Review of plasma rotation in tokamak

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Abstract

We review the most common aspects of generation and effects of the plasma rotation in tokamak.

Contents

1 Plasma rotation

Introduction.

2 Rotation with speed close to thermal

Paper **kinetic th flowing Hinton Hazeltine**.

3 Coriolis and Centrifugal momentum transport

Paper **centrifugal force Peeters 2009**

3.1 Basic equations in the comoving frame

Toroidal momentum transport is NOT purely diffusive.

Standard gyro-kinetic ordering

$E \times B$ velocity
is of the same order as
v_* diamagnetic velocity

For toroidal rotation,

$$v_{tor} \sim v_{th} \sim c_s$$

Spatial scales are much larger than ρ_s. Therefore *drift-kinetic* theory is good.

Spatial scales include ρ_s, then *gyro-kinetic* theory.

Equations.

The variables are $(\mathbf{x}, \mathbf{v})$ and there is a differential one-form with the schematic structure

$$p_\mu dx^\mu = \mathbf{p} \cdot \mathbf{dx} - E\ dt$$

using generalized momentum $m\mathbf{v} + Ze\mathbf{A}$ and the energy, we have explicitly

$$\gamma = (m\mathbf{v} + Ze\mathbf{A}) \cdot \mathbf{dx} - \left(Ze\phi + \frac{1}{2}mv^2 \right) dt$$

Consider toroidal rotation

$$\mathbf{\Omega} = (\mathbf{\nabla}R \times \mathbf{\nabla}\varphi)\ R\Omega$$

along the main axis of the torus, $\mathbf{\nabla}R = \widehat{\mathbf{e}}_R$, $\mathbf{\nabla}\varphi = \frac{1}{R}\widehat{\mathbf{e}}_\varphi$, $\widehat{\mathbf{e}}_R \times \widehat{\mathbf{e}}_\varphi = \widehat{\mathbf{e}}_Z$ (Z is here vertical, parallel with the *main* axis of the torus). There is a rigid toroidal rotation

$$\begin{aligned} \mathbf{u}_0 &= \mathbf{\Omega} \times \mathbf{x} \\ &= R^2\Omega\ \mathbf{\nabla}\varphi \end{aligned}$$

It is adopted as referential the comoving frame with rigid rotation with constant Ω. In this comoving frame there is variation of the rotation frequencies for magnetic surfaces near the surface where there is no relative rotation. Distances are also affected by the relative motion on neighbor surfaces.

$$\mathbf{v} \rightarrow \mathbf{v} + \mathbf{u}_0$$

$$\frac{d\mathbf{x}}{dt} \rightarrow \frac{d\mathbf{x}}{dt} + \mathbf{u}_0$$
$$d\mathbf{x} \rightarrow d\mathbf{x} + \mathbf{u}_0 dt$$

$$\begin{aligned} \mathbf{E} &= -\boldsymbol{\nabla}\phi \\ &\rightarrow \mathbf{E} + \mathbf{u}_0 \times \mathbf{B} \end{aligned}$$

The magnetic field is unchanged.
The magnetic potential is

$$\begin{aligned} \mathbf{A} = \boldsymbol{\nabla} \quad & g(R, \psi) \times \boldsymbol{\nabla}\varphi \\ & + \psi \boldsymbol{\nabla}\varphi \end{aligned}$$

$$\mathbf{A} \cdot \mathbf{u}_0 = \psi \Omega$$

The electric potential

$$\phi \rightarrow \phi + \mathbf{A} \cdot \mathbf{u}_0$$

Then to retain, for the new variables

$$\begin{aligned} \mathbf{x} &\equiv \text{space variable in co-moving feame} \\ \mathbf{v} &\equiv \text{velocity in co-moving frame} \\ \mathbf{E} &\equiv \text{electric field in the co-moving frame} \\ \mathbf{A} &\equiv \text{magnetic potential in co-moving frame} \\ \phi &\equiv \text{potential in co-moving frame} \end{aligned}$$

Returning to the differential one-form γ, we replace

$$d\mathbf{x} \ \text{ by } \ d\mathbf{x} + \mathbf{u}_0 dt$$

$$\mathbf{v} \ \text{ by } \ \mathbf{v} + \mathbf{u}_0$$

$$\phi \text{ by } \ \phi + \mathbf{A} \cdot \mathbf{u}_0$$

Then

$$\begin{aligned} & \gamma^{transf} \\ = \ & [Ze\mathbf{A} + m(\mathbf{v} + \mathbf{u}_0)] \cdot (d\mathbf{x} + \mathbf{u}_0 dt) \\ & - \left[Ze(\phi + \mathbf{A} \cdot \mathbf{u}_0) + \frac{1}{2} m (\mathbf{v} + \mathbf{u}_0)^2 \right] dt \end{aligned}$$

or

$$\begin{aligned}
& \left[ZeA + m\left(\mathbf{v}+\mathbf{u}_0\right)\right]\cdot d\mathbf{x} + \left[Ze\mathbf{A} + m\left(\mathbf{v}+\mathbf{u}_0\right)\right]\cdot \mathbf{u}_0 dt \\
& - \left[Ze\phi + Ze\mathbf{A}\cdot\mathbf{u}_0 + \frac{1}{2}m\mathbf{v}^2 + m\mathbf{v}\cdot\mathbf{u}_0 + \frac{1}{2}m\mathbf{u}_0^2\right] dt \\
= & \left[Ze\mathbf{A} + m\left(\mathbf{v}+\mathbf{u}_0\right)\right]\cdot d\mathbf{x} \\
& + Ze\mathbf{A}\cdot\mathbf{u}_0 dt + m\mathbf{v}\cdot\mathbf{u}_0 dt + m\mathbf{u}_0^2 dt \\
& - Ze\phi dt - Ze\mathbf{A}\cdot\mathbf{u}_0 dt - \frac{1}{2}m\mathbf{v}^2 dt - m\mathbf{v}\cdot\mathbf{u}_0 dt - \frac{1}{2}m\mathbf{u}_0^2 dt
\end{aligned}$$

We note that terms with $\mathbf{A}\cdot\mathbf{u}_0$ cancel. And terms $m\mathbf{v}\cdot\mathbf{u}_0 dt$ occur with $+$ and $-$ and are suppressed.

Collecting the remaining terms

$$\begin{aligned}
& \gamma^{transf} \\
= & \left[Ze\mathbf{A} + m\left(\mathbf{v}+\mathbf{u}_0\right)\right]\cdot d\mathbf{x} \\
& - \left(Ze\phi + \frac{1}{2}m\mathbf{v}^2 - \frac{1}{2}m\mathbf{u}_0^2\right) dt
\end{aligned}$$

Then the final form of the one-form is

$$\begin{aligned}
\gamma = & \left[Ze\mathbf{A} + m\left(\mathbf{v}+\mathbf{u}_0\right)\right]\cdot d\mathbf{x} \\
& - \left(Ze\phi + \frac{1}{2}m\mathbf{v}^2 - \frac{1}{2}m\mathbf{u}_0^2\right) dt
\end{aligned}$$

The Lagrangian is, after a Lie transformation

$$\begin{aligned}
\Gamma = & \left[Ze\mathbf{A} + m\left(v_{\parallel}\widehat{\mathbf{n}} + \mathbf{u}_0\right)\right]\cdot d\mathbf{X} \\
& + \mu\ d\zeta \\
& - H\ dt
\end{aligned}$$

$$\begin{aligned}
\mu &= \frac{mv_{\perp}^2}{2B} \\
\zeta &\equiv \text{gyro-angle}
\end{aligned}$$

where

$$H = Ze\left\langle\phi\right\rangle + \frac{1}{2}mv_{\parallel}^2 + \mu B - \frac{1}{2}mu_0^2$$

this is the energy $Ze\left\langle\phi\right\rangle + \frac{1}{2}m\left(v_{\parallel}^2 + v_{\perp}^2\right) - \frac{1}{2}mu_0^2$.

The set of variables is

$$\left(\mathbf{X}, \zeta, v_{\|}, \mu\right)$$

The equations, derived from the Lagrangian

$$\begin{aligned}
\frac{d\mathbf{X}}{dt} &= \{\mathbf{X}, H\} \\
&= \frac{1}{ZeB_{\|}^{*}}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} H + \frac{\mathbf{B}^{*}}{B_{\|}^{*}} \frac{\partial H}{\partial\left(m v_{\|}\right)}
\end{aligned}$$

$$\begin{aligned}
\frac{dv_{\|}}{dt} &= \left\{v_{\|}, H\right\} \\
&= -\frac{1}{m} \frac{\mathbf{B}^{*}}{B_{\|}^{*}} \cdot \boldsymbol{\nabla} H
\end{aligned}$$

To obtain an equivalent equation, one starts from multiplying $\frac{d\mathbf{X}}{dt}$ by $\boldsymbol{\nabla} H$,

$$\begin{aligned}
\frac{d\mathbf{X}}{dt} \cdot \boldsymbol{\nabla} H &= \left[\frac{1}{ZeB_{\|}^{*}}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} H + \frac{\mathbf{B}^{*}}{B_{\|}^{*}} \frac{\partial H}{\partial\left(m v_{\|}\right)}\right] \cdot \boldsymbol{\nabla} H \\
&= \frac{\mathbf{B}^{*}}{B_{\|}^{*}} \frac{\partial H}{\partial\left(m v_{\|}\right)} \cdot \boldsymbol{\nabla} H \\
&= \frac{1}{m} \frac{\mathbf{B}^{*}}{B_{\|}^{*}} m v_{\|} \cdot \boldsymbol{\nabla} H \\
&= -m\ v_{\|} \left(-\frac{1}{m} \frac{\mathbf{B}^{*}}{B_{\|}^{*}} \cdot \boldsymbol{\nabla} H\right) = -m v_{\|} \frac{dv_{\|}}{dt}
\end{aligned}$$

then it has been derived an equivalent form

$$m v_{\|} \frac{dv_{\|}}{dt} = -\frac{d\mathbf{X}}{dt} \cdot \boldsymbol{\nabla} H$$

Now the new field

$$\begin{aligned}
\mathbf{B}^{*} &= \mathbf{B} + \frac{m}{Ze} \boldsymbol{\nabla} \times \left(v_{\|} \widehat{\mathbf{n}} + \mathbf{u}_{0}\right) \\
B_{\|}^{*} &= \widehat{\mathbf{n}} \cdot \mathbf{B}^{*}
\end{aligned}$$

We **NOTE** that the new field defined as $\mathbf{B}^*$ has the structure $\mathbf{B}+\boldsymbol{\omega}$. Here $\boldsymbol{\omega}$ is the vorticity associated to the velocity $v_\parallel\widehat{\mathbf{n}}+\mathbf{u}_0$. Another form for this definition is

$$\begin{aligned} \boldsymbol{\Omega}^* &= \boldsymbol{\Omega}+\boldsymbol{\omega} \\ \text{where}\ \ \boldsymbol{\Omega}^* &= \frac{Ze\mathbf{B}^*}{m} \\ \boldsymbol{\Omega} &= \frac{Ze\mathbf{B}}{m} \end{aligned}$$

NOTE

This combination $(\mathbf{B},\boldsymbol{\omega})$ is not usual, but it is one of the suggested associations.

- another, is Clebsch variable

$$\mathbf{B}\pm\mathbf{v}$$

- still another is

$$(\boldsymbol{\omega},\mathbf{j})$$

that seems compatible with the preceeding one, $\boldsymbol{\nabla}\times(\mathbf{B}\pm\mathbf{v})=\mu_0\mathbf{j}\pm\boldsymbol{\omega}$.

However $\mathbf{B}$ and $\boldsymbol{\omega}$ have in common the fact that they are differential two-forms, i.e. fluxes. $\boldsymbol{\omega}$ is a flux but $\mathbf{j}$ is a differential three-form. Wedge-product with dt must be a scalar in $4D$,

$$j\wedge dt\sim\rho\quad\text{(charge density)}$$

END

An expression for the new field $\mathbf{B}^*$ separates the parallel and perpendicular parts, using

$$\begin{aligned} &\widehat{\mathbf{n}}\ \ \text{and} \\ &\widehat{\mathbf{n}}\times(\widehat{\mathbf{n}}\times...) \end{aligned}$$

$$\frac{\mathbf{B}^*}{B_\parallel^*}=\widehat{\mathbf{n}}-\frac{m}{Ze}\frac{1}{B_\parallel^*}\widehat{\mathbf{n}}\times\left\{\widehat{\mathbf{n}}\times\left[\boldsymbol{\nabla}\times\left(v_\parallel\widehat{\mathbf{n}}+\mathbf{u}_0\right)\right]\right\}$$

$$\begin{aligned} \boldsymbol{\nabla}\times\left(v_\parallel\widehat{\mathbf{n}}+\mathbf{u}_0\right) &= v_\parallel\boldsymbol{\nabla}\times\widehat{\mathbf{n}} \\ &\quad +2\boldsymbol{\Omega} \end{aligned}$$

returning, we get

$$\begin{aligned} \frac{\mathbf{B}^*}{B_\parallel^*} &= \widehat{\mathbf{n}}+\frac{m}{Ze}\frac{1}{B_\parallel^*}v_\parallel\widehat{\mathbf{n}}\times(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla})\,\widehat{\mathbf{n}} \\ &\quad +2\frac{m}{Ze}\frac{1}{B_\parallel^*}\boldsymbol{\Omega}_\perp \end{aligned}$$

Assuming rigid rotation

$$\boldsymbol{\nabla} u_0^2 = 2R\Omega^2 \boldsymbol{\nabla} R$$

Then the velocity in the comoving frame is

$$\begin{aligned}
\frac{d\mathbf{X}}{dt} &= v_{\parallel}\widehat{\mathbf{n}} \\
&+\frac{1}{B_{\parallel}^{*}}\frac{1}{Ze/m}\,\widehat{\mathbf{n}}\times v_{\parallel}^{2}\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}+\frac{1}{B_{\parallel}^{*}}\frac{\mu}{Ze}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}B+\frac{1}{B_{\parallel}^{*}}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\left\langle\phi\right\rangle \\
&+\frac{1}{B_{\parallel}^{*}}2\frac{1}{Ze/m}\boldsymbol{\Omega}_{\perp} \\
&-\frac{1}{B_{\parallel}^{*}}\frac{1}{Ze/m}\Omega^{2}R\;\widehat{\mathbf{n}}\times\boldsymbol{\nabla}R
\end{aligned}$$

with the components

$$\begin{aligned}
&\frac{1}{B_{\parallel}^{*}}\frac{1}{Ze/m}\,\widehat{\mathbf{n}}\times v_{\parallel}^{2}\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\;\;\text{curvature drift, }\mathbf{v}_{curvature} \\
&\frac{1}{B_{\parallel}^{*}}\frac{\mu}{Ze}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}B\quad\text{grad-B drift, }\mathbf{v}_{\boldsymbol{\nabla}B} \\
&\frac{1}{B_{\parallel}^{*}}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\left\langle\phi\right\rangle\quad\text{ExB drift, }\mathbf{v}_{ExB}
\end{aligned}$$

To understand the meaning of the two last terms due to rotation, one starts from the most general equation:

when there is a force acting on the particle that gyrates in a strong magnetic field there will result a DRIFT of the center of gyration of the particle in a direction that is perpendicular on the force and on the magnetic field, i.e. $\sim \mathbf{F}\times\mathbf{B}$,

$$\mathbf{v}_{drift} = \frac{1}{ZeB}\frac{\mathbf{F}\times\mathbf{B}}{B_{\parallel}^{*}}$$

Now, in the rotating reference system there is the Coriolis force

$$\mathbf{F}_{cor} = mv_{\parallel}\widehat{\mathbf{n}}\times 2\boldsymbol{\Omega}$$

and there is also a centrifugal force

$$\mathbf{F}_{cf} = m\Omega^2 R\;\boldsymbol{\nabla} R$$

Then there will be two new components of the drift velocity

$$\begin{aligned}\mathbf{v}_{cor} &= \frac{1}{ZeB}\frac{\mathbf{F}_{cor}\times\mathbf{B}}{B_{\|}^{*}} \\ &= \frac{1}{ZeB}\frac{\left(mv_{\|}\widehat{\mathbf{n}}\times 2\boldsymbol{\Omega}\right)\times\mathbf{B}}{B_{\|}^{*}}\end{aligned}$$

separately

$$\begin{aligned}(\widehat{\mathbf{n}}\times\boldsymbol{\Omega})\times\widehat{\mathbf{n}} &= -\left[\widehat{\mathbf{n}}\left(\boldsymbol{\Omega}\cdot\widehat{\mathbf{n}}\right)-\boldsymbol{\Omega}\left(\widehat{\mathbf{n}}\cdot\widehat{\mathbf{n}}\right)\right] \\ &= \boldsymbol{\Omega}-\widehat{\mathbf{n}}\ \left(\widehat{\mathbf{n}}\cdot\boldsymbol{\Omega}\right) \\ &\equiv \boldsymbol{\Omega}_{\perp}\end{aligned}$$

and

$$\mathbf{v}_{coriolis} = 2\frac{1}{ZeB}\frac{mv_{\|}}{B_{\|}^{*}}\boldsymbol{\Omega}_{\perp}$$

$$\mathbf{v}_{cf} = -\frac{1}{Ze}m\Omega^{2}R\frac{\widehat{\mathbf{n}}\times\nabla R}{B_{\|}^{*}}$$

These expressions for drift velocities of various sources can be used in

$$\begin{aligned}\frac{d\mathbf{X}}{dt} &= v_{\|}\widehat{\mathbf{n}}+\mathbf{v}_{curvature}+\mathbf{v}_{\boldsymbol{\nabla}B}+\mathbf{v}_{ExB} \\ &\quad +\mathbf{v}_{coriolis}+\mathbf{v}_{cf}\end{aligned}$$

to give a more detailed form to the parallel-energy formula

$$mv_{\|}\frac{dv_{\|}}{dt} = -\frac{d\mathbf{X}}{dt}\cdot\boldsymbol{\nabla}H$$

where

$$\boldsymbol{\nabla}H = Ze\boldsymbol{\nabla}\left\langle\phi\right\rangle+\mu\boldsymbol{\nabla}B-m\Omega^{2}R\ \boldsymbol{\nabla}R$$

leading to

$$\begin{aligned}mv_{\|}\frac{dv_{\|}}{dt} &= -\frac{d\mathbf{X}}{dt}\cdot\left[Ze\boldsymbol{\nabla}\left\langle\phi\right\rangle+\mu\boldsymbol{\nabla}B-m\Omega^{2}R\ \boldsymbol{\nabla}R\right] \\ &= -\left[v_{\|}\widehat{\mathbf{n}}+\mathbf{v}_{curvature}+\mathbf{v}_{\boldsymbol{\nabla}B}+\mathbf{v}_{ExB}+\mathbf{v}_{coriolis}+\mathbf{v}_{cf}\right] \\ &\quad \cdot\left[Ze\boldsymbol{\nabla}\left\langle\phi\right\rangle+\mu\boldsymbol{\nabla}B-m\Omega^{2}R\ \boldsymbol{\nabla}R\right]\end{aligned}$$

Separately, few terms from the RHS

$$\begin{aligned}-\mathbf{v}_{\boldsymbol{\nabla}B}\cdot Ze\boldsymbol{\nabla}\left\langle\phi\right\rangle &= -\left(\frac{1}{B_{\|}^{*}}\frac{\mu}{Ze}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}B\right)\cdot Ze\boldsymbol{\nabla}\left\langle\phi\right\rangle \\ &= \frac{-\boldsymbol{\nabla}\left\langle\phi\right\rangle\times\widehat{\mathbf{n}}}{B_{\|}^{*}}\cdot\mu\boldsymbol{\nabla}B \\ &= \mathbf{v}_{ExB}\cdot\mu\boldsymbol{\nabla}B\end{aligned}$$

$$
\begin{aligned}
-\mathbf{v}_{cf} \cdot Ze\boldsymbol{\nabla}\left\langle \phi \right\rangle &= -\left(-\frac{1}{Ze} m\Omega^2 R \frac{\widehat{\mathbf{n}} \times \nabla R}{B_{\|}^{*}}\right) \cdot Ze\boldsymbol{\nabla}\left\langle \phi \right\rangle \\
&= -\frac{-\boldsymbol{\nabla}\left\langle \phi \right\rangle \times \widehat{\mathbf{n}}}{B_{\|}^{*}} \cdot \; m\Omega^2 R \; \boldsymbol{\nabla} R \\
&= -\mathbf{v}_{ExB} \cdot m\Omega^2 R \; \boldsymbol{\nabla} R
\end{aligned}
$$

$$
\begin{aligned}
-\mathbf{v}_{cf} \cdot \mu\boldsymbol{\nabla} B &= -\left(-\frac{1}{Ze} m\Omega^2 R \frac{\widehat{\mathbf{n}} \times \nabla R}{B_{\|}^{*}}\right) \cdot \mu\boldsymbol{\nabla} B \\
&= -\frac{1}{Ze}\frac{\widehat{\mathbf{n}} \times \mu\boldsymbol{\nabla} B}{B_{\|}^{*}} \cdot \; m\Omega^2 R \; \boldsymbol{\nabla} R \\
&= -\mathbf{v}_{\boldsymbol{\nabla} B} \cdot m\Omega^2 R \; \boldsymbol{\nabla} R
\end{aligned}
$$

These are terms that result from the product in the RHS.
Consider other terms from the expanded RHS

$$
\begin{aligned}
& (\mathbf{v}_{\boldsymbol{\nabla} B} + \mathbf{v}_{ExB}) \cdot \left[Ze\boldsymbol{\nabla}\left\langle \phi \right\rangle + \mu\boldsymbol{\nabla} B - m\Omega^2 R \; \boldsymbol{\nabla} R\right] \\
= \; & \mathbf{v}_{\boldsymbol{\nabla} B} \cdot Ze\boldsymbol{\nabla}\left\langle \phi \right\rangle \quad \text{this is } -\mathbf{v}_{ExB} \cdot \mu\boldsymbol{\nabla} B \\
& +\mathbf{v}_{\boldsymbol{\nabla} B} \cdot \mu\boldsymbol{\nabla} B \quad \text{this is } 0 \\
& -\mathbf{v}_{\boldsymbol{\nabla} B} \cdot m\Omega^2 R \; \boldsymbol{\nabla} R \\
& +\mathbf{v}_{ExB} \cdot Ze\boldsymbol{\nabla}\left\langle \phi \right\rangle \quad \text{this is } 0 \\
& +\mathbf{v}_{ExB} \cdot \mu\boldsymbol{\nabla} B \quad \text{this will cancel with the 2nd line} \\
& -\mathbf{v}_{ExB} \cdot m\Omega^2 R \; \boldsymbol{\nabla} R
\end{aligned}
$$

This part consists of

$$
\begin{aligned}
& (\mathbf{v}_{\boldsymbol{\nabla} B} + \mathbf{v}_{ExB}) \cdot \left[Ze\boldsymbol{\nabla}\left\langle \phi \right\rangle + \mu\boldsymbol{\nabla} B - m\Omega^2 R \; \boldsymbol{\nabla} R\right] \\
= \; & -\mathbf{v}_{\boldsymbol{\nabla} B} \cdot m\Omega^2 R \; \boldsymbol{\nabla} R \; - \mathbf{v}_{ExB} \cdot m\Omega^2 R \; \boldsymbol{\nabla} R
\end{aligned}
$$

Continue with other terms from the product

$$
\begin{aligned}
& \mathbf{v}_{cf} \cdot \left[Ze\boldsymbol{\nabla}\left\langle \phi \right\rangle + \mu\boldsymbol{\nabla} B - m\Omega^2 R \; \boldsymbol{\nabla} R\right] \\
= \; & \mathbf{v}_{cf} \cdot Ze\boldsymbol{\nabla}\left\langle \phi \right\rangle \quad \text{this is } \mathbf{v}_{ExB} \cdot m\Omega^2 R \; \boldsymbol{\nabla} R \\
& +\mathbf{v}_{cf} \cdot \mu\boldsymbol{\nabla} B \quad \text{this is } \mathbf{v}_{\boldsymbol{\nabla} B} \cdot m\Omega^2 R \; \boldsymbol{\nabla} R \\
& -\mathbf{v}_{cf} \cdot m\Omega^2 R \; \boldsymbol{\nabla} R \;\; \text{or } -\left(-\frac{1}{Ze} m\Omega^2 R \frac{\widehat{\mathbf{n}} \times \nabla R}{B_{\|}^{*}}\right) \cdot m\Omega^2 R \; \boldsymbol{\nabla} R \text{ is } 0
\end{aligned}
$$

from this part of the product we get

$$
\begin{aligned}
& \mathbf{v}_{cf} \cdot \left[Ze\boldsymbol{\nabla}\left\langle \phi \right\rangle + \mu\boldsymbol{\nabla} B - m\Omega^2 R \; \boldsymbol{\nabla} R\right] \\
= \; & \mathbf{v}_{ExB} \cdot m\Omega^2 R \; \boldsymbol{\nabla} R + \mathbf{v}_{\boldsymbol{\nabla} B} \cdot m\Omega^2 R \; \boldsymbol{\nabla} R
\end{aligned}
$$

Before proceeding further we notice that the two partial calculations of the terms in the full product lead to

$$\begin{aligned}
& \left(\mathbf{v}_{\boldsymbol{\nabla} B}+\mathbf{v}_{ExB}\right)\cdot\left[Ze\boldsymbol{\nabla}\left\langle\phi\right\rangle+\mu\boldsymbol{\nabla}B-m\Omega^2R\ \boldsymbol{\nabla}R\right] \\
& +\mathbf{v}_{cf}\cdot\left[Ze\boldsymbol{\nabla}\left\langle\phi\right\rangle+\mu\boldsymbol{\nabla}B-m\Omega^2R\ \boldsymbol{\nabla}R\right] \\
= & -\mathbf{v}_{\boldsymbol{\nabla} B}\cdot m\Omega^2R\ \boldsymbol{\nabla}R\ -\mathbf{v}_{ExB}\cdot m\Omega^2R\ \boldsymbol{\nabla}R \\
& +\mathbf{v}_{ExB}\cdot m\Omega^2R\ \boldsymbol{\nabla}R+\mathbf{v}_{\boldsymbol{\nabla} B}\cdot m\Omega^2R\ \boldsymbol{\nabla}R \\
= & \ 0
\end{aligned}$$

It results that the terms arising from $(\mathbf{v}_{\boldsymbol{\nabla} B}+\mathbf{v}_{ExB}+\mathbf{v}_{cf})\cdot[...]$ have zero contribution to the final form.

