes a cuasi-two-dimensional one. By contrast, in **Horton Sugama** the operator acting on the non-adiabatic part of the ions is

$$\left(\frac{\partial}{\partial t} + v_\parallel \frac{1}{qR} \frac{\partial}{\partial \theta} + i\omega_D \right) h^{ion}$$

and shows that the parallel term $v_\parallel \nabla_\parallel$ with $\nabla_\parallel = \frac{1}{qR}\frac{\partial}{\partial \theta}$ is present. This is however a *kinetic* model, while **Carreras et al.** is a *fluid* model. The term

$$v_\parallel \frac{1}{qR} \frac{\partial}{\partial \theta} = v_\theta \frac{\partial}{r \partial \theta}$$

is the poloidal variation of the distribution function h^{ion}. And ω_D is the frequency associated to the drift motion of ions, $\mathbf{v}_D^{ion}$.

In one of the models of **Carreras Diamond** et al. it is considered first the non-adiabatic response of the ELECTRONS in a model of *trapped electron drift instability* with the identification of the $i\delta_k$ term. The relationship between the density and potential is then *inverted*, and in the problem the potential is replaced everywhere by the fluctuating ion density.

The model equation is written for the perturbed density

$$n = \widetilde{n}/n_0$$

and becomes

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(1 - \rho_s^2 \boldsymbol{\nabla}_\perp^2\right) n \\
&+ V_{*n} \frac{\partial n}{\partial y} + \text{ (the diamagnetic poloidal advection of the perturbation)} \\
&+ D_0 \frac{\partial^2 n}{\partial y^2} \text{ (the drive, effective } i\delta \text{ , or } ik_y^2 D_0 \text{ and } D_0 \sim \frac{1}{\nu_{eff}}) \\
&- L_n D_0 \left[\boldsymbol{\nabla}_\perp \left(\frac{\partial n}{\partial y} \right) \times \widehat{\mathbf{e}}_z \right] \cdot \boldsymbol{\nabla}_\perp n \text{ (the } \mathbf{E} \times \mathbf{B} \text{ convection of the nonadiabatic e-density)} \\
&+ \rho_s c_s \left(\boldsymbol{\nabla}_\perp n \times \widehat{\mathbf{e}}_z \right) \cdot \boldsymbol{\nabla}_\perp \left(\rho_s^2 \boldsymbol{\nabla}_\perp^2 \right) n \text{ (the ion polarization nonlinearity)} \\
&= 0
\end{aligned}$$

where the normalized ion density is

$$n \equiv \frac{\widetilde{n}}{n_0} \text{ (ions)}$$

the ion diamagnetic drift velocity is

$$V_{*n} = \frac{c_s \rho_s}{L_n}$$

and the notation is introduced

$$D_0 = \alpha \sqrt{\varepsilon} \frac{(\rho_s c_s)^2}{L_T L_n \nu_{eff}}$$

This comes from the electron *parallel* momentum conservation, involving the parallel gradient of the pressure, parallel gradient of the potential and the *collisions.*

Note the difference compared with the model of **Carreras Sidikman Diamond Terry**:

1. the Kelvin-Helmholtz term

   $$\rho_s c_s \frac{\partial \widetilde{n}_i}{\partial y} \frac{\partial}{\partial x} \left(\rho_s^2 \frac{\partial^2 \langle n \rangle}{\partial x^2} \right)$$

   We **note** that the structure of this term is similar to the classical vectorial non-linearity of Euler, Hasegawa-Mima, etc.,

   $$\left[\left(-\boldsymbol{\nabla} \phi \times \widehat{\mathbf{e}}_z \right) \cdot \boldsymbol{\nabla} \right] \boldsymbol{\nabla}^2 \phi$$

   advection of the vorticity field by its own velocity field. The replacements are: instead of $\boldsymbol{\nabla} \phi$ we havehere $\partial \widetilde{n}_i / \partial y$ and instead of the vorticity $\Delta \phi$, we have here $\partial^2 \langle n \rangle / \partial x^2$. Note that the adevected (by the fluctuation) quantity is the *average* density.

2. the absence of the parallel damping term,

   $$-\frac{c_s^2}{\nu_i} \nabla_{\parallel}^2 \widetilde{n}_i$$

3. the absence of the total derivative

   $$\frac{d \widetilde{n}_i}{dt} = \frac{\partial}{\partial t} + \langle \mathbf{V}_E \rangle \cdot \boldsymbol{\nabla} \widetilde{n}_i$$

End.

Without the non-adiabatic electrons [the third: $D_0 \frac{\partial^2 n}{\partial y^2}$ and fourth: $-L_n D_0 \left[\boldsymbol{\nabla}_{\perp} \left(\frac{\partial n}{\partial y} \right) \times \widehat{\mathbf{e}}_z \right] \cdot \boldsymbol{\nabla}_{\perp} n$ (which is the $E \times B$ nonlinearity) terms] the equation reduces to the original Hasegawa-Mima equation. An energy sink can be modelled by adding a hyperviscosity term in the model equation. This leads to a finite band of unstable drift modes with a high k cutoff. Let's note $\mathbf{k}_\perp = \mathbf{k}$.

In Fourier space

$$i\frac{\partial}{\partial t}n_{\mathbf{k}} - \frac{\omega_{*\mathbf{k}} + ik_y^2 D_0}{1+k^2\rho_s^2}\, n_{\mathbf{k}} + \frac{i}{1+k^2\rho_s^2}\left(N_{\mathbf{k}}^{\mathbf{E}\times\mathbf{B}} + N_{\mathbf{k}}^{POL}\right) = 0$$

where the nonlinearities are

$$N_{\mathbf{k}}^{\mathbf{E}\times\mathbf{B}} = -L_n D_0 \left[\boldsymbol{\nabla}_\perp \left(\frac{\partial n}{\partial y}\right) \times \widehat{\mathbf{e}}_z\right] \cdot \boldsymbol{\nabla}_\perp n$$

or

$$N_{\mathbf{k}}^{\mathbf{E}\times\mathbf{B}} = -i\frac{1}{2}\, L_n D_0 \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \left[\left(\mathbf{k}\times\mathbf{k}'\right)\cdot\widehat{\mathbf{e}}_z\right]\, \left(k_y'' - k_y'\right) n_{\mathbf{k}'}\, n_{\mathbf{k}''}$$

and, the *vectorial nonlinearity*, of the type Hasegawa-Mima, coming from the ion polarization velocity

$$N_{\mathbf{k}}^{POL} = \frac{1}{2}\rho_s c_s \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \left[\left(\mathbf{k}\times\mathbf{k}'\right)\cdot\widehat{\mathbf{e}}_z\right]\, \rho_s^2\, \left(k''^2 - k'^2\right)\, n_{\mathbf{k}'}\, n_{\mathbf{k}''}$$

The linear dispersion relation $i\frac{\partial}{\partial t} = \omega_{\mathbf{k}}$

$$\omega_{\mathbf{k}} = \omega_{\mathbf{k}}^{(0)} + i\gamma_{\mathbf{k}}^{(0)} = \frac{\omega_{*\mathbf{k}}}{1+k^2\rho_s^2} + i\frac{k_y^2 D_0}{1+k^2\rho_s^2}$$

which means that the term

$$\frac{k_y^2 D_0}{1+k^2\rho_s^2} \text{ is the drive.}$$

(this must be compared with the *dispersion relation* for the Rossby waves in **Nezlin Sutyrin**).

For comparison, **Hasegawa-Mima** equation, starting from the polarization drift of the ions

$$\frac{\partial}{\partial t}\left(1-\rho_s^2\boldsymbol{\nabla}_\perp^2\right) n + V_{*n}\frac{\partial n}{\partial y} + \rho_s c_s \left(\boldsymbol{\nabla}_\perp n \times \widehat{\mathbf{e}}_z\right)\cdot\boldsymbol{\nabla}_\perp\left(\rho_s^2\boldsymbol{\nabla}_\perp^2\right) n = 0$$

where the electron density is adiabatic, which gives an equation for the potential φ. The problem is two-dimensional and named "quasi-three dimensional" because the electrons are Boltzmannian along the magnetic line, which vanishes the radial $E\times B$ convection of the perturbed density $\widetilde{n}/n_0 \sim e\widetilde{\phi}/T_e$.

To examine the conserved quantities we ignore the **drive** and the **sink** (damping). If only the **polarization (vectorial) nonlinearity**

$$N_{\mathbf{k}}^{POL} = \rho_s c_s \left(\boldsymbol{\nabla}_\perp n \times \widehat{\mathbf{e}}_z\right)\cdot\boldsymbol{\nabla}_\perp\left(\rho_s^2\,\boldsymbol{\nabla}_\perp^2 n\right)$$

is retained **the system has two conserved quantities** :

$$
\begin{aligned}
\text{the energy } E &= \frac{1}{2}\int dV\,\left(|n|^2+\rho_s^2\,|\boldsymbol{\nabla}_\perp n|^2\right) \\
&= \frac{1}{2}\sum_{\mathbf{k}}\left(1+k^2\rho_s^2\right)|n_{\mathbf{k}}|^2 \\
\text{the enstrophy } \Omega &= \frac{1}{2}\int dV\,\left(\left|\rho_s^2\boldsymbol{\nabla}_\perp^2 n\right|^2+\rho_s^2\,|\boldsymbol{\nabla}_\perp n|^2\right) \\
&= \frac{1}{2}\sum_{\mathbf{k}} k^2\rho_s^2\,\left(1+k^2\rho_s^2\right)|n_{\mathbf{k}}|^2
\end{aligned}
$$

The statistical mechanics prediction for the density fluctuation spectrum in an equilibrium state

$$
\left\langle |n_{\mathbf{k}}|^2\right\rangle = \frac{1}{\left(1+k^2\rho_s^2\right)\left(a+b\,k^2\rho_s^2\right)}
$$

where a and b are Lagrange multipliers.

The isotropic energy spectrum is

$$
E_{\mathbf{k}} = \pi\,k\rho_s\left(1+k^2\rho_s^2\right)\left\langle |n_{\mathbf{k}}|^2\right\rangle = \frac{\pi\,k\rho_s}{a+b\,k^2\rho_s^2}
$$

and the system pushes the energy to large spatial scales.

The isotropic spectrum of the enstrophy is

$$
\Omega_{\mathbf{k}} = k^2\rho_s^2\,E_{\mathbf{k}} = \frac{\pi\,k^3\rho_s^3}{a+b\,k^2\rho_s^2}
$$

and the system pushes enstrophy to large scales. This is the **dual cascade**. The energy going to the large scales is the **inverse cascade**.

When there is also the $\mathbf{E}\times\mathbf{B}$ nonlinearity

$$
N^{\mathbf{E}\times\mathbf{B}} = -L_n D_0\left[\boldsymbol{\nabla}_\perp\left(\frac{\partial n}{\partial y}\right)\times\widehat{\mathbf{e}}_z\right]\cdot\boldsymbol{\nabla}_\perp n
$$

the system has only one conserved quantity, the energy. The equilibrium density fluctuation spectrum is

$$
\left\langle |n_{\mathbf{k}}|^2\right\rangle = \frac{c}{1+k^2\rho_s^2}
$$

Note the comments made by **Horton 1999 RMP** on the distinct role of the

1. $E\times B$-drift nonlinearity, scalar
2. Hasegawa-Mima or vectorial nonlinearity: $E\times B$ - velocity advection of the *vorticity*. It comes from *polarization drift.*

There are numerical simulations showing that the $\mathbf{E}\times\mathbf{B}$ nonlinearity is the mechanism for saturation, at the mesoscale level. Without this the HM (vectorial) nonlinearity tolerates continuous increase of energy.

2.0.2 Nonlinear dispersion relation

Write the model in the dimensionless form using the space and $(time)^{-1}$ units:

$$
\begin{aligned}
& \rho_s \\
\Omega_i &= \frac{c_s}{\rho_s}
\end{aligned}
$$

$$\left(i\frac{\partial}{\partial t} - \frac{\omega_{*\mathbf{k}} + ik_y^2 \widetilde{D}_0}{1+k^2} \right) n_{\mathbf{k}} + \frac{i}{1+k^2} N_{\mathbf{k}} = 0 \tag{1}$$

where the nonlinear term is

$$
\begin{aligned}
N_{\mathbf{k}} &= N_{\mathbf{k}}^{\mathbf{E}\times\mathbf{B}} + N_{\mathbf{k}}^{POL} \qquad (2)\\
&= -i\frac{1}{2}\xi \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \left[\left(\mathbf{k}\times\mathbf{k}'\right)\cdot\widehat{\mathbf{e}}_z \right] \left(k_y'' - k_y'\right) n_{\mathbf{k}'} n_{\mathbf{k}''} \\
&\quad + \frac{1}{2} \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \left[\left(\mathbf{k}\times\mathbf{k}'\right)\cdot\widehat{\mathbf{e}}_z \right] \rho_s^2 \left(k''^2 - k'^2\right) n_{\mathbf{k}'} n_{\mathbf{k}''}
\end{aligned}
$$

where the dimensionless parameter ξ is

$$\xi \equiv \frac{L_n D_0}{\rho_s c_s} \equiv \alpha\sqrt{\varepsilon} \left(\frac{\rho_s}{L_T}\right) \left(\frac{\Omega_i}{\nu_{eff}}\right)$$

and

$$\widetilde{D}_0 \equiv \xi \left(\frac{\rho_s}{L_n}\right)$$

In order to find the **nonlinear dispersion relation** we need to carry out **the one-point renormalization** and find the renormalized eigenvalue equation for $n_{\mathbf{k}}$. The method of renormalization is EDQNM = eddy damped quasi-normal Markovian closure scheme. This is effectively an iterative closure method with the use of eddy damping to represent incoherent or higher order wave correlations.

1. the nonlinearity is written in terms of **driven modes**: in the nonlinear term one of the functions $n_{\mathbf{k}}$ is obtained by iteration, with retaining only the direct interaction:

$$N_{\mathbf{k}} = \sum_{\mathbf{k}'=\mathbf{k}''-\mathbf{k}} \left[\left(\mathbf{k}\times\mathbf{k}'\right)\cdot\widehat{\mathbf{e}}_z \right] \left[i\xi\left(k_y' + k_y''\right) + \left(k'^2 - k''^2\right) \right] n_{-\mathbf{k}'} n_{\mathbf{k}''}^{(2)} \tag{3}$$

2. and the driven fluctuations are solutions of

$$
\begin{aligned}
& \left(\frac{\partial}{\partial t} + \Delta\omega_{\mathbf{k}''}\right) n_{\mathbf{k}''}^{(2)} + i\left(\frac{\omega_{*\mathbf{k}''}}{1+k''^2} + i\frac{k_y''^2 \widetilde{D}_0}{1+k''^2} \right) n_{\mathbf{k}''}^{(2)} \qquad (4)\\
&= \frac{\left(\mathbf{k}\times\mathbf{k}'\right)\cdot\widehat{\mathbf{e}}_z}{1+k''^2} \left[-i\xi\left(k_y' - k_y\right) + \left(k'^2 - k^2\right) \right] n_{\mathbf{k}'} n_{\mathbf{k}}
\end{aligned}
$$

which is the combinination of the Equations (1) and (2) with the occurence of a new term, a frequency shift $\Delta\omega$. The most important aspect is the suppression of all the terms from the sum in the nonlinear term excepting the term relating the **direct interaction**

$$\mathbf{k}'' = \mathbf{k}' + \mathbf{k}$$

the term $\Delta\omega_{\mathbf{k}''}$ is the **eddy damping rate**.

3. in these equations for $n_{\mathbf{k}''}^{(2)}$ only those interacting waves i.e. the $\mathbf{k}$ and $\mathbf{k}'$ waves are kept. The equation is solved

$$\begin{aligned} n_{\mathbf{k}''}^{(2)} &= \frac{\left(\mathbf{k}\times\mathbf{k}'\right)\cdot\widehat{\mathbf{e}}_z}{1+k''^2}\left[-i\xi\left(k_y'-k_y\right)+\left(k'^2-k^2\right)\right] \\ &\times\int^t dt'\exp\left[\left(-i\omega_{\mathbf{k}''}^{(0)}+\gamma_{\mathbf{k}''}^{(0)}-\Delta\omega_{\mathbf{k}''}\right)(t-t')\right]n_{\mathbf{k}'}(t')\,n_{\mathbf{k}}(t') \end{aligned}$$

4. Replacing in the Equation (1) we shall find the triple product

$$n_{-\mathbf{k}'}\,n_{\mathbf{k}''}^{(2)} \rightarrow n_{-\mathbf{k}'}(t)\,n_{\mathbf{k}'}(t')\,n_{\mathbf{k}}(t')$$

Here we make the ansatz for $t > t'$:

$$n_{-\mathbf{k}}(t)\,n_{\mathbf{k}}(t') \rightarrow \langle n_{-\mathbf{k}}(t)\,n_{\mathbf{k}}(t')\rangle = |n_{\mathbf{k}}(0)|^2\exp\left[\left(-i\omega_{\mathbf{k}}^{(0)}+\gamma_{\mathbf{k}}^{(0)}-\Delta\omega_{\mathbf{k}}\right)(t-t')\right]$$

5. These form is inserted into the equation (3) for $N_{\mathbf{k}}$.

$$\begin{aligned} N_{\mathbf{k}} &= \sum_{\mathbf{k}'=\mathbf{k}''-\mathbf{k}}\frac{\left[\left(\mathbf{k}\times\mathbf{k}'\right)\cdot\widehat{\mathbf{e}}_z\right]^2}{1+k''^2}R_{\mathbf{k},\mathbf{k}',\mathbf{k}''} \\ &\times\left\{\left[\xi^2\left(k_y'^2+k_y''^2-k_y^2\right)+k^2\left(k^2-k'^2\right)\right]+i\xi k_y\left(k'^2-2k^2\right)\right\}\left|n_{\mathbf{k}'}\right|^2 n_{\mathbf{k}} \end{aligned}$$

where the propagator is

$$R_{\mathbf{k},\mathbf{k}',\mathbf{k}''} = \frac{1}{\left(\omega_{\mathbf{k}}^{(0)}+\omega_{\mathbf{k}'}^{(0)}-\omega_{\mathbf{k}''}^{(0)}\right)+i\left(\Delta\omega_{\mathbf{k}}+\Delta\omega_{\mathbf{k}'}+\Delta\omega_{\mathbf{k}''}-\gamma_{\mathbf{k}}^{(0)}-\gamma_{\mathbf{k}'}^{(0)}-\gamma_{\mathbf{k}''}^{(0)}\right)}$$

The non-adiabatic response of the ELECTRONS in a model of trapped electron drift instability allows the identification of the $i\delta_k$ term. The relationship between the density and potential is then *inverted*, and in the problem the potential is replaced everywhere by the fluctuating ion density.

The model equation becomes

$$
\begin{aligned}
&\frac{d}{dt}\left(1-\rho_s^2\boldsymbol{\nabla}_\perp^2\right)\widetilde{n}_i \quad \text{(the total time-derivative contains convection by } \langle \mathbf{V}_E\rangle\text{)}\\
&+V_{*n}\frac{\partial \widetilde{n}_i}{\partial y}+\\
&+D_0\frac{\partial^2\widetilde{n}_i}{\partial y^2} \text{ (the drive, effective } i\delta \text{ , or } ik_y^2 D_0\text{)}\\
&-L_n D_0\left[\boldsymbol{\nabla}_\perp\left(\frac{\partial \widetilde{n}_i}{\partial y}\right)\times\widehat{\mathbf{e}}_z\right]\cdot\boldsymbol{\nabla}_\perp\widetilde{n}_i \text{ (the } \mathbf{E}\times\mathbf{B} \text{ convection of the nonadiabatic term)}\\
&+\rho_s c_s\left(\boldsymbol{\nabla}_\perp\widetilde{n}_i\times\widehat{\mathbf{e}}_z\right)\cdot\boldsymbol{\nabla}_\perp\left(\rho_s^2\boldsymbol{\nabla}_\perp^2\right)\widetilde{n}_i \text{ (the ion polarization nonlinearity)}\\
&+\rho_s c_s\frac{\partial \widetilde{n}_i}{\partial y}\frac{\partial}{\partial x}\left(\rho_s^2\frac{\partial^2\langle n\rangle}{\partial x^2}\right) \quad \text{(the Kelvin-Helmholtz term)}\\
&-\frac{c_s^2}{\nu_i}\nabla_\parallel^2\widetilde{n}_i \quad \text{(the parallel dissipation term)}\\
=&\quad 0
\end{aligned}
$$

3 Model Horton RMP 1999

The model is

$$
\begin{aligned}
&\frac{\partial n_i}{\partial t}+\mathbf{v}_E\cdot\boldsymbol{\nabla}_\perp n_i\\
&+\boldsymbol{\nabla}_\perp\cdot\left(\frac{n_i m_i}{e_i B^2}\frac{d\mathbf{E}_\perp}{dt}\right)\\
&+\nabla_\parallel\left(n_i u_\parallel\right)\\
=&\quad 0
\end{aligned}
$$

$$
n_i m_i\left(\frac{\partial u_\parallel}{\partial t}+\mathbf{v}_E\cdot\boldsymbol{\nabla}u_\parallel\right)=-\boldsymbol{\nabla}_\parallel p_i+e_i n_i E_\parallel
$$

$$
\begin{aligned}
&\frac{3}{2}n_i\left(\frac{\partial T_i}{\partial t}+\mathbf{v}_E\cdot\boldsymbol{\nabla}T_i\right)\\
=&\quad -\boldsymbol{\nabla}\cdot\mathbf{q}_i\\
&-n_i T_i\left(\boldsymbol{\nabla}_\perp\cdot\mathbf{v}_E+\nabla_\parallel u_\parallel\right)
\end{aligned}
$$

4 ITG, renormalization DIA, Diamond, Lee

We follow the paper: [**?**].

Hydrodynamic equations for electrostatic slab model (**Lee and Diamond 1986**)

The electrons can be assumed Boltzmannian.

Continuity equation for the ion density

$$\frac{\partial n_i}{\partial t} + \boldsymbol{\nabla} \cdot (n_i \mathbf{v}_{\perp i}) + \nabla_{\parallel} \left(n_i \widetilde{v}_{\parallel i} \right) = 0$$

The perturbation of the ion density along the line, due to the perturbation of the *ion* parallel velocity $\widetilde{v}_{\parallel i}$. The electrons, very fast, are Boltzmannian. The ions accumulate in some places along the magnetic field line.

