Part I
Drift waves and drift turbulence

1 Phenomenology

propagation of drift waves:	*electron diamagnetic* direction

The physical mechanism of the drift waves is presented by **Horton** in the **RMP 1999**.

A perturbation of potential induces local rotation of plasma by $E \times B$ motion, with E being oriented radially outward from the center of the perturbation ϕ , *i.e.* from the maximum of the potential ϕ. This rotation takes plasma from the higher density region (closer to the magnetic axis) and brings it to regions where the density is lower. And takes plasma of lower density (from the region far from the magnetic axis) and brings it by rotation in the zone where the initial density is higher. It results a displacement of the maximum of the potential ϕ in the poloidal direction. The velocity is $v_{e,dia}$.

We **note** that this explanation does not make use of the parallel dynamics and does not include magnetic shear.

Similarly, the explanation given by **Catto Rosenbluth Liu 1973 parallel sheared velocity** is:

the perturbation of the ion density at the reference surface is due to the fact that the motion induced by the potential perturbation (the velocity is u_x radial) transports more ions from the regions where there is higher density (closer to the magnetic axis along minor radius x) toward the $x = 0$ ("resonant"-) surface and transports fewer ions from the regions where the equilibrium density is lower (further from magnetic axis, higher $x > 0$), toward $x = 0$. Then

$$\delta n^{(3)} = -\left(u_x \delta t\right) \frac{dN}{dr}$$

and integrating

$$n^{(3)} = -\int dt u_x \frac{dN}{dr} = -\frac{u_x}{\omega} \frac{dN}{dr}$$

which is neutralised by the density of electrons that takes the Boltzmann equilibrium distribution in ϕ along the magnetic field line

$$n^{(3)} = -\frac{u_x}{\omega}\frac{dN}{dr} = N\frac{e\phi}{T_e}$$

and this gives

$$|\omega| \approx k_y \frac{k_B T_e}{m_i \Omega_i}\frac{1}{N}\frac{dN}{dr} \rightarrow k_y \frac{T_e}{m_i \Omega_i}\frac{d}{dr}\ln n$$

since the radial velocity induced by ϕ is

$$u_x = \frac{-ik_y\phi}{B}$$

(for an expansion

$$\mathbf{E} = \left(-ik_y\widehat{\mathbf{e}}_y - ik_z\widehat{\mathbf{e}}_z\right)\phi$$

with electric field poloidal and $\approx$ parallel direction). Then

$$|\omega| = |\omega_{e,dia}|$$

The time scale associated with changes in density due to the presence of a gradient $\frac{dN}{dr}$ *is just the time required for the electrons, and therefore* $\mathbf{E}$ *to drift a wavelength* $2\pi/k_y$.

This means that in a time $2\pi\left|\omega_{e,dia}\right|^{-1}$ the pattern of the wave drifts a full wavelength in the minus y direction.

It has been assumed that $T_e \gg T_i$.

The time scale for the process of building-up the perturbation of the ion density due to the $E \times B$ velocity is given by the *frequency* of the wave ω. In this description of the drift wave, with identification of the typical wave frequency ω_{*e}, the only assumption is the Boltzmannian distribution of the electrons, resulting from the very high parallel (thermal) electron velocity.

From the paper **Numerical solution drift kinetic eq. Santarius Hinton**.

"Drift waves require the presence of two components in the plasma:

1. an adiabatic species which can come to equilibrium with the wave in a time short compared with a wave period, and

2. a hydrodynamic species which $\mathbf{E} \times \mathbf{B}$ drifts in the electrostatic field of the wave. "

In the *dissipative trapped electron* mode,

- all ions are hydrodynamic
- electrons are:
  - partly *adiabatic* and
  - there is a component which is *non-adiabatic*

In the absence of dissipation electrons and ions oscillate in phase under the influence of the $\mathbf{E} \times \mathbf{B}$ drift.

When the particles are driven out-of-phase by dissipative mechanisms such as *collisions* or *wave-particle resonances*, a growing or damped mode can occur.

For *electrons*.

The drift kinetic equation is obtained from

$$\begin{aligned} \frac{df_e}{dt} &= C(f_e) \\ \frac{\partial f_e}{\partial t} + \frac{\partial f_e}{\partial \mathbf{x}} \cdot \frac{d\mathbf{x}}{dt} + \frac{\partial f_e}{\partial q_\alpha}\frac{dq_\alpha}{dt} &= C(f_e) \end{aligned}$$

where q_α is a generalized phase-space variable.

$$\begin{aligned} &\frac{\partial f_e}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} f_e \\ &+\frac{\partial f_e}{\partial \mu}\frac{d\mu}{dt} + \frac{\partial f_e}{\partial \varepsilon}\frac{d\varepsilon}{dt} + \frac{\partial f_e}{\partial \alpha}\frac{d\alpha}{dt} \\ = \; & C(f_e) \end{aligned}$$

where

$$\begin{aligned} \mu &\equiv \frac{v_\perp^2}{2B} \\ \varepsilon &= \frac{v^2}{2} - \frac{|e|}{m_e}\Phi \quad \text{(for electrons)} \\ \alpha &\equiv \text{gyrophase angle} \end{aligned}$$

For

1. electrostatics wave instabilities, and
2. after *gyrophase averaging*,

one obtains

$$\frac{\partial \overline{f}_e}{\partial t} + \left(\mathbf{v}_{\parallel} + \mathbf{v}_D + \mathbf{v}_E\right) \cdot \boldsymbol{\nabla} \overline{f}_e - \frac{|e|}{m_e} \frac{\partial \Phi}{\partial t} \frac{\partial \overline{f}_e}{\partial \varepsilon} = C\left(f_e\right)$$

The *bar* means *gyrophase averaged*.

$$\mathbf{v}_{\parallel} = \frac{\left(\mathbf{v} \cdot \mathbf{B}\right) \mathbf{B}}{B^2}$$

$$\mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$

$$\mathbf{v}_{De} = -\frac{1}{\Omega_{ce}} \left(\frac{v_{\perp}^2}{2} + v_{\parallel}^2\right) \frac{1}{B} \widehat{\mathbf{n}} \times \boldsymbol{\nabla} B$$

$$\Omega_{ce} = \frac{|e| B}{m_e}$$

The equilibrium distribution function is Maxwellian

$$f_{Me} = \frac{n}{\left(\sqrt{\pi} v_{th,e}\right)^3} \exp\left(-\frac{v^2}{v_{th,e}^2}\right)$$

$$v_{th,e} = \sqrt{\frac{2T_e}{m_e}}$$

The first order distribution function is a correction to the *adiabatic* component

$$f_e = \frac{|e| \Phi}{T_e} f_{Me} + f_{e1}$$

The variables adopted from now on

$$\left(\mathbf{r}, t, \mu, \frac{v^2}{2}\right)$$

The linearized equation is

$$\begin{aligned} & \frac{\partial f_{e1}}{\partial t} + \left(\mathbf{v}_{\parallel} + \mathbf{v}_D\right) \cdot \boldsymbol{\nabla} f_{e1} \\ = \quad & C\left(f_{e1}\right) - \frac{|e|}{T_e} f_{Me} \frac{\partial \Phi}{\partial t} \\ & -\mathbf{v}_E \cdot \boldsymbol{\nabla} f_{Me} \end{aligned}$$

The last term is the drift of the Maxwellian (equilibrium) distribution function by the electric velocity.

$$
\begin{aligned}
-\mathbf{v}_E \cdot \boldsymbol{\nabla} f_{Me} &= -\frac{-\boldsymbol{\nabla}\Phi \times \widehat{\mathbf{n}}}{B} \cdot \boldsymbol{\nabla} f_{Me} \\
&= \frac{1}{B}\frac{\partial \Phi}{r\partial \theta}\frac{\partial f_{Me}}{\partial r}
\end{aligned}
$$

where the variation of the potential corresponds to a wave propagating in the poloidal variation. The potential acts here by its variation along the poloidal θ direction.

Now it is assumed that all terms have time variation as

$$\exp\left(-i\omega t\right)$$

and the equation becomes

$$
\begin{aligned}
& -i\omega f_{e1} + \left(\mathbf{v}_{\|} + \mathbf{v}_D\right) \cdot \boldsymbol{\nabla} f_{e1} \\
= \; & C\left(f_{e1}\right) \\
& + \frac{|e|}{T_e}\left\{ i\omega\Phi + \frac{T_e}{|e|\,B}\frac{\partial \Phi}{r\partial \theta}\frac{d\ln n}{dr}\left[1 + \eta_e\left(\frac{v^2}{v_{th,e}^2} - \frac{3}{2}\right)\right]\right\} f_{Me}
\end{aligned}
$$

Two observations can be helpful.

First, the toroidal variation (on the toroidal angle φ) must be considered also periodic and expanded as

$$\exp\left(-il\varphi\right)$$

Second the correction function f_{e1} is linear in the potential Φ. Then both can be expanded on the poloidal angle in a way that is compatible with this linear relationship

$$\Phi\left(r,\theta,\varphi\right) = \sum_{m=-\infty}^{\infty} a_{\,m} \exp\left(im\theta - il\varphi\right)$$

and the f_{1e} is expanded as

$$f_{e1} = \sum_{m=-\infty}^{\infty} f_m a_m$$

NOTE this expansion must be retained. In the Fourier expansion of the potential Φ the coefficients a_m are not necessarily small, their amplitude is *not* ordered according to the indice m. Then the second expansion, or the distribution function f_{e1} in terms that each contains an amplitude a_m of Fourier components of the potential does not imply that it is an expansion in order of magnitude. **END**

The drift kinetic equation becomes

$$\begin{aligned} & -i\omega f_m + \left(\mathbf{v}_{\|} + \mathbf{v}_D\right) \cdot \boldsymbol{\nabla} f_m \\ = & \; C\left(f_m\right) \\ & + i\left(\omega - \omega_{Te}^*\right) \frac{|e|}{T_e} f_{Me} \exp\left(im\theta - il\varphi\right) \end{aligned}$$

where

$$\omega_{Te}^* = \omega_{em}^* \left[1 + \eta_e \left(\frac{v^2}{v_{th}^2} - \frac{3}{2}\right)\right]$$

$$\begin{aligned} \omega_{em}^* &= -k_y \frac{T_e}{|e|\, B} \frac{d \ln n}{dr} \\ &= -\frac{m}{r} \frac{T_e}{|e|\, B} \frac{d \ln n}{dr} \\ &= -\frac{m}{r} \frac{T_e}{|e|\, B} \frac{1}{L_n} \end{aligned}$$

and

$$\eta_e \equiv \frac{d \ln T_e}{d \ln n} = \frac{L_n}{L_T}$$

The collision operator is *Lorentz*, necessary for taking into account the *pitch angle scattering*, with no energy change.

$$C\left(f\right) = \nu_{ei}\left(v\right) \frac{1}{2} \frac{\partial}{\partial \xi} \left(1 - \xi^2\right) \frac{\partial f}{\partial \xi}$$

where

$$\xi \equiv \frac{v_{\|}}{v}$$

$$\nu_{ei}\left(v\right) = \frac{3\sqrt{\pi}}{4} \frac{1}{\tau_e} \frac{v_{th,e}^3}{v^3}$$

$$\begin{aligned} \tau_e &= \frac{3}{16\sqrt{\pi}} \frac{m_e^2 v_{th,e}^3}{e^4 n_i \ln \Lambda} \\ &= \text{Braginskii momentum transfer collision time} \end{aligned}$$

The magnetic field

$$\begin{aligned}\mathbf{B} &= \frac{r}{qR}\frac{B_0}{1+(r/R)\cos\theta}\widehat{\mathbf{e}}_\theta \\ &+\frac{B_0}{1+(r/R)\cos\theta}\widehat{\mathbf{e}}_\varphi\end{aligned}$$

Approximating

$$\frac{\varepsilon}{q^2} \ll 1$$

one has

$$\begin{aligned}|\mathbf{B}| &\approx \frac{B_0}{h(\theta)} \\ &= \frac{B_0}{1+(r/R)\cos\theta}\end{aligned}$$

the guiding center velocity is

$$\begin{aligned}\frac{d\mathbf{r}}{dt} &= v_\parallel \widehat{\mathbf{n}} \\ &-\frac{m_e}{|e|\,B}\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)\frac{1}{B}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}B\end{aligned}$$

Adopting as coordines

$$(r,\theta,\varphi)$$

the velocirty components

$$\begin{aligned}\frac{rd\theta}{dt} &= \frac{r}{qR}v_\parallel + \frac{1}{\Omega_{ce}}\frac{v_\perp^2/2+v_\parallel^2}{R}\cos\theta \\ \frac{Rd\varphi}{dt} &= v_\parallel - \frac{r}{Rq}\frac{1}{\Omega_{ce}}\frac{v_\perp^2/2+v_\parallel^2}{R}\cos\theta \\ \frac{dr}{dt} &= \frac{1}{\Omega_{ce}}\frac{v_\perp^2/2+v_\parallel^2}{R}\sin\theta\end{aligned}$$

where

$$\Omega_{ce} = \frac{|e|\,B}{m_e}$$

Now it is extracted by factorization from the unknown functions f_m a periodic space dependence

$$\widetilde{f}_m(r,\theta) = f_m \exp\left[il\left(\varphi - q\theta\right)\right]$$

We express the equation in terms of $\widetilde{f}_m(r,\theta)$,

$$f_m = \widetilde{f}_m(r,\theta)\exp\left[-il(\varphi - q\theta)\right]$$

and in the derivatives that we must calculate in the drift-kinetic equation we act also on the exponential factor

$$\begin{aligned}
\frac{d\theta}{dt}\frac{\partial f_m}{\partial\theta} &= \frac{d\theta}{dt}\frac{\partial}{\partial\theta}\left\{\widetilde{f}_m(r,\theta)\exp\left[-il(\varphi-q\theta)\right]\right\} \\
&= \frac{1}{r}\left[\frac{r}{qR}v_\parallel + \frac{1}{\Omega_{ce}}\frac{v_\perp^2/2+v_\parallel^2}{R}\cos\theta\right] \\
&\quad\times\left\{\frac{\partial\widetilde{f}_m}{\partial\theta}\exp\left[-il(\varphi-q\theta)\right]\right. \\
&\quad\left.+\widetilde{f}_m\frac{\partial}{\partial\theta}\exp\left[-il(\varphi-q\theta)\right]\right\}
\end{aligned}$$

or

$$\begin{aligned}
\frac{d\theta}{dt}\frac{\partial f_m}{\partial\theta} &= \exp\left[-il(\varphi-q\theta)\right]\left(\frac{r}{qR}v_\parallel + \frac{1}{\Omega_{ce}}\frac{v_\perp^2/2+v_\parallel^2}{R}\cos\theta\right) \\
&\times\frac{1}{r}\left(\frac{\partial\widetilde{f}_m}{\partial\theta} + ilq\widetilde{f}_m\right)
\end{aligned}$$

Next

$$\begin{aligned}
\frac{d\varphi}{dt}\frac{\partial f_m}{\partial\varphi} &= \frac{d\varphi}{dt}\frac{\partial}{\partial\varphi}\left\{\widetilde{f}_m(r,\theta)\exp\left[-il(\varphi-q\theta)\right]\right\} \\
&= \frac{1}{R}\left(v_\parallel - \frac{r}{Rq}\frac{1}{\Omega_{ce}}\frac{v_\perp^2/2+v_\parallel^2}{R}\cos\theta\right) \\
&\quad\times\left\{\frac{\partial\widetilde{f}_m}{\partial\varphi}\exp\left[-il(\varphi-q\theta)\right]\right. \\
&\quad\left.+\widetilde{f}_m(r,\theta)\frac{\partial}{\partial\varphi}\exp\left[-il(\varphi-q\theta)\right]\right\}
\end{aligned}$$

or

$$\begin{aligned}
\frac{d\varphi}{dt}\frac{\partial f_m}{\partial\varphi} &= \exp\left[-il(\varphi-q\theta)\right]\left(v_\parallel - \frac{r}{Rq}\frac{1}{\Omega_{ce}}\frac{v_\perp^2/2+v_\parallel^2}{R}\cos\theta\right) \\
&\times\frac{1}{R}\left(\frac{\partial\widetilde{f}_m}{\partial\varphi} - il\widetilde{f}_m\right)
\end{aligned}$$

Finally the last term contains the derivation to the radial coordinate

$$\begin{aligned}
\frac{dr}{dt}\frac{\partial f_m}{\partial r} &= \frac{dr}{dt}\frac{\partial}{\partial r}\left\{\widetilde{f}_m(r,\theta)\exp\left[-il(\varphi-q\theta)\right]\right\} \\
&= \left(\frac{1}{\Omega_{ce}}\frac{v_\perp^2/2+v_\parallel^2}{R}\sin\theta\right) \\
&\quad\times\left\{\frac{\partial\widetilde{f}_m(r,\theta)}{\partial r}\exp\left[-il(\varphi-q\theta)\right]\right. \\
&\quad\left.+\widetilde{f}_m(r,\theta)\exp(-il\varphi)\frac{\partial}{\partial r}\exp\left[ilq\theta\right]\right\}
\end{aligned}$$

The last term is

$$\frac{\partial}{\partial r}\exp(ilq\theta) = il\theta\frac{dq}{dr}\exp(ilq\theta)$$

and the result is

$$\begin{aligned}
\frac{dr}{dt}\frac{\partial f_m}{\partial r} &= \exp\left[-il(\varphi-q\theta)\right]\left(\frac{1}{\Omega_{ce}}\frac{v_\perp^2/2+v_\parallel^2}{R}\sin\theta\right) \\
&\quad\times\left(\frac{\partial\widetilde{f}_m(r,\theta)}{\partial r}+il\theta q'\widetilde{f}_m(r,\theta)\right)
\end{aligned}$$

We express the term produced by the grad B and curvature in the equations of motion of a particle by the variables ξ and v,

$$\left(v_\perp, v_\parallel\right) \rightarrow (v,\xi)$$

as

$$\begin{aligned}
v_\perp^2/2+v_\parallel^2 &= \frac{1}{2}\left(v_\perp^2+2v_\parallel^2\right) = \frac{1}{2}\left(v^2+v_\parallel^2\right) \\
&= \frac{v^2}{2}\left(1+\frac{v_\parallel^2}{v^2}\right) \\
&= \frac{v^2}{2}\left(1+\xi^2\right)
\end{aligned}$$

$$v_\parallel = v\xi$$

This is an important quantity

$$\xi = \frac{v_\parallel}{v}$$

since it can be used to characterize the *trapping*. For small ξ the particle is trapped. However $\lambda = \frac{v_\perp^2}{v^2}\frac{1}{B(\mathbf{x})}$ is more useful for this.

Now we collect the terms.
The term with $\frac{\partial}{\partial\theta}$.

$$\exp\left[il\left(\varphi - q\theta\right)\right]\frac{d\theta}{dt}\frac{\partial f_m}{\partial\theta}$$
$$= \left(\frac{r}{qR}v_\parallel + \frac{1}{\Omega_{ce}}\frac{v_\perp^2/2 + v_\parallel^2}{R}\cos\theta\right)\frac{1}{r}\left(\frac{\partial\widetilde{f}_m}{\partial\theta} + ilq\widetilde{f}_m\right)$$
$$= \left[\frac{rv\xi}{qR} + \frac{v^2}{2\Omega_{ce}R}\left(1+\xi^2\right)\cos\theta\right]\left(\frac{\partial\widetilde{f}_m}{r\partial\theta} + \frac{ilq}{r}\widetilde{f}_m\right)$$

The term with $\frac{\partial}{\partial\varphi}$.

$$\exp\left[il\left(\varphi - q\theta\right)\right]\frac{d\varphi}{dt}\frac{\partial f_m}{\partial\varphi}$$
$$= \left[v\xi - \frac{rv^2}{2R^2q\Omega_{ce}}\left(1+\xi^2\right)\cos\theta\right]\left(\frac{\partial\widetilde{f}_m}{R\partial\varphi} - \frac{il}{R}\widetilde{f}_m\right)$$

the term with $\frac{\partial}{\partial r}$.

$$\exp\left[il\left(\varphi - q\theta\right)\right]\frac{dr}{dt}\frac{\partial f_m}{\partial r}$$
$$= \left[\frac{v^2}{2\Omega_{ce}R}\left(1+\xi^2\right)\sin\theta\right]\left(\frac{\partial\widetilde{f}_m\left(r,\theta\right)}{\partial r} + il\theta\frac{dq}{dr}\widetilde{f}_m\left(r,\theta\right)\right)$$

Collecting the terms that will contribute to the drift of the first order distribution function we take separately those that multiply $\widetilde{f}_m\left(r,\theta\right)$,

$$\left[\frac{rv\xi}{qR} + \frac{v^2}{2\Omega_{ce}R}\left(1+\xi^2\right)\cos\theta\right]\frac{ilq}{r}\widetilde{f}_m$$
$$+\left[v\xi - \frac{rv^2}{2R^2q\Omega_{ce}}\left(1+\xi^2\right)\cos\theta\right]\left(-\frac{il}{R}\widetilde{f}_m\right)$$
$$+\left[\frac{v^2}{2\Omega_{ce}R}\left(1+\xi^2\right)\sin\theta\right]il\theta q'\widetilde{f}_m$$

We note that the first terms in the first and second paranthesis can be combined

$$\frac{rv\xi}{qR}\frac{ilq}{r} + v\xi\left(-\frac{il}{R}\right) = \frac{ilv\xi}{R} - \frac{ilv\xi}{R} = 0$$

Then the other terms lead to

$$
\begin{aligned}
& \frac{v^2}{2\Omega_{ce}R}\left(1+\xi^2\right)\cos\theta\frac{ilq}{r} \\
& -\frac{rv^2}{2R^2q\Omega_{ce}}\left(1+\xi^2\right)\cos\theta\left(-\frac{il}{R}\right) \\
& +\frac{v^2}{2\Omega_{ce}R}\left(1+\xi^2\right)\sin\theta il\theta q' \\
= \; & il\frac{v^2}{2\Omega_{ce}Rr}\left(1+\xi^2\right)\left(q\cos\theta+\frac{\varepsilon^2}{q}\cos\theta+q'r\theta\sin\theta\right)
\end{aligned}
$$

which multiplies $\widetilde{f}_m$.

The drift term becomes

$$
\begin{aligned}
& \exp\left[il\left(\varphi-q\theta\right)\right]\left(\mathbf{v}_{\parallel}+\mathbf{v}_D\right)\cdot\boldsymbol{\nabla}f_m \\
= \; & \exp\left[il\left(\varphi-q\theta\right)\right]\left(\frac{d\theta}{dt}\frac{\partial f_m}{\partial\theta}+\frac{d\varphi}{dt}\frac{\partial f_m}{\partial\varphi}+\frac{dr}{dt}\frac{\partial f_m}{\partial r}\right) \\
= \; & \left[\frac{r}{qR}\xi v+\frac{1}{\Omega_{ce}}\frac{v^2}{2}\left(1+\xi^2\right)\frac{1}{R}\cos\theta\right]\frac{\partial\widetilde{f}_m}{r\partial\theta} \\
& +\frac{il}{r}\frac{1}{\Omega_{ce}}\frac{v^2}{2}\left(1+\xi^2\right)\frac{1}{R}\left(q\cos\theta+\frac{\varepsilon^2}{q}\cos\theta+\frac{dq}{dr}r\theta\sin\theta\right)\widetilde{f}_m \\
& +\frac{1}{\Omega_{ce}}\frac{v^2}{2}\left(1+\xi^2\right)\frac{1}{R}\sin\theta\frac{\partial\widetilde{f}_m}{\partial r}
\end{aligned}
$$

The equation is

$$
\begin{aligned}
& -i\omega\widetilde{f}_m+\left[\frac{rv\xi}{qR}+\frac{v^2\cos\theta}{2\Omega_{ce}R}\left(1+\xi^2\right)\right]\frac{\partial\widetilde{f}_m}{r\partial\theta} \\
& +\frac{ilv^2}{2\Omega_{ce}Rr}\left(q\cos\theta+rq'\theta\sin\theta\right)\left(1+\xi^2\right)\widetilde{f}_m \\
& +\frac{v^2\sin\theta}{2\Omega_{ce}R}\left(1+\xi^2\right)\frac{\partial\widetilde{f}_m}{\partial r} \\
& +(\text{new term}) \\
& -\frac{\nu_{ei}}{2}\frac{\partial}{\partial\xi}\left(1-\xi^2\right)\frac{\partial\widetilde{f}_m}{\partial\xi} \\
= \; & i\left(\omega-\omega_{Te}^{*}\right)\frac{|e|}{T_e}f_{Me}\exp\left[i\left(m-lq\right)\theta\right]
\end{aligned}
$$

The "new term" is a term of energy and comes from the variation of the perturbed distribution function in the space of the velocity. It actually originates from the first steps of derivation of the drift kinetic equation, before the averaging over the gyration motion. It comes from the term

$$\frac{d\mu}{dt}\frac{d}{d\mu}f$$

which will take into account the variation of the distribution function with μ or $v_\perp$ when the magnetic momentum has a time variation. The magnetic momentum can have time variation during the motion of the particle along the magnetic line. This is described by the equations of motion of the particle $d\mathbf{x}/dt$. The term is

$$-\frac{v\varepsilon\sin\theta}{qRh\left(\theta\right)}\frac{1-\xi^2}{2}\frac{\partial\widetilde{f}_m}{\partial\xi}$$

NOTE

The equations in **Wong Burrell** are

$$\begin{aligned}\frac{dv_\parallel}{dt} &= -\frac{v_\perp^2}{2}\nabla_\parallel\ln B+v_\parallel\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}\ln B-\frac{e}{m}\nabla_\parallel\phi\\ \frac{d}{dt}\left(\frac{v_\perp^2}{2}\right) &= \frac{v_\perp^2}{2}v_\parallel\boldsymbol{\nabla}_\parallel\ln B+\frac{v_\perp^2}{2}\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}\ln B\end{aligned}$$

and including also the equations for the trajectories

$$\phi=\phi_0\left(r\right)+\phi_1\left(r,\theta\right)$$

$$\frac{dx_\theta}{dt}=v_\parallel\frac{B_\theta}{B_T}+\frac{1}{B_0}\frac{d\phi_0}{dr}$$

$$\frac{dx_r}{dt}=-\frac{1}{B_0}\frac{\partial\phi_1}{r\partial\theta}-\frac{1}{\Omega_{ci}}\frac{v_\perp^2/2+v_\parallel^2}{R}\sin\theta$$

$$\frac{d}{dt}\left(\frac{v_\perp^2}{2}\right)=\left(\frac{v_\perp^2}{2}\right)v_\parallel\frac{B_\theta}{B_T}\frac{\sin\theta}{R}+\left(\frac{v_\perp^2}{2}\right)\frac{1}{B_0}\left(\frac{d\phi_0}{dr}\right)\frac{\sin\theta}{R}$$

$$\frac{dv_\parallel}{dt}=-\left(\frac{v_\perp^2}{2}\right)\frac{B_\theta}{B_T}\frac{\sin\theta}{R}+v_\parallel\frac{1}{B_0}\left(\frac{d\phi_0}{dr}\right)\frac{\sin\theta}{R}-\frac{e}{m}\frac{B_\theta}{B_T}\frac{\partial\phi_1}{r\partial\theta}$$

and in **Novakovskii** is

$$\begin{aligned}\frac{dv_\parallel}{dt} &= \left(-\frac{v_\perp^2}{2}\widehat{\mathbf{n}}+v_\parallel\mathbf{v}_E\right)\cdot\boldsymbol{\nabla}\ln B\\ \frac{d}{dt}\left(\frac{v_\perp^2}{2}\right) &= \frac{v_\perp^2}{2}\left(v_\parallel\widehat{\mathbf{n}}+\mathbf{v}_E\right)\cdot\boldsymbol{\nabla}\ln B\end{aligned}$$

See **Rotation**.tex.

Waves sustained by the gradient of the density and propagating in the poloidal direction. The frequency is comparable to ω_* an the poloidal wavenumber is small compared to ρ_s^{-1} (large perpendicular wavelengths compared to ρ_s).

There are two branches of drift waves in the tokamak plasma:

1. **Slab-like branch**, which is the toroidal version of the classical sheared slab model of Pearlstein and Berk. It is characterized by rapid variation of the wave field along the magnetic line (not too small $k_{\|}$). The eigenmodes are not bound modes and the energy is radiated in the radial direction to the point of the **ion Landau absorbtion.**

2. **The toroidicity induced branch** which results from the inclusion of the **ion magnetic drifts ∇B** and *curvature*. It has a bound state with slow variation along the magnetic line (slow variation on the connection length scale qR). The toroidicity induced modes do not experience shear damping. It is destabilized by any inverse electron dissipation mechanism (*i.e.* the electrons feed the wave).

From **Horton RMP 1999**

The drift wave couples to the ion-acoustic wave.

The ion-acoustic wave occurs as oscillations parallel to the magnetic field

$$\omega_{ac}(\mathbf{k}) = \pm \left(\frac{k_{\|}^2 c_s^2}{1 + k_{\perp}^2 \rho_s^2} \right)^{1/2}$$

These waves consist of *compressions* and *rarefaction* of the ion fluid along the magnetic line.

The electrons are *adiabatic.*

There is a *dispersion* of these waves, which means that the frequency depends on the wavenumber.

the dispresion of ω with the perpendicular wavenumber $k_{\perp}$ is due to the polarization current

$$\mathbf{J}_{pol} = \frac{\sum_a m_a n_a}{B^2} \frac{d\mathbf{E}_{\perp}}{dt}$$

According to **Horton** the reversible polarization currents are associated to the work

$$\mathbf{J}_{pol} \cdot \mathbf{E}$$

done to create the energy of the motion $E \times B$,

$$n_a \frac{m_a v_E^2}{2}$$

In the absence of the gradient of density $\omega_{*e} = k_y v_{dia,e} \rightarrow 0$ and the ion acoustic waves are absorbed by parallel ion Landau damping

$$\frac{\gamma_k^{(i)}}{\omega_{\mathbf{k}}} = c_s^2 \int d^3v \quad \pi\delta\left(\omega_{\mathbf{k}} - k_{\|}v_{\|}\right) k_{\|} \frac{\partial f_i(\mathbf{v})}{\partial v_{\|}}$$

due to the fact that the parallel phase velocity

$$\frac{\omega_{\mathbf{k}}}{k_{\|}} \simeq c_s$$

lies in the thermal ion velocity distribution.