Then we write the remaining terms as

$$\begin{aligned}
& -\left(v_{\parallel}\widehat{\mathbf{n}}+\mathbf{v}_{curvature}+\mathbf{v}_{\boldsymbol{\nabla} B}+\mathbf{v}_{ExB}+\mathbf{v}_{coriolis}+\mathbf{v}_{cf}\right)\cdot[...] \\
= & -\left(v_{\parallel}\widehat{\mathbf{n}}+\mathbf{v}_{curvature}+\mathbf{v}_{coriolis}\right)\cdot[...]
\end{aligned}$$

This is the result

$$mv_{\parallel}\frac{dv_{\parallel}}{dt}=-\left(v_{\parallel}\widehat{\mathbf{n}}+\mathbf{v}_{curvature}+\mathbf{v}_{coriolis}\right)\cdot\left[Ze\boldsymbol{\nabla}\left\langle\phi\right\rangle+\mu\boldsymbol{\nabla}B-m\Omega^2R\ \boldsymbol{\nabla}R\right]$$

The authors underlie that the two equations for a particle motion in the comoving frame

- the drifts

$$\frac{d\mathbf{X}}{dt}=v_{\parallel}\widehat{\mathbf{n}}+\mathbf{v}_{curvature}+\mathbf{v}_{\boldsymbol{\nabla} B}+\mathbf{v}_{ExB}+\mathbf{v}_{coriolis}+\mathbf{v}_{cf}$$

  with their explicit form, and

- the energetic expression

$$\begin{aligned}
mv_{\parallel}\frac{dv_{\parallel}}{dt} = & -\left(v_{\parallel}\widehat{\mathbf{n}}+\mathbf{v}_{curvature}+\mathbf{v}_{coriolis}\right) \\
& \cdot\left[Ze\boldsymbol{\nabla}\left\langle\phi\right\rangle+\mu\boldsymbol{\nabla}B-m\Omega^2R\ \boldsymbol{\nabla}R\right]
\end{aligned}$$

must be used for the gyro-kinetic equations.

3.2 Connection between the comoving frame and the laboratory frame

In the Laboratory frame there are no *Coriolis* or *centrifugal* terms.

Then - where these two physical effects, which are easily derived in the co-moving frame, are to be found in the Laboratory description ?

Answer: *in the curvature drift*

The basic equations, $L \equiv$ laboratory frame

$$v_{\parallel L} = u_{\parallel} + v_{\parallel}$$

where the rigid rotation is projected on $\parallel$ direction

$$u_{\parallel} = \frac{B_{tor}}{B} R\Omega$$

It is considered

$$B_{tor} = \frac{B_0}{h}$$

then

$$u_{\parallel} = \frac{B_0}{hB} R\Omega$$

This is the parallel component of the rotation velocity of the frame

$$\begin{aligned}\mathbf{u}_0 &= R\Omega \; R\boldsymbol{\nabla}\varphi \\ &= R\Omega \; \widehat{\mathbf{e}}_{\varphi}\end{aligned}$$

and its projection on parallel direction is

$$\begin{aligned}(\mathbf{u}_0)_{\parallel} &= u_0 \; \frac{B_{tor}}{B} \\ &= \frac{B_{tor}}{B} R\Omega\end{aligned}$$

and

$$\frac{d\mathbf{X}_L}{dt} = \mathbf{u}_0 + \frac{d\mathbf{X}}{dt}$$

Consider the curvature drift.

It is taken into account that

$$\begin{aligned}&\widehat{\mathbf{n}} \times [(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \, \widehat{\mathbf{n}}] \\ = \; &\frac{\widehat{\mathbf{n}} \times \boldsymbol{\nabla} B}{B} + \frac{\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p}{B^2/\mu_0}\end{aligned}$$

$$\begin{aligned}
&\frac{1}{\left(\frac{ZeB_\parallel^*}{m}\right)} v_\parallel^2 \widehat{\mathbf{n}} \times \left[\left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right) \widehat{\mathbf{n}}\right] \\
= \quad & \frac{1}{\left(\frac{ZeB_\parallel^*}{m}\right)} v_\parallel^2 \left[\frac{\widehat{\mathbf{n}} \times \boldsymbol{\nabla} B}{B} + \frac{\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p}{B^2/\mu_0}\right]
\end{aligned}$$

The last term is $\sim \beta$ and will be neglected.

NOTE
Below **electric field separatrix** it will be used a similar formula

$$\widehat{\mathbf{n}} \times \boldsymbol{\nabla} \psi = I \widehat{\mathbf{n}} - RB\ \widehat{\mathbf{e}}_\varphi$$

and

$$\begin{aligned}
\mathbf{u}_{i\perp} \quad &= \quad \frac{1}{B} \widehat{\mathbf{n}} \times \left(\frac{1}{n_i e} \boldsymbol{\nabla} p_i + \boldsymbol{\nabla} \phi\right) \\
&\approx \quad \omega \left(-\widehat{\mathbf{n}}\ \frac{I}{B} + R\ \widehat{\mathbf{e}}_\varphi\right)
\end{aligned}$$

where

$$\omega = -\left(\frac{1}{n_i e} \frac{\partial p}{\partial \psi} + \frac{\partial \phi}{\partial \psi}\right)$$

The final form is

$$\left[\left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right)\ \widehat{\mathbf{n}}\right] \cdot \mathbf{u}_{i\perp} \approx \omega\ I\ \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right) \left(\frac{1}{B}\right)$$

in **Kim**.
END

This allows to write the evolution of the parallel component of the velocity in the laboratory frame

$$m \frac{dv_{\parallel L}}{dt} = m \left(\frac{dv_\parallel}{dt} + \frac{du_\parallel}{dt}\right)$$

where

$$u_\parallel = \frac{B_{tor}}{B} R\Omega$$

and we remember that

$$\frac{dv_\parallel}{dt} = -\frac{1}{m}\ \frac{\mathbf{B}^*}{B_\parallel^*} \cdot \boldsymbol{\nabla} H$$

Then

$$
\begin{aligned}
m\frac{dv_{\| L}}{dt} &= -Ze\ \nabla_{\|} \left\langle \phi_L + \phi \right\rangle \\
&\quad -\frac{1}{\left(\frac{ZeB_{\|}^{*}}{m}\right)} v_{\| L}^{2} \frac{\widehat{\mathbf{n}} \times \boldsymbol{\nabla} B}{B} \cdot \boldsymbol{\nabla} \left\langle \phi_L + \phi \right\rangle \\
&\quad -\mu \nabla_{\|} B
\end{aligned}
$$

The toroidal rotation

$$
\begin{aligned}
\mathbf{u}_0 &= R\Omega\ R\boldsymbol{\nabla}\varphi \\
&= \widehat{\mathbf{n}}\ u_{\|} + \mathbf{v}_{E,L}
\end{aligned}
$$

We will use the relation $v_{\| L} = u_{\|} + v_{\|}$, where we know $dv_{\| L}/dt$. It is also taken into account the electric drift in the L frame

$$
\mathbf{v}_{E,L} = \frac{-\boldsymbol{\nabla} \left\langle \phi_L \right\rangle \times \widehat{\mathbf{n}}}{B}
$$

Using

$$
\begin{aligned}
\mathbf{u}_0 &= \mathbf{v}_{E,L} + u_{\|}\widehat{\mathbf{n}} \\
&= R\Omega\ R\boldsymbol{\nabla}\varphi
\end{aligned}
$$

the projection of the *laboratory* velocity along grad-B is obtained by multiplying the above equation with $\boldsymbol{\nabla} B$ and taking into account that $\boldsymbol{\nabla} B \cdot \boldsymbol{\nabla}\varphi = 0$, then

$$
\mathbf{v}_{E,L} \cdot \boldsymbol{\nabla} B = -u_{\|}\ \nabla_{\|} B
$$

The derivative of $v_{\|}$, the velocity in the moving frame

$$
\begin{aligned}
m\frac{dv_{\|}}{dt} &= -Ze\nabla_{\|} \left\langle \phi \right\rangle \\
&\quad -Ze\frac{1}{\left(\frac{ZeB_{\|}^{*}}{m}\right)} \left(v_{\|} + u_{\|}\right) \left(\widehat{\mathbf{n}} \times \boldsymbol{\nabla} \ln B\right) \cdot \boldsymbol{\nabla} \left\langle \phi \right\rangle \\
&\quad -\mu \nabla_{\|} B \\
&\quad -\frac{m}{B} u_{\|} \left(v_{\|} + u_{\|}\right)\ \nabla_{\|} B \\
&\quad -m\frac{du_{\|}}{dt}
\end{aligned}
$$

Now we need $\frac{du_{\parallel}}{dt}$,

$$u_{\parallel}=\frac{B_{tor}}{B}R\Omega$$

The simple time derivative of this expression must still be corrected to reflect the time variation of the spatial coordinates

$$\frac{du_{\parallel}}{dt}=\frac{d\mathbf{X}_L}{dt}\cdot\boldsymbol{\nabla}\left(\frac{B_{tor}}{B}R\Omega\right)$$

which is

$$\frac{du_{\parallel}}{dt}=-u_{\parallel}\left[\left(v_{\parallel}+u_{\parallel}\right)\widehat{\mathbf{n}}+\mathbf{v}_{E,L}+\mathbf{v}_E\right]\cdot\boldsymbol{\nabla}\ln B$$

This is the last term in the expression for $\frac{dv_{\parallel}}{dt}$. Replacing,

$$\begin{aligned}
m\frac{dv_{\parallel}}{dt} &= -Ze\ \nabla_{\parallel}\left\langle\phi\right\rangle \\
&\quad -Ze\frac{1}{\left(\frac{ZeB_{\parallel}^{*}}{m}\right)}\left(v_{\parallel}+2u_{\parallel}\right)\left(\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\ln B\right)\cdot\left\langle\boldsymbol{\nabla}\phi\right\rangle \\
&\quad -\mu\nabla_{\parallel}B \\
&\quad -mu_{\parallel}^{2}\ \nabla_{\parallel}\ln B
\end{aligned}$$

The velocity of the particle

$$\begin{aligned}
\frac{d\mathbf{X}}{dt} &= v_{\parallel}\widehat{\mathbf{n}} \\
&\quad +\mathbf{v}_E \\
&\quad +\frac{1}{\left(\frac{ZeB_{\parallel}^{*}}{m}\right)}\left[\left(v_{\parallel}^{2}+2v_{\parallel}u_{\parallel}+u_{\parallel}^{2}\right)+\mu B\right]\ \widehat{\mathbf{n}}\times\boldsymbol{\nabla}\ln B
\end{aligned}$$

The conclusion of coparison of the derived expressions is the possibility to approximate the velocities as follows

The Coriolis velocity

$$\mathbf{v}_{coriolis}=\frac{1}{\left(\frac{ZeB_{\parallel}^{*}}{m}\right)}\ 2v_{\parallel}u_{\parallel}\ \widehat{\mathbf{n}}\times\boldsymbol{\nabla}\ln B$$

The centrifugal drift

$$\mathbf{v}_{centrifug}=\frac{1}{\left(\frac{ZeB_{\parallel}^{*}}{m}\right)}\ u_{\parallel}^{2}\ \widehat{\mathbf{n}}\times\boldsymbol{\nabla}\ln B$$

And an approximation for the dependence of the magnetic field magnitude

$$B \sim \frac{1}{R}$$

Approximations

$$\begin{aligned} B_{tor} &= B + O\left(\varepsilon^2\right) \\ &= \frac{1}{R} + O\left(\varepsilon^2\right) \end{aligned}$$

$$\begin{aligned} u_{\|} &= R\Omega \\ &\quad +O\left(\varepsilon^2\right) \end{aligned}$$

$$\begin{aligned} \widehat{\mathbf{n}} \times \boldsymbol{\nabla} \ln B &= -\widehat{\mathbf{n}} \times \boldsymbol{\nabla} \ln R \\ &\quad +O\left(\varepsilon^2\right) \end{aligned}$$

$$\begin{aligned} \boldsymbol{\Omega}_{\perp} &= -\Omega\, \widehat{\mathbf{n}} \times \boldsymbol{\nabla} R \\ &\quad +O\left(\varepsilon^2\right) \end{aligned}$$

Now we have two different direction for the calculation of velocities
- starting from the differential one-form, then Lagrangian and equations of motion, or
- starting from the equations of motion in the Laboratory frame and adopting a transformation leading to the co-moving frame

Terms of Coriolis and Centrifugal drift that are present in the co-moving frame
are recovered in
terms in the Laboratory frame

It is the *curvature drift* of the Laboratory frame that produces the terms that correspond to the Coriolis and centrifugal terms.
Consider the curvature drift in the Laboratory frame

$$\frac{1}{\left(\frac{ZeB_{\|}^{*}}{m}\right)} v_{\|L}^{2}\, \widehat{\mathbf{n}} \times \boldsymbol{\nabla} \ln B$$

and the connection with the velocity in the co-moving frame

$$v_{\parallel L} = v_\parallel + u_\parallel$$

which must be replaced above

$$\begin{aligned}
& \frac{1}{\left(\frac{ZeB_\parallel^*}{m}\right)} \; v_{\parallel L}^2 \; \widehat{\mathbf{n}} \times \boldsymbol{\nabla} \ln B \\
= & \frac{1}{\left(\frac{ZeB_\parallel^*}{m}\right)} \; \left(v_\parallel + u_\parallel\right)^2 \; \widehat{\mathbf{n}} \times \boldsymbol{\nabla} \ln B \\
= & \frac{1}{\left(\frac{ZeB_\parallel^*}{m}\right)} \; v_\parallel^2 \; \widehat{\mathbf{n}} \times \boldsymbol{\nabla} \ln B \quad \text{curvature drift in co-moving} \\
& + \frac{1}{\left(\frac{ZeB_\parallel^*}{m}\right)} \; 2 v_\parallel u_\parallel \; \widehat{\mathbf{n}} \times \boldsymbol{\nabla} \ln B \quad \text{Coriolis drift in co-moving} \\
& + \frac{1}{\left(\frac{ZeB_\parallel^*}{m}\right)} \; u_\parallel^2 \; \widehat{\mathbf{n}} \times \boldsymbol{\nabla} \ln B \quad \text{centrifugal drift in co-moving}
\end{aligned}$$

The results are practically the same, within the approximations mentioned above.

3.3 Particle in a rotating plasma

Paper **centrifugal force gyrokinetic Casson**.

Physical effect on ions

- The detrapping effect of the potential Φ ; This is produced by electric field $E_r = (-\boldsymbol{\nabla}\Phi) \cdot \widehat{\mathbf{e}}_r$, and $E_r \times B_\theta \sim v_\varphi$ that changes the state of a particle: trapped ($v_\parallel$ small) or circulating ($v_\parallel$ large); accelerating the particles by parallel electric field means to helps detrapping;

- the trapping effect of centrifugal force v^2/R, since the centrifugal force changes the perpendicular velocity and creates better situation for trapping;*

almost cancel.

* there is a drift produced by the centrifugal force

$$\mathbf{v}_{cf} = \frac{\mathbf{F} \times \mathbf{B}}{B^2}$$

where

$$\begin{aligned}\mathbf{F}_{cf} &= m\Omega^2 R\ \boldsymbol{\nabla} R \\ &= m\Omega^2 R\ \left(-\widehat{\mathbf{e}}_R\right)\end{aligned}$$

and this has a component that is radial $\{[(-\widehat{\mathbf{e}}_R)\times\mathbf{B}]\cdot\widehat{\mathbf{e}}_r\neq 0\}$. It can modify the trapping (**Peeters**).

Physical effect on electrons

The electrostatic potential Φ produces an effect of trapping for the electrons.

Take

$$\begin{aligned}Z &\equiv \text{coordinate along the major axis of the torus} \\ &\ \text{(vertical)}\end{aligned}$$

It is adopted a *frame* that rotates in toroidal direction

$$\begin{aligned}\boldsymbol{\Omega} &= -\Omega\ \boldsymbol{\nabla} Z \\ &\ \text{(vertical)}\end{aligned}$$

Plasma rotates toroidally with the angular velocity

$$\begin{aligned}\boldsymbol{\omega}_\varphi(\psi) &= -\omega_\varphi(\psi)\ \boldsymbol{\nabla} Z \\ \text{where }\ \psi &= \frac{r}{R_A}\end{aligned}$$

where

$$[R_A] = m\ \ \text{(distance)}$$

There are these two angular velocities which are defined to be coincident on a certain magnetic surface ψ

$$\begin{aligned}&\text{surface } \psi \\ \Omega &= \omega_\varphi(\psi)\end{aligned}$$

Here the local plasma toroidal velocity is

$$\begin{aligned}\mathbf{u}_{tor} &= R\ \boldsymbol{\Omega}\times\boldsymbol{\nabla} R \\ &\sim \mathbf{B}_{tor} \\ &\ \text{it has the direction of } \mathbf{B}_{tor}\end{aligned}$$

Define Mach number

$$\begin{aligned} u &\equiv \frac{R_A \Omega}{v_{th}} \quad \left(m \frac{1/s}{(m/s)} \right) \quad \text{nondimensional} \\ v_{th} &= \sqrt{\frac{2T_i}{m_i}} \quad \text{on the surface taken as reference} \end{aligned}$$

The equations (see **Peeters**)

$$m v_{\|} \frac{d v_{\|}}{dt} = \frac{d\mathbf{X}}{dt} \left[-Ze \, \mathbf{\nabla} \left\langle \phi + \Phi \right\rangle - \mu \mathbf{\nabla} B + m \Omega^2 R \, \mathbf{\nabla} R \right]$$

$$\begin{aligned} \frac{d\mathbf{X}}{dt} &= v_{\|} \widehat{\mathbf{n}} \\ &\quad + \frac{1}{Ze} \left(\frac{m v_{\|}^2}{B} + \mu \right) \frac{\widehat{\mathbf{n}} \times \mathbf{\nabla} B}{B} \quad (\text{drift } \mathbf{v}_D) \\ &\quad + \frac{-\mathbf{\nabla} \left\langle \phi + \Phi \right\rangle \times \widehat{\mathbf{n}}}{B} \quad (\text{electric}) \\ &\quad + \frac{2 m v_{\|}}{ZeB} \mathbf{\Omega}_{\perp} \quad \text{Coriolis} \\ &\quad - \frac{m \Omega^2 R}{ZeB} \widehat{\mathbf{n}} \times \mathbf{\nabla} R \quad \text{centrifugal} \end{aligned}$$

where

$$\mathbf{\Omega}_{\perp} = \mathbf{\Omega} - \left(\mathbf{\Omega} \cdot \widehat{\mathbf{n}} \right) \widehat{\mathbf{n}}$$

The equilibrium distribution if Maxwellian in velocity space

$$F_M = \frac{n}{\left(2\pi T/m \right)^{3/2}} \exp \left[-\frac{\left(v_{\|} - u_{\|} \right)^2}{2T/m} - \frac{\mu B}{T} \right]$$

where

$$u_{\|} = \frac{B_{tor}}{B} \left[R \omega_{\varphi} \left(\psi \right) - R \Omega \right]$$

As the surface of reference has been defined $\left(\omega_{\varphi} \left(\psi_{ref} \right) = \Omega \right)$ it results that

$$u_{\|} \left(\psi = \psi_{ref} \right) = 0$$

but it has variation for nearby surfaces (radial distance relative to ψ_{ref}). This variation with ψ leads to gradients that arise when the equilbrium distribution function is diferentiated

$$u' = -\frac{R}{v_{th}} \left(\frac{\partial \omega_{\varphi}}{\partial \psi} \right)$$

The phase-space conservation of the gyrocenter distribution

$$F\left(\mathbf{X},\mu,v_{\parallel}\right)$$

is expanded in

$$\rho_{*}=\frac{\rho_{i}}{R}$$

$$\begin{aligned}
& v_{\parallel}\ \nabla_{\parallel}F_{M} \\
& +\left[\frac{Ze}{m}\left(-\nabla_{\parallel}\left\langle\Phi\right\rangle\right)\right. \\
& \qquad -\frac{\mu}{m}\nabla_{\parallel}B \\
& \qquad \left.+\Omega^{2}R\ \nabla_{\parallel}R\right]\ \frac{\partial F_{M}}{\partial v_{\parallel}} \\
= \quad & 0
\end{aligned}$$

The presence of Φ is requested by *neutrality* since the centrifugal force will impose it. Φ detraps the ions contrary to the centrifugal force that tends to trap them.

The density, $s=e,i$

$$\begin{aligned}
n_{s}\left(\theta\right) \quad = \quad & n_{R_{0},s}\exp\left\{-\frac{Ze\left\langle\Phi\right\rangle}{T_{s}}+\frac{\left(\Omega^{2}R^{2}-\Omega^{2}R_{0}^{2}\right)}{2T_{s}/m_{s}}\right\} \\
& \text{depends on } \theta
\end{aligned}$$

where $R_{0,s}$ is chosen where $n\left(\theta\right)=n_{R_{0},s}$.

The potential with veriation on large spatial scale

$$\left\langle\Phi\right\rangle\approx\Phi$$

The neutrality condition, in the rotating frame

$$e\Phi=\frac{1}{\frac{1}{T_{i}}+\frac{1}{T_{e}}}\left(\frac{m_{i}}{T_{i}}-\frac{m_{e}}{T_{e}}\right)\frac{1}{2}\left(\Omega^{2}R^{2}-\Omega^{2}R_{0}^{2}\right)$$

This is for ions+electrons.

When there are impurities neutrality is solved numerically.

One can define the "centrifugal and potential energy"

$$\mathcal{E}\left(\theta\right)=Ze\Phi-\frac{1}{2}m\left(\Omega^{2}R^{2}-\Omega^{2}R_{0}^{2}\right)$$

The gradient-length of the density is modified by the rotation.

It is measured by R/L_n.

The expression contains R_0 and this is a choice. It also demends on the radial coordinate ψ like R.

Taking the variation of the density on surface

$$n_s(\theta,\psi) = n_{R_0,s}(\psi)\ \exp\left(-\frac{\mathcal{E}_s(\theta,\psi)}{T_s(\psi)}\right)$$

such that the equilibrium Maxwellian is

$$F_M = \frac{n_{R_0}}{(2\pi T_s/m_s)^{3/2}} \exp\left[-\frac{\left(v_\parallel - u_\parallel\right)^2}{T_s/m_s} - \frac{\mu B}{T_s} - \frac{\mathcal{E}}{T_s}\right]$$

Two choices

$$\begin{aligned} R_0 &= R_{axis} \\ R_0 &= R_{LFS} \end{aligned}$$

Now define

$$\left.\frac{R}{L_n}\right|_{R=R_0} = -\left.\frac{\partial \ln n}{\partial \psi}\right|_{R_0}$$

then

$$\begin{aligned} \frac{R}{L_n^E}(\theta) &= -\frac{\partial \ln n}{\partial \psi} \\ &= \left.\frac{R}{L_n}\right|_{R=R_0} \\ &\quad +\frac{\mathcal{E}}{T}\frac{R}{L_T} \\ &\quad +\frac{1}{T}\frac{\partial \mathcal{E}}{\partial \psi} \end{aligned}$$

For a plasma of electrons and ions

$$\frac{\mathcal{E}_e}{T_e} = \frac{\mathcal{E}_i}{T_i}$$

They are simplified to (after neglecting the electron mass)

$$\mathcal{E}_s = -\frac{T_s}{T_e + T_i} m_i \frac{1}{2}\left(\Omega^2 R^2 - \Omega^2 R_0^2\right)$$

The change in the gradient density is

$$
\begin{aligned}
& \frac{R}{L_n^E} - \left.\frac{R}{L_n}\right|_{R_0} \\
= \quad & \frac{\frac{\partial T_e}{\partial \psi} + \frac{\partial T_i}{\partial \psi}}{\left(T_e + T_i\right)^2} m_i \frac{1}{2} \left(\Omega^2 R^2 - \Omega^2 R_0^2\right) \\
& + \frac{1}{T_e + T_i} m_i \frac{1}{2} \left[\Omega^2 2R \left.\frac{\partial R}{\partial \psi}\right|_\theta - \Omega^2 2R_0 \left.\frac{\partial R_0}{\partial \psi}\right|_\theta \right]
\end{aligned}
$$

The "drift kinetic equation" in the presence of rotation

$$
\begin{aligned}
& \frac{\partial f}{\partial t} + \frac{d\mathbf{X}}{dt} \cdot \boldsymbol{\nabla} f \\
& - \widehat{\mathbf{n}} \cdot \frac{1}{m} \left(\mu \boldsymbol{\nabla} B + \boldsymbol{\nabla} \mathcal{E}\right) \frac{\partial f}{\partial v_\parallel} \\
= \quad & S \quad \text{(source)}
\end{aligned}
$$

The source consists of the terms that comes from the advection of the equilibrium distribution function.

$$
\begin{aligned}
S \quad = \quad & -\mathbf{v}_E \cdot \left(\frac{\boldsymbol{\nabla} n_{R_0}}{n_{R_0}} - \frac{1}{T} m \frac{1}{2} \, \Omega^2 2R_0 \frac{\partial R_0}{\partial \psi} \, \boldsymbol{\nabla} \psi \right. \\
& \quad + \left[\frac{v_\parallel^2}{v_{th}^2} + \frac{\mu B + \mathcal{E}}{T} - \frac{3}{2} \right] \frac{\boldsymbol{\nabla} T}{T} \\
& \quad \left. + \frac{B_{tor}}{B} \frac{1}{T} m v_\parallel R \; \boldsymbol{\nabla} \omega_\varphi \right) F_M \\
& + \frac{d\mathbf{X}}{dt} \cdot \frac{-Ze \boldsymbol{\nabla} \left\langle \phi \right\rangle}{T} F_M
\end{aligned}
$$

The equation of neutrality

$$
\begin{aligned}
& \sum_s Z_s n_{R_0,s} \left[2\pi B \int d\mu dv_\parallel \; J_0\left(k_\perp \rho_s\right) \; \widehat{f}_s \right. \\
& \quad \left. + \frac{Z_s}{T_s} \left[\Gamma\left(b_s\right) - 1\right] \exp\left(-\frac{\mathcal{E}_s}{T_s}\right) \widehat{\phi} \right] \\
= \quad & 0
\end{aligned}
$$

where

$$
\begin{aligned}
b_s \quad & = \quad \frac{1}{2} \frac{k_\perp^2}{\left(Z_s e B / m_s\right)^2} v_{th,s}^2 \\
& = \quad \frac{1}{2} \frac{k_\perp^2}{\Omega_s^2} v_{th,s}^2
\end{aligned}
$$

The centrifugal drift

$$\mathbf{v}_{cf} = -\frac{1}{2}\frac{1}{ZeB} m\ \Omega^2\ 2R\ \widehat{\mathbf{n}} \times \boldsymbol{\nabla} R$$

or

$$-\frac{1}{2}\frac{1}{\Omega_{cicl}} \widehat{\mathbf{n}} \times \boldsymbol{\nabla}\left(\Omega^2 R^2\right)$$

the drift

$$\mathbf{v}_{coriolis} = 2\ \frac{v_{\|}}{\Omega_{cicl}}\ \boldsymbol{\Omega}_{\perp}$$

4 Rotation in mirrors

The paper is **rotation in mirrors Bekhtenev**. It is in *research, plasma, general, rotation.*

Apart gyromotion there is a drift

$$\mathbf{v}_E = \frac{\mathbf{F} \times \mathbf{B}}{B^2}$$

and the angular moment

$$\boldsymbol{\Omega}_E = \frac{\mathbf{r} \times \mathbf{v}_E}{r^2}$$

where

$$\mathbf{F} = e\mathbf{E} + m\Omega_E^2\ \mathbf{r}$$

Isorotation law

$$\Omega_E \approx \frac{E}{B}\frac{1}{r} \approx \text{const}$$

Centrifugal inertial force

$$m\Omega_E^2\ r$$

See also **Horton mirrors**.