Momentum equation for ions, along the magnetic line

$$m_i n_i \left(\frac{\partial \widetilde{v}_{\parallel i}}{\partial t} + (\mathbf{v}_E \cdot \boldsymbol{\nabla}) \widetilde{v}_{\parallel i} \right) = -\boldsymbol{\nabla}_{\parallel} P_i - e n_i \boldsymbol{\nabla}_{\parallel} \Phi + \mu_i \boldsymbol{\nabla}_{\parallel}^2 \widetilde{v}_{\parallel i}$$

The momentum dissipation is determined by the *viscosity* μ_i. This can be: (1) collisional; or (2) Landau damping. See below.

The "density of energy" (*i.e.* pressure) balance for ions

$$\frac{\partial P_i}{\partial t} + (\mathbf{v}_E \cdot \boldsymbol{\nabla}) P_i + \Gamma P_i \left(\nabla_{\parallel} \widetilde{v}_{\parallel i} \right) = 0$$

where Γ is the ratio of the specific heat coefficients. $\mathbf{v}_E \sim E \times B$ is the radial electric velocity that advects cold ions in places where they already are accumulated, thus *reducing* the pressure.

As explined in the physical picture (in **Cowley, Sudan, Kulsrud**) the advection of *cold ions* from other radius positions, to the current point, is made by the electric velocity $\mathbf{v}_E$. The balance of pressure include the *compressibility* due to the accumulation along the magnetic line, *i.e.* $\widetilde{v}_{\parallel i}$.

4.0.3 Energy-like quantities

Wave energy

$$E^W = \frac{1}{2} \int d^3x \left(|\Phi|^2 + |\boldsymbol{\nabla}\Phi|^2 \right)$$

Kinetic energy

$$E^K = \frac{1}{2} \int d^3x \left| \widetilde{v}_{\parallel i} \right|^2$$

Density of energy (pressure) fluctuation energy

$$E^I = \frac{1}{2} \frac{1}{\Gamma} \int d^3x \left| \widetilde{P}_i \right|^2$$

The time evolution of these energy contents

$$\frac{\partial E^W}{\partial t} = -\int d^3x \left[\Phi \ \left(\nabla_{\|}\widetilde{v}_{\|i}\right) + \left\langle \Phi \ \left(-\boldsymbol{\nabla}\Phi \times \widehat{\mathbf{n}}\right) \cdot \boldsymbol{\nabla}_{\perp} \left(\boldsymbol{\nabla}_{\perp}^2 \Phi\right) \right\rangle \right]$$

$$\frac{\partial E^K}{\partial t} = -\int d^3x \left[\widetilde{v}_{\|i} \ \nabla_{\|}\Phi + \widetilde{v}_{\|i} \ \nabla_{\|}\widetilde{P}_i + \left\langle \widetilde{v}_{\|i} \ \left(-\boldsymbol{\nabla}\Phi \times \widehat{\mathbf{n}}\right) \cdot \boldsymbol{\nabla}_{\perp}\widetilde{v}_{\|i} \right\rangle + \mu_i \left\langle \left(\nabla_{\|}\widetilde{v}_{\|i}\right)^2 \right\rangle \right]$$

$$\frac{\partial E^I}{\partial t} = -\int d^3x \left[\widetilde{P}_i \ \left(\nabla_{\|}\widetilde{v}_{\|i}\right) - \frac{1}{\Gamma}\left\langle \widetilde{v}_r \widetilde{P}_i \right\rangle \frac{d\left\langle P_0\right\rangle}{dr} + \frac{1}{\Gamma}\left\langle \widetilde{P}_i \ \left(-\boldsymbol{\nabla}\Phi \times \widehat{\mathbf{n}}\right) \cdot \boldsymbol{\nabla}_{\perp}\widetilde{P}_i \right\rangle \right]$$

The authors classify the couplings

$$\begin{array}{l} \Phi \ \left(\nabla_{\|}\widetilde{v}_{\|i}\right) \\ \widetilde{v}_{\|i} \ \nabla_{\|}\widetilde{P}_i \\ \widetilde{P}_i \ \left(\nabla_{\|}\widetilde{v}_{\|i}\right) \end{array}$$

These are *linear energy couplings* = equipartitioning, by *sound wave propagation.*

And

$$\begin{array}{l} \left\langle \Phi \ \left(-\boldsymbol{\nabla}\Phi \times \widehat{\mathbf{n}}\right) \cdot \boldsymbol{\nabla}_{\perp} \left(\boldsymbol{\nabla}_{\perp}^2 \Phi\right) \right\rangle \\ \left\langle \widetilde{v}_{\|i} \ \left(-\boldsymbol{\nabla}\Phi \times \widehat{\mathbf{n}}\right) \cdot \boldsymbol{\nabla}_{\perp}\widetilde{v}_{\|i} \right\rangle \\ \left\langle \widetilde{P}_i \ \left(-\boldsymbol{\nabla}\Phi \times \widehat{\mathbf{n}}\right) \cdot \boldsymbol{\nabla}_{\perp}\widetilde{P}_i \right\rangle \end{array}$$

are *nonlinear energy transfer*.

And

$$\begin{array}{ll} \frac{1}{\Gamma}\left\langle \widetilde{v}_r \widetilde{P}_i \right\rangle \frac{d\left\langle P_0\right\rangle}{dr} & \text{source} \\ -\mu_i \left\langle \left(\nabla_{\|}\widetilde{v}_{\|i}\right)^2 \right\rangle & \text{sink} \end{array}$$

Finding the typtical space-time scales

The renormalization of the $E \times B$ convective turbulebce

$$\begin{array}{rcl} \frac{1}{B_0}\left(-\boldsymbol{\nabla}\widetilde{\Phi} \times \widehat{n}\right) \cdot \boldsymbol{\nabla}_{\perp} & \rightarrow & \frac{D_{\mathbf{k}}}{\Delta_k^2} \\ D_{\mathbf{k}} & \equiv & \text{radial diffusion} \end{array}$$

The spatial scale is

$$\begin{array}{rcl} \Delta_{\mathbf{k}} & \equiv & \text{radial mixing length} \\ & = & \left[\frac{D_{\mathbf{k}}}{\left(k_{\|}'\right)^2}\right]^{1/4} \end{array}$$

and results by balancing the ion sound term with the vorticity convection term.

Time scales

The time coherence scale

$$\tau_{c,\mathbf{k}} = \left[\frac{D_{\mathbf{k}}}{\Delta_{\mathbf{k}}^2} \right]^{-1}$$

The dissipation time

$$\tau_{d,\mathbf{k}} = \left[\mu_{\parallel} \int dx \frac{\left\langle \left(\nabla_{\parallel} \widetilde{v}_{\parallel} \right)^2 \right\rangle_{\mathbf{k}}}{E_{\mathbf{k}}^K} \right]^{-1}$$

The equipartitioning time scale

$$\tau_{eq,\mathbf{k}} = \left[\int dx \frac{\left\langle \widetilde{p}_i \ \nabla_{\parallel} \widetilde{v}_{\parallel} \right\rangle_{\mathbf{k}}}{E_{\mathbf{k}}^I} \right]^{-1}$$

It is defined a Reynolds number

$$\mathrm{Re} = \frac{\tau_{d,\mathbf{k}}}{\tau_{c,\mathbf{k}}}$$

the relative importance of the *decorrelation time* and of the *dissipative* time.

At saturation

$$\begin{aligned} & \text{the nonlinear transfer of energy to dissipation} \\ = & \\ & \text{growth rate} \end{aligned}$$

At saturation we have

$$\Delta\omega_{\mathbf{k}} = \left(\frac{1+\eta_i}{\tau} \right)^2 k_{\parallel}' \ \frac{\Delta_{\mathbf{k}}^2}{\Delta\omega_{\mathbf{k}}}$$

$$D_{\mathbf{k}} \sim \left(\frac{1+\eta_i}{\tau} \right)^{1/2} k_{\parallel}' \ \Delta_{\mathbf{k}}^3$$

from where we obtain

$$D_{\mathbf{k}} = \left(\frac{1+\eta_i}{\tau} \right)^2 k_y \rho_s \frac{\rho_s^2 c_s}{L_s}$$

$$\Delta_{\mathbf{k}} = \left(\frac{1+\eta_i}{\tau} \right)^{1/2} \rho_s$$

And the basic time scales

$$\tau_{eq,\mathbf{k}}^{-1} = \frac{\Gamma}{\tau} \; \frac{1+\eta_i}{\tau} \; k_y \frac{c_s \rho_s}{L_s}$$

$$\tau_{coherent,\mathbf{k}}^{-1} = \frac{1+\eta_i}{\tau} \; k_y \frac{c_s \rho_s}{L_s}$$

$$\begin{aligned} \tau_{dissipation,\mathbf{k}}^{-1} &= \mu_i \frac{k_y^2}{k_{0x}^2 L_s^2} \\ &= \frac{1}{\tau} \; k_y \frac{c_s \rho_s}{L_s} \end{aligned}$$

The time of equilibration is smaller than the time of coherency which is further smaller than the time of dissipation.

The time of equilibration is short. Then the time for nonlinear transfer of energy is longer.

4.1 Basic model

The equations

$$k_{\parallel}(x) \approx k_y \frac{x - x_0}{L_s}$$

the perturbations

$$\widetilde{f}(x) \; \exp\left[-i\omega t + i k_y y + i k_{\parallel} z\right]$$

The three fluid-like equations are modified by normalizations

$$\widetilde{\phi} = \frac{e\widetilde{\Phi}}{T_e}$$

$$\widetilde{v}_{\parallel} = \frac{\widetilde{v}_{\parallel,i}}{c_s}$$

$$\widetilde{p} = \frac{\widetilde{p}_i}{\langle P_{i0} \rangle} \frac{T_i}{T_e}$$

with the total ion pressure

$$P_i = \langle P_{i0} \rangle + \widetilde{p}_i$$

and

$$n_i = n_0 + \widetilde{n}_i$$

The equations become

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(1-\nabla_\perp^2\right)\widetilde{\phi}+ \\
&+v_D\left[1+\frac{1+\eta_i}{\tau}\nabla_\perp^2\right]\nabla_y\widetilde{\phi} \\
&+\left(-\boldsymbol{\nabla}\widetilde{\phi}\times\widehat{\mathbf{n}}\right)\cdot\boldsymbol{\nabla}_\perp\left(\boldsymbol{\nabla}_\perp^2\widetilde{\phi}\right) \\
&+\nabla_\parallel\widetilde{v}_\parallel \\
=\;\; & 0
\end{aligned}
$$

$$
\frac{\partial\widetilde{v}_\parallel}{\partial t}-\left(-\boldsymbol{\nabla}_\perp\widetilde{\phi}\times\widehat{\mathbf{n}}\right)\cdot\boldsymbol{\nabla}_\perp\widetilde{v}_\parallel=-\nabla_\parallel\widetilde{p}_i-\nabla_\parallel\widetilde{\phi}+\mu\;\nabla_\parallel^2\widetilde{v}_\parallel
$$

$$
\frac{\partial\widetilde{p}}{\partial t}+v^{dia}\left(\frac{1+\eta_i}{\tau}\right)\nabla_y\widetilde{\phi}-\left(-\nabla_\parallel\widetilde{\phi}\times\widehat{\mathbf{n}}\right)\cdot\boldsymbol{\nabla}\widetilde{p}=-\Upsilon\;\nabla_\parallel\widetilde{v}_\parallel
$$

where

$$
\begin{aligned}
v^{dia}&=-\frac{T_e}{eB}\frac{d}{dx}\ln n\\
\tau&=\frac{T_i}{T_e}\\
\eta_i&=\frac{d\ln T_i}{d\ln n_0}\\
\Upsilon&=\frac{\Gamma}{\tau}\\
\mu&=\mu_\parallel\frac{\Omega_{ci}}{c_s^2}
\end{aligned}
$$

4.1.1 Renormalization and time scales

A typical renormalization of the nonlinear convection of the fluctuation

$$
\begin{aligned}
\frac{-\boldsymbol{\nabla}\Phi\times\widehat{\mathbf{n}}}{B_0}\cdot\boldsymbol{\nabla}\quad\rightarrow\quad & \frac{D_{\mathbf{k}}}{\Delta_{\mathbf{k}}^2}\\
& \left[\frac{1}{s}\right]
\end{aligned}
$$

i.e. replace the nonlinear convection with a diffusion term.

The **nonlinear coherence (correlation) time**

$$
\tau_{c,\mathbf{k}}=\left(\Delta_{\mathbf{k}}\omega\right)^{-1}=\left(\frac{D_{\mathbf{k}}}{\Delta_{\mathbf{k}}^2}\right)^{-1}
$$

The **dissipation time** (note that it is dissipation of the parallel fluctuations, equivalent to absorbtion of the wave energy that is radiated toward the *ion turning point*)

$$\tau_{d,\mathbf{k}} = \left(\mu_{\|} \int dx \frac{\left\langle \left(\nabla_{\|} \widetilde{v}_{\| i}\right)^2 \right\rangle_{\mathbf{k}}}{E^K_{\mathbf{k}}}\right)^{-1}$$

It comes from parallel compressibility but with parallel viscosity.

The **energy equipartition time**

$$\tau_{eq,\mathbf{k}} = \left(\int dx \frac{\langle \widetilde{p} \nabla_{\|} \widetilde{v}_{\| i}\rangle_{\mathbf{k}}}{E^I_{\mathbf{k}}}\right)^{-1}$$

The **Reynold number**: ratio of the dissipative and correlation times

$$\mathbf{R} = \frac{\tau_{d,\mathbf{k}}}{\tau_{c,\mathbf{k}}} = \frac{\text{dissipation time}}{\text{correlation time}}$$

Saturation condition: the level of turbulence is such that the nonlinear transfer of the fluctuation energy to the dissipation spectral region equals the fluctuation growth.

4.1.2 Gyro-ordering and the gyro-kinetic equation

The gyro-ordering

$$\frac{\omega}{\Omega_i} \sim \varepsilon^1$$

$$\begin{aligned} \frac{\rho_i}{\lambda_{\|}} &\ll 1 \quad \text{or} \\ k_{\|}\,\rho_i &\sim \varepsilon^1 \end{aligned}$$

$$\begin{aligned} \lambda_{\perp} &\sim \rho_i \quad \text{or} \\ k_{\perp}\rho_i &\sim \varepsilon^0 \end{aligned}$$

with $(\mathbf{k}, \omega)$ a typical wave-vector pair. The small parameter is

$$\varepsilon = \frac{\rho_i}{a}$$

The problem of the gyro-kinetic regime is the following:

- modes can contain perpendicular wavelengths down to the scale of the ion Larmor radius
- the analytical treatment usually is developed with *guiding center* theory.

The **gyro-kinetic equation** is derived from the Vlasov equation.

5 ITG in sheared rotation Hamaguchi and Horton

A derivation is in **PPCF1992**.

A paper by **Hamaguchi Horton ITG 1990**. The ITG in slab geometry with *sheared plasma rotation*.

In the paper of **HamaguchiHorton1992** it is developed an aplication of the system of equation derived by them previously (in PPCF1992). The electrostatic ITG-driven mode (or η_i-mode) is derived from the **two-fluid** equations of Braginski under the assumptions

charge neutrality	$n_e = n_i$
constant electron temperature	$T_e = const$
zero resistivity	$\eta = 0$
zero electron inertia	$\frac{d\mathbf{v}_e}{dt} = 0$

The equations are written for

electric potential	ϕ
parallel *ion* velocity	$v_{\parallel i}$
ion pressure	p_i

This is a *fluid* model for the ITG and it is similar to those developed by **Diamond, Carreras**, etc. The quantities defined for introducing the equations

$$
\begin{aligned}
\mathbf{v}_E &= \frac{-\boldsymbol{\nabla}\phi \times \widehat{\mathbf{n}}}{B} \\
\mathbf{v}_{dia,i} &= \frac{1}{n}\frac{1}{m_i \Omega_{ci}} \widehat{\mathbf{n}} \times \boldsymbol{\nabla} p_i \\
\mathbf{v}_p &= -\frac{1}{\Omega_{ci} B}\left(\frac{\partial}{\partial t} + \frac{-\boldsymbol{\nabla}\phi \times \widehat{\mathbf{n}}}{B} \cdot \boldsymbol{\nabla}_\perp\right) \boldsymbol{\nabla}_\perp \phi
\end{aligned}
$$

The fluctuations are, for *adiabatic* electrons

$$
\begin{aligned}
n &= n_0(x) + \widetilde{n} \\
&= n_0(x)\left(1 + \frac{|e|\,\widetilde{\phi}}{T_e}\right) \\
p_i &= p_{i0}(x) + \widetilde{p}_i \\
\phi &= \widetilde{\phi} \\
v_\parallel &= \widetilde{v}_\parallel
\end{aligned}
$$

Physical parameters

$$
\begin{aligned}
c_s &= \left(\frac{T_e}{m_i}\right)^{1/2} \\
\rho_s &= \frac{c_s}{\Omega_{ci}} \\
L_n &= -\left(\frac{d}{dx}\ln n_0\left(x\right)\right)^{-1} \\
L_T &= -\left(\frac{d}{dx}\ln T_{i0}\left(x\right)\right)^{-1}
\end{aligned}
$$

Notations

$$
\begin{aligned}
\eta_i &= \frac{L_n}{L_T} \\
K &= \frac{T_i}{T_e}\left(1+\eta_i\right) \\
\Gamma &= \gamma\frac{T_i}{T_e}
\end{aligned}
$$

Space-time variables

$$
\begin{aligned}
\widetilde{x} &= \frac{x-x_0}{\rho_s} \\
\widetilde{y} &= \frac{y}{\rho_s} \\
\widetilde{z} &= \frac{z}{L_n} \\
\tau &= t\frac{c_s}{L_n}
\end{aligned}
$$

Normalization of the fluctuating quantities

$$
\begin{aligned}
&\frac{\widetilde{n}}{n_0} \\
&\frac{\widetilde{p}_i}{p_0} \\
&\frac{\widetilde{v}_\parallel}{c_s} \\
&\frac{\left|e\right|\widetilde{\phi}}{T_e}
\end{aligned}
$$

are all of the order of

$$
\frac{\rho_s}{L_n} \ll 1
$$

Scaling of the normalized quantities (this is typical for **Horton**)

$$
\begin{aligned}
\phi &\equiv \frac{|e|\,\widetilde{\phi}}{T_e}\frac{L_n}{\rho_s} \\
v_\parallel &\equiv \frac{\widetilde{v}_\parallel}{c_s}\frac{L_n}{\rho_s} \\
p &\equiv \frac{\widetilde{p}_i}{p_{i0}}\frac{L_n}{\rho_s}\frac{T_i}{T_e}
\end{aligned}
$$

The equations are (supressing the tildas)

$$
\begin{aligned}
\left(1-\boldsymbol{\nabla}_\perp^2\right)\frac{\partial \phi}{\partial \tau} &= -\left(1+K\boldsymbol{\nabla}_\perp^2\right)\frac{\partial \phi}{\partial y} + \left[\left(-\boldsymbol{\nabla}_\perp \phi \times \widehat{\mathbf{n}}\right)\cdot \boldsymbol{\nabla}_\perp\right]\boldsymbol{\nabla}_\perp^2 \phi \\
&\quad -\nabla_\parallel v_\parallel \\
&\quad -\mu_\perp \boldsymbol{\nabla}_\perp^4 \phi
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial v_\parallel}{\partial \tau} + \left[\left(-\boldsymbol{\nabla}_\perp \phi \times \widehat{\mathbf{n}}\right)\cdot \boldsymbol{\nabla}_\perp\right] v_\parallel &= -\nabla_\parallel p - \nabla_\parallel \phi \\
&\quad +\mu_\perp \boldsymbol{\nabla}_\perp^2 v_\parallel + \mu_\parallel \boldsymbol{\nabla}_\parallel^2 v_\parallel
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial p}{\partial \tau} + \left[\left(-\boldsymbol{\nabla}_\perp \phi \times \widehat{\mathbf{n}}\right)\cdot \boldsymbol{\nabla}_\perp\right] p - K\frac{\partial \phi}{\partial y} &= -\Gamma \nabla_\parallel v_\parallel \\
&\quad +\chi_\perp \boldsymbol{\nabla}_\perp^2 p + \chi_\parallel \boldsymbol{\nabla}_\parallel^2 p
\end{aligned}
$$

The sheared magnetic field is

$$
\mathbf{B} = B\widehat{\mathbf{n}} + \frac{x-x_0}{L_s}\widehat{\mathbf{e}}_y
$$

and

$$
s \equiv \frac{L_n}{L_s}
$$

The parallel derivative is

$$
\nabla_\parallel = \frac{\partial}{\partial z} + xs\frac{\partial}{\partial y}
$$

(where it is contained

$$
\begin{aligned}
k_y &\rightarrow \frac{\partial}{\partial y} \\
k_\parallel &= k_y \frac{x}{L_s} \\
x &\rightarrow \frac{x-x_0}{\rho_s}
\end{aligned}
$$

and a factor $1/L_n$ is extracted from z and, then, from the second term too.)

The parallel colisionless dissipations

$$\mu_{\|} \text{ and } \chi_{\|}$$

are Landau damping. [**note** this is because $k_{\|}$ is smaller than the value where the phase velocity equals the ion thermal velocity, leading to turning point absorbtion].

There is a *fluid* treatment of Landau damping made by **Hammet and Perkins**.

6 Thershold of the ITG instability Hassam, Antonsen, Drake, Guzdar

This model clearly emphasizes the hard growth close to the threshold

$$\eta_i = 2$$

so fast that it is supposed that the plasma will remain close of marginal stability.

NOTE. For the term

$$v_D \left(1 + \frac{1+\eta_i}{\tau} \boldsymbol{\nabla}_{\perp}^2\right) \frac{\partial \phi}{\partial y}$$

see **Kim Horton Hamaguchi**. The term also appears in **Wang Diamond Rosenbluth 1992** where $v_D \equiv v_{e,dia} = -\frac{T_e}{|e|B} \frac{d}{dx} \ln n_0$.

In the paper of **Hassam Antonsen Drake Guzdar** it is first time shown that the dispersion relation can be very steep and there is a soft onset.

7 Numerical studies of ITG Terry, Diamond, Thayer, Leboeuf et al. (1988)

The numerical simulations carried out by **Terry, Diamond, Thayer, Leboeuf et al.** (**1988**) show saturation of the ITG by quasilinear flattening of the ion temperature over a radial extension corresponding to the larger radial mode.