An inhomogeneity due to the gradient of pressure makes $\omega_{*e} \neq 0$ and the coupling of the drift waves and the ion-acoustic waves leads to a different dispersion relation for the drift waves for the *fast waves*

$$\omega_{\mathbf{k}} = \frac{k_y v_{dia,e}}{1 + k_\perp^2 \rho_s^2} + \frac{k_{\|}^2 c_s^2}{k_y v_{dia,e}\left(1 + k_\perp^2 \rho_s^2\right)}$$

The *fast waves* have a phase velocity that is higher than the *sound* velocity hence they avoid the absorbtion by Landau damping

$$\frac{\omega}{k_{\|}} \gg c_s$$

The *fast waves* propagate along the magnetic field with high velocity

$$v_{g\|} = \frac{\partial\omega}{\partial k_{\|}} = 2c_s \frac{k_{\|}c_s}{\omega_{*e}}$$

and propagate perpendicular to the magnetic field lines with *low* velocity

$$v_{g\perp} = \frac{\partial\omega}{\partial k_x} = -2c_s\left(\frac{\rho_s}{L_n}\right)\left(k_x k_y \rho_s^2\right)$$

Due to the rapid propagation along the magnetic field lines, the eigenmodes are formed rapidly along the lines.

the ion-acoustic-drift waves have gradients (*i.e.* spatial variations)

1. perpendicular to the lines

$$k_\perp \sim \frac{1}{\rho_s}\left(\frac{\omega_{*e}}{\omega} - 1\right)^{1/2}$$

2. parallel to the lines

$$\left|k_{\|}\right| \leq \frac{|\omega_{*e}|}{c_s} = \frac{1}{L_n}\left|k_y\rho_s\right| < \frac{1}{L_n}$$

The fluctuations are long, thin filaments aligned with the helical magnetic field.

NOTE the fact that the eigenmode is formed very rapidly along the field line si due to the fast propagation of the wave in the parallel direction. If the perpendicular polarization current has zero-divergence, then the coupling with the parallel direction is broken. Or, the polarization current has zero divergence when $\rho_{eff} \to \infty$,

$$\nabla \cdot \mathbf{J}_{pol} \to 0 \quad \text{when} \quad \rho_{eff} \to \infty$$

END

Mechanism of saturation. Nonlinear ion-wave interaction (ion-Compton scattering): balance of electron-induced growth with transfer of the wave energy to long wavelengths by the ion Compton scattering process.

2 Estimations

The level of fluctuations [**?**]

$$\frac{e\varphi}{T_e} \sim \frac{1}{k_\perp L_n}$$

This is typically of the order of $1\% - 10\%$. Other estimation [**?**]:

$$\frac{e\varphi}{T_e} \sim \frac{\rho_s}{L_n}$$

The quantitative estimation of the potential fluctuation

$$\frac{e\varphi}{T} \sim \frac{\widetilde{n}}{n_0}$$

Then

$$\varphi = (T/e)\,\frac{\widetilde{n}}{n_0}$$

The velocity in an eddy

$$\begin{aligned} v_{eddy} &= \left|\frac{-\nabla\varphi \times \widehat{\mathbf{n}}}{B}\right| \\ &= k_y \frac{\varphi}{B} \end{aligned}$$

For example, for a fluctuation of the density of 1% at a temperature

$$T = 1000 \ (eV)$$

$$\frac{\widetilde{n}}{n_0} \sim 1\%$$

$$\begin{aligned} \varphi &\sim (T/e) \frac{\widetilde{n}}{n_0} \\ &= 1000 \times 10^{-2} \\ &= 10 \ (V) \end{aligned}$$

Take

$$\begin{aligned} k_y &= \frac{m}{r} = \frac{50}{1} \left(m^{-1}\right) \\ \lambda_y &= \frac{2\pi}{k_\perp} = \frac{1}{50} \ (m) = 0.02 \ (m) \\ &= 2 \ (cm) \end{aligned}$$

and

$$B = 3 \ (T)$$

The velocity in an *eddy* is

$$\begin{aligned} v_{eddy} &= k_y \frac{\varphi}{B} \quad \left[\frac{1}{m} \frac{V}{T}\right] \\ &= 50 \frac{10}{3} = 166 \ (m/s) \end{aligned}$$

In other cases one obtains much greater velocity

$$v_{eddy} \sim 10^5 \, (m/s)$$

The scaling of the diffusion coefficient

$$D \sim n^{2/3} T_e^{1/6}$$

The wavenumber spectrum decays as

$$S(k) \sim k^{-17/6}$$

The frequency spectrum decays as

$$S(\omega) \sim \omega^{-2}$$

The typical wavenumber

$$k_\perp \rho_s \lesssim 0.1 \ \text{ or } \ \lambda_\perp \gtrsim 10 \ \rho_s$$

3 General Theory

3.1 Phenomenology of Electron Drift Waves

The discussion of **Horton** in **RMP**. The physical picture starts from the existence of a local maximum of density of ions. Then there is rotation of the plasma (ions) in the field

$$\frac{\mathbf{E}\times\mathbf{B}}{B^2}\rightarrow\widehat{\mathbf{e}}_x\left(-\frac{d\phi}{dy}\frac{1}{B}\right)$$

The rotation induces a displacement of the maximum due to the variation of the density along the radial x direction,

$$\frac{dN}{dx}$$

The displacement is made in a time interval and it results a velocity which is the *electron diamagnetic velocity.* The direction of displacement is to the right in the picture and is the *electron* diamagnetic flow. However nothing was specifically related to electrons: the initial local maximum of potential is positive and is due to a local positive perturbation of the density of ions, an agglomeration of ions. From this local maximum of density of ions there are locally radial electric fields, pointing toward the exterior from the center of the perturbation. Combining with the equilibrium magnetic field this electric field gives a local rotation $\mathbf{E}\times\mathbf{B}/B^2$ which is locally counterclockwise. Now this counter-clockwise rotation takes from up (lower equilibrium density) a mass of ions and places it to the left point of the circle of rotation - eddy. nd takes from down (higher equilibrium density) a mass of ions and places it to the right of the circle of the eddy. The we get a new distribution of the density of the perturbation: the high density is now somewhere to the right, where the parcel of high density has been transported by the eddy. This means that the density perturbation has effectively moved to the right. For a magnetic field that points along the line of sight, away from the reader, this is clockwise within a magnetic surface. The *clockwise* direction is the direction of the *diamagnetic drift of the electrons.*

The drift waves with rotation are described by

$$\begin{aligned}&\frac{\rho_s^2}{r}\frac{\partial}{\partial r}\left(rN\left(r\right)\frac{\partial\phi}{\partial r}\right)\\&+N\left(r\right)\left[\frac{\omega_e^{dia}}{\omega-\mathbf{k}\cdot\mathbf{u}}-1+i\delta_k-k_y^2\rho_s^2+\frac{k_\parallel^2c_s^2}{\left(\omega-\mathbf{k}\cdot\mathbf{u}\right)^2}\right]\phi\\=\;&0\end{aligned}$$

where

$$\begin{aligned} \omega_e^{dia} &= k_y \frac{T_e}{-|e|B} \frac{1}{N} \frac{\partial N}{\partial r} \\ &= k_y v_e^{dia} \end{aligned}$$

(when the magnetic field is pointing from the reader toward the page, the electron diamagnetic flow is in the clockwise direction, or *negative* y axis). Magnetic shear

$$k_\parallel^2 = k_y^2 \frac{x^2}{L_s^2}$$

It results

$$\left[\rho_s^2 \frac{\partial^2}{\partial x^2} + \left(\lambda(\omega) + k_y^2 \frac{c_s^2 x^2}{\omega^2 L_s^2} \right) \right] \phi = 0$$

$$\lambda(\omega) = \frac{\omega_e^{dia}}{\omega} - 1 - k_y^2 \rho_s^2 + i\delta_k$$

The dispersion relation

$$\lambda_k(\omega) = \frac{\omega_e^{dia}}{\omega} - 1 - k_y^2 \rho_s^2 + i\delta_k = i \frac{L_n}{L_s} (2l+1) \frac{\left|\omega_e^{dia}\right|}{\omega}$$

Since we examine the state

$$\begin{aligned} \omega &= \omega^{lab} - k_y u \rightarrow -k_y u \\ \frac{\omega_e^{dia}}{\omega} &\rightarrow \frac{v_e^{dia}}{u} \lesssim 1 \\ & k_y \rightarrow 0 \end{aligned}$$

then

$$i\delta_k \approx i \frac{L_n}{L_s} (2l+1)$$

where

$$\frac{L_s}{L_n} \simeq 20...100$$

(**Tang Rewoldt Frieman**) is very large.

By contrast, when the density has fast spatial variation and the magnetic shear has slow spatial variation

$$\frac{L_n}{L_s} \ll 1$$

it results a small value for δ_k which means that the excitation is done very easily in this region of the parameters.

Let us take

$$\begin{aligned} L_s^{-1} &= \frac{\widehat{s}}{Rq} \\ &= \frac{rq'}{q}\frac{1}{Rq} \end{aligned}$$

It may be possible to have

$$L_s = \frac{Rq}{\widehat{s}} \sim \frac{Rq}{1/2} \sim \text{very large}$$

and

$$L_s \gg L_n$$

if L_n is calculated in the pedestal. Then δ_k can be very small.

The regime is:

1. large wavelengths along the poloidal direction, $k_y \to 0$;

2. slow drift wave fluctuations, $\omega \to 0$;

3. the electron drift velocity v_e^{dia} is opposite and almost equal to the rotation speed

$$\begin{aligned} \left| v_e^{dia} \right| &\lesssim |u| \quad \text{and} \\ v_e^{dia} &\sim \widehat{\mathbf{e}}_y \\ \mathbf{u} &\sim -\widehat{\mathbf{e}}_y \end{aligned}$$

   according to the picture of **Horton RMP**.

4. easy excitation

$$\delta_k \to 0$$

   in the pedestal, where $L_n \ll L_s$.

In physical terms this means that displacement to the right of the maxima of the local accumulation of ions (a perturbation of the density of ions) due to the $E \times B$ rotation induced by the electrostatic potential is *cancelled* by the flow of plasma that is contrary to this displacement.

3.2 Electrostatic drift waves in *slab* geometry

The magnetic field is along the $\widehat{\mathbf{e}}_z$ direction

$$\mathbf{B}_0 = B_0 \widehat{\mathbf{e}}_z$$

and the uniform gravitational field is taken along the $\widehat{\mathbf{e}}_x$ direction

$$\mathbf{G} = G\widehat{\mathbf{e}}_x.$$

The equations of motion are

$$\frac{d\mathbf{r}'}{dt} = \mathbf{v}'$$

$$\frac{d\mathbf{v}'}{dt} = \frac{q}{m}\mathbf{v}' \times \mathbf{B}_0 + G\widehat{\mathbf{e}}_x$$

with the initial condition

$$\mathbf{r}'(t' = t) = \mathbf{r} \qquad \text{and} \qquad \mathbf{v}'(t' = t) = \mathbf{v}.$$

The solution is:

$$\begin{aligned} v'_x(t') &= v_\perp \cos\left(\theta + \Omega(t' - t)\right) \\ v'_y(t') &= v_\perp \sin\left(\theta + \Omega(t' - t)\right) + \tfrac{G}{\Omega} \\ v'_z(t') &= v_z \end{aligned}$$

and

$$\begin{aligned} x'(t') &= x + \tfrac{v_\perp}{\Omega}\left[\sin\left(\theta + \Omega(t' - t)\right) - \sin\theta\right] \\ y'(t') &= y - \tfrac{v_\perp}{\Omega}\left[\cos\left(\theta + \Omega(t' - t)\right) - \cos\theta\right] + \tfrac{G}{\Omega}(t' - t) \\ z'(t') &= z + v_z(t' - t) \end{aligned}$$

where the *initial* vector velocity is represented by components

$$\mathbf{v} = (v_\perp \cos\theta, v_\perp \sin\theta, v_z)$$

and $\Omega = -qB_0/m$.

Vlasov's equation in a collisionless plasma for the species j of charge q_j and mass m_j :

$$\frac{\partial}{\partial t} f_j(\mathbf{r}, \mathbf{v}, t) + \mathbf{v} \cdot \boldsymbol{\nabla}_{\mathbf{r}} f_j(\mathbf{r}, \mathbf{v}, t) + \left(\frac{q_j}{m_j}\mathbf{E} + \frac{q_j}{m_j}\mathbf{v} \times \mathbf{B} + \mathbf{G}\right) \cdot \boldsymbol{\nabla}_{\mathbf{v}} f_j(\mathbf{r}, \mathbf{v}, t) = 0.$$

The Maxwell's equations are

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \sum_j q_j \int f_j d\mathbf{v}$$

$$\nabla \times \mathbf{B} = \mu_0 \sum_j q_j \int \mathbf{v} f_j d\mathbf{v} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

where we have denoted the particle species by the index j.

The drift kinetic equation:

$$\frac{d}{dt}\widetilde{f}_e = \left(-\left|e\right| v_\parallel \widehat{n} \cdot \nabla\varphi + i\varpi_{*e}\left|e\right|\varphi\right)\frac{\partial F_{0e}}{\partial \epsilon}$$

where $\epsilon = m_e v^2/2$. The time-change of $\widetilde{f}_e$ along the particle orbit is given by the balance of the two terms in the right hand side:

$\left(-\left|e\right| v_\parallel \widehat{n} \cdot \nabla\varphi\right)\frac{\partial F_{0e}}{\partial \epsilon}$ = time-change in the energy distribution function due to the acceleration of the electrons in the parallel electric field of the wave

This is because

$$-\left|e\right| v_\parallel \widehat{n} \cdot \nabla\varphi = v_\parallel \left(\left|e\right| E_\parallel\right) = v_\parallel F_\parallel^{elec}$$

is the mechanical work done by electrons against the parallel electric field.

$\left(i\varpi_{*e}\left|e\right|\varphi\right)\frac{\partial F_{0e}}{\partial \epsilon}$ = time-change in the energy due to motion of density elements driven in radial convection with potential energy $\left|e\right|\varphi$

This energy variation is connected with the diamagnetic flow and the term comes from the left-hand side, after linear expansion in the small amplitudes of the perturbations

$$\widetilde{f} \text{ and } \varphi$$

It is composed of the energy of an electron in the potential of the wave, $\left|e\right|\varphi$ which is "changed" in a time given by the inverse diamagnetic frequency $1/\omega_{*e}$. This would be incorrect since we know that the diamagnetic flow does not displace the particles. The term however cannot come from

$$\mathbf{v} \cdot \nabla\left(F_0 + \widetilde{f}\right)$$

since the diamagnetic velocity is a *fluid* velocity not a *particle* velocity therefore cannot be included in $\mathbf{v}$. In $\mathbf{v}$ onlt $v_\parallel$ and the drifts of the particle can enter. In particular the

$$\frac{-\boldsymbol{\nabla}\varphi\times\widehat{\mathbf{n}}}{B}\rightarrow\widetilde{v}_x$$

particle oscillations in the field of the electrostatic perturbation of the wave.

Important remark

The three-dimensional character and the toroidicity are different things.

The *toroidicity* usually means *ion drifts* v_{Di} and the necessity of representing quantities in the **ballooning representation**. This involves in the equation of the mode m the modes $m+1$ and $m-1$ but the coupling is linear (it is caused by the inclusion of the **ion drifts**, which contains the trigonometric functions cos and sin in the expression of ω_D, where also the **shear** parameter appears: $\widehat{s}$). It is not the cause of the energy spectral cascade. (Only the nonlinearitry can do that.)

The *three-dimensional character of the dynamics* of a particular mode means that the divergence of the transversal electrostatic fluctuating flow due to the $\mathbf{E}\times\mathbf{B}$ velocity is *non-zero* and then it must be completed with the consideration of the *parallel* component of the flow. And possibly the electron parallel dynamics is taken into account, in particular the electron response is taken adiabatic $|e|\,\varphi/T_e$ (as in Hasegawa-Mima eq.). And, possibly, the Landau damping is an active damping mechanism for the ion energy.

The equation for the wave amplitude of the **electron drift mode** (EDM):

$$\frac{d^2\phi\left(x\right)}{dx^2}+Q^{EDM}\left(x\right)\phi\left(x\right)=0$$

where

$$\begin{aligned}Q^{EDM}\left(x\right) &= \left(-1+\frac{1+1/\tau}{\Gamma_0}\frac{\omega}{\omega-\omega_{*i}}\right)+x^2\left[-\frac{1+1/\tau}{\Gamma_0}\frac{\omega}{\omega-\omega_{*i}}\left(\frac{k_y v_{thi}}{L_s\omega}\right)^2\right]\\ &\equiv a+x^2 b\end{aligned}$$

The eigenvalue is obtained from the condition

$$a=-i\sqrt{b}\left(2n+1\right)$$

and the corresponding eigenmodes are expressed in terms of Hermite polynomials

$$\phi\left(x\right)=H_n\left(\sqrt{i\sigma}\,x\right)\exp\left(-\frac{1}{2}i\sigma x^2\right)$$

with

$$\sigma = \frac{1}{L_s \overline{\rho}_i} \left(\frac{1+1/\tau}{\Gamma_0} \frac{\omega}{\omega - \omega_{*i}} \right)^{1/2} \frac{k_y v_{thi}}{\omega}.$$

The following notation has been introduced

$$\overline{\rho}_i^2 = \rho_i^2 \frac{d}{db} \ln \Gamma_0$$

This solution connects smoothly in the asymptotic region to the eikonal solution, $\phi(x) \sim \exp\left(i \int^x k_x dx\right)$. For large x the outgoing wave which was oscillatory for smaller x becomes spatially damped by *ion Landau damping.*

The form of the solution also permits to estimate the radial extension of the eigenmode,

$$x_T \simeq \frac{\omega}{k_y v_{thi} L_n^{-1}}.$$

sometimes called *ion-turning point*, where the phase velocity of the electrostatic perturbation, propagating along the z direction, reaches a value which is equal to the thermal velocity of the ions. Then there is absorbtion of the electrostatic perturbation energy by the ions and this puts a limit to the expansion of the wave in the radial (x) direction.

4 Integral equation for the drift wave eigenmodes (Tang Rewoldt Frieman)

The distribution function obeys the Vlasov equation. Two distribution functions are mentioned: F and f. It is cited **Tang NF 1978**.

The perturbed distribution function is f and the equation for it is

$$\begin{aligned} & \frac{\partial f}{\partial t} \\ & +\mathbf{v} \cdot \boldsymbol{\nabla} f \\ & -\frac{e}{m} \boldsymbol{\nabla} \Phi \cdot \left(\mathbf{v} \frac{\partial F}{\partial E} + \mathbf{v}_\perp \frac{1}{B} \frac{\partial F}{\partial \mu} + \frac{\widehat{\mathbf{n}} \times \mathbf{v}_\perp}{v_\perp^2} \frac{\partial F}{\partial \zeta} \right) \\ & -\Omega \frac{\partial f}{\partial \zeta} \\ = & \ 0 \end{aligned} \tag{1}$$

where

$$
\begin{aligned}
E &\equiv \frac{v^2}{2} \text{ (kinetic energy per unit mass)} \\
\mu &\equiv \frac{v_\perp^2}{2mB} \text{ (magnetic moment on unit mass)} \\
\zeta &\equiv \text{gyro-phase}
\end{aligned}
\tag{2}
$$

$$
\mathbf{B} = B_z \widehat{\mathbf{e}}_z + B_y(x)\,\widehat{\mathbf{e}}_y \tag{3}
$$

$$
\begin{aligned}
\frac{k_\parallel}{k_\perp} &= O(\varepsilon) \\
\frac{B_y}{B_z} &= O(\varepsilon)
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
\widehat{\mathbf{n}} &= \widehat{\mathbf{n}}^{(0)} + \widehat{\mathbf{n}}^{(1)} \\
&= \widehat{\mathbf{e}}_z + \frac{B_y}{B}\widehat{\mathbf{e}}_y
\end{aligned}
\tag{5}
$$

We **NOTE** that this form of the *drift-kinetic* equation results from the change of variable (see **neoclassic**)

$$
(\mathbf{x}, \mathbf{v}, t) \rightarrow (\mathbf{x}, \varepsilon, \mu, \zeta, t)
$$

and leads to

$$
\frac{\partial}{\partial \mathbf{v}} = \mathbf{v}\frac{\partial}{\partial \varepsilon} + \mathbf{v}_\perp \frac{\partial}{B\partial \mu} + \frac{\widehat{\mathbf{n}} \times \mathbf{v}_\perp}{v_\perp^2}\frac{\partial}{\partial \zeta}
$$

END.

The distribution function at equilibrium F contains the

The distribution function F is composed of the Maxwellian part plus the correction

$$
F = F_M + F^{(1)} \tag{6}
$$

and it has been shown that

$$
F^{(1)} = \frac{v_\perp}{\Omega}\sin\zeta \frac{\partial F_M}{\partial(\varepsilon x)} \tag{7}
$$

Note that we can recognize here

$$
\boldsymbol{\rho} \cdot \boldsymbol{\nabla} F_M \tag{8}
$$

the first order correction to the Maxwellian, due to the Finite Larmor Radius (FLR). The spatial derivation is made over distances comparable with ρ. This is the reason of taking the space in the derivation (εx) which is a small scale, while x is large. **End**.

Expansion in ε. To lowest order we have

$$f^{(0)} \tag{9}$$

with the equation

$$\begin{aligned} & \mathbf{v}_{\perp} \cdot \boldsymbol{\nabla}^{(0)} f^{(0)} \\ & -\frac{e}{m} \boldsymbol{\nabla}^{(0)} \Phi^{(0)} \cdot \mathbf{v}_{\perp} \frac{\partial F_M}{\partial E} \\ & -\Omega \frac{\partial f^{(0)}}{\partial \zeta} \\ = & \ 0 \end{aligned} \tag{10}$$

The notations

$$\begin{aligned} \boldsymbol{\nabla}_{\perp}^{(0)} &= \frac{\partial}{\partial \mathbf{x}} \\ \boldsymbol{\nabla}^{(1)} &= \widehat{\mathbf{e}}_x \frac{\partial}{\partial (\varepsilon x)} \end{aligned} \tag{11}$$

and

$$\nabla_z = \nabla_z^{(1)} = \widehat{\mathbf{e}}_z \frac{\partial}{\partial z} \tag{12}$$

We **NOTE** the presence of the term

$$-\frac{e}{m} \boldsymbol{\nabla}^{(0)} \Phi^{(0)} \cdot \mathbf{v}_{\perp} \frac{\partial F_M}{\partial E}$$

which can be transformed writting

$$\mathbf{v} \cdot \boldsymbol{\nabla} \Phi = \frac{d\Phi}{dt} - \frac{\partial \Phi}{\partial t}$$

and further

$$\begin{aligned} \frac{d\Phi}{dt} &\rightarrow \text{adiabatic term, } \frac{e\Phi}{T} \\ \frac{\partial \Phi}{\partial t} &\rightarrow -i\omega \Phi \end{aligned}$$

END.

It is time to separate in $f^{(0)}$ the adiabatic part which comes from the balance between the two terms in the 0-th order equation.

$$f^{(0)} = \frac{e}{m}\Phi^{(0)}\frac{\partial F_M}{\partial E} + h^{(0)} \tag{13}$$

$$\mathbf{v}_\perp \cdot \boldsymbol{\nabla}^{(0)} h^{(0)} - \Omega \frac{\partial}{\partial \zeta} h^{(0)} = 0 \tag{14}$$

Now $h^{(0)}$ is expanded in Fourier series: in y and z it is periodic. But we also expand in x with the Fourier variable k_x,

$$\begin{aligned} h^{(0)} &= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dk_x \overline{h}\left(E, \mu, \zeta, k_x\right) \\ &\quad \times \exp\left(ik_x x + ik_y y + ik_z z - i\omega t\right) \end{aligned} \tag{15}$$

Note that this is rather unusual.**End.**

Then the equation in 0-th order for the non-adiabatic part of the distribution function now reads

$$\begin{aligned} &\left(ik_x v_\perp \cos\zeta + ik_y v_\perp \sin\zeta - \Omega\frac{\partial}{\partial\zeta}\right)\overline{h} \\ &\times \exp\left[i\left(k_x x + k_y y\right)\right] \\ =\; & 0 \end{aligned} \tag{16}$$

The solution of this equation is obtained after *assuming* separation of the variable gyro-angle ζ from the others (E, μ, k_x),

$$h^{(0)} = \widehat{h}^{(0)}\left(E, \mu, k_x\right) g\left(\zeta\right) \tag{17}$$

and the operator acts only on the the function g:

$$\frac{ik_x v_\perp}{\Omega}\cos\zeta + \frac{ik_y v_\perp}{\Omega}\sin\zeta = \frac{1}{g}\frac{\partial g}{\partial \zeta} \tag{18}$$

$$g = \exp\left[\frac{iv_\perp}{\Omega}\left(k_x \cos\zeta - k_y \sin\zeta\right)\right] \tag{19}$$

Returning to $h^{(0)}$ we have

$$\begin{aligned} h^{(0)} &= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dk_x \widehat{h}^{(0)}\left(E, \mu, k_x\right) \\ &\quad \times \exp\left[i\left(k_x x + k_y y + k_z z - \omega t + L\right)\right] \end{aligned} \tag{20}$$

$$L \equiv \frac{v_\perp}{\Omega}\left(k_x \cos\zeta - k_y \sin\zeta\right) \tag{21}$$

To determine the unknown factor $\widehat{h}^{(0)}(E,\mu,k_x)$ we have to go to higher order in the expansion in ε and *average* the equation over the gyro-phase.

$$\begin{aligned}
&\frac{\partial f^{(0)}}{\partial t} \\
&+v_\parallel \widehat{\mathbf{n}}^{(0)}\cdot\boldsymbol{\nabla}^{(1)} f^{(0)} + v_\parallel \widehat{\mathbf{n}}^{(1)}\cdot\boldsymbol{\nabla}^{(0)} f^{(0)} \\
&+\mathbf{v}_\perp\cdot\boldsymbol{\nabla}^{(1)} f^{(0)} + \mathbf{v}_\perp\cdot\boldsymbol{\nabla}^{(0)} f^{(1)} \\
&-\frac{e}{m}\boldsymbol{\nabla}^{(0)}\Phi^{(0)}\cdot\left(\mathbf{v}\frac{\partial F^{(1)}}{\partial E}+\mathbf{v}_\perp\frac{1}{B}\frac{\partial F^{(1)}}{\partial \mu}+\frac{\widehat{\mathbf{n}}^{(0)}\times\mathbf{v}_\perp}{v_\perp^2}\frac{\partial F^{(1)}}{\partial\zeta}\right) \\
&-\frac{e}{m}\frac{\partial F_M}{\partial E}\left[\boldsymbol{\nabla}^{(1)}\Phi^{(0)}\cdot\mathbf{v}_\perp + v_\parallel\left(\widehat{\mathbf{n}}^{(1)}\cdot\boldsymbol{\nabla}^{(0)}+\widehat{\mathbf{n}}^{(0)}\cdot\boldsymbol{\nabla}^{(1)}\right)\Phi^{(0)}\right] \\
&-\Omega\frac{\partial f^{(1)}}{\partial\zeta} \\
=\;\; & 0
\end{aligned} \tag{22}$$

We **NOTE** the presence of the perturbation of the direction of the magnetic field

$$\widehat{\mathbf{n}}\rightarrow\widehat{\mathbf{n}}^{(0)}+\widehat{\mathbf{n}}^{(1)}$$

which is due to the *magnetic shear*

$$B_y=\frac{x}{L_s}B_0$$

Alternatively, it effectively means that *electromagnetic* effects are taken into consideration.

We **NOTE** that the *gradient* operator is perturbed too, which means that there are at least two spatial scales

$$\boldsymbol{\nabla}\rightarrow\boldsymbol{\nabla}^{(0)}+\boldsymbol{\nabla}^{(1)}$$

As has been explained above, the 0-th order distribution function $f^{(0)}$ can be separated in

- adiabatic part
$$\frac{e}{m}\Phi^{(0)}\frac{\partial F_M}{\partial E} \tag{23}$$
- and the non-adiabatic part, $h^{(0)}$.

This form of $f^{(0)}$ is introduced in the equation above, and this equation becomes an equation for the nonadiabaticpart

$$\begin{aligned}
& -i\omega h^{(0)} \\
& +\frac{e}{m}\frac{\partial \Phi^{(0)}}{\partial t}\frac{\partial F_M}{\partial E} \\
& +v_{\parallel}\widehat{\mathbf{n}}^{(0)}\cdot\boldsymbol{\nabla}^{(1)}h^{(0)}+v_{\parallel}\widehat{\mathbf{n}}^{(1)}\cdot\boldsymbol{\nabla}^{(0)}h^{(0)} \\
& +\mathbf{v}_{\perp}\cdot\boldsymbol{\nabla}^{(1)}h^{(0)}+\mathbf{v}_{\perp}\cdot\boldsymbol{\nabla}^{(0)}f^{(1)} \\
& +\frac{e}{m}\Phi^{(0)}\mathbf{v}_{\perp}\cdot\boldsymbol{\nabla}^{(1)}\frac{\partial F_M}{\partial E} \\
& -\frac{e}{m}\boldsymbol{\nabla}^{(0)}\Phi^{(0)}\cdot\left(\mathbf{v}\frac{\partial F^{(1)}}{\partial E}+\mathbf{v}_{\perp}\frac{1}{B}\frac{\partial F^{(1)}}{\partial \mu}+\frac{\widehat{\mathbf{n}}^{(0)}\times\mathbf{v}_{\perp}}{v_{\perp}^{2}}\frac{\partial F^{(1)}}{\partial \zeta}\right) \\
& -\Omega\frac{\partial f^{(1)}}{\partial \zeta} \\
= & \ 0
\end{aligned} \tag{24}$$

Now we expand the perturbation $\Phi^{(0)}$ potential in Fourier in all dimensions, therefore inclusiv k_x.

$$\Phi^{(0)}(x,y,z,t)=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}dk_x\widehat{\phi}(k_x)\exp(ik_xx+ik_yy+ik_zz-i\omega t) \tag{25}$$

The dependence of $\Phi^{(0)}(x,y,z,t)$ on (y,z) and on time (t) is *assumed* periodic. This means that the Fourier amplitude $\widehat{\phi}(k_x)$ remains to be only dependent on the Fourier variable on x, *i.e.* k_x.