5 Magnetic field generation in sheared rotation relative to neutrals

The paper is **SelfGeneratedMagneticFieldShearedFlow**

The domain is ionosphere and the paper is in *biblio, ionosphere.*

The equations

$$\frac{\partial n_\alpha}{\partial t} + \nabla\cdot (n_\alpha \mathbf{V}_\alpha) = 0$$

$$\begin{aligned} \frac{d\mathbf{V}_\alpha}{dt} &= -\frac{1}{n_\alpha m_\alpha}\nabla p_\alpha + \frac{q_\alpha}{m_\alpha}\mathbf{E} + \frac{q_\alpha}{m_\alpha}\mathbf{V}_\alpha \times \mathbf{B} \\ & \quad -\nu_{\alpha n}(\mathbf{V}_\alpha - \mathbf{V}_n) \\ & \quad -\nu_{\alpha\beta}(\mathbf{V}_\alpha - \mathbf{V}_\beta) \end{aligned}$$

where

$$\mathbf{V}_n \equiv \text{neutral fluid velocity}$$

$$\alpha, \beta \equiv \text{electrons and ions}$$

$$\begin{aligned} q_e &= -e \\ q_i &= +e \\ & \quad (\text{where } e > 0) \end{aligned}$$

Adding the equation (one-fluid)

$$\begin{aligned} n_i m_i \frac{d\mathbf{V}_i}{dt} &= -\nabla p + \mathbf{J} \times \mathbf{B} \\ & \quad -n_i (m_i v_{in} + m_e v_{en}) \mathbf{V}_i \\ & \quad +\frac{m_e \nu_{en}}{e}\mathbf{J} \\ & \quad +n_i (m_i v_{in} + m_e v_{en}) \mathbf{V}_n \end{aligned}$$

where

$$\mathbf{J} = e n_i \mathbf{V}_i - e n_e \mathbf{V}_e$$

$$n_i = n_e$$

$$P = p_e + p_i$$

$$m_e \nu_{ei} = m_i \nu_{ie}$$

and

$$n_n \gg n_i$$

Consider also the Ampere's law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

then

$$\begin{aligned}
n_i m_i \frac{d\mathbf{V}_i}{dt} &= -\nabla \left(\frac{B^2}{2\mu_0} + p \right) + \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \mathbf{B} \\
& -n_i (m_i v_{in} + m_e v_{en}) (\mathbf{V}_i - \mathbf{V}_n) \qquad \text{(collisional)} \\
& + \frac{m_e}{\mu_0 e} \nu_{en} \ \nabla \times \mathbf{B} \qquad \left(\text{collisional, comes from } \frac{m_e \nu_{en}}{e} \mathbf{J} \right)
\end{aligned}$$

The electron momentum equation is

$$\begin{aligned}
\mathbf{E} &= -\frac{1}{n_e e} \nabla p_e - \mathbf{V}_e \times \mathbf{B} \\
& + \frac{m_e \nu_{en}}{e} (\mathbf{V}_e - \mathbf{V}_n) \\
& + \frac{m_e}{n_e e^2} \nu_{ei} \ \mathbf{J}
\end{aligned}$$

The Faraday law

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

and one can eliminate the electron velocity using

$$\mathbf{V}_e = \mathbf{V}_i - \frac{1}{n_e e} \mathbf{J}$$

It results

$$\begin{aligned}
\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{V}_i \times \mathbf{B}) \qquad \text{convective} \\
& -\frac{m_e}{e} \nabla \times \left[\nu \frac{\mathbf{J}}{en} \right] \qquad \text{diffusion} \\
& -\nabla \times \frac{\mathbf{J} \times \mathbf{B}}{ne} \qquad \text{Hall} \\
& + \frac{m_e}{e} \nabla \times [\nu_{en} (\mathbf{V}_i - \mathbf{V}_n)] \qquad \text{source}
\end{aligned}$$

where

$$\nu \equiv \nu_{ei} + \nu_{en}$$

There are now applications.

For a flow of neutrals that is along y direction with velocity that varies (shear) along the x direction

$$\mathbf{V}_n = V_n(x)\,\widehat{\mathbf{e}}_y$$

the magnetic field generated is along the z direction

$$\mathbf{B} = B(x,t)\,\widehat{\mathbf{e}}_z$$

The problem is treated as initial value equation, with Laplace transform on time.

Note
Neutrals are necessary.
The plasma is weakly ionized, the density of neutrals is high.
The velocity of ions and neutrals are sheared.
End

6 Rotation in tokamak

6.1 Asymptotic poloidal rotation

From **Novakovskii Liu Sagdeev Rosenbluth**

In *solution drift equation.*See below.

In a plasma with density and temperature gradients an initially Maxwellian distribution inevitably evolves to a state with a finite RADIAL electric field.

- [Hazeltine dT/dr].
- In **Novakovskii Galeev Sagdeev Hassam**

$$v_{E\infty} = -V_n - \frac{3}{2}V_T - \frac{B_\theta}{B_T}U_0$$

6.2 Inertia factor to poloidal rotation

The inertia factor for the poloidal rotation

$$\left(1+q^2\Lambda\right)\frac{\partial V_E}{\partial t}$$

is discussed in

- **Novakovskii Sagdeev Hassam Galeev**

- **Hassam Drake (**Stringer)
- **Hirshman** paradox ambipolarity

This subject is examined in *000_reference_plasma*. The derivation of **Hassam** is adopted.

6.3 Radial polarization electric current

In **Novakovskii Galeev Hassam**

This is

$$\begin{aligned} \langle j_r \rangle &= \langle n V_{ir} \rangle \\ &\sim \left(1 + q^2 + \widehat{\nu}^{-1/3} q^2\right) \frac{mc^2}{B^2} \left(\frac{\partial E_r}{\partial t}\right) \end{aligned}$$

The current is of ions.

About the complicated problem of saturation.

We inject without stop new fast ions, NBI.

The polarization should increase.

The return current should also increase.

There is a parallel current that arises from the zero-divergence of the current.

The parallel current is saturated by collisions.

In **Honda** it is mentioned that some *parallel* dissipative mechanims can saturate the *perpendicular* electric field.

6.4 From Kim Burrell.

- NBI counter injection (counter-current)

- first orbit ion loss, a radial outward current (massive loss, approx half of fast ions, $\sim 40\%$)

- thermal ions respond by an inward current (to mainatin neutrality) $\mathbf{J}_r^{orbit}$; return current. The return current is the current of *all plasma*, while the loss of fast ions is only of the small population of fast ions.

- the return current (involving the bulk plasma) AND **B** create a torque

$$\mathbf{J}_r \times \mathbf{B}$$

applied on plasma; the toroidal part of the torque $J_r^{orbit} B_\theta$ is directed counter-current (as is NBI)

- rotation is generated by this torque: poloidal v_θ and toroidal v_φ;

- the rotation produces the *radial electric field* E_r; this is because there is NO equilibrium balance between the diamagnetic and the poloidal and toroidal flows. An electric field is necessary to ensure the force balance.

They obtain

$$E_r = \frac{1}{n_i Z_i e}\left(\frac{dP_i}{dr}\right) - U_{i\theta} B_\varphi + U_{i\varphi} B_\theta$$

At counter-NBI (where the loss is 38% of NBI ions)

- the pressure $\frac{1}{n_i Z_i e}\left(\frac{dP_i}{dr}\right)$ [diamagnetic] and
- the toroidal rotation $U_{i\varphi} B_\theta$

terms are *negative* globally.
For this reason the electric field is negative $E_r < 0$.
At the edge the pressure term, *negative*, makes E_r strongly negative.

Regarding the toroidal torque, for counter-NBI.
It has two components: one is $\mathbf{J}_r \times \mathbf{B}$ and the other is the torque created by the *fast NBI ions* which remain inside plasma.

The equation for the toroidal momentum

$$\frac{\partial}{\partial t}\left(U_\varphi n_i\right) - \frac{1}{r}\frac{\partial}{\partial r}\left[r\ \chi n_i \left(\frac{\partial U_\varphi}{\partial r}\right)\right] = F$$

where

$$\chi \equiv \text{coeff. of radial transport of the momentum}$$

$$F = F_{NBI} + F_{cx}$$

Normalization using reference values

$$n_0 = 10^{19}\ \left(m^{-3}\right)$$

$$U_{\varphi 0} = 10\ \ (km/s)$$

$$t_0 = 100\ \ (ms)$$

$$a = 0.6\ \ (m)$$

$$\chi_0 = \frac{a^2}{t_0} = 3.6\ \left(\frac{m^2}{s}\right)$$

$$F_0 = \frac{m_i n_0 U_{\varphi 0}}{t_0} = 3.34 \times 10^{-3} \quad \left(\frac{N}{m^3}\right)$$

with $m_i = 1.67 \times 10^{-27}\ (kg) \times 2\ (Deuteriu) = 3.34 \times 10^{-27}\ (kg)$.

The force resulting from charge exchange-induced change of the toroidal momentum (loss)

$$F_{cx} = -\frac{n_i U_0\ \nu_{cx}}{t_0} \quad (?)$$

$$\nu_{cx} = N^{neut} \left\langle \sigma\ v \right\rangle_{cx}$$

$$\begin{aligned} N_0^{neut} &\equiv \text{density of neutrals} \\ &= 10^{16} \left(\frac{1}{m^3}\right) \end{aligned}$$

In **Stringer1969 driftwithradialelectricfield**

See *025_ stringer*.

The statement that the static inertia is involved.

This corresponds to the treatment of **Rosenbluth hose** (see below for detailes discussion) where

$$h\mathbf{B} \cdot \left[\left(\mathbf{v} \cdot \boldsymbol{\nabla} \right) \mathbf{v} \right] = -\frac{c_s^2}{\rho} h\mathbf{B} \cdot \boldsymbol{\nabla} \rho$$

shows that the static inertia of plasma (ions) is balanced by parallel gradient of pressure (isothermal).

[for a similar hose-like situation, **Robertson transient Alfven** in *Alfven*]

It appears the factor ρ_{eff}.

Hinton Wong rotation. ρ_{eff}.

6.5 Burrell changes in electric field at L to H transition

The paper is in **1994 parametricdependenceelectric field edge**.

And **poloidal rotation near the edge in H mode Hinton**. For ρ_{eff}. Squeezing effect. Two treatments: heuristic and kinetic (distribution function).

6.6 The frequency of poloidal rotation Shaing Hazeltine

This paper **Shaing Hazeltine effect of orbit squeezing**.

The kinetic equation

$$\begin{aligned} & \left(v_{\parallel}\widehat{\mathbf{n}} + \mathbf{V}_E\right) \cdot \boldsymbol{\nabla}\theta \; \frac{\partial f_1}{\partial \theta} + \mathbf{v}_D \cdot \boldsymbol{\nabla}\psi \frac{\partial f_1}{\partial \psi} \\ & + \mathbf{v}_D \cdot \boldsymbol{\nabla}\psi \frac{\partial f_0}{\partial \psi} \\ = \; & C\left[f_1\right] \end{aligned}$$

Estimations

$$\begin{aligned} \frac{\partial f_1}{\partial \psi} & \sim \frac{f_1}{\psi_p} \\ \psi_p & \equiv \text{typical width of a particle orbit} \end{aligned}$$

The radial advection $\mathbf{v}_D \cdot \boldsymbol{\nabla}\psi$ of the radial gradient of the distribution function $\partial f_0 / \partial \psi$,

$$\frac{\partial f_0}{\partial \psi} \sim \frac{f_0}{\psi_e}$$

On the other hand the order of magnitude

$$f_1 \sim \frac{\psi_p}{\psi_e} f_0$$

Then

$$\frac{\partial f_1}{\partial \psi} \sim \frac{\partial f_0}{\partial \psi}$$

Conclusion: f_1 has sharp variation in the narrow radial zone.

The frequency of rotation

$$\begin{aligned} \omega & = \left(v_{\parallel}\widehat{\mathbf{n}} + \mathbf{V}_E\right) \cdot \boldsymbol{\nabla}\theta \\ & = \left(v_{\parallel} + I \frac{1}{B} \frac{d\phi}{d\psi}\right) \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta \end{aligned}$$

here $I = RB_T$, $\frac{I}{B}\frac{d\phi}{d\psi} \sim \frac{RB_T}{B}\frac{1}{|\boldsymbol{\nabla}\psi|}\frac{d\phi}{dr} \sim \frac{RB_T}{B}\frac{1}{RB_\theta}\frac{d\phi}{dr} = \frac{B_T}{B_\theta}\left(\frac{-E_r}{B}\right)$ is the parallel projection of the poloidal velocity of electric origin. Then $V_{\parallel} = v_{\parallel} + \frac{B_T}{B_\theta} V_E$ and $\nabla_{\parallel}\theta = \frac{1}{qR}\frac{\partial}{\partial \theta}\theta = \frac{1}{qR}$.

[the following part is similar to **poloidal rotation H mode Hinton**]. The invariants of guiding center

$$E = \frac{1}{2}v_{\|}^{2} + \mu B\left(\theta\right) + \frac{e}{M}\phi\left(\theta\right)$$

$$\begin{aligned} P &= \psi - I\frac{v_{\|}}{\Omega_c} \\ & \quad \text{(generalized momentum)} \end{aligned}$$

One has

$$R^2 \mathbf{v} \cdot \boldsymbol{\nabla}\varphi \approx I\frac{v_{\|}}{B}$$

for small flow velocities.

$$\omega = -\frac{I}{\Omega_c}\left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta\right)\frac{\frac{\partial P}{\partial \psi}}{\frac{\partial P}{\partial E}}$$

The potential in the energy is expanded

$$\phi = \phi_0\left(\psi_0\right) + \left.\frac{d\phi}{d\psi}\right|_{\psi_0}\left(\psi - \psi_0\right) + \left.\frac{d^2\phi}{d\psi^2}\right|_{\psi_0}\left(\psi - \psi_0\right)^2$$

The squeezing factor is generated by the *radial variation of the radial electric field.*

This means that for a trapped particle a branch (half) of banana travels in a radial electric field and the other half in a different radial electric field.

The reference point is adopted

$$\theta_0 = \pi$$

Then the parallel velocity in that point is a reference too

$$\begin{aligned} v_{\|0} &= v_{\|}\left(\psi_0, \theta_0\right) \\ &= \sqrt{2\left[E - \mu B\left(\psi_0, \theta_0\right) - \frac{e}{m}\phi\left(\psi_0\right)\right]} \end{aligned}$$

For large $|S|$,

$$|S| > 1$$

the trapped particles reside on the *high field side* of the torus (smaller major radius).

Then the choice

$$\theta_0 = \pi$$

The expression

$$S = 1 + \frac{I^2}{\Omega_c B_0} \frac{\partial^2 \phi}{\partial \psi^2}$$

Now define

$$h_0 = h(\psi_0, \theta_0)$$
$$B = \frac{B_0}{h(\psi, \theta)}$$
$$V_{E_0} = \frac{I}{B_0} \left.\frac{d\phi}{d\psi}\right|_0$$

The reference surface is

$$\psi_0$$

during the motion the particle is at

$$\psi(\theta, P, E, \mu)$$

and the particle orbit has a departure

$$\Delta\psi = \psi(\theta; P, E, \mu) - \psi_0$$

Then

$$\omega = (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta) \left[\left(\frac{h_0}{h} \right) v_{\|0} + h V_{E_0} + \left(\frac{1}{h} + S(h-1) \right) \left(\frac{\Omega_0}{I} \right) \Delta\psi \right]$$

The variable $\Delta\psi$ is replaced here from the equation that shows the departure of the particle's ψ from the reference surface ψ_0.

Then

$$\begin{aligned} \omega = (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta) \Bigg\{ & \left(h V_{E_0} + \frac{h_0}{h} v_{\|0} \right)^2 \\ & + \left[1 + h^2 (S-1) \right] \left(\frac{h_0}{h} - 1 \right) \\ & \times \left[2 \left(E - \frac{e\phi_0}{M} \right) \left(\frac{h_0}{h} - 1 \right) - \frac{2\mu B_0}{h} \right] \Bigg\}^{1/2} \end{aligned}$$

We have

$$\omega = \pm \widehat{\omega} \sqrt{1 - \kappa \sin^2 \left(\frac{\theta}{2} \right)}$$

$$\widehat{\omega} = \left|\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta\right| \sqrt{\left(V_{E0} + v_{\|0}\right)^2 + 4S\varepsilon\left(v_{\|0}^2 + \mu B_0\right)}$$

$$\kappa = 4S\varepsilon \frac{v_{\|0}^2 + \mu B_0}{\left(V_{E0} + v_{\|0}\right)^2 + 4S\varepsilon\left(v_{\|0}^2 + \mu B_0\right)}$$

The limit trapped/passing

$$\kappa = 1$$

The limit

$$\kappa \;\; \rightarrow \;\; \infty \;\; \text{means} \;\; \widehat{\omega} \rightarrow 0$$
$$\text{deep trapped}$$

Additional expansion parameter

$$\frac{\frac{\nu_i}{S\varepsilon}}{\frac{v_{th,i}}{Rq}\sqrt{\varepsilon S}} \ll 1$$

The collision operator (pitch angle)

$$C\left[f\right] = \nu^{defl} \; \frac{v_{\|}}{B} \frac{\partial}{\partial \mu} \mu v_{\|} \frac{\partial f}{\partial \mu}$$

for the independent variables

$$\left(E, \mu\right)$$

The next steps

- it is changed

$$\left(E, \mu\right) \rightarrow \left(E, \omega\right)$$

- it is assumed that

$$\sqrt{\varepsilon \left|S\right|} \ll 1$$

- neglect $\partial f / \partial \omega$;
- neglect a term for

$$M_p < 1$$

then one obtains

$$C[f] = \nu^{delf} \frac{v^2}{R^2 q^2} \left(\frac{\partial^2 f}{\partial \omega^2} \right)$$

[good for small fraction of trapped particles]

Change

$$(E, \omega) \rightarrow (E, \kappa)$$

and obtain

$$C[f] = \nu^{delf} \frac{v^2}{R^2 q^2} \frac{2\kappa\omega}{\widehat{\omega}^2} \frac{\partial}{\partial \kappa} \left(\frac{2\kappa\omega}{\widehat{\omega}^2} \frac{\partial f}{\partial \kappa} \right)$$

6.7 Rotation and bootstrap Peeters review

From the text *bootstrap.tex*.

The fluid formulation.
In parallel projection, the full momentum conservation is

$$\begin{aligned} n m_i \mathbf{B} \cdot \frac{d\mathbf{u}_i}{dt} &= \mathbf{B} \cdot (-\boldsymbol{\nabla} p) + \mathbf{B} \cdot (-\boldsymbol{\nabla} \cdot \boldsymbol{\pi}) \\ &\quad + en \ \mathbf{B} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{F} \end{aligned}$$

This is first adapted for *circulating* ions.

Then it is assumed that the build-up of a parallel velocity of circulating ions by the moment transferred from trapped ions can be represented as anisotropy of the pressure tensor, projected on the magnetic field

$$n m_i \mathbf{B} \cdot \frac{d\mathbf{u}_i}{dt} = \mathbf{B} \cdot (-\boldsymbol{\nabla} \cdot \boldsymbol{\pi})$$

Here, therefore one replaces the LHS with the formula above

$$n m_i \frac{\partial u_{i\|}^{circ}}{\partial t} = m_i \frac{\nu_{ii}}{\varepsilon} \varepsilon^{3/2} \frac{T_i}{e B_\theta} \frac{dn}{dr} - \sqrt{\varepsilon} \nu_{ii} m_i n \ u_{i\|}^{circ}$$

and obtain

$$\begin{aligned} B \ m_i \nu_{ii} \sqrt{\varepsilon} \left(\frac{T_i}{e B_\theta} \frac{dn}{dr} - n \ u_{i\|}^{circ} \right) &= -\mathbf{B} \cdot \boldsymbol{\nabla} \cdot \overline{\boldsymbol{\pi}} \\ B \ m_i \nu_{ii} \sqrt{\varepsilon} \ n \ \left(\frac{T_i}{e B_\theta} \frac{d}{dr} \ln n - u_{i\|}^{circ} \right) &= -\mathbf{B} \cdot \boldsymbol{\nabla} \cdot \overline{\boldsymbol{\pi}} \end{aligned}$$

The first term in the brackets is the diamagnetic velocity of ions

$$\begin{aligned} & \frac{T_i}{eB_\theta}\frac{d}{dr}\ln n - u_{i\|}^{circ} \\ = & \frac{B_{tor}}{B_\theta}\left(-v_{i,dia}\right) - u_{i\|}^{circ} \end{aligned}$$

since

$$v_{i,dia} = -\frac{T_i}{eB_{tor}}\frac{d}{dr}\ln n$$

From these two terms it can be formed the *poloidal* velocity of ions. First one multiplies

$$\begin{aligned} & \left[\frac{T_i}{eB_\theta}\frac{d}{dr}\ln n - u_{i\|}^{circ}\right] \times \frac{B_\theta}{B} \\ = & \frac{B_{tor}}{B} \times \frac{T_i}{eB_{tor}}\frac{d}{dr}\ln n - u_{i\|}^{circ}\frac{B_\theta}{B} \\ = & \frac{B_{tor}}{B} \times \left(-v_{i,dia}\right) - u_{i,pol}^{circ} \\ = & -V_{pol} \end{aligned}$$

this is the poloidal velocity of the circulating ions. The ratio B_{tor}/B close to 1 corrects the diamagnetic velocity which is perpendicular on the magnetic line to be projected on the poloidal direction.

$$V_{pol} = \left(v_{i,dia}\right)_{pol} + u_{i,pol}^{circ}$$

Now the paranthesis $\left(\frac{T_i}{eB_\theta}\frac{d}{dr}\ln n - u_{i\|}^{circ}\right)$ is replaced by

$$\left(\frac{T_i}{eB_\theta}\frac{d}{dr}\ln n - u_{i\|}^{circ}\right) = \frac{B}{B_\theta}\left(-V_{pol}\right)$$

and the momentum equation is written

$$\begin{aligned} B\ m_i\nu_{ii}\sqrt{\varepsilon}\ n\ \left(\frac{T_i}{eB_\theta}\frac{d}{dr}\ln n - u_{i\|}^{circ}\right) &= -\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\overline{\boldsymbol{\pi}} \\ \frac{B^2}{B_\theta}m_i\nu_{ii}\sqrt{\varepsilon}n\ V_{pol} &= \mathbf{B}\cdot\boldsymbol{\nabla}\cdot\overline{\boldsymbol{\pi}} \end{aligned}$$

The factors $m_i\nu_{ii}\sqrt{\varepsilon}n$ are dynamical viscosity, μ_i

$$\mu_i B^2\frac{V_{pol}}{B_\theta} = \mathbf{B}\cdot\boldsymbol{\nabla}\cdot\overline{\boldsymbol{\pi}}$$

Returning

$$nm_i \mathbf{B} \cdot \frac{d\mathbf{u}_i^{circ}}{dt} = -\mu_i B^2 \frac{V_{pol}}{B_\theta}$$

The interpretation given by Peeters

"The density gradient leads to a diamagnetic velocity in the surface which has a poloidal component. In this direction, however, the magnetic field strength changes which leads to a viscous force which damps the poloidal rotation through a build-up of the parallel velocity until the poloidal component of this velocity cancels the poloidal component of the diamagnetic velocity. The total velocity is then in the direction of the symmetry of the system [toroidal, axisymmetric] *and, therefore, no longer 'feels' the variation of the field strength. The viscous force that appears in the fluid theory can be traced back to the friction between trapped and passing. By definition, a trapped particle cannot rotate in the poloidal direction"*

Comment

- there is the diamagnetic flow
- the diamagnetic flow is perpendicular $\perp$ on the magnetic field line
- then there is a poloidal θ projection of the diamagnetic flow
- this poloidal projection (of the diamagnetic flow) is heavily damped by TTMP
- the process of damping of the poloidal projection of the diamagnetic flow necessarily induce a *parallel* flow
- this parallel flow also has a poloidal projection
- the poloidal projection of the parallel flow and the poloidal projection of the diamagnetic flow *cancel* each other
- there will not be any poloidal rotation.
- all rotation is toroidal

Note that the gradient of temperature is not taken into account. And the possibility of a *drive* for the poloidal rotation, e.g. the polarization separation of charge.