The equations used in the numerical simulations are

Equation of *continuity for the ion density*

$$\begin{aligned} & \frac{\partial}{\partial t}\left(1 - \boldsymbol{\nabla}_{\perp}^2\right)\phi - \left[\left(-\boldsymbol{\nabla}_{\perp}\phi \times \widehat{\mathbf{e}}_z\right) \cdot \boldsymbol{\nabla}_{\perp}\right] \boldsymbol{\nabla}_{\perp}^2 \phi \\ & \quad + v_{e,Dia}\left(1 + \frac{1+\eta_i}{\tau} \boldsymbol{\nabla}_{\perp}^2\right) \frac{\partial \phi}{\partial y} \\ & \quad + \nabla_{\|} v_{\| i} \\ = & \; 0 \end{aligned} \tag{5}$$

Momentum conservation in parallel direction, for the IONS

$$
\begin{aligned}
& \frac{\partial v_{\parallel i}}{\partial t} + \left[\left(-\boldsymbol{\nabla}_{\perp}\phi \times \widehat{\mathbf{e}}_z \right) \cdot \boldsymbol{\nabla}_{\perp} \right] v_{\parallel i} \\
= \quad & -\nabla_{\parallel} p - \nabla_{\parallel}\phi \\
& +\mu \nabla_{\parallel}^2 v_{\parallel i}
\end{aligned}
\tag{6}
$$

Energy conservation for ions

$$
\begin{aligned}
& \frac{\partial p}{\partial t} + \left[\left(-\boldsymbol{\nabla}_{\perp}\phi \times \widehat{\mathbf{e}}_z \right) \cdot \boldsymbol{\nabla}_{\perp} \right] p \\
& \qquad +v_{e,Dia} \frac{1+\eta_i}{\tau} \frac{\partial p}{\partial y} \\
= \quad & -\Upsilon \nabla_{\parallel} v_{\parallel i}
\end{aligned}
\tag{7}
$$

where v_D is the *diamagnetic* velocity of the electrons

$$
v_{e,Dia} = -\frac{T_e}{|e|\, B} \frac{1}{n_0} \frac{dn_0}{dx}
$$

$$
\tau = \frac{T_e}{T_i}
$$

$$
\mu = \mu_{\parallel} \frac{\Omega_i}{c_s^2}
$$

$$
\Upsilon = \frac{\Gamma}{\tau}
$$

Here Γ is the ratio of the specific heats.

The compressibility of the parallel ion flow $-\Upsilon \nabla_{\parallel} v_{\parallel i}$ and the convection of the pressure (density of ion energy) equilibrium gradient are energy processes.

The role of the term $-\Upsilon \nabla_{\parallel} v_{\parallel i}$ is to exchange energy between the driven pressure fluctuations and the parallel velocity which is dissipated at high k through the viscosity term $\mu \nabla_{\parallel}^2 v_{\parallel i}$. This trem plays the role of the Landau damping.

The equations are nonlinear but the nonlinearity is renormalised analytically by DIA, resulting in diffusive terms.

The numerical simulations show that the fluctuations are radially extended and that the more extended radially modes are more easily excited.

8 A slab model for the η_i mode: Kim, Horton, Hamaguchi

This model is a correction of the **Horton Hamaguchi** model with a more careful consideration of the *divergence of the ion polarization drift*.

The ITG dynamics places on equal foot the potential fluctuations φ and the pressure fluctuations p and suggests the variable

$$\psi = \varphi + p$$

$$\begin{aligned}
& \frac{\partial}{\partial t}\left(\varphi - \boldsymbol{\nabla}_\perp^2 \psi\right) - \left[\left(-\boldsymbol{\nabla}_\perp \varphi \times \widehat{\mathbf{n}}\right) \cdot \boldsymbol{\nabla}_\perp\right] \boldsymbol{\nabla}_\perp^2 \psi \\
& \qquad + \frac{\partial \varphi}{\partial y} \\
& \qquad + \boldsymbol{\nabla}_\parallel v_\parallel \\
& \qquad + \mu_\perp \boldsymbol{\nabla}_\perp^4 \psi \\
= & \quad 0
\end{aligned}$$

Here the following *units* have been used

$$\begin{aligned}
\rho_s &= \frac{c_s}{\Omega_c} \ \text{ for the perpendicular scale} \\
& \quad L_n \ \text{ for the parallel scale} \\
c_s &= \left(\frac{T_e}{m_i}\right)^{1/2} \ \text{ for the velocity} \\
t_s &= \frac{L_n}{c_s} \ \text{ for the time} \\
& \quad T_i \ \text{ for the temperatures} \\
& \quad \frac{T_e}{|e|} \ \text{ for the electric potential} \\
& \quad n_0 \ \text{ for the density}
\end{aligned}$$

The second equation is the parallel momentum conservation for *electrons* and for *ions*

$$\begin{aligned}
& \frac{\partial v_\parallel}{\partial t} + \left[\left(-\boldsymbol{\nabla}_\perp \varphi \times \widehat{\mathbf{n}}\right) \cdot \boldsymbol{\nabla}_\perp\right] v_\parallel \\
= & \quad -\boldsymbol{\nabla}_\parallel \psi \\
& \quad + \nu_\perp \boldsymbol{\nabla}_\perp^2 v_\parallel + \nu_\parallel \boldsymbol{\nabla}_\parallel^2 v_\parallel
\end{aligned}$$

and the Equation for the pressure

$$\begin{aligned}
& \frac{\partial}{\partial t}\left(p - \tau\gamma n\right) + \left[\left(-\boldsymbol{\nabla}_\perp \varphi \times \widehat{\mathbf{n}}\right) \cdot \boldsymbol{\nabla}_\perp\right] \left(p - \gamma\tau n\right) \\
= & \quad -\tau\left(1 + \eta_i - \gamma\right) \frac{\partial \varphi}{\partial y} \\
& \quad + \chi_\perp \boldsymbol{\nabla}_\perp^2 p + \chi_\parallel \boldsymbol{\nabla}_\parallel^2 p
\end{aligned}$$

Here

$$\psi \equiv \varphi + p$$

It is approximated

$$\frac{\partial}{\partial l_{\|}} \equiv \boldsymbol{\nabla}_{\|} = sx\frac{\partial}{\partial y}$$

which means that the parallel variations $\partial/\partial l_{\|}$ are due to the magnetic shear s, where

$$s \equiv \frac{L_n}{L_s}$$

NOTE.

The *parallel velocity* $v_{\|}$ and the *parallel compressibility* $\boldsymbol{\nabla}_{\|}v_{\|} \neq 0$ are essential for the ITG mode.

The $E \times B$ radial fluctuating convection of the *equilibrium* gradient dp_0/dr of the pressure are absorbed by the parallel compressibility of the parallel velocity, which is also an energetic term. But the parallel divergence of the parallel velocity is also a term in the ion continuity equation and so it constraints the fluctuations of the electrostatic potential ϕ. What is new and specific is the necessity to take into account the polarization of the medium: there is a polarization flow $\mathbf{v}_p$ of the ions. This flow has non-zero divergence, its inflow in the transversal plane being taken by the flow along the field line (parallel direction).

In this paper it is calculated the *divergence of the polarization drift*

$$\begin{aligned}\boldsymbol{\nabla} \cdot \mathbf{v}_{pol} \quad \approx \quad & -\left(\frac{1}{\Omega_{ci}B}\right)\left\{\frac{\partial}{\partial t}\boldsymbol{\nabla}_{\perp}^{2}\left(\varphi+\frac{p}{n\,|e|}\right)\right. \\ & \left. +\frac{1}{B}\left[\left(-\boldsymbol{\nabla}_{\perp}\varphi\times\widehat{\mathbf{n}}\right)\cdot\boldsymbol{\nabla}_{\perp}\right]\boldsymbol{\nabla}_{\perp}^{2}\left(\varphi+\frac{p}{n\,|e|}\right)\right\}\end{aligned}$$

$$\boldsymbol{\nabla} \cdot \mathbf{v}_{pol} = -\frac{1}{\Omega_{ci}B}\boldsymbol{\nabla}_{\perp}\cdot\frac{d}{dt}\boldsymbol{\nabla}_{\perp}\left(\varphi+\frac{p}{n\,|e|}\right)$$

End NOTE.

Note

The ITG is determined by

1. the compressibility of the perpendicular ion flow of the *polarization* drift type, $\boldsymbol{\nabla}_{\perp} \cdot \mathbf{v}_{pol} \neq 0$.
2. the compressibility in the parallel direction of the *parallel flow*, $\boldsymbol{\nabla}_{\|}v_{\|} \neq 0$. This is a source of heat and appears in the equation for the pressure fluctuations.

End

9 A slab model of the ITG instability. The *barotropic* equation

The plasma model consists of the continuity equations and the equations of motion for electrons and ions:

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{v}_\alpha) = 0 \tag{8}$$

$$m_\alpha n_\alpha \left(\frac{\partial \mathbf{v}_\alpha}{\partial t} + (\mathbf{v}_\alpha \cdot \nabla) \mathbf{v}_\alpha \right) = -\nabla p_\alpha - \nabla \cdot \boldsymbol{\pi}_\alpha + e_\alpha n_\alpha (\mathbf{E} + \mathbf{v}_\alpha \times \mathbf{B}) + \mathbf{R}_\alpha \tag{9}$$

where $\alpha = e, i$. The friction forces $\mathbf{R}_e = -\mathbf{R}_i = ne\mathbf{J}_\parallel / \sigma_\parallel$ are not important for very low plasma resistivity, which we will assume. The collisional viscosity $\boldsymbol{\pi}_{e,i}$ will be neglected as well. However we will need to include it later when we will consider the balance of the forces contributing to the poloidal rotation. The electron and ion temperatures are considered constant along the magnetic lines $\nabla_\parallel T_{e,i} = 0$.

The momentum conservation equations are used to determine the perpendicular velocities of the electrons and ions. The parallel momentum conservation equation for electrons, in the absence of *dissipation* or *drifts* gives the adiabatic distribution of the density fluctuation. The velocities are introduced in the continuity equations to find the dynamical equations for the density and electric potential.

From the equations of motion for the *ions* the velocities are obtained in the form:

$$\mathbf{v}_i = \mathbf{v}_{\perp i} = \mathbf{v}_{dia,i} + \mathbf{v}_E + \mathbf{v}_{pol,i}$$

where the ion diamagnetic velocity is

$$\mathbf{v}_{dia,i} = \frac{1}{n_i} \frac{1}{m_i \Omega_i} \widehat{\mathbf{n}} \times \nabla p_i$$

The **ion-polarization velocity** is:

$$\begin{aligned} \mathbf{v}_{pol,i} &= -\frac{1}{|e| n_i B} n_i m_i \left(\frac{\partial \mathbf{v}_{E,i}}{\partial t} + (\mathbf{v}_{E,i} \cdot \nabla) \mathbf{v}_{E,i} \right) \times \widehat{\mathbf{n}} \\ &= -\frac{1}{B \Omega_i} \left(\frac{\partial}{\partial t} + (\mathbf{v}_E \cdot \nabla_\perp) \right) \nabla_\perp \phi \end{aligned} \tag{10}$$

since

$$\begin{aligned} \mathbf{v}_{E,i} \times \widehat{\mathbf{n}} &= \frac{-\nabla \phi \times \widehat{\mathbf{n}}}{B} \times \widehat{\mathbf{n}} = -\left\{ (-\nabla \phi)(\widehat{\mathbf{n}} \cdot \widehat{\mathbf{n}}) - \widehat{\mathbf{n}} \left[(-\nabla \phi) \cdot \widehat{\mathbf{n}} \right] \right\} \\ &= \nabla \phi \end{aligned}$$

Finally the perpendicular *ion* velocity is

$$\begin{aligned} \mathbf{v}_{\perp,i} &= \mathbf{v}_{dia,i} + \mathbf{v}_{pol,i} + \frac{-\nabla \phi \times \widehat{\mathbf{n}}}{B} \\ &= \frac{T_i}{|e| B} \frac{1}{n_i} \frac{dn_i}{dr} \widehat{\mathbf{e}}_y + \frac{1}{\Omega_i} \frac{d}{dt} \left(\frac{-\nabla_\perp \phi}{B} \right) + \frac{-\nabla \phi \times \widehat{\mathbf{n}}}{B} \end{aligned} \tag{11}$$

The equation for the velocity of the *electrons* is

$$\mathbf{v}_e = \mathbf{v}_{\perp,e} + \mathbf{v}_{\parallel,e}$$

$$\begin{aligned}
\mathbf{v}_{\perp e} &= \mathbf{v}_E + \mathbf{v}_{dia,e} \\
&= \frac{-\boldsymbol{\nabla}\phi \times \widehat{\mathbf{n}}}{B} + \frac{1}{n}\frac{1}{m_e \Omega_e}\left(\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p_e\right) \\
&= \frac{-\boldsymbol{\nabla}\phi \times \widehat{\mathbf{n}}}{B} + \frac{T_e}{-\left|e\right| B}\frac{1}{n_0}\frac{dn_0}{dr}\widehat{\mathbf{e}}_y
\end{aligned} \tag{12}$$

In the absence of friction the electron response is *adiabatic*.

We assume neutrality $n_e = n_i = n$ and introduce the expressions of the velocities in the continuity equations for ions and for electrons.

The ion continuity equation is

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot \left(n \mathbf{v}_{\perp,i}\right) = 0$$

The electron continuity equation is

$$\begin{aligned}
& \frac{\partial n}{\partial t} + \boldsymbol{\nabla}_\perp \cdot \left(n \mathbf{v}_{\perp,e}\right) + \\
& + \nabla_\parallel \left(n v_{\parallel e}\right) \\
= \; & 0
\end{aligned}$$

The equations are **substracted** , to obtain

$$\begin{aligned}
& -\nabla_\parallel \left(n v_{\parallel e}\right) \\
& -\frac{1}{B \Omega_i}\boldsymbol{\nabla}_\perp \cdot \left(n \frac{d}{dt}\boldsymbol{\nabla}_\perp \phi\right) \\
& + \boldsymbol{\nabla}_\perp \cdot \left[n \mathbf{v}_{dia,i}\right] - \boldsymbol{\nabla}_\perp \cdot \left[n \mathbf{v}_{dia,e}\right] \\
= \; & 0
\end{aligned} \tag{13}$$

where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}_E \cdot \boldsymbol{\nabla}$$

Now we consider the perturbation of the density, $n = n_0 + \widetilde{n}$. From the last term in the left we get:

$$\begin{aligned}
\frac{dn_0}{dr}\widehat{\mathbf{e}}_r \cdot \mathbf{v}_{dia,i} + \left(\boldsymbol{\nabla}_\perp \widetilde{n}\right) \cdot \mathbf{v}_{dia,i} &= \\
&= 0 - \frac{T_i}{T_e}\left(\frac{T_e}{\left|e\right| B}\frac{1}{n_0}\frac{dn_0}{dr}\right)\frac{\partial \widetilde{n}}{\partial y}
\end{aligned}$$

Including the similar term for the electrons, and returning to the equation (13), we obtain

$$
\begin{aligned}
& -\nabla_{\|}\left(n v_{\| e}\right) \\
& -\frac{1}{B \Omega_i} \boldsymbol{\nabla}_{\perp} \cdot\left(n \frac{d}{d t} \boldsymbol{\nabla}_{\perp} \phi\right) \\
& -\frac{T_i}{T_e}\left(\frac{T_e}{|e| B} \frac{1}{n_0} \frac{d n_0}{d r}\right) \frac{\partial \widetilde{n}}{\partial y}-\left(\frac{T_e}{|e| B} \frac{1}{n_0} \frac{d n_0}{d r}\right) \frac{\partial \widetilde{n}}{\partial y} \\
= \quad & 0
\end{aligned}
$$

NOTE that relative to **Terry Diamond Thayer** it is absent here the derivative to the *equilibrium temperature* which would introduce $(1+\eta_i)$,

$$
\text{Terry, } +v_{e, Dia}\left(1+\frac{1+\eta_i}{\tau} \boldsymbol{\nabla}_{\perp}^2\right) \frac{\partial \phi}{\partial y}
$$

END.

We write the momentum conservation for the electrons in the *parallel* direction, with the usual approximations: no *inertia*, $d/dt \approx 0$, balance of parallel gradient of the pressure $\nabla_{\|} p$, of the parallel electric force $n|e| E_{\|}$ and of the collisional friction force $n \nu_{ei} v_{e\|}$.

Now the system of equation which is obtained:

$$
\frac{d}{d t} n+\nabla_{\|}\left(n v_{\| e}\right)=0 \tag{14}
$$

$$
-\left(1+\frac{T_i}{T_e}\right)\left(\frac{T_e}{|e| B} \frac{1}{n_0} \frac{d n_0}{d r}\right) \frac{\partial \widetilde{n}}{\partial y}+\frac{1}{B \Omega_i} \boldsymbol{\nabla}_{\perp} \cdot\left(n \frac{d}{d t} \boldsymbol{\nabla}_{\perp} \phi\right)+\nabla_{\|}\left(n v_{\| e}\right)=0 \tag{15}
$$

$$
\mathbf{v}_{\| e}=-\frac{\sigma_{\|}}{e^2} \boldsymbol{\nabla}_{\|}\left(|e| \phi+T_e \ln \left(n / n_0\right)\right) \tag{16}
$$

This system is a basic model for the ITG instability and turbulence.

9.1 Detailed form

The equation of continuity for the electrons can be written

$$
\frac{d}{d t} n+\nabla_{\|}\left(n v_{\| e}\right)=\frac{\partial n}{\partial t}+\boldsymbol{\nabla}_{\perp} \cdot\left(n \mathbf{v}_{\perp}\right)+\boldsymbol{\nabla}_{\|} \cdot\left(n \mathbf{v}_{\|}\right)=0 \tag{17}
$$

and since we know that at equlibrium n_0 is constant on the surface and varies only in the radial direction, we take

$$
n=n_0(r)+\widetilde{n}
$$

The electron perpendicular velocity is composed of the $E \times B$ and the electron diamagnetic velocity.

$$\begin{aligned} & \frac{\partial \widetilde{n}}{\partial t} + \left(\mathbf{v}_E + \mathbf{v}_{dia,e}\right) \cdot \left(\widehat{\mathbf{e}}_r \frac{dn_0}{dr}\right) + \left(\mathbf{v}_E + \mathbf{v}_{dia,e}\right) \cdot \mathbf{\nabla}_\perp \widetilde{n} \\ & + \left(n_0 + \widetilde{n}\right) \mathbf{\nabla}_\perp \cdot \left(\mathbf{v}_E + \mathbf{v}_{dia,e}\right) \\ & + n_0 \left(\mathbf{\nabla}_\parallel \cdot \mathbf{v}_\parallel\right) + v_\parallel \nabla_\parallel \widetilde{n} \\ = \; & 0 \end{aligned}$$

The convective nonlinearity

$$\mathbf{\nabla}_\perp \cdot \left[\left(\frac{-\mathbf{\nabla}\phi \times \widehat{\mathbf{n}}}{B}\right) \widetilde{n}\right]$$

vanishes for adiabatic electron response, $\widetilde{n}/n_0 = -\left|e\right| \phi / T_e$ and stright, uniform $\mathbf{B}$. This is because of the mixed scalar-vectorial product

$$\begin{aligned} \left(\frac{-\mathbf{\nabla}\phi \times \widehat{\mathbf{n}}}{B}\right) \cdot \mathbf{\nabla}_\perp \widetilde{n} &= \text{(adiabatic electrons)} = \left(\frac{-\mathbf{\nabla}\phi \times \widehat{\mathbf{n}}}{B}\right) \cdot \mathbf{\nabla}_\perp \phi \\ &= 0 \end{aligned}$$

We replace:

$$\mathbf{v}_{dia,e} \cdot \widehat{\mathbf{e}}_r = 0$$

and

$$\mathbf{\nabla}_\perp \cdot \mathbf{v}_E = 0$$

The transversal divergence of the *diamagnetic* velocity in cylindrical geometry with constant $\mathbf{B}$ is

$$\begin{aligned} \mathbf{\nabla}_\perp \cdot \mathbf{v}_{dia,e} &= \mathbf{\nabla}_\perp \cdot \left(\frac{1}{n_0} \frac{T_e}{-\left|e\right| B} \widehat{\mathbf{n}} \times \mathbf{\nabla}\left(n_0 + \widetilde{n}\right)\right) = \\ &= \frac{\partial}{\partial r}\left[\frac{1}{n_0} \frac{T_e}{-\left|e\right| B} \frac{\partial n_0}{\partial r}\right] + \mathbf{\nabla}_\perp \cdot \left(\frac{T_e}{-\left|e\right| B} \widehat{\mathbf{n}} \times \mathbf{\nabla}\widetilde{n}\right) \\ &\simeq \frac{\partial}{\partial r}\left[\frac{T_e}{-\left|e\right| B L_n}\right] + \frac{T_e}{-\left|e\right| B} \mathbf{\nabla}_\perp \cdot \left(\widehat{\mathbf{n}} \times \mathbf{\nabla}\widetilde{n}\right) \end{aligned}$$

Consider the last term. For example, $\widetilde{n} = a\left(r\right) \cos m\theta$ would imply

$$\widehat{\mathbf{n}} \times \mathbf{\nabla}\widetilde{n} = \begin{pmatrix} \widehat{\mathbf{e}}_r & \widehat{\mathbf{e}}_\theta & \widehat{\mathbf{n}} \\ 0 & 0 & 1 \\ \frac{\partial \widetilde{n}}{\partial r} & \frac{\partial \widetilde{n}}{\partial y} & \nabla_\parallel \widetilde{n} \end{pmatrix} = \left(-\frac{\partial \widetilde{n}}{\partial y}\right) \widehat{\mathbf{e}}_r + \left(\frac{\partial \widetilde{n}}{\partial r}\right) \widehat{\mathbf{e}}_\theta$$

The transversal divergence of this vector field is

$$\begin{aligned} \mathbf{\nabla}_\perp \cdot \left[\left(-\frac{\partial \widetilde{n}}{\partial y}\right) \widehat{\mathbf{e}}_r + \left(\frac{\partial \widetilde{n}}{\partial r}\right) \widehat{\mathbf{e}}_\theta\right] &= \\ &= \frac{\partial}{\partial r}\left(-\frac{\partial \widetilde{n}}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \widetilde{n}}{\partial r}\right) \\ &= 0 \end{aligned}$$

which is trivial: the divergence of a rotational is zero.