A similar procedure is applied to the function $f^{(1)}$ with the difference that *we extract as a separate factor from the Fourier conjugate an exponential of the function L*. This is connected with the structure of the Fourier expansion obtained before for the (0)-th order

$$\begin{aligned}
h^{(0)} &= \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}dk_x\widehat{h}^{(0)}(E,\mu,k_x) \\
&\quad\times\exp\left[i\left(k_xx+k_yy+k_zz-\omega t+L\right)\right]
\end{aligned}$$

Then for the first order

$$\begin{aligned}
f^{(1)} &= \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}dk_x\widehat{f}^{(1)}(E,\mu,\zeta,k_x) \\
&\quad\times\exp(ik_xx+ik_yy+ik_zz-i\omega t) \\
&\quad\times\exp(iL)
\end{aligned} \tag{26}$$

The Fourier representations of the functions

$$\begin{aligned} & h^{(0)} \\ & \Phi^{(0)} \\ & f^{(1)} \end{aligned} \tag{27}$$

are introduced in the equation and it is taken the average over the gyrophase

$$\begin{aligned} & -i\omega\widehat{h}^{(0)} \\ & +ik_z v_\parallel \widehat{h}^{(0)} + ik_y v_\parallel \frac{B_y}{B}\widehat{h}^{(0)} \\ & +i\omega\frac{e\widehat{\phi}^{(0)}}{T}F_M J_0\left(\frac{k_\perp v_\perp}{\Omega}\right) \\ & +\frac{e}{m}\widehat{\phi}^{(0)}\left\langle \exp\left(-iL\right)\left\{ v_\perp\cos\zeta\frac{\partial^2 F_M}{\partial E\partial\left(\varepsilon x\right)} \right.\right. \\ & -ik_x\left[v_\perp\cos\zeta\left(\frac{\partial F^{(1)}}{\partial E}+\frac{1}{v_\perp}\frac{\partial F^{(1)}}{\partial v_\perp}\right)-\frac{1}{v_\perp}\sin\zeta\frac{\partial F^{(1)}}{\partial\zeta}\right] \\ & \left.\left. +ik_y\left[v_\perp\sin\zeta\left(\frac{\partial F^{(1)}}{\partial E}+\frac{1}{v_\perp}\frac{\partial F^{(1)}}{\partial v_\perp}+\frac{1}{v_\perp}\cos\zeta\frac{\partial F^{(1)}}{\partial v_\perp}\right)\right]\right\}\right\rangle \\ = & \ 0 \end{aligned} \tag{28}$$

with the notations

$$k_\perp \equiv \left(k_x^2+k_y^2\right)^{1/2} \tag{29}$$

$$\langle ...\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\zeta\left(...\right) \tag{30}$$

The average over the gyrophase of the expression L is

$$\begin{aligned} & \int_0^{2\pi} d\zeta \exp\left[\frac{v_\perp}{\Omega}\left(k_x\cos\zeta - k_y\sin\zeta\right)\right] \\ = & \ 2\pi J_0\left(\frac{k_\perp v_\perp}{\Omega}\right) \end{aligned}$$

where

$$k_\perp = \sqrt{k_x^2+k_y^2}$$

We are going to use the explicit form of the first correction to the Maxwellian distribution function, which is due to the *Finite Larmor Radius*

$$F^{(1)} = \frac{v_\perp}{\Omega}\sin\zeta\frac{\partial F_M}{\partial\left(\varepsilon x\right)} \tag{31}$$

The last term in the equation, containing the average over the gyrophase, can be simplified

$$\begin{aligned} & \frac{e}{m}\widehat{\phi}^{(0)}\left\langle \exp\left(-iL\right)\left[v_{\perp}\cos\zeta\frac{\partial^2 F_M}{\partial E\partial\left(\varepsilon x\right)}\right.\right. \\ & -ik_x\frac{v_{\perp}^2}{\Omega}\cos\zeta\sin\zeta\frac{\partial^2 F_M}{\partial E\partial\left(\varepsilon x\right)} \\ & \left.\left. -ik_y\left(\frac{v_{\perp}^2}{\Omega}\sin^2\left(\zeta\right)\frac{\partial^2 F_M}{\partial E\partial\left(\varepsilon x\right)}+\frac{1}{\Omega}\frac{\partial F_M}{\partial\left(\varepsilon x\right)}\right)\right]\right\rangle \end{aligned} \tag{32}$$

This equation can be written, taking separately the last term in the last paranthesis since it is not multiplied with trigonometricfunctions of the gyrophase ζ and leaving the other separately

$$\begin{aligned} = & -\frac{e}{m}\widehat{\phi}^{(0)}\frac{ik_y}{\Omega}\frac{\partial F_M}{\partial\left(\varepsilon x\right)}\left\langle\exp\left(-iL\right)\right\rangle \\ & +\frac{e}{m}\widehat{\phi}^{(0)}\frac{\partial^2 F_M}{\partial E\partial\left(\varepsilon x\right)}\left\langle\exp\left(-iL\right)\left[v_{\perp}\cos\zeta-i\left(k_x\cos\zeta+k_y\sin\zeta\right)\frac{v_{\perp}^2}{\Omega}\sin\zeta\right]\right\rangle \end{aligned} \tag{33}$$

Further, we note that the average of the factor $\langle\exp\left(-iL\right)\rangle$ gives a Bessel function.

$$\begin{aligned} = & -\frac{e}{m}\widehat{\phi}^{(0)}\frac{ik_y}{\Omega}\frac{\partial F_M}{\partial\left(\varepsilon x\right)}J_0\left(\frac{k_{\perp}v_{\perp}}{\Omega}\right) \\ & +\frac{e}{m}\widehat{\phi}^{(0)}\frac{\partial^2 F_M}{\partial E\partial\left(\varepsilon x\right)}v_{\perp}\left\langle\frac{\partial}{\partial\zeta}\left\{\sin\zeta\exp\left[i\frac{v_{\perp}}{\Omega}\left(k_y\cos\zeta-k_x\sin\zeta\right)\right]\right\}\right\rangle \end{aligned} \tag{34}$$

Since the average over the gyrophase of the second part is trivially zero, the final form of this term is

$$=-\frac{e}{m}\widehat{\phi}^{(0)}\frac{ik_y}{\Omega}\frac{\partial F_M}{\partial\left(\varepsilon x\right)}J_0\left(\frac{k_{\perp}v_{\perp}}{\Omega}\right) \tag{35}$$

We **NOTE** the nature of this term: the factors

$$\begin{aligned} -\frac{e}{m}\widehat{\phi}^{(0)}\frac{ik_y}{\Omega} & = -ik_y\widehat{\phi}^{(0)}\frac{1}{B} \\ & = \text{velocity } E\times B \text{ in the radial } (x) \text{ direction, in the wave field} \end{aligned}$$

and this advects the background distribution function. The Bessel function represents the effect of the *finite Larmor radius.* This is the first factor Bessel function and is due to the gyro-phase average

$$\langle\exp\left(-iL\right)\rangle$$

Later there will be another factor Bessel function, from

$$\langle \exp(+iL) \rangle$$

END

With this result we return to the equation for $\widehat{h}^{(0)}$ and obtain

$$\widehat{h}^{(0)} = \frac{e\widehat{\phi}^{(0)}}{T} F_M J_0 \left(\frac{k_\perp v_\perp}{\Omega}\right) \left(\frac{\omega - \omega_*^T}{\omega - k_\| v_\|}\right) \tag{36}$$

with the notations

$$\begin{aligned} k_\| &\equiv k_z + k_y \frac{B_y}{B} \\ \omega_*^T &\equiv \omega_* \left[1 + \eta \left(\frac{E}{T/m} - \frac{3}{2}\right)\right] \end{aligned} \tag{37}$$

$$\eta \equiv \frac{d \ln T}{d \ln n} = \frac{L_n}{L_T} \tag{38}$$

$$\omega_{*e} \equiv -k_y \frac{T}{|e|\, B} \frac{1}{L_n} \tag{39}$$

$$L_n^{-1} \equiv -\frac{1}{n} \frac{dn}{dx} \tag{40}$$

We **NOTE** that the *non-adiabatic* part of the perturbed distribution function exists due to a

1. drive

$$\omega - \omega_*^T$$

showing that the perturbation only can exist when the wave does not follow the diamagnetic frequency

2. a propagator

$$\frac{1}{\omega - k_\| v_\|}$$

representing the "integration along the particle trajectory " operator, applied on the potential of the wave $\phi(\mathbf{x}, t)$.

END

Up to here we have determined the Fourier transform of the non-adiabatic part of the potential $\widehat{h}^{(0)}$. Now we return to real space and perform the inverse Fourier transform

$$\begin{aligned} h^{(0)} &= \frac{e}{T}F_M\left(\frac{\omega-\omega_*^T}{\omega-k_\parallel v_\parallel}\right)\exp\left(ik_y y+ik_z z-i\omega t\right) \\ &\times\frac{1}{\left(2\pi\right)^{1/2}}\int_{-\infty}^{\infty}dk_x\widehat{\phi}^{(0)}\left(k_x\right)J_0\left(\frac{k_\perp v_\perp}{\Omega}\right)\exp\left(ik_x x+iL\right) \end{aligned} \tag{41}$$

We **note** that at this moment we have NOT taken the average over the gyrophase in the equation for $\widehat{h}^{(0)}$.

The non-adiabatic function $\widehat{h}^{(0)}$ is introduced in the expression of $f^{(0)}$ and the gyrophase average is taken

$$\begin{aligned} \left\langle f^{(0)}\right\rangle &= -\frac{e\Phi^{(0)}}{T}F_M \\ &+\frac{e}{T}F_M\left(\frac{\omega-\omega_*^T}{\omega-k_\parallel v_\parallel}\right)\exp\left(ik_y y+ik_z z-i\omega t\right) \\ &\times\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dk_x\widehat{\phi}^{(0)}\left(k_x\right)J_0^2\left(\frac{k_\perp v_\perp}{\Omega}\right)\exp\left(ik_x x\right) \end{aligned} \tag{42}$$

We **NOTE** the generation of the *second* factor Bessel function, coming from the average over the gyro-phase of the factor $\exp\left(iL\right)$ that is now a part of the *particular form* of the Fourier expansion assumed for the functions $h^{(0)}$ and $f^{(1)}$. **END.**

We have, due to the *magnetic shear*

$$\frac{B_y}{B}=\frac{x}{L_s} \tag{43}$$

It is approximated

$$k_z=0 \tag{44}$$

from which

$$k_\parallel=k_y\frac{x}{L_s} \tag{45}$$

After taking $k_z=0$ the Fourier phases remain to be composed of

$$\exp\left(ik_y y-i\omega t\right) \tag{46}$$

if the integration over k_x is carried out. This integration is formally done and the functions are formally replaced with functions depending on x

$$\Phi^{(0)}=\widetilde{\phi}\left(x\right)\exp\left(ik_y y-i\omega t\right) \tag{47}$$

$$n_j^{(0)} = \widetilde{n}_j(x)\exp(ik_y y - i\omega t) \tag{48}$$

The species are

$$\begin{aligned} j &\equiv i \text{ for ions} \\ j &\equiv e \text{ for electrons} \end{aligned} \tag{49}$$

The velocity-space integrations of the result obtained above $\left\langle f^{(0)}\right\rangle$ leads to

$$\begin{aligned} \widetilde{n}_i &= -\frac{|e|\,n}{T_i}\widetilde{\phi}(x) \\ &-\frac{|e|\,n}{T_i}\left(\frac{1}{v_{th,i}k_y\frac{|x|}{L_s}}Z(\xi_i)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dk_x\exp(ik_x x)\right. \\ &\quad\times\left\{\left[\omega-\omega_{*i}\left(1-\frac{3}{2}\eta_i\right)\right]\Gamma_0-\omega_{*i}\eta_i\left[\Gamma_0+b\left(\Gamma_1-\Gamma_0\right)\right]\right\} \\ &\quad\left.-\frac{\omega_{*i}\eta_i}{v_{th,i}k_y\frac{|x|}{L_s}}\left[\xi_i+\xi_i^2 Z(\xi_i)\right]\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dk_x\exp(ik_x x)\,\Gamma_0\right) \\ &\quad\times\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dk_x\exp(-ik_x x)\,\widetilde{\phi}(x) \end{aligned} \tag{50}$$

and for the electrons

$$\begin{aligned} \widetilde{n}_e &= \frac{|e|\,n}{T_e}\widetilde{\phi}(x) \\ &+\frac{|e|\,n}{T_e}\widetilde{\phi}(x)\left(\frac{\omega-\omega_{*e}}{v_{th,e}k_y\frac{|x|}{L_s}}Z(\xi_e)\right. \\ &\left.+\frac{\omega_{*e}\eta_e}{v_{th,e}k_y\frac{|x|}{L_s}}\left[\frac{1}{2}Z(\xi_e)-\xi_e-\xi_e^2 Z(\xi_e)\right]\right) \end{aligned} \tag{51}$$

The notations are

$$\begin{aligned} v_{th,i} &= \left(\frac{2T_i}{m_i}\right)^{1/2} \\ v_{th,e} &= \left(\frac{2T_e}{m_e}\right)^{1/2} \end{aligned} \tag{52}$$

$$\begin{aligned} \xi_i &= \frac{\omega}{v_{th,i}k_y\frac{|x|}{L_s}} = \frac{\left(\frac{\omega}{k_\parallel(x)}\right)}{v_{th,i}} = \frac{\text{parallel phase-velocity}}{\text{ion thermal velocity}} \\ \xi_e &= \frac{\omega}{v_{th,e}k_y\frac{|x|}{L_s}} = \frac{\left(\frac{\omega}{k_\parallel(x)}\right)}{v_{th,e}} = \frac{\text{parallel phase-velocity}}{\text{electron thermal velocity}} \end{aligned} \tag{53}$$

$$\begin{aligned}\Gamma_0 &= I_0 \exp(-b) \\ \Gamma_1 &= I_1 \exp(-b)\end{aligned} \tag{54}$$

$$\begin{aligned}b &= \frac{1}{2} k_\perp^2 \rho_i^2 \\ &= \frac{1}{2}\left(k_x^2 + k_y^2\right) \rho_i^2\end{aligned} \tag{55}$$

$$\rho_i = \frac{v_{th,i}}{\Omega_i} \tag{56}$$

$$k_\perp \rho_e = \frac{k_\perp v_{th,e}}{\Omega_e} \simeq \text{negligible} \tag{57}$$

4.1 Method of solution of the integral equation for the drift wave eigenfunction

The equation is quasi-neutrality

$$\widetilde{n}_e - \widetilde{n}_i \simeq 0 \tag{58}$$

The method of solution is the Ritz method, using a basis of functions for the perturbed potential and reducing the integral equation to an algebraic system.

$$\widetilde{\phi}(x) = \sum_{n=0}^{\infty} \widetilde{\phi}_n h_n(x) \tag{59}$$

$$h_n(x) \equiv \frac{1}{\left(\sqrt{\frac{\pi}{\sigma}} 2^n n!\right)^{1/2}} H_n\left(\sqrt{\sigma} k_y x\right) \exp\left(-\frac{1}{2}\sigma k_y^2 x^2\right) \tag{60}$$

where H_n is the Hermite polynomial of order n.

NOTE

In the paper **Blaizot** which starts from random matrices, one uses the Hrmite polynomials defined as

$$H_k(x) = (-1)^k \exp\left(\frac{Nx^2}{2}\right) \frac{d^k}{dx^k} \exp\left(-\frac{Nx^2}{2}\right)$$

with the Integral Representation

$$H_k(x) = (-iN)^k \sqrt{\frac{N}{2\pi}} \exp\left(\frac{Nx^2}{2}\right) \int_{-\infty}^{\infty} dq\ q^k \exp\left(-\frac{N}{2q^2} + iqxN\right)$$

With this set of polynomials it is also defined another set, *monic polynomials*, with the coefficient of the highest power equal to 1.

$$
\begin{aligned}
\pi_k(x) &= \frac{H_k(x)}{N^k} \\
&= \prod_{i=1}^{k} (x - \overline{x}_i)
\end{aligned}
$$

where

$$\overline{x}_i \equiv \text{the real zeros of the Hermite polynomials}$$

These *monic polynomials* π_k verify the equation

$$\pi_k(x,\tau) = (-i)^k \sqrt{\frac{N}{2\pi\tau}} \int_{-\infty}^{\infty} dq\; q^k \exp\left[-\frac{N}{2\tau}(q - ix)^2\right]$$

$$\int_{-\infty}^{\infty} dx \exp\left[-\frac{Nx^2}{2\tau}\right] \pi_k(x,\tau)\, \pi_m(x,\tau) = \delta_{km} \times n!\sqrt{2\pi}\left(\frac{\tau}{N}\right)^{n+\frac{1}{2}}$$

and

$$
\begin{aligned}
\frac{\partial}{\partial\tau}\pi_k(x,\tau) &= -\nu_s \frac{\partial^2}{\partial x^2}\pi_k(x,\tau) \\
\nu_s &\equiv \frac{1}{2N}
\end{aligned}
$$

END

Use of the Ritz 's technique means simply to

1. insert this expansion of $\widetilde{\phi}$ in series of basis functions $h_n(x)$ with coefficients $\widetilde{\phi}_n$.
2. multiply with one element of the basis
$$h_{n'}(x) \tag{61}$$
3. integrate over x
4. multiply (just for constants) with
$$\frac{T_e k_y}{|e|\, n} \tag{62}$$

It results

$$\sum_{n=0}^{\infty} L_{nn'}(\omega)\widetilde{\phi}_n = 0 \tag{63}$$

The coefficients of this system of algebraic equations are

$$\begin{aligned}
L_{nn'} &= \left(1+\frac{T_e}{T_i}\right)\delta_{nn'} \qquad (64)\\
&+k_y \int_{-\infty}^{\infty} dx h_{n'}(x)\left[\frac{\omega-\omega_{*e}}{v_{th,e}k_y\frac{|x|}{L_s}}Z(\xi_e)+\frac{\omega_{*e}\eta_e}{v_{th,e}k_y\frac{|x|}{L_s}}\left[\frac{1}{2}Z(\xi_e)-\xi_e-\xi_e^2 Z(\xi_e)\right]\right.\\
&+\frac{T_e}{T_i}\left(\frac{1}{v_{th,i}k_y\frac{|x|}{L_s}}Z(\xi_i)\int_{-\infty}^{\infty}\frac{dk_x}{(2\pi)^{1/2}}\exp(ik_x x)\left\{\left[\omega-\omega_{*i}\left(1-\frac{3}{2}\eta_i\right)\right]\Gamma_0\right.\right.\\
&\qquad\qquad\left.-\omega_{*i}\eta_i\left[\Gamma_0+b(\Gamma_1-\Gamma_0)\right]\right\}\\
&\left.\left.-\frac{\omega_{*i}\eta_i}{v_{th,i}k_y\frac{|x|}{L_s}}\left[\xi_i+\xi_i^2 Z(\xi_i)\right]\int_{-\infty}^{\infty}\frac{dk_x}{(2\pi)^{1/2}}\exp(ik_x x)\Gamma_0\right)\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}dx\exp(-ik_x)\right]h_n(x)
\end{aligned}$$

The particular advantage of the Hermite functions is that the Fourier transform can be expressed by the same Hermite functions. The following relationship exists

$$\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}dx\exp(-ik_x x)h_n(x)=\frac{(-i)^n}{\sigma^{1/2}k_y}h_n\left(\frac{k_x}{\sigma k_y^2}\right) \tag{65}$$

The collisionless electron drift waves are called *universal waves*.

5 The solution of the Drift-kinetic equation (Santarius Hinton)

The geometry is

$$(r,\theta,\varphi)$$

with

$$\theta = 0 \ \text{ on the equatorial plane, ouside}$$

The variables are

$$(\epsilon,\mu,\zeta,\mathbf{r},t)$$

$$
\begin{aligned}
\epsilon &= \frac{v^2}{2} - |e|\,\Phi \\
\mu &= \frac{v_\perp^2}{2B}
\end{aligned}
$$

and the Fokker-Planck equation for *electrons* is

$$
\begin{aligned}
& \frac{\partial f_e}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} f_e \\
& + \frac{\partial f_e}{\partial \epsilon}\frac{d\epsilon}{dt} + \frac{\partial f_e}{\partial \mu}\frac{d\mu}{dt} + \frac{\partial f_e}{\partial \zeta}\frac{d\zeta}{dt} \\
= \; & C f_e
\end{aligned}
$$

Approach based on the gyro-averaging followed by expansion

Assume

1. electrostatic waves

2. small gyroradius

and performing the gyrophase average

$$
\begin{aligned}
& \frac{\partial \overline{f}_e}{\partial t} + \left(v_\parallel \widehat{\mathbf{n}} + \mathbf{v}_{dr} + \mathbf{v}_E\right) \cdot \boldsymbol{\nabla} \overline{f}_e \\
& - \frac{e}{m_e}\frac{\partial \Phi}{\partial t}\frac{\partial \overline{f}_e}{\partial \epsilon} \\
= \; & C f_e
\end{aligned}
$$

where

$$
e \equiv |e|
$$

$$
\mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2}
$$

$$
\mathbf{v}_{dr} = -\frac{1}{\Omega_{ce}}\frac{1}{B}\left(\frac{v_\perp^2}{2} + v_\parallel^2\right)\widehat{\mathbf{n}} \times \boldsymbol{\nabla} B
$$

the $-$ sign comes from

$$
\frac{e_e B}{m_e} = -\frac{|e|\,B}{m_e} = -\Omega_{ce} < 0
$$

The equilibrium distribution function

$$
f_{Me} = \frac{1}{\left(\sqrt{\pi} v_{th,e}\right)^3} \exp\left(-\frac{v^2}{v_{th,e}^2}\right)
$$

$$v_{th,e} = \sqrt{\frac{2T_e}{m_e}}$$

We carry out an expansion in the *drift wave* small parameter, the amplitude of the electrostatic perturbation $n_e |e| \phi$ divided to the plasma density of energy $n_e T_e$, which is $e\Phi/T_e$.

The first order correction to the equilibrium distribution function is $f_e^{(1)}$.

This first order correction is composed of the *adiabatic* part and of the *non-adiabatic* part

$$f_e^{(1)} = \frac{e\Phi}{T_e} f_{Me} + f_{e1}$$

The dirft kinetic equation is *linearized* in the variables that describe the motion of the particles and the distribution function is inserted in the equation. The non-adiabatic part verifies the equation

$$\begin{aligned} & \frac{\partial f_{e1}}{\partial t} + \left(v_{\|} \widehat{\mathbf{n}} + \mathbf{v}_{dr} \right) \cdot \boldsymbol{\nabla} f_{e1} \\ & -C f_{e1} \\ = \; & -\frac{e}{T_e} f_{Me} \frac{\partial \Phi}{\partial t} - \mathbf{v}_E \cdot \boldsymbol{\nabla} f_{Me} \end{aligned}$$

The last term is

$$-\mathbf{v}_E \cdot \boldsymbol{\nabla} f_{Me} = -\frac{-\boldsymbol{\nabla}\Phi \times \widehat{\mathbf{n}}}{B} \cdot \boldsymbol{\nabla} f_{Me} \approx \frac{1}{B} \frac{\partial \Phi}{r \partial \theta} \frac{\partial f_{Me}}{\partial r}$$

Now one introduces the Fourier representation on the time variable

$$\exp\left(-i\omega t\right)$$

and obtain

$$\begin{aligned} & -i\omega f_{e1} \\ & + \left(v_{\|} \widehat{\mathbf{n}} + \mathbf{v}_{dr} \right) \cdot \boldsymbol{\nabla} f_{e1} \\ & -C f_{e1} \\ = \; & \frac{e}{T_e} f_{Me} \left\{ i\omega\Phi + \frac{T_e}{eB} \frac{\partial \Phi}{r \partial \theta} \frac{d \ln n}{dr} \left[1 + \eta_e \left(\frac{v^2}{v_{th,e}^2} - \frac{3}{2} \right) \right] \right\} \end{aligned}$$

The dependence on the other spatial variables.

We take

$$\exp\left(-il\varphi\right)$$

on the toroidal angle.

The potential is expanded in θ.

$$\Phi = \exp\left(-il\varphi\right) \sum_{m=-\infty}^{m=\infty} a_m \exp\left(im\theta\right)$$

and obtain

$$f_{e1} = \sum_{m=-\infty}^{m=\infty} f_m a_m$$

$$\begin{aligned} & -i\omega f_{em} \\ & + \left(v_{\parallel}\widehat{\mathbf{n}} + \mathbf{v}_{dr}\right) \cdot \boldsymbol{\nabla} f_m \\ & -C f_m \\ = \quad & i\left(\omega - \omega_{T*e,m}\right) \frac{e}{T_e} f_{Me} \exp\left(im\theta - il\varphi\right) \end{aligned}$$

where

$$\begin{aligned} \omega_{*Te,m} & \equiv \omega_{*e,m} \left[1 + \eta_e \left(\frac{v^2}{v_{th,e}^2} - \frac{3}{2}\right)\right] \\ \omega_{*e,m} & = -\frac{m}{r} \frac{T_e}{|e| B} \frac{d\ln n}{dr} \\ \eta_e & = \frac{d\ln T_e}{d\ln n} \end{aligned}$$

Comment on the expansion of the distribution function

The equilibrium distribution function is obtained from the drift-kinetic equation to zero order in the gyroradius.

There would be a first order correction in the *poloidal* gyroradius. This is the *neoclassical* correction to the Maxwellian.

5.0.1 Alternative development, *neoclassic*.

Start from the Fokker-Planck equation.

Find the first correction to the Maxwellian equilibrium distribution, *before averaging over the gyrophase*.

The result will contain the ρ_θ which is characteristic to *neoclassic*.

The term

$$\frac{e}{m_e} \boldsymbol{\nabla}\Phi \cdot \frac{\partial \widehat{f_e}}{\partial \mathbf{v}}$$

will produce the (thermal-) diamagnetic frequency $\omega_{Te,m}$.

This is because it will use the expression of $\widehat{f}_e$ which is

$$\widehat{f}_e = f_{Me}\left\{1 + \left[\frac{d\ln n}{dr} + \frac{d\ln T_e}{dr}\left(\frac{v^2}{v_{th,e}^2} - \frac{3}{2}\right)\right]\frac{m_e}{eB_\theta}v_\varphi\right\}$$

Note that a simplified form of this is

$$\begin{aligned}\widehat{f} &\sim f_{Me}\left\{1 + \frac{1}{L_n}\frac{v_\varphi}{(eB/m_e)}\right\} \\ &= f_{Me}\left(1 + \frac{1}{L_n}\frac{v_\varphi}{-\left|\Omega_{ci}^\theta\right|}\right) \\ &\approx f_{Me}\left(1 + \frac{\rho_\theta}{L_n}\right)\end{aligned}$$

End

In this approach, we have

$$\begin{aligned}&\frac{e}{m_e}\boldsymbol{\nabla}\Phi \cdot \frac{\partial \widehat{f}_e}{\partial \mathbf{v}} \\ =\ &\frac{1}{R}\frac{\partial \Phi}{\partial \varphi}\frac{1}{B_\theta}\left[\frac{d\ln n}{dr} + \frac{d\ln T_e}{dr}\left(\frac{v^2}{v_{th,e}^2} - \frac{3}{2}\right)\right] f_{Me}\end{aligned}$$

Taking

$$\Phi \sim \exp(-il\varphi)$$

we obtain

$$\begin{aligned}&\frac{e}{m_e}\boldsymbol{\nabla}\Phi \cdot \frac{\partial \widehat{f}_e}{\partial \mathbf{v}} \\ =\ &-i\omega_e^*\left[1 + \eta_e\left(\frac{v^2}{v_{th,e}^2} - \frac{3}{2}\right)\right]\frac{e\Phi}{T_e}f_{Me}\end{aligned}$$

where a *new* notation is introduced

$$\omega_e^* \equiv -\frac{lq}{r}\frac{T_e}{eB}\frac{d\ln n}{dr}$$

NOTE of **Santarius Hinton**:

Using the *first-order distribution function* as results from the *neoclassical-type* approach, *i.e.* solving the drift-kinetic equation without first averaging over the gyrophase, is probably a bad approximation:

the neoclassical distribution function in the first order is valid only for *trapped particles* and in particular for deeply trapped particles.

However the approximation that consists in replacing the expansion of the averaged Fokker-Plank equation by the use of the fisrt order non-averaged Fokker-Planck equation (neoclassical approach) is tolerable near resonant surfaces

$$ql - m \approx 0$$

END

5.0.2 The collision operator

Is Lorentz

$$Cf = \frac{1}{2}\nu_{ei}(v)\ \frac{\partial}{\partial \xi}\left(1-\xi^2\right)\frac{\partial f}{\partial \xi}$$

where

$$\xi = \frac{v_\parallel}{v}$$

$$\nu_{ei}(v) = \frac{3\sqrt{\pi}}{4}\frac{1}{\tau_e}\frac{v_{th,e}^3}{v^3}$$

and the Braginskii momentum transfer collision time

$$\tau_e = \frac{3}{16\sqrt{\pi}}\frac{m_e^2}{e^4}\frac{1}{\ln\Lambda}\frac{v_{th,e}^3}{n_i}$$

again we **note**

$$\nu \sim \tau_e^{-1} \sim \frac{n}{T^{3/2}}$$

The magnetic field

$$\begin{aligned}
\mathbf{B} &= \frac{B_0}{1+\varepsilon\cos\theta}\widehat{\mathbf{e}}_\varphi + \frac{rB_0}{qR\left(1+\varepsilon\cos\theta\right)}\widehat{\mathbf{e}}_\theta \\
|\mathbf{B}| &\approx \frac{B_0}{h(\theta)}
\end{aligned}$$

The guiding centre equations

$$\frac{d\mathbf{r}}{dt} = v_\parallel \widehat{\mathbf{n}} - \frac{1}{\Omega_e}\left(\frac{v_\perp^2}{2} + v_\parallel^2\right)\widehat{\mathbf{n}} \times \boldsymbol{\nabla}\ln B$$

and in the coordinates

$$(r, \theta, \varphi)$$

the equations are

$$
\begin{aligned}
\frac{dr}{dt} &= \frac{1}{\Omega_{e0}}\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{R}\sin\theta \\
\frac{rd\theta}{dt} &= \frac{r}{qR}v_\parallel + \frac{1}{\Omega_{e0}}\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{R}\cos\theta \\
\frac{Rd\varphi}{dt} &= v_\parallel - \frac{r}{qR}\frac{1}{\Omega_{e0}}\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{R}\cos\theta
\end{aligned}
$$

In Ω_{e0} it is used B_0.