See **Stringer** for the necessity of the parallel flow.

7 Toroidal rotation

7.1 Introduction toroidal rotation

State of the problem. *Toroidal* rotation is

L-mode	counter-current
H-mode	co-current

In some situations (**Rice PoP2012**) the increase in the density leads at a threshold to the reversal of the toroidal rotation from *co-* to *counter-* current, but with improvement (!) in confinement.

From **poloidal rotation H mode Hinton**
"well after the L to H transition, the main ions rotate
in the ion diamagnetic direction, at a speed somewhat less
than the ion diamagnetic speed. This is in the opposite
direction to the measured impurity ion poloidal rotation. It is in the same direction as the main ion poloidal rotation predicted by neoclassical theory in the banana
regime [gradient of temperature, Hinton]*, but much larger for the experimental parameters. "*

7.2 Wong Burrell 1982 toroidal rotation

The text and equations are commented in *drift kinetic equation.tex.*

The parameter of neoclassical transport

$$\delta \equiv \frac{\rho_\theta}{R}$$

The parallel plasma flow is nearly the same as the toroidal rotation

The parallel plasma flow is order

$$\delta$$

i.e. first order relative to the ion thermal velocity

$$\begin{aligned} v_{\parallel}^{plasma} &\sim v_{toroidal}^{plasma} \\ &\sim \delta \times v_{th,i} \end{aligned}$$

When there is NBI, the *plasma toroidal flow velocity* is much higher, a high fraction of $v_{th,i}$.

For heavy impurities, without NBI, the velocity is order δ from $v_{th,i}$,

$$v_{\parallel}^{heavy-impurity} \sim \delta \times v_{th,i}$$

How the toroidicity occurs in neoclassical drift theory

- the mirror force
- the curvature and $\mu \mathbf{\nabla} B$ neoclassical particle drift

both must be included in the drift wave equations.
One obtains a modified Maxwellian.
If we want it to be *steady* even when the magnetic pumping acts to damp the rotation, one has to impose a certain ordering to the radial electric field

$$E_r \sim -\frac{d\phi}{dr}$$

$$\begin{aligned} & \frac{\left(\frac{d\phi}{dr}\right)}{B_\theta} \text{ (toroidal electric flow)} \\ \sim\ & v_{th,i} \end{aligned}$$

Ambipolarity must be imposed and it will determine the radial electric field.

The steady state

$$\begin{aligned} & \text{toroidal rotation} \\ \rightarrow\ & \text{centrifugal force} \\ \rightarrow\ & \text{large poloidal electric field } E_\theta \end{aligned}$$

Another effect

$$\begin{array}{rl} & \text{toroidal rotation} \\ \to & \text{centrifugal force} \\ \to & \text{drift } \perp \text{ , due to } \mathbf{F}_R \times \widehat{\mathbf{n}} \end{array}$$

8 Variational principle for guiding center equations Littlejohn

NOTE several coments are in **Hahm Fong**.

The variational equation

$$\delta \int L \; dt = 0$$

$$L = \frac{1}{\varepsilon} \mathbf{A}^* \cdot \frac{d\mathbf{X}}{dt} + \epsilon\mu \frac{d\zeta}{dt} - H$$

$$\begin{aligned} H &= \frac{1}{2} U^2 + \mu B + \Phi \\ & \text{hamiltonian} \end{aligned}$$

where $\mu = \frac{v_\perp^2}{2B}$, $\zeta \equiv$gyroangle, $\mathbf{A}$ is the magnetic potential.

The notation ϵ is more complicated.

It is

$$\epsilon = \frac{\rho_L}{a}$$

where ρ_L is the Larmor gyration radius and a is either the minor radius or the typical length of variation of the equilibrium, $\epsilon \sim L_n$ for example.

ϵ will be considered a small quantity and used for expansion.

Here

$$U \equiv \text{parallel velocity of the guiding center (usually } v_\parallel)$$

$$\begin{aligned} \frac{d\mu}{dt} &= 0 \\ \mu &\equiv \text{constant of motion} \end{aligned}$$

$$\frac{d\zeta}{dt} = \frac{B}{\epsilon}$$

The reason to use ϵ, a measure of weak spatial variation $\sim \rho/L$, is to introduce new variables $\mathbf{E}$ instead of the electric field, $\mathbf{E}^{phys} = \epsilon\ \mathbf{E}$. And electric potential $\phi^{phys} = \epsilon\ \phi$.

In addition, the time variation will be "expanded"

$$\begin{aligned} t &\rightarrow \tau \\ \tau &= \epsilon t \text{ slow time scale} \end{aligned}$$

The modified vector potential

$$\mathbf{A}^{*} = \mathbf{A} + \epsilon U\ \widehat{\mathbf{n}}$$

and for magnetic field

$$\mathbf{B}^{*} = \mathbf{B} + \epsilon U\ \boldsymbol{\nabla}\times\widehat{\mathbf{n}}$$

and for the electric field

$$\begin{aligned} \mathbf{E}^{*} &= -\frac{\partial \mathbf{A}^{*}}{\partial \tau} - \boldsymbol{\nabla}\Phi \\ &= \mathbf{E} - \epsilon U\ \frac{d\widehat{\mathbf{n}}}{d\tau} \end{aligned}$$

NOTE

This modification of the magnetic potential (and, as a consequence, of the magnetic and electric vectors) must be seen as the identification of the "drift magnetic surfaces" $\mathbf{A} \rightarrow \mathbf{A}^{*} = \mathbf{A} + \frac{v_{\parallel}\mathbf{B}}{\Omega}$. They are explained in **Hazeltine Hinton** Eq.(3.25) and in **Catto Kagan** (more recently).

The magnetic surfaces that correspond to

$$\begin{aligned} \mu &= \text{ct} \\ \epsilon &= \text{ct} \\ \psi^{*} &= \text{ct} \end{aligned}$$

are called *drift surfaces* [**Morozov Solovev**]

$$\mathbf{B}^{*} \cdot \boldsymbol{\nabla}\psi^{*} = 0$$

The guiding center motion is confined to this surface ψ^{*}.

$$\mathbf{v}_{guiding-center} = \frac{\mathbf{B}^{*}}{B} v_{\parallel}$$

$$\psi^* = \psi - \frac{I}{\Omega} v_\parallel$$

where $I \sim RB_T$.

END

The equation derived by Littlejohn

$$\mathbf{E}^* + \frac{1}{\varepsilon}\frac{d\mathbf{X}}{dt} \times \mathbf{B}^* = \frac{dU}{d\tau}\,\widehat{\mathbf{n}} + \mu \boldsymbol{\nabla} B$$

Since U is the parallel velocity

$$U = \widehat{\mathbf{n}} \cdot \frac{d\mathbf{X}}{dt}$$

the time variation of the position vector can be found from the two equations above

$$\frac{1}{\varepsilon}\frac{d\mathbf{X}}{dt} \times \mathbf{B}^* = \frac{dU}{d\tau}\widehat{\mathbf{n}} + (\mu \boldsymbol{\nabla} B - \mathbf{E}^*)$$

This is vector-multiplied by $\widehat{\mathbf{n}}$,

$$\begin{aligned} \widehat{\mathbf{n}} \times \frac{1}{\varepsilon}\left(\frac{d\mathbf{X}}{dt} \times \mathbf{B}^*\right) &= \widehat{\mathbf{n}} \times (\mu \boldsymbol{\nabla} B - \mathbf{E}^*) \\ \frac{1}{\varepsilon}\frac{d\mathbf{X}}{dt}(\widehat{\mathbf{n}} \cdot \mathbf{B}^*) - \mathbf{B}^*\left(\widehat{\mathbf{n}} \cdot \frac{1}{\varepsilon}\frac{d\mathbf{X}}{dt}\right) &= \widehat{\mathbf{n}} \times (\mu \boldsymbol{\nabla} B - \mathbf{E}^*) \end{aligned}$$

we now make explicit in the left hand side

$$(\widehat{\mathbf{n}} \cdot \mathbf{B}^*) = B_\parallel^*$$

and

$$-\mathbf{B}^*\left(\widehat{\mathbf{n}} \cdot \frac{1}{\varepsilon}\frac{d\mathbf{X}}{dt}\right) = -\mathbf{B}^*\frac{1}{\varepsilon}U$$

then

$$\begin{aligned} \frac{1}{\epsilon}\frac{d\mathbf{X}}{dt}\, B_\parallel^* &= \frac{1}{\epsilon}U\ \mathbf{B}^* + \widehat{\mathbf{n}} \times (\mu \boldsymbol{\nabla} B - \mathbf{E}^*) \\ \frac{d\mathbf{X}}{dt} &= \frac{1}{B_\parallel^*}\left[U\ \mathbf{B}^* + \epsilon\ \widehat{\mathbf{n}} \times (\mu \boldsymbol{\nabla} B - \mathbf{E}^*)\right] \end{aligned}$$

It is noted

$$B_\parallel^* = B + \epsilon U\ \widehat{\mathbf{n}} \cdot (\boldsymbol{\nabla} \times \widehat{\mathbf{n}})$$

which is obtained by

$$B_\parallel^* = \widehat{\mathbf{n}} \cdot \mathbf{B}^*$$

The formula to be used further is

$$\frac{d\mathbf{X}}{dt} = \frac{1}{B_{\parallel}^{*}} \left[U \ \mathbf{B}^{*} + \epsilon \ \widehat{\mathbf{n}} \times \left(\mu \boldsymbol{\nabla} B - \mathbf{E}^{*} \right) \right]$$

where we take

$$\mathbf{E}^{*} \equiv 0$$

then we have

$$\frac{d\mathbf{X}}{dt} = \frac{1}{B_{\parallel}^{*}} \left[U \ \mathbf{B}^{*} + \epsilon \ \widehat{\mathbf{n}} \times \mu \boldsymbol{\nabla} B \right]$$

Actually, the last term in the formula above must be written in the extended form, including all parts of the drift velocity (what is here is only the *magnetic drift*)

$$\begin{aligned} & \frac{1}{B_{\parallel}^{*}} \left[\epsilon \ \widehat{\mathbf{n}} \times \mu \boldsymbol{\nabla} B \right] \\ \rightarrow \quad & \frac{1}{B_{\parallel}^{*}} \left[\mathbf{v}_{D} \right] \end{aligned}$$

and $\mathbf{v}_D$ is calculated above.

And the first term is

$$\begin{aligned} \frac{1}{B_{\parallel}^{*}} \left[U \ \mathbf{B}^{*} \right] \quad & \rightarrow \quad U \widehat{\mathbf{n}} \\ & \sim \quad \mathbf{v}_{\parallel} \end{aligned}$$

This means to use as drift velocity (magnetic and curvature drifts)

$$\begin{aligned} & \frac{m}{e} \frac{1}{B^{2}} \mu \ \widehat{\mathbf{B}} \times \boldsymbol{\nabla} B \\ & + \frac{m}{e} \frac{1}{B} v_{\parallel}^{2} \ \frac{1}{B^{2}} \left\{ \widehat{\mathbf{B}} \times \boldsymbol{\nabla} B + B \left(\boldsymbol{\nabla} \times \mathbf{B} \right) - \frac{\mathbf{B}}{B} \left[\mathbf{B} \cdot \left(\boldsymbol{\nabla} \times \mathbf{B} \right) \right] \right\} \end{aligned}$$

which is introduced in the expression of $\frac{d\mathbf{X}}{dt}$.

Before doing that, we must recall the explanations regarding the necessity to define the *drift surfaces* by modifying the poloidal flux function (surface function) ψ to ψ^{*}, as shown by **Morozov Solovev, Littlejohn, Hazeltine Hinton, Catto Kagan**.

It is found that

$$\mathbf{v}_{guiding-center} = \frac{\mathbf{B}^{*}}{B} v_{\parallel}$$

Then, everytime we will have a combination of parallel velocity $v_\parallel$ and of guiding-centre velocity, $\mathbf{v}_D$ there will be a coefficient

$$\frac{B}{B_\parallel^*}$$

in front of the guiding centre drift.

This is

$$\begin{aligned} & \frac{d\mathbf{X}}{dt} \\ = & \ \mathbf{v}_\parallel + \frac{B}{B_\parallel^*}\left\{\frac{m}{e}\frac{1}{B^2}\mu\,\widehat{\mathbf{B}} \times \boldsymbol{\nabla} B\right. \\ & +\frac{m}{e}\frac{1}{B}v_\parallel^2\,\frac{1}{B^2}\left(\widehat{\mathbf{B}} \times \boldsymbol{\nabla} B + B\left(\boldsymbol{\nabla} \times \mathbf{B}\right) - \frac{\mathbf{B}}{B}\left[\mathbf{B} \cdot \left(\boldsymbol{\nabla} \times \mathbf{B}\right)\right]\right) \\ & \left. +\text{electric drift}\right\} \end{aligned}$$

This is the equation used by **Jenko2008**.

9 Lie transform. Brizzard

Objective: derive the gyro-kinetic equation in a referential that co-moves with the toroidally rotating plasma.

Transformation of variables

$$(\mathbf{x}, \mathbf{v}) \rightarrow (\mathbf{r}, \mathbf{v} - \mathbf{u}_s)$$

Method: Lie transformation.
Elimination of the fast gyromotion time scale.

The equilibrium force balance

$$(\mathbf{u}_s \cdot \boldsymbol{\nabla})\,\mathbf{u}_s = -\frac{1}{m_s N_s}\boldsymbol{\nabla} p_s - \Omega_s \mathbf{u}_s \times \widehat{\mathbf{n}} + \Omega_s \frac{1}{B}\boldsymbol{\nabla}\Phi$$

Taking out explicitley the velocity

$$\begin{aligned} \mathbf{u} \ = & \ u_\parallel \widehat{\mathbf{n}} \\ & + \frac{1}{\Omega}\widehat{\mathbf{n}} \times \left[\frac{1}{eN}\boldsymbol{\nabla} p + \frac{e}{m}\boldsymbol{\nabla}\Phi + \frac{m}{e}\left(\mathbf{u} \cdot \boldsymbol{\nabla}\right)\mathbf{u}\right] \end{aligned}$$

The equation of continuity

$$\boldsymbol{\nabla} \cdot (N\mathbf{u}) = 0$$

Velocities are ordered according to the parameter $1/\Omega_c$.

Zero order

$$\mathbf{u}^{(0)} = u_{\parallel}^{(0)} \widehat{\mathbf{n}} + \frac{-\boldsymbol{\nabla}\Phi^{(0)} \times \widehat{\mathbf{n}}}{B}$$

The first order

$$\mathbf{u}^{(1)} = u_{\parallel}^{(1)} \, \widehat{\mathbf{n}} + \frac{1}{B} \widehat{\mathbf{n}} \times \left[\boldsymbol{\nabla}\Phi^{(1)} + \frac{B}{\Omega_c} \left(\frac{1}{mN} \boldsymbol{\nabla} p + \left(\mathbf{u}^{(0)} \cdot \boldsymbol{\nabla} \right) \mathbf{u}^{(0)} \right) \right]$$

where

$$\nabla_{\parallel} \Phi^{(1)} \neq 0$$

The equilibrium of a rotating plasma requires

$$\mathbf{u} \cdot \left(\boldsymbol{\nabla} \ln N - \frac{3}{2} \boldsymbol{\nabla} \ln T \right) = 0$$

This implies that $\frac{N}{T^{3/2}}$ is uniform along the rotation $\mathbf{u}$. In particular this is the combination of plasma parameters that occurs in the frequency of collisions.

$$\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \ln T = 0$$

no temperature variation along the magnetic field.

$$\widehat{\mathbf{n}} \cdot \left[\boldsymbol{\nabla} \ln N + \frac{e}{T} \boldsymbol{\nabla} \Phi + \frac{m}{T} \left(\mathbf{u} \cdot \boldsymbol{\nabla} \right) \mathbf{u} \right] = 0$$

The first two terms can form the Boltzmann distribution, for constant T,

$$N \sim \exp \left(-\frac{e\Phi}{T} \right)$$

but it is perturbed by the inertial nonlinearity.

$$\mathbf{u} \cdot \frac{3}{2} \boldsymbol{\nabla} \ln T + \boldsymbol{\nabla} \cdot \mathbf{u} = 0$$

This is the equation of continuity, $\boldsymbol{\nabla} \cdot (N\mathbf{u}) = 0$, using the first condition.

$$\frac{1}{3} \boldsymbol{\nabla} \cdot \mathbf{u} + \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \mathbf{u} \cdot \widehat{\mathbf{n}} = 0$$

This is a tensorial contraction.

Consider the equilibrium potentials

$$\mathbf{A},\ \Phi$$

Consider a system of reference that moves with velocity $\mathbf{u}$ of the plasma. The phase space of a particle in this frame has the following coordinates

$$\mathbf{z} \equiv (\mathbf{r},\ w_0,\ \mu_0,\ \alpha_0)$$

The coordinates are noncanonical.

The differential one-form consists of the *simplectic* part, of the type $\mathbf{p} \cdot \mathbf{dr}$ and the Hamiltonian part, $h\ dt$.

$$\begin{aligned}\gamma &= [e\mathbf{A} + m(\mathbf{u} + \mathbf{c})] \cdot \mathbf{dr} \\ &\quad - \left(e\Phi + \frac{m}{2}\left|\mathbf{u} + \mathbf{c}\right|^2\right)\ dt\end{aligned}$$

It has been introduced the velocity $\mathbf{c}$ of the particle in the frame that moves with the fluid (with velocity $\mathbf{u}$). This particle velocity has components relative to the magnetic field

$$\mathbf{c} = w_0\ \widehat{\mathbf{n}} + \mathbf{c}_\perp$$

and

$$\mu_0 = \frac{mc_\perp^2}{2B}$$

lowest order magnetic moment.

$$\alpha_0 \equiv \text{gyroangle}$$

The procedure to go from particle phase space coordinates to *guiding center* coordinates consists of using the Lagrangian γ in an expansion in a small parameter

$$\epsilon \equiv \frac{m}{e}$$

To first order, the Lagrangian of the guiding center

$$\Gamma = \mathbf{A}^* \cdot d\mathbf{X} + \frac{1}{\Omega_c}\mu B\ d\alpha - H\ dt$$

[the reason to retain the second order term $\frac{1}{\Omega_c}\mu B\ d\alpha$ is to obtain later a term in the Poisson bracket associated to this Lagrangian]

The new coordinates are

$$\mathbf{X},\ W,\ \mu,\ \alpha$$

$$\begin{array}{rcl}
\mathbf{X} & \equiv & \text{guiding center coordinates} \\
W & \equiv & \text{parallel guiding center VELOCITY} \\
 & & \text{in the moving frame} \\
\mu & \equiv & \text{magnetic moment of the guiding center} \\
\alpha & \equiv & \text{guiding center gyroangle}
\end{array}$$

$$\mathbf{A}^* = \mathbf{A} + \frac{m}{e}\left(\mathbf{u}^{(0)} + W\ \widehat{\mathbf{n}}\right)$$

and the guiding center Hamiltonian

$$H = e\Phi + \mu B + \frac{m}{2}\left|\mathbf{u}^{(0)} + W\ \widehat{\mathbf{n}}\right|^2$$

The energy. Remember $\mathbf{u}^{(0)}$ consists of a 0-order parallel velocity $u_\parallel^{(0)}\widehat{\mathbf{n}}$ plus the electric poloidal velocity of order-0, $\Phi^{(0)}$. To this it is added the velocity $W\widehat{\mathbf{n}}$ which is the *guiding center* particle velocity.

Using the *simplectic* part of the Lagrangian

$$\mathbf{A}^* \cdot d\mathbf{X} + \frac{1}{\Omega_c}\mu B\ d\alpha$$

it is derived a Poisson bracket

$$\begin{aligned}
\{F, G\} &= \frac{\Omega_c}{B}\left[\frac{\partial F}{\partial \alpha}\frac{\partial G}{\partial \mu} - \frac{\partial F}{\partial \mu}\frac{\partial G}{\partial \alpha}\right] \\
&\quad - \frac{1}{eB_\parallel^*}\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} F \times \boldsymbol{\nabla} G \\
&\quad + \frac{1}{mB_\parallel^*}\mathbf{B}^* \cdot \left(\frac{\partial G}{\partial W}\boldsymbol{\nabla} F - \frac{\partial F}{\partial W}\boldsymbol{\nabla} G\right)
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{B}^* &= \boldsymbol{\nabla} \times \mathbf{A}^* \\
&= \mathbf{B} + \frac{B}{\Omega_c}\boldsymbol{\nabla} \times \mathbf{u}^{(0)*}
\end{aligned}$$

$$\begin{aligned}
B_\parallel^* &= \widehat{\mathbf{n}} \cdot \mathbf{B}^* \\
&= B\left(1 + \frac{1}{\Omega_c}\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \times \mathbf{u}^{(0)*}\right)
\end{aligned}$$

$$\mathbf{u}^{(0)*} = \mathbf{u}^{(0)} + W\ \widehat{\mathbf{n}}$$

Using the Poisson bracket, the equation for the guiding center position

$$\begin{aligned}\dot{\mathbf{X}} &= \{\mathbf{X}, H\} \\ &= \frac{1}{eB_{\parallel}^{*}}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} H \\ &\quad + \frac{\mathbf{B}^{*}}{mB_{\parallel}^{*}}\frac{\partial H}{\partial W}\end{aligned}$$

and for the parallel velocity of the guiding center

$$\begin{aligned}\dot{W} &= \{W, H\} \\ &= -\frac{\mathbf{B}^{*}}{mB_{\parallel}^{*}} \cdot \boldsymbol{\nabla} H\end{aligned}$$

The equation for the guiding center moment

$$\dot{\mu} = \{\mu, H\} = 0$$

and for the guiding center gyroangle

$$\dot{\alpha} = \{\alpha, H\} = 0$$

In detail

$$\begin{aligned}\dot{\mathbf{X}} &= \mathbf{u}^{(0)*} + \frac{1}{eB_{\parallel}^{*}}\widehat{\mathbf{n}} \times \left[e\boldsymbol{\nabla}\Phi^{(1)} + \mu\boldsymbol{\nabla}B + m\left(\mathbf{u}^{(0)*} \cdot \boldsymbol{\nabla}\right)\mathbf{u}^{(0)*}\right] \\ &= \mathbf{u}^{(0)*} + \mathbf{V}_D\end{aligned}$$

$$\dot{W} = -\frac{1}{mB_{\parallel}^{*}}\mathbf{B}^{*} \cdot \left[e\boldsymbol{\nabla}\Phi^{(1)} + \mu\boldsymbol{\nabla}B + m\left(\mathbf{u}^{(0)*} \cdot \boldsymbol{\nabla}\right)\mathbf{u}^{(0)*}\right]$$

The velocity $\mathbf{u}^{(0)*}$ consists of a part that is $\mathbf{u}^{(0)}$ and the parallel velocity W of the guiding center in the moving frame

$$\begin{aligned}\mathbf{u}^{(0)*} &= \mathbf{u}^{(0)} + W\ \widehat{\mathbf{n}} \\ &= u_{\parallel}^{(0)}\widehat{\mathbf{n}} + \frac{-\boldsymbol{\nabla}\Phi^{(0)} \times \widehat{\mathbf{n}}}{B}\end{aligned}$$

Further, $\mathbf{u}^{(0)}$ consists of the parallel velocity $u_{\parallel}^{(0)}\widehat{\mathbf{n}}$ plus the poloidal electric velocity due to the potential $\Phi^{(0)}$. The electric field in this zeroth order, $-\boldsymbol{\nabla}\Phi^{(0)}$ is related with the velocity and the magnetic field by

$$\boldsymbol{\nabla}\Phi^{(0)} = \mathbf{u}^{(0)*} \times \mathbf{B}$$

The convective part of the inertial term is

$$
\begin{aligned}
\left(\mathbf{u}^{(0)*}\cdot\boldsymbol{\nabla}\right)\mathbf{u}^{(0)*} &= \left[\left(\mathbf{u}^{(0)}+W\widehat{\mathbf{n}}\right)\cdot\boldsymbol{\nabla}\right]\left(\mathbf{u}^{(0)}+W\widehat{\mathbf{n}}\right) \\
&= \left(\mathbf{u}^{(0)}\cdot\boldsymbol{\nabla}\right)\mathbf{u}^{(0)} \quad \text{centrifugal} \\
&\quad +W^2\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}} \quad \text{acceleration by curvature} \\
&\quad +W\ \ \nabla_{\|}\mathbf{u}^{(0)} \qquad\qquad (3) \\
&\quad +W\ \left(\mathbf{u}^{(0)}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}} \qquad\quad (4)
\end{aligned}
$$

The terms (3) and (4) can be written

$$
\begin{aligned}
& W\ \ \nabla_{\|}\mathbf{u}^{(0)}+W\ \left(\mathbf{u}^{(0)}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}} \\
= \ & 2W\ \ \nabla_{\|}\mathbf{u}^{(0)} \quad \text{Coriolis} \\
& -W\ \ \nabla_{\|}\mathbf{u}^{(0)}+W\ \left(\mathbf{u}^{(0)}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}
\end{aligned}
$$

The last two terms are

$$
\begin{aligned}
& -W\ \ \nabla_{\|}\mathbf{u}^{(0)}+W\ \left(\mathbf{u}^{(0)}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}} \\
= \ & -W\ \widehat{\mathbf{n}}\ \left(\mathbf{u}^{(0)}\cdot\boldsymbol{\nabla}\right)\ln B - W\widehat{\mathbf{n}}\ \left(\boldsymbol{\nabla}\cdot\mathbf{u}^{(0)}\right)
\end{aligned}
$$

10 Poloidal rotation

The drift turbulence rotates poloidally
in the **electron diamagnetic** direction

With **B** pointing away along the line of sight (*i.e.* into the page) , the drift turbulence rotates in the clockwise direction.

The rotation of plasma along the magentic field lines is NOT uniform, it is modulated like the magnitude of the magnetic field. For this reason it is defined

$$U=\frac{u_{\|}}{B}$$

Note also that the neoclassical intrinsic flows, at equilibrium, in plasma exist

- without resistivity: *hose-like*; poloidal, toroidal, zero radial; with $\cos\theta$ variation in the surface, for u, ρ, v and for the currents ; no need to mention the diamagnetic flow but a poloidal rotation at equilibrium MUST be assumed.