Then if we do not consider the spatial variation of the temperature T_e in the region where the strong variation of the mode takes place, we have that the *perpendicular divergence of the diamagnetic velocity* is zero. Then

$$\begin{aligned}
\nabla_\perp \cdot \mathbf{v}_{dia,e} &= \frac{\partial}{\partial r}\left[\frac{T_e}{-|e|\,BL_n}\right] \\
&= \frac{\partial}{\partial r}\left[\frac{\omega_{*e}}{k_y}\right]
\end{aligned}$$

Acting first on the temperature

$$\begin{aligned}
\nabla_\perp \cdot \mathbf{v}_{dia,e} &= \frac{1}{-|e|\,B}\left[\frac{\partial T_e}{\partial r}\frac{1}{L_n} + T_e\frac{\partial}{\partial r}\left(\frac{1}{L_n}\right)\right] \\
&= \frac{T_e}{-|e|\,B}\left[\frac{1}{L_{T_e}}\frac{1}{L_n} + \frac{\partial}{\partial r}\left(\frac{1}{L_n}\right)\right] \\
&= \frac{T_e}{-|e|\,B}\left[\frac{1}{L_{T_e}}\frac{1}{L_n} - \frac{1}{L_n^2}\frac{dL_n}{dr}\right] \\
&= \frac{T_e}{-|e|\,B}\frac{1}{L_n}\left(\kappa_T - \kappa_n\kappa_n'\right)
\end{aligned}$$

The radial variation of the phase velocity of the drift waves occurs due to the radial variation of the density and temperature gradient lengths. If we neglect this second derivative of the density profile,

$$\nabla_\perp \cdot \mathbf{v}_{dia,e} = 0$$

Returning to the electron continuity equation, we have

$$\begin{aligned}
&\frac{\partial \widetilde{n}}{\partial t} + \frac{1}{B}\frac{\partial \phi}{\partial y}\frac{dn_0}{dr} + (\mathbf{v}_E + \mathbf{v}_{dia,e}) \cdot \nabla_\perp \widetilde{n} + \\
&+ n_0\left(\nabla_\parallel \cdot \mathbf{v}_\parallel\right) + v_\parallel \nabla_\parallel \widetilde{n} \\
=\;& 0
\end{aligned} \tag{18}$$

The component of diamagnetic advection of the density fluctuations

$$\mathbf{v}_{dia,e} \cdot \nabla_\perp \widetilde{n} = \frac{T_e}{-|e|\,B}\frac{1}{n_0}\frac{dn_0}{dr}\frac{\partial \widetilde{n}}{\partial y}$$

The second term in the equation (18) is

$$\begin{aligned}
\frac{1}{B}\frac{\partial \phi}{\partial y}\frac{dn_0}{dr} &= -\frac{\partial}{\partial y}\left(\frac{|e|\,\phi}{T_e}\right) n_0 \frac{T_e}{-|e|\,B}\frac{1}{n_0}\frac{dn_0}{dr} \\
&= v_{dia,e}\frac{\partial}{\partial y}\left(\frac{|e|\,\phi}{T_e}\right) n_0
\end{aligned} \tag{19}$$

We take separately two terms in the continuity equation: (19) and the diamagnetic advection of the density fluctuation, since in uniform $\mathbf{B}$, slab geometry they cancel.

$$
\begin{aligned}
\frac{1}{B}\frac{\partial\phi}{\partial y}\frac{dn_0}{dr}+\mathbf{v}_{dia,e}\cdot\boldsymbol{\nabla}_{\perp}\widetilde{n} &= \\
&= n_0 v_{dia,e}\frac{\partial}{\partial y}\left(\frac{|e|\,\phi}{T_e}\right)+v_{dia,e}\frac{\partial\widetilde{n}}{\partial y} \\
&= n_0 v_{dia,e}\left[\frac{\partial}{\partial y}\left(\frac{|e|\,\phi}{T_e}\right)+\frac{\partial}{\partial y}\frac{\widetilde{n}}{n_0}\right] \\
&= n_0 v_{dia,e}\frac{1}{T_e}\frac{\partial}{\partial y}\left(|e|\,\phi+T_e\ln\left(n/n_0\right)\right)
\end{aligned}
$$

We take into account the Eq.(16) : $\mathbf{v}_{\parallel e}=-\frac{\sigma_{\parallel}}{e^2}\boldsymbol{\nabla}_{\parallel}\left(|e|\,\phi+T_e\ln\left(n/n_0\right)\right)$ and the simplified relation

$$
k_{\parallel}=k_y\frac{x}{L_s}\text{ or }\nabla_{\parallel}=\frac{x}{L_s}\frac{\partial}{\partial y}
$$

and write

$$
\begin{aligned}
n_0 v_{dia,e}\frac{1}{T_e}\frac{\partial}{\partial y}\left(|e|\,\phi+T_e\ln\left(n/n_0\right)\right) &= n_0 v_{dia,e}\frac{1}{T_e}\frac{L_s}{x}\boldsymbol{\nabla}_{\parallel}\left(|e|\,\phi+T_e\ln\left(n/n_0\right)\right) \\
&= -\frac{e^2}{\sigma_{\parallel}}v_{\parallel e}n_0 v_{dia,e}\frac{1}{T_e}\frac{L_s}{x} \\
&= -\frac{n_0}{T_e}\frac{e^2}{\sigma_{\parallel}}\omega_{*e}\frac{v_{\parallel e}}{k_{\parallel}}
\end{aligned}
$$

where $\omega_{*e}=k_y v_{dia,e}$. Slightly upper the resonant surface, where $k_{\parallel}>0$, the condition of infinite parallel conductivity

$$
\sigma_{\parallel}\rightarrow\infty
$$

leads to the vanishing of the total quantity

$$
\frac{1}{B}\frac{\partial\phi}{\partial y}\frac{dn_0}{dr}+\mathbf{v}_{dia,e}\cdot\boldsymbol{\nabla}_{\perp}\widetilde{n}=0
$$

a result which could also be obtained from the condition of adiabaticity of the electrons.

The Eq.(17) can be written

$$
\frac{\partial\widetilde{n}}{\partial t}+\mathbf{v}_E\cdot\boldsymbol{\nabla}_{\perp}\widetilde{n}+n_0\left(\boldsymbol{\nabla}_{\parallel}\cdot\mathbf{v}_{\parallel}\right)+v_{\parallel}\nabla_{\parallel}\widetilde{n}=0.
$$

We now turn to the second equation, Eq.(15). To develop separately the ion-polarization drift term, we note:

$$
\begin{aligned}
W &\equiv \frac{1}{B\Omega_i}\boldsymbol{\nabla}_{\perp}\cdot\left[\left(n_0+\widetilde{n}\right)\left(\frac{\partial}{\partial t}+\mathbf{v}_E\cdot\boldsymbol{\nabla}_{\perp}\right)\boldsymbol{\nabla}_{\perp}\phi\right] \\
&= \frac{1}{B\Omega_i}\boldsymbol{\nabla}_{\perp}\cdot\left[\left(n_0+\widetilde{n}\right)\mathbf{I}\right]
\end{aligned}
$$

where we make explicit the electric potential associated to the initial plasma poloidal rotation, ϕ_0.

$$\mathbf{I} \equiv \left(\frac{\partial}{\partial t} + (V_0 \widehat{\mathbf{e}}_y + \widetilde{\mathbf{v}}) \cdot \boldsymbol{\nabla}_\perp \right) \boldsymbol{\nabla}_\perp \left(\phi_0 + \widetilde{\phi} \right)$$

We have

$$\begin{aligned}
\mathbf{I} &= \frac{\partial}{\partial t} \left(\boldsymbol{\nabla}_\perp \phi_0 \right) + \\
&+ \frac{\partial}{\partial t} \left(\boldsymbol{\nabla}_\perp \widetilde{\phi} \right) + \\
&+ \left[V_0 \frac{\partial}{\partial y} + \left(\widetilde{\mathbf{v}} \cdot \boldsymbol{\nabla}_\perp \right) \right] \left(\boldsymbol{\nabla}_\perp \phi_0 + \boldsymbol{\nabla}_\perp \widetilde{\phi} \right)
\end{aligned}$$

or

$$\begin{aligned}
\mathbf{I} &= \frac{\partial}{\partial t} \left(\widehat{\mathbf{e}}_r B V_0 \right) + \\
&+ \frac{\partial}{\partial t} \left(\boldsymbol{\nabla}_\perp \widetilde{\phi} \right) + \\
&+ \left[V_0 \frac{\partial}{\partial y} + \left(\widetilde{\mathbf{v}} \cdot \boldsymbol{\nabla}_\perp \right) \right] \left(\boldsymbol{\nabla}_\perp \phi_0 + \boldsymbol{\nabla}_\perp \widetilde{\phi} \right) \\
&= \frac{\partial}{\partial t} \left(\widehat{\mathbf{e}}_r B V_0 \right) + V_0 B \frac{\partial V_0}{\partial y} \widehat{\mathbf{e}}_r + \\
&+ \frac{\partial}{\partial t} \left(\boldsymbol{\nabla}_\perp \widetilde{\phi} \right) + \left(\widetilde{\mathbf{v}} \cdot \boldsymbol{\nabla}_\perp \right) B V_0 \widehat{\mathbf{e}}_r + \\
&+ V_0 \frac{\partial}{\partial y} \left(\boldsymbol{\nabla}_\perp \widetilde{\phi} \right) + \left(\widetilde{\mathbf{v}} \cdot \boldsymbol{\nabla}_\perp \right) \left(\boldsymbol{\nabla}_\perp \widetilde{\phi} \right)
\end{aligned}$$

with the relations

$$\begin{aligned}
\boldsymbol{\nabla}_\perp \widetilde{\phi} &= B \left(\widetilde{\mathbf{v}} \times \widehat{\mathbf{n}} \right) \\
&= \left(B \widetilde{v}_y \right) \widehat{\mathbf{e}}_r + \left(-B \widetilde{v}_r \right) \widehat{\mathbf{e}}_y
\end{aligned}$$

$$\widetilde{v}_y = \frac{1}{B} \frac{\partial \widetilde{\phi}}{\partial r} \text{ and } \widetilde{v}_r = -\frac{1}{B} \frac{\partial \widetilde{\phi}}{\partial y}$$

After very simple calculations we obtain:

$$\begin{aligned}
\frac{1}{B} I_r &= \frac{\partial V_0}{\partial t} + \widetilde{v}_y \frac{\partial V_0}{\partial y} + V_0 \frac{\partial V_0}{\partial y} \\
&+ \frac{\partial \widetilde{v}_y}{\partial t} + \widetilde{v}_r \frac{\partial V_0}{\partial r} + V_0 \frac{\partial \widetilde{v}_y}{\partial y} \\
&+ \widetilde{v}_r \frac{\partial \widetilde{v}_y}{\partial r} + \widetilde{v}_y \frac{\partial \widetilde{v}_y}{\partial y}
\end{aligned} \tag{20}$$

and

$$\frac{1}{B}I_y = -\frac{\partial\widetilde{v}_r}{\partial t} - V_0\frac{\partial\widetilde{v}_r}{\partial y} - \widetilde{v}_r\frac{\partial\widetilde{v}_r}{\partial r} - \widetilde{v}_y\frac{\partial\widetilde{v}_r}{\partial y} \tag{21}$$

It will be useful to calculate the derivaties of these quantities

$$\begin{aligned}\frac{1}{B}\frac{\partial I_r}{\partial r} &= \frac{\partial}{\partial r}\frac{\partial V_0}{\partial t} + \frac{\partial\widetilde{v}_y}{\partial r}\frac{\partial V_0}{\partial y} + \widetilde{v}_y\frac{\partial^2 V_0}{\partial y\partial r} + \frac{\partial V_0}{\partial r}\frac{\partial V_0}{\partial y} + V_0\frac{\partial^2 V_0}{\partial r\partial y} \\ &+\frac{\partial\widetilde{v}_y}{\partial t}\frac{\partial V_0}{\partial r} + \widetilde{v}_r\frac{\partial^2 V_0}{\partial r^2} + \frac{\partial}{\partial t}\frac{\partial\widetilde{v}_y}{\partial r} + \\ &+\frac{\partial V_0}{\partial r}\frac{\partial\widetilde{v}_y}{\partial y} + V_0\frac{\partial^2\widetilde{v}_y}{\partial r\partial y} + \\ &+\frac{\partial\widetilde{v}_r}{\partial r}\frac{\partial\widetilde{v}_r}{\partial y} + \widetilde{v}_r\frac{\partial^2\widetilde{v}_y}{\partial r^2} + \\ &+\frac{\partial\widetilde{v}_y}{\partial r}\frac{\partial\widetilde{v}_y}{\partial y} + \widetilde{v}_y\frac{\partial^2\widetilde{v}_y}{\partial r\partial y}\end{aligned} \tag{22}$$

and

$$\begin{aligned}\frac{1}{B}\frac{\partial I_y}{\partial y} &= -\frac{\partial V_0}{\partial y}\frac{\partial\widetilde{v}_r}{\partial y} \\ &-\frac{\partial}{\partial t}\frac{\partial\widetilde{v}_r}{\partial y} - V_0\frac{\partial^2\widetilde{v}_y}{\partial y^2} - \\ &-\frac{\partial\widetilde{v}_r}{\partial y}\frac{\partial\widetilde{v}_r}{\partial r} - \widetilde{v}_r\frac{\partial^2\widetilde{v}_r}{\partial r\partial y} - \\ &-\frac{\partial\widetilde{v}_y}{\partial y}\frac{\partial\widetilde{v}_r}{\partial y} - \widetilde{v}_y\frac{\partial^2\widetilde{v}_r}{\partial y^2}\end{aligned} \tag{23}$$

The quantity denoted by W takes the form

$$\begin{aligned}W &= \frac{1}{B\Omega_i}\left(\frac{dn_0}{dr}\right)I_r + \frac{1}{B\Omega_i}n_0\left(\frac{\partial I_r}{\partial r}\right) + \\ &+\frac{1}{B\Omega_i}\left(\frac{\partial\widetilde{n}}{\partial r}\right)I_r + \frac{1}{B\Omega_i}\widetilde{n}\left(\frac{\partial I_r}{\partial r}\right) + \\ &+\frac{1}{B\Omega_i}n_0\left(\frac{\partial I_y}{\partial y}\right) + \frac{1}{B\Omega_i}\left(\frac{\partial\widetilde{n}}{\partial y}\right)I_y + \frac{1}{B\Omega_i}\widetilde{n}\left(\frac{\partial I_y}{\partial y}\right)\end{aligned}$$

9.2 The mode evolution in a fixed plasma rotation profile

We will assume that the mode evolves initially without perturbing the equilibrium profiles, in particular the seed poloidal rotation. This allows us to simplify the expressions above, taking:

$$\begin{aligned}\frac{\partial V_0}{\partial y} &= 0 \\ \frac{\partial V_0}{\partial t} &= 0\end{aligned}$$

Then the first lines in the formulas Eqs.(20), (22), (23) are zero. Let us consider in the expression of W the part W_0 which **does not contain the fluctuating density** $\widetilde{n}$. Writting

$$W = W_0 + \widetilde{W}$$

we have

$$\begin{aligned}W_0 &\equiv \frac{1}{\Omega_i}\left(\frac{dn_0}{dr}\right)\left[\widetilde{v}_r\frac{\partial V_0}{\partial r} + V_0\frac{\partial \widetilde{v}_y}{\partial y} + \widetilde{v}_r\frac{\partial \widetilde{v}_y}{\partial r} + \widetilde{v}_y\frac{\partial \widetilde{v}_y}{\partial y} + \frac{\partial \widetilde{v}_y}{\partial t}\right] + \\ &+ \frac{1}{\Omega_i} n_0 \boldsymbol{\nabla}_\perp \cdot \left(I_r \widehat{\mathbf{e}}_r + I_y \widehat{\mathbf{e}}_y\right)\end{aligned}$$

and

$$\widetilde{W} = \frac{1}{B\Omega_i}\left(\frac{\partial \widetilde{n}}{\partial r}\right) I_r + \frac{1}{B\Omega_i}\widetilde{n}\left(\frac{\partial I_r}{\partial r}\right) + \frac{1}{B\Omega_i}\left(\frac{\partial \widetilde{n}}{\partial y}\right) I_y + \frac{1}{B\Omega_i}\widetilde{n}\left(\frac{\partial I_y}{\partial y}\right)$$

Replacing the perturbed velocity by the perturbed potential, writting all terms and summing, we get:

$$\begin{aligned}W_0 &= \left(\frac{\partial}{\partial t} + V_0\frac{\partial}{\partial y}\right)\frac{1}{B}\frac{\partial \widetilde{\phi}}{\partial r} + \left(-\frac{1}{B}\frac{\partial \widetilde{\phi}}{\partial y}\right)\frac{dV_0}{dr} + \left(\frac{-\boldsymbol{\nabla}_\perp \phi \times \widehat{\mathbf{n}}}{B} \cdot \boldsymbol{\nabla}_\perp\right)\frac{\partial \widetilde{\phi}}{\partial r} + \\ &+ \frac{1}{\Omega_i} n_0 \left\{ \frac{1}{B} V_0 \frac{\partial}{\partial y} \nabla_\perp^2 \widetilde{\phi} + \right. \\ &\qquad + \frac{1}{B}\left(\frac{-\boldsymbol{\nabla}_\perp \phi \times \widehat{\mathbf{n}}}{B} \cdot \boldsymbol{\nabla}_\perp\right)\nabla_\perp^2 \widetilde{\phi} - \\ &\qquad - \frac{1}{B}\frac{\partial \widetilde{\phi}}{\partial y}\frac{d^2 V_0}{dr^2} + \\ &\qquad \left. + \frac{1}{B}\frac{\partial}{\partial t}\left(\nabla_\perp^2 \widetilde{\phi}\right)\right\}\end{aligned}$$

Collecting all what we have at this moment the ion continuity equation (15) becomes:

$$
\begin{aligned}
& \nabla_{\|}\left(n v_{\| e}\right) + \\
& -\left(1+\frac{T_i}{T_e}\right)\left(\frac{T_e}{|e| B} \frac{1}{n_0} \frac{d n_0}{d r}\right) \frac{\partial \widetilde{n}}{\partial y} + \\
& +\frac{1}{B \Omega_i} n_0\left\{\left(\frac{\partial}{\partial t}+V_0 \frac{\partial}{\partial y}\right) \nabla_{\perp}^2 \widetilde{\phi}-V_0^{\prime \prime} \frac{\partial \widetilde{\phi}}{\partial y}+\left(\frac{-\boldsymbol{\nabla}_{\perp} \phi \times \widehat{\mathbf{n}}}{B} \cdot \boldsymbol{\nabla}_{\perp}\right) \nabla_{\perp}^2 \widetilde{\phi}\right\} \\
& \qquad +\frac{1}{B \Omega_i}\left(\frac{d n_0}{d r}\right)\left[\left(\frac{\partial}{\partial t}+V_0 \frac{\partial}{\partial y}\right) \widetilde{v}_y+\widetilde{v}_r \frac{\partial \widetilde{v}_y}{\partial r}+\widetilde{v}_y \frac{\partial \widetilde{v}_y}{\partial y}+\widetilde{v}_r \frac{\partial V_0}{\partial r}\right] \\
& \qquad +\widetilde{W}+\{\text{terms of the first line in the expressions of } I_r \text{ and derivatives}\} \\
= & \ 0
\end{aligned}
$$

In the above equation (which is exact) we shall make the following approximations:

- neglect the term containing the parallel electron velocity, assuming infinite electric conductivity;
- neglect the term which contains dn_0/dr since it is in the ratio $k : 1/L_n$ with the other terms, and we consider

$$
kL_n \ll 1
$$

- neglect $\widetilde{W}$; these terms are in the ratio $\widetilde{n}/n_0$ with the terms which are retained;
- neglect of the terms in the first lines of the expressions for I_r and dI_r/dr, dI_y/dy. (These are the terms in the curly brakets, the last line). As explained above, we assume that the mode evolves in a background of fixed rotation profile, $V_0(r)$.

The resulting equation is

$$
\begin{aligned}
& -B \Omega_i\left(1+\frac{T_i}{T_e}\right)\left(\frac{T_e}{|e| B} \frac{1}{n_0} \frac{d n_0}{d r}\right) \frac{\partial}{\partial y}\left(\frac{\widetilde{n}}{n_0}\right) \\
& +\left(\frac{\partial}{\partial t}+V_0 \frac{\partial}{\partial y}\right) \nabla_{\perp}^2 \widetilde{\phi}-V_0^{\prime \prime} \frac{\partial \widetilde{\phi}}{\partial y}+\left(\frac{-\boldsymbol{\nabla}_{\perp} \phi \times \widehat{\mathbf{n}}}{B} \cdot \boldsymbol{\nabla}_{\perp}\right) \nabla_{\perp}^2 \widetilde{\phi} \\
= & \ 0
\end{aligned}
$$

and, replace the adiabatic form of the density perturbation

$$
\frac{\widetilde{n}}{n_0}=-\frac{|e| \widetilde{\phi}}{T_e}
$$

We define

$$\beta \equiv B\Omega_i \left(1+\frac{T_i}{T_e}\right)\left(\frac{T_e}{|e|\,B}\frac{1}{n_0}\frac{dn_0}{dr}\right)\frac{|e|}{T_e} = \Omega_i \left(1+\frac{T_i}{T_e}\right)\frac{1}{L_n}$$

and obtain

$$\beta\frac{\partial\widetilde{\phi}}{\partial y} + \left(\frac{\partial}{\partial t} + V_0\frac{\partial}{\partial y}\right)\nabla_\perp^2\widetilde{\phi} - V_0^{''}\frac{\partial\widetilde{\phi}}{\partial y} + \left(\frac{-\boldsymbol{\nabla}_\perp\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}_\perp\right)\nabla_\perp^2\widetilde{\phi} = 0$$

which is the barotropic equation.

9.3 Nondimensional form of the equation

Let us make the replacements

$$\begin{aligned} y &\rightarrow Ly \\ t &\rightarrow t\frac{L}{U_0} \\ \widetilde{\phi} &\rightarrow \phi\frac{T_e}{|e|} \\ V_0 &\rightarrow UU_0 \end{aligned}$$

such that from now on y, t, ϕ and U are nondimensional quantities. We also change the radial coordinate to the dimensionless variable x

$$r \rightarrow Lx$$

$$\begin{aligned} & \left(\Omega_i\left(1+\frac{T_i}{T_e}\right)\frac{1}{L_n}\frac{L^2}{U_0}\right)\frac{\partial\phi}{\partial y} + \\ & + \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial y}\right)\nabla_\perp^2\phi \\ & -\frac{d^2U}{dx^2}\frac{\partial\phi}{\partial y} + \\ & + \left(\frac{1}{U_0}\frac{T_e}{|e|}\frac{1}{B}\frac{1}{L}\right)\left[\left(-\boldsymbol{\nabla}_\perp\phi\times\widehat{\mathbf{n}}\right)\cdot\boldsymbol{\nabla}_\perp\right]\nabla_\perp^2\phi \\ = \;& 0 \end{aligned}$$

The coefficients are

$$\beta' \equiv \Omega_i\left(1+\frac{T_i}{T_e}\right)\frac{1}{L_n}\frac{L^2}{U_0} \quad \text{non-dimensional}$$

$$\varepsilon \equiv \frac{1}{U_0}\frac{T_e}{|e|}\frac{1}{B}\frac{1}{L} \quad \text{non-dimensional}$$

For an order of magnitude, ε is the ratio of the diamagnetic electron velocity to the rotation velocity U_0 multiplied by the ratio of the density gradient length to the length of the spatial domain. This quantity, ε is in general small.