The distribution function which is a wave perturbation is assumed to vary on the toroidal direction as

$$\exp\left(-il\varphi\right)$$

In order to simplify the space variation and to get a smooth function, one extracts another factor

$$\exp\left(ilq\theta\right)$$

Then the factor that involves the two angles and represents the fast oscillation is

$$\exp\left(il\varphi - ilq\theta\right)$$

The convective part is

$$
\begin{aligned}
& \exp\left[il\left(\varphi - q\theta\right)\right]\left(\mathbf{v}+\mathbf{v}_D\right)\cdot\boldsymbol{\nabla} f_m \\
= & \exp\left[il\left(\varphi - q\theta\right)\right]\left(\frac{dr}{dt}\frac{\partial f_m}{\partial r} + \frac{d\theta}{dt}\frac{\partial f_m}{\partial \theta} + \frac{d\varphi}{dt}\frac{\partial f_m}{\partial \varphi}\right) \\
= & \left[\frac{1}{qR}\xi v + \frac{1}{r}\frac{1}{\Omega_0}\frac{\frac{v^2}{2}\left(1+\xi^2\right)}{R}\right]\frac{\partial \widetilde{f}_m}{\partial\theta} \\
& + il\frac{1}{r}\left(q\cos\theta + \frac{\varepsilon^2}{q}\cos\theta + \frac{dq}{dr}r\theta\sin\theta\right)\frac{1}{\Omega_0}\frac{\frac{v^2}{2}\left(1+\xi^2\right)}{R}\widetilde{f}_m \\
& + \frac{1}{\Omega_0}\frac{\frac{v^2}{2}\left(1+\xi^2\right)}{R}\sin\theta\ \widetilde{f}_m
\end{aligned}
$$

where

$$\widetilde{f}_m\left(r,\theta\right) = f_m\left(r,\theta,\varphi\right)\exp\left[il\left(\varphi - q\theta\right)\right]$$

The term ε^2/q is neglected. Also the term with $\partial f_m/\partial r$.

The boundary between trapped and untrapped particles is

$$\xi^2_{boundary} = \frac{\varepsilon\cos^2\left(\theta/2\right)}{1+\varepsilon\cos\theta}$$

6 Dispersion relation for the drift waves

In **Petviashvili** and in **Pokhotelov Petviashvili** the relation of dispersion for the drift waves is

$$1 + k_\perp^2 \rho_s^2 + \frac{k_y v_*}{\omega} - \frac{\omega^2}{\Omega_c^2} = 0$$

The branch that has frequency much smaller than Ω is

$$\omega = -\frac{k_y v_*}{1 + k_\perp^2 \rho_s^2}$$

The dispersion relation for the drift waves is in **Current Carrying Drift waves Hatekayama**

$$\begin{aligned}
\varepsilon\left(\mathbf{k},\omega\right) &= 1 + \frac{k_{Di}^2}{k^2}\left[\eta + (1-\eta)\,\Gamma\left(b\right) + \Gamma\left(b\right)\frac{\omega + \omega_{*i}}{k_\parallel v_{i\parallel}} Z\left(\frac{\omega}{k_\parallel v_{i\parallel}}\right)\right] \\
&\quad + \frac{k_{De}^2}{k^2}\left[1 + \frac{\omega - k_\parallel u - \omega_{*e}}{k_\parallel v_e} Z\left(\frac{\omega - k_\parallel u}{k_\parallel v_e}\right)\right]
\end{aligned}$$

where

$$\begin{aligned}
k_{De} &= \frac{4\pi n_e e^2}{T_{e\parallel}} \\
k_{Di} &= \frac{4\pi n_i e^2}{T_{i\parallel}}
\end{aligned}$$

$$\begin{aligned}
\omega_{*e} &= k_y \frac{cT_{e\parallel}}{eB}\kappa \\
\omega_{*i} &= k_y \frac{cT_{i\parallel}}{eB}\kappa
\end{aligned}$$

$$\kappa \equiv -\frac{d}{dr}\ln n$$

$$\Gamma\left(b\right) = \exp\left(-b\right) I_0\left(b\right)$$

$$b \equiv \frac{k_y^2 \rho_i^2}{2}$$

$$\eta \equiv \frac{T_{i\parallel}}{T_{i\perp}}$$

7 Dissipative trapped electron mode

7.1 Basic trapped electron mode theory (Rewoldt Tang Frieman)

This is the paper by **Rewoldt Tang Frieman 1976**.

The definition of parameters.

the magnetic field

$$\mathbf{B} = \frac{B_0}{h(\theta)}\widehat{\mathbf{e}}_\varphi + \frac{B_\theta^0(r)}{h(\theta)}\widehat{\mathbf{e}}_\theta$$

where

$$h(\theta) = 1 + \varepsilon\cos\theta$$

$$B \equiv |\mathbf{B}| \approx \frac{B_0}{h(\theta)}$$

The differential along a magnetic field line

$$\begin{aligned} \frac{dl_\parallel}{B} &= \frac{Rd\varphi}{B_\varphi} \\ &= \frac{rd\theta}{B_\theta} \end{aligned}$$

where

$$R = R_0 h(\theta)$$

$$\begin{aligned} q(r) &= \frac{rB_0}{R_0 B_\theta^0} = \frac{r\frac{B_0}{h(\theta)}h(\theta)}{R_0\frac{B_\theta^0}{h(\theta)}h(\theta)} = \frac{rB_\varphi}{R_0 B_\theta} \\ &\approx \left.\frac{d\varphi}{d\theta}\right|_{along\ \mathbf{B}} \end{aligned}$$

In the velocity space

$$\left(v_\perp, v_\parallel\right) \longleftrightarrow (\epsilon, \Lambda)$$

$$\begin{aligned} \epsilon &= \frac{m_j}{2}\left(v_\perp^2 + v_\parallel^2\right) \\ \Lambda &= \frac{\mu B_0}{\epsilon} \end{aligned}$$

where

$$\mu \equiv \frac{m_j v_\perp^2}{2B}$$

$$
\begin{aligned}
v_{\parallel} &= \sigma_{\parallel} \sqrt{\frac{2}{m_j} (\epsilon - \mu B)} \\
&= \sigma_{\parallel} \, v \sqrt{1 - \frac{\Lambda}{h(\theta)}}
\end{aligned}
$$

where

$$\sigma_{\parallel} = sign\left(v_{\parallel}\right)$$

Then the *pitch angle variable*

$$
\begin{aligned}
\Lambda &= \frac{\mu B_0}{\epsilon} = \frac{m_j v_{\perp}^2}{2B} B_0 \frac{1}{\frac{m_j v^2}{2}} \\
&= \frac{v_{\perp}^2}{v^2} \frac{B_0}{\frac{B_0}{h}} = \frac{v_{\perp}^2}{v^2} h
\end{aligned}
$$

We see that *small* Λ means small $v_{\perp}$ relative to the full kinetic energy, which characterizes the *circulating* particles.

Conversely, the *trapped* particles have Λ large.

The type of trajectories

For

$$
\begin{aligned}
0 &\leq \Lambda < 1 - \varepsilon_0 \\
& \quad \text{circulating}
\end{aligned}
$$

For

$$
\begin{aligned}
1 - \varepsilon_0 &< \Lambda \leq 1 + \varepsilon_0 \\
& \quad \text{trapped}
\end{aligned}
$$

and the turning points are

$$\pm\theta_0 = \pm \arccos\left(\frac{\Lambda - 1}{\varepsilon_0}\right)$$

We **NOTE** that this is a narrow region in the velocity space.

The invariants of the particle motion

$$
\begin{aligned}
\epsilon &\equiv \text{kinetic energy} \\
\mu &\equiv \text{magnetic moment} \\
P_{\varphi} &\equiv \text{toroidal canonical angular momentum}
\end{aligned}
$$

$$\begin{aligned}
P_\varphi &= m_j R \left(v_\varphi - r\Omega_{\theta j} \right) \\
&\approx m_j R_0 \left(v_\parallel - r\Omega_{\theta j}^0 \right)
\end{aligned}$$

$$\begin{aligned}
\Omega_{\theta j} &= \frac{e_j B_\theta}{m_j} \\
\Omega_{\theta j}^0 &= \frac{e_j B_\theta^0}{m_j}
\end{aligned}$$

The bounce period for trapped particles

$$\begin{aligned}
\tau_{bounce} &= \oint dt = \oint \frac{d\theta}{v_\parallel} \frac{rB}{B_\theta} \\
&= qR_0 \oint \frac{d\theta}{v_\parallel(\theta)} \\
&= 2qR_0 \int_{-\theta_0}^{\theta_0} \frac{d\theta}{\left| v_\parallel(\theta) \right|}
\end{aligned}$$

and for circulating

$$\tau_{circ} = qR_0 \int_{-\pi}^{\pi} \frac{d\theta}{v_\parallel(\theta)}$$

The integration over the velocity space

$$\int d^3v = \frac{\pi}{2} \left(\frac{2}{m_j} \right)^{3/2} \sum_{\sigma_\parallel} \int_0^\infty d\epsilon \; \epsilon^{1/2} \; \int_0^{h(\theta)} d\Lambda \; \frac{1}{h(\theta)} \frac{1}{\sqrt{1 - \frac{\Lambda}{h(\theta)}}}$$

at a fixed poloidal angle θ.

Let us **note** that

$$\sqrt{1 - \frac{\Lambda}{h(\theta)}} = \frac{v_\parallel}{v} \frac{1}{\sigma_\parallel}$$

and

$$\frac{1}{h} = \frac{B_0}{h} \frac{1}{B_0} = \frac{B}{B_0}$$

then the last integration is

$$\frac{1}{B_0} v \sigma_\parallel \int_0^{h(\theta)} d\Lambda \frac{B}{v_\parallel}$$

and recall the change of variables in the derivation of the drift kinetic equation.

The surface of reference

$$q(r_0) = \frac{m_0}{l}$$

The Fourier expansion of the electrostatic potential $\widetilde{\Phi}$ separates

- a fast poloidal variation $\exp(-im_0\theta)$ and
- a slow poloidal variation, due to the envelope, $\widetilde{\phi}(\theta, r)$ of the eigenmode

Then

$$\widetilde{\Phi}(r, \theta, \varphi, t) = \widetilde{\phi}_l \exp(-i\omega t + il\varphi - im_0\theta)$$

The kinetic equation for

$$f = f^{(0)} + f^{(1)} + ...$$

where

$$f_j^{(0)} = f_{Mj}\left(1 + \widehat{f}_j\right)$$

with the Maxwellian

$$f_{Mj} = \frac{n_j(r)}{\left(\pi \frac{2T_j}{m_j}\right)^{3/2}} \exp\left[-\frac{\epsilon}{T_j(r)}\right]$$

and the first correction

$$\widetilde{f}_j = -\frac{v_\varphi}{\Omega_{\theta j}}\left[\frac{1}{n}\frac{dn}{dr} - \frac{1}{T_j}\frac{dT_j}{dr}\left(\frac{3}{2} - \frac{\epsilon}{T_j}\right)\right]$$

The radial distance is measured by a new variable, the *eikonal* $S(r)$. this is the physical distance $r - r_0$ measured in units of the spacing between two rational surfaces

$$\Delta r_s = \frac{1}{lq'}$$

The new radial variable is

$$S(r) = \frac{r - r_0}{\Delta r_s}$$

With the new radial variable we return to the Fourier expansion

$$\widetilde{\Phi}(r, \theta, \varphi, t) = \widetilde{\phi}(\theta, S) \exp\{il[\varphi - q(r)\theta]\} \exp(-i\omega t)$$

RTF observe that the combination

$$\begin{aligned} & \varphi - q(r)\theta \\ \approx \; & \text{const along the magnetic line} \end{aligned}$$

Then the variation of the mode along the magnetic field line is located to the factor

$$\widetilde{\phi}(S,\theta)\exp(iS\theta)$$

this is important since one of the operators that occur in the drift kinetic equation is the derivation along the magnetic line

$$\frac{\partial}{\partial l_{\parallel}}$$

and this leaaves approximately constant that combination

$$\frac{\partial}{\partial l_{\parallel}}\exp\left[il\left(\varphi - q(r)\theta\right)\right] \approx 0$$

Remember that we have an *electrostatic wave*, $\widetilde{\Phi}$.

In the first order

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + \left(\mathbf{v}_{\parallel} + \mathbf{v}_{De}\right)\cdot\boldsymbol{\nabla}\right) f_e^{(1)} \\ = \; & -\mathbf{v}_E^{(1)}\cdot\boldsymbol{\nabla} f_e^{(0)} \\ & -|e|\left(\mathbf{v}_{\parallel} + \mathbf{v}_{De}\right)\cdot\boldsymbol{\nabla}\widetilde{\Phi}\;\frac{\partial f_e^{(0)}}{\partial\epsilon} \\ & -\nu_f(\epsilon)\left(f_e^{(1)} - \frac{|e|\,\widetilde{\Phi}}{T_e} f_{Me}\right) \end{aligned}$$

The right hand side tells us that the perturbation produced by the advection of the main part (order zero) of the function $f_e^{(0)}$ in the fluctuating electric field $\mathbf{v}_E^{(1)}$ is balanced by the energy perturbation produced by the work on the distribution function done by the electric field: mainly parallel, $\nabla_{\parallel}\widetilde{\Phi}$ but including also the component along the drift velocity $\mathbf{v}_{De}$. And by the collisional friction acting upon the *non-adiabatic* part.

The zero order function contains the Maxwellian and also a "zero-order" - perturbation

$$f_e^{(0)} = f_{Me} + f_{Me}\widehat{f}_e$$

Then

$$\begin{aligned}
& \mathbf{v}_E^{(1)} \cdot \boldsymbol{\nabla} \left(f_{Me} + f_{Me} \widehat{f}_e \right) \\
= \ & \frac{-\boldsymbol{\nabla} \widetilde{\Phi} \times \widehat{\mathbf{n}}}{B_0} \cdot \boldsymbol{\nabla} f_{Me} \\
& + \frac{-\boldsymbol{\nabla} \widetilde{\Phi} \times \widehat{\mathbf{n}}}{B_0} \cdot \boldsymbol{\nabla} f_{Me} \ \widehat{f}_e \\
& + f_{Me} \ \frac{-\boldsymbol{\nabla} \widetilde{\Phi} \times \widehat{\mathbf{n}}}{B_0} \cdot \boldsymbol{\nabla} \widehat{f}_e
\end{aligned}$$

The last term is of higher order $\sim \widetilde{\Phi} \widehat{f}_e$. And the second term contains these two small factors. They are neglected

$$\begin{aligned}
\mathbf{v}_E^{(1)} \cdot \boldsymbol{\nabla} \left(f_{Me} + f_{Me} \widehat{f}_e \right) &= \frac{-\boldsymbol{\nabla} \widetilde{\Phi} \times \widehat{\mathbf{n}}}{B_0} \cdot \boldsymbol{\nabla} f_{Me} \\
&= \frac{1}{B_0} \frac{\partial \widetilde{\Phi}}{r \partial \theta} \frac{\partial f_{Me}}{\partial r}
\end{aligned}$$

The next contribution is energetic. The wave field produces an electric force $-|e| \left(-\boldsymbol{\nabla} \widetilde{\Phi} \right)$ that has a parallel projection, along the line. Then it works with the velocity $\mathbf{v}_\parallel$ and produces a perturbation of the distribution on energy of the particles. The wave field accelerates or decelerates particles.

$$\begin{aligned}
& - |e| \left(\mathbf{v}_\parallel + \mathbf{v}_{De} \right) \cdot \boldsymbol{\nabla} \widetilde{\Phi} \ \frac{\partial f_e^{(0)}}{\partial \epsilon} \\
= \ & - |e| \left(\mathbf{v}_\parallel + \mathbf{v}_{De} \right) \cdot \boldsymbol{\nabla} \widetilde{\Phi} \ \frac{\partial}{\partial \epsilon} \left(f_{Me} + f_{Me} \widehat{f}_e \right) \\
= \ & - |e| \left(\mathbf{v}_\parallel + \mathbf{v}_{De} \right) \cdot \boldsymbol{\nabla} \widetilde{\Phi} \left(-\frac{1}{T_e} \right) f_{Me} \\
& + \text{term from } d\mathbf{r}/d\epsilon
\end{aligned}$$

The last term is given in **rewoldt tang frieman** as

$$\begin{aligned}
& \text{term from } d\mathbf{r}/d\epsilon \text{ is} \\
& - |e| \left(\mathbf{v}_\parallel + \mathbf{v}_{De} \right) \cdot \boldsymbol{\nabla} \widetilde{\Phi} \ \frac{\partial f_e^{(0)}}{\partial \epsilon} \\
= \ & - |e| \left(\mathbf{v}_\parallel + \mathbf{v}_{De} \right) \cdot \boldsymbol{\nabla} \widetilde{\Phi} \ \frac{1}{\Omega_{\theta e}} \left(\frac{v_\parallel}{T_e} - \frac{1}{m_e v_\parallel} \right) \frac{\partial f_{Me}}{\partial r}
\end{aligned}$$

suggesting that

$$\left. \frac{\partial r}{\partial \epsilon} \right|_{banana} = \frac{1}{\Omega_{\theta e}} \left(\frac{v_\parallel}{T_e} - \frac{1}{m_e v_\parallel} \right)$$

NOTE

Since the total time derivative is taken along the orbit of the trapped particle (banana)

$$\left.\frac{df_e^{(0)}}{dt}\right|_{banana} = \left[\frac{\partial}{\partial t} + \left(\mathbf{v}_\parallel + \mathbf{v}_{De}\right)\cdot\boldsymbol{\nabla}\right] f_e^{(0)}\Big|_{banana}$$

we must consider in the term

$$\left.\frac{\partial f_{Me}}{\partial \epsilon}\right|_{banana}$$

the trajectory of the particle, *i.e.* the banana

$$\frac{\partial f_{Me}}{\partial \epsilon} = \left.\frac{\partial f_{Me}}{\partial r}\frac{\partial r}{\partial \epsilon}\right|_{banana}$$

The trajectory is (see **neoclassic**)

$$\begin{aligned}
\frac{dr}{dt} &= -\frac{\frac{\mu B_0}{m} + v_\parallel^2}{\Omega}\frac{\sin\theta}{R_0} \\
\frac{rd\theta}{dt} &= \frac{\varepsilon}{q}v_\parallel - \frac{\frac{\mu B_0}{m} + v_\parallel^2}{\Omega R_0}\cos\theta + \frac{1}{B_z}\frac{d\Phi_0}{dr}
\end{aligned}$$

and the equations are integrated

$$r - r_0 = \left[\frac{\mu B_0 + m v_\parallel^2}{m\Omega_c R}\frac{1}{\Delta v_\parallel}\right] r_0\left(\cos\theta - 1\right)$$

Here

$$\begin{aligned}
\frac{\partial r}{\partial \epsilon} &= \frac{\partial}{\partial \epsilon}\frac{1}{m\Omega_{cj}R}\frac{1}{\Delta v_\parallel} m v_\parallel^2 r_0\left(\cos\theta - 1\right) \\
&= \frac{1}{m\Omega_{cj}R}\frac{1}{\Delta v_\parallel} r_0\left(\cos\theta - 1\right) m 2 v_\parallel \frac{\partial v_\parallel}{\partial \epsilon}
\end{aligned}$$

where

$$\frac{\partial v_\parallel}{\partial \epsilon} = \frac{1}{v_\parallel}$$

END NOTE

We continue with this equation. We have

$$\frac{\partial f_{Me}}{\partial r} = f_{Me}\left[\frac{1}{n_e}\frac{dn_e}{dr} - \frac{1}{T_e}\frac{dT_e}{dr}\left(\frac{3}{2} - \frac{\epsilon}{T_e}\right)\right]$$

Regarding the two terms that occur is $\frac{\partial r}{\partial \epsilon} = \frac{1}{\Omega_{\theta e}} \left(\frac{v_{\|}}{T_e} - \frac{1}{m_e v_{\|}} \right)$:

the first term, $\frac{v_{\|}}{T_e}$ is smaller than the second term $\frac{1}{m_e v_{\|}}$

$$\frac{v_{\|}}{T_e} \ll \frac{1}{m_e v_{\|}}$$

for the trapped and for the barely circulating particles. This is because for trapped particles the parallel velocity (and energy) is much smaller than the perpendicular one, - this is why they are trapped.

On the other hand between the two velocities that occur in the factor, we have

$$\left| \mathbf{v}_{\|} \right| \gg \left| \mathbf{v}_D \right|$$

and $\mathbf{v}_D$ is neglected. Now that term is

$$\begin{aligned}
& -\left|e\right| \left(\mathbf{v}_{\|} + \mathbf{v}_{De} \right) \cdot \boldsymbol{\nabla} \widetilde{\Phi} \; \frac{1}{\Omega_{\theta e}} \left(\frac{v_{\|}}{T_e} - \frac{1}{m_e v_{\|}} \right) \frac{\partial f_{Me}}{\partial r} \\
\approx \quad & -\left|e\right| v_{\|} \nabla_{\|} \widetilde{\Phi} \frac{1}{\Omega_{\theta e}} \left(-\frac{1}{m_e v_{\|}} \right) \frac{\partial f_{Me}}{\partial r} \\
= \quad & -\left[\frac{-\left|e\right|}{m_e \Omega_{\theta e}} \nabla_{\|} \widetilde{\Phi} \right] \frac{\partial f_{Me}}{\partial r}
\end{aligned}$$

The term of advection of the zeroth-order distribution function by the electric field velocity, which is first order

$$\begin{aligned}
& -\widetilde{\mathbf{v}}_E^{(1)} \cdot \boldsymbol{\nabla} f_e^{(0)} \\
= \quad & -\widetilde{v}_{Er}^{(1)} \frac{\partial f_e^{(0)}}{\partial r}
\end{aligned}$$

Further, we can make a replacement that will make the equation easier to integrate. Essentially, we note that the convective part of the full derivation operator $\mathbf{v} \cdot \boldsymbol{\nabla}$ can be replaced by the difference $d/dt - \partial/\partial t$. Then

$$\frac{\left|e\right|}{T_e} \left(\mathbf{v}_{\|} + \mathbf{v}_{De} \right) \cdot \boldsymbol{\nabla} \widetilde{\Phi} \; f_{Me} = \frac{\left|e\right|}{T_e} f_{Me} \left(\left. \frac{d}{dt} \right|_{banana} - \frac{\partial}{\partial t} \right) \widetilde{\Phi}$$

and to this we will add the friction term

Then we can return to the equation

$$\begin{aligned}\left.\frac{df_e^{(1)}}{dt}\right|_{banana} &= \left(\frac{\partial}{\partial t}+\left(\mathbf{v}_{\|}+\mathbf{v}_{De}\right)\cdot\boldsymbol{\nabla}\right)f_e^{(1)} \\ &= \frac{|e|}{T_e}f_{Me}\left(\left.\frac{d}{dt}\right|_{banana}-\frac{\partial}{\partial t}+i\nu_f\right)\widetilde{\Phi} \\ &\quad -\left[-\frac{|e|}{m_e\Omega_{\theta e}}\nabla_{\|}\widetilde{\Phi}+\widetilde{v}_{Er}^{(1)}\right]\frac{\partial f_{Me}}{\partial r} \\ &\quad -\nu_f f_e^{(1)}\end{aligned}$$

where we replace the operator $\nabla_{\|}=\frac{1}{qR}\frac{\partial}{\partial\theta}$ and introduce the time Fourier expansion,

$$\begin{aligned}&\left.\frac{df_e^{(1)}}{dt}\right|_{banana} \\ &= \frac{|e|}{T_e}f_{Me}\left(\left.\frac{d}{dt}\right|_{banana}+i\omega+i\nu_f\right)\widetilde{\Phi} \\ &\quad -\frac{1}{RB_\theta}\frac{\partial f_{Me}}{\partial r}\frac{\partial\widetilde{\Phi}}{\partial\varphi} \\ &\quad -\nu_f f_e^{(1)}\end{aligned}$$

One replaces

$$\frac{\partial\widetilde{\Phi}}{\partial\varphi}=il\widetilde{\Phi}$$

$$\left(\left.\frac{d}{dt}\right|_{banana}+\nu_f\right)f_e^{(1)}=\frac{|e|}{T_e}f_{Me}\left[\left(\left.\frac{d}{dt}\right|_{banana}+\nu_f\right)+i\left(\omega-\omega_{*e}^T\right)\right]\widetilde{\Phi}$$

where

$$\omega_{*e}^T=\omega_*\left[1+\eta_e\left(\frac{\epsilon}{T_e}-\frac{3}{2}\right)\right]$$

$$\begin{aligned}\omega_{*e} &= \frac{l}{R_0}\frac{T_e}{|e|\,B_\theta^{(0)}}\frac{1}{n}\frac{dn}{dr} \\ &= \frac{m_0}{r_0}\frac{T_e}{|e|\,B_0}\frac{d}{dr}\ln n\end{aligned}$$

because

$$\begin{aligned}\frac{l}{R_0} &= \frac{m_0}{r_0}\frac{B_\theta^{(0)}}{B_0} \\ m_0-q\left(r_0\right)l &= 0\end{aligned}$$

It is defined the *eikonal* like in optics

$$\begin{aligned} S(r) &= q(r)\,l - m \\ &= l\,[q(r) - q(r_0)] \end{aligned}$$

We note that this is zero at the resonant surface

$$S(r_0) = 0$$

and

$$S(r) \approx \frac{r - r_0}{\Delta r_s}$$

where the distance between two resonant surfaces is

$$\Delta r_s = \left(l \frac{dq}{dr} \right)^{-1} = \frac{1}{lq'}$$

A Fourier expansion of the fluctuation of the potential is expressed in terms of these variables

$$\begin{aligned} \widetilde{\Phi}(r, \theta, \varphi) &= \widetilde{\phi}(S, \varphi) \exp[iS(r)\,\theta] \\ &\quad \times \exp[il\varphi - ilq(r)\,\theta] \\ &\quad \times \exp[-i\omega t] \end{aligned}$$

It is noted that the combination

$$il\,[\varphi - q(r)\,\theta] = \text{ const along a magnetic line}$$

Then the variation of $\widetilde{\Phi}$ along a field line is due to the first factor $\widetilde{\phi}(S, \varphi) \exp[iS(r)\,\theta]$.

With this form it is possible to obtain an approximation of the variation of $\widetilde{\Phi}$ in the parallel direction

$$\begin{aligned} \frac{\partial \widetilde{\Phi}}{\partial l_{\|}} &= \exp[il\varphi - ilq(r)\,\theta] \quad \text{(no parallel variation)} \\ &\quad \times \exp(-i\omega t) \\ &\quad \times \frac{\partial}{\partial l_{\|}} \widetilde{\phi}[S(r), \theta] \exp[iS(r)\,\theta] \quad \text{(here is parallel variation)} \end{aligned}$$

Since we have

$$\begin{aligned} \frac{\partial}{\partial l_{\|}} &= \frac{1}{qR_0} \frac{\partial}{\partial \theta} = \frac{B_\theta}{B_T} \frac{\partial}{r \partial \theta} \\ &= \text{parallel projection of the variation on poloidal direction} \end{aligned}$$

we have

$$
\begin{aligned}
\frac{\partial\widetilde{\Phi}}{\partial l_{\parallel}} &= \exp\left[il\varphi - ilq\left(r\right)\theta\right]\exp\left(-i\omega t\right)\frac{1}{qR_0}\frac{\partial}{\partial\theta}\left(\widetilde{\phi}\left[S\left(r\right),\theta\right]\exp\left[iS\left(r\right)\theta\right]\right) \\
&= \exp\left[il\varphi - ilq\left(r\right)\theta\right]\exp\left(-i\omega t\right)\frac{1}{qR_0}\exp\left[iS\left(r\right)\theta\right]\left(iS+\frac{\partial}{\partial\theta}\right)\widetilde{\phi}\left[S\left(r\right),\theta\right]
\end{aligned}
$$

the friction parameter can be absorbed in an expression of exponential multiplying the two function $f_e^{(1)}$ respectively $\widetilde{\Phi}$.

$$
\left(\left.\frac{d}{dt}\right|_{banana}\right) f_e^{(1)}\exp\left(\nu_f t\right) = \frac{|e|}{T_e}f_{Me}\left[\left.\frac{d}{dt}\right|_{banana} + i\left(\omega-\omega_{*e}^T\right)\right]\widetilde{\Phi}\exp\left(\nu_f t\right)
$$

This equation can be integrated along the trajectory *banana*,

$$
\begin{aligned}
f_e^{(1)} &= \frac{|e|\,\widetilde{\Phi}}{T_e}f_{Me} \\
&\quad + i\left(\omega-\omega_{*e}^T\right)\frac{|e|}{T_e}f_{Me}\ \exp\left(-\nu_f t\right)\int_{-\infty}^{t}dt'\widetilde{\Phi}\exp\left(\nu_f t'\right)
\end{aligned}
$$

In the first term we recognize the *adiabatic term*.
The second is an integration along the orbits.
The orbit is a banana with the conditions

$$
\begin{aligned}
\theta\left(t'=t\right) &= \theta \\
r\left(t'=t\right) &= r
\end{aligned}
$$

The first order function $f_e^{(1)}$ is represented as a Fourier series

$$
f_e^{(1)} = \widetilde{f}_e\exp\left[-i\omega t + il\varphi - im_0\theta\right]
$$

and similar for the potential $\widetilde{\Phi}$, as above

$$
\begin{aligned}
\widetilde{\Phi}\left(r,\theta,\varphi\right) &= \widetilde{\phi}\left(S,\varphi\right)\exp\left[iS\left(r\right)\theta\right] \\
&\quad\times\exp\left[il\varphi - ilq\left(r\right)\theta\right] \\
&\quad\times\exp\left[-i\omega t\right]
\end{aligned}
$$

Then we transfer the Fourier factors from the left hand side to the right hand side, since they are not involved in the integration they combine with the factors over wich we integrate and exhibit those parts that indeed represent the time integration.

$$\begin{aligned}
\widetilde{f}_e &= \frac{|e|}{T_e} f_{Me} \left\{ \widetilde{\phi}(S,\theta) \right. \\
&+ i\left(\omega - \omega_{*e}^T\right) \int_{-\infty}^{t} dt' \widetilde{\phi}\left[S\left[r\left(t'\right)\right], \theta\left(t'\right)\right] \exp\left[-i\left(\omega + i\nu_f\right)\left(t - t'\right)\right] \\
&\times \exp\left[il\varphi\left(t'\right) - im_0\theta\left(t'\right) - il\varphi + im_0\theta\right]
\end{aligned}$$

It is time to use the trajectory expressions. For the angle positions (θ, φ) to reach the values at the time t' after starting from (θ, φ) at the initial time t it is necessary for the particle to have travelled according to the equations

$$\begin{aligned}
l\varphi\left(t'\right) - m_0\theta\left(t'\right) &= l\varphi - m_0\theta \qquad \text{(initial condition)} \\
&+ l\int_t^{t'} dt'' \left[\frac{d\varphi}{dt''} - q\left(r_0\right)\frac{d\theta}{dt''}\right]
\end{aligned}$$

where we should insert the equations of motion for φ and for θ.

The displacements relative to the magnetic line are due to the drifts.