- with resistivity: the radial velocity appears $\sim \eta$. Pfirsch Schluter

see below and **Stringer.tex, equilibrium_flows.tex**.

NOTE

About the connection between the poloidal and toroidal rotations. From several (Russian) sources and **Rozhansky Tendler**.

Main contribution to the radial transport comes from particles with poloidal velocities

$$V_E + \Theta V_{\|} \approx 0$$

where $V_E = (-d\phi/dr)/B$ is the poloidal velocity due to the radial electric field. $\Theta \equiv B_\theta/B_T$. This relation is on the poloidal direction. *When the poloidal electric velocity V_E increases the parallel velocity $V_{\|}$ must also increase and with such a sign to maintain $V_E + \Theta V_{\|} \approx 0$.* This can be so important that a number of trapped particles actually become circulating. There are two consequences:

- there is a possible *instability*. This comes from the fact that any conversion from trapped to circulating leads to a change in the position of the centers of the trajectories and then to a radial current. This radial current may enhance the poloidal rotation and the process is amplified
- the number of trapped particle decreases, which makes a smaller *bootstrap* current

NOTE that there is physical mechanism that connects the two rotations, V_E and $V_{\|}$. We just say that it is necessary.

10.1 Main ion and impurity rotation Kim 1994

The bootstrap current is carried by both electron and ion species roughly at an equal amount.

The parallel flow

$$\begin{aligned} n_i u_{\|i} &= -n_i I \frac{1}{B}\frac{\partial \Phi}{\partial \psi} \quad \text{electric} \\ &\quad -\frac{B}{\langle B^2\rangle}\frac{I}{e_i S}\frac{\partial p}{\partial \psi} \quad \text{projection of diamagnetic} \\ &\quad -\left(1-\frac{B^2}{\langle B^2\rangle}\right)\frac{I}{e_i B}\frac{\partial p_i}{\partial \psi} \quad \text{Pfirsch Schluter} \end{aligned}$$

and the poloidal flow

$$u_\theta = \frac{B_\theta}{\langle B^2\rangle}\frac{I}{e_i n_i}\left(1-\frac{1}{S}\right)\frac{\partial p_i}{\partial \psi}$$

NOTE
For

$$I = RB_{tor}$$
$$|\boldsymbol{\nabla}\psi| = 2\pi \ RB_\theta$$

then

$$\begin{aligned} I\frac{\partial}{\partial\psi} &\rightarrow \frac{B_{tor}}{B_\theta}\frac{\partial}{\partial r} \\ &= (\text{projection on } \|) \times \frac{\partial}{\partial r} \end{aligned}$$

END

For a trace impurity

$$u_\theta^I = \frac{1}{2}v_{th,i}\rho_i\left(1-\frac{1}{S}\frac{1}{L_{p_i}}+\frac{Z_i}{T_i}\frac{T_I}{Z_I}\frac{1}{L_{p_I}}\right)\frac{B\ B_{tor}}{\langle B^2\rangle}$$

The difference in the toroidal velocity, between ions and impurities

$$u_\varphi^i - u_\varphi^I = \frac{1}{2}v_{th,i}\rho_{\theta i}\left(-\frac{1}{L_{p_i}}+\frac{Z_i}{T_i}\frac{T_I}{Z_I}\frac{1}{L_{p_I}}\right)\left(1-\frac{I^2}{R^2\langle B^2\rangle}\right)$$

It is a large difference between the two toroidal velocities of He^{++} and of C^{+6}.

10.2 Hinton Kim Kim Brizard Burrell poloidal rotation 1995

This is interesting only because the flow of ions is sustained by the trapped ions, from unbalanced local fluxes arising from density gradients.
In *bootstrap.tex.*

From **Hinton Kim poloidal rotation**
Strong electric field. Squeezing S is important.

"measurements show that, shortly
after the low (L)- to H-mode transition, the main ions are
rotating in the ion diamagnetic direction, at a speed that is
somewhat less than the ion diamagnetic speed."
The impurities rotate in opposite direction.

The direction of ion poloidal rotation is that which is imposed by the trapped ions, via the imbalance of fluxes in the presence of a gradient of pressure (diamagnetism of trapped ion bananas, projected on the poloidal section).

In Theory: Poloidal rotation driven by loss of ions should be in the electron diamagnetic direction.

In the paper **Hinton Kim Kim poloidal rotation 1995**
"Consider the detailed balance of momentum transfers in
collisional trapping and detrapping. A steady state requires
that momentum be gained by the trapped ions in the process
of collisional trapping at the same rate that it is removed by
collisional detrapping:"

$$n_{untr} u_{\|untr} = \left(\frac{\nu_{tr}}{\nu_{untr}} \right) n_{tr} \; u_{\|tr}$$

Note see also **Cordey**. **End.**

NOTE
This equilibrium is only interesting for the two populations of ions:

- trapped, and
- untrapped

And the equilibrium just expresses preservation of the total momentum.

The events that bring circulating ions into trapped state bring (to the population of trapped ions) a momentum per unit of time

$$n_{untr} u_{\|untr} \; m_i \times \nu_{untr}$$

The events that convert a trapped ion into a circulating one will remove from the "total momentum" of the population of trapped ions an amount of moment, per unit of time

$$n_{tr} u_{\|tr} \; m_i \times \nu_{tr}$$

For equilibrium, the total momentum "contained" in the population of trapped ions must remain constant.

Then, what is introduced must be equal with what is taken off, per unit of time

$$n_{untr} u_{\parallel untr}\ m_i\ \nu_{untr} = n_{tr}\ u_{\parallel tr}\ m_i\ \nu_{tr}$$

END

NOTE

However here we have two different processes.

One is the conversion of a circulating ion into a trapped state. This is done by collision. At this event a certain amount of momentum, per unit time, is brought to the population of trapped ions, $n_{untr} u_{\parallel untr}\ m_i \times \nu_{untr}$.

The other process is the collisional transfer of momentum from the trapped ions to the circulating ions, but NOT necessarily a conversion from trapped to circulating. In this case the mechanism is just like the bootstrap. And there is loss of momentum from the population of trapped ions: or, relaxation of the gradient of pressure of the trapped particles.

Then the balance expressed by $n_{untr} u_{\parallel untr}\ m_i\ \nu_{untr} = n_{tr}\ u_{\parallel tr}\ m_i\ \nu_{tr}$ is actually a balance of fluxes of momentum at the *boundary trapped/circulating* in the velocity space.

[not clear why this should be in equilibrium]

This balance does NOT mention a gradient of pressure.

It is simply a question of fluxes across the boundary in velocity space separating trapped from circulating.

But

we have ONE mechanism of input of momentum from circulating to trapped, i.e. at an event of conversion of a circulating into trapped, and

we have TWO mechanisms for loss of momentum from the population of trapped particles:

(1) by collisional conversion of a trapped ion into a circulating one (i.e. across the boundary in velocity space), and

(2) by the collisional transfer of momentum from the "unbalanced flows of bananas due to a gradient of pressure" to a circulating particle, i.e. by relaxation of the gradient of pressure.

END

NOTE

This balance is similar to the formula used by **Cordey** to explain the bootstrap current. There, the momentum gained by circulating electrons by collisions from the trapped ions is further saturated (i.e. controlled) by collisional friction with background ions.

END

NOTE

However in ELMs the population of trapped ions increases.

The layer of poloidal velocity is destroyed, which means that the parallel flow is also reduced. The parallel flow was carried by the circulating ions, and this was partly due to the momentum they received collisionally from the trapped ions. Many circulating ions will be converted into trapped ions (decrease of their parallel velocity). The amount of momentum that is injected this way in the population of trapped ions is high.

It is not possible to the opposite process, of conversion from trapped to circulating, - to be equally efficient.

END

The ratio of collision frequencies

$$\frac{\nu_{tr}}{\nu_{untr}} = \left(\frac{v_{th,i}}{\delta u_{\|}}\right)^2$$

$$\begin{aligned}
\delta u_{\|} &= (\varepsilon S)^{1/2} \ v_{th,i} \\
&= \text{range of parallel velocities} \\
&\quad \text{for the trapped particles}
\end{aligned}$$

Counting the flux of trapped particles in a point, from the two directions

$$\begin{aligned}
n_{tr} \ u_{\| tr} &= \frac{1}{2}\left[n\left(r-\frac{\delta r}{2}\right)-n\left(r+\frac{\delta r}{2}\right)\right] \\
&\quad \times \delta u_{\|} \\
&= -\frac{1}{2} \ \delta r \ \delta u_{\|} \ \frac{\partial n}{\partial r}
\end{aligned}$$

Here δr is taken the banana width

$$\delta r = \sqrt{\frac{\varepsilon}{S}} \frac{v_{th,i}}{\Omega_\theta}$$

"identifying the mean ion flow with the untrapped ion flow,"

$$\left(n u_{\|}\right)_{untr} = -\frac{v_{th,i}^2}{2S\Omega_\theta} \frac{\partial n}{\partial r}$$

Note the mean ion flow is what really can move, the other being fixed on bananas. They are exclusively the untrapped ions. **End.**

See also *bootstrap.tex*.

The calculation was done in a referential that move with $\frac{E_r}{B_\theta}$ velocity in toroidal direction, reasonable choice for discussing banana motions.

Now we return to the laboratory frame

$$nu_\parallel = n\frac{E_r}{B_\theta} - \frac{1}{S}\frac{T}{eB_\theta}\frac{\partial n}{\partial r}$$

Further, one uses

$$u_\theta = u_\perp + \frac{B_\theta}{B}u_\parallel$$

and

$$u_\perp = -\frac{E_r}{B} + \frac{1}{n_i}\frac{1}{e_i B}\frac{\partial p_i}{\partial r}$$

it results

$$u_\theta = \frac{1}{n_i}\frac{T_i}{e_i B}\left(1 - \frac{1}{S}\right)\frac{\partial n_i}{\partial r}$$

"In DIII-D H-mode plasmas, the resulting
ion poloidal flow velocity is predicted to be roughly half the
ion diamagnetic velocity near the separatrix,"

NOTE
the persistent experimental observation is that
- the plasma rotation is in the ion diamagnetic direction
- the ion poloidal flow is smaller (half) of the diamagnetic rotation
END

NOTE

This work does NOT assume the presence of a mechanism that supports E_r.

The parallel motion of ions is strictly due to the collisional transfer of momentum from the trapped ions to the circulating ions.

This transfer is such that the unbalanced flux of trapped ions (as results from the gradient of density of trapped ions) is fully transferred collisionally to the circulating ions, with corresponding collision ferquencies.

This provides an expression for $u_{\parallel i}$ in terms of $\frac{\partial n_i}{\partial r}$.

The radial electric field contributes (however) to the parallel flow (first the calculations are done in a referential moving with the parallel electric velocity, then return to laboratory).

But when one considers the projection of $u_{\parallel i}$ such as to obtain u_θ, the electric field disappears. This is because the perp component of flow contains already E_r/B but with opposite sign.

In conclusion, here the poloidal flow u_θ does NOT depend on the radial electric field.

What is the meaning of that cancellation of the electric field ?

This means that as much E_r offers a poloidal flow, the same amount is lost by the projection of the E_r contribution to parallel flow. The usual constraint to exclude effective poloidal rotation, since it is damped. What however remains on θ is the diamagnetic flow.

This is what **Hinton Kim** find at the end.

The poloidal rotation is like diamagnetic, but contains a factor S and is suppressed if $S = 1$.

There is NO poloidal rotation in this treatment.

See **Peeters bootstrap**.

END

10.3 Electric field separatrix Kim Hinton 1994

The formula

$$\widehat{\mathbf{n}} \times \boldsymbol{\nabla}\psi = I\widehat{\mathbf{n}} - RB\,\widehat{\mathbf{e}}_\varphi$$

Then neglecting

$$\frac{\partial p_i}{\partial \theta} \text{ and } \frac{\partial \phi}{\partial \theta}$$

i.e. the variation in surfaces of the pressure and of the electric potential, an approximation of the perpendicular velocity is

$$\begin{aligned}
\mathbf{u}_{i\perp} &= \frac{1}{B}\widehat{\mathbf{n}} \times \left(\frac{1}{n_i e}\boldsymbol{\nabla} p_i + \boldsymbol{\nabla}\phi\right) \\
&\approx \omega\left(-\widehat{\mathbf{n}}\,\frac{I}{B} + R\,\widehat{\mathbf{e}}_\varphi\right)
\end{aligned}$$

where

$$\omega = -\left(\frac{1}{n_i e}\frac{\partial p}{\partial \psi} + \frac{\partial \phi}{\partial \psi}\right)$$

The final form is

$$\left[(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla})\ \widehat{\mathbf{n}}\right] \cdot \mathbf{u}_{i\perp} \approx \omega\ I\ \ (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \left(\frac{1}{B}\right)$$

Comment.
We must calculate

$$\begin{aligned} & \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\ \widehat{\mathbf{n}} \cdot \mathbf{u}_{i\perp} \\ = & \ \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\ \widehat{\mathbf{n}} \cdot \ \left[\omega \left(-\widehat{\mathbf{n}}\ \frac{I}{B} + R\ \widehat{\mathbf{e}}_{\varphi} \right) \right] \end{aligned}$$

In *reference* we find

$$\begin{aligned} & (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}} \cdot \widehat{\mathbf{e}}_{\varphi} R \\ = & \ \boldsymbol{\kappa} \cdot \widehat{\mathbf{e}}_{\varphi} R \quad \text{(toroidal projection of curvature)} \\ = & \ I \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \left(\frac{1}{B} \right) \end{aligned}$$

This suggests that the writting is conform to the following group of contraction

$$\begin{aligned} & \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\ \widehat{\mathbf{n}} \cdot \ \left[\omega \left(-\widehat{\mathbf{n}}\ \frac{I}{B} + R\ \widehat{\mathbf{e}}_{\varphi} \right) \right] \\ = & \ \boldsymbol{\kappa} \cdot \left(-\omega \widehat{\mathbf{n}}\ \frac{I}{B} \right) + \boldsymbol{\kappa} \cdot (\omega R\ \widehat{\mathbf{e}}_{\varphi}) \\ = & \ 0 + \omega\ I\ \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \left(\frac{1}{B} \right) \end{aligned}$$

and this gives the final result, conform to **electric field separatrix Kim**

$$(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\ \widehat{\mathbf{n}}) \cdot \mathbf{u}_{i\perp} \approx \omega\ I\ \ (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \left(\frac{1}{B} \right)$$

This is the projection of the perpendicular velocity of ions on the curvature vector

$$\boldsymbol{\kappa} \cdot \mathbf{u}_{i\perp}$$

The writting is ambiguous. We have four vector and we obtain a scalar. Then there are two contractions. If there is contraction between the first $\widehat{\mathbf{n}}$ and the $\widehat{\mathbf{n}}$ after the nabla, then this is zero

$$\begin{aligned} \overset{\downarrow}{\widehat{\mathbf{n}}} \cdot \boldsymbol{\nabla} \overset{\downarrow}{\widehat{\mathbf{n}}} \cdot \mathbf{u}_{i\perp} \ \rightarrow & \ \widehat{n}_k\ \partial_j \widehat{n}_k\ u_{i\perp,j} \\ = & \ (\widehat{n}_k\ \partial_j \widehat{n}_k)\ \ u_{i\perp,j} + \widehat{n}_k\ \widehat{n}_k\ \ (\partial_j u_{i\perp,j}) \\ = & \ 0 + \boldsymbol{\nabla} \cdot \mathbf{u}_{i\perp} = \boldsymbol{\nabla} \cdot \left[\omega \left(-\widehat{\mathbf{n}}\ \frac{I}{B} + R\ \widehat{\mathbf{e}}_{\varphi} \right) \right] \end{aligned}$$

the first term

$$\begin{aligned}\widehat{n}_k \, \partial_k \widehat{n}_j \, u_{i\perp,j} &\sim \widehat{n}_k \, \partial_k \, \widehat{n}_j \left[\omega \left(-\widehat{n}_j \frac{I}{B} \right) \right] \\ &= -\omega \, I \, \left(\widehat{n}_k \, \partial_k \right) \left(\frac{1}{B} \right)\end{aligned}$$

END Comment

After neglecting $\nabla_{\|}\omega$, it is obtained the divergence of the perpendicular velocity of the ions

$$\boldsymbol{\nabla} \cdot \mathbf{u}_{i\perp} \approx -2 \, \omega \, I \, \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \right) \left(\frac{1}{B} \right)$$

10.4 Radial electric field LH Groebner DIII

From the text

"For discharges with the toroidal field B_T in the standard direction, which is clockwise as viewed from the top of the tokamak, $v_\perp$ is directed up at the outside edge of the plasma. This geometry implies that the $\mathbf{V} \times \mathbf{B}$ term of Eq. (1) makes a negative contribution to E_r"

10.5 Poloidal rotation Turnianski

Text

"For the discharge of interest the maximum diamagnetic drift velocity is estimated to be 40 km/s and in the opposite direction to the $E \times B$ drift velocity."

Increase in v_θ is observed immediately after an ELM terminates and falls more rapidly back to the initial velocity around the end of an inter-ELM period.

The v_θ attains its maximum just before the ELM. ~ -9 km/s.

"Simulations show that the diamagnetic drift veloc-

ity contribution to the observed poloidal velocity is at most 2 km/s in the same direction as the observed rotation but small compared with that at the end of the inter-ELM period."

10.6 Rotation counter current Alcator C Rice

"The internal inductance and core rotation velocity both dropped at first, but as the MHD activity developed at 0.93 s, as seen on the magnetics trace, the rotation velocity halted and there was an increase in l_i."

10.7 Poloidal rotation Stacey

The paper is **poloidal rotation Stacey 2002.**

The equations
Continuity for j,

$$\mathbf{\nabla} \cdot (n_j \mathbf{v}_j) = 0$$

Momentum cnservation in stationarity

$$\begin{aligned} n_j m_j (\mathbf{v}_j \cdot \mathbf{\nabla}) \mathbf{v}_j &= -\mathbf{\nabla} p_j - \mathbf{\nabla} \cdot \boldsymbol{\pi}_j \\ &\quad + e_j n_j \mathbf{\nabla} \phi + e_j n_j \mathbf{v}_j \times \mathbf{B} \\ &\quad + \mathbf{M}_j + \mathbf{R}_j \\ &\quad - m_j S_j \mathbf{v}_j \end{aligned}$$

where

$$\mathbf{M}_j \equiv \text{external momentum source for species } j$$

$$\begin{aligned} \mathbf{R}_j &\equiv \text{interspecies collisional momentum transfer} \\ &= -n_j m_j \sum_{k \neq j} \nu_{jk} \ (\mathbf{v}_j - \mathbf{v}_k) \qquad \text{(Lorentz collisions)} \end{aligned}$$

$$S_j \equiv \text{particle source for species } j$$

The neoclassical parallel viscosity tensor
using neoclassical parallel viscosity coefficient in banana-plateau

$$\eta_{0j} = n_j m_j v_{th,j} q R \ \varepsilon^{-3/2} \frac{\nu_{jj}^*}{\left(1 + \varepsilon^{-3/2} \nu_{jj}^*\right) \left(1 + \nu_{jj}^*\right)}$$

where the normalized collision frequency

$$\nu_{jk}^{*} = \frac{\nu_{jk}}{v_{th,j} / (qR)}$$

Since we intend to project the momentum equation on the poloidal diraction, we will calculate

the poloidal component of the divergence of the parallel viscosity tensor

$$\widehat{\mathbf{e}}_{\theta} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_{j} = \eta_{0j} \left(\frac{1}{2} A_{0j} \right) \left[\frac{\partial}{r \partial \theta} \ln \left(\eta_{0j} A_{0j} \right) - 3 \frac{\sin \theta}{R} \right]$$

where

$$\begin{aligned} \frac{1}{2} A_{0j} &= -\frac{1}{3} \frac{\partial}{r \partial \theta} v_{\theta j} \\ &\quad + v_{\theta j} \left(\frac{\partial}{r \partial \theta} \ln R + \frac{1}{3} \frac{\partial}{r \partial \theta} \ln B_{\theta} \right) \\ &\quad + \left(\frac{B_{\theta}}{B_{\varphi}} \right) R \, \frac{\partial}{r \partial \theta} \left(\frac{v_{\varphi j}}{R} \right) \end{aligned}$$

The toroidal velocity results from the *radial* projection of the momentum equation (like $j_r = 0$)

$$\begin{aligned} v_{\varphi,j} (r, \theta) &= \left(\frac{B_{\varphi}}{B_{\theta}} \right) v_{\theta,j} (r, \theta) \\ &\quad + \frac{1}{B_{\theta}} \left[-\frac{\partial \phi (r, \theta)}{\partial r} \right] \\ &\quad + \frac{1}{n_j} \frac{1}{e_j B_{\theta}} \left[-\frac{\partial p_j (r, \theta)}{\partial r} \right] \end{aligned}$$

The average

$$\langle \boldsymbol{\nabla} \cdot n_j \mathbf{v}_j \rangle = \langle S_j \rangle$$

This is subtracted from the equation of continuity

$$\begin{aligned} & \frac{\partial}{r \partial \theta} \left[(1 + \varepsilon \cos \theta) \;\; n_j v_{\theta,j} \right] \\ = & \;\; (1 + \varepsilon \cos \theta) \left(S_j - \langle S_j \rangle \right) \end{aligned}$$

Integrated in θ this equation it must be added a constant

$$K_j (\theta)$$

Then

$$\begin{aligned} & v_{\theta,j}(r,\theta) \\ = \; & \frac{1}{1+\varepsilon\cos\theta}\overline{n}_j\ \overline{v}_{\theta,j} \\ & +\frac{1}{1+\varepsilon\cos\theta}\int_0^{\theta} d\theta'\ \left(1+\varepsilon\cos\theta'\right)\left[S_j(r,\theta)-\overline{S}_j(r,\theta)\right] \end{aligned}$$

Now

$$\begin{aligned} & \widehat{\mathbf{e}}_\theta\cdot\left[\left(\mathbf{v}_j\cdot\boldsymbol{\nabla}\right)\mathbf{v}_j\right] \\ \approx \; & \frac{1}{2}\frac{\partial}{r\partial\theta}v_{\theta,j}^2+\frac{v_{\varphi,j}^2}{R}\sin\theta \end{aligned}$$

It is assumed the following form

$$\begin{aligned} & n_j(r,\theta) \\ = \; & 1+n_j^c(r)\cos\theta \\ & \quad +n_j^s(r)\sin\theta \\ & +... \end{aligned}$$

Moments

$$\left\langle\frac{1}{n_j m_j}\frac{1}{1+\varepsilon\cos\theta}\left(\widehat{\mathbf{e}}_\theta\cdot[\text{momentum eq.}]\right)\ X\right\rangle$$

for

$$\begin{aligned} X \; = \; & 1, \\ & \sin\theta, \\ & \cos\theta \end{aligned}$$

11 Experiments and observations

The paper **core flows transitions lebschy 2017.**

"The core poloidal rotation of the plasma around mid-radius is found to be always in the

ion diamagnetic direction, in disagreement with neoclassical (NC) predictions. The edge *rotation is found to be electron-directed and consistent with NC codes."*

"This paper shows
that the reversal of the core toroidal rotation occurs clearly after the LOC–SOC transition and
concomitant with the peaking of the electron density."

$$\begin{array}{rcl} LOC & \rightarrow & TEM \\ SOC & \rightarrow & ITG \end{array}$$

"The phase velocity is directed in the electron diamagnetic direction for TEM and in the ion diamagnetic direction for ITG turbulence and v_{ph} is, therefore, indicative of the dominant turbulent mode and a change in the sign of v_{ph} would be expected assuming that there is a change from TEM to ITG."

"Ohmic L-mode discharges feature another interesting phenomenon: the intrinsic core toroidal is observed to spontaneously reverse from the co- to the counter-current direction when the electron density is increased. In this paper, intrinsic rotation refers to the toroidal rotation of the plasma established in the absence of externally applied torque"

In the paper **Advanced Simulation Hayashi JT60upgrade** it is mentioned that Tungsten accumulates in the center in the case of toroidal rotation in the *counter-current* direction, which is in general associated to L-mode. It is also proposed an explanation for the high Z inward pinch, based on the fact that the toroidal rotation is transmitted to them via friction, they increase the energy and hence the orbit width (departure from the magnetic surface) they traverse zones with different electron temperatures and change the average degree of ionization (effective Z) and the toroidal velocity is inversely proportional with the charge. Therefore it appears an inward pinch.

(But, other people seem to say that the accumulation of impurities in the center is typical for H-mode. To check).

See also **Isler**:

the influx of Fe toward the centre is very fast in NBI *counter-injection* , i.e. in the direction contrary to the current. Plasma begins to cool in the centre rapidly.

From the paper **Rotation Ohmically Heated Tokamak TEXT**.

The toroidal rotation velocity at the perifery in He is much higher than the toroidal rotation velocity in H. This looks like a *atomic mass* effect.

The ion thermal velocity was

$$v_{thi} = 90 \ \ (km/s)$$

At the plasma perifery the rotation is in the direction of I_p, the plasma current. *Co-current*, as in H mode of **Alcator C-mod Rice.**

In the center, the toroidal rotation is opposite to I_p. (*Counter* like in the case of Alcator in L mode)

This reminds the problem of **DIPOLE** flows. The cases:

1. Two-dimensional flow consisting of two lobes of flow, the upper monopole of one sign and the lower monopole of opposite sign. This structure is unstable and can lead to a new *dipolar* arrangement, where there are two monopoles: one compact in the center of the disk and the other one ring-type at the periphery of the disk, with opposite direction of flows. This time it is question of dipolar *toroidal* flow. Another example is dipolar poloidal flow.