The quantity β' is the ratio of the ion cyclotron frequency to the inverse of the time required to cross the spatial domain with the typical flow velocity. Since the later (U_0/L) involves macroscopic quantities this ratio can be large. It is multiplied by the ratio of the spatial length to the density gradient length (these quantities can be comparable and the ratio not too different of unity).

We change the notations eliminating the primes. The equation becomes

$$\beta\frac{\partial\phi}{\partial y}+\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial y}\right)\nabla_\perp^2\phi-\frac{d^2U}{dx^2}\frac{\partial\phi}{\partial y}+\varepsilon\left[\left(-\boldsymbol{\nabla}_\perp\phi\times\widehat{\mathbf{n}}\right)\cdot\boldsymbol{\nabla}_\perp\right]\nabla_\perp^2\phi=0$$

This is the *barotropic* equation, known in the physics of atmosphere.

10 Kinetic description of ITG mode Brunner

Slab.

This is from **Brunner**.

The Fourier transform of the potential

$$\phi=\widehat{\phi}\exp\left(i\mathbf{k}.\mathbf{r}\ -\ i\omega t\right)$$

and the Gyro-kinetic equation where the Guiding centre trajectories reduce to the parallel motion, gives the following non-adiabatic ion response

$$\widetilde{g}_i=\frac{q_i}{T_i}\frac{\omega-\omega_{*i}}{\omega-k_\parallel v_\parallel}F_{Mi}\,J_0\left(\frac{k_\perp v_\perp}{\Omega_i}\right)\widehat{\phi}\exp\left(i\mathbf{k}.\mathbf{r}\ -\ i\omega t\right)$$

where

$$\omega_{*i}=\omega_n\left(1+\eta\,T\frac{\partial}{\partial T}\right)$$

and

$$\eta_i=\frac{L_n}{L_T}$$

NOTE comparisons

Kim Horton Tajima toroidal resonance.

The neutrality condition

$$\begin{aligned}&1+\tau\\=&\int_{-\infty}^{\infty}\frac{dv_\parallel}{\sqrt{\pi}}\int_0^\infty 2v_\perp dv_\perp\exp\left(-v^2\right)\\&\times\frac{\omega-\omega_{*i}\left[1+\eta_i\left(v^2-3/2\right)\right]}{\omega-\omega_{Di}\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)-\sqrt{2}k_\parallel v_\parallel}J_0^2\left(\sqrt{2}k_\perp v_\perp\right)\end{aligned}$$

where

$$
\begin{aligned}
\omega_{*i} &= k_y \\
\omega_{Di} &= 2\epsilon_n \omega_{*i} \\
\epsilon_n &= \frac{L_n}{R} \\
\eta_i &= \frac{L_n}{L_{T_i}}
\end{aligned}
$$

$$
L_n^{-1} = -\frac{d \ln n}{dr}
$$

Normalizations

$$
\begin{aligned}
v_\perp &\leftarrow \frac{v_\perp}{\sqrt{2} v_{th,i}} \\
v_\| &\leftarrow \frac{v_\|}{\sqrt{2} v_{th,i}}
\end{aligned}
$$

$$
v_{th,i} = \sqrt{\frac{T_i}{m_i}}
$$

$$
\begin{aligned}
\omega &\leftarrow \frac{\omega}{v_{th,i}/L_n} \\
\omega_{*i} &\leftarrow \frac{\omega_{*i}}{v_{th,i}/L_n}
\end{aligned}
$$

$$
\begin{aligned}
k_\perp &\leftarrow \frac{k_\perp}{v_{th,i}/\Omega_{ci}} \\
k_\| &\leftarrow \frac{k_\|}{L_n}
\end{aligned}
$$

Positive real ω represents ion diamagnetic direction.

Conclusion:

1. the drive is the difference between the frequency ω and the frequency associated to the radial convection of the equilibrium gradient $\omega_{*i}\left[1+\eta_i\left(v^2-3/2\right)\right]$.
2. the propagator shows the resonances:
   (a) parallel resonance of the ion velocity with the mode
   (b) drift of the ions in the $\boldsymbol{\nabla} B$ and curvature of the lines.
3. there is an effect of the Larmor radius, by the presence of the factor Bessel function squared J_0^2.

END

With adiabatic electrons, the quasineutrality relation for the electrostatic ITG mode is

$$\begin{aligned} \frac{1}{\tau}+1+\left[1-\frac{\omega_{ni}}{\omega}\left(1-\frac{\eta_i}{2}\right)\right](W-1)\,\Gamma_0 & \\ -\frac{\omega_{ni}}{\omega}\eta_i\left[\frac{1}{2}\left(\frac{\omega}{k_{\|}v_{th,i}}\right)W\Gamma_0+(W-1)\left(k_{\perp}\rho_i\right)^2\,(\Gamma_1-\Gamma_0)\right] & \\ = \quad 0 & \end{aligned}$$

The plasma dispersion function W has argument

$$z_i=\frac{\omega}{\left|k_{\|}\right|v_{thi}}$$

For the cold ions, and using the following approximation for the function W:

$$W(z)=-\frac{1}{z^2}-\frac{3}{z^4}-\frac{15}{z^6}-...$$

the dispersion relation is obtained

$$\left[1+\left(k_{\perp}\rho_s\right)^2\right]\omega^2-\omega\omega_{*e}-\left(k_{\|}c_s\right)^2=0 \tag{24}$$

where

$$c_s^2=\frac{T_e}{m_i}$$

and

$$\rho_s=\frac{c_s}{\Omega_i}.$$

Note the perpendicular dispersion , which is related to the **ion polarization drift** , the last effect of the **Finite Larmor Radius** of the ions, in this approximation.

Observatia mea:(31 aug.98) Ce este atunci curentul de polarizare: este strict legat de Larmor radius, sau este solicitat de divergenta nula a densitatii de curent, in conditiile in care toate celelalte componente ale densitatii de curent: wave $\mathbf{E}\times\mathbf{B}$ sunt zero ?.

For $\left|k_{\|}c_s\right|\gg|\omega_{*e}|$, the dispersion relation becomes

$$\omega^2\simeq\frac{\left(k_{\|}c_s\right)^2}{1+\left(k_{\perp}\rho_s\right)^2}$$

which is the dispersion relation of the **ion-sound** wave.

For $\left|k_{\|}c_s\right|\ll|\omega_{*e}|$ one of the two roots of the dispersion relation Eq.(24) becomes

$$\omega\simeq\omega_{*e}$$

This is the *electron drift mode* which can be destabilized by a non-adiabatic electron response.

If the ions cannot be considered cold, i.e. there is a finite $T_i \sim T_e$ ion temperature, there exists an instability *driven by the ions*. One must take

$$\left|\frac{k_\parallel v_{thi}}{\omega}\right| \ll 1$$

in the dispersion relation and expand to second order in $1/|z_i|$. Negelcting Finite Larmor Radius (FLR) effects and taking $L_n \to \infty$ (flat density profile) the dispersion relation becomes:

$$1 - \left(\frac{k_\parallel v_{thi}}{\omega}\right)^2 \left(1 - \frac{\omega_{Ti}}{\omega}\right) = 0$$

For $\left|k_\parallel v_{thi}\right| \ll \omega_{Ti}$ this equation gives an unstable root $\omega = \omega_r + i\gamma$, $\gamma > 0$. This is the **slab-ITG instability ,** or the η_i - instability:

$$\omega \simeq \frac{1}{2}\left(1 + i\sqrt{3}\right)\left[\left(k_\parallel v_{thi}\right)^2 \omega_{Ti}\right]^{1/3}$$

Approximative values are:

$$\omega_r \sim \gamma \sim \omega_{Ti}$$

Maximal growth rates appear for

- longitudinal phase velocities $\omega/k_\parallel$ of the order of the **ion thermal velocity**, v_{thi}; [close to absorbtion by ions at turning point in the sheared magnetic field; $k_\parallel$ is very small ≈ 0].
- perpendicular wavelengths of the order of the **ion Larmor radius**, ρ_i. (explaining the presence of J_0^2); the *ion polarization drift* must be present: the vectorial (HM) nonlinearity acts at this level.

There are two conditions for the existence of the instability:

One is for η_i. Here $\xi_i = (k_\perp \rho_i)^2$ is the argument of Λ_0 and Λ_1, while $z_i = \omega/\left(\left|k_\parallel\right| v_{thi}\right)$ is the argument of W, plasma dispersion function.

$$\eta_i > \eta_{ic} = \frac{2}{1 + 2\xi_i \frac{\Lambda_0 - \Lambda_1}{\Lambda_0}} \text{ or } \eta_i < 0$$

The other condition is

$$\left|k_\parallel\right| < k_{\parallel \lim}$$

where, for $k_\perp \rho_i \ll 1$,

$$k_{\parallel \lim} = \frac{1}{2}\left(1 - \frac{2}{\eta_i}\right)^{1/2} \frac{|\omega_{Ti}|}{v_{thi}} = \frac{1}{2}\left(1 - \frac{2}{\eta_i}\right)^{1/2} \frac{\rho_i}{|L_{Ti}|} k_y$$

For $\left|k_\parallel\right| > k_{\parallel\lim}$ the instability is suppressed by Landau damping.

$$\left.\begin{array}{l}\gamma \\ k_{\parallel\lim}\end{array}\right\} \text{are maximum for } \; k_\perp \rho_i \simeq 1$$

For shorter perpendicular wavelengths the potential felt by the ions tends to be averaged out over their Larmor gyration and the drive becomes less effective.

The potential structure for the periodicity numbers (m, n)

$$\phi \simeq \phi\left(\rho\right) e^{i(m\theta + n\varphi)}.$$

The parallel wavenumber is obtained from

$$\mathbf{n} \cdot \boldsymbol{\nabla}\phi \simeq i\frac{1}{Rq}\left(nq - m\right)\phi \;\Rightarrow\; k_\parallel\left(\rho\right) = \frac{1}{Rq\left(\rho\right)}\left(nq\left(\rho\right) - m\right).$$

The mode is localized around the surface where

$$q\left(\rho_r\right) = \frac{m}{n} \text{ and } k_\parallel\left(\rho_r\right) = 0.$$

At a distance $\Delta\rho$ from the resonant surface teh parallel wavenumber is modified by the shear

$$\left|k_\parallel\right| = \left|k_\theta\right| \frac{\Delta\rho}{L_s}$$

with

$$L_s = \frac{Rq}{\widehat{s}}$$

the shear length and $\widehat{s} = rq'/q$ and $k_\theta = m/\rho$ is the poloidal wavenumber.

Using the expression for $\omega_{Ti} = -k_\theta \frac{T_i}{|e|B}\frac{1}{L_T}$ we obtain the radial extension of the mode (for $k_\perp\rho_i \ll 1$):

$$\Delta\rho \simeq \frac{1}{2}\left(1 - \frac{2}{\eta_i}\right)^{1/2} \frac{L_s}{L_{Ti}}\rho_i$$

estimation which is obatined on the limit of $k_\parallel$ for Landau damping.

The condition of overlap of the potential structures from neighbouring resonant surfaces is

$$\Delta\rho \gtrsim \frac{1}{nq'} \Rightarrow \frac{L_{Ti}}{R} \lesssim \left(k_\theta\rho_i\right) q$$

For the same fixed toroidal wavenumber n several potential structures can overlap to form an extended radial potential structure.

The dispersion relation for the ITG mode is obtained as usual by deriving the particle answer to the wave field and imposing charge neutrality:

$$Q_e = -\tau\, Q_i.$$

When the ion Larmor radius is smaller than the perpendicular wavelength, the ion term in the above equality can be expanded:

$$Q_e = -\tau \left(Q_i + \left(\frac{dQ_i}{db} \right) b_x + ... \right)$$

Here the notations are

$$b_x \equiv \frac{k_x^2 \rho_i^2}{2} = \frac{\tau\, Q_i + Q_e}{-\tau \left(\frac{dQ_i}{db} \right)}. \tag{25}$$

This equality becomes an equation if we assume as usual that the radial wavelength is a radial derivative if it cannot be considered constant on the spatial extension of the mode:

$$\frac{\rho_i^2}{2} \frac{d^2 \phi\,(x)}{dx^2} + b_x\,(x)\,\phi\,(x) = 0$$

The notations are:

$$\rho_i = \frac{v_{thi}}{\Omega_i}$$

$$v_{thi} = \left(\frac{2T_i}{m_i} \right)^{1/2}$$

$$\Omega_i = \frac{|e|\, B}{m_i}$$

$$\tau = \frac{T_e}{T_i}$$

$$b = \frac{k_y^2 \rho_i^2}{2}.$$

The ion and electron particle responses are obtained from the Vlasov equations and takes into account the effect of the electric filed and of trapped electrons:

$$\begin{aligned} Q_i \quad \equiv \quad & 1 + \left\{ \left(1 - \frac{\omega_{*i}}{\omega + \omega_E} \right) \Gamma_0 + \frac{\eta_i}{2} \frac{\omega_{*i}}{\omega + \omega_E} \left[\Gamma_0 - 2b_i \left(\Gamma_1 - \Gamma_0 \right) \right] \right\} \xi_i Z_0 \left(\xi_i \right) \\ & - \eta_i \frac{\omega_{*i}}{\omega + \omega_E} \Gamma_0 \xi_i Z_2 \left(\xi_i \right) \end{aligned} \tag{26}$$

and

$$\begin{aligned} Q_e \quad = \quad & 1 + \left\{ \left[1 - \frac{\omega_{*e}}{\omega + \omega_E - k_\parallel u} \left(\frac{1 - \eta_e}{2} \right) \right] \xi_e Z_0^+ \left(\overline{\xi}_e \right) - \eta_e \frac{\omega_{*e}}{\omega + \omega_E} \xi_e Z_2^+ \left(\overline{\xi}_e \right) \right. \\ & \left. + \left(1 - \frac{\omega_{*e}}{\omega + \omega_E} \right) \sqrt{\varepsilon} A_2 - \eta_e \frac{\omega_{*e}}{\omega + \omega_E} \sqrt{\varepsilon} A_1 \right\} \\ & \times \left[1 + \frac{i \nu_e}{\omega + \omega_E - k_\parallel u} \xi_e Z_0^+ \left(\overline{\xi}_e \right) \right]^{-1} \end{aligned} \tag{27}$$

A simpler expression for the electron density perturbation is given in the "$i\delta$" model,

$$\delta n_e = n_e \frac{e\phi}{T_e} \left(1 - i\delta_e\right)$$

where δ_e is calculated for the trapped electron mode, or, in a simplified model, is taken zero assuming that the electrons are adiabatic. The following notations have been used:

$$\omega_{*j} = -k_y \frac{T_j}{e_j B} \frac{1}{L_n}$$

$$\eta_j = \frac{d \ln T_j}{d \ln n}$$

$$\Gamma_n = I_n\left(b\right) \exp\left(-b\right)$$

$$Z_n\left(\xi\right) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^n \frac{\exp\left(-t^2\right)}{t - \xi} dt$$

$$\omega_E = k_y \frac{E\left(x\right)}{B}$$

$$\xi_i = \frac{\omega + \omega_E}{\left|k_{\parallel}\right| v_{thi}}$$

$$\xi_e = \frac{\omega + \omega_E - k_{\parallel} u}{\left|k_{\parallel}\right| v_{the}}$$

$$\overline{\xi}_e = \frac{\omega + \omega_E - k_{\parallel} u + i\nu_e}{\left|k_{\parallel}\right| v_{the}}$$

$$k_{\parallel} = k_y \frac{x}{L_s}$$

where u is the drift velocity of the circulating electrons and ν_e is the electron-ion collision frequency.

The trapped electron integrals are

$$A_{1,2} = -\frac{2}{\sqrt{\pi}} \int_{\sqrt{\nu_*}}^{\infty} dt \frac{\sqrt{t}\exp(t)}{1 - \frac{\omega_{de}}{\omega+\omega_E} t + \frac{i\nu_e/\varepsilon}{\omega+\omega_E} t^{-3/2}} \left\{ \begin{array}{c} t - 3/2 \\ 1 \end{array} \right.$$

The effective collision frequency of the trapped particles is (for the inverse aspect ratio $\varepsilon = r/R$)

$$\nu_* = \frac{\sqrt{2} v_{the} \left(Rq\right)}{\varepsilon^{3/2} \nu_e}$$

(?) and the electron curvature drift frequency is

$$\omega_{de} = \frac{\omega_{*e} L_n}{R}.$$

In this version of the theory of the ITG modes the equation for the potrntial can be written schematically

$$\frac{d^2\phi(x)}{dx^2} + Q^{ITG}(x)\,\phi(x) = 0 \tag{28}$$

where $Q^{ITG}(x)$ is obtained from the expression of $b_x(x)$ Eq.(25) after replacing the expressions Eqs.(26) and (27).

Using the boundary condition of energy propagation toward the shear damping radial region, the eigenfunction can be obtained. A typical spatial extention of the eigenmodes is several units of ρ_s $(= c_s/\Omega_i)$ which is in general larger than the rational surface spacing $\Delta = (k_y \widehat{s})^{-1}$.

A similar form of the equation for the ITG eigenmode is obtained in the ballonning representation of in toroidal geometry. In this case the asymptotic form of the function $f(\eta)$ which is introduced by the balooning representation is

$$f_+(\eta) = A_+ \exp\left(\frac{iqk_\perp \rho \widehat{s} \omega L_n}{2\varepsilon_n c_s}\eta^2\right)$$

with $\widehat{s} = rq'/q$ and $\varepsilon_n = L_n/R$. This is the condition of outward energy propagation (radiation) identical to the asymptotic condition imposed for the slab drift mode. The shear damping of these modes is effective except for strong toroidal localization in the bad curvature region of the torus. The solution is [**?**]

$$f_n(\eta) = H_n(\sigma_n \eta) \exp\left(-\frac{1}{2}\sigma_n \eta^2\right)$$

where

$$\sigma_n = \frac{q\omega L_n}{\varepsilon_n c_s}\left[\frac{2\varepsilon_n k_\perp \rho c_s}{\omega L_n}\left(\frac{1}{2} - \widehat{s}\right) - k_\perp^2 \rho^2 \widehat{s}^2\right]^{1/2}.$$

NOTE. We can comapre this parameter σ with the result of **Pearlstein Berk**

$$\sigma^{P-B} = \frac{1}{L_s \overline{\rho}_i}\sqrt{\frac{1+\eta_e}{I_0 \exp(-b)}\frac{\omega}{\omega - \omega_i^*}}\left(\frac{v_{th,i}}{\omega/k_\parallel}\right)$$

where

$$b \equiv k_y^2 \rho_i^2$$

$$\overline{\rho}_i^2 \equiv \rho_i^2 \frac{d}{db}\ln\left[I_0 \exp(-b)\right]$$

END

11 Unstable universal drift eigenmodes ITG

The work of **Cheng Chen, 1980.**

The perturbation

$$\Psi(r,\theta,\varphi,t)=\sum_l \widehat{\phi}_l(s)\exp(im_0\theta+il\theta-in\varphi-i\omega t)$$

We extract the time and a major harmonic part

$$\exp(-i\omega t)\times\exp(im_0\theta-in\varphi)$$

but this still leaves a variation over the surface (in poloidal angle θ) which is further expanded

$$\exp(il\theta)$$

The radial coordinate is now the *eikonal*

$$s=\frac{r-r_0}{\Delta r_s}$$

a distance from the resonant surface r_0 normalized to the standard radial interval between two resonant surfaces

$$\Delta r_s=\frac{1}{k_\theta\widehat{s}}$$

where

$$\begin{aligned}\widehat{s}&=\frac{rq'}{q}\\ k_\theta&=\frac{m_0}{r}\end{aligned}$$

and the relation between the wavenumbers is

$$m_0=nq(r_0)$$

which we use to see as $q(r_0)=m_0/n$.

It is assumed that the few harmonis with indice l that will be needed in the Fourier expansion after extracting $m_0\theta-nq(r)$ are much less than m_0,

$$|l|\ll m_0$$

in other words only few other Fourier modes will be necessary to describe the electric perturbation beside the basic harmonic one, m_0.

The eigenmode equation

$$\begin{aligned}&\left[b_\theta\left(\widehat{s}^2\frac{d^2}{ds^2}-1\right)\right.\\ &+\overline{Q}(s,l)\\ &\left.-\varepsilon_n\frac{T}{\Omega}\right]\widehat{\phi}_l(s)\\ =&\ 0\end{aligned}$$

where the Landau damping is present through

$$\begin{aligned}\overline{Q} &= \frac{1-\Omega}{\Omega+\frac{1}{\tau}}\left[1+\xi_e Z\left(\xi_e\right)\right] \\ &\quad -\tau\left[1+\xi_i Z\left(\xi_i\right)\right]\end{aligned}$$

The coupling between different l modes is introduced through the *operator* T, whose expression is derived from *ion drifts* in ∇B and *curvature*.

$$\begin{aligned}T\widehat{\phi}_l\left(s\right) &= \widehat{\phi}_{l+1}+\widehat{\phi}_{l-1} \\ &\quad +\widehat{s}\frac{\partial}{\partial r}\left[\widehat{\phi}_{l+1}\left(s\right)-\widehat{\phi}_{l-1}\left(s\right)\right]\end{aligned}$$

And

$$b_\theta = k_\theta^2\rho_s^2$$

$$\tau = \frac{T_e}{T_i}$$

$$\rho_s = \frac{c_s}{\Omega_{ci}}$$

$$c_s^2 = \frac{T_e}{m_i}$$

$$\varepsilon_n = \frac{L_n}{R}$$

$$\xi_i = \left(\frac{\tau}{2}\right)^{1/2}\frac{\Omega}{|s-l|}\eta_s$$

$$\xi_e = \frac{\xi_i}{d}$$

$$d = \left(\frac{m_e}{m_i}\frac{T_i}{T_e}\right)^{1/2} = \frac{\sqrt{T_i/m_i}}{\sqrt{T_e/m_e}} = \frac{v_{th,i}}{v_{th,i}}$$

$$\begin{aligned}\eta_s &= \frac{q}{\varepsilon_n}\sqrt{b_\theta} \\ &= q\frac{R}{L_n}k_\theta\rho_s\end{aligned}$$

The plasma dispersion function is $Z\left(\xi\right)$.

$$\Omega = \frac{\omega}{\omega_{*e}}$$

the condition is that the frequency associated to the ion's drift is much smaller than the frequency of the mode

$$\frac{L_n}{R}\frac{\omega_{*e}}{\omega} \ll 1$$

An order of magnitude

$$
\begin{aligned}
|m_0| &\sim |n| \sim \frac{L_n}{\rho_s} \gg 1 \\
&\sim O\left(10^2\right)
\end{aligned}
$$

A new spatial (normalized) variable is introduced

$$z = s - l$$

$$
\begin{aligned}
\widehat{\phi}_l(s) &= \Phi(z) \\
\widehat{\phi}_{l\pm 1}(s) &= \Phi(z \mp 1)
\end{aligned}
$$

It is assumed that there is *no phase shift* between eigenmodes residing on neighbor resonant surfaces: $\Phi(z+1)$ and $\Phi(z-1)$.