The integration on θ is sufficiently accurate to be done with taking into account the change of the safety factor

$$\begin{aligned}
l\varphi\left(t'\right) - m_0\theta\left(t'\right) &= l\varphi - m_0\theta \\
&+ l\left[q\left(r^{(0)}\right) - q\left(r_0\right)\right]\left(\theta' - \theta\right) \\
&+ \int_t^{t'} dt'' \omega_{De}\left(t''\right)
\end{aligned}$$

where

$$\begin{aligned}
\omega_{De}\left(t''\right) &\equiv l\left[\frac{d\varphi}{dt''} - q\left(r_0\right)\frac{d\theta}{dt''}\right] \\
&+ l\left[q(r) - q\left(r^0\right)\right]\frac{d\theta}{dt''}
\end{aligned}$$

and it has been introduced the average radius of the banana trajectory

$$r^{(0)} \equiv \frac{1}{\tau_b}\oint d\tau \; r$$

We **note** that $\omega_{De}\left(t''\right)$ is composed of two parts

- one part is an evolution centered on a line situated on the surface r_0, a line with the ratio poloidal/toroidal (m/n) given by $q\left(r_0\right)$.

- a part which evolves only on θ, at a distance $r - r^{(0)}$ from the center-line of the banana.

Other averages are introduced to simplify the representation of the periodic banana or circulating orbits

$$\begin{aligned} & [l\varphi(t') - m_0\theta(t')] - [l\varphi - m_0\theta] \\ = \; & S^{(0)}(\theta' - \theta) \\ & + \omega_{De}^{(0)}(t' - t) \\ & + w_{De}^{(\tilde{})} \end{aligned}$$

where

$$S^{(0)} \equiv l\left[q\left(r^{(0)}\right) - q(r_0)\right]$$

This is a simple notation, since the term was $l\left[q\left(r^{(0)}\right) - q(r_0)\right](\theta' - \theta)$ and now the factor is named $S^{(0)}$.

It is introduced the *average* of the frequency associated to the motion along the banana.

$$\omega_{De}^{(0)} \equiv \frac{1}{\tau_b}\oint dt'' \; \omega_{De}(t'')$$

and the rest is

$$w_{De}^{(\tilde{})} \equiv \int_t^{t'} dt'' \left[\omega_{De}(t'') - \omega_{De}^{(0)}\right]$$

so it has the character of an extraction of a bulk, averaged, quantity $\omega_{De}^{(0)}$ which is a constant. What remains is function of time and reflects the departure, in time, from this average.

There will be differences between the trapped and untrapped responses.

For trapped particles

$$\begin{aligned} & [l\varphi(t') - m_0\theta(t')] - [l\varphi - m_0\theta] \\ = \; & \omega_{De}^{(0)}(t' - t) \\ & + P_b^{(\tilde{})}(t') \\ & \text{for TRAPPED particles} \end{aligned}$$

where

$$\begin{aligned} P_b^{(\tilde{})}(t') \; = \; & S^{(0)}(\theta' - \theta) \\ & + w_{De}^{(\tilde{})} \end{aligned}$$

For circulating particles, the term with $S^{(0)}$ which multiplies a difference of angles θ travelled between t and t' now can be integrated since we know the orbit: it is a full circle. Then this term becomes explicitely proportional with the time difference and is taken apart.

$$\begin{aligned}
& [l\varphi(t') - m_0\theta(t')] - [l\varphi - m_0\theta] \\
= & \left[\omega_{De}^{(0)} + S^{(0)}\omega_t\right](t'-t) \\
& + P_t^{(\tilde{})}(t')
\end{aligned}$$

where the last term is defined taking into account of this extraction of the term proportional with $(t'-t)$

$$\begin{aligned}
P_t^{(\tilde{})}(t') &= S^{(0)}(\theta'-\theta) - S^{(0)}\omega_t(t'-t) \\
& + w_{De}^{(\tilde{})} \\
& \text{for CIRCULATING particles}
\end{aligned}$$

The two functions are periodic

$$\begin{aligned}
P_b^{(\tilde{})}(t') &\equiv \text{periodic with } \omega_b \\
P_t^{(\tilde{})}(t') &\equiv \text{periodic with } \omega_t
\end{aligned}$$

where ω_t is the frequency of *transit* for circulating particles.

Then it is possible to expand in Fourier series

$$\begin{aligned}
& \widetilde{\phi}[S(t'), \theta(t')] \exp\left[iP_{b,t}^{(\tilde{})}(t')\right] \\
= & \sum_{p=-\infty}^{\infty} \widetilde{\Phi}^{(p)} \exp\left\{ip\omega_{b,t}\, \widehat{t}[\theta(t')]\right\}
\end{aligned}$$

where

$$\widehat{t}(\theta) = \int_0^\theta \frac{d\theta'}{\frac{v_\parallel(\theta')}{R_0 q}} + \text{const}$$

The first order function is

$$\begin{aligned}
\widetilde{f}_e &= \frac{|e| f_{Me}}{T_e}\widetilde{\phi}(S,\theta) \\
& + \frac{|e| f_{Me}}{T_e} i\left(\omega - \omega_{*e}^T\right) \int_{-\infty}^t dt' \sum_{p=-\infty}^{\infty} \widetilde{\Phi}^{(p)} \exp\left\{ip\omega_{b,t}\, \widehat{t}[\theta(t')]\right\} \\
& \times \exp\left\{-i\left[\omega + i\nu_f - \omega_{De}^{(0)} - S^{(0)}\omega_t \Theta(\Lambda_c - \Lambda)\right](t'-t)\right\}
\end{aligned}$$

where

$$\Lambda_c = 1 - \varepsilon_0$$

is the limit that separates the trapped from the circulating particles, in the parameter $\Lambda = v_\perp^2/v^2 \times h$.

$$\Theta(\Lambda_c - \Lambda) = 0 \quad \text{for } \Lambda > \Lambda_c \quad \text{(trapped particles)}$$

Large Λ means large perpendicular energy , $v_\perp^2$, and this implies trapping. From the total energy v^2, a large fraction $v_\perp^2/v^2$ means most of the energy is in perpendicular (gyration) motion.

$$\Theta(\Lambda_c - \Lambda) = 1 \quad \text{for } \ \Lambda < \Lambda_c \quad \text{(circulating particles)}$$

The integration on t' can be made

$$\begin{aligned}
\widetilde{f}_e &= \frac{|e|\, f_{Me}}{T_e}\widetilde{\phi}(S,\theta) \\
&\quad - \frac{|e|\, f_{Me}}{T_e}\left(\omega - \omega_{*e}^T\right)\sum_{p=-\infty}^{\infty}\frac{\widetilde{\Phi}^{(p)}\exp\left[ip\omega_{b,t}\,\widehat{t}(\theta)\right]}{\omega + i\nu_f - \omega_{De}^{(0)} - \left[p + S^{(0)}\Theta(\Lambda_c - \Lambda)\right]\omega_{b,t}}
\end{aligned}$$

Now we can calculate the perturbation to the density

$$\begin{aligned}
\widetilde{n}_e &= \int d^3v\ \widetilde{f}_e \\
&= \frac{|e|\, n}{T_e}\widetilde{\phi}(S,\theta) \\
&\quad - \frac{|e|}{T_e}\int d^3v\ f_{Me}\left(\omega - \omega_{*e}^T\right)\sum_{p=-\infty}^{\infty}\frac{\widetilde{\Phi}^{(p)}\exp\left[ip\omega_{b,t}\,\widehat{t}(\theta)\right]}{\omega + i\nu_f - \omega_{De}^{(0)} - \left[p + S^{(0)}\Theta(\Lambda_c - \Lambda)\right]\omega_{b,t}}
\end{aligned}$$

The velocity space integration will be expressed in terms of the variables adequate for the trapping/circulating particles.

$$\begin{aligned}
\widetilde{n}_e &= \frac{|e|}{T_e}n\widetilde{\phi}(S,\theta) \\
&\quad - \frac{|e|}{T_e}\frac{\pi}{2}\left(\frac{2}{m_e}\right)^{3/2}\sum_{\sigma=\pm}\int_0^{h(\theta)} d\Lambda\frac{1}{h(\theta)\sqrt{1 - \frac{\Lambda}{h(\theta)}}} \\
&\qquad\qquad \times \int_0^{\epsilon} d\epsilon\ \sqrt{\epsilon} f_{Me}(\epsilon)\left[\omega - \omega_{*e}^T(\epsilon)\right] \\
&\quad \times \sum_{p=-\infty}^{\infty}\frac{\widetilde{\Phi}^{(p)}(S,\theta,\Lambda)\exp\left[ip\omega_{b,t}\,\widehat{t}(\theta)\right]}{\omega + i\nu_f - \omega_{De}^{(0)}(\Lambda,\epsilon) - \left[p + S^{(0)}\Theta(\Lambda_c - \Lambda)\right]\omega_{b,t}(\Lambda,\epsilon)}
\end{aligned}$$

The separation of the 0 term

$$
\begin{aligned}
\widetilde{n}_e &= \frac{|e|}{T_e} n \widetilde{\phi}(S,\theta) \\
&\quad - \frac{|e|}{T_e}\frac{\pi}{2}\left(\frac{2}{m_e}\right)^{3/2} \frac{1}{h(\theta)} \sum_{\sigma=\pm} \int_0^\infty d\epsilon \, \sqrt{\epsilon} f_{Me}(\epsilon) \left[\omega - \omega_{*e}^T(\epsilon)\right] \\
&\quad \times \left[\int_{1-\varepsilon_0}^{h(\theta)} d\Lambda \frac{1}{\sqrt{1-\frac{\Lambda}{h(\theta)}}} \times \qquad\qquad\qquad \text{(trapped)} \right. \\
&\quad \times \left(\frac{\widetilde{\Phi}^{(0)}}{\omega + i\nu_f - \omega_{De}^{(0)}} - \sum_{p\neq 0} \widetilde{\Phi}^{(p)} \exp\left(ip\omega_b \widehat{t}\right) \left\{ \frac{1}{p\omega_b} + \frac{\omega + i\nu_f - \omega_{De}^{(0)}}{p^2\omega_b^2} + i\pi\delta\left[\omega + i\nu_f - \omega_{De}^{(0)} - \right.\right.\right. \\
&\quad - \int_0^{1-\varepsilon_0} d\Lambda \frac{1}{\sqrt{1-\frac{\Lambda}{h(\theta)}}} \times \qquad\qquad\qquad \text{(circulating)} \\
&\quad \times \sum_p \widetilde{\Phi}^{(p)} \exp\left(ip\omega_t \widehat{t}\right) \left\{ \frac{1}{(p+S)\,\omega_t} + \frac{\omega + i\nu_f - \omega_{De}^{(0)}}{(p+S)^2\,\omega_t^2} + i\pi\delta\left[\omega + i\nu_f - \omega_{De}^{(0)} - (p+S)\,\omega_t\right] \right\}
\end{aligned}
$$

We stop to comment how simple was, comparatively, to solve for the perturbation of the distribution function $f_j^{(1)}$ in the neoclassical problem. There are striking differences;

- here we have waves; this means that the particles are moving in the geometry of the field (neoclassical drifts) and also in the potential of the wave
- by contrast, the neoclassical problem treated by **Stringer** (or **Taguchi**) is *static*. It is described an equilibrium. The drift kinetic equation expresses the balance of the various effects of the motion of particles: both geometric (neoclassical drift) and due to flows;
- in the theory of trapped-electron drift instabilities the objective is the perturbation of the two densities (ions, electrons) in the field of the wave
- in the purely neoclassical theory **Stringer** the objective is to determine the variation of the *potential* and of the *density* in the magnetic surface $n_j^{(1)}(r,\theta)$ and $\phi^{(1)}(r,\theta)$. the reslt is that the neutrality condition becomes the equation for the electrostatic potential.

- the **Stringer** purely neoclassical theory is continued with calculations of the
    - parallel and perpendicular currents , in particular Pfirsch Schluter
    - parallel and perpendicular and radial velocity, part of the radial one being associated to transport.
    - if the transport flux is not uniform on the poloidal circumference, then there is a coupling with the toroidal flows with harmonic distributions in poloidal section, and it results an instability of the poloidal rotation

7.2 Model for the strongly dissipative trapped-electron mode in sheared poloidal flow

This model is **Carreras, Sidikman, Diamond, Terry 1992**.

The dissipative trapped electron regime is described by a *non-adiabatic* response of the *electrons* essentially due to strong collisions of the trapped electrons. The non-adiabatic part of the distribution function of the *trapped electrons* is

$$
\begin{aligned}
-i\left(\omega-\omega_E-\omega_e^{drift}+i\nu_{eff}\right)\widetilde{g}_{\mathbf{k}} &= i\frac{|e|}{T_{e0}}\sqrt{\varepsilon}f_0\left\{\omega-\omega_E-\omega_{e*}\left[1+\eta_e\left(E-\frac{1}{2}\right)\right]\right\}\widetilde{\phi}_{\mathbf{k}} \\
&\quad \text{(for TRAPPED ELECTRONS)}
\end{aligned}
$$

where

$$
\eta_e \equiv \frac{d\ln T_e}{d\ln n_0} = \frac{L_n}{L_{Te}}
$$

$$
\begin{aligned}
\omega_{e*} &= k_y V_{e*n} \\
&= k_y \frac{c_s\rho_s}{L_n}
\end{aligned}
$$

$$
\omega_e^{drift} \equiv \text{ frequency of the electron drift due to curvature and } \boldsymbol{\nabla} B
$$

$$
\begin{aligned}
c_s^2 &= \frac{T_e}{m_i} \\
\rho_s &= \frac{c_s}{\Omega_i}
\end{aligned}
$$

$$
\omega_E = k_y V_0\left(x\right) \quad \text{(there is rotation)}
$$

$$\sqrt{\varepsilon} = \text{fraction of trapped electrons}$$

$$\nu_{eff} = \text{collision frequency of electrons}$$

The condition is

$$\begin{aligned} \nu_{eff} &\gg \omega - \omega_E \\ \nu_{eff} &\gg \omega_e^{drift} \end{aligned}$$

NOTE the absence of the parallel resonance, $-i\omega + ik_{\|}v_{\|}$. The Landau *contour* does not seem applicable. How to apply the complex-plane generalization of the Landau contour problem, developed by **Dong, Horton, Sugama, Kishimoto**?
END

NOTE regarding the calculation of $\widetilde{g}_{\mathbf{k}}$, the non-adiabatic response of the electrons.

The equation for $\widetilde{g}_{\mathbf{k}}$ involves an inversion of an operator which is applied on $\widetilde{g}_{\mathbf{k}}$ representing the derivation of this function along the trajectory. In the linearization of the Vlasov equation the right hand side comes from the combination of equilibrium parameters, f_0=Maxwellian, with pertubation of the electrostatic potential $\widetilde{\phi}_{\mathbf{k}}$. The operator is inverted as

$$\left(\left. \frac{d}{dt} \right|_{trajectory} \right)^{-1} = \int dt' \left. \{...\} \right|_{trajectory \ \mathbf{x}(t')}$$

and due to the Fourier representation and after some approximations, we get an integration of an exponential and this leads to the inverse of the factor at the exponent. This means that

$$\begin{aligned} & \left(\left. \frac{d}{dt} \right|_{trajectory} \right)^{-1} \\ = & \frac{1}{-i \left(\omega - \omega_E - \omega_e^{drift} + i\nu_{eff} \right)} \end{aligned}$$

and this explains the factor multiplying $\widetilde{g}_{\mathbf{k}}$. This is the propagator or Green function, inverse of the operator of derivation along the trajectory.
END

This propagator is simplified according to the strong dissipative regime.

After integration on the velocity space one obtains

$$\delta n^{NA} \approx i\frac{|e|}{T_{e0}}\sqrt{\varepsilon}\widetilde{\phi}_{\mathbf{k}}\frac{\omega - \omega_E - \omega_{e*}\left(1+\alpha\eta_e\right)}{\nu_{eff}}$$

For the *strongly dissipative trapped electron drift instability* the dispersion relation will be

$$\omega_{\mathbf{k}} = \omega_{E\mathbf{k}} + \frac{\omega_{e*}}{1+k_y^2\rho_s^2} \quad \text{(future dispersion relation, used now)}$$

and this is assumed in order to obtain an approximation of the non-adiabatic response

$$\begin{aligned}\delta n^{NA} &\approx i\frac{|e|}{T_{e0}}\sqrt{\varepsilon}\widetilde{\phi}_{\mathbf{k}}\frac{\frac{\omega_{e*}}{1+k_y^2\rho_s^2} - \omega_{e*}\left(1+\alpha\eta_e\right)}{\nu_{eff}} \\ &= i\frac{|e|}{T_{e0}}\sqrt{\varepsilon}\widetilde{\phi}_{\mathbf{k}}\left(\frac{-\omega_{e*}k_y^2\rho_s^2}{1+k_y^2\rho_s^2} - \omega_{e*}\alpha\eta_e\right)\frac{1}{\nu_{eff}}\end{aligned}$$

and since

$$\eta_e = \frac{L_n}{L_T} \gg 1$$

one should retain only the last term in paranthesis

$$\delta n^{NA} \approx -i\frac{|e|}{T_{e0}}\sqrt{\varepsilon}\widetilde{\phi}_{\mathbf{k}}k_y V_{en}\alpha\frac{L_n}{L_{T_e}}\frac{1}{\nu_{eff}}$$

but

$$V_{e*}\frac{L_n}{L_{T_e}} = V_{*eT_e}$$

$$\delta n^{NA} \approx -i\alpha\frac{|e|}{T_{e0}}\sqrt{\varepsilon}\widetilde{\phi}_{\mathbf{k}}V_{*eT_e}\frac{1}{\nu_{eff}}k_y$$

We replace

$$ik_y \rightarrow \frac{\partial}{\partial y}$$

$$\delta n^{NA} \approx -\alpha\sqrt{\varepsilon}\frac{V_{*eT_e}}{\nu_{eff}}\frac{\partial}{\partial y}\left(\frac{|e|\,\widetilde{\phi}_{\mathbf{k}}}{T_{e0}}\right)$$

The *neutrality* requires

$$\begin{aligned}\frac{\widetilde{n}_i}{n_0} &= \frac{n_0\frac{|e|\widetilde{\phi}_{\mathbf{k}}}{T_e} + n_0\delta n^{NA}}{n_0} \\ &= \frac{|e|\,\widetilde{\phi}_{\mathbf{k}}}{T_e} - \alpha\sqrt{\varepsilon}\frac{V_{*eT_e}}{\nu_{eff}}\frac{\partial}{\partial y}\left(\frac{|e|\,\widetilde{\phi}_{\mathbf{k}}}{T_{e0}}\right) \\ &= \left(1 - \alpha\sqrt{\varepsilon}\frac{V_{*eT_e}}{\nu_{eff}}\frac{\partial}{\partial y}\right)\frac{|e|\,\widetilde{\phi}_{\mathbf{k}}}{T_e}\end{aligned}$$

This is a perturbation expansion and we can invert the operator

$$\frac{|e|\,\widetilde{\phi}_{\mathbf{k}}}{T_e} \approx \left(1 + \alpha\sqrt{\varepsilon}\frac{V_{*eT_e}}{\nu_{eff}}\frac{\partial}{\partial y}\right)\frac{\widetilde{n}_i}{n_0}$$

In this way the electrostatic potential is replaced by the perturbation of the density.

8 The parallel friction versus the Landau damping

The parallel friction is essential in the formation of the drift eigenmodes.

If there are no collisions then the parallel dissipative mechanism used to balance the

- parallel gradient of the pressure
- the parallel electric field force

is the Landau damping.

9 Electrostatic drift waves in *slab* geometry including *NOISE*

This is from the text **STOGAM**.

9.1 The particle trajectory and the Fokker-Planck equation

The magnetic field is

$$\mathbf{B}_0 = B_0\widehat{\mathbf{e}}_z \tag{66}$$

and the uniform gravitational field is

$$\mathbf{G} = G\widehat{\mathbf{e}}_x . \tag{67}$$

The equations of motion are

$$\frac{d\mathbf{r}'}{dt} = \mathbf{v}' \tag{68}$$

$$\frac{d\mathbf{v}'}{dt} = \frac{q}{m}\mathbf{v}' \times \mathbf{B}_0 + G\widehat{\mathbf{e}}_x \tag{69}$$

with the initial condition

$$\mathbf{r}'(t' = t) = \mathbf{r} \qquad \text{and} \qquad \mathbf{v}'(t' = t) = \mathbf{v}. \tag{70}$$

The solution is:

$$\begin{array}{rcl} v'_x(t') & = & v_\perp \cos\left(\theta + \Omega(t'-t)\right) \\ v'_y(t') & = & v_\perp \sin\left(\theta + \Omega(t'-t)\right) + \frac{G}{\Omega} \\ v'_z(t') & = & v_z \end{array} \tag{71}$$

and

$$\begin{array}{rcl} x'(t') & = & x + \frac{v_\perp}{\Omega}\left[\sin\left(\theta + \Omega(t'-t)\right) - \sin\theta\right] \\ y'(t') & = & y - \frac{v_\perp}{\Omega}\left[\cos\left(\theta + \Omega(t'-t)\right) - \cos\theta\right] + \frac{G}{\Omega}(t'-t) \\ z'(t') & = & z + v_z(t'-t) \end{array} \tag{72}$$

where

$$\mathbf{v} = (v_\perp \cos\theta, v_\perp \sin\theta, v_z) \tag{73}$$

and

$$\Omega_j = -\frac{q_j B_0}{m_j} \tag{74}$$

Vlasov's equation in a collisionless plasma:

$$\frac{\partial}{\partial t} f_j(\mathbf{r},\mathbf{v},t) + \mathbf{v}\cdot\boldsymbol{\nabla}_{\mathbf{r}} f_j(\mathbf{r},\mathbf{v},t) + \left(\frac{q_j}{m_j}\mathbf{E} + \frac{q_j}{m_j}\mathbf{v}\times\mathbf{B} + \mathbf{G}\right)\cdot\boldsymbol{\nabla}_{\mathbf{v}} f_j(\mathbf{r},\mathbf{v},t) = 0. \tag{75}$$

The Maxwell's equations are

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{1}{\varepsilon_0}\sum_j q_j \int f_j d\mathbf{v} \tag{76}$$

$$\frac{1}{\mu_0}\boldsymbol{\nabla}\times\mathbf{B} = \varepsilon_0\frac{\partial\mathbf{E}}{\partial t} + \sum_j q_j \int \mathbf{v} f_j d\mathbf{v} \tag{77}$$

$$\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{78}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \tag{79}$$

where we have denoted the particle species by the index j.

The drift kinetic equation:

$$\frac{d}{dt}\widetilde{f}_e = \left(-\left|e\right| v_\parallel \widehat{n}\cdot\nabla\varphi + i\varpi_{*e}\left|e\right|\varphi\right)\frac{\partial F_{0e}}{\partial\epsilon} \tag{80}$$

where $\epsilon = m_e v^2/2$. The time-change of $\widetilde{f}_e$ along the particle orbit is given by the balance of the two terms in the right hand side:

$$\left(-\left|e\right| v_{\parallel}\widehat{n}\cdot\nabla\varphi\right)\frac{\partial F_{0e}}{\partial\epsilon} = \begin{array}{l}\text{time-change in the energy distribution function}\\ \text{due to the acceleration of the electrons}\\ \text{in the electric field of the wave}\end{array} \tag{81}$$

$$\left(i\varpi_{*e}\left|e\right|\varphi\right)\frac{\partial F_{0e}}{\partial\epsilon} = \begin{array}{l}\text{time-change in the local density}\\ \text{of particles due to the convection}\\ \text{of the density by the drift}\end{array} \tag{82}$$

The equations with perturbation

$$\begin{aligned}\frac{d\mathbf{r}'}{dt'} &= \mathbf{v}' \qquad (83)\\ \frac{d\mathbf{v}'}{dt'} &= \frac{q_j}{m_j}\mathbf{v}'\times\mathbf{B}_0 + \frac{q_j}{m_j}\widetilde{\mathbf{E}}\end{aligned}$$

The last term is a fluctuation that can be represented also by a fluctuating potential

$$\widetilde{\mathbf{E}} = -\boldsymbol{\nabla}\widetilde{\phi} \tag{84}$$

or we can consider fluctuations of velocity

$$\begin{aligned}v'_x &= v'_\perp\cos\left[\xi\left(t'\right)\right] + \widetilde{v}_x \qquad (85)\\ v'_y &= v'_\perp\sin\left[\xi\left(t'\right)\right] + \widetilde{v}_y\end{aligned}$$

but we have to consider that the perpendicular velocity is, on the average, the same as in the absence of the fluctuations

$$v'_\perp \simeq v_\perp \tag{86}$$

The equation for $\mathbf{v}'$ is expressed in terms of the two components of the perpendicular velocity

$$\frac{d}{dt'}\left\{v'_\perp\cos\left[\xi\left(t'\right)\right] + \widetilde{v}_x\right\} = \frac{q_j}{m_j}\begin{pmatrix}\widehat{\mathbf{e}}_x & \widehat{\mathbf{e}}_y & \widehat{\mathbf{n}}\\ v'_x & v'_y & v'_z\\ 0 & 0 & B_0\end{pmatrix}_x + \widetilde{v}_x \tag{87}$$

or, neglecting the effect of fluctuations on the phase

$$-v'_\perp\sin\left[\xi\left(t'\right)\right]\frac{d\xi\left(t'\right)}{dt'} = \frac{q_j}{m_j}v'_\perp\sin\left[\xi\left(t'\right)\right]B_0 \tag{88}$$

resulting the equation for the phase

$$\frac{d\xi(t')}{dt'} = \Omega_j \tag{89}$$

$$\xi(t') = \Omega_j (t' - t) + \theta \tag{90}$$

with the definition

$$\Omega_j \equiv -\frac{q_j B_0}{m_j} \tag{91}$$

or

$$\begin{aligned} \Omega_e &= \frac{|e| B_0}{m_e} \\ \Omega_i &= -\frac{|e| B_0}{m_i} \end{aligned} \tag{92}$$

The equations of trajectory are

$$\begin{aligned} x'(t') &= x + \frac{v_\perp}{\Omega} \{\sin[\theta + \Omega(t' - t)] - \sin\theta\} + \delta x(t') \\ y'(t') &= y - \frac{v_\perp}{\Omega} \{\cos[\theta + \Omega(t' - t)] - \cos\theta\} + \delta y(t') \\ z'(t') &= z + v_z (t' - t) + \delta z(t') \end{aligned} \tag{93}$$

where

$$\begin{aligned} \delta x(t') &\equiv \frac{1}{B_0} \int^{t'} d\tau \left(-\frac{\partial \widetilde{\phi}}{\partial y} \right) \\ \delta y(t') &\equiv \frac{1}{B_0} \int^{t'} d\tau \left(+\frac{\partial \widetilde{\phi}}{\partial x} \right) \\ \delta z(t') &\equiv \frac{q_j}{m_j} \int^{t'} d\tau \left(-\frac{\partial \widetilde{\phi}}{\partial z} \right) \end{aligned} \tag{94}$$

are fluctuations of the position of the guiding center.

The response of the electrons to the wave potential plus the fluctuating potential

$$\begin{aligned} \widetilde{f}_e &= \frac{|e|}{T_e} \left(\phi + \widetilde{\phi} \right) F_{Me} \\ &\quad -i|e|(\omega - \omega_{*e}) \int_{-\infty}^{t} dt' \left(\phi + \widetilde{\phi} \right) \left. \frac{\partial F_{Me}}{\partial \varepsilon} \right|_{x(t')} \end{aligned} \tag{95}$$

The Maxwellian function is

$$F_{Me} = \left(\frac{m_e}{2\pi T_e} \right)^{3/2} \exp\left(-\frac{\varepsilon}{T_e} \right) \tag{96}$$

and

$$\frac{\partial F_{Me}}{\partial \varepsilon} = -\frac{1}{T_e} F_{M_e} \tag{97}$$

Then the perturbation to the distribution function is

$$\begin{aligned} \widetilde{f}_e &= \frac{|e|}{T_e} \left(\phi + \widetilde{\phi} \right) F_{Me} \\ &\quad + i \frac{|e|}{T_e} \left(\omega - \omega_{*e} \right) \int_{-\infty}^{t} dt' \left(\phi + \widetilde{\phi} \right) \left. F_{Me} \right|_{x(t')} \end{aligned} \tag{98}$$

The integral operator is the inverse of the time derivative along the trajectory $x(t')$,

$$\begin{aligned} \widetilde{f}_e &= \frac{|e|}{T_e} \left(\phi + \widetilde{\phi} \right) F_{Me} \\ &\quad + i \frac{|e|}{T_e} \left(\omega - \omega_{*e} \right) F_{Me} \left(\frac{d}{dt} \right)^{-1} \left(\phi + \widetilde{\phi} \right) \end{aligned} \tag{99}$$

Now we introduce the Fourier representation of the wave potential

$$\begin{aligned} \phi(x, y, z, t) &= \phi(x) \exp(-i\omega t) \\ &\quad \times \exp(i k_y y + i k_z z) \end{aligned} \tag{100}$$

and the same for $\widetilde{f}_e$ and for $\widetilde{\phi}$.

We have

$$\begin{aligned} \left(\frac{d}{dt} \right)^{-1} \phi &= \int_{-\infty}^{t} dt' \phi\left[x(t') \right] \exp(-i\omega t') \\ &\quad \times \exp\left[i k_y y(t') + i k_z z(t') \right] \\ &= \int_{-\infty}^{t} dt' \phi\left[x(t') \right] \exp(-i\omega t') \\ &\quad \times \exp\left\{ i k_y \left[y - \frac{v_\perp}{\Omega} \left(\cos\left(\theta + \Omega (t' - t) - \cos\theta \right) \right) + \delta y(t') \right] \right\} \\ &\quad \times \exp\left\{ i k_z \left[z + v_z (t' - t) + \delta z(t') \right] \right\} \end{aligned} \tag{101}$$

$$\left(\frac{d}{dt}\right)^{-1}\phi = \exp\left(ik_y y + ik_z z\right)\int_{-\infty}^{t} dt'\phi\left[x\left(t'\right)\right]\exp\left(-i\omega t'\right) \tag{102}$$
$$\times\exp\left\{-i\frac{k_y v_\perp}{\Omega}\cos\left(\theta+\Omega\left(t'-t\right)\right)\right\}\exp\left\{i\frac{k_y v_\perp}{\omega}\cos\theta\right\}$$
$$\times\exp\left[ik_y\delta y\left(t'\right)\right]$$
$$\times\exp\left[ik_z v_z\left(t'-t\right)\right]$$
$$\times\exp\left[ik_z\delta z\left(t'\right)\right]$$

Now we can make the expansions in terms of Bessel functions.