2. Three-dimensional flow in a toroidal geometry consisting of a flow occupying half-torus and having one sign and the other half-torus being occupied by a flow with opposite direction. The configuration of flow is discovered by **Montgomery** for a *no-slip* boundary condition of the flow with viscosity (MHD equilibria in tokamak with poloidal flow and viscosity). This configuration is unstable and leads to a new distribution of flows in the torus with a different structure: in the center there is flow on a compact torus and having one direction and in the outer part of the original torus there is a flow with a geometry of toric-shell and having opposite direction. The transition from the first state to the second takes place by a large scale motion where one of the flows is captured inside the other. The original state is similar to what we have from Pfirsch-Schluter currents in tokamaks. The PS distribution is *harmonic* due to the $\sin\theta$ term in the toroidal flow that accompanies the poloidal flow.

In the paper **Space potential in the tokamak TEXT** PFB3 (1991) 3448 there is a description of the experiments of measuring the radial profile of the electrostatic potential in TEXT using Heavy Ions Beam Probe HIBP.

They mention the experiments done on **TM4** by Russians where they see a negative potential in the plasma centre and a small region of positive potential at the edge. They invoke Ware pinch for electrons toward the centre and a loss of ions from the local bananas to the limiter. However there is a region of positive φ at the edge and the width of this region is larger when the *density* is increased. They invoke the *intrinsic stochasticity* as a source of loss of electrons from that region, leading to a higher density for the *ions.*

The conclusion for TEXT. In the central region, with $\varphi < 0$:

Normal Ohmic discharges are approximately explained by ion momentum balance equation using an assumed neoclassical *poloidal* rotation velocity and **NO** significant toroidal rotation velocity.

In **TEXT**

$$\begin{aligned} E_r &= -\frac{\partial \varphi}{\partial r} \\ &= \frac{1}{|e| n_i} \frac{\partial p_i}{\partial r} + \langle v_{Ti} B_\theta \rangle - \langle v_{\theta i} B_T \rangle \end{aligned}$$

where $p_i = n_i T_i$ is the ion pressure and $\langle \rangle$ is over the flux surface. For TEXT the following inequalities exist

$$\langle v_{Ti} B_\theta \rangle \ll \langle v_{\theta i} B_T \rangle \ll \frac{1}{e n_i} \frac{\partial p_i}{\partial r}$$

Then the toroidal velocity part can be neglected. Strangely enough, the *diamagnetic* velocity is higher than the poloidal rotation velocity.

The poloidal rotation is assumed to be *neoclassical*

$$v_{\theta i} = -g \frac{1}{e B_T} \frac{\partial T_i}{\partial r}$$

where g depends on collisionality,

$$g \simeq \frac{1}{2} \text{ for plateau}$$

see **Hazeltine** for PS and banana, see **Ware Wiley** for the demonstration of the stationarity of poloidal rotation in neoclassics.

Then

$$E_r = \frac{T_i}{e n_i} \frac{\partial n_i}{\partial r} \left[1 + \eta_i \left(1 + g \right) \right]$$

where

$$\eta_i = \frac{d \ln \left(T_i \right)}{d \ln \left(n_i \right)} = \frac{L_{n_i}}{L_{T_i}}$$

Predictions for the *central ion temperature* according to Artsimovich

$$T_i^{Arts} \left(0 \right) = \kappa \frac{\sqrt[3]{B_T I_p \overline{n}_e R^2}}{\sqrt{A_i}}$$

where

$$\begin{aligned}
\kappa &= 2.8 \times 10^{-6} \\
[B_T] &= \text{tesla} \\
[R] &= m \\
[I_p] &= \text{amperes} \\
[\overline{n}_e] &= m^{-3} \\
\left[T_i^{Arts}(0)\right] &= eV
\end{aligned}$$

Then the potential in the centre is

$$\varphi(0) = -(2.5 \pm 0.5) \frac{T_i^{Arts}(0)}{e}$$

It is applied in TEXT a Ergodic Magnetic Limiter effect. The electric field at the edge becomes more positive, on a wider radial interval.

The theoretical value of the expected radial electric field that is determined in a region of magnetic stochasticity in a collisonless plasma is

$$E_r^{st} = \frac{T_e}{e} \frac{\partial \left[\ln\left(n_e \sqrt{T_e}\right)\right]}{\partial r}$$

See also **Rozhansky Tendler**.

In the stochastic region, the full *magnetic island width* is

$$w_i = 2\sqrt{\frac{4q^2 |b_{mn}| R}{m B_T (\partial q/\partial r)}}$$

and the *distance* between adiacent island centers w_s separated in q by Δ_q is

$$w_s = \frac{\Delta_q}{\partial q/\partial r}$$

Condition of reaching stochsaticity (defined as overlap of magnetic islands)

$$\frac{|b_{rmn}|}{B_T} \geq \frac{r_{mn}}{8Rmq\left(\beta + \sqrt{\beta}\right)^2}$$

To calculate the radial component of the magnetic field

$$b_r = \sum_{m,n} (b_{rmn})_{r=r_{mn}} \left(\frac{r}{r_{mn}}\right)^{-(m+1)} \cos(m\theta + n\phi + \delta_{mn})$$

where δ_{mn} are random phases.

From the paper **Stability Radial Electric field Shaing PF27 (1984) 1567.**

The equation for E_r can be obtained from the *surface average* of the *radial* component of the Ampère law:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

where

$$\varepsilon_0 \mu_0 = \frac{1}{c^2},$$

but with $\mathbf{E}$ coming from an electrostatic potential.

$$\frac{\partial}{\partial t} \langle \mathbf{E} \cdot \nabla \psi \rangle = -\frac{1}{\varepsilon_0} \langle \mathbf{j} \cdot \nabla \psi \rangle$$

This is because the radial component of the rotational of the magnetic field $(\nabla \times \mathbf{B})|_r$ is zero. The radial current $\langle \mathbf{j} \cdot \nabla \psi \rangle$ consists of two pieces:

1. the <u>conduction</u> current, or the *non-ambipolar* current, driven by the gradients of pressure, temperature, electrostatic potential

$$\underbrace{\sum_a e_a \langle \mathbf{\Gamma}_a \cdot \nabla \psi \rangle}_{\text{radial currents}}$$

2. the <u>polarization</u> current, driven by the time-varying radial electric field:

$$\sum_a \frac{n_a m_a}{B^2} \frac{\partial}{\partial t} \langle \mathbf{E} \cdot \nabla \psi \rangle$$

The equation is

$$\frac{\partial E_r}{\partial t} = \frac{(1/\varepsilon_0) |\nabla \psi|^{-2}}{1 + c^2/v_A^2} \sum_a e_a \langle \mathbf{\Gamma}_a \cdot \nabla \psi \rangle$$

The equation cannot be purely local, *i.e.* connected strictly to each magnetic surface.

The radial excursion of the particles due to their drifts extends over several surfaces and this *finite-size orbit* effect induces the diffusion of the radial electric field

$$\begin{aligned}\frac{\partial E_r}{\partial t} &= \frac{(1/\varepsilon_0)\left|\boldsymbol{\nabla}\psi\right|^{-2}}{1+c^2/v_A^2}\sum_a e_a \left\langle \boldsymbol{\Gamma}_a\cdot\boldsymbol{\nabla}\psi\right\rangle \\ &+\frac{1}{V'}\frac{\partial}{\partial\psi}\left(V'D\frac{\partial E_r}{\partial\psi}\right)\end{aligned}$$

where D is a diffusion coefficient.

It should be written

$$\begin{aligned}\frac{\partial E_r}{\partial t} &= \frac{1}{\varepsilon_0\left(1+\frac{c^2}{v_A^2}\right)}\frac{1}{\left|\boldsymbol{\nabla}\psi\right|^2}\sum_a e_a \left\langle \boldsymbol{\Gamma}_a\cdot\boldsymbol{\nabla}\psi\right\rangle \\ &+\frac{1}{V'}\frac{\partial}{\partial\psi}\left(V'D\frac{\partial E_r}{\partial\psi}\right)\end{aligned}$$

This is also in **Hastings**, for stellerators, in general non-axisymmetric devices.

See the paper of **Hinton and Robertson** on the neoclassical polarization drift that modifies the factor

$$1+\frac{c^2}{v_A^2} \text{ to } 1+\frac{c^2}{v_{A\theta}^2}$$

which is much higher. The time variation of the radial electric field $\dot{E}_r$ leads to radial velocity of the trapped ions $\overline{v}_r$. The *untrapped* ions are NOT moving radially, in the average over their transit motion.

In the paper **ChangeEr NBI Burrell** (see above) it is mentioned an experiment where the fast neutrals of 75 KeV have been injected in a plasma, *counter* to I_p (like in L-mode). In very short time, within their first banana orbit (much less than 1 ms), 38% of these ions have been lost to the border.

The *counter-NBI* is contemplated as a method to change the radial electric field E_r to facilitate the transition LH. This is because it was considered that the cause of the L to H transition was the loss of ions from the large bananas, to the limiter (see also **Shaing and Crume**).

The colisionless loss of fast ions inforce an *inward* flow of ions to compensate for the lost charge.

The equation

$$E_r = -v_{\theta i}B_\varphi + v_{\varphi i}B_\theta + \frac{1}{Z_i n_i\left|e_i\right|}\frac{dp_i}{dr}$$

is valid for *any* ion (bulk or impurity) separately.

When the **NBI** is counter-current (like L-mode) the term with the pressure gradient (*diamagnetic*) and the term with the poloidal magnetic field (containing the toroidal velocity) are negative

$$\begin{aligned}\frac{1}{Z_i n_i |e_i|}\frac{dp_i}{dr} &< 0 \\ v_{\varphi i} B_\theta &< 0\end{aligned}$$

which means that they contribute to the radial electric field with *negative* amounts

$$E_r < 0$$

Very little orbit loss in the co-injection case.

Co-injection (like in H-mode) means that NBI ions are sent parallel with the plasma current I_p, which means contrary to the motion of electrons that carry the current.

The profile of the toroidal velocity is affected by the momentum diffusion and by the charge-exchange collisional damping. [See **Fulop Catto Helander** for this effect.]

The torque applied to plasma in the <u>toroidal</u> direction due to the radial loss of ion (on fast orbits) is calculated as

$$J_r^{orbit} B_\theta R dR$$

and the evolution of the torque is given by the equation

$$\frac{\partial}{\partial t}(n_i U_\varphi) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\chi n_i \frac{\partial U_\varphi}{\partial r}\right) + F_{nb} + F_{cx}$$

where χ is the diffusivity of the toroidal momentum.

The force produced by the *charge-exchange* with the neutrals is

$$\begin{aligned}F_{cx} &= n_i U_\varphi \nu_{cx} t_0 \\ \nu_{cx} &= N \langle \sigma v \rangle_{cx} \quad \text{rate of charge exchange} \\ & \qquad\qquad\quad \text{number of cx events per unit of time} \\ N &\equiv \text{density of neutrals} = N_0 \exp\left(-\frac{1-r}{\delta}\right)\end{aligned}$$

where t_0 is a characteristic time, $N_0 = 10^{16}$ neutrals/m^3 , $\delta = 0.05$.

The radial profile of the coefficient of transport of *toroidal momentum* is

$$\begin{aligned}\chi &= \begin{cases} k(0.05 + r^2) & \text{for} \quad r < r_c \\ 0.2k & \text{for} \quad r > r_c \end{cases} \\ \text{with } k &= 3.5 \\ r_c &= 0.8\end{aligned}$$

There is a time lag between the fast ion loss and the rise of the torque $J_r \times B$. Note that at the **Hinton-Rosenbluth** mechanism there is no time lag.

The torque applied in the toroidal direction was very substantial in the edge region.

However precisely there the toroidal rotation was not very important. Probably due to the fact that the torque had to compete with *CHARGE-EXCHANGE COLLISIONAL DAMPING* and with momentum diffusion.

The loss of fast trapped ions is prompt.

But it takes time for the torque $\mathbf{J} \times \mathbf{B}$ to induce toroidal rotation.

Conclusion of **Burrell** for *counter-NBI*: the idea of driving the toroidal rotation U_φ using $\mathbf{J} \times \mathbf{B}$ torque is not straightforward after all. It depends on momentum diffusion.

Reversal of direction of toroidal rotation.

In the experiments of density ramp up (Alcator) by **Bortolon Duval Scarabosio** it is observed a change in the direction of toroidal rotation when the density increases. No additional momentum input.

The total angular momentum of the plasma column carried by the *carbon* plasma component n_C evolves

$$\text{from} \quad -10 \times 10^{-5} \quad \text{to} \quad 1 \times 10^{-5} \quad (J \ s)$$

Burrell 1996 fast E_r transitions.

It is revealed that the poloidal velocity plays an important role.

The poloidal velocity

$$v_{\theta i} \quad \text{is in the direction of } \underline{\text{ion}} \text{ diamagnetic rotation } v_{dia,i}$$

This also means that the *electron diamagnetic* velocity is approximately equal (for $T_e = T_i$) and is *opposite* in direction to the ion diamagnetic velocity since

$$\begin{aligned}
\mathbf{v}_{dia,e} &= \frac{1}{n_e m_e \Omega_e} \widehat{\mathbf{n}} \times \boldsymbol{\nabla} p_e \\
\mathbf{v}_{dia,i} &= \frac{1}{n_i m_i \Omega_i} \widehat{\mathbf{n}} \times \boldsymbol{\nabla} p_i
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}_{dia,e} &= \frac{1}{n_e \left(-\left|e\right|\right) B_\varphi} \left|\frac{dp_e}{dr}\right| \widehat{\mathbf{e}}_\theta \\
\mathbf{v}_{dia,i} &= \frac{1}{n_i \left|e\right| B_\varphi} \left|\frac{dp_i}{dr}\right| \widehat{\mathbf{e}}_\theta
\end{aligned}$$

For $n_e = n_i$ and $T_e \simeq T_i$ we have

$$\mathbf{v}_{dia,e} \simeq -\mathbf{v}_{dia,i}$$

The effect of $v_{\theta i}$ is to produce an E_r which is **positive**, *i.e.* directed toward the exterior of the plasma (since the term is $-v_{\theta,i}B_\varphi$).

The contribution from the term with ∇p_i (*diamagnetic*) to E_r is **negative**

$$\begin{aligned}
0 &= -\frac{1}{n_i e_i}\nabla p_i + \mathbf{E} + \mathbf{v}_i \times \mathbf{B} \\
\mathbf{E} &= -\mathbf{v}_i \times \mathbf{B} + \frac{1}{n_i e_i}\nabla p_i \\
E_r &= -v_{\theta,i}B_\varphi + v_{\varphi,i}B_\theta + \frac{1}{n_i e_i}\frac{dp_i}{dr}
\end{aligned}$$

$$\begin{aligned}
-v_{\theta,i}B_\varphi &> 0 \\
\frac{1}{n_i e_i}\frac{dp_i}{dr} &< 0 \text{ for normal radially decreasing profile of } p_i
\end{aligned}$$

Burrell observes that the

1) positive contribution of $v_{\theta,i}$ and

2) negative contribution of $\frac{dp_i}{dr}$

are comparable

and we **note** that, if they are comparable

$$\begin{aligned}
-v_{\theta,i}B_\varphi + \frac{1}{n_i e_i}\frac{dp_i}{dr} &\simeq 0 \\
v_{\theta,i} &\simeq \frac{1}{n_i n_i B_\varphi}\frac{dp_i}{dr} \\
v_{\theta,i} &\simeq v_{dia,i}
\end{aligned}$$

and conclude that the ion rotation is in the direction of the ion diamagnetic flow and they have very close magnitudes. Then the effective Larmor radius is very large

$$1 - \frac{v_{dia,i}}{v_{\theta,i}} \to 0$$

Burrell finds *two-steps* in the transition from L to H mode:

1. first part is the rise of E_r; **note** that this consists of *charge-separation* as in polarizable medium. Probably this charge separation affects the drift waves, not by velocity shear but directly as charge separation.

2. second part is the rise of $\boldsymbol{\nabla} p_i$. The diamagnetic velocity begin to increase due to barrier to density transport.

Note that this can be interpreted as follows: the change of the rotation $v_{\theta,i}$ represents the transition L to H. The pressure p_i reacts such as to generate the ρ_s^{eff} very large, $\rho_s^{eff} \rightarrow \infty$. This is achieved by $v_{dia,i} \simeq v_{\theta,i}$. The question would be *why*.

A direct explanation does not imply ρ_s^{eff}, but the fact that sheared rotation leads to suppression of turbulence/transport and then the density and temperature are no more lost to the edge. Their gradients rise. This will end up when the gradients are so strong that the residual transport (induced by these gradients) balances the input density and heat and an equilibrium is reached.

Another explanation would involve the Ertel's theorem but the shear of the poloidal velocity is still much smaller than $\Omega_{c,i}$.

We cannot say what physical process makes $v_{dia,i}$ to rise such as to become comparable to $v_{\theta,i}$. Except that the density gradient increases due to the transport barrier.

But we can say that the change in $v_{dia,i}$ *will stop* when $\rho_s^{eff} \rightarrow \infty$.

Later. We need however that the rise of the density leads ultimately to an increase of the *diamagnetic* flow beyond the poloidal rotation. Because in this way we can explain the change of the sign of the coefficient in the elliptic differential equation which is obtained after the term with potential ϕ is retained, the term with square potential ϕ^2 is zero due to the maximum of the radial profile of the diamagnetic velocity (**Spatschek, Horton, Petviashvili**) and the third-order term ϕ^3 is kept. The solution expressed as elliptic functions shows localized vortices that can trigger ELMs.

On the other hand we have the observation about the effect that large ρ_{eff} can have in the dynamics of the drift wave instability. Only long poloidal λ can be excited and the frequencies will be low, similar to poloidal rotation.

End note.

Burrell also note that the evolution of E_r is on a time scale of $20 - 30\ \mu s$, which is extremely fast.

[**Note** *Later*: this is the time of ionization and of the generation of torque from ionization].

The suppression of the drift wave fluctuations is extremely fast, on the time scale that has been mentioned.

There is a possibility that the suppression of the Cerenkov radiation of the drift waves to be the reason.

There is a contradiction in the paper of Burrell: it is said that the term due to ion rotation $-v_{\theta,i}B_\varphi$ is comparable with the term due to the gradient of pressure $[1/(n_i e_i) dp/dr]$. But they have opposite signs and then E_r should be small. Or the figures show it is very large. This contradiction is solved if we assume that the **positive** contribution from the poloidal rotation is compensated and overcomed by the **negative** contribution from the ∇p_i term. Then E_r results negative.

We note however that in the treatment of **Nycander** of large scale vortices with density of electrons Boltzmannian (plus neutrality) the diamagnetic velocity which is compared with the velocity of the vortical structure is the *electronic* $v_{dia,e}$. This gives:

1. the direction of the propagation of drift waves
2. the maximum phase velocity of the drift waves

and is therefore the reference physical process that can affect by coupling and radiation of energy the structure of the vortex.

In other works however it is found that the plasma poloidal rotation takes place in the *ion diamagnetic* direction.

This paper should be compared with the article of **Stacey 2002** about the gyroviscosity of neoclassical origin, in DIIID.

The paper **Toroidal rotation saw teeth MAST** gives radial profiles of toroidal velocity of rotation in MAST. Stability of magnetic modes against the new condition, in presence of rotation, are discussed.

Experiment on zonal flow PoP10 2003 1712. Possible connection with the oscillating solution of the type *breather* for *sine*-Gordon equation. And with **Guido** who sees layers of poloidal velocity. And with the **Japanese** from Marseille CIRM who sees a breathing of large scale.

Sine-Gordon has a breather.

NOTE the breather-like oscillation in a large part of the plasma cross section may be an oscillator precursor to the transition provoked by an instability where two halves of toroidal flow, one in one direction and the other in the opposite direction are transformed into a *toroidal dipole* of flows. **Montgomery Bates**. **End**.

Paper by **Rice NF50 2010** on Alcator C-mod Ion Transport Barrier in the ICRF heating.

ICRH off axis: generates ITB. A significant slowing of the intrinsic toroidal central rotation during the formation of the ITB. **Note** that we expect that before ICRH the plasma has equilibrium toroidal rotation in *counter* current sense, because we are in the L-mode. When ICRH is applied an ITB is formed and the regime becomes more similar to a H-mode. Then the toroidal rotation should be in *co*-current sense. At least we see a decrease of the *counter*-current flow, as a tendency to reverse the toroidal rotation.

However in the paper **Observations anomalous momentum Rice** it is explained the sequence:

1. L-mode
2. EDA (enhanced D_α) H-mode
3. ELM-free H mode
4. ITB

It results that ITB is placed beyond H-mode and then the basic toroidal rotation should be *co-current*.

End.

ITB lasts at least $10 \times \tau_e$.

The density is stronlgy peaked.

The impurities begin to accumulate in the core, often triggering a disruption of the H-mode.

The paper **Co-current Alcator C Mode Rice high frequency** is about the effect of injecting ICRWaves in Alcator C mode and observe that there is an effect of co-current toroidal rotation. Broken *omnigeneity* of trapped ion bananas under ICRH. See **White**.

In L mode the core toroidal rotation is *counter*-current.

At the transition to the H-mode the rotation becomes *co*-current.

When the ITB forms, the toroidal rotation in the core begins to slow down. It approaches zero and even can create a <u>well</u> of *counter*-current rotation in the center (*hallow rotation profile*). This is a local *reversal* of toroidal rotation.

When the ICRH is applied off-axis to a H-mode and an ITB is formed, the radial electric field E_r is close to zero in the core and even slightly *counter*-current, as if that part would be in L-mode. It rises to a high value when it reaches high value and strong shear. We **Note** that the figure 7a from this paper shows that the toroidal velocity in the core, inside the ITB, is very small (or even *counter*) late after the ITB formation. On the other hand we see in figure 8b that E_r is also very small. Therefore from the expression of E_r only the poloidal rotation term and the diamagnetic term survive. They are almost equal. **End.**

When ITB is formed via ICRH off-axis the toroidal rotation, which in H-mode is *co*-current, begins to slow down and reverses while the density is peaking.

In the paper **ICRH generation rotation** it is found that the ICRH produces a strong toroidal *counter* rotation, (which we would be tempted to associate to L-mode). This occurs since most of the ICRH-born minority ions exist in a trapped regime. If, on the contrary, the minority ions are passing then the induced toroidal rotation can be in any sense: *co-* or *counter*.

They also say that the strong *co*-rotation obtained by fast waves ICRH in Alcator C-Mod is paradoxical. They say that the absorbtion of the wave energy is preferential for ions moving in one direction.

In the paper **Neoclassical Poloidal Rotation Stacey** it is mentioned that experiments on ISX-B with NBI have shown that there is a <u>low concentration</u> of impurities in the center of the tokamak when

- the *magnetic field*
- the *current*, and
- the *neutral beam*

are all in the same direction.

In general the H-mode is known to *accumulate* Fe in the center, by redistribution inside the plasma NOT by inflow. If now it is observed low concentration this may be a signature that there is no H-mode.

The paper **Rice Obs anomalous momentum transport** in Alcator C-mod without momentum input.

The first transition is from

L-mode	to	EDA (enhanced D_α) H-mode

At this transition it results a *co*-current toroidal rotation which propagates in from the edge towards the centre. The time scale is longer than the energy confinement time and much less than the neoclassical transport of momentum.

NOTE that is compatible with the model of neoclassical polarization: a fast variation of the radial electric field $\dot{E_r}$ induces a toroidal precession of the bananas. They drag the untrapped ions on a time fixed by collisions. **END.**

In EDA H-mode the toroidal rotation profile is flat ~ 50 (km/s). The rotation is *co*-current.

The rotation *cannot* be connected with the ICRH wave or fast particle effects.

$$\Delta v_{tor} \sim \frac{W_{stored\ energy}}{I_p}$$

The velocity profiles

The velocity profiles

L-mode	counter rotation
EDA H-mode	nearly flat profile
ELM-free H-mode	centrally peaked (*inward* pinch?)
ITB	hallow in the centre

The electron density

The density profiles

L-mode	$n_e(r)$ centrally peaked
EDA H-mode	$n_e(r)$ flat profile, constant in time
ELM-free H-mode	$n_e(r)$ flat profile, steadily rising
ITB	$n_e(r)$ strong central peaking

Going from EDA H-mode to ITB the rotation has been much reduced and eventually reversed direction, with strong peaking of the electron density in the centre. At the barrier foot there is a positive gradient of E_r:

$$\begin{aligned} \frac{\delta E_r}{\delta r} &= \frac{(+8\ kV/m)_{r/a=0.6} - (-8\ kV/m)_{r/a=0.3}}{\delta r} \\ &> 0 \end{aligned}$$

In ITB there is a strong concentration of impurities in the core.

In ITB there is a negative E_r well in the core.

In L mode the rotation is *mainly* in the *counter*-current direction. The density has strong effect on the rotation.

The paper **Rice spontaneous core toroidal rotation**.

Reversal of toroidal rotation from *counter* to *co*, following small changes of the electron density and plasma current.

In L mode

$$-60\ km/s < v_{tor} < +20\ km/s$$

As the density falls, the toroidal rotation reverses to *co*current. **NOTE** that this suggests that the plasma has accessed the H-mode.**END.**

Similar to TCV: when the density increases there is inversion of the rotation from *co* to *counter*, which means transition from the H-mode to the L-mode. This means that high density is detrimental to H-mode.

When the magnetic field is lowered the toroidal velocity reduces from strongly counter-current to zero velocity. **NOE** that we can interpret that in the sense that B low is favorable to the H-mode. **END.**

When the toroidal velocity is close to 0 (plasma is almost stagnant) there is transition to the H-mode.

Co-current central rotation is indicative of a positive core radial electric field

$$\text{co-current rotation } \rightarrow E_r > 0 \text{ in the core}$$

The evolution

L-mode, $E_r = -15\ kV/m < 0$, *counter* rotation
$\rightarrow L$- to H- mode transition$\rightarrow$
H-mode, $E_r = +40\ kV/m > 0$, *co* rotation

In the core:

1. no measurable poloidal rotation

2. weak diamagnetic contribution

The paper **Observation of inward outward particle convection in the core TCV Furno Weisen**.

Observation of *inward* and *outward* particle convection correlated with the mode activity during ECH and ECCD.

In the presence of sufficiently high central ECH and ECCD power and kink distortion of the plasma core the direction of the particle convection can reverse, bocoming *outward* directed.