The equation becomes

$$
\begin{aligned}
& b_\theta \left(\widehat{s}^2 \frac{d^2}{dz^2} - 1 \right) \Phi(z) \\
& + \overline{Q} \Phi(z) \\
& - \varepsilon_n \frac{1}{\Omega} \left\{ \Phi(z+1) + \Phi(z-1) + \widehat{s} \frac{d}{dz} \left[\Phi(z-1) - \Phi(z+1) \right] \right\} \Phi(z) \\
= {} & 0
\end{aligned}
$$

There is aboundary condition: for large z the outgoing wave decays asymptotically because of the onset of the ion Landau damping.

(actually this is at the *ion turning point* on the radius, up and down relative to the resonant surface).

The approximation to the plasma dispersion function

$$
\begin{aligned}
& 1 + \xi_i Z(\xi_i) \\
= {} & -\frac{1}{2\xi_i^2} + i\sqrt{\pi}\xi_i \exp\left(-\xi_i^2\right) \\
\text{for } \xi_i \gg {} & 1
\end{aligned}
$$

the equation becomes

$$
\begin{aligned}
& b_\theta \left(\widehat{s}^2 \frac{d^2}{dz^2} - 1 \right) \Phi(z) \\
& + \left(\frac{1-\Omega}{\Omega + \frac{1}{\tau}} + \frac{z^2}{\Omega^2 \eta_s^2} \right) \Phi(z) \\
& - \varepsilon_n \frac{1}{\Omega} \left\{ \Phi(z+1) + \Phi(z-1) + \widehat{s} \frac{d}{dz} \left[\Phi(z-1) - \Phi(z+1) \right] \right\} \Phi(z) \\
& - \left(\frac{1-\Omega}{\Omega + \frac{1}{\tau}} \xi_e Z(\xi_e) - i\sqrt{\pi}\tau \xi_i \exp\left(-\xi_i^2\right) \right) \Phi(z) \\
= & \ 0
\end{aligned}
$$

A Fourier transformation which involves the damping term

$$
\begin{aligned}
W\widehat{\phi}(\eta) = & \ \Omega^2 \\
& \times \int \eta_s^2 \left(\frac{1-\Omega}{\Omega + \frac{1}{\tau}} \xi_e Z(\xi_e) - i\sqrt{\pi}\tau \xi_i \exp\left(-\xi_i^2\right) \right) \Phi(z) \\
& \times \exp(-i\eta z) \ dz
\end{aligned}
$$

We understand why the parathesis which is the notation in **Cheng Chen**

$$
\overline{W}(z) = \frac{1-\Omega}{\Omega + \frac{1}{\tau}} \xi_e Z(\xi_e) - i\sqrt{\pi}\tau \xi_i \exp\left(-\xi_i^2\right)
$$

is considered a function of z,

$$
\begin{aligned}
z &= s - l \\
&\sim \text{radius } r
\end{aligned}
$$

It is because the ion parameter

$$
\xi_i = \left(\frac{\tau}{2} \right)^{1/2} \frac{\Omega}{|s-l|} \eta_s
$$

depends on $s - l$. And

$$
\begin{aligned}
\eta_s &= \frac{q}{\varepsilon_n} \sqrt{b_\theta} \\
&= q \frac{R}{L_n} k_\theta \rho_s
\end{aligned}
$$

is a function of the distance on the minor radius, through q.

The other term in the Fourier transformation is

$$
\begin{aligned}
Q(\Omega, \eta) = & \ \Omega^2 \eta_s^2 \left[b_\theta \left(\widehat{s}^2 \eta_s^2 + 1 \right) \right. \\
& - \frac{1-\Omega}{\Omega + 1/\tau} \\
& \left. + 2 \frac{\varepsilon_n}{\Omega} \left(\cos\eta + \widehat{s} \sin\eta \right) \right]
\end{aligned}
$$

and, after the Fourier transformation the equation becomes

$$\left[\frac{d^2}{d\eta^2}+Q\left(\Omega,\eta\right)-W\right]\widehat{\phi}\left(\eta\right)=0$$

If the dissipation term is ignored the equation

$$\left[\frac{d^2}{d\eta^2}+Q\left(\Omega_0,\eta\right)\right]\widehat{\phi}_0\left(\eta\right)=0$$

The boundary condition for the function is

$$\begin{aligned} \widehat{\phi}\left(\eta\right) &\rightarrow \exp\left[i\Omega_0\eta_s\sqrt{b_\theta}\widehat{s}\frac{\eta^2}{2}\right] \\ \text{as} \quad \left|\eta\right| &\rightarrow \infty \end{aligned}$$

This means that the energy is radiated outward.

For strong shear

$$\widehat{s}>\frac{1}{2}$$

there is possibility of bounded eigenmodes in the local minima of the potential $Q\left(\eta\right)$

$$\frac{dQ\left(\eta\right)}{d\eta}=0$$

$$\begin{aligned} & b_\theta\widehat{s}^2\eta_0+\frac{\varepsilon_n}{\Omega}\left[\left(\widehat{s}-1\right)\sin\eta_0+\widehat{s}\eta_0\cos\eta_0\right] \\ = & \ 0 \end{aligned}$$

This effectively determines a value

$$\eta_0\sim\text{Fourier conjugate of } z=s-l\sim r$$

The departure relative to this variable η_0 is

$$\eta-\eta_0\equiv\theta \quad \text{(a new variable)}$$

Now the potential is expanded around η_0,

$$\left[\frac{d^2}{d\theta^2}+Q\left(\eta_0\right)+\frac{d^2Q\left(\eta\right)}{d\eta^2}\frac{\theta^2}{2}-W\right]\widehat{\phi}\left(\eta\right)=0$$

This is transformed Fourier back to the variable t which is conjugate to θ.

$$\begin{aligned} \phi\left(t\right) &= \frac{1}{2\pi}\int d\theta\ \widehat{\phi}\left(\eta\right)\exp\left(i\theta t\right) \\ &= \Phi\left(t\right)\exp\left(-i\eta_0 t\right) \end{aligned}$$

then

$$\left(-\frac{1}{2}\frac{d^2Q(\eta)}{d\eta^2}\frac{d^2}{dt^2}+Q(\eta_0)-t^2-\Omega^2\eta_s^2\overline{W}(t)\right)\phi(t)=0$$

The following normalizations are made

$$y=t\left(-\frac{1}{2}\frac{d^2Q(\eta_0)}{d\eta^2}\right)^{-1/4}$$

$$\lambda=Q(\eta_0)\left(-\frac{1}{2}\frac{d^2Q(\eta_0)}{d\eta^2}\right)^{-1/2}$$

$$\Lambda=\Omega^2\eta_s^2\overline{W}(t)\left(-\frac{1}{2}\frac{d^2Q(\eta_0)}{d\eta^2}\right)^{-1/2}$$

The equation is

$$\left(\frac{d^2}{dy^2}+\lambda-y^2-\Lambda\right)\phi(y)=0$$

First approximation consists of taking Λ smaller than y and λ.
The remaining form of the equation is streated paertcurbatively

$$\phi_0(y)=\exp\left(-\frac{y^2}{2}\right)$$

and the dispersion relation

$$Q(\eta_0)\left(-\frac{1}{2}\frac{d^2Q(\eta_0)}{d\eta^2}\right)^{-1/2}=1$$

The dispersion relation must be analysed around the point of minimum (local) of the potential.
This is the solution of the equation that expresses the fact that the first derivative is zero

$$b_\theta\widehat{s}^2\eta_0+\frac{\varepsilon_n}{\Omega}\left[(\widehat{s}-1)\sin\eta_0+\widehat{s}\eta_0\cos\eta_0\right]=0$$

The range of parameters exhibits the following inequalities

$$\begin{aligned}\left|\frac{\varepsilon_n}{\Omega}\right| &> |b_\theta\widehat{s}|\\ |\eta_0| &> 1\end{aligned}$$

The estimation for the root is

$$\begin{aligned}\eta_0 &\approx \frac{\pi}{2}\\ \text{for }\widehat{s} &\sim 1\end{aligned}$$

This allows to calculate the potential

$$Q(\eta_0) = \Omega^2 \eta_s^2 \left\{ 1 + b_\theta \left[1 + \widehat{s}^2 \left(\frac{\pi}{2} \right)^2 \right] - \frac{1 + \frac{1}{\tau}}{\Omega + \frac{1}{\tau}} + \widehat{s}\pi \frac{\varepsilon_n}{\Omega} \right\}$$

and it is now possible to calculate the factor

$$\left(-\frac{1}{2} \left. \frac{d^2 Q}{d\eta^2} \right|_{\eta_0} \right)^{1/2} \approx \Omega \eta_s \left[\widehat{s} \frac{\varepsilon_n}{\Omega} \frac{\pi}{2} - b_\theta \widehat{s}^2 \right]^{1/2}$$

When the last factor is small

$$\widehat{s} \frac{\varepsilon_n}{\Omega} \frac{\pi}{2} - b_\theta \widehat{s}^2 \ll 1$$

and there is a large difference of temperature, we can obtain the solution to the equation of dispersion

$$\Omega_0 = \frac{(1 - \widehat{s}\varepsilon_n \pi)}{1 + b_\theta \left[1 + \widehat{s}^2 \left(\frac{\pi}{2} \right)^2 \right]}$$

12 Kinetic ITG: numeric Xu Rosenbluth

The Vlasov equation

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + v_\parallel \frac{\partial}{\partial l_\parallel} + i\omega_{D,j} \right) h(l) \\ = \quad & \left(\frac{\partial}{\partial t} + i\omega_{*,j}^T \right) \frac{q_j}{T_j} F_{M,j} J_0 \phi \\ & + \langle \exp(iL) \;\; C(h_j) \;\; \exp(-iL) \rangle \end{aligned}$$

The finite Larmor radius

$$J_0 \equiv J_0 \left(k_\perp \rho_j \right)$$

and

$$\begin{aligned} & \langle \exp(iL) \;\; C(h_j) \;\; \exp(-iL) \rangle \\ \equiv \quad & \text{gyro-averaged Fokker Planck collision operator} \end{aligned}$$

The distribution function consists of the zeroth order, adiabatic, plus the perturbation

$$f = -\frac{q_j \phi(l)}{T_j} F_{M,j} + h_j$$

This function is also represented by

$$\widetilde{f} = f \;\; \exp\left[inS(r, \chi, \varphi) - i\omega t \right]$$

where

$$
\begin{aligned}
\varphi &\equiv \text{toroidal angle} \\
\chi &= \frac{l}{qR} \equiv \text{the poloidal angle}
\end{aligned}
$$

$$
\begin{aligned}
S &\equiv \text{eikonal function} \\
\mathbf{k}_\perp &= \boldsymbol{\nabla}_\perp S
\end{aligned}
$$

$$
\begin{aligned}
\omega_{*,j}^T &= -\frac{T_j}{q_j B}\frac{1}{F_{M,j}}\left(\widehat{\mathbf{n}} \times \mathbf{k}\right) \cdot \boldsymbol{\nabla} F_{M,j} \\
&= \omega_{*,j}\left[1 + \eta_j\left(\frac{\epsilon}{T_j} - \frac{3}{2}\right)\right]
\end{aligned}
$$

$$\omega_{*,j} = -k_\chi \frac{\rho_j v_{th,j}}{L_{j,n}}$$

$$L_{n,j}^{-1} = -\frac{1}{n_j}\frac{dn_j}{dr}$$

$$v_{th,j} = \sqrt{\frac{T_j}{m_j}}$$

$$\eta_j = \frac{d\ln n_j}{d\ln T_j}$$

$$\omega_{D,j} = \widehat{\omega}_j \frac{2\epsilon - \mu B}{T_j}$$

$$\widehat{\omega}_j = -\frac{T_j}{q_j B}\ \left(\widehat{\mathbf{n}} \times \boldsymbol{\kappa}\right) \cdot \mathbf{k}$$

$$\boldsymbol{\kappa} = \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{n}}$$

$$\mu_j = \frac{m_j v_\perp^2}{2B}$$

$$\epsilon = \frac{m_j v^2}{2}$$

Other variables

$$
\begin{aligned}
\mathbf{k}_\perp &= n\boldsymbol{\nabla} S \\
&= -k_\chi\left[\widehat{\mathbf{e}}_\chi + \widehat{s}\left(\chi - \chi_0\right)\widehat{\mathbf{e}}_r\right]
\end{aligned}
$$

$$\left(\widehat{\mathbf{n}} \times \boldsymbol{\kappa}\right) \cdot \mathbf{k} = \frac{k_\chi}{R_0}\left[\cos\chi + \widehat{s}\left(\chi - \chi_0\right)\sin\chi\right]$$

$$\widehat{s} = \frac{d\ln q}{d\ln r} = \frac{rq'}{q}$$

$$\frac{\partial}{\partial l_{\|}} = \frac{1}{qR}\frac{\partial}{\partial \theta}$$

The collisions are important.

The transition from trapped to passing can have a destabilizing effect on the wave.

$$\begin{aligned}
& \langle \exp(iL)\ \ C(h_j)\ \ \exp(-iL) \rangle \\
= \quad & \frac{\partial}{\partial v_\perp^2}\left(\nu_{s\perp}\ v^2 h_j\right) + \frac{\partial}{\partial v_\|}\left(\nu_{s\|}\ v_\| h_j\right) + \frac{1}{2}\frac{\partial^2}{\partial\left(v_\perp^2\right)^2}\left(\nu_\perp\ v^4 h_j\right) + \frac{1}{2}\frac{\partial^2}{\partial\left(v_\|\right)^2}\left(\nu_\|\ v^2 h_j\right) \\
& + \frac{\partial^2}{\partial\left(v_\perp^2\right)\partial\left(v_\|\right)}\left(\nu_{\|\perp}\ v^3 h_j\right) - \frac{k_\perp^2}{2\Omega_{c,j}^2}\left(2G + \frac{v_\perp^2}{v^2} H\right) h_j \\
& + F_{M,j}\frac{1}{n_0 v_{th,j}^2}\left[v_\| J_0 \int d^3v\ v_\| J_0\ \ F h_j + v_\perp J_1 \int d^3v\ v_\perp J_1\ F h_j\right] \\
& + \left(\frac{v^2}{2v_{th,j}^2} - \frac{3}{2}\right)\frac{2}{3}F_{M,j}\frac{1}{n_0 v_{th,j}^2}\ J_0 \int d^3v \left(v^2 F - 3G - H\right) J_0 h_j
\end{aligned}$$

The notations

$$x \equiv x^{\alpha/\beta} = \frac{v^2\ \text{ (of particle } \alpha)}{2v_{th,\beta}^2}$$

(remember that $v_{th,j}$ does not contain 2)

$$F = \nu_0^{\alpha/\beta}\left[\left(1 + \frac{m_\alpha}{m_\beta}\right)\Psi\left(x^{\alpha/\beta}\right)\right]$$

$$G = \nu_0\ \frac{1}{2}v^2\left[\Psi(x)\left(1 - \frac{1}{2x}\right) + \frac{d\Psi(x)}{dx}\right]$$

$$H = -\nu_0\ \frac{1}{2}v^2\left[\Psi(x)\left(1 - \frac{3}{2x}\right) + \frac{d\Psi(x)}{dx}\right]$$

$$\nu_0^{\alpha/\beta} = \frac{4\pi q_\alpha q_\beta}{m_\alpha^2}\ln\Lambda_{\alpha\beta}\frac{n_\beta}{v^3}$$

Cylindrical coordinates in the velocity space

$$\begin{aligned}
& v_\perp, v_\|, \zeta \\
\zeta \quad \equiv \quad & \text{gyro-angle}
\end{aligned}$$

$$\frac{\partial}{\partial \mathbf{v}}\ \frac{\mathbf{v}}{v^3} = 4\pi\delta(\mathbf{v})$$

$$\frac{\partial}{\partial \mathbf{v}}\frac{\mathbf{v}\ \mathbf{v}}{v^4} = \frac{\mathbf{v}}{v}4\pi\delta(\mathbf{v})$$

$$
\begin{aligned}
\Delta_{\mathbf{v}} &= \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp\frac{\partial}{\partial v_\perp}+\frac{\partial^2}{\partial v_\parallel^2}+\frac{1}{v_\perp^2}\frac{\partial^2}{\partial\zeta^2} \\
&= 4\frac{\partial^2}{\partial\left(v_\perp^2\right)}\left[v_\perp^2\ \ (...)\right]-4\frac{\partial}{\partial\left(v_\perp^2\right)} \\
&\quad+\frac{\partial^2}{\partial v_\parallel^2}+\frac{1}{v_\perp^2}\frac{\partial^2}{\partial\zeta^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial v} &= \frac{\mathbf{v}}{v}\frac{\partial}{\partial\mathbf{v}} \\
&= 2\frac{v_\perp^2}{v}\frac{\partial}{\partial\left(v_\perp^2\right)}+\frac{v_\parallel}{v}\frac{\partial}{\partial v_\parallel}
\end{aligned}
$$

The collisions are simulated numerically.
Compare with **Dimits**, etc.

13 Trapped ion modes Xu Rosenbluth

This is *temperature gradient drive trapped ion instabilities.*

There is a perturbation of the magnetic field in the parallel direction

$$
\delta B_\parallel
$$

This occurs through a perturbation of the perpendicular (on $\mathbf{B}$) component of the *magnetic potential.*

The parallel projection of the relationship

$$
-\frac{\partial\mathbf{A}}{\partial t}-\boldsymbol{\nabla}\phi=\mathbf{E}
$$

is

$$
\begin{aligned}
-\left(-i\omega\right)A_\parallel-\nabla_\parallel\Lambda &= 0 \\
A_\parallel &= \frac{1}{i\omega}\nabla_\parallel\Lambda
\end{aligned}
$$

This component creates magnetic field perturbations that are perpendicular to the magnetic surface.

There is a substitution

$$
\widetilde{f}=f\exp\left[inS\left(r,\theta,\xi\right)-i\omega t\right]
$$

where

$$
\begin{aligned}
f &= -\frac{e_j\phi}{T_j}F_{Mj} \\
&\quad+g_jJ_0\left(k_\perp\rho_i\right)
\end{aligned}
$$

and the *eikonal* is

$$\mathbf{k}_\perp = \boldsymbol{\nabla} S$$

The equation for the new function g_j,

$$\begin{aligned} & \left(\omega - \omega_{Di} + i v_\parallel \frac{\partial}{\partial l_\parallel} \right) g_j \\ = \quad & (\omega - \omega_{*T}) \frac{e_j}{T_j} F_{Mj} \left[J_0 \left(\phi - v_\parallel A_\parallel \right) - i v_\perp A_\perp J_1 \right] \\ & + i C \left(g_j \right) \end{aligned}$$

where

$$J_l \left(k_\perp \rho_j \right)$$

The collisional operator is the gyroaveraged *pitch angle* scattering

$$C \left(g_j \right) = \nu \frac{\left| v_\parallel \right|}{B} \frac{\partial}{\partial \mu} \left(\left| v_\parallel \right| \mu \frac{\partial}{\partial \mu} g_j \right)$$

The parameters are

$$\begin{aligned} \omega_{*T} \quad &= \quad \frac{T_j}{e_j B} \frac{1}{F_{Mj}} \left(\widehat{\mathbf{n}} \times \boldsymbol{\nabla} F_{Mj} \right) \cdot \mathbf{k} \\ &= \quad \omega_{*j} \left[1 + \eta_j \left(\frac{\epsilon}{T_j} - \frac{3}{2} \right) \right] \end{aligned}$$

$$\omega_{Dj} = \omega_{kj} - \frac{\mu}{e_j B^2} \sum_\alpha \left(\widehat{\mathbf{n}} \times \boldsymbol{\nabla} P_\alpha \right) \cdot \mathbf{k}$$

$$\epsilon = \frac{m_j v^2}{2}$$

$$\mu = \frac{m_j v_\perp^2}{2B}$$

$$\begin{aligned} \omega_{kj} \quad &= \quad \widehat{\omega}_{kj} \frac{2\epsilon - \mu B}{T_j} \\ &= \quad \widehat{\omega}_{kj} \frac{v^2 - v_\perp^2/2}{T_j/m_j} = \widehat{\omega}_{kj} \frac{v_\parallel^2 + v_\perp^2/2}{T_j/m_j} \end{aligned}$$

$$\widehat{\omega}_{kj} = -\frac{T_j}{e_j B} \left(\widehat{\mathbf{n}} \times \boldsymbol{\kappa} \right) \cdot \mathbf{k}$$

$$\begin{aligned} \boldsymbol{\kappa} \quad &= \quad \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} \\ &= \quad \text{curvature} \end{aligned}$$

Collisions

$$\nu_{ei}^{(E)} = 2 \left(\frac{T_e}{\epsilon} \right)^{3/2} \frac{\omega_{pe}^2 e^2 \sqrt{m_e}}{\left(2 T_e \right)^{3/2}} \ln \Lambda \left\{ 1 + H \left(\sqrt{\frac{\epsilon}{T_e}} \right) \right\}$$

$$\nu_{ii}^{(E)} = 2\left(\frac{T_i}{\epsilon}\right)^{3/2} \frac{\omega_{pi}^2 e^2 \sqrt{m_i}}{(2T_i)^{3/2}} \ln\Lambda \; H\left(\sqrt{\frac{\epsilon}{T_i}}\right)$$

and

$$\begin{aligned} H(z) &= \frac{1}{\sqrt{\pi}}\frac{1}{z}\exp\left(-z^2\right) \\ &\quad + \left(1 - \frac{1}{2z^2}\right)\frac{2}{\sqrt{\pi}}\int_0^z dt \exp\left(-t^2\right) \end{aligned}$$

$$\begin{aligned} \omega_{*j} &= -k_\theta \frac{\rho_j v_{th,j}}{2}\frac{1}{L_{nj}} \\ v_{th,j} &= \sqrt{\frac{2T_j}{m_j}} \\ \rho_j &= \frac{v_{th,j}}{\Omega_j} \end{aligned}$$

$$\frac{1}{L_{nj}} = -\frac{d}{dr}\ln n_j$$

$$k_\theta = -\frac{m}{r}$$

$$\eta_j = \frac{L_{nj}}{L_{Tj}}$$

It is made a substitution

$$\begin{aligned} g_j &= \left(1 - \frac{\omega_{*j}^T}{\omega}\right)\frac{e_j}{T_j} F_{Mj} \; J_0 \Lambda \\ &\quad + h_j \end{aligned}$$

remember that Λ is introduced by substituting $A_\parallel$ by Λ. The new function

$$\psi = \phi - \Lambda$$

and the change

$$\begin{aligned} &\left(\omega + i v_\parallel \frac{\partial}{\partial l} - \omega_{Dj}\right) h_j \\ = &\left(\omega - \omega_{*j}^T\right)\frac{e_j}{T_j} F_{Mj}\left[J_0\left(\psi - \frac{\omega_D}{\omega}\Lambda\right) + \frac{v_\perp^2}{2\Omega_{cj}}\delta B_\parallel\right] \\ &+ iC\left(h_j\right) \end{aligned}$$

Two approximations have been made

$$\begin{aligned} J_1 &= \frac{k_\perp \rho_i}{2}\frac{v_\perp}{v_{th,i}} \\ J_0 &= 1 - \left(\frac{k_\perp \rho_i}{2}\right)^2 \left(\frac{v_\perp}{v_{th,i}}\right)^2 \end{aligned}$$

The function h_j is obtained from this equation then integrated over the velocity space and the result is used in the neutrality

$$\begin{aligned}
& -n_0 \frac{e^2}{T_i}\left(1-\frac{1}{\tau}\right)\psi \\
& -n_0 \frac{e^2}{T_i}\frac{k_\perp^2\rho_i^2}{2}\left[1-\frac{\omega_{*i}}{\omega}\left(1+\eta_i\right)\right]\Lambda \\
& +\sum_j e_j \int d^3v \; H_j J_0 \\
= \; & 0
\end{aligned}$$

where

$$\tau \equiv \frac{T_e}{T_i}$$

The projection of the Ampere law on the direction $\widehat{\mathbf{e}}_\perp$, after the introduction of the function h_j,

$$\begin{aligned}
\delta B_\parallel &= -ik_\perp A_\perp \\
&= \sum_j \mu_0 \frac{n_j e_j}{B}\frac{\omega_{*j}}{\omega}\left(1+\eta_j\right)\Lambda \\
& \quad -B_1
\end{aligned}$$

The component B_1 is

$$B_1 = \mu_0 \sum_j e_j \int d^3v \frac{v_\perp^2}{2\Omega_{cj}} h_j$$

the same equation for h_j written for all species j is multiplied with the charge e_j and it is integrated over velocity space, followed by adding them. The result will also include the *neutrality* condition and the parallel projection of the Ampere's law.