The term from Eq.(102) that contains the Larmor frequency

$$\exp\left\{-i\frac{k_y v_\perp}{\Omega}\cos\left(\theta+\Omega\left(t'-t\right)\right)\right\} \tag{103}$$
$$= \sum_{n=-\infty}^{\infty}\left(-i\right)^n J_n\left(\frac{k_\perp v_\perp}{\Omega_{ci}}\right)\exp\left[-in\left(\theta+\Omega\left(t'-t\right)\right)\right]$$

The second factor containing $\cos\theta$ has a similar Bessel expansion

$$\exp\left\{i\frac{k_y v_\perp}{\omega}\cos\theta\right\} \tag{104}$$
$$= \sum_{m=-\infty}^{\infty}\left(i\right)^n J_m\left(\frac{k_\perp v_\perp}{\Omega_{ci}}\right)\exp\left(-im\theta\right)$$

The product of these two expansions is

$$\exp\left\{-i\frac{k_y v_\perp}{\Omega}\cos\left(\theta+\Omega\left(t'-t\right)\right)\right\}\exp\left\{i\frac{k_y v_\perp}{\omega}\cos\theta\right\} \tag{105}$$
$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left(-i\right)^n\left(i\right)^n J_n\left(\frac{k_\perp v_\perp}{\Omega_{ci}}\right)J_m\left(\frac{k_\perp v_\perp}{\Omega_{ci}}\right)$$
$$\times\exp\left[-in\left(\theta+\Omega\left(t'-t\right)\right)+im\theta\right]$$

We retain the

$$m=n=0$$

term in the double sum

$$\left[J_0\left(\frac{k_y v_\perp}{\Omega}\right)\right]^2$$

We note that the two signs of Ω_{ci} (+ for the electrons and − for the ions) lead to the same sign in the final expression, eventually after two changes. Then we have

$$\Omega \rightarrow |\Omega|$$

in the denominator of the argument of the Bessel function.

Let us return to the operator "inverse of the Vlasov operator"

$$\begin{aligned}\left(\frac{d}{dt}\right)^{-1}\phi &= \int_{-\infty}^{t} dt'\phi\left[x\left(t'\right)\right]\left[J_0\left(\frac{k_y v_\perp}{\Omega}\right)\right]^2 \\ &\quad\times \exp\left(-i\omega t'\right)\exp\left(ik_y y + ik_z z\right)\exp\left[ik_z v_z\left(t'-t\right)\right] \\ &\quad\times \exp\left(ik_y \delta y\left(t'\right)\right)\exp\left(ik_z\delta z\left(t'\right)\right)\end{aligned}$$

We extract a factor that represents the Fourier basis function

$$\exp\left(-i\omega t + ik_y y + ik_z z\right)$$

Then

$$\begin{aligned}&\exp\left(+i\omega t - ik_y y - ik_z z\right)\left(\frac{d}{dt}\right)^{-1}\phi \\ =& \left[J_0\left(\frac{k_y v_\perp}{\Omega}\right)\right]^2 \int_{-\infty}^{t} dt'\phi\left[x\left(t'\right)\right]\exp\left[-i\omega\left(t'-t\right) + ik_z v_z\left(t'-t\right)\right] \\ &\times \exp\left(ik_y\delta y\left(t'\right)\right)\exp\left(ik_z\delta z\left(t'\right)\right)\end{aligned}$$

9.1.1 Note on the random change of the $x\left(t'\right)$-trajectory

We see that the potential remains to be a function of $x\left(t'\right)$ after the Fourier expansion on the coordinates y and z. Since there is a scattering of the trajectory we should write the trajectory as a deterministic part (the one that exists in the absence of the additional turbulence) and the fluctuating part that is connected with the additional turbulence

$$\begin{aligned}\phi\left[x\left(t'\right)\right] &= \phi\left[x + \delta r\left(t'\right)\right] \\ &\simeq \phi\left(x\right) + \frac{1}{2}\left[\delta r\left(t'\right)\right]^2\frac{d^2\phi}{dx^2}\end{aligned}$$

The first order term should give 0 after statistical averaging and for this reason it has not been written. In this case however we neglect cross correlations with potential appearing at the exponent, in $\delta y\left(t'\right)$ and respectively $\delta z\left(t'\right)$.

The equation of Poisson

$$\frac{d^2\phi}{dx^2} - \left(k_y^2 + k_z^2\right)\phi = -\frac{1}{\varepsilon_0}\sum_{j=e,i}\int d^3v\widetilde{f}_j$$

The supplementary term that contains the second derivative to x of ϕ is

$$
\begin{aligned}
& \sum_{j=e,i} \int d^3v \widetilde{f}_j \\
= \; & -\sum_{j=e,i} \frac{q_j}{T_j} \left(\phi + \widetilde{\phi} \right) \int d^3v F_M \\
& -\sum_{j=e,i} \int d^3v \frac{iq_j}{T_j} F_M \left(\omega - \omega_{*j} \right) \left[J_0 \left(\frac{k_\perp v_\perp}{|\Omega_j|} \right) \right]^2 \int_{-\infty}^{t} dt' \phi \left[x \left(t' \right) \right] \\
& \times \exp \left[-i\omega \left(t' - t \right) + ik_z v_z \left(t' - t \right) \right] \exp \left[ik_y \delta y \left(t' \right) + ik_z \delta z \left(t' \right) \right] \\
& + \left(\frac{d^2\phi}{dx^2} \right) \sum_{j=e,i} \int d^3v \frac{iq_j}{T_j} F_M \left(\omega - \omega_{*j} \right) \left[J_0 \left(\frac{k_\perp v_\perp}{|\Omega_j|} \right) \right]^2 \int_{-\infty}^{t} dt' \frac{1}{2} \left[\delta r \left(t' \right) \right]^2 \\
& \times \exp \left[-i\omega \left(t' - t \right) + ik_z v_z \left(t' - t \right) \right] \exp \left[ik_y \delta y \left(t' \right) + ik_z \delta z \left(t' \right) \right]
\end{aligned}
$$

There are correlation terms
$$\widetilde{\phi}\widetilde{\phi} \exp \left(\widetilde{\phi} \right)$$
the two $\widetilde{\phi}$ come from $\left[\delta r \left(t' \right) \right]^2 /2$ and the exponent contains the fluctuations $\delta y \left(t' \right)$ and $\delta z \left(t' \right)$.

9.1.2 Note on the Hirschman-Molvig treatment of the term $\left[\delta r \left(t' \right) \right]^2$

In Hirschman Molvig the term

$$\delta r \left(t' \right) = \frac{1}{B_0} \int^{t'} dt'' \left(-\frac{\partial \widetilde{\phi}}{\partial y} \right)$$

is considered a stochastic variable and avergaes are done over its correlations. This is because in their case there is another $\delta r \left(t' \right)$ at the exponent, from the shear
$$ik_\parallel v_\parallel t'$$
is actually a time integral, in which the linear variation of the parallel wavenumber, due to shear, is taken into account

$$
\begin{aligned}
ik_\parallel v_\parallel t' \; & \rightarrow \; \int^{t'} dt'' k_\parallel \left[\delta r \left(t'' \right) \right] v_\parallel \\
& = \; \int^{t'} dt'' \left[\frac{dk_\parallel}{dx} \delta r \left(t'' \right) \right] v_\parallel
\end{aligned}
$$

Then, since this appears at the exponent

$$\begin{aligned}
\left\langle \exp\left(ik_{\|}v_{\|}t'\right)\right\rangle &\to \left\langle \exp\left\{\int^{t'} dt'' k_{\|}\left[\delta r\left(t''\right)\right] v_{\|}\right\}\right\rangle \\
&= \exp\left\{-\frac{1}{2}\left(k_{\|}'v_{\|}\right)^2 \left\langle \left(\int^{t'} dt''\delta r\left(t''\right)\right)^2\right\rangle\right\}
\end{aligned}$$

The expansion

$$\left(\frac{d^2\phi}{dx^2}\right)\frac{1}{2}\left[\delta r\left(t'\right)\right]^2$$

is also averaged assuming that the δr variation is diffusive

$$\left\langle \left(\frac{d^2\phi}{dx^2}\right)\frac{1}{2}\left[\delta r\left(t'\right)\right]^2\right\rangle = \frac{1}{2}\left(2D_{rr}t'\right)\frac{d^2\phi}{dx^2}$$

Since they neglect the correlations between the exponential and the factors, they have to calculate

$$\begin{aligned}
&\int^{t} dt'\left[\phi\left(x\right)+D_{rr}t'\frac{d^2\phi}{dx^2}\right]\times \\
&\times \exp\left\{...\left\langle\delta\theta^2\right\rangle-\frac{1}{2}\left(k_{\|}'v_{\|}\right)^2\left\langle\left(\int^{t'} dt''\delta r\left(t''\right)\right)^2\right\rangle\right\}
\end{aligned}$$

with a Gaussian distribution function on $\delta r\left(t'\right)$

$$\frac{1}{\sqrt{2\pi\left\langle\delta r\left(t'\right)^2\right\rangle}}\exp\left[-\frac{\left(\delta r\right)^2}{2\left\langle\delta r^2\right\rangle}\right]$$

where

$$\left\langle\delta r\left(t'\right)^2\right\rangle = D_{rr}t'$$

The average is however simplified

$$\begin{aligned}
&\frac{1}{2}\left(\delta r\right)^2\frac{d^2\phi}{dx^2}\times\exp\left[i\int^{t'} dt'' k_{\|}'v_{\|}\delta r\left(t''\right)+im\left\langle\delta\theta\right\rangle\right] \\
=\ &\frac{1}{2}\left\langle\left(\delta r\right)^2\right\rangle\frac{d^2\phi}{dx^2}\left\langle\exp\left[i\int^{t'} dt'' k_{\|}'v_{\|}\delta r\left(t''\right)+im\left\langle\delta\theta\right\rangle\right]\right\rangle
\end{aligned}$$

the factors and the exponential are not correlated.

9.2 The electron response

The distribution function for the electrons

$$
\begin{aligned}
&\widetilde{f}_e(\mathbf{x},\mathbf{v},t) \\
= \;& \frac{|e|}{T_e}\left(\phi+\widetilde{\phi}\right)F_{Me} \\
&+\left[J_0\left(\frac{k_\perp v_\perp}{|\Omega_j|}\right)\right]^2 \frac{i|e|}{T_e}(\omega-\omega_{*e})F_{Me} \\
&\times\int_{-\infty}^{t}dt'\left\{\phi\left[x(t')\right]+\widetilde{\phi}\left[x(t')\right]\right\}\exp\left[-i\omega(t'-t)+ik_zv_z(t'-t)\right] \\
&\times\exp\left[ik_y\delta y_e(t')+ik_z\delta z_e(t')\right]
\end{aligned}
$$

In this expression, as explained, one makes an expansion of the potential calculated on the trajectory $x(t')$ in a value calculated on the deterministic trajectory plus a second derivative of the potential multiplied with the square of the fluctuation of the trajectory, due to the additional scattering.

The first term (the potential calculated on the deterministic trajectory) is subject to the Markovian approximation, *i.e.* the potential on the deterministic trajectory is considered as contributing on only the actual point (x,t) and therefore can be taken out from the integration over t'. The same is applied to the second derivative of the potential, which is also taken out from the time integration. By this approximation we reproduce the most elementary propagator exhibiting the resonance in the parallel direction.

$$
\begin{aligned}
&\widetilde{f}_e(\mathbf{x},\mathbf{v},t) \\
= \;& \frac{|e|}{T_e}\left(\phi+\widetilde{\phi}\right)F_{Me} \\
&+\left[J_0\left(\frac{k_\perp v_\perp}{|\Omega_j|}\right)\right]^2 \frac{i|e|}{T_e}(\omega-\omega_{*e})F_{Me} \\
&\times\left\{\phi(x)\int_{-\infty}^{t}dt'\exp\left[-i\omega(t'-t)+ik_zv_z(t'-t)\right]\exp\left[ik_y\delta y_e(t')+ik_z\delta z_e(t')\right]\right. \\
&\left.+\frac{d^2\phi}{dx^2}\int_{-\infty}^{t}dt'\frac{1}{2}\left[\delta x(t')\right]^2\exp\left[-i\omega(t'-t)+ik_zv_z(t'-t)\right]\exp\left[ik_y\delta y_e(t')+ik_z\delta z_e(t')\right]\right\}
\end{aligned}
$$

We will make a series of assumptions.

1. We take $\delta y(t')$ and $\delta z(t')$ as resulting from fluctuating velocities that are approximatively constant over the time integral involved by the inversion

$\left(\frac{d}{dt}\right)^{-1}$.

$$\begin{aligned}\delta y\left(t'\right) &= \widetilde{v}_y\left(t'-t\right)\\ \delta z\left(t'\right) &= \widetilde{v}_z\left(t'-t\right)\end{aligned}$$

We take boundary conditions

$$\begin{aligned}\delta y\left(t\right) &= 0\\ \delta z\left(t\right) &= 0\end{aligned}$$

2. In the term that contains the second order derivative of ϕ at x, we make the average like Hirschman-Molvig: separately the factor and the exponential, neglecting cross correlations.

The propagator, with these approximations

$$\phi\left(x\right)\int_{-\infty}^{t}dt'\exp\left[-i\omega\left(t'-t\right)+ik_zv_z\left(t'-t\right)+ik_y\widetilde{v}_y\left(t'-t\right)+ik_z\widetilde{v}_z\left(t'-t\right)\right]$$

and taking

$$\tau\equiv t'-t$$

$$\begin{aligned}&\phi\left(x\right)\int_{-\infty}^{0}d\tau\exp\left[-i\omega\tau+ik_zv_z\tau+ik_y\widetilde{v}_y\tau+ik_z\widetilde{v}_z\tau\right]\\ =\;&\phi\left(x\right)\left.\frac{\exp\left(-i\omega+ik_zv_z+ik_y\widetilde{v}_y+ik_z\widetilde{v}_z\right)\tau}{-i\omega+ik_zv_z+ik_y\widetilde{v}_y+ik_z\widetilde{v}_z}\right|_{-\infty}^{0}\\ =\;&\phi\left(x\right)\frac{1}{-i\omega+ik_zv_z+ik_y\widetilde{v}_y+ik_z\widetilde{v}_z}\\ =\;&\phi\left(x\right)\frac{1}{ik_z}\frac{1}{v_z+\left(-i\omega+k_y\widetilde{v}_y\right)/k_z+\widetilde{v}_z}\\ =\;&\phi\left(x\right)\frac{1}{ik_z}\frac{1}{v_z-\left[\left(i\omega-k_y\widetilde{v}_y\right)/k_z-\widetilde{v}_z\right]}\end{aligned}$$

9.2.1 The electron response in the absence of external fluctuations

The electron distribution function simplifies to

$$\begin{aligned}&\widetilde{f}_e\left(\mathbf{x},\mathbf{v},t\right)\\ =\;&\frac{|e|}{T_e}\phi F_{Me}\\ &+\left[J_0\left(\frac{k_\perp v_\perp}{\Omega_{ce}}\right)\right]^2\frac{i\left|e\right|}{T_e}\left(\omega-\omega_{*e}\right)F_{Me}\phi\left(x\right)\frac{1}{ik_z}\frac{1}{v_z-\left[\left(i\omega-k_y\widetilde{v}_y\right)/k_z-\widetilde{v}_z\right]}\end{aligned}$$

and the density

$$
\begin{aligned}
\widetilde{n}_e &= \int d^3v \widetilde{f}_e\left(\mathbf{x},\mathbf{v},t\right) \\
&= \int 2\pi v_\perp dv_\perp dv_z \widetilde{f}_e\left(\mathbf{x},\mathbf{v},t\right) \\
&= \frac{|e|\,\phi}{T_e} n_0 \\
&\quad + n_0 \int_0^\infty 2\pi v_\perp dv_\perp \left[J_0\left(\frac{k_\perp v_\perp}{\Omega_{ce}}\right)\right]^2 \left(\frac{m_e}{2\pi T_e}\right)^{\frac{1}{2}\times 2} \exp\left(-\frac{m_e v_\perp^2}{2T_e}\right) \frac{i\,|e|}{T_e} \left(\omega - \omega_{*e}\right) \phi\left(x\right) \\
&\quad \times \frac{1}{ik_z} \int_{-\infty}^{\infty} dv_z \frac{1}{v_z - \left[\left(i\omega - k_y \widetilde{v}_y\right)/k_z - \widetilde{v}_z\right]} \left(\frac{m_e}{2\pi T_e}\right)^{\frac{1}{2}} \exp\left(-\frac{m_e v_z^2}{2T_e}\right)
\end{aligned}
$$

Taken separately, the integration on the perpendicular velocity will involve the squared Bessel function

$$
A_1 \equiv 2\pi \int_0^\infty v_\perp dv_\perp \left[J_0\left(\frac{k_\perp v_\perp}{\Omega_{ce}}\right)\right]^2 \left(\frac{m_e}{2\pi T_e}\right) \exp\left(-\frac{m_e v_\perp^2}{2T_e}\right)
$$

We introduce the notation

$$
b \equiv \left(\frac{k_\perp}{\Omega_{ce}}\sqrt{\frac{T_e}{m_e}}\right)
$$

and with the substitution

$$
x \equiv v_\perp \sqrt{\frac{m_e}{T_e}}
$$

we have the integration

$$
\begin{aligned}
A_1 &= \int_0^\infty x dx \left[J_0\left(\sqrt{b}x\right)\right]^2 \exp\left(-\frac{x^2}{2}\right) \\
&= I_0\left(b\right) \exp\left(-b\right)
\end{aligned}
$$

Now, separately, the integration over the z-velocity.

$$
A_2 \equiv \frac{1}{ik_z} \int_{-\infty}^{\infty} dv_z \frac{1}{v_z - \left[\left(i\omega - k_y \widetilde{v}_y\right)/k_z - \widetilde{v}_z\right]} \left(\frac{m_e}{2\pi T_e}\right)^{\frac{1}{2}} \exp\left(-\frac{m_e v_z^2}{2T_e}\right)
$$

Use the substitution

$$
\sqrt{\frac{m_e}{2T_e}} v_z \equiv \beta
$$

and the notation

$$\sqrt{\frac{m_e}{2\pi T_e}}\left(\frac{\omega - k_y\widetilde{v}_y}{k_z} - \widetilde{v}_z\right) \equiv \widetilde{\xi}$$

and the expression reads

$$A_2 = \sqrt{\frac{m_e}{2\pi T_e}}\frac{1}{ik_z}\int_{-\infty}^{\infty} d\beta \frac{1}{\beta - \widetilde{\xi}} \exp\left(-\beta^2\right)$$

9.2.2 Digression on the difference relative to the Krall-Rosenbluth model

We have started from the same structure of drift wave calculation like Hirschman-Molvig:

$$\begin{aligned}
\frac{d\widetilde{f}_j}{dt} &= \left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\right)\widetilde{f}_j \\
&= -\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B_0}\cdot\boldsymbol{\nabla}F_{Mj} - \frac{q_j}{m_j}\left(-\boldsymbol{\nabla}\phi\right)\cdot\frac{\partial}{\partial\mathbf{v}}F_{Mj}
\end{aligned}$$

the last term being the energetic interaction between the potential perturbation with the background Maxwellian distribution function. This last term can be written

$$-\frac{q_j}{m_j}\left(-\boldsymbol{\nabla}\phi\right)\cdot\frac{\partial}{\partial\mathbf{v}}F_{Mj} = -\frac{q_j}{m_j}\left(-\boldsymbol{\nabla}\phi\right)\cdot\left(-\frac{m_j}{2T_j}\right)2\mathbf{v}F_{Mj}$$

and the product

$$\mathbf{v}\cdot\boldsymbol{\nabla}\phi = \left(\frac{d}{dt} - \frac{\partial}{\partial t}\right)\phi$$

Then the distribution function will have the equation

$$\begin{aligned}
\frac{d\widetilde{f}_j}{dt} &= -\frac{-ik_y\phi}{B_0}\frac{dF_{Mj}}{dr} - \frac{q_j}{m_j}\left(\frac{d\phi}{dt} - \frac{\partial\phi}{\partial t}\right)F_{Mj} \\
&= -\frac{q_j}{m_j}\frac{d\phi}{dt}F_{Mj} + \frac{ik_yT_j}{q_jB_0}\left(\frac{1}{n_0}\frac{dn_0}{dr}\right)\frac{q_j\phi}{T_j}F_{Mj} + \frac{q_j}{T_j}\frac{\partial\phi}{\partial t}F_{Mj}
\end{aligned}$$

or

$$\begin{aligned}
\frac{d\widetilde{f}_j}{dt} &= \frac{d}{dt}\left(-\frac{q_j\phi}{T_j}\right)F_{Mj} + i\omega_{*j}\frac{q_j\phi}{T_j}F_{Mj} - i\omega\frac{q_j\phi}{T_j}F_{Mj} \\
&= \frac{d}{dt}\left(-\frac{q_j\phi}{T_j}\right)F_{Mj} - i\left(\omega - \omega_{*j}\right)\frac{q_j\phi}{T_j}F_{Mj}
\end{aligned}$$

$$\widetilde{f}_j = -\frac{q_j \phi}{T_j} F_{Mj} - i\left(\omega - \omega_{*j}\right)\left(\frac{d}{dt}\right)^{-1} \frac{q_j \phi}{T_j} F_{Mj}$$

and this was the starting point of our calculations.

By comparison, in Krall and Rosenbluth the distribution function is expressed in terms of invariants

$$\begin{aligned}
\alpha &\equiv v_x^2 + v_y^2 - 2Gx \\
\beta &\equiv \left(v_z - V_0\right)^2 \\
\gamma &\equiv x - \frac{v_y}{\Omega}
\end{aligned}$$

The energetic term is expressed via the derivatives of the Maxwellian distribution to these quantities that characterize the motion of a particle

$$\begin{aligned}
\frac{\partial F_M}{\partial \mathbf{v}} &= 2\left(\mathbf{v} - v_z \widehat{e}_z\right) \frac{\partial F_M}{\partial \alpha} \\
&\quad + 2\left(v_z - V_0\right) \widehat{\mathbf{e}}_z \frac{\partial F_M}{\partial \beta} \\
&\quad + \left(-\frac{1}{\Omega}\right) \widehat{\mathbf{e}}_y \frac{\partial F_M}{\partial \gamma}
\end{aligned}$$

One can see that, if

$$V_0 \equiv 0 \ , \ \ G \equiv 0$$

and the temperatures are such that

$$\frac{\partial F_M}{\partial \alpha} = \frac{\partial F_M}{\partial \beta}$$

then the two terms

$$\begin{aligned}
&-2v_z \widehat{\mathbf{e}}_z \frac{\partial F_M}{\partial \alpha} \text{ and} \\
&+2v_z \widehat{\mathbf{e}}_z \frac{\partial F_M}{\partial \beta}
\end{aligned}$$

cancel and in place of

$$\int v_z dv_z ...$$

that would produce the W-function, we have

$$\int dv_z ...$$

that produces the Z-function.

9.2.3 The dispersion function

Now we define

$$Z\left(\widetilde{\xi}\right) \equiv \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\beta \frac{\exp\left(-\beta^2\right)}{\beta - \widetilde{\xi}}$$

where

$$\begin{aligned} \widetilde{\xi} &= \frac{1}{v_{the}} \left(\frac{\omega - k_y \widetilde{v}_y}{k_z} - \widetilde{v}_z \right) \\ &= \frac{\omega}{v_{the} k_z} - \frac{k_y}{k_z} \left(\frac{\widetilde{v}_y}{v_{the}} \right) - \frac{\widetilde{v}_z}{v_{the}} \end{aligned}$$

Since

$$\frac{k_y}{k_z} \gg 1$$

for drift waves, it results that the second term might be important.

We then have

$$\begin{aligned} \widetilde{n}_e &= \frac{|e|}{T_e} \phi n_0 \\ &+ \frac{|e|}{T_e} \phi n_0 \frac{\omega - \omega_{*e}}{k_z v_{the}} Z\left(\widetilde{\xi}_e\right) I_0\left(b_e\right) \exp\left(-b_e\right) \end{aligned}$$

where

$$b_e \equiv \frac{1}{2} k_\perp^2 \left(\frac{v_{the}}{\Omega_{ce}} \right)^2$$

(pag 30 from handwritten notes).

9.3 The ion response

The calculation is developed similarly for ions.

$$\frac{d}{dt} \widetilde{f}_{1i} = -\frac{q_i}{m_i} \left(\mathbf{E}_1 + \widetilde{\mathbf{E}}_1 \right) \frac{\partial}{\partial \mathbf{v}} f_{0i}$$

$$\begin{aligned} \widetilde{f}_i\left(\mathbf{x}, \mathbf{v}, t\right) &= -\frac{|e|}{T_i} \left(\phi + \widetilde{\phi} \right) F_{Mi} \\ &+ \left[J_0 \left(\frac{k_\perp v_\perp}{|\Omega_i|} \right) \right]^2 \frac{-i\,|e|}{T_i} \left(\omega - \omega_{*i} \right) F_{Mi} \\ &\times \int_{-\infty}^{t} dt' \left\{ \phi\left[x\left(t'\right)\right] + \widetilde{\phi}\left[x\left(t'\right)\right] \right\} \\ &\times \exp\left[-i\omega\left(t' - t\right) + ik_z v_z \left(t' - t\right) + ik_y \delta y_i\left(t'\right) + ik_z \delta z_i\left(t'\right) \right] \end{aligned}$$

where

$$\omega_{*i} = -k_y \frac{T_i}{|e| B_0} \frac{d}{dr} \ln n_i = -\frac{T_i}{T_e} \omega_{*e}$$

NOTE

We can repeat the same approximation as for electron response:

1. neglect $\widetilde{\phi}$ in the time integration (inverse of the time derivative along trajectory);
2. expand the potential ϕ calculated on the trajectory (under t' integration) up to second order;

$$\begin{aligned}
\widetilde{f}_i (\mathbf{x}, \mathbf{v}, t) &= -\frac{|e|}{T_i} \left(\phi + \widetilde{\phi} \right) F_{Mi} \\
&+ \left[J_0 \left(\frac{k_\perp v_\perp}{|\Omega_i|} \right) \right]^2 \frac{-i |e|}{T_i} (\omega - \omega_{*i}) F_{Mi} \\
&\times \left\{ \phi (x) \int_{-\infty}^{t} dt' \exp \left[-i\omega (t' - t) + i k_z v_z (t' - t) + i k_y \delta y_i (t') + i k_z \delta z_i (t') \right] \right. \\
&\left. + \frac{d^2 \phi}{dx^2} \int_{-\infty}^{t} dt' \frac{1}{2} \left[\delta x_i (t') \right]^2 \exp \left[-i\omega (t' - t) + i k_z v_z (t' - t) + i k_y \delta y_i (t') + i k_z \delta z_i (t') \right] \right\}
\end{aligned}$$

END of Note

Proceeding as in the case of electrons we get

$$\begin{aligned}
\widetilde{f}_i (\mathbf{x}, \mathbf{v}, t) &= -\frac{|e|}{T_i} \phi F_{Mi} \\
&+ \left[J_0 \left(\frac{k_\perp v_\perp}{|\Omega_i|} \right) \right]^2 \frac{-i |e|}{T_i} (\omega - \omega_{*i}) F_{Mi} \\
&\times \phi (x) \frac{1}{i k_z} \frac{1}{v_z - \left(\frac{\omega - k_y \widetilde{v}_{yi}}{k_z} - \widetilde{v}_{zi} \right)}
\end{aligned}$$

and after introducing the *ion* plasma dispersion function

$$\begin{aligned}
\widetilde{n}_i &= \frac{-|e|}{T_i} \phi n_{0i} \\
&\quad \frac{-|e|}{T_i} \phi n_{0i} \frac{\omega - \omega_{*i}}{k_z v_{thi}} Z \left(\widetilde{\xi}_i \right) I_0 (b_i) \exp (-b_i)
\end{aligned}$$

where

$$b_i \equiv \left(\frac{k_\perp v_{thi}}{|\Omega_i|} \right)^2$$

(where is 1/2 ?)