The structures observed by X rays emmissivity show

$$m = 1 \quad , \quad n = 1$$

The density:

Normal sawtooth:	Collapse of peaked density profile	
With $P_{ECH} \geqslant 0.8\ MW$	$\left\{ \begin{array}{c} \text{Sudden flattening} \\ \text{of hollow density} \\ \text{due to sawtooth crash} \end{array} \right\}$	$\left\{ \begin{array}{c} \text{The hollow profile} \\ \text{is due to outward flux} \\ \text{during ramp-up} \end{array} \right\}$

Note: when P_{ECH} is increased beyond a threshold, there is a high Temeperature gradient in the center. In the same time there is a periodic perturbation in the form of rotating coherent convective structures at a surface of inversion. These structures act to the transfer of energy between parts of plasma: core and outward. We have the conditions for *inverse Ranque-Hilsch* effect and the two elements ($\mathbf{\nabla} T_e$ and convective structures) induce and sustain a poloidal rotation, like in a vortex tube. The poloidal rotation will modify the density profile. The example is the Japanese machine HYPER-I, where there is a hollow density profile and a strong poloidal rotation. It is possible that the inner core of the plasma to rotate poloidally in the opposite direction compared with the poloidal rotation of plasma of the edge. Then the factor that will model the density profile is

$$\frac{1}{\left(\rho_s^{eff}\right)^2} = \frac{1}{\rho_s^2}\left(1 - \frac{v_{dia}}{u}\right) \to 0$$

which means that the gradient of the density is correlated with the poloidal velocity of rotation. **End.**

They find that there is a threshold for the power of ECH for reversing the direction of the particle convection.

Note. This is probably a threshold in the *temperature gradient*, like in the case of Ranque-Hilsch vortex tube. **End**.

There is a rotating $m = 1$ structure with frequency $\approx 5\ kHz$.

In Fig.5 of their article it is shown

1. an inverted density profile, indicating outward-directed particle convection flux
2. a highly peaked temperature profile
3. a dispalcement indicating the existence of a structure (mode)

In other machines: *flattening of density profiles in the presence of localized core heating*. **Note** the same physical picture can be an explanation: poloidal rotation arising from inverse Ranque-Hilsch effect.

They note correlation between density gradients and Temperature gradients.

Outward convection has been seen only in the presence of $m = 1$ mode.

The angular momentum transport is a *front propagation*

In **Burrell 1996** it is discussed the change in the turbulence when the input power is rised or decreased. Experimental results show that the effect of $E \times B$ rotation propagates from the core to the boundary when the input power is increased. It is known that the shear of the $E \times B$ rotation has an influence on the linear growth rate of the instabilities and also has an effect on the radial correlation length of the turbulence. The rotation transported (which is a spatial = radial redistribution of the vorticity profile in the plasma cross-section) modifies the background while it propagates. Therefore the propagation is nonlinear.

The transport of angular momentum is a *front propagation*.

The first part of the radially-propagating profile of the vorticity affects the background and modifies the local fluctuations. The parts of the propagating profile which come after that will find more convenient conditions to induce locally rotations in the background.

There may be also a synergetic effect:

the arrival of the "rotation state" will induce a change in the instability growth rate and only waves with long scale on poloidal direction can be excited. They have very small ω and this means that they can be easily converted into rotation of the plasma.

In **Rice NF2010** it is explained that inside the ITB that is formed by off-axis ICRH the temperature profile and the density profile (more peaked) are changed and the Ion Temperature Gradient instability is no more excited. The effect propagates from the core to the exterior when the point of resonance ICRH is moved toward exterior.

In the paper **Ambipolarons Morrison** it is shown for a bumpy torus that there is propagation of a *kink* of electric field. The equation is derived from **Hastings** approach. However there it is question of two states that the electric field can have. The front is just the transition between these two states.

12 Equilibrium poloidal rotation (Hazeltine)

The equilibrium poloidal rotation is determined by the gradient of temperature.

The equation is

$$\frac{\partial f}{\partial t}+(u\widehat{\mathbf{n}}+\mathbf{v}_D)\cdot\boldsymbol{\nabla}f+\frac{e_a}{m_a}\frac{\partial\phi}{\partial t}\frac{\partial f}{\partial\epsilon}=C(f)$$

where the term with time dependent potential ϕ is NOT related to the waves in plasmas but to decay of rotation.

13 Fluid equations for plasma rotation

13.1 The Pfirsch Schluter current

The parallel current arising from the non-zero divergence of the diamagnetic flow

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{j} &= 0\\ \boldsymbol{\nabla}_\perp\cdot\mathbf{j}_\perp+\boldsymbol{\nabla}_\parallel\cdot\mathbf{j}_\parallel &= 0\end{aligned}$$

the taking the perpendicular current as resulting from the *diamagnetic* flows of electrons and ions, the parallel gradient can be written as

$$\begin{aligned}\boldsymbol{\nabla}_\parallel\cdot\mathbf{j}_\parallel &= -\boldsymbol{\nabla}_\perp\cdot\mathbf{j}_\perp\\ \frac{1}{qR}\frac{\partial}{\partial\theta}j_\parallel &= -\boldsymbol{\nabla}_\perp\cdot\left(e\frac{1}{m\Omega}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}p\right)\\ &= -\boldsymbol{\nabla}_\perp\cdot\left(e\frac{1}{|e|\,B}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}p\right)\end{aligned}$$

Let us look to the last term. It is the perpendicular divergence of the *diamagnetic* flow (this reminds us of the Pfirsch-Schluter current).

Note that the operator of parallel derivative is

$$\nabla_\parallel\sim\frac{1}{qR}\frac{\partial}{\partial\theta}$$

and that the perpendicular current $\mathbf{j}_\perp$ is the *diamagnetic current, of ions + electrons.* **End.**

This is a *neoclassical* effect.

It is the magnetic field that has a space variation in the perpendicular direction. First we have

$$\begin{aligned}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}p &= \left|\frac{dp}{dr}\right|(\widehat{\mathbf{n}}\times\widehat{\mathbf{e}}_r)\\ &= -\widehat{\mathbf{e}}_\theta\left|\frac{dp}{dr}\right|\end{aligned}$$

Then, restricting to the gradient of the part that contains B, we use the expression of the gradient operator expressed in the geometry of the toroidal region.

This part is repeated later in this text.

$$B = \frac{B_0}{1+\varepsilon\cos\theta}$$

$$\boldsymbol{\nabla}\cdot\left(\widehat{\mathbf{e}}_\theta\frac{B_0}{B}\right) = \boldsymbol{\nabla}\cdot\left[\widehat{\mathbf{e}}_\theta\left(1+\varepsilon\cos\theta\right)\right]$$

In the orthogonal coordinates (r, θ, φ) we have the element of distance:

$$dl^2 = (dr)^2 + r^2 (d\theta)^2 + (R_0 + r\cos\theta)^2 d\varphi^2$$

which gives the coefficients

$$\begin{aligned} h_1 &= 1 \\ h_2 &= r \\ h_3 &= R_0 + r\cos\theta \end{aligned}$$

Then the divergence of a vector $\mathbf{a}$ is written

$$\boldsymbol{\nabla}\cdot\mathbf{a} = \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial r}(h_2h_3a_1) + \frac{\partial}{\partial\theta}(h_1h_3a_2) + \frac{\partial}{\partial\varphi}(h_1h_2a_3)\right)$$

which gives

$$\begin{aligned} \boldsymbol{\nabla}\cdot\left[\widehat{\mathbf{e}}_\theta\left(1+\varepsilon\cos\theta\right)\right] &= \frac{1}{r\left(R_0+r\cos\theta\right)}\frac{\partial}{\partial\theta}\left(\left(R_0+r\cos\theta\right)\left(1+\varepsilon\cos\theta\right)\right) \\ &= \frac{1}{r\left(R_0+r\cos\theta\right)}R_0\frac{\partial}{\partial\theta}\left[\left(1+\varepsilon\cos\theta\right)^2\right] \\ &= \varepsilon\frac{(-2\sin\theta)}{r} \end{aligned}$$

From this result we get

$$
\begin{aligned}
-\boldsymbol{\nabla}_\perp \cdot \mathbf{j}_\perp &= -\boldsymbol{\nabla}_\perp (dia) = \\
&= -\boldsymbol{\nabla}_\perp \cdot \left(e\frac{1}{m\Omega}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p \right) \\
&= -\boldsymbol{\nabla}_\perp \left[\left(e\frac{1}{m\Omega} \right) (-\widehat{\mathbf{e}}_\theta) \left| \frac{dp}{dr} \right| \right] \\
&= \boldsymbol{\nabla}_\perp \cdot \left(\widehat{\mathbf{e}}_\theta \frac{B_0}{B} \right) \frac{1}{B_0} \left| \frac{dp}{dr} \right| \\
&= \varepsilon \frac{(-2\sin\theta)}{r} \frac{1}{B_0} \left| \frac{dp}{dr} \right| \\
&= \frac{r}{RB_0} \left| \frac{dp}{dr} \right| \frac{\partial}{r\partial\theta} (2\cos\theta)
\end{aligned}
$$

NOTE ON An alternative calculation

$$
-\boldsymbol{\nabla}_\perp \cdot \left(e\frac{1}{m\Omega}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p \right) = -\boldsymbol{\nabla}_\perp \cdot \left[\left(e\frac{1}{m\Omega} \right) \left(-\widehat{\mathbf{e}}_\theta \left| \frac{dp}{dr} \right| \right) \right]
$$

Taking factor $|dp/dr|$ we have to calculate

$$
\begin{aligned}
\boldsymbol{\nabla}_\perp \cdot \left(e\frac{1}{m\Omega}\widehat{\mathbf{e}}_\theta \right) &= \boldsymbol{\nabla}_\perp \cdot \left(\frac{1}{B}\widehat{\mathbf{e}}_\theta \right) \\
&= \boldsymbol{\nabla}_\perp \left(\frac{1}{B} \right) \cdot \widehat{\mathbf{e}}_\theta + \frac{1}{B} \left(\boldsymbol{\nabla}_\perp \cdot \widehat{\mathbf{e}}_\theta \right)
\end{aligned}
$$

The first term is

$$
\begin{aligned}
\boldsymbol{\nabla}_\perp \left(\frac{1}{B} \right) &= -\frac{1}{B^2} \boldsymbol{\nabla}_\perp B \\
&= -\frac{1}{B^2} \boldsymbol{\nabla}_\perp \left(B_0 \frac{R_0}{R} \right) \\
&= -\frac{1}{B^2} B_0 R_0 \left(-\frac{1}{R^2} \boldsymbol{\nabla}_\perp R \right) \\
&= \frac{B_0}{B^2} \frac{R_0}{R^2} \widehat{\mathbf{e}}_R
\end{aligned}
$$

We take separately

$$\begin{aligned}
\nabla_\perp R &= \left(\widehat{\mathbf{e}}_r \frac{1}{h_r}\frac{\partial}{\partial r} + \widehat{\mathbf{e}}_\theta \frac{1}{h_\theta}\frac{\partial}{\partial \theta} + \widehat{\mathbf{e}}_\varphi \frac{1}{h_\varphi}\frac{\partial}{\partial \varphi}\right)(R_0 + r\cos\theta) \\
&= \left(\widehat{\mathbf{e}}_r \frac{\partial}{\partial r} + \widehat{\mathbf{e}}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \widehat{\mathbf{e}}_\varphi \frac{1}{R_0 + r\cos\theta}\frac{\partial}{\partial \varphi}\right)(R_0 + r\cos\theta) \\
&= \widehat{\mathbf{e}}_r \cos\theta + \widehat{\mathbf{e}}_\theta (-\sin\theta) \\
&= \widehat{\mathbf{e}}_R
\end{aligned}$$

Here we should decide if the angle θ is measured from the equatorial plane or from the symmetry axis of the torus. Above it was considered that θ is measured from the equatorial plane towards the higher z direction.

$$\frac{1}{B^2}\frac{B_0 R_0}{R^2} = B_0 R_0 \frac{(1+\varepsilon\cos\theta)^2}{B_0^2}\frac{1}{(R_0 + r\cos\theta)^2} = \frac{1}{B_0 R_0}$$

The first term is then

$$\nabla_\perp \left(\frac{1}{B}\right)\cdot \widehat{\mathbf{e}}_\theta = \frac{1}{B_0 R_0}\widehat{\mathbf{e}}_R \cdot \widehat{\mathbf{e}}_\theta = \frac{1}{B_0 R_0}(-\sin\theta) = \frac{1}{B_0 R_0}\frac{\partial}{\partial\theta}(\cos\theta)$$

The second term is $\frac{1}{B}(\nabla_\perp \cdot \widehat{\mathbf{e}}_\theta)$ and contains the *divergence* of the versor

$$\begin{aligned}
&\nabla_\perp \cdot \widehat{\mathbf{e}}_\theta \\
=\ & \frac{1}{h_r h_\theta h_\varphi}\left\{\frac{\partial}{\partial r}\left[h_\theta h_\varphi (\widehat{\mathbf{e}}_\theta)_r\right] + \frac{\partial}{\partial \theta}\left[h_r h_\varphi (\widehat{\mathbf{e}}_\theta)_\theta\right] + \frac{\partial}{\partial \varphi}\left[h_r h_\theta (\widehat{\mathbf{e}}_\theta)_\varphi\right]\right\} \\
=\ & \frac{1}{r(R_0 + r\cos\theta)}\left\{\frac{\partial}{\partial\theta}\left[h_r h_\varphi (\widehat{\mathbf{e}}_\theta)_\theta\right]\right\} \\
=\ & \frac{1}{r(R_0 + r\cos\theta)}\frac{\partial}{\partial\theta}\left[(R_0 + r\cos\theta)\right] \\
=\ & \frac{1}{R_0 + r\cos\theta}\frac{\partial}{\partial\theta}(\cos\theta)
\end{aligned}$$

The perpendicular divergence of the diamagnetic current is

$$\begin{aligned}
-\boldsymbol{\nabla}_{\perp}\cdot\mathbf{j}_{\perp} &= -\boldsymbol{\nabla}_{\perp}\cdot\left(e\frac{1}{m\Omega}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}p\right)\\
&= -\boldsymbol{\nabla}_{\perp}\left(e\frac{1}{m\Omega}\right)\cdot\left(-\widehat{\mathbf{e}}_{\theta}\left|\frac{dp}{dr}\right|\right)\\
&= \left|\frac{dp}{dr}\right|\left[\frac{1}{B_0R_0}\frac{\partial}{\partial\theta}\left(\cos\theta\right)+\right.\\
&\quad\left.+\frac{1}{B}\frac{1}{R_0+r\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\right)\right]\\
&= \left|\frac{dp}{dr}\right|\frac{1}{B_0R_0}\frac{\partial}{\partial\theta}\left(2\cos\theta\right)
\end{aligned}$$

and obtain the same result.

END OF NOTE on the alternative calculation

Equalizing the two sides of the *current conservation* equation

$$\frac{1}{qR}\frac{\partial}{\partial\theta}j_{\|}=-\frac{r}{RB}e\left(\frac{dp}{dr}\right)\frac{\partial}{r\partial\theta}\left(2\cos\theta\right)$$

Integrating on the poloidal angle θ:

$$J_{\|}=-\varepsilon\frac{2}{B_{\theta}}\frac{dp}{dr}\cos\theta$$

There is a poloidal electric field related to this current

$$\begin{aligned}
E_{\|} &= \eta j_{\|}^{PS}\\
\text{from this we want to find }\; E_{\theta} &= \frac{B}{B_{\theta}}E_{\|}
\end{aligned}$$

$$E_{\theta}=\frac{1}{\sigma_{\|}}\frac{B}{B_{\theta}}\left(-\varepsilon\frac{2}{B_{\theta}}\frac{dp}{dr}\cos\theta\right)$$

It is the projection on θ (poloidal) of the relationship $E_{\|}=J_{\|}/\sigma_{\|}$, with the factor of projection

$$E_{\|}\left(B/B_{\theta}\right)=E_{\theta}$$

Formula

$$\widehat{\mathbf{n}}\times\widehat{\mathbf{e}}_r=\frac{B_{\varphi}}{B}\widehat{\mathbf{e}}_{\theta}+\frac{B_{\theta}}{B}\widehat{\mathbf{e}}_{\varphi}$$

Formula

$$\frac{B_\theta}{B_\varphi} = \frac{\varepsilon}{q}$$

$$\left(\frac{B_\varphi}{B}\right)^2 = 1 - \frac{\varepsilon^2}{q^2}$$

$$qR = \frac{rB_\varphi}{B_\theta}$$

Formula

$$\mathbf{v}_{neo}^{ion} \cdot \boldsymbol{\nabla}\psi = -\frac{m_i}{|e|}\frac{v_\parallel}{qR}\frac{\partial}{\partial\theta}\left[v_\parallel R\left(1 - \frac{\varepsilon^2}{q^2}\right)\right]$$

Now, since

$$\boldsymbol{\nabla}\psi = 2\pi R B_\theta \widehat{\mathbf{e}}_r$$

we have the radial component of the drift velocity

$$\begin{aligned}
v_{D,r} &\equiv \left.\mathbf{v}_{neo}^{ion}\right|_r \\
&= \left(\frac{1}{2\pi R B_\theta}\right)(-)\frac{m_i}{|e|}\frac{v_\parallel}{qR}\frac{\partial}{\partial\theta}\left[v_\parallel R\left(1 - \frac{\varepsilon^2}{q^2}\right)\right] \\
&\simeq -\frac{m_i}{|e|}\frac{v_\parallel}{2\pi R B_\varphi}\frac{\partial}{r\partial\theta}\left(v_\parallel R\right)
\end{aligned}$$

$$\frac{\partial v_\parallel}{\partial\theta} = -\frac{1}{m_i v_\parallel}\mu\varepsilon B_0 \sin\theta$$

The factor $\sin\theta$ appears in this expression due to the *projection* of the vertical drift velocity of the ions due to the *curvature* and $\boldsymbol{\nabla}B$ terms, on the local radial direction perpendicular on the circular flux surface. This is the left-hand side term, where it is the drift of particles $v_{D,r}$.

In **Rosenbluth Hinton alpha** the following formula is used

$$\mathbf{v}_d \cdot \boldsymbol{\nabla}\psi = I v_\parallel \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta \frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{\Omega}\right)$$

where

$$I = RB_\varphi$$

and we recognize

$$\begin{aligned}
\frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{\Omega_i}\right) &= \frac{m_i}{|e|}\frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{B_0/h}\right) \\
&= \frac{m_i}{|e|}\frac{1}{R_0 B_0}\frac{\partial}{\partial\theta}\left(v_\parallel R\right)
\end{aligned}$$

We have

$$\begin{aligned}
\nabla_{\|}\theta &= \widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta = \frac{1}{r}\cos\left(\text{angle of } \boldsymbol{\nabla}\theta \text{ and } \widehat{\mathbf{n}}\right) \\
&= \frac{1}{r}\frac{B_\theta}{B} = \frac{RB_\theta}{rB}\frac{1}{R} \\
&= \frac{1}{qR}
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}_d\cdot\boldsymbol{\nabla}\psi &= Iv_{\|}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta\frac{\partial}{\partial\theta}\left(\frac{v_{\|}}{\Omega}\right) = I\frac{v_{\|}}{qR}\frac{\partial}{\partial\theta}\left(\frac{v_{\|}}{\Omega}\right) \\
v_{d,r} &= \frac{1}{RB_\varphi}RB_\varphi\frac{v_{\|}}{qR}\frac{\partial}{\partial\theta}\left(\frac{v_{\|}}{\Omega}\right) = \frac{v_{\|}}{qR}\frac{\partial}{\partial\theta}\left(\frac{v_{\|}}{\Omega}\right)
\end{aligned}$$

The *bounce average* of the radial component of the drift velocity is zero

$$\overline{(\mathbf{v}_d\cdot\boldsymbol{\nabla}\psi)} = 0$$

Shats. In the paper **PRL 2002** Shats uses the equation

$$\frac{\partial v_\theta}{\partial t} = -\frac{\partial}{\partial r}\left(\widetilde{v}_r\widetilde{v}_\theta\right) - \frac{J_r B_\varphi}{mn} - \mu v_\theta$$

where the *Reynolds stress* is the first term. This was probably for zonal flows.

13.2 Su Yushmanov Horton Dong: The poloidal rotation

The sheared rotation due to combined effect of

1. Reynolds stress created by drift waves
2. Stringer mechanism created by θ-dependent Reynolds stress
3. damping by Magnetic Pumping

Here we must add:
Stringer torque due to the flux of ions from NBI
torque due to $j \times B$ where j comes from loss of NBI ions and from expansion of bananas.

The following formulas are from **Su Yushmanov Dong Horton** PoP 1 (1994) 1905.

The two components of the plasma rotation are defined such as to be function of the magnetic surface label, ψ. They are obtained first by projection on *parallel* $\widehat{\mathbf{n}} = \mathbf{B}/B$ and on *perpendicular* direction $\widehat{\mathbf{e}}_r \times \widehat{\mathbf{n}}$, where the latter versors are obtained from $\boldsymbol{\nabla}\psi / (2\pi R B_p)$ and from $\mathbf{B}/B$,

$$u_{\parallel} = \frac{\langle \mathbf{B} \cdot \mathbf{v} \rangle}{B_0}$$

$$u_{\perp} = \frac{B_0}{2\pi R_0 B_\theta} \left\langle \left(\frac{\boldsymbol{\nabla}\psi \times \mathbf{B}}{B^2} \right) \cdot \mathbf{v} \right\rangle$$

where

$$\mathbf{B} = R_0 B_\varphi \boldsymbol{\nabla}\varphi + \frac{1}{2\pi} \left(\boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi \right)$$

$$B_0 = \left\langle B^2 \right\rangle^{1/2} = \left\langle B_\theta^2 + B_T^2 \right\rangle^{1/2}$$

The average on the magnetic surface is

$$\langle A \rangle = \frac{\int A \frac{d\theta}{\mathbf{B} \cdot \boldsymbol{\nabla}\theta}}{\int \frac{d\theta}{\mathbf{B} \cdot \boldsymbol{\nabla}\theta}}$$

NOTE The same is used by **Hinton Rosenbluth toroidal momentum input**)

The **mean flow velocity is**

$$\mathbf{v} = \frac{\mathbf{B}}{B_0} \left(u_{\parallel} - \frac{B_T}{B_\theta} u_{\perp} \right) + u_{\perp} \frac{B_0}{R_0 B_\theta} R^2 \boldsymbol{\nabla}\varphi$$

The first term is the incompressible plasma flow along the magnetic field lines.

The second term is the **rigid body rotation** of the plasma in the magnetic surface. This term does not contribute to the plasma viscosity. The plasma viscosity force is function of only the combination

$$u_{\parallel} - \frac{B_T}{B_\theta} u_{\perp}$$

The connection between the perpendicular plasma velocity $\mathbf{u}_{\perp}$ and the **radial electric field** is given by

$$u_{\perp} = -\left(\frac{2\pi B_0}{R_0 B_\theta} \right) \left\langle \frac{R^2 B_\theta^2}{B^2} \right\rangle \left(\frac{d\Phi}{d\psi} + \frac{1}{en} \frac{dP}{d\psi} \right)$$

The velocity is defined such that positive E_r gives positive $u_{\perp}$ which is the rotation in the ion diamagnetic direction.

The factor $2\pi R B_\theta$ comes from $\boldsymbol{\nabla}\psi$ which is introduced to make possible the derivatives of Φ and p to ψ instead of r.

From momentum balance (this is **Su, Yushmanov** et Austin)

$$nm\left(1+2\widehat{q}^2\right)\frac{\partial u_\perp}{\partial t} = -F_\perp^R - F_{\tilde{}}^R - F^{neo} - F_\perp^a$$

$$nm\frac{\partial u_\parallel}{\partial t} = \frac{B_\theta}{B_T}F^{neo} - F_\parallel^R - F_\parallel^a$$

where the forces with superscript R are due to the Reynolds stress in a turbulence.

The toroidal geometry is represented in the value of the quantity $\widehat{q}$

$$2\widehat{q}^2 = \left(\frac{B_T}{B_\theta}\right)^2\left(1-\frac{1}{\langle R^2\rangle\langle R^{-2}\rangle}\right)$$

For large aspect ration, $\varepsilon \ll 1$,

$$2\widehat{q}^2 \approx 2q^2$$

We also note

$$2\widehat{q}^2 = \left(\frac{q}{\varepsilon}\right)^2\left(1-\frac{1}{\langle h^2\rangle\left\langle\frac{1}{h^2}\right\rangle}\right)$$

Also used by **ambipolarity Hirshman**.
See *variation on surface NOTES*.
See **Novakovskii** for kinetic poloidal damping and GAM modes.
See **Hassam** for the fluid derivation of the inertia factor $(1+2q^2)$.

The velocity is taken as a sum of the mean velocity

$$\begin{aligned}\mathbf{v} &= \frac{\mathbf{B}}{B_0}\left(u_\parallel - \frac{B_T}{B_{pol}}u_\perp\right)\\ &\quad +u_\perp\frac{B_0}{R_0 B_{pol}}R^2\boldsymbol{\nabla}\phi\end{aligned}$$

and a fluctuating velocity, $\widetilde{\mathbf{v}}$ which will give the Reynolds stress.

In the expression of the mean velocity flow there are two terms. The first

$$\frac{\mathbf{B}}{B_0}\left(u_\parallel - \frac{B_T}{B_{pol}}u_\perp\right)$$

represents the *incompressible* plasma flow along the magnetic field lines.

The second

$$u_{\perp}\frac{B_0}{R_0 B_{pol}}R^2\boldsymbol{\nabla}\phi$$

is a *rigid body rotation* within the magnetic surface. It has a direction $\widehat{\mathbf{e}}_{\varphi}$ which is toroidal. This picture is also mentioned by **Hassam Kulsrud** who say that the motion of plasma consists of

1. the free motion along the magnetic flux tube
2. the motion of the flux tubes themselves

The second part of the velocity (the rigid body rotation) does not participate to the viscosity which finally will damp the poloidal rotation.

the viscosity will only depend on the combination exhibited in the first term

$$u_{\parallel}-\frac{B_T}{B_{pol}}u_{\perp}$$

NOTE that the expressions for velocity components are simpler compared with those of the paper **Yushmanov Horton** electric field generation at the edge due to loss of hot ions.