Then

$$\begin{aligned}
& \frac{1}{\mu_0}\frac{1}{\omega^2}\frac{\partial}{\partial l_\parallel}\left(k_\perp^2 \frac{\partial \Lambda}{\partial l_\parallel}\right) \\
= \; & -n_0 \frac{e^2}{T_i}\frac{k_\perp^2\rho_i^2}{2}\left[1-\frac{\omega_{*i}}{\omega}\left(1+\eta_i\right)\right](\psi+\Lambda) \\
& -\sum_j n_j \frac{e_j^2}{T_j}\frac{\omega_{*j}\widehat{\omega}_{Dj}}{\omega^2}\left(1+\eta_j\right)\Lambda \\
& -\sum_j n_j \frac{e_j}{B}\frac{\omega_{*j}}{\omega}\left(1+\eta_j\right)\delta B_\parallel \\
& +\sum_j e_j \int d^3v \frac{\omega_{Dj}}{\omega} J_0 h_j
\end{aligned}$$

The equation for h_j is

$$\begin{aligned}
& \left(\omega-\omega_{Dj}+iv_{\|}\frac{\partial}{\partial_{\|}}\right)h_j \\
= \quad & \left(\omega-\omega_{*j}^{T}\right)\frac{e_j}{T_j}F_{Mj}\left[J_0\left(\psi+\frac{\omega_k}{\omega}\Lambda\right)-\frac{v_{\perp}^2}{2\Omega_{cj}}B_1\right] \\
& +iC\left(h_j\right)
\end{aligned}$$

The function h_j is constant along the magnetic field line

$$v_{\|}\frac{\partial h_j}{\partial l_{\|}}\approx 0$$

For trapped particles, one takes the bounce average

$$\begin{aligned}
\left(\omega-\overline{\omega}_{Dj}\right)h_j \quad = \quad & \left(\omega-\omega_{*}^{T}\right)\frac{e_j}{T_j}F_{Mj} \\
& \times\left[J_0\left(\overline{\psi}+\overline{\frac{\omega_k}{\omega}\Lambda}\right)-\overline{\frac{v_{\perp}^2}{2\Omega_{cj}}B_1}\right] \\
& +i\overline{C\left(h_j\right)}
\end{aligned}$$

where the *bounce* average is

$$\overline{G}=\frac{\int\frac{dl}{\left|v_{\|}\right|}G}{\int\frac{dl}{\left|v_{\|}\right|}}$$

About the boundary condition.

The distribution function is *continuous* at the limit between the trapped and untrapped particles.

The limit between the *trapped* and *untrapped* particles is

$$\mu=\frac{\epsilon}{B_{\max}}$$

At this limit the nonadiabatic response of the *circulating* particles is zero.

This condtion, that the *nonadiabatic* part of the distribution function is negligible can be written

$$h_j\sim O\left(\frac{\omega}{\omega_{Trapped,j}}\right)\ll 1$$

The boundary condition is

$$\left.h_j\left(\epsilon,\mu\right)\right|_{\mu=\epsilon/B_{\min}}=0$$

The boundary condition for deeply trapped aprticles, $v_{\|}\ll v_{th,i}$,

$$\left|\frac{\partial h_j}{\partial\mu}\right|_{\mu=\epsilon/B_{\min}}<\infty$$

The parallel derivation

$$\frac{\partial}{\partial l_{\parallel}} = \frac{1}{qR}\frac{\partial}{\partial \theta}$$

Xu and Rosenbluth solve the problem of the eigenmode by constructing a functional and applying a variational method.

- the quasineutrality equation

  $$-n_0 \frac{|e|^2}{T_i}\left(1+\frac{1}{\tau}\right)\psi - ...$$

  is multiplied by

  $$\frac{\psi}{B_0}$$

- the definition of the term B_1 (appearing in the equation for $\delta B_{\parallel}$) ,

  $$B_1 = \mu_0 \sum_j e_j \int d^3v \; \frac{v_{\perp}^2}{2\Omega_{cj}} h_j$$

  is mutiplied by

  $$\frac{B_1}{B}$$

- the equation for Λ the scalar "potential" that replaces the parallel component of the magnetic potential $A_{\parallel} \sim \nabla_{\parallel}\Lambda$,

  $$\frac{1}{\mu_0}\frac{1}{\omega^2}\frac{\partial}{\partial l_{\parallel}}\left(k_{\perp}^2 \frac{\partial \Lambda}{\partial l_{\parallel}}\right) = ...$$

  is multiplied by

  $$\frac{\Lambda}{B_0}$$

- the equation for the non-adiabatic components h_j averaged over a bounce

  $$(\omega - \overline{\omega}_{Dj})\, h_j = \left(\omega - \omega_*^T\right)\frac{e_j}{T_j} F_{Mj} \times [...]$$

  is multiplied with

  $$\frac{h_j}{F_{Mj}\left(\omega - \omega_*^T\right)}$$

The new equations are integrated over a magnetic field line and over the velocity space.

Finally they are added.

The following quantity is the functional

$$
\begin{aligned}
Q \quad = \quad & \frac{1}{\mu_0}\frac{1}{\omega^2}\int \frac{dl}{B}\left(k_\perp \frac{\partial \Lambda}{\partial l_\parallel}\right)^2 \\
& - n_i \frac{|e|^2}{T_i}\left[1 - \frac{\omega_{*i}}{\omega}\left(1+\eta_i\right)\right]\int \frac{dl}{B}\left(\frac{k_\perp^2 \rho_i^2}{2}\right)\left(2\psi\Lambda + \Lambda^2\right) \\
& - \sum_j n_j \frac{e_j^2}{T_j}\frac{\omega_{*j}}{\omega}\left(1+\eta_j\right)\int \frac{dl}{B}\widehat{\omega}_{kj}\Lambda^2 \\
& - n_0 \frac{|e|^2}{B}\left(1+\frac{1}{\tau}\right)\int \frac{dl}{B}\psi^2 \\
& - \frac{\mu_0}{2}\sum_j \frac{T_j}{m_j^2}\int_{trapped} \frac{dl}{|v_\parallel|} d\epsilon d\mu \left[\frac{h_j^2\left(\omega - \overline{\omega}_{Dj}\right)}{F_{Mj}\left(\omega - \omega_{*j}^T\right)} - i\frac{h_j \overline{C}\left(h_j\right)}{F_{Mj}\left(\omega - \omega_{*j}^T\right)}\right] \\
& + \frac{1}{\mu_0}\int \frac{dl}{B}\ \left(B_1\right)^2 \\
& + 2\sum_j e_j \int \frac{dl}{B}\int_{trapped} d^3v\ h_j \left[J_0\left(\psi + \frac{\omega_{kj}}{\omega}\Lambda\right) - \frac{v_\perp^2}{2\Omega_{cj}}B_1\right]
\end{aligned}
$$

The quantity Q is variational and taking variations to h_j, ψ, Λ and B_1 we re-obtain the equations we used to construct Q.

It is possible to simplify this form by taking into account the equation for h_j averaged over the bounce.

Then the last term is replaced with the other components of that equation which further will be absorbed in the term that has the similar structure in Q.

The result is then another expression for Q. The terms have the meaning

- The first
$$\frac{1}{\mu_0}\frac{1}{\omega^2}\int \frac{dl}{B}\left(k_\perp \frac{\partial \Lambda}{\partial l_\parallel}\right)^2$$
energy required to bend the magnetic field lines
- the second
$$\frac{1}{\mu_0}\int \frac{dl}{B}\ \left(B_1\right)^2$$
is the work done to *compress* the magnetic field and the plasma.
- the third
$$-\sum_j n_j \frac{e_j^2}{T_j}\frac{\omega_{*j}}{\omega}\left(1+\eta_j\right)\int \frac{dl}{B}\widehat{\omega}_{kj}\Lambda^2$$
drives the *balloonning* and the *interchange* modes.
- the fourth
$$-n_0 \frac{|e|^2}{B}\left(1+\frac{1}{\tau}\right)\int \frac{dl}{B}\psi^2$$

the energy needed to produce charge separation due to longitudinal electric field.

- the fifth

$$-\frac{\mu_0}{2}\sum_j \frac{T_j}{m_j^2}\int_{trapped}\frac{dl}{\left|v_\parallel\right|}d\epsilon d\mu\left[\frac{h_j^2\left(\omega-\overline{\omega}_{Dj}\right)}{F_{Mj}\left(\omega-\omega_{*j}^T\right)}-i\frac{h_j\overline{C}\left(h_j\right)}{F_{Mj}\left(\omega-\omega_{*j}^T\right)}\right]$$

energy required to compress the plasma in the MHD model. This term describes the interaction between MHD modes, mediated by the plasma.

- the sixth

$$-n_i\frac{\left|e\right|^2}{T_i}\left[1-\frac{\omega_{*i}}{\omega}\left(1+\eta_i\right)\right]\int\frac{dl}{B}\left(\frac{k_\perp^2\rho_i^2}{2}\right)\left(2\psi\Lambda+\Lambda^2\right)$$

is the *polarization* and the Finite Larmor Radius effect for the ions.

14 ITG and sheared velocity Artun and Tang

The paper **Gyrokinetic ITG Artun Tang** (1992) provides a kinetic derivation of the linear dispersion relation for the ITG and includes *sheared flows*.

15 ITG in sheared rotation Dong Horton

This is a model where the background flow is introduced via a potential $\Phi_0\left(x\right)$.

The flow consists of

1. toroidal flow with radial shear

$$v_{0\parallel}\left(x\right)\equiv v_\parallel\left(r\right)$$

2. poloidal flow with radial shear

$$v_E\left(x\right)\equiv v_\theta\left(r\right)$$

The model is **quasi-toroidal** and consists of

1. assuming that the drift of ions **curvature** and **magnetic** can be approximated with constant values, those taken for the position on the equatorial plane, at

$$\theta=0\quad\text{(outside of the torus)}$$

which is the maximum value of the drift

2. the **mode-coupling** induced by the drift term can be neglected

The parameters of the problem

$$
\begin{aligned}
\widehat{v}'_E &= \frac{L_n}{v_{th,i}} \frac{dv_E}{dx} \quad \text{sheared } E \times B \text{ imposed velocity} \\
\widehat{v}'_{0\|} &= \frac{L_n}{v_{th,i}} \frac{dv_{0\|}}{dx} \quad \text{sheared parallel imposed velocity} \\
\widehat{L} &= \frac{L_n}{L_s} \quad (\text{should be } \ll 1) \\
\eta_i &= \frac{L_n}{L_{Ti}} \\
\varepsilon_n &= \frac{L_n}{R} \quad \text{usually arises due to the curvatue and } \boldsymbol{\nabla} B \text{ drift of ions} \\
\tau &= \frac{T_e}{T_i}
\end{aligned}
$$

The equation for the motion of particles

$$
\begin{aligned}
m\frac{d^2\mathbf{x}}{dt^2} &= e\mathbf{E} + e\mathbf{v} \times \mathbf{B} \\
\frac{d^2\mathbf{x}}{dt^2} &= \frac{e}{m}\mathbf{E} + \Omega\mathbf{v}\times\widehat{\mathbf{n}}
\end{aligned}
$$

The constants of motion

$$
\begin{aligned}
\alpha &\equiv \frac{1}{2}\left(v_x^2 + v_y^2\right) + \frac{e}{m}\Phi_0(x) \quad \left\{ \begin{array}{c} \text{the total energy} \\ \text{of the perpendicular motion} \end{array} \right. \\
\beta &\equiv \frac{1}{2}\left[v_\| - v_{0\|}(x)\right]^2 \quad \text{the energy of the THERMAL parallel motion} \\
X_g &= x + \frac{v_y - v_E(X_g)}{\Omega} \quad \text{position of the guiding centre}
\end{aligned}
$$

The velocity of the guiding centre is $E \times B$

$$
v_E(X_g) = -\frac{E_0(X_g)}{B}
$$

The distribution function in equilibrium is

$$
\begin{aligned}
f_0(\alpha, \beta, X_g) = \frac{n(X_g)}{\pi^{3/2} v_{th}^3} g(X_g) \exp \Bigg[&-\frac{\left(v_x^2 + v_y^2\right)}{v_{th}^2} \\
&-\frac{\left(v_\| - v_{0\|}\right)^2}{v_{th}} \\
&-\frac{e}{T(X_g)}\Phi(x)\Bigg]
\end{aligned}
$$

where the function $g(X_g)$ is introduced for normalization

$$\int f_0 d^3v = n$$

It will result, aftre canculating the normalization

$$f_0\left(w_\perp, v_\parallel, X_g\right) = \frac{n(X_g)}{\pi^{3/2} v_{th,i}^3 (X_g)} \exp\left[-\frac{w_\perp^2}{v_{th,i}^2 (X_g)} - \frac{\left(v_\parallel - v_{0\parallel}(X_g)\right)^2}{v_{th,i}^2 (X_g)}\right]$$

where

$$w_\perp^2 \equiv v_x^2 + \left[v_y - v_E(X_g)\right]^2$$

NOTE for comparison the article of **Catto Rosenbluth Liu** on the *sheared velocity of the plasma flow along* **B** induced by the NBI. The equations of motion of the ion particles

$$\begin{aligned} \frac{d\mathbf{r}'}{dt} &= \mathbf{v}' \\ \text{with initial condition } \mathbf{r}'(t'=t) &= \mathbf{r} \end{aligned}$$

and

$$\begin{aligned} \frac{d\mathbf{v}'}{dt} &= \frac{|e|}{m_i}\mathbf{v}' \times \mathbf{B}_0(x') \\ \text{with initial condition } \mathbf{v}'(t'=t) &= \mathbf{v} \end{aligned}$$

From these equations result the <u>*invariants of the motion*</u>

$$|\mathbf{v}'|^2 = |\mathbf{v}|^2 = v^2$$

$$x' + \frac{v_y'}{\Omega_i} = x + \frac{v_y}{\Omega_i}$$

$$v_z' + \frac{v_y'}{L_s}\left(x' + \frac{v_y'}{2\Omega_i}\right) = v_z + \frac{v_y}{L_s}\left(x + \frac{v_y}{2\Omega_i}\right)$$

With these invariants they construct an *equilibrium distribution function* as

$$\begin{aligned} f_0 &= \frac{N_0}{\left(2\pi v_{th}^2\right)^{3/2}} \exp\left(\frac{x + \frac{v_y}{\Omega_{ci}}}{L_n}\right) \\ &\times \exp\left\{\frac{1}{2v_{th}^2}\left[v^2 - \right.\right. \\ &-2\left(v_z + \frac{x}{L_s}v_y + \frac{v_y^2}{2\Omega_i L_s}\right)\left[u + U'\left(x + \frac{v_y}{\Omega_i}\right)\right] \\ &\left.\left. \qquad + \left[u + U'\left(x + \frac{v_y}{\Omega}\right)\right]^2\right]\right\} \end{aligned}$$

Where u is the *sheared part of the mean velocity* of the particles. It is considered that for ions $u = 0$.

15.1 Integral equations in the presence of sheared flows

The following expression for the magnetic field is assumed

$$\mathbf{B} = B_0 \widehat{\mathbf{n}} + B_0 \frac{x}{L_s} \widehat{\mathbf{e}}_y$$

and for the **ION** drift velocity

$$v_D^i = \frac{1}{R\Omega} \left(\frac{v_\perp^2}{2} + v_\parallel^2 \right)$$

The value of the **ION** drift velocity is taken at $\theta = 0$ on the equatorial plane, at the outermost point. Otherwise we should consider the full expression of the frequency $\omega_{D,i}$ associated with the variation on the poloidal angle of the drifts.

The expansion of the distribution function

$$f = f_0 + f_1$$

with the first order distribution function obtained from the equation

$$\begin{aligned} f_1 (X_g) &= -\frac{e\widetilde{\phi}}{T} f_0 \\ &+ \left(i (\omega - \mathbf{v}_d \cdot \mathbf{k}) f_0 - i k_\theta \frac{T}{m\Omega} \frac{\partial f_0}{\partial X_g} \right) \times \\ &\times \int_{-\infty}^{t} dt' \left(-\frac{e\widetilde{\phi}(x')}{T} \right) \exp \left[-i\omega (t' - t) + i k_\theta (y' - y) \right. \\ &\qquad \left. + i k_z (z' - z) \right] \end{aligned}$$

where

$$\mathbf{v}_d = \mathbf{v}_E + v_{0\parallel} \widehat{\mathbf{n}}$$

(**note** there is no curvature or $\boldsymbol{\nabla} B$ drift velocity).

Comment related to the ρ_{eff} and to the regime where the effective Larmor radius can become infinite

From the equation written above we see where the ρ_{eff} can be manifested.

The factor in front of the integration of the potential along the particle's trajectory is

$$i (\omega - \mathbf{v}_d \cdot \mathbf{k}) f_0 - i k_\theta \frac{T}{m\Omega} \frac{\partial f_0}{\partial X_g}$$

and is actually the inversion of the operator acting on the perturbation of the distribution function f_1.

We can ask what happens in the regime characterized by the following parameters

1. the mode frequency is very low, as it is in the zonal flows

$$\omega \rightarrow 0$$

2. The velocity in the parallel direction $v_{0\|}$ does not intervene since the wavenumber along the field is very close to zero

$$k_{\|} v_{0\|} \rightarrow 0$$

3. Only the poloidal velocity $v_E \equiv u$ is important and remains in the term that multiplies f_0, $(\omega - \mathbf{v}_d \cdot \mathbf{k})$.

4. The last term gives us the diamagnetic velocity of the electrons multiplied by the wavenumber in the poloidal $\theta \equiv y$ direction

$$\begin{aligned} -ik_\theta \frac{T}{eB_0} \frac{1}{n} \frac{dn}{dr} f_0 &= +ik_\theta v_{*e} f_0 \\ &= +i\omega_{*e} \end{aligned}$$

Then what remains from this factor is

$$\begin{aligned} & i\left(-v_E k_\theta\right) f_0 + ik_\theta v_{*e} f_0 \\ = \; & -ik_\theta v_E \left(1 - \frac{v_{*e}}{v_E}\right) f_0 \end{aligned}$$

Therefore we see that there is a critical involvement of the quantity

$$1 - \frac{v_{*e}}{v_E}$$

in the expression of the *drive* of the mode. The integration will produce the propagator, *i.e.* the inverse of the operator of derivation along the particle trajectory.

When this factor is almost zero, the factor in front of the integral of the potential $\widetilde{\phi}\left[x\left(t'\right)\right]$ along the trajectories goes to zero and the response of the ion distribution function is adiabatic and is incompatible with the adiabatic response of the electrons.

The wave does not exist.

this means that the critical situation is when the plasma poloidal rotation is in the electron DIAMAGNETIC direction and it is slightly higher than the electron diamagnetic velocity

$$u \gtrsim v_{e,dia}$$

Another expression of this situation is based on the observation that the factor in front of the integral over particle trajectory is the *inverse* of an operator which initially acted upon the perturbation f_1 of the distribution function. If the factor is zero, then the initial operator is *not defined*. This is the propagator and becomes singular, like

$$\sim \frac{1}{i\omega}$$

There is a singularity in the initial operator.