9.4 The Poisson equation

We have

$$\frac{d^2\phi}{dx^2} - k_y^2\phi - k_z^2\phi = -\frac{1}{\varepsilon_0}\left|e\right|\left(\widetilde{n}_i - \widetilde{n}_e\right)$$

which is written as a stationary Schrodinger-type equation

$$\frac{d^2\phi}{dx^2} + Q\left(x\right)\phi = 0$$

with the notation

$$\begin{aligned} & Q\left(x\right) \\ \equiv\ & -k_y^2 - k_z^2 - \frac{\left|e\right|^2}{\varepsilon_0} n_0 \left(\frac{1}{T_i} + \frac{1}{T_e}\right) \\ & -\frac{\left|e\right|^2}{\varepsilon_0}\frac{n_0}{T_i}\frac{\omega - \omega_{*i}}{k_z v_{thi}} Z\left(\widetilde{\xi}_i\right) I_0\left(b_i\right)\exp\left(-b_i\right) \\ & -\frac{\left|e\right|^2}{\varepsilon_0}\frac{n_0}{T_e}\frac{\omega - \omega_{*e}}{k_z v_{the}} Z\left(\widetilde{\xi}_e\right) I_0\left(b_e\right)\exp\left(-b_e\right) \end{aligned}$$

In view of normalization with physically relevant scales for the drift problem, we write

$$\begin{aligned} & \frac{d^2\phi}{dx^2}\frac{\varepsilon_0 T_i}{\left|e\right|^2 n_0} \\ & -\left(k_y^2 + k_z^2\right)\frac{\varepsilon_0 T_i}{\left|e\right|^2 n_0}\phi \\ & -\left[1 + \frac{T_i}{T_e}\right. \\ & \left. + \frac{\omega - \omega_{*i}}{k_z v_{thi}} Z\left(\widetilde{\xi}_i\right) I_0\left(b_i\right)\exp\left(-b_i\right) + \frac{T_i}{T_e}\frac{\omega - \omega_{*e}}{k_z v_{the}} Z\left(\widetilde{\xi}_e\right) I_0\left(b_e\right)\exp\left(-b_e\right)\right] \end{aligned}$$

The normalization is made on the basis of the following physical parameters

$$\begin{aligned} c_s^2 &= \frac{T_e}{m_i} \\ \rho_s &= \frac{c_s}{\Omega_i} \end{aligned}$$

$$\rho_i^2 = \frac{2T_i m_i}{\left|e\right|^2 B_0^2}$$

$$\lambda_D^2 = \frac{\varepsilon_0 T_e}{|e|^2 n_0}$$

Then

$$\begin{aligned}
& \lambda_D^2 \frac{d^2\phi}{dx^2} - \lambda_D^2 \left(k_y^2 + k_z^2\right)\phi - \left\{1 + \frac{T_e}{T_i}\right. \\
& \left. + \frac{T_e}{T_i}\frac{\omega - \omega_{*i}}{k_z v_{thi}} Z\left(\widetilde{\xi}_i\right) I_0\left(b_i\right)\exp\left(-b_i\right) + \frac{T_i}{T_e}\frac{\omega - \omega_{*e}}{k_z v_{the}} Z\left(\widetilde{\xi}_e\right) I_0\left(b_e\right)\exp\left(-b_e\right)\right\}\phi \\
& = 0
\end{aligned}$$

The parameters can be evaluated approximatively

$$\begin{aligned}
b_i &\equiv \frac{1}{2}k_\perp^2\rho_i^2 \simeq 1 \\
b_e &\equiv \frac{1}{2}k_\perp^2\rho_e^2 \ll 1 \ \text{(will be neglected)}
\end{aligned}$$

$$\begin{aligned}
I_0\left(b_e\right)\exp\left(-b_e\right) &\simeq 1 - b_e \simeq 1 \\
I_0\left(b_i\right)\exp\left(-b_i\right) &\equiv \Gamma_0
\end{aligned}$$

We also have

$$\begin{aligned}
\omega_{*i} &= k_\perp \frac{T_i}{|e| B_0}\left(\frac{1}{n_i}\frac{dn_i}{dr}\right) \\
&= -k_\perp \frac{T_i}{|e| B_0}\frac{1}{|L_n|} < 0
\end{aligned}$$

and

$$\begin{aligned}
\omega_{*e} &= -k_\perp \frac{T_e}{|e| B_0}\left(\frac{1}{n_0}\frac{dn_0}{dr}\right) \\
&= k_\perp \frac{T_e}{|e| B_0}\frac{1}{L_n} > 0
\end{aligned}$$

$$\omega_{*i} = -\frac{T_i}{T_e}\omega_{*e}$$

after defining

$$\frac{1}{n_0}\frac{dn_0}{dr} = -\frac{1}{L_n} \ \text{with } L_n > 0$$

We also introduce

$$\tau \equiv \frac{T_e}{T_i}$$

The formula is at this moment

$$\begin{aligned}
&\lambda_D^2 \frac{d^2\phi}{dx^2} - \lambda_D^2 \left(k_y^2 + k_z^2\right)\phi - \\
&- \left[1 + \tau + \Gamma_0 \frac{\tau\omega + \omega_{*e}}{k_z v_{thi}} Z\left(\widetilde{\xi}_i\right) + \frac{\omega - \omega_{*e}}{k_z v_{the}} Z\left(\widetilde{\xi}_e\right)\right]\phi \\
= \;& 0
\end{aligned}$$

Note
Concerning the article of **Hirschmann-Molvig**

$$d \equiv \Gamma_0 \left(\tau + \frac{\omega_{*e}}{\omega}\right)$$

$$\Lambda \equiv \frac{1}{d}\left[1 + \tau\left(1 - \Gamma_0\right) - \Gamma_0 \frac{\omega_{*e}}{\omega}\right]$$

The latter parameter can be written

$$\begin{aligned}
\Lambda &= \frac{\omega\left(1+\tau\right) - \Gamma_0\left(\omega\tau + \omega_{*e}\right)}{\Gamma_0\left(\tau\omega + \omega_{*e}\right)} \\
&= \frac{\omega\left(1+\tau\right)}{\Gamma_0\left(\tau\omega + \omega_{*e}\right)} - 1
\end{aligned}$$

and the equation from Hirshmann-Molvig is

$$\begin{aligned}
&\frac{d^2\phi}{dx^2} - \left[\frac{\omega\left(1+\tau\right)}{\Gamma_0\left(\tau\omega + \omega_{*e}\right)} - 1 + \tau^2\left(\frac{L_n}{L_s}\right)^2\left(\frac{\omega_{*e}}{\omega}\right)^2 x^2\right. \\
&\left.+\frac{1}{x}\left(\frac{\omega}{\omega_{*e}} - 1\right)\sqrt{\frac{\tau m_e}{2m_i}}\frac{L_s}{L_n}\frac{1}{\frac{\Gamma_0}{\omega}\left(\tau\omega + \omega_{*e}\right)}\right]\phi \\
= \;& 0
\end{aligned}$$

and we inquire from where their equation has been derived. We have to infer from their expressions

$$\begin{aligned}
&\frac{d^2\phi}{dx^2} - \frac{1}{\frac{\Gamma_0}{\omega}\left(\tau\omega + \omega_{*e}\right)}\left[1 + \tau - \frac{\Gamma_0}{\omega}\left(\tau\omega + \omega_{*e}\right)\right. \\
&\qquad + \left(\frac{L_n}{L_s}\right)^2\left(\frac{\omega_{*e}}{\omega}\right)^2 x^2 \frac{1}{\tau^2}\frac{\Gamma_0}{\omega}\left(\tau\omega + \omega_{*e}\right) \\
&\qquad \left.+\frac{\omega - \omega_{*e}}{\omega_{*e}}\frac{1}{x/L_s}\frac{1}{L_n}\sqrt{\frac{\tau m_e}{m_i}} Z\right]\phi \\
= \;& 0
\end{aligned}$$

or

$$
\begin{aligned}
& \frac{d^2\phi}{dx^2} - \frac{1}{\frac{\Gamma_0}{\omega}\left(\tau\omega + \omega_{*e}\right)}\left[1 + \tau - \frac{\Gamma_0}{\omega}\left(\tau\omega + \omega_{*e}\right)\right. \\
& + \frac{1}{4}k_z^2\rho_i^2 v_{thi}^2 \frac{\Gamma_0}{\omega}\left(\tau\omega + \omega_{*e}\right) \\
& \left. + \frac{2}{\rho_i}\frac{\omega - \omega_{*e}}{k_z v_{the}} Z\left(\xi_e\right)\right]\phi \\
= \; & 0
\end{aligned}
$$

then the differences come from

1. in H-M ρ_i is defined without the factor 2; then 4 and 2 disappear;
2. distances are normalized to ρ_i hence ρ_i disappears from the equation;

It results that the following approximation is made

$$
Z\left(\xi_i\right) \simeq -\frac{1}{\xi_i}
$$

because

$$
\xi_i \gg 1
$$

END of Note concerning Hirschman-Molvig article

The approximation for the dispersion function

$$
Z\left(\xi\right) = \mathrm{Re}\left[Z\left(\xi\right)\right] + i\frac{k_z}{\left|k_z\right|}\sqrt{\pi}\exp\left(-\xi^2\right)
$$

The argument

$$
\widetilde{\xi}_i = \frac{\omega}{v_{thi}k_z} - \frac{k_y}{k_z}\left(\frac{\widetilde{v}_y}{v_{thi}}\right) - \frac{\widetilde{v}_z}{v_{thi}}
$$

where the quantities with $\widetilde{}$ are due to the perturbation. They have a real and an imaginary part, coming from

$$
\omega = \omega_r + i\gamma
$$

$$
\begin{aligned}
\mathrm{Re}\,\widetilde{\xi}_i &= \frac{\omega_r}{v_{thi}k_z} - \frac{k_y}{k_z}\left(\frac{\widetilde{v}_y}{v_{thi}}\right) - \frac{\widetilde{v}_z}{v_{thi}} \\
\mathrm{Im}\,\widetilde{\xi}_i &= \frac{\widetilde{\gamma}}{v_{thi}k_z}
\end{aligned}
$$

Let us start from the definition

$$Z\left(\widetilde{\xi}\right) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\beta \frac{\exp\left(-\beta^2\right)}{\beta - \widetilde{\xi}}$$

and consider a small shift of the singularity

$$\widetilde{\xi} = \xi_0 + \delta\widetilde{\xi}$$

Then

$$Z\left(\widetilde{\xi}\right) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\beta \left[\frac{\exp\left(-\beta^2\right)}{\beta - \xi_0} + d\widetilde{\xi} \frac{\exp\left(-\beta^2\right)}{\left(\beta - \xi_0\right)^2}\right]$$

9.5 Perturbation of the dispersion function. The approximative expression.

We use an approximation of the dispersion function. This is appropriate for the very large values of the argument

$$\widetilde{\xi}_i = \xi_{i0} + \delta\widetilde{\xi}_i \gg 1$$

as for ions, and for small Landau damping, when the phase velocity of the wave in the parallel direction is much higher than the thermal velocity of the ions.

$$\begin{aligned} Z\left(\widetilde{\xi}_i\right) &= i\frac{k_z}{|k_z|}\sqrt{\pi}\exp\left(-\widetilde{\xi}_i^2\right) \\ & -\frac{1}{\widetilde{\xi}_i}\left(1 + \frac{1}{2\widetilde{\xi}_i^2} + ...\right) \end{aligned}$$

An approximation is obtained by first keeping only the first power in the inverse of the argument and discarding the higher powers; then, inserting the shifted value of the argument and expanding the exponential and the inverse of the variable to the first order in $\delta\widetilde{\xi}_i$.

$$Z\left(\widetilde{\xi}\right) \simeq i\sqrt{\pi}\exp\left(-\xi_{i0}^2\right) - \frac{1}{\xi_{i0}} - 2i\sqrt{\pi}\exp\left(-\xi_{i0}^2\right)\xi_{i0}\delta\widetilde{\xi}_i + \frac{\delta\widetilde{\xi}_i}{\xi_{i0}^2}$$

NOTE

The fact that the phase velocity in parallel direction is much higher than thermal ion velocity means that in the calculation of Z we do not correctly take into account the resonance and the Landau damping of the ions.

For this reason we cannot use the expansion of the dispersion function Z as starting point for the perturbation $\delta\widetilde{\xi}$.

We will use the full expression but we will expand as analytic function, introducing the derivatives of Z.

$$Z\left(\xi_0 + \delta\widetilde{\xi}\right) = Z\left(\xi_0\right) + \delta\widetilde{\xi}\left.\frac{dZ}{d\xi}\right|_{\xi_0} + \frac{1}{2}\left(\delta\widetilde{\xi}\right)^2 \left.\frac{d^2 Z}{d\xi^2}\right|_{\xi_0} + \ldots$$

END

There are three small parameters that must be used for expansions:

1. the ratio between the phase velocity and the thermal velocity
$$\xi \equiv \frac{\omega}{k_z v_{th}} \ll 1$$
when this is possible.

2. the perturbation due to the fluctuating field represented by the scattering in velocities
$$\frac{\widetilde{v}_y}{v} \ll 1$$
where the comparison is made with the velocity along the same direction as the perturbation due to the additional turbulence.

3. the growth rate γ (imaginary part of ω), compared with the real part of ω
$$\frac{\mathrm{Im}\,\omega}{\mathrm{Re}\,\omega} = \frac{\gamma}{\omega_r} \ll 1$$

These small quantities will be used for expansions.

9.5.1 Calculation with Z after expansion in a small shift of the argument

By replacing explicitely the frequency

$$\omega = \omega_R + i\gamma$$

the variable of the dispersion function becomes

$$\xi_0 = \xi_{0R} + i\xi_{0I}$$

We have

$$\begin{aligned} Z\left(\widetilde{\xi}_i\right) &= i\sqrt{\pi}\exp\left[-\left(\xi_{i0R}+i\xi_{i0I}\right)^2\right] \\ &-\frac{1}{\xi_{i0R}+i\xi_{i0I}} \\ &-2i\sqrt{\pi}\exp\left[-\left(\xi_{i0R}+i\xi_{i0I}\right)^2\right]\left(\xi_{i0R}+i\xi_{i0I}\right)\delta\widetilde{\xi}_i \\ &+\frac{\delta\widetilde{\xi}_i}{\left(\xi_{i0R}+i\xi_{i0I}\right)^2}+... \end{aligned}$$

The factorization of a larger part at the exponents

$$\begin{aligned} Z\left(\widetilde{\xi}_i\right) &= i\sqrt{\pi}\exp\left[-\xi_{i0R}^2\left(1+2i\frac{\xi_{i0I}}{\xi_{i0R}}\right)\right] \\ &-\frac{1}{\xi_{i0R}}\left(1-i\frac{\xi_{i0I}}{\xi_{i0R}}\right) \\ &-2i\sqrt{\pi}\exp\left[-\xi_{i0R}^2\left(1+2i\frac{\xi_{i0I}}{\xi_{i0R}}\right)\right]\left(\xi_{i0R}+i\xi_{i0I}\right)\delta\widetilde{\xi}_i \\ &+\delta\widetilde{\xi}_i\frac{1}{\xi_{i0R}^2}\left(1-2i\frac{\xi_{i0I}}{\xi_{i0R}}\right) \end{aligned}$$

The form is

$$Z\left(\widetilde{\xi}_i\right)=A_i+\xi_{i0I}B_i+\left(\xi_{i0I}\right)^2C_i$$

where

$$\begin{aligned} A_i &\equiv i\sqrt{\pi}\exp\left(-\xi_{i0R}^2\right)\xi_{i0R} \\ &-\frac{1}{\xi_{i0R}} \\ &-2i\sqrt{\pi}\exp\left(-\xi_{i0R}^2\right)\xi_{i0R}\delta\widetilde{\xi}_i \\ &+\frac{\delta\widetilde{\xi}_i}{\xi_{i0R}^2} \end{aligned}$$

$$\begin{aligned} B_i &\equiv \xi_{i0I}\left[2\sqrt{\pi}\exp\left(-\xi_{i0R}^2\right)\xi_{i0R}\right. \\ &+\frac{i}{\xi_{i0R}^2} \\ &-4\sqrt{\pi}\exp\left(-\xi_{i0R}^2\right)\xi_{i0R}\delta\widetilde{\xi}_i \\ &2\sqrt{\pi}\exp\left(-\xi_{i0R}^2\right)\delta\widetilde{\xi}_i \\ &\left.-\frac{2i}{\xi_{i0R}^3}\delta\widetilde{\xi}_i\right] \end{aligned}$$

$$C_i = (\xi_{i0I})^2 \left[4i\sqrt{\pi}\exp\left(-\xi_{i0R}^2\right)\xi_{i0R}\delta\widetilde{\xi}_i\right]$$

10 Electrostatic drift waves in the toroidal geometry

The electron-ion collision frequency exceeds the inverse transit time

$$\nu_{ei} > \omega_{T_e} = \frac{v_{the}}{Rq}$$

then the equations are: **the electron continuity equation**

$$\frac{\partial n}{\partial t} + n\nabla_{\|}v_{\|e} + \boldsymbol{\nabla}\cdot(n\mathbf{v}_{\perp e}) = 0$$

and the **electron parallel equation of motion**

$$\frac{dv_{\|e}}{dt} = v_{the}^2\nabla_{\|}\left(\frac{|e|\varphi}{T_e} - n\right) - \nu_{ei}v_{\|e}$$

The perpendicular velocity of *electrons* is

$$\begin{aligned}\mathbf{v}_{\perp e} &= \mathbf{v}_{De} + \mathbf{v}_{\mathbf{E}} \\ &= -\frac{T_e}{|e|B_0}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\ln B + \frac{-\boldsymbol{\nabla}\varphi\times\widehat{\mathbf{n}}}{B_0}\end{aligned}$$

composed of the $\mathbf{E}\times\mathbf{B}$ convection in the field of the wave and the electron magnetic drift velocity $\mu\boldsymbol{\nabla}B$. *No polarization* velocity is included, since it is of higher order and in any case *small* for electrons.

The second equation (**equation of electron-parallel-momentum conservation**) expresses the balance of the parallel electric field and the parallel gradient of the pressure, with a contribution from electro-ion collisions (parallel viscosity). Assuming that the inertia term for the electrons is negligible ($\frac{dv_{\|e}}{dt} = 0$)the parallel balance involves only

1. the parallel pressure gradient, which is only due to the *density* parallel variation, $\nabla_{\|}p = T_e\nabla_{\|}\widetilde{n}$.
2. the parallel electric force $-\nabla_{\|}\varphi = E_{\|}$, and
3. the parallel friction force, $n_0\nu_{ei}v_{\|}$

This momentum balance allows us to obtain the parallel electron velocity

$$v_{\parallel e} = \frac{v_{the}^2}{\nu_{ei}} \nabla_{\parallel} h(r, \theta, \varphi, t)$$

where h is the non-adiabatic part of the electron density fluctuation, *i.e.* $\left(\frac{|e|\varphi}{T_e} - n\right)$. The equation for h is the equation of continuity with this replacement

$$\frac{\partial h}{\partial t} - \frac{v_{the}^2}{\nu_{ei}} \nabla_{\parallel}^2 h + \mathbf{v}_{De} \cdot \boldsymbol{\nabla} h - \frac{-\boldsymbol{\nabla}\varphi \times \widehat{\mathbf{n}}}{B_0} \cdot \boldsymbol{\nabla} h = (\omega - \omega_{*e}) \frac{i\,|e|\,\varphi}{T_e}$$

We **NOTE** the extremely important role of the friction represented by the collisionality ν_{ei}, which ensures the force balance in the parallel direction. This ν_{ei} will occur at the numitor.

11 Coupling of poloidal modes

Some reference **atomic phys effects Beer Hahm**

"*in a toroidal system the divergence of the $E \times B$ drift in the ion continuity equation does not vanish and leads to coupling between different poloidal harmonics*"

$$\boldsymbol{\nabla}_{\perp} \cdot \mathbf{v}_{E \times B} = -2i\omega_{de} \left(\cos\theta - \frac{i}{k_\theta} \sin\theta \, \frac{\partial}{\partial r} \right) \frac{e\Phi}{T}$$

where

$$\omega_{de} = k_\theta \frac{\rho_s c_s}{R}$$

12 Drift waves in flat density profile

The paper by **Tang Rewoldt, Liu Chen, 1986**.

It is specific to H-mode.

Large

$$\eta_j = \frac{d \ln T_j}{d \ln n_j} = \frac{L_n}{L_T}$$

When $\boldsymbol{\nabla} n_0 = 0$ the drift waves evolve from dissipative trapped electron drift wave instability to the ion fluid-like modes like trapped ion modes or η_i mode or ITG. See **Beer thesis**.

13 Drift waves in the presence of the radial electric field (rotation)

It will frequently be used the quantity:

$$\omega_E = k_y V_0(x)$$

where $V_0(x)$ arises from the $E \times B$ flow of the plasma.

This velocity is strictly refering to particles and not to the fluid.

This velocity must be used in kinetic calculations, where the drifts of the particles are important for the equation of Vlasov.

13.0.2 Strongly dissipative drift electron mode

This case allow the use of the $i\delta$ model.**Carreras Sidikman Diamond Terry 1992** [**?**].

The magnetic field

$$\mathbf{B} = B_0 \left(\widehat{\mathbf{e}}_z + \frac{x}{L_s} \widehat{\mathbf{e}}_y \right)$$

and the rotation velocity fro the sheared equilibrium radial electric field

$$V_0 = -\frac{E_{0x}}{B_0}$$

The parallel flow damping can be assumed to be collisional (and not determined by the Landau damping).

1. For the long-wavelength modes

$$\begin{aligned} k_\perp \rho_s &\ll 1 \\ \rho_s &\ll \lambda_\perp \end{aligned}$$

   the nonlinearity is from the $E \times B$ term.

2. For the short-wavelength - high k - end of the spectrum,

$$\lambda_\perp \sim \rho_s$$

   the ion-polarization current nonlinearity should be retained.

The *dissipative trapped electron regime.*

For the non-adiabatic part of the **trapped electron response**:

$$-i\left(\omega - \omega_E - \omega_{De} + i\nu_{eff}\right) \widetilde{g}_{\mathbf{k}} = i\frac{|e|}{T_e}\varphi_{\mathbf{k}}\sqrt{\varepsilon}\left\{\omega - \omega_E - \omega_{*e}\left[1 + \eta_e\left(\epsilon - \frac{1}{2}\right)\right]\right\} F_{0e}$$

where

ν_{eff}	effective collision frequency	
$\widetilde{g}_{\mathbf{k}}$	k-component of the non-adiabatic trapped electron response	
$\eta_e = \frac{L_n}{L_T}$	the η-factor for electrons	
$\sqrt{\varepsilon}$	fraction of trapped electrons	
$\omega_{*e} = k_y \frac{\rho_s c_s}{L_n}$	electron diamagnetic frequency	$c_s = \left(\frac{T_{e0}}{m_i}\right)^{1/2}$
$\omega_E = k_y V_0 (x)$	Doppler shift of the frequency	drift due to rotation

At the end it will be estimated the frequency of the mode as

$$\omega_{\mathbf{k}} \approx \omega_{E\mathbf{k}} + \frac{\omega_{*e}}{1 + k_y^2 \rho_s^2}$$

For the strongly dissipative trapped electron drift mode the following approximation can be made:

$$\nu_{eff} \gg \left\{ \begin{array}{c} \omega - \omega_E \quad \text{and} \\ \omega_{De} \end{array} \right.$$

Taking into account these inequalities, and integrating the function $\widetilde{g}_{\mathbf{k}}$ over velocity space, we obtain the non-adiabatic electron density response:

$$\delta n_{e,k}^{NA} \approx i \frac{|e|}{T_e} \varphi_{\mathbf{k}} \sqrt{\varepsilon} \frac{\omega - \omega_E - \omega_{*e} (1 + \alpha \eta_e)}{\nu_{eff}}$$

where $\alpha = 1.5$ is a constant.

It can be used the estimation of the mode frequency:

$$\omega_{\mathbf{k}} - \omega_{E\mathbf{k}} \approx \frac{\omega_{*e}}{1 + k_y^2 \rho_s^2}$$

where $\omega_{E\mathbf{k}}$ is the Doppler shift at the singular surface of the mode $\mathbf{k}$. Replacing in the density response we get:

$$\delta n_{e,k}^{NA} \approx i \frac{|e|}{T_e} \varphi_{\mathbf{k}} \sqrt{\varepsilon} \frac{-k_y^2 \rho_s^2 \omega_{*e} / \left(1 + k_y^2 \rho_s^2\right) - \omega_{*e} \alpha \eta_e}{\nu_{eff}}.$$

Since it is assumed that

$$\eta_e \equiv \frac{L_n}{L_{T_e}} \gg 1$$

and

$$k_y \rho_s < 1$$

the expression can be further simplified by neglecting $-\frac{k_y^2\rho_s^2}{1+k_y^2\rho_s^2}\omega_{*e}$. (Obs. only the second term at the numerator is retained, and: $V_{*e} = k_y\omega_{*e}$):

$$\delta n_e^{NA} = -\alpha\sqrt{\varepsilon}\frac{V_{*e}}{\nu_{eff}}\eta_e\frac{\partial}{\partial y}\left(\frac{|e|}{T_e}\varphi\right)$$

Note that the expression of the diamagnetic electron frequency is here factorized and the 'inverse Fourier transform' is performed such that the k_y is replaced by the partial derivative at y (i.e. $k_y \rightarrow \frac{1}{i}\frac{\partial}{\partial y}$) .

The quasineutrality

$$\frac{\widetilde{n}_i}{n_0} = \frac{|e|}{T_e}\varphi - \alpha\sqrt{\varepsilon}\frac{V_{*eT_e}}{\nu_{eff}}\eta_e\frac{\partial}{\partial y}\left(\frac{|e|}{T_e}\varphi\right).$$

Since the nonadiabatic density of the electrons can be treated perturbatively, the equation above can be *inverted* to obtain the potential φ in function of the ion density response (the mathematical operation of inversion can be done in k space where it is easier to apply evaluations and approximations):

$$\frac{|e|}{T_e}\varphi \approx \left(1 + \alpha\sqrt{\varepsilon}\frac{V_{*eT_e}}{\nu_{eff}}\eta_e\frac{\partial}{\partial y}\right)\frac{\widetilde{n}_i}{n_0}.$$

In this way we get an expression for the potential perturbation in terms of the *ion density* perturbation, after applying the neutrality requirement.

Basically we have calculated the *non-adiabatic* part of the response of the *electron* density to a perturbation. The non-adiabatic electron response is due to the *trapping* on electron banana orbits.This affects a population of $\sqrt{\varepsilon}$ from the total electron density. The drive remains, as usual the resonance between the oscillations of the wave (shifted by the rotation) $\omega - \omega_E$ and the electron diamagnetic trequency ω_{*e}.

The connection between the perturbed density $\widetilde{n}$ and perturbed potential φ is of the type

$$1 + i\delta$$

which means that the non-adiabatic part is shifted in phase as

$$\exp(i\pi)$$

The **ions are treated as a fluid** (hydrodynamic ions).

The equation of continuity for the ions is

$$\begin{aligned} & \frac{\partial\widetilde{n}_i}{\partial t} + \langle\mathbf{V}_E\rangle\cdot\boldsymbol{\nabla}\widetilde{n}_i + \widetilde{V}_{Ex}\frac{d\langle n\rangle}{dx} + \widetilde{V}_E\cdot\boldsymbol{\nabla}\widetilde{n}_i \\ = \; & -n_0\left(\boldsymbol{\nabla}_\perp\cdot\widetilde{\mathbf{V}}_E + \nabla_\parallel\widetilde{V}_\parallel\right) \end{aligned}$$

The perpendicular ion flow is due to the

1. $E \times B$ velocity
2. the ion polarization drift flow

The parallel ion momentum balance is determined by friction due to collisions and parallel pressure:

$$0 \approx -c_s^2 \nabla_{\|} \widetilde{n}_i - n_0 \nu_i \widetilde{V}_{\|}$$

The divergence of this part of the velocity :

$$\nabla_{\|} \widetilde{V}_{\|}$$

appears in the *equation of continuity* of the ions. Through the balance of the parallel momentum, as written above, this term will give a second-order parallel derivative of the fluctuating ion density $\widetilde{n}_i$, $(\nabla_{\|}^2 \widetilde{n}_i)$ and this gives finally an x^2 dependent term in the equation, due to relation $k_{\|} \sim x$, as results from *magnetic shear.*

Then the equation of continuity for the ions, after replacing the parallel velocity from the parallel balance and the perpendicular velocity from the $E \times B$ and the ion polarization drift

$$\begin{aligned}
& \frac{\partial \widetilde{n}_i}{\partial t} + \langle \mathbf{V}_E \rangle \cdot \boldsymbol{\nabla} \widetilde{n}_i + V_{*n} \frac{\partial \widetilde{n}_i}{\partial y} \text{ ion density advection} \\
& -L_n D_0 \left[\boldsymbol{\nabla}_{\perp} \left(\frac{\partial \widetilde{n}_i}{\partial y} \right) \times \widehat{\mathbf{e}}_z \right] \cdot \boldsymbol{\nabla}_{\perp} \widetilde{n}_i \text{ the } E \times B \text{ nonlinearity} \\
& +D_0 \frac{\partial^2 \widetilde{n}_i}{\partial y^2} \text{ instability driving term } (i\delta \text{ drive}) \\
& -\rho_s^2 \left[\frac{\partial}{\partial t} + \langle \mathbf{V}_E \rangle \cdot \boldsymbol{\nabla} \right] \boldsymbol{\nabla}_{\perp}^2 \widetilde{n}_i \\
& +\rho_s c_s \left[\boldsymbol{\nabla}_{\perp} \widetilde{n}_i \times \widehat{\mathbf{e}}_z \right] \cdot \boldsymbol{\nabla}_{\perp} \left(\rho_s^2 \nabla_{\perp}^2 \widetilde{n}_i \right) \text{ the vorticity nonlinearity} \\
& +\rho_s c_s \frac{\partial \widetilde{n}_i}{\partial y} \frac{\partial}{\partial x} \left(\rho_s^2 \frac{\partial^2 \langle n \rangle}{\partial x^2} \right) \text{ the Kelvin-Helmholtz term} \\
& -\frac{c_s^2}{\nu_i} \nabla_{\|}^2 \widetilde{n}_i \text{ parallel flow damping} \\
= \; & 0
\end{aligned}$$

Here

$$D_0 = \sqrt{\varepsilon} \frac{(\rho_s c_s)^2 \alpha}{L_T L_n \nu_{eff}}$$

The average means integration over the y and z coordinates, poloidal and parallel directions.

Simplifications

1. the convective nonlinearity is more important than the vorticity nonlinearity
$$k_\perp^2 \rho_s^2 \ll \frac{k_y^2 D_0}{\omega_{*e}}$$
2. the Kelvin-Hlemholtz term is neglected since it is assumed that there is no *inflection* point in the sheared velocity profile. If the K-H term were kept without the presence of the *inflection* point then the only change would have been
$$\omega = \frac{\omega_{*e} + k_y \rho_s^2 V_y''}{1 + k_\perp^2 \rho_s^2}$$

What is the specific content of this model?

$$\begin{aligned}
& \frac{d\widetilde{n}_i}{dt} \quad \text{the electric } \langle \mathbf{V}_E \rangle \cdot \boldsymbol{\nabla} \text{ advection of the pert. density} \\
& + V_{*i} \frac{\partial \widetilde{n}_i}{\partial y} \quad \text{usual diamagnetic term, but now the pert. is driven by } V_{*i} \\
& - L_n D_0 \left[\boldsymbol{\nabla}_\perp \left(\frac{\partial \widetilde{n}_i}{\partial y} \right) \times \widehat{\mathbf{e}}_z \right] \cdot \boldsymbol{\nabla}_\perp \widehat{n}_i \quad \text{non-linear, HM type, with } \varphi \text{ replaced} \\
& + D_0 \frac{\partial^2 \widetilde{n}_i}{\partial y^2} \\
& - \rho_s^2 \frac{d}{dt} \boldsymbol{\nabla}_\perp^2 \widetilde{n}_i \\
& - \frac{c_s^2}{\nu_i} \nabla_\parallel^2 \widetilde{n}_i \quad \text{comes from parallel ion-coll. balance and density cont. eq.} \\
= \; & 0
\end{aligned}$$

14 The two-nonlinearity model of dissipative drift

In the paper [?]

$$-i \left(\omega_{\mathbf{k}} - \omega_{\mathbf{k}}^{De} + i \nu_{eff} \right) \widetilde{g}_{\mathbf{k}} = i \frac{|e| \widetilde{\phi}_{\mathbf{k}}}{T_{e0}} \sqrt{\varepsilon} f_0 \left\{ \omega_{\mathbf{k}} - \omega_{*\mathbf{k}} \left[1 + \eta_e \left(\frac{v^2}{v_e^2} - \frac{3}{2} \right) \right] \right\}$$

where:

$$\nu_{eff} = \text{ effective collision frequency}$$

$$\omega_{\mathbf{k}}^{De} = \text{is the electron curvature drift frequency}$$
$$\sqrt{\varepsilon} = \text{fraction of the trapped electrons}$$

The strongly *dissipative trapped electron* modes

$$\nu_{eff} \gg \omega_{\mathbf{k}}^{De} \ , \ \omega_{*\mathbf{k}}$$

After taking this limit in the equation and integrating over the velocity space

$$\frac{\delta n_{\mathbf{k}}^{NA}}{n_0} = i\frac{|e|\,\widetilde{\phi}_{\mathbf{k}}}{T_{e0}}\sqrt{\varepsilon}\frac{\omega_{\mathbf{k}} - \omega_{*\mathbf{k}}\left(1+\alpha\eta_e\right)}{\nu_{eff}}$$

where $\alpha = 3/2$.