End of Comment

The equations of motion of the particle are

$$\begin{aligned}
\frac{d}{dt} v_x &= \Omega \left(v_y - v_E\right) \\
\frac{d}{dt} \left(v_y - v_E\right) &= -\Omega v_x
\end{aligned}$$

One introduces the notation

$$u_y \equiv v_y - v_E \left(X_g\right)$$

and the equations are

$$\begin{aligned}
\dot{v}_x &= \Omega u_y \\
\dot{u}_y &= -\Omega v_x
\end{aligned}$$

$$\begin{aligned}
\ddot{v}_x + \Omega^2 v_x &= 0 \\
v_x &= v_{x0} \cos\left(\delta + \Omega t\right)
\end{aligned}$$

then

$$\begin{aligned}
\dot{v}_x &= v_{x0} \left(-\Omega\right) \sin\left(\delta + \Omega t\right) = \Omega u_y \\
u_y &= -v_{x0} \sin\left(\delta + \Omega t\right)
\end{aligned}$$

$$v_y = v_E \left(X_g\right) - v_{x0} \sin\left(\delta + \Omega t\right)$$

The equations of motion must be integrated

$$\begin{aligned}
\frac{dx}{dt} &= \dot{v}_x = \Omega u_y = -\Omega v_{x0} \sin\left(\delta + \Omega t\right) \\
x &= x_0 + v_{x0} \cos\left(\delta + \Omega t\right)
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dt} &= v_E \left(X_g\right) - v_{x0} \sin\left(\delta + \Omega t\right) \\
y &= y_0 + v_E \left(X_g\right) t + \frac{v_{x0}}{\Omega} \cos\left(\delta + \Omega t\right)
\end{aligned}$$

The argument of the exponential

$$-i\omega \left(t' - t\right) + ik_\theta \left(y' - y\right) + ik_z \left(z' - z\right)$$

is added to the argument of the exponential of the Maxwellian function f_0 and is transformed. After the notations

$$\begin{aligned} \tau &\equiv t' - t \\ w_\perp^2 &\equiv v_x^2 + (v_y - v_E)^2 = v_x^2 + u_y^2 \end{aligned}$$

and introducing

$$\delta' = \delta + \Omega\tau$$

the *phase* of the gyromotion at time t'. The exponential 's argument becomes

$$\begin{aligned} \Theta \equiv & \quad \text{(phase)} \\ & -i\left(\omega - k_\theta v_E - k_\parallel v_{0\parallel}\right)\tau \\ & -i\frac{L_n}{R}\frac{\omega_{*e}}{T_e/T_i}\left[\frac{w_\perp^2}{v_{th,i}^2}\tau + \frac{v_E^2}{v_{th,i}^2}\tau + 2\frac{w_\perp v_E}{v_{th,i}^2\Omega_i}\left(\cos\delta' - \cos\delta\right) + 2\frac{v_{0\parallel}^2}{v_{th,i}^2}\tau\right] \\ & +i\frac{k_\theta w_\perp}{\Omega}\left(\sin\delta' - \sin\delta\right) \end{aligned}$$

Two notations are introduced

$$\begin{aligned} \varepsilon_n &\equiv \frac{L_n}{R} \\ \omega_{*e} &\equiv k_\theta \frac{T_e}{m_e\Omega_e}\frac{1}{L_n} \end{aligned}$$

and the distribution function at equilibrium

$$f_0(\alpha, \beta, X_g) = \frac{n(X_g)}{\pi^{3/2}\left[v_{th,i}(X_g)\right]^3}\exp\left[-\frac{w_\perp^2}{\left[v_{th,i}(X_g)\right]^2} - \frac{\left(v_\parallel - v_{0\parallel}(X_g)\right)^2}{\left[v_{th,i}(X_g)\right]^2}\right]$$

and

$$\begin{aligned} \frac{\partial f_0(X_g)}{\partial X_g} = & \; f_0\left[\frac{n'}{n} + \frac{T_i'}{T_i}\left(\frac{w_\perp^2}{(v_{th,i})^2} - \frac{3}{2} + \frac{\left(v_\parallel - v_{0\parallel}\right)^2}{(v_{th,i})^2}\right)\right. \\ & \left. + v_{0\parallel}'\frac{\left(v_\parallel - v_{0\parallel}\right)}{T_i/m_i} + v_E'\frac{u_y}{T_i/m_i}\right] \end{aligned}$$

with the derivative being

$$()' \equiv \frac{d}{dX_g}$$

The perturbation to the distribution function is

$$\begin{aligned} f_1^{ions}(x,t) = & -\frac{e}{T_i}\left\{f_0\widetilde{\phi}(x)\right. \\ & \left. + \left[i(\omega - \mathbf{k}\cdot\mathbf{v}_d)f_0 - ik_\theta\frac{T_i}{m\Omega}\frac{\partial f_0}{\partial X_g}\right]\int_{-\infty}^{0} d\tau\widetilde{\phi}(x')e^{\Theta}\right\} \end{aligned}$$

For neutrality we use the perturbed densities of ions and electrons.

$$\widehat{n}_i = \frac{1}{\sqrt{2\pi}} \int \exp\left(-ikx\right) dx \int f_1 d\mathbf{v}$$

and

$$\widehat{n}_e = \frac{en}{T_e} \widehat{\phi}\left(k\right)$$

where $\widehat{\phi}\left(k\right)$ is the Fourier component of the potential.

The equation of continuity

$$\begin{aligned}
\widehat{n}_i\left(k\right) &= -\frac{ne}{T_i}\widehat{\phi}\left(k\right) \\
&\quad -i\int \frac{dk'}{\sqrt{2\pi}} \frac{e\widehat{\phi}\left(k'\right)}{T_i} \int \frac{dx}{\sqrt{2\pi}} \exp\left[i\left(k'-k\right)x\right] \\
&\quad \times \int d\tau \exp\left[-i\omega\tau + ik_\theta v_E \tau - i\frac{\varepsilon_n v_E^2}{\left(T_e/T_i\right)\left(v_{th,i}\right)^2}\omega_{*e}\tau\right] H\left(\tau, x\right)
\end{aligned}$$

Introducing the normalized components of velocity

$$\begin{aligned}
\widehat{w}_\perp &\equiv \frac{w_\perp}{v_{th,i}} \\
\widehat{v}_\parallel &\equiv \frac{v_\parallel}{v_{th,i}}
\end{aligned}$$

and

$$\begin{aligned}
\widehat{v}_{0\parallel} &\equiv \frac{v_{0\parallel}}{v_{th,i}} \\
\widehat{u}_y &\equiv \frac{u_y}{v_{th,i}}
\end{aligned}$$

then the function $H\left(\tau, x\right)$ can be written

$$\begin{aligned}
H\left(\tau, x\right) \equiv & \frac{n}{\pi^{3/2}} \int d\widehat{v}_\parallel d\widehat{w}_\perp^2 d\delta \\
& \times J_0\left(\frac{w_\perp k_\perp}{\Omega}\right) J_0\left(\frac{w_\perp k'_\perp}{\Omega}\right) \\
& \times F \\
& \times \exp\left[-\left(\widehat{u}_\parallel^2 + \widehat{w}_\perp^2\right)\right. \\
& \qquad + ik_\parallel v_\parallel \tau \\
& \qquad \left. -i\frac{\varepsilon_n}{T_e/T_i}\omega_{*e}\tau\left(\widehat{w}_\perp^2 + \widehat{v}_\parallel^2\right)\right]
\end{aligned}$$

The function F is obtained from

$$\begin{aligned} F &\equiv \omega_{*e}\left[\widehat{\omega}-\frac{k_{\|}v_{0\|}}{\omega_{*e}}-\frac{k_\theta v_E}{\omega_{*e}}\right. \\ &+\frac{1}{\tau_e}+\frac{\eta_i}{\tau_e}\left(\widehat{w}_\perp^2-\frac{3}{2}+\widehat{u}_\|^2\right) \\ &\left.-2\widehat{v}_{0\|}'\widehat{u}_\|-2\widehat{u}_y\widehat{v}_E'\right] \end{aligned}$$

with the normalizations

$$\begin{aligned} \widehat{\omega} &\equiv \frac{\omega}{\omega_{*e}} \\ \widehat{w}_\perp &\equiv \frac{w_\perp}{v_{th,i}} \\ \widehat{u}_\| &\equiv \frac{u_\|}{v_{th,i}} \\ \widehat{v}_{0\|} &\equiv \frac{v_{0\|}}{v_{th,i}} \end{aligned}$$

The integrations in velocity space are done over the parallel velocity $\widehat{v}_\|$, the perpendicular velocity $\widehat{w}_\perp^2=\widehat{v}_\perp^2+\widehat{v}_y^2$, and the gyrophase δ.

The result of integration over the velocity space is

$$\begin{aligned} H(\tau,x) &= \frac{2n\omega_{*e}}{\sqrt{a}(1+a)}\Gamma_0\left(k_\perp,k_\perp'\right)\exp\left(-c+\frac{b^2}{4a}\right) \\ &\times\left(\widehat{\omega}-\frac{k_\|v_{0\|}(x)}{\omega_{*e}}-\frac{k_\theta v_E(x)}{\omega_{*e}}\right. \\ &+\frac{1}{\tau_e} \\ &+\frac{\eta_i}{\tau_e}\left\{\frac{2}{1+a}F_0-\frac{3}{2}+\frac{1}{2a}+\left[\left(\frac{1}{a}-1\right)\widehat{v}_{0\|}(x)+\frac{ik_\|v_{th,i}}{2a}\tau\right]^2\right\} \\ &\left.-2\frac{v_{0\|}'}{\tau_e}\left[\left(\frac{1}{a}-1\right)\widehat{v}_{0\|}(x)+\frac{ik_\|v_{th,i}}{2a}\tau\right]\right) \end{aligned}$$

The notations are

$$\begin{aligned} c &\equiv \widehat{v}_{0\|}^2=\frac{v_{0\|}^2(x)}{v_{th,i}^2} \\ b &\equiv -\left(2\widehat{v}_{0\|}+ik_\|v_{th,i}\tau\right) \\ a &\equiv 1+\frac{i2\varepsilon_n}{\tau_e}\omega_{*e}\tau \\ F_0 &\equiv 1-\frac{k_\perp^2+k_\perp'^2}{2(1+a)}+\frac{k_\perp k'}{(1+a)}\frac{I_1}{I_0} \end{aligned}$$

15.2 Toroidal kinetic η_i mode Dong Horton Kim

This is similar with the Romanelli model.

In the paper of **Dong Horton** it is treated an approximation of the ITG analytical model. This consists of taking the *quasitoroidal* model in which:

1. the ion curvature and
2. ion magnetic gradient

drifts are considered **constant** over a flux surface and equal to the maximum value at $\theta = 0$, *i.e.* at the outside of the torus.

In addition, in the *quasitoroidal* model the **mode coupling** introduced by the toroidal feature of the equilibrium magnetic configuration is neglected.

The paper **Kinetic resonance Kim Horton Kishimoto Tajima**.

The *contour integration* in the complex velocity plane, like in Landau damping.

The *slab* model for the η_i mode:

coupling of the gradient of the ion temperature with the parallel transit drift

The *toroidal* model:

coupling of the gradient of the ion temperature with the toroidal magnetic drift, i.e. with the drift resulting from the $\boldsymbol{\nabla} B$-curvature.

Relation of dispersion

$$1 + \tau = P(\omega)$$

$$\tau \equiv \frac{T_i}{T_e}$$

$$\begin{aligned}
P(\omega) &\equiv \text{the dispersion function} \\
&= \int_{-\infty}^{\infty} \frac{dv_\parallel}{\sqrt{\pi}} \int_0^{\infty} 2v_\perp dv_\perp \exp\left(-v^2\right) \\
&\quad \times \frac{\omega - \omega_{*i}\left[1 + \eta_i\left(v^2 - \frac{3}{2}\right)\right]}{\omega - \omega_i^{drift}\left(\frac{v_\perp^2}{2} + v_\parallel\right) - \sqrt{2}k_\parallel v_\parallel} J_0^2\left(\sqrt{2}k_\perp v_\perp\right)
\end{aligned}$$

where

$$\begin{aligned}
\omega_{*i} &= k_y \\
\omega_i^{drift} &= 2\epsilon_n \omega_{*i} \\
\epsilon_n &= \frac{L_n}{R} \\
\eta_i &= \frac{L_n}{L_{T_i}} \\
L_n^{-1} &= -\frac{d}{dr}\ln n
\end{aligned}$$

Normalizations

$$
\begin{aligned}
v_\perp, v_\parallel &\rightarrow \sqrt{2} v_{th,i} \\
v_{th,i} &= \sqrt{\frac{T_i}{m_i}}
\end{aligned}
$$

The resonance in the space (plane) of the velocity is now an *ellipse*.

16 ITG model with sheared rotation: Wang, Diamond, Rosenbluth

The model has:

1. adiabatic electrons
$$\frac{\widetilde{n}_e}{n_0} \approx \frac{|e| \varphi}{T_e}$$
2. hydrodynamic ions

The equations are
$$\mathbf{B} = B_0 \widehat{\mathbf{e}}_z + B_y \widehat{\mathbf{e}}_y$$
where
$$B_y = B_0 \frac{x}{L_s}$$
The radial electric field is
$$\mathbf{E} = E_0 \frac{x}{L_s} \widehat{\mathbf{e}}_x$$

$$
\begin{aligned}
\mathbf{V} &= V_0 \frac{x}{L_s} \widehat{\mathbf{e}}_y \\
\text{where } V_0 &= -\frac{E_0}{B_0}
\end{aligned}
$$

Now, the equations

$$
\begin{aligned}
& \frac{D}{Dt} \left(1 - \boldsymbol{\nabla}_\perp^2 \right) \phi + \\
& + v_{e,dia} \left[1 + \left(\frac{1+\eta_i}{\tau} \right) \boldsymbol{\nabla}_\perp^2 \right] \frac{\partial \phi}{\partial y} \\
& + \nabla_\parallel v_\parallel \\
= \; & 0
\end{aligned}
$$

$$\frac{D}{Dt} v_\parallel = -\nabla_\parallel p - \nabla_\parallel \phi$$

$$\begin{aligned} & \frac{D}{Dt}p \\ & +v_{e,dia}\left(\frac{1+\eta_i}{\tau}\right)\frac{\partial\phi}{\partial y} \\ = & -\frac{\Gamma}{\tau}\nabla_{\parallel}v_{\parallel} \end{aligned}$$

where

$$\begin{aligned} v_{e,dia} &= -\frac{T_e}{|e|\,B}\frac{d}{dx}\ln n_0 \\ \tau &\equiv \frac{T_e}{T_i} \\ \eta_i &\equiv \frac{d\ln T_i}{d\ln n_0} \end{aligned}$$

The operator of total time derivation is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{V}\cdot\boldsymbol{\nabla} = \frac{\partial}{\partial t} + \left(V_0\frac{x}{L_E}\right)\frac{\partial}{\partial y}$$

Normalizations

$$\begin{aligned} \phi &\leftarrow \frac{e\phi}{T_e} \\ v_{\parallel} &\leftarrow \frac{v_{\parallel}}{c_s} \\ p &\leftarrow \frac{p}{n_0T_e} \end{aligned}$$

Aftre Fourier transformation

$$\begin{aligned} \nabla_{\parallel} &= ik_{\parallel} = ik_y\frac{x}{L_s} \\ \frac{\partial}{\partial y} &= ik_y \\ \frac{\partial}{\partial t} &= -i\omega \end{aligned}$$

The equation is

$$\frac{\partial^2\phi}{\partial x^2} + U\left(x,\overline{\omega}\right)\phi = 0$$

where

$$\begin{aligned} U\left(x,\overline{\omega}\right) = & -k_y^2 \\ & +\frac{1-\overline{\omega}}{\overline{\omega}+\frac{1+\eta_i}{\tau}} \\ & +\frac{\left(\frac{L_n}{L_s}\right)^2x^2}{\overline{\omega}^2-\frac{\Gamma}{\tau}\left(\frac{L_n}{L_s}\right)^2x^2} \end{aligned}$$

The new frequency is

$$\overline{\omega} \equiv \frac{\omega - k_y V_0 \frac{x}{L_E}}{\omega_{e,dia}}$$
$$\omega_{e,dia} = k_y v_{e,dia}$$

17 Resonance in the velocity space Kim Horton

The equation of dispersion for the toroidal ITG

$$1 + \tau = P(\omega)$$

where

$$\tau \equiv \frac{T_e}{T_i}$$

$$\begin{aligned} P(\omega) &= \int_{-\infty}^{\infty} \frac{dv_\parallel}{\sqrt{\pi}} \int_0^\infty 2v_\perp dv_\perp \exp\left(-v^2\right) \\ &\times \frac{\omega - \omega_{*i}\left[1 + \eta_i\left(v^2 - 3/2\right)\right]}{\omega - \omega_{Di}\left(\frac{v_\perp^2}{2} + v_\parallel^2\right) - \sqrt{2}k_\parallel v_\parallel} J_0^2\left(\sqrt{2}k_\perp v_\perp\right) \end{aligned}$$

The notations have been explained in a preceding Section.

We note the difference between the numerator and denominator:

in the numerator we have the *drive* of the wave. The *drive* is based on the mismatch between the wave frequency ω and the intrinsic frequency $\omega_{*i}\left[1 + \eta_i\left(v^2 - \frac{3}{2}\right)\right]$ which for a given k_y results from the plasma parameters, basically from the gradient of the equilibrium pressure. This is the reason for the presence of the *diamagnetic* frequency ω_{*i}.

One over the denominator represents the inverse of the operator "derivation along the particle trajectory $x(t)$ " which is equivalent with the operator "integration along the particle trajectory ". We can say that one over the denominator is the Green function of the operator "derivation along the particle trajectory $x(t)$ " or, the *propagator*. Here the trajectory of the particle is essential and for this reason it appears the term with the ion *drift* frequency

$$2\omega_{*i}\frac{L_n}{R}\left(\frac{v_\perp^2}{2} + v_\parallel^2\right)$$

But also it appears the term related to the *parallel* motion of the ions

$$k_\parallel v_\parallel$$

Regarding the resonances that are possible in the dispersion relation, when the integration over the velocity space has to traverse singular integrand

$$\left(v_\parallel + \frac{k_\parallel}{\sqrt{2}\omega_{Di}}\right)^2 + \frac{v_\perp^2}{2} = \frac{\omega}{\omega_{Di}} + \frac{k_\parallel^2}{2\omega_{Di}^2}$$

This is a **shifted ellipse** in the space $\left(v_{\parallel}, v_{\perp}\right)$.

The condition that a resonance is possible is

$$\frac{\omega}{\omega_{Di}} + \frac{k_{\parallel}^2}{2\omega_{Di}^2} \geq 0$$

or

$$\omega \geq \omega_{branch} = -\frac{k_{\parallel}^2}{2\omega_{Di}}$$

This means that the phase velocity of the wave in the parallel direction $\omega/k_{\parallel}$ to be greater than $(-)$ the *inverse* of the phase velocity associated with the drift of the ions.

The *fraction of resonant particles*

$$r(\omega)$$

at a given frequency ω is calculated by integrating over the velocity space with a constraint of resonance introduced by a δ function

$$\begin{aligned} r(\omega) &= \int_{-\infty}^{\infty} \frac{dv_{\parallel}}{\sqrt{\pi}} \int_{0}^{\infty} 2v_{\perp} dv_{\perp} \exp\left(-v_{\parallel}^2 - v_{\perp}^2\right) \\ &\quad \times \delta\left(\omega - \omega_{Di}\left(\frac{v_{\perp}^2}{2} + v_{\parallel}^2\right) - \sqrt{2}k_{\parallel}v_{\parallel}\right) \end{aligned}$$

It results

$$r(\omega) = \frac{2}{\sqrt{\pi}\omega_{Di}} \int_{-\widetilde{\omega}^{1/2}}^{\widetilde{\omega}^{1/2}} dv_{\parallel} \exp\left[-\left(v_{\parallel} - u\right)^2 - 2\left(v_{\parallel}^2 - \widetilde{\omega}\right)\right]$$

with

$$\widetilde{\omega} = \frac{\omega - \omega_{branch}}{\omega_{Di}}$$

$$u \equiv -\frac{k_{\parallel}}{\sqrt{2}\omega_{Di}}$$

It can be seen that

$$r(\omega) \equiv 0 \quad \text{for } \omega < \omega_{branch}$$

The fraction depends on $k_{\parallel}$. It increases very sharply at the threshold of ω.

The problem of contour of integration in the complex plane is solved for a more general expression than $P(\omega)$:

$$\overline{P}(\omega) = \int_{0}^{\infty} v_{\perp} dv_{\perp} \int_{-\infty}^{\infty} dv_{\parallel} \frac{f\left(\omega, v_{\perp}, v_{\parallel}\right)}{\omega - av_{\perp}^2 - bv_{\parallel}^2 - cv_{\parallel}}$$

where f is an analytic function. To extend from

$$\begin{aligned} \text{growing } \gamma &= \operatorname{Im}\omega > 0 \ \text{ to} \\ \text{damped } \gamma &= \operatorname{Im}\omega < 0 \ \text{ modes,} \end{aligned}$$

one needs to make an analytical prolongation in the domain $\mathrm{Im}(\omega) < 0$, which is done by deforming the contour of integration. The resonance condition

$$\omega = a v_{\perp}^2 + b v_{\parallel}^2 + c v_{\parallel}$$

the resonant poles appear on this curve. The residue is

$$R(\omega) = 2\pi i \int_C f\left[\omega, v_{\perp}\left(\omega, v_{\parallel}\right), v_{\parallel}\right]$$

where $v_{\perp}$ is obtained by solving the algebraic equation of resonance.

A change of variables is made, such as to reduce the contour to be described by a *single* variable

$$\begin{aligned}
v'_{\perp} &= \sqrt{a} v_{\perp} \\
v'_{\parallel} &= \sqrt{b}\left(v_{\parallel} + \frac{c}{2b}\right) \\
v' &= \sqrt{\left(v'_{\perp}\right)^2 + \left(v'_{\parallel}\right)^2} \\
\mu' &= \frac{v'_{\parallel}}{v'}
\end{aligned}$$

The condition of resonance is

Landau	ITG
$v_{\parallel} = \frac{\omega}{c}$	$(v')^2 = \omega + \frac{c}{2b}$
$0 \leq v_{\perp} \leq \infty$	$-1 \leq \mu' \leq +1$

The residue

$$R(\omega) = 2\pi i \sqrt{\omega + \frac{c^2}{2b}} \int_{-1}^{+1} dy\ f\left[\omega,\ v_{\perp}(\omega, y),\ v_{\parallel}(\omega, y)\right]$$

The *dispersion relation for the damped case is obtained by adding the residue to the to the direct double integrtion in the expression of* $P(\omega)$.

18 Possibility of non-normal mode excitation from a sheared flow

Non-Hermitian.

The works of **Farrell Ioannou** in atmosphere.

The Orr-Sommerfeld instability of a layer of sheared flow.

The layer of sheared flow is a transport barrier of a zonal flow in tokamak.

The perturbations are ITG modes.

The perturbations extract randomly energy from the basic sheared flow.

This picture is an extension of the classical balance of opposite tendencies that involve:

- Reynolds stress of turbulence
- rotation, sheared, supported by the Reynolds stress
- suppression of ITG linearly by the sheared flow; suppression of the turbulence
- reduction or suppression of the Reynolds stress, which reduces or suppresses the torque that sustain sheared rotation

Now we have a new actor.

The sheared rotation does not destroys completely the ITG.

This is because the sheared rotation actually is able to feed ITG pertrubations, by random transfers of energy from the sheared flow to the perturbations.

Then the rotation will *reduce* the ITG modes but this reduction will halt at the level of manifestation of non-normal modes.

This level may be the one that is necessary for the Reynolds stress to continue to be the torque.