The quasineutrality equation:

$$\frac{|e|\,\widetilde{\phi}_{\mathbf{k}}}{T_{e0}} = \left(1 + \alpha\sqrt{\varepsilon}\frac{V_{*T}}{\nu_{eff}}ik_y\right)\frac{\widetilde{n}_{i\mathbf{k}}}{n_0}$$

About ions.

$$\frac{\partial \widetilde{n}_i}{\partial t} + \widetilde{V}_x\frac{dn_0}{dx} + \widetilde{\mathbf{V}}\cdot\boldsymbol{\nabla}\widetilde{n}_i = -n_0\left(\boldsymbol{\nabla}_{\perp}\cdot\widetilde{\mathbf{V}}_{\perp} + \boldsymbol{\nabla}_{\parallel}\cdot\widetilde{\mathbf{V}}_{\parallel}\right)$$

where the perpendicular ion velocity is due to the $\mathbf{E}\times\mathbf{B}$ motion and polarization drift.

In these works the variation of the drift waves along the magnetic filed is neglected

$$\boldsymbol{\nabla}_{\parallel} = 0$$

and the model becomes a cuasi-two-dimensional one.

The model equation becomes

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(1-\rho_s^2\boldsymbol{\nabla}_{\perp}^2\right)n \\
&+V_{*n}\frac{\partial n}{\partial y} + \quad \text{(the diamagnetic flow on poloidal } y\text{)} \\
&+D_0\frac{\partial^2 n}{\partial y^2} \text{ (the drive, effective } i\delta \text{ , or } ik_y^2 D_0\text{)} \\
&-L_n D_0\left[\boldsymbol{\nabla}_{\perp}\left(\frac{\partial n}{\partial y}\right)\times\mathbf{e}_z\right]\cdot\boldsymbol{\nabla}_{\perp}n \text{ (the } E\times B \text{ convection of the nonadiabatic term)} \\
&+\rho_s c_s\left(\boldsymbol{\nabla}_{\perp}n\times\mathbf{e}_z\right)\cdot\boldsymbol{\nabla}_{\perp}\left(\rho_s^2\boldsymbol{\nabla}_{\perp}^2\right)n \text{ (the polarization nonlinearity)} \\
&= 0
\end{aligned}$$

where the normalized ion density is

$$n \equiv \frac{\widetilde{n}}{n_0}$$

the ion diamagnetic drift velocity is

$$V_{*n} = \frac{c_s \rho_s}{L_n}$$

and the notation is introduced

$$D_0 = \alpha \sqrt{\varepsilon} \frac{(\rho_s c_s)^2}{L_T L_n \nu_{eff}}$$

Without the non-adiabatic electrons (the third: $D_0 \frac{\partial^2 n}{\partial y^2}$ and fourth: $-L_n D_0 \left[\boldsymbol{\nabla}_\perp \left(\frac{\partial n}{\partial y} \right) \times \mathbf{e}_z \right] \cdot \boldsymbol{\nabla}_\perp n$ terms) the equation reduces to the original Hasegawa-Mima equation. An energy sink can be modelled by adding a hyperviscosity term in the model equation. This leads to a finite band of unstable drift modes with a high k cutoff. Let's note $\mathbf{k}_\perp = \mathbf{k}$.

In Fourier space

$$i \frac{\partial}{\partial t} n_{\mathbf{k}} - \frac{\omega_{*\mathbf{k}} + i k_y^2 D_0}{1 + k^2 \rho_s^2} n_{\mathbf{k}} + \frac{i}{1 + k^2 \rho_s^2} \left(N_{\mathbf{k}}^{\mathbf{E} \times \mathbf{B}} + N_{\mathbf{k}}^{POL} \right) = 0$$

where the nonlinearities are

$$N_{\mathbf{k}}^{\mathbf{E} \times \mathbf{B}} = -i \frac{1}{2} L_n D_0 \sum_{\mathbf{k}' + \mathbf{k}'' = \mathbf{k}} \left[(\mathbf{k} \times \mathbf{k}') \cdot \mathbf{e}_z \right] \left(k_y'' - k_y' \right) n_{\mathbf{k}'} n_{\mathbf{k}''}$$

and

$$N_{\mathbf{k}}^{POL} = \frac{1}{2} \rho_s c_s \sum_{\mathbf{k}' + \mathbf{k}'' = \mathbf{k}} \left[(\mathbf{k} \times \mathbf{k}') \cdot \mathbf{e}_z \right] \rho_s^2 \left(k''^2 - k'^2 \right) n_{\mathbf{k}'} n_{\mathbf{k}''}$$

The linear dispersion relation $i \frac{\partial}{\partial t} = \omega_{\mathbf{k}}$

$$\omega_{\mathbf{k}} = \omega_{\mathbf{k}}^{(0)} + i \gamma_{\mathbf{k}}^{(0)} = \frac{\omega_{*\mathbf{k}}}{1 + k^2 \rho_s^2} + i \frac{k_y^2 D_0}{1 + k^2 \rho_s^2}$$

which means that the term

$$\frac{k_y^2 D_0}{1 + k^2 \rho_s^2} \text{ is the drive.}$$

To examine the conserved quantities we ignore the **drive** and the **sink** (damping). If only the **polarization nonlinearity**

$$N_{\mathbf{k}}^{POL} = \rho_s c_s \left(\boldsymbol{\nabla}_\perp n \times \mathbf{e}_z \right) \cdot \boldsymbol{\nabla}_\perp \left(\rho_s^2 \boldsymbol{\nabla}_\perp^2 n \right)$$

is retained **the system has two conserved quantities** :

$$\begin{aligned}
\text{the energy } E &= \frac{1}{2}\int dV\ \left(|n|^2 + \rho_s^2\,|\boldsymbol{\nabla}_\perp n|^2\right) \\
&= \frac{1}{2}\sum_{\mathbf{k}}\ \left(1 + k^2\rho_s^2\right)|n_{\mathbf{k}}|^2 \\
\text{the enstrophy } \Omega &= \frac{1}{2}\int dV\ \left(\left|\rho_s^2\boldsymbol{\nabla}_\perp^2 n\right|^2 + \rho_s^2\,|\boldsymbol{\nabla}_\perp n|^2\right) \\
&= \frac{1}{2}\sum_{\mathbf{k}} k^2\rho_s^2\ \left(1 + k^2\rho_s^2\right)|n_{\mathbf{k}}|^2
\end{aligned}$$

The statistical mechanics prediction for the density fluctuation spectrum in an equilibrium state

$$\left\langle |n_{\mathbf{k}}|^2\right\rangle = \frac{1}{(1 + k^2\rho_s^2)\,(a + b\,k^2\rho_s^2)}$$

where a and b are Lagrange multipliers.

The isotropic energy spectrum is

$$E_{\mathbf{k}} = \pi\, k\rho_s\left(1 + k^2\rho_s^2\right)\left\langle |n_{\mathbf{k}}|^2\right\rangle = \frac{\pi\, k\rho_s}{a + b\,k^2\rho_s^2}$$

and the system pushes the energy to large scales.

The isotropic spectrum of the enstrophy is

$$\Omega_{\mathbf{k}} = k^2\rho_s^2\, E_{\mathbf{k}} = \frac{\pi\, k^3\rho_s^3}{a + b\,k^2\rho_s^2}$$

and the system pushes enstrophy to large scales. This is the **dual cascade**. The energy going to the large scales is the**inverse cascade**.

When there is also the $\mathbf{E}\times\mathbf{B}$ nonlinearity

$$N^{\mathbf{E}\times\mathbf{B}} = -L_n D_0\left[\boldsymbol{\nabla}_\perp\left(\frac{\partial n}{\partial y}\right)\times\mathbf{e}_z\right]\cdot\boldsymbol{\nabla}_\perp n$$

the system has only one conserved quantity, the energy. The equilibrium density fluctuation spectrum is

$$\left\langle |n_{\mathbf{k}}|^2\right\rangle = \frac{c}{1 + k^2\rho_s^2}$$

14.0.3 Nonlinear dispersion relation

Write the model in the dimensionless form using the space and $(time)^{-1}$ units:

$$\begin{aligned} & \rho_s \\ \Omega_i &= \frac{c_s}{\rho_s} \end{aligned}$$

$$\left(i\frac{\partial}{\partial t} - \frac{\omega_{*\mathbf{k}} + ik_y^2 \widetilde{D}_0}{1+k^2}\right) n_{\mathbf{k}} + \frac{i}{1+k^2} N_{\mathbf{k}} = 0 \tag{106}$$

where the nonlinear term is

$$\begin{aligned} N_{\mathbf{k}} &= N_{\mathbf{k}}^{\mathbf{E}\times\mathbf{B}} + N_{\mathbf{k}}^{POL} \\ &= -i\frac{1}{2}\xi \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \left[(\mathbf{k}\times\mathbf{k}')\cdot\mathbf{e}_z\right] \left(k_y'' - k_y'\right) n_{\mathbf{k}'} n_{\mathbf{k}''} \\ &\quad + \frac{1}{2} \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \left[(\mathbf{k}\times\mathbf{k}')\cdot\mathbf{e}_z\right] \rho_s^2 \left(k''^2 - k'^2\right) n_{\mathbf{k}'} n_{\mathbf{k}''} \end{aligned} \tag{107}$$

where the dimensionless parameter ξ is

$$\xi \equiv \frac{L_n D_0}{\rho_s c_s} \equiv \alpha\sqrt{\varepsilon}\left(\frac{\rho_s}{L_T}\right)\left(\frac{\Omega_i}{\nu_{eff}}\right)$$

and

$$\widetilde{D}_0 \equiv \xi \left(\frac{\rho_s}{L_n}\right)$$

In order to find the **nonlinear dispersion relation** we need to carry out **the one-point renormalization** and find the renormalized eigenvalue equation for $n_{\mathbf{k}}$. The method of renormalization is EDQNM = eddy damped quasinormal Markovian closure scheme. This is effectively an iterative closure method with the use of eddy damping to represent incoherent or higher order wave correlations.

1. the nonlinearity is written in terms of **driven modes**: in the nonlinear term one of the functions $n_{\mathbf{k}}$ is obtained by iteration, with retaining only the direct interaction:

$$N_{\mathbf{k}} = \sum_{\mathbf{k}'=\mathbf{k}''-\mathbf{k}} \left[(\mathbf{k}\times\mathbf{k}')\cdot\mathbf{e}_z\right] \left[i\xi\left(k_y' + k_y''\right) + \left(k'^2 - k''^2\right)\right] n_{-\mathbf{k}'} n_{\mathbf{k}''}^{(2)} \tag{108}$$

2. and the driven fluctuations are solutions of

$$\left(\frac{\partial}{\partial t} + \Delta\omega_{\mathbf{k}''}\right) n_{\mathbf{k}''}^{(2)} + i\left(\frac{\omega_{*\mathbf{k}''}}{1+k''^2} + i\frac{k_y''^2 \widetilde{D}_0}{1+k''^2}\right) n_{\mathbf{k}''}^{(2)} \tag{109}$$
$$= \frac{(\mathbf{k}\times\mathbf{k}')\cdot\mathbf{e}_z}{1+k''^2}\left[-i\xi\left(k_y'-k_y\right)+\left(k'^2-k^2\right)\right] n_{\mathbf{k}'} n_{\mathbf{k}}$$

which is the combinination of the Equations (106) and (107) with the occurence of a new term, a frequency shift $\Delta\omega$. The most important aspect is the suppression of all the terms from the sum in the nonlinear term excepting the term relating the **direct interaction**

$$\mathbf{k}'' = \mathbf{k}' + \mathbf{k}$$

the term $\Delta\omega_{\mathbf{k}''}$ is the **eddy damping rate**.

3. in these equations for $n_{\mathbf{k}''}^{(2)}$ only those interacting waves i.e. the $\mathbf{k}$ and $\mathbf{k}'$ waves are kept. The equation is solved

$$n_{\mathbf{k}''}^{(2)} = \frac{(\mathbf{k}\times\mathbf{k}')\cdot\mathbf{e}_z}{1+k''^2}\left[-i\xi\left(k_y'-k_y\right)+\left(k'^2-k^2\right)\right]$$
$$\times \int^t dt' \exp\left[\left(-i\omega_{\mathbf{k}''}^{(0)} + \gamma_{\mathbf{k}''}^{(0)} - \Delta\omega_{\mathbf{k}''}\right)(t-t')\right] n_{\mathbf{k}'}(t')\, n_{\mathbf{k}}(t')$$

4. Replacing in the Equation (106) we shall find the triple product

$$n_{-\mathbf{k}'}\, n_{\mathbf{k}''}^{(2)} \longrightarrow n_{-\mathbf{k}'}(t)\, n_{\mathbf{k}'}(t')\, n_{\mathbf{k}}(t')$$

Here we make the ansatz for $t > t'$:

$$n_{-\mathbf{k}}(t)\, n_{\mathbf{k}}(t') \rightarrow \langle n_{-\mathbf{k}}(t)\, n_{\mathbf{k}}(t')\rangle = |n_{\mathbf{k}}(0)|^2 \exp\left[\left(-i\omega_{\mathbf{k}}^{(0)} + \gamma_{\mathbf{k}}^{(0)} - \Delta\omega_{\mathbf{k}}\right)(t-t')\right]$$

5. These form is inserted into the equation (108) for $N_{\mathbf{k}}$.

$$N_{\mathbf{k}} = \sum_{\mathbf{k}'=\mathbf{k}''-\mathbf{k}} \frac{\left[(\mathbf{k}\times\mathbf{k}')\cdot\mathbf{e}_z\right]^2}{1+k''^2} R_{\mathbf{k},\mathbf{k}',\mathbf{k}''}$$
$$\times \left\{\left[\xi^2\left(k_y'^2 + k_y''^2 - k_y^2\right) + k^2\left(k^2 - k'^2\right)\right] + i\xi k_y\left(k'^2 - 2k^2\right)\right\} |n_{\mathbf{k}'}|^2 n_{\mathbf{k}}$$

where the propagator is

$$R_{\mathbf{k},\mathbf{k}',\mathbf{k}''} = \frac{1}{\left(\omega_{\mathbf{k}}^{(0)} + \omega_{\mathbf{k}'}^{(0)} - \omega_{\mathbf{k}''}^{(0)}\right) + i\left(\Delta\omega_{\mathbf{k}} + \Delta\omega_{\mathbf{k}'} + \Delta\omega_{\mathbf{k}''} - \gamma_{\mathbf{k}}^{(0)} - \gamma_{\mathbf{k}'}^{(0)} - \gamma_{\mathbf{k}''}^{(0)}\right)}$$

15 Nonlinear drift waves

From the paper **Horton, Meiss PF 26 (1983) 990**.

Electrostatic electron drift waves and ion acoustic waves in inhomogeneous plasma slab

$$n_0(x), \quad x \equiv \text{radial}$$

with sheared magnetic field

$$\begin{aligned}\mathbf{\Omega}_i &= \frac{|e|\,\mathbf{B}}{m_i} \\ &= \frac{|e|\,B_0}{m_i}\left(\widehat{\mathbf{n}} + \frac{x}{L_s}\widehat{\mathbf{e}}_y\right)\end{aligned}$$

The equations

1. the pressureless ion fluid (*i.e.* the ions are cold, there is no ion temperature $T_i \sim 0$ therefore the pressure in the equation of momentum conservation is neglected. This also excludes the η_i modes.)
2. almost adiabatic electron fluid equation
3. quasineutrality

The ion fluid equation

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\mathbf{\nabla}\right)\ln\left(\frac{n(\mathbf{x},t)}{n_0}\right) - \frac{v_x}{L_n} + \mathbf{\nabla}\cdot\mathbf{v} = 0$$

For comparison in **Laedke Spatschek PF 31 (1988) 1492** this equation is written

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\mathbf{\nabla}\right)\ln n + \mathbf{\nabla}_{\perp}\cdot\mathbf{v} = 0$$

where the divergence of velocity is in the plane transversal to the magnetic field. The term equivalent to $-v_x/L_n$ is separated later when the velocity is specified as an electric $E\times B$ term plus the ion polarization term.

The equation of ion momentum conservation

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\mathbf{\nabla}\right)\mathbf{v} = -c_s^2\mathbf{\nabla}\phi + \mathbf{v}\times\mathbf{B}$$

where $c_s^2 = T_e/m_i$. We note that in the term of force in the right, the gradient of the ion pressure *does not exist.* The term

$$-c_s^2\mathbf{\nabla}\phi$$

comes from the *electric force* resulting from the presence of a potential in plasma

$$\begin{aligned} |e|\, n_i \mathbf{E} &= -|e|\, n_i \boldsymbol{\nabla}\Phi \\ &= -|e|\, n_i \frac{T_e}{|e|} \boldsymbol{\nabla} \frac{|e|}{T_e}\Phi \\ &= -|e|\, n_i \frac{T_e}{|e|} \boldsymbol{\nabla}\phi \end{aligned}$$

This term is divided to the *mass* $m_i n_i$ coefficient of the convective derivative from the left:

$$\begin{aligned} \frac{1}{m_i n_i}\left(-|e|\, n_i \frac{T_e}{|e|}\boldsymbol{\nabla}\phi\right) &= -\frac{T_e}{m_i}\boldsymbol{\nabla}\phi \\ &= -c_s^2 \boldsymbol{\nabla}\phi \end{aligned}$$

Therefore this term is related with an *electric disequilibrium* in plasma. It is **not** connected with the diamagnetic velocity, generated by the gradient of density of the ions, since their temperature is zero (cold ions) and they have no pressure.

It is interesting to note that assuming *cold ions* we neglect the ion temperature but the Larmor radius is still present, as ρ_s, from T_e and Ω_{ci}. The ion diamagnetic velocity does not exist but the electric field from a potential perturbation in plasma, ϕ, generates the first order $E \times B$ part of the velocity.

The ordering is

$$\begin{aligned} \lambda_{\|} &\sim L_n \\ \lambda_{\perp} &\sim \rho_s \end{aligned}$$

or

$$k_{\|} L_n \sim k_{\perp}\rho_s \sim \mathcal{O}(1)$$

Note that in the case of Hasegawa-Mima the parallel direction can be neglected because it is assumed that $k_{\|} L_n \sim \mathcal{O}(\varepsilon)$, or, effectively, that $k_{\|} \to 0$.

The ordering assumed by **Meiss and Horton** *retains* the third direction and requires the normalizations

$$\begin{aligned} \mathbf{x}_{\perp} &\to \rho_s \mathbf{x}_{\perp} \\ z &\to L_n z \end{aligned}$$

therefore the characteristics lengths ρ_s and respectively L_n are adopted as units of length along the perpendicular and respectively parallel directions.

$$\begin{aligned}
\phi &\rightarrow \frac{\rho_s}{L_n}\phi = \varepsilon\phi \\
\frac{\mathbf{v}}{c_s} &\rightarrow \frac{\rho_s}{L_n}\mathbf{v} = \varepsilon\mathbf{v} \\
t &\rightarrow \frac{L_n}{c_s}t
\end{aligned}$$

We measure the velocity $\mathbf{v}$ not in units of ion sound velocity c_s but in much smaller units, the ion sound velocity scaled with the small factor ρ_s/L_n

$$\begin{aligned}
\text{unit of velocity} &= c_s\frac{\rho_s}{L_n} \\
&\ll c_s
\end{aligned}$$

as if we would expect that the variation of the velocity to be very fast, on a very short time scale.

The following quantity is introduced to facilitate comparison between the shear and the parallel direction

$$S \equiv \frac{L_n}{L_s} \ll 1$$

The perpendicular velocity contains

1. the $E \times B$ term $\frac{-\boldsymbol{\nabla}_\perp\phi\times\widehat{\mathbf{n}}}{B}$
2. the ion polarization term $\frac{\rho_s}{L_n}\left(\frac{\partial}{\partial t} + \frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}\right)\boldsymbol{\nabla}_\perp\phi = \frac{c_s}{\Omega_i}\frac{1}{L_n}\frac{d}{dt}\left(\boldsymbol{\nabla}_\perp\phi\right)$
3. the effect of the motion in z direction combined with the magnetic shear

The velocity is

$$\begin{aligned}
\mathbf{v}_\perp &= \frac{-\boldsymbol{\nabla}_\perp\phi\times\widehat{\mathbf{n}}}{B} \\
&\quad +\frac{\rho_s}{L_n}\left(\frac{\partial}{\partial t} + \frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}\right)\boldsymbol{\nabla}_\perp\phi \\
&\quad +\frac{\rho_s}{L_n}Sxv_z
\end{aligned}$$

The equation for the z component v_z is

$$\left(\frac{\partial}{\partial t} + \frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}\right)v_z = -\frac{\partial\phi}{\partial z} + Sxv_z$$

We **note** the absence of the collisional friction $\nu_i v_\parallel$ and of the gradient of pressure $\nabla_\parallel p$.

Introducing the expression of the perpendicular velocity ($E \times B$ plus polarization plus parallel v_z contribution) into the ion continuity equation we get

$$\begin{aligned}
& \left(\frac{\partial}{\partial t} + \frac{-\boldsymbol{\nabla}\phi \times \widehat{\mathbf{n}}}{B} \cdot \boldsymbol{\nabla} \right) \left(1 + \mathcal{L} - \boldsymbol{\nabla}_\perp^2\right) \phi \\
& + v_* \frac{\partial \phi}{\partial y} \\
& + \nabla_\parallel v_z \\
= \; & 0
\end{aligned}$$

where $\mathcal{L}$ is the non-adiabatic part of the electron density response. The parallel derivative is

$$\begin{aligned}
\nabla_\parallel & \equiv \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \\
& = \frac{\partial}{\partial z} + Sx \frac{\partial}{\partial y}
\end{aligned}$$

NOTE

From **Streamer formation ITG Drake** the derivative on direction z

$$\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} = \frac{\partial}{\partial z} + \boldsymbol{\nabla}\psi \times \widehat{\mathbf{n}}$$

Neglecting the parallel velocity contributions we obtain the Hasegawa-Mima equation

$$\left(\frac{\partial}{\partial t} + \frac{-\boldsymbol{\nabla}\phi \times \widehat{\mathbf{n}}}{B} \cdot \boldsymbol{\nabla} \right) \left(1 - \boldsymbol{\nabla}_\perp^2\right) \phi + v_* \frac{\partial \phi}{\partial y} = 0$$

16 Convective cells and drift waves (Sagdeev, Shapiro, Shevchenko)

There is a rapid generation of convective cells out of the drift waves, in inhomogeneous plasma.

The convective cells are signalized by

$$\begin{aligned}
\mathrm{Re}\,(\omega) & = 0 \\
k_\parallel & = 0
\end{aligned}$$

i.e. perfect alignement to the magnetic field and no oscillation in time.

the particle convection

$$\mathbf{v}^{\alpha} = \frac{1}{B_0}\left[\mathbf{E}+\frac{T^{\alpha}}{e^{\alpha}n},\mathbf{B}\right]$$

It is studied the nonlinear interaction between *drift waves* and *convective cells.*

It is taken into account the *transversal ion inertia* and the *ion viscosity.*

$$\begin{aligned}
& \frac{\partial n}{\partial t}+\frac{1}{B_0}\left(-\frac{\partial\phi}{\partial y}\frac{\partial n}{\partial x}+\frac{\partial\phi}{\partial x}\frac{\partial n}{\partial y}\right) \\
& -\frac{1}{B_0}\frac{n_0}{\Omega_{ci}}\left(\frac{\partial}{\partial t}-\mu\Delta\right)\Delta U \\
& -\frac{1}{B_0^2}\frac{1}{\Omega_{ci}}\left[\frac{\partial}{\partial x}n_0\left(-\frac{\partial\phi}{\partial y}\frac{\partial}{\partial x}+\frac{\partial\phi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial U}{\partial x}\right. \\
& \qquad\qquad \left.+\frac{\partial}{\partial y}n_0\left(-\frac{\partial\phi}{\partial y}\frac{\partial}{\partial x}+\frac{\partial\phi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial U}{\partial y}\right] \\
= {} & 0
\end{aligned}$$

Here

$$U\equiv\phi+\frac{T_i}{e}\ln n$$

Note that in **Horton** it appears

$$\varphi=\phi+p$$

which seems to have been derived after expansion

$$\begin{aligned}
\ln n &= \ln\left(n_0+\widetilde{n}\right)=\ln n_0+\ln\left(1+\frac{\widetilde{n}}{n_0}\right) \\
&\approx \ln n_0+\frac{\widetilde{n}}{n_0}
\end{aligned}$$

and the last term is the *perturbed pressure*

$$T_i\widetilde{n}\rightarrow p$$

End

The viscosity is

$$\mu=\frac{3}{10}\frac{T_i}{m_i\Omega_{ci}^2}\nu_{ei}$$

The electron continuity equation

$$
\begin{aligned}
& \frac{\partial n}{\partial t}+\frac{1}{B_0}\left(-\frac{\partial \phi}{\partial y}\frac{\partial n}{\partial x}+\frac{\partial \phi}{\partial x}\frac{\partial n}{\partial y}\right) \\
& +n_0\frac{\partial v_{ez}}{\partial z} \\
= & \ 0
\end{aligned}
$$

17 Transport induced by drift waves

17.1 The ion heat conduction, paper by Bolton and Ware 1983

the formula

$$Q=-\int \frac{d\theta}{2\pi}\frac{h}{|e|\,B_\theta}G_{ion}$$

where G_{ion} is the energy-weighted friction force

$$\mathbf{G}=-\int d^3v\left(\frac{mv^2}{2}\right)m\mathbf{v}\ C\left(f\right)$$

The energy-weighted momentum flux carried by the collisions $\mathbf{G}$ is a vector.

It has three components

$$
\begin{aligned}
& \text{neoclassic} \\
Q^{nc} & = -\frac{1}{2\pi}\int d\theta\frac{1}{|e|\,B_{\theta 0}}\overline{G}_{\parallel}
\end{aligned}
$$

$$
\begin{aligned}
& \text{Pfirsch-Schluter} \\
Q^{ps} & = -\frac{1}{2\pi}\int d\theta\frac{2\varepsilon}{|e|\,B_{\theta 0}}\widetilde{G}_{\parallel}\cos\theta
\end{aligned}
$$

$$Q^{classic}=-\frac{1}{2\pi}\int d\theta\frac{1}{|e|\,B}G_{\perp}\left(1+\varepsilon\cos\theta\right)$$

the drift kinetic equation for the gyrophase-averaged part of the distribution

$$\frac{B_\theta}{B}v_{\parallel}\frac{\partial f_1}{r\partial\theta}=-v_{Dr}\frac{\partial f_{Mi}}{\partial r}+C\left(f_1\right)$$

Now there is a change of variables

$$\mathbf{r},v_x,v_y,v_z \quad\rightarrow\quad \mathbf{r},v,\frac{v_{\parallel}}{v}$$

with the notation

$$\xi \equiv \frac{v_\parallel}{v}$$

$$\xi\frac{\partial f_1}{\partial\theta}-\frac{\varepsilon}{2}\left(1-\xi^2\right)\left(\sin\theta-\frac{\varepsilon}{2}\sin 2\theta\right)\frac{\partial f_1}{\partial\xi}=\varepsilon\frac{m\left(v_\parallel^2+\frac{v_\perp^2}{2}\right)\sin\theta}{|e|\,B_{\theta 0}v}\frac{\partial f_{Mi}}{\partial r}+\frac{B}{B_{\theta 0}}\frac{r}{v}C\left(f_1\right)$$

Note that the second term, which includes $\partial f_1/\partial\xi$ is an *energetic* term since the distribution function is derivated at the parallel velocity $v_\parallel$.

Note Recall a similar result for the velocity of poloidal and toroidal rotation in the **Stringer spontaneous spin-up** (**Hassam Antonsen Drake Liu**).

$$\begin{aligned} v_\varphi &\approx V_t - 2qV_p\cos\theta \\ &\quad +\varepsilon\left[V_t\cos\theta + 2qV_p\left(1+\frac{1}{4}\cos 2\theta\right)\right] \end{aligned}$$

However here it is question of the fluid flow velocities and not of particle velocities. The equation mentioned here comes from fluid conservation equations, with consideration of neoclassical geometric effect.

See **Stringer notes.**

End.

18 Notes

18.1 General notes from papers

The paper **DriftWaves _Rewoldt** explains that the density in the H-mode is fequently uniform.

In the spirit of our idea that

$$u_\theta \sim v_i^{dia}$$

which means that the density evolves such that the density gradient generates a diamagnetic flow which is slightly smaller than the rotation speed. The latter is determined by the radial electric field, which, if $n\left(r\right)\approx$constant, means $v_e^{dia}\approx 0$. And further this means

$$u_\theta \approx v_e^{dia} \approx 0$$

and this means

$$\frac{1}{B}\left(-\frac{\partial\phi}{\partial y}\right)\approx 0$$

or, the constancy of the potential across the small radius.

We conclude that a uniform profile of density in the H-mode means that there almost NO poloidal rotation of the plasma.

The energy per wavenumber mode k is, in **Current Carrying Drift waves Hatekayama**.

In this work it is question of drift waves in Q-machine.

$$W_k = \frac{|\mathbf{E}_k|^2}{8\pi}\omega\frac{\partial\varepsilon}{\partial\omega}$$

or

$$W_k = \frac{|\mathbf{E}_k|^2}{8\pi}\left[1+\frac{k_{De}^2}{k^2}+\frac{k_{Di}^2}{k^2}\eta\left(1-\Gamma\right)+\frac{k_{Di}^2}{k^2}\Gamma\frac{\omega_e'+2\omega_i^*}{2\omega_e'}\left(\frac{k_{\parallel}v_{i\parallel}}{\omega_e'}\right)^2\right]$$

where

$$\begin{aligned}
\frac{|\mathbf{E}_k|^2}{8\pi} &= \text{electric field energy } \rightarrow \frac{|\mathbf{E}_k|^2}{2\varepsilon_0} \\
\frac{|\mathbf{E}_k|^2}{8\pi}\frac{k_{De}^2}{k^2} &= \text{electron Debye shielding} \rightarrow \frac{|\mathbf{E}_k|^2}{2\varepsilon_0}\frac{k_{De}^2}{k^2} \\
\frac{|\mathbf{E}_k|^2}{8\pi}\frac{k_{Di}^2}{k^2}\eta\left(1-\Gamma\right) &= \text{ion perpendicular kinetic energy, } \frac{E\times B}{B^2} \\
\frac{|\mathbf{E}_k|^2}{8\pi}\frac{k_{Di}^2}{k^2}\Gamma\frac{\omega_e'+2\omega_i^*}{2\omega_e'}\left(\frac{k_{\parallel}v_{i\parallel}}{\omega_e'}\right)^2 &= \text{ion parallel motion energy}
\end{aligned}$$

and

$$\omega_e' \equiv \omega_{*e}\frac{\Gamma}{1+\left(1-\Gamma\right)\eta_{\perp}}$$

for

$$\eta_{\perp} \equiv \frac{T_e}{T_{i\perp}}$$