The Reynolds stress produces terms that are marked by the upperscript R

$$F_{\parallel}^{R}=\left\langle nm\frac{\mathbf{B}}{B_0}\cdot\left[\left(\widetilde{\mathbf{v}}\cdot\boldsymbol{\nabla}\right)\widetilde{\mathbf{v}}\right]\right\rangle$$

$$F_{\perp}^{R}=\left\langle nm\frac{B_0\left(\boldsymbol{\nabla}\psi\times\mathbf{B}\right)}{2\pi R_0 B_{pol}B^2}\cdot\left[\left(\widetilde{\mathbf{v}}\cdot\boldsymbol{\nabla}\right)\widetilde{\mathbf{v}}\right]\right\rangle$$

$$F_{\sim}^{R}=\left\langle nm\frac{B_T}{B_{pol}}\left(\frac{B_0^2}{B^2}-1\right)\frac{\mathbf{B}}{B_0}\cdot\left[\left(\widetilde{\mathbf{v}}\cdot\boldsymbol{\nabla}\right)\widetilde{\mathbf{v}}\right]\right\rangle$$

The term of neoclassical damping effect on poloidal rotation is

$$F^{neo}=-\frac{B_T}{B_0 B_{\theta}}\left\langle B^2\left(\mathbf{B}\cdot\boldsymbol{\nabla}\right)\frac{P_{\parallel}-P_{\perp}}{2B^2}\right\rangle$$

is the viscosity of the plasma when it is pushed along nonuniform magnetic field (magnetic pumping effect).

This expression of the damping force as proportional with the difference between the two pressures $P_{\parallel}-P_{\perp}$ is also used by **Rosenbluth Hinton** for the *alpha-particle - induced rotation.*

The additional forces are related to the ripple or to the atomic processes. The two components are

$$F_{\perp}^{a}=\left\langle \left[\frac{B_0}{2\pi R_0 B_\theta}\frac{\boldsymbol{\nabla}\psi\times\mathbf{B}}{B^2}+\frac{B_T}{B_\theta}\left(\frac{B_0^2}{B^2}-1\right)\frac{\mathbf{B}}{B_0}\right]\cdot\mathbf{F}^a\right\rangle$$

$$F_{\parallel}^{a}=\left\langle \frac{\mathbf{B}}{B_0}\cdot\mathbf{F}^a\right\rangle$$

The neoclassical viscosity is

$$F^{neo}=-3\mu^{neo}\frac{B_T}{B_0^2}\left\langle \left(\frac{(\mathbf{B}\cdot\boldsymbol{\nabla})\,B}{B}\right)^2\right\rangle \left(u_{\parallel}-\frac{B_T}{B_\theta}u_{\perp}-k_{\nu_*}\frac{1}{eB_\theta}\frac{dT_i}{dr}\right)$$

where the neoclassical viscosity coefficient is (for velocities which are much less then the sound velocity)

$$\mu^{neo}\approx R_0 q\frac{nmv_{th}\nu_*}{1+\nu_*}\frac{1}{1+\varepsilon^{3/2}\nu_*}$$

$$\nu_*=\nu\,\varepsilon^{3/2}\frac{1}{v_{th}/\left(qR\right)}$$

and ν is the ion collision frequency.This is also in the preprint of **Stacey**. See also **Novakovskii**.

The coefficient k_{ν_*} describes the relative effect of the **parallel heat flux** on the longitudinal viscosity

$$\begin{array}{lll} k_{\nu_*}=1.17 & \text{for} & \nu_*\ll 1 \\ k_{\nu_*}=-0.5 & \text{for} & 1\ll\nu_*\ll\varepsilon^{-3/2} \\ k_{\nu_*}=-2.1 & \text{for} & \nu_*\gg\varepsilon^{-3/2} \end{array}$$

The equilibrium value of the poloidal rotation

$$\begin{aligned} u_\theta &= -u^{neo} \\ &= -k_{\nu_*}\frac{1}{eB_T}\frac{dT_i}{dr}\left(\frac{L_n}{\rho_s c_s}\right) \end{aligned}$$

The equilibrium poloidal velocity is determined by the ion temperature gradient and it is

ion diamagnetic direction	for	$\nu_*>1,\quad k_{\nu_*}<0$
electron diamagnetic direction	for	$\nu_*<1,\quad k_{\nu_*}>0$

There are forces due to the nonambipolar processes (like: charge exchange, ion-direct-loss, neutral beams). They are represented in a simplified form:

$$F_\perp^a = nm\nu_\perp^a \left(u_\perp - u_\perp^a\right)$$
$$F_\parallel^a = nm\nu_\parallel^a \left(u_\parallel - u_\parallel^a\right)$$

The resulting equations governing the plasma **perpendicular** and **parallel** motions are

$$\begin{aligned}\left(1+2q^2\right)\frac{\partial u_\perp}{\partial t} &= -\nu^{nc}\left(u_\perp - \Theta u_\parallel + u^{nc}\right) - \\ &-\nu_\perp^a\left(u_\perp - u_\perp^a\right) - \\ &-\frac{\partial}{\partial x}\left\langle \widetilde{v}_x \widetilde{v}_\perp\right\rangle - \\ &-2q\left\langle \cos\theta \frac{\partial}{\partial x}\left(\widetilde{v}_x\widetilde{v}_\parallel\right)\right\rangle\end{aligned}$$

$$\begin{aligned}\frac{\partial u_\parallel}{\partial t} &= -\nu^{nc}\Theta\left(-u_\perp + \Theta u_\parallel - u^{nc}\right) - \\ &-\nu_\parallel^a\left(u_\parallel - u_\parallel^a\right) - \\ &-\frac{\partial}{\partial x}\left\langle \widetilde{v}_x\widetilde{v}_\parallel\right\rangle\end{aligned}$$

where

$$\Theta = \frac{r}{qR} = \frac{B_\theta}{B_\varphi} \ll 1$$
$$u^{nc} = k_{\nu_*}\frac{1}{eB_\varphi}\frac{dT_i}{dr}\left(\frac{L_n}{\rho_s c_s}\right)$$

is the equilibrium velocity of the **poloidal velocity**, which is

$$u_\theta = u_\perp - \Theta u_\parallel$$

The **toroidal velocity** is

$$u_\varphi = u_\parallel + \Theta u_\perp$$

From the same paper of **Su, Yushmanov**,

$$\begin{aligned}\nu^{nc} &= \frac{3\mu^{nc}}{nm}\frac{B_T^2}{B_\theta^2}\frac{1}{B_0^2}\left\langle\left(\frac{(\mathbf{B}\cdot\boldsymbol{\nabla})B}{B}\right)^2\right\rangle \\ &\approx \frac{3}{2}\frac{B_T^2}{B_\theta^2}\frac{\sqrt{\varepsilon}}{1+\nu_*}\left(\frac{\nu L_n}{c_s}\right)\end{aligned}$$

or

$$v^{nc} = 3\mu^{nc} \frac{1}{nm} \left(\frac{q}{\varepsilon}\right)^2 \frac{1}{B_0^2} \left\langle \left(\nabla_\| B\right)^2 \right\rangle = \frac{3}{2} \left(\frac{q}{\varepsilon}\right)^2 \frac{\sqrt{\varepsilon}}{1+\nu_*} \left(\frac{\nu L_n}{c_s}\right)$$

$$S_0 = \frac{B_\theta}{B_T}$$

$$u^{nc} = k_{\nu_*} \frac{c}{eB_T} \frac{dT_i}{dr} \left(\frac{L_n}{\rho_s c_s}\right)$$

A useful approximation

$$\frac{\partial}{\partial \psi} = \frac{1}{2\pi R B_\theta} \frac{\partial}{\partial x}$$

for $\varepsilon \ll 1$.

A conclusion from this paper by **Su Yushmanov Dong Horton**. The ratio of

1. the mean poloidal velocity

$$u_\perp = 0.44 \ v_{dia,e}$$

and

2. the mean toroidal velocity

$$u_\| = 3.5 \ v_{dia,e}$$

is about 1/9.

Possible rephrase: a radial electric field has been established at equilibrium. It produces poloidal and toroidal rotations. Even if the toroidal magnetic field ($\sim$ poloidal rotation) is much higher than the poloidal magnetic field ($\sim$toroidal rotation) the inertia of plasma against poloidal rotation is high and the ratio is actually reversed, favorable for toroidal rotation. Plus, the magnetic damping acting against the poloidal rotation.

13.3 Toroidal and poloidal projections of the plasma velocities (Shaing)

This set is used by Shaing in [17].

Apparently this includes the *resonance broadening* (of the resonance $u_{\parallel} = 0$) due to the time variation of the radial electric field, when the poloidal rotation is damped, on a time scale of τ_{ii}. This also appears in the treatment of **Lebedev, Diamond et al.**

The **toroidal** component:

$$\begin{aligned}
\frac{\partial}{\partial t}\left\langle R^2 \boldsymbol{\nabla}\varphi \cdot n\mathbf{V}\right\rangle + \frac{1}{M}\left\langle \frac{\partial \mathbf{E}}{\partial t} \cdot \boldsymbol{\nabla}\psi \right\rangle &= -\nu_{eff}\left\langle R^2 \boldsymbol{\nabla}\varphi \cdot n\mathbf{V}\right\rangle \\
&\quad - \left\langle R^2 \boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}\right\rangle - \frac{1}{M}\left\langle R^2 \boldsymbol{\nabla}\varphi \cdot \mathbf{J}_r \times \mathbf{B}\right\rangle
\end{aligned}$$

and the **poloidal** component

$$\begin{aligned}
M_{eff}\frac{\partial}{\partial t}\frac{\left\langle n\mathbf{V} \cdot \mathbf{B}_{\theta}\right\rangle}{\left\langle n\right\rangle} - \frac{I}{\left\langle R^2 n\right\rangle}\frac{1}{M}\left\langle \frac{\partial \mathbf{E}}{\partial t} \cdot \boldsymbol{\nabla}\psi \right\rangle &= \frac{I}{\left\langle R^2 n\right\rangle}\frac{1}{M}\left\langle \mathbf{J}_r \cdot \boldsymbol{\nabla}\psi\right\rangle \\
&\quad - \left\langle \mathbf{B} \cdot \mathbf{V} \cdot \boldsymbol{\nabla}\mathbf{V}\right\rangle - \left\langle \frac{\mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}}{nM}\right\rangle \\
&\quad -\nu_{eff}\left\langle \mathbf{B}_{\theta} \cdot \mathbf{V}\right\rangle
\end{aligned}$$

NOTE

The term

$$\frac{1}{M}\left\langle \frac{\partial \mathbf{E}}{\partial t} \cdot \boldsymbol{\nabla}\psi \right\rangle$$

is the time variation of the *radial* electric field.

$$\varepsilon_0\left(1 + \frac{c^2}{v_A^2}\right)\frac{\partial E_r}{\partial t} = e\left(\Gamma_i - \Gamma_e\right) + ...$$

This is *polarization* of plasma.

END OF NOTE

Here the magnetic field is

$$\mathbf{B} = \boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi + I\boldsymbol{\nabla}\varphi$$

$$\mathbf{B}_{\theta} = \boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi$$

$$\begin{aligned}
I &= R^2 \mathbf{B} \cdot \boldsymbol{\nabla}\varphi \\
&= RB_{\varphi}
\end{aligned}$$

The **effective poloidal inertia**

$$M_{eff} = 1 + Cq^2$$

where

$$C = \begin{cases} 2 & \text{for} \quad M_p \ll 1 \\ 2/\varepsilon & \text{for} \quad M_p \simeq 1 \end{cases}$$

and the Mach number is defined as:

$$M_p = \frac{V_\theta}{v_{th,i}} \frac{B}{B_\theta}$$

NOTE

Since B_θ has radial variation, there is a radius where M_p is close to 1. There the inertia against the poloidal rotation grows unlimitted. It is the shock. **Rosenbluth Hazeltine Lee** 1971 and **Friedberg**.

13.4 Fluid rotation, radial current and poloidal variation of parameters (Rozhansky Tendler)

The full plasma momentum balance can use the ion velocity in the inertail term

$$n_i m_i \frac{d\mathbf{u}_i}{dt} = -\mathbf{\nabla}(p_i + p_e) - \mathbf{\nabla} \cdot \boldsymbol{\pi}_i + \mathbf{j} \times \mathbf{B} + \mathbf{F}$$

and two projections are made

parallel, and
toroidal

- parallel with the magnetic field line

$$\begin{aligned} \left\langle \mathbf{B} \cdot n_i m_i \frac{d\mathbf{u}_i}{dt} \right\rangle &= -\left\langle \mathbf{B} \cdot \mathbf{\nabla} \cdot \boldsymbol{\pi}_i \right\rangle^{NEO} \\ &\quad -\left\langle \mathbf{B} \cdot \mathbf{\nabla} \cdot \boldsymbol{\pi}_i \right\rangle^{AN} \end{aligned}$$

- toroidal projection

$$\begin{aligned} \langle j \rangle &= \left\langle \frac{1}{B^2} \mathbf{B}_T \cdot n_i m_i \frac{d\mathbf{u}_i}{dt} \right\rangle \\ &\quad + \left\langle \frac{1}{B^2} \mathbf{B}_T \cdot \mathbf{\nabla} \cdot \boldsymbol{\pi}_i \right\rangle^{NEO} + \left\langle \frac{1}{B^2} \mathbf{B}_T \cdot \mathbf{\nabla} \cdot \boldsymbol{\pi}_i \right\rangle^{AN} \end{aligned}$$

and it is surface averaged

$$\begin{aligned}\langle f \rangle &= \frac{1}{2\pi}\int_0^{2\pi} h d\theta \ f \\ \text{where } h &= 1+\varepsilon\cos\theta\end{aligned}$$

At stationarity and without anomalous transport

$$\langle \mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_i\rangle^{NEO} = 0$$

here

$$\boldsymbol{\pi}_i = \left(p_{\|i} - p_{\perp i}\right)\left(\widehat{\mathbf{n}}\widehat{\mathbf{n}} - \frac{1}{3}\mathbf{I}\right)$$

(Chow Goldberger Law).

The divergence of this tensor

$$(\boldsymbol{\nabla}\cdot\boldsymbol{\pi})_k = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}\pi^i_k\right) - \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\pi^{ij}$$

$$\begin{aligned} g_{11} &= 1 \\ g_{22} &= r^2 \\ g_{33} &= (1+\varepsilon\cos\theta)^2\end{aligned}$$

It is assumed that the anisotropy of the pressure is only dependent on the poloidal angle θ

$$p_{\|i} - p_{\perp i} \sim \text{function of } \theta$$

and

$$\mathbf{B}_T = B_T\frac{\widehat{\mathbf{e}}_\varphi}{\sqrt{|g|}}$$

it is obtained

$$\begin{aligned}\langle j\rangle &= \left\langle \frac{1}{B^2}\mathbf{B}_T\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_i\right\rangle^{NEO} \\ &= 0\end{aligned}$$

This is a purely neoclassical result: *there is no radial current at stationarity and in the absence of anomalous contributions to transport.*

The equation of continuity

$$\boldsymbol{\nabla}\cdot(n\mathbf{u}_i) = 0$$

leads to

$$
\begin{aligned}
u_{i\theta} &= \frac{V_\theta}{1+\varepsilon\cos\theta} \\
u_{i\varphi} &= \overline{u}_\varphi - 2qV_\theta\cos\theta \\
&\quad +\varepsilon\overline{u}_\varphi\cos\theta + 2\varepsilon q V_\theta \\
u_{i\|} &= u_{i\varphi} + \Theta u_{i\theta}
\end{aligned}
$$

This is for fluid.

Kinetic.

$$
\begin{aligned}
&\frac{\partial f_j}{\partial t} + \mathbf{V}_j \cdot \boldsymbol{\nabla} f_j \\
&+\dot{V}_{\|j}\left(\frac{\partial f_j}{\partial V_{\|j}}\right) + \dot{V}_{\perp j}\left(\frac{\partial f_j}{\partial V_{\perp j}}\right) \\
=\ & St\left(f_j\right)
\end{aligned}
$$

for $j = e, i$.

The velocities are

$$
\begin{aligned}
\mathbf{V}_j &= -\frac{1}{\Omega_j}\frac{1}{R}\left(\frac{V_{\perp j}^2}{2} + V_{\|j}^2\right)\left(\widehat{\mathbf{e}}_\theta\cos\theta + \widehat{\mathbf{e}}_r\sin\theta\right) \\
&+\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B} \\
&+V_{\|j}\widehat{\mathbf{e}}_\| \\
&+u_r\widehat{\mathbf{e}}_r
\end{aligned}
$$

where u_r is the fluid radial velocity of the ions, due to diffusion and convection.

In this u_r enters also the prompt loss of NBI ions.

The guiding center velocities are

$$
\frac{dV_{\|j}}{dt} = \Theta\frac{1}{B}\left(-\frac{d\phi}{rd\theta}\right) - \Theta\frac{1}{R}\frac{V_{\perp j}^2}{2}\sin\theta + \frac{1}{R}V_0V_{\|j}\sin\theta
$$

$$
\frac{dV_{\perp j}}{dt} = \Theta\frac{1}{R}\frac{V_{\|j}V_{\perp j}}{2}\sin\theta + \frac{1}{R}\frac{V_0V_{\perp j}}{2}\sin\theta
$$

where, according to a definition with negative electric field

$$
V_0 = \frac{1}{B}\frac{d\Phi_0}{dr}
$$

Note we should recognize the structure: the term $\Theta\frac{1}{B}\left(-\frac{d\phi}{rd\theta}\right)$ is the projection *along the magnetic line* : $()_\theta \times \frac{B_\theta}{B_T}$ of the poloidal velocity produced by the radial electric field. But it is question of the variation of the potential on the magnetic surface, $\phi(\theta)$.

To solve the drift-kinetic equation we expand f_j in two terms, the equilibrium function and the correction that shows the variation on the poloidal direction

$$f_j = f_{0j}(r) + f_{1j}(r,\theta)$$

and

$$\Phi = \Phi_0 + \Phi_1(r,\theta)$$

The equation is linearized and it is taken

$$\nu_i \rightarrow 0$$

then

$$f_{1j} = \left[PV\left(\frac{1}{V_0+\Theta V_{\|j}}\right) + \pi\delta\left(V_0+\Theta V_{\|j}\right)\frac{\partial}{\partial\theta}\right]\ \widehat{A}_j\ f_{0j}$$

the operator is

$$\begin{aligned}
\widehat{A}_j \quad \equiv \quad & \left\{\frac{e_j}{m_j}\Phi_1\left[\Theta\frac{\partial}{\partial V_{\|j}} + \frac{1}{\Omega_j}\frac{\partial}{\partial r}\right]\right. \\
& -\left(\frac{V_{\perp j}^2}{2} + V_{\|j}^2\right)\frac{1}{\Omega_j}\varepsilon\cos\theta \\
& \left. -m_j\left(\frac{V_{\perp j}^2}{2} + V_{\|j}^2\right)\varepsilon\frac{1}{T_j}\cos\theta\right\}
\end{aligned}$$

and the equilibrium distribution function is shifted by the toroidal velocity

$$f_{0j} = \frac{n_0}{\left(2\pi T_j/m_j\right)^{3/2}}\exp\left[-\frac{\left(V_{\|j}-u_\varphi\right)^2}{2T_j/m_j} - \frac{V_{\perp j}^2}{2T_j/m_j}\right]$$

With this distribution function we calculate the density correction in order 1 and impose neutrality

$$\begin{aligned}
n_1 \quad &= \quad \int f_{1e} d\mathbf{V} \\
&= \quad \int f_{1i} d\mathbf{V}
\end{aligned}$$

To calculate the pressure tensor

$$p_{\|i} = \int 2\pi V_{\|i} V_{\perp i} dV_{\|i} dV_{\perp i} \; V_{\|i} \; f_i$$
$$p_{\perp i} = \int \pi V_{\perp i}^2 dV_{\|i} dV_{\perp i} \;\; V_{\perp i} \; f_i$$

Then the parallel viscosity in neoclassical expression is

$$\begin{aligned} & \langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_i \rangle^{NEO} \\ = & \sqrt{\pi} n \varepsilon^2 \sqrt{m_i} \sqrt{T_i} B_0 \frac{V_\theta - V_\theta^{NEO}}{\sqrt{2} r} \end{aligned}$$

where

$$V_\theta^{NEO} = \left[\frac{1}{2} \right] \times \frac{1}{eB_0} \frac{dT_i}{dr} \quad \text{(Hazeltine)}$$

The condition of neutrality imposed to the first order correction to the *density variation on the magnetic surface* leads to

$$\frac{e\phi_1}{T_e} = \sqrt{\pi} n \varepsilon \sqrt{m_i} \sqrt{T_i} \sin\theta \sqrt{2} B_0 \Theta \left(1 + \frac{T_e}{T_i} \right) \left(V_\theta + V_\theta^{NEO} \right)$$

See for comparison the result of **Rosenbluth Hazeltine Lee**.

Now there is a variation of the electric potential on the magnetic surface, along the poloidal direction, $\phi_1(\theta)$. This makes a contribution to the drift velocity

$$(V_D)_r = -\frac{1}{\Omega_{ci}} \frac{1}{R} \left(\frac{V_{\perp i}^2}{2} + V_{\|i}^2 \right) - \frac{1}{e} \frac{1}{B_0} \frac{\partial \phi_1}{r \partial \theta}$$

and the radial current is

$$\langle j \rangle = \left\langle \int 2\pi dV_{\|i} V_{\perp i} dV_{\perp i} \;\; (V_D)_r \;\; (f_{0i} + f_{1i}) \right\rangle$$

the condition

$$\frac{|V_\theta|}{\Theta} \ll \sqrt{\frac{T_e}{m_e}}$$

means that the parallel velocity resulted from poloidal rotation $\left(V_\theta + \Theta V_\| \sim 0 \right)$ is much smaller than the thermal velocity. Then the response of the electrons is adiabatic

$$\frac{n_1}{n_e} = \frac{|e| \, \phi_1}{T_e}$$

Then one can use the distribution function to order 1 and obtain the density n_1 perturbation on the magnetic surface and equalize it with the Boltzmann distribution. It results the *harmonic* components of the density, *i.e.* the $\cos\theta$ and the $\sin\theta$ components.

$$\frac{n_1}{n_e} = n_c^* \cos\theta + n_s^* \sin\theta$$

After integration of f over the velocity space

$$\begin{aligned} n_c^* &= \frac{1}{\Delta}\left(-2\varepsilon\left\{\frac{2\alpha^2+1}{2\alpha}D(\alpha)-\alpha\right\}\right. \\ &\qquad \times\left\{1+\frac{T_e}{T_i}\left[1-D(\alpha)\right]\right\} \\ &\qquad \left.+\frac{T_e}{T_i}\pi\varepsilon\alpha^2\left(2\alpha^2+1\right)\exp\left(-2\alpha^2\right)\right) \end{aligned}$$

$$\begin{aligned} n_s^{**} &= \frac{1}{\Delta}\left(\left\{1+\frac{T_e}{T_i}\left[1-D(\alpha)\right]\right\}\left(2\alpha^2+1\right)\right. \\ &\qquad \left.+2\frac{T_e}{T_i}\left[\frac{2\alpha^2+1}{2\alpha}D(\alpha)-\alpha\right]\right) \\ &\qquad \times\sqrt{\pi}\varepsilon\alpha\exp\left(-\alpha^2\right) \end{aligned}$$

where

$$\begin{aligned} \Delta &= \left\{1+\frac{T_e}{T_i}\left[1-D(\alpha)\right]\right\}^2 \\ &\qquad +\pi\alpha^2\exp\left(-2\alpha^2\right)\frac{T_e^2}{T_i^2} \end{aligned}$$

$$\alpha \equiv \frac{V_0}{\Theta c_s}$$

This variable α is the ratio of the poloidal velocity V_0 to the poloidal projection of the sound speed. It is defined also in **Rosenbluth Hazeltine LEE**. There it defines the critical state where the acoustic waves in parallel direction have the same velocity as the plasma which rotates poloidally. Shock is expected.

And

$$c_s = \sqrt{\frac{2T_i}{m_i}}$$

In the **Reviews of Plasma Physics 19 Rozhansky Tendler** the Dawson integral is

$$D(\alpha) = 2\alpha \exp\left(-\alpha^2\right) \int_0^\alpha \exp\left(t^2\right) dt$$

Now, because

$$V_0 \equiv \frac{1}{B_0}\frac{d\Phi}{dr} = V_E$$

is the *poloidal* velocity due to the electric field, and Θc_s is much larger than $\Theta V_\parallel$ the parallel velocity projected on the poloidal direction, we have

$$\alpha \equiv \frac{V_E}{\Theta c_s} \ll \frac{V_E}{\Theta V_\parallel} \sim 1$$

then

$$\alpha \ll 1$$

If however

$$\alpha \sim 1$$

then

$$\frac{n_1}{n} \sim \varepsilon$$

When

$$\alpha \gg 1$$

which means that the poloidal rotation is much higher than the projected parallel velocity, then

$$\frac{n_1(\theta)}{n_0} = -2\varepsilon \cos\theta \left[1 + \frac{1}{\alpha^2}\left(1 + \frac{T_e}{2T_i}\right)\right]$$

when $\alpha \gg 1$.

Now one can calculate the *harmonic* component of the parallel flow velocity

$$u_{\parallel i}^{(1)} = \frac{1}{n} \int 2\pi V_{\parallel i} dV_{\parallel i}\ V_{\perp i} dV_{\perp i}\ f_{1i}$$

and in first order in ε one obtains

$$u_{\parallel i}^{(1)}(\theta) = -\frac{V_0}{\Theta}\left(2\varepsilon\cos\theta + \frac{n_1}{n}\right)$$

13.5 Neoclassical ion transport in rotating plasmas (Hinton Wong)

This is **Hinton Wong PF1985**

The text is also in *plasma general derivation of drift kinetic eq*.

The transport of heat is OK but of momentum is more than an order of magnitude in error.

The theory must be improved.

Effects: Coriolis and Centrifugal Force.

A temperature gradient drives an angular momentum flux.

A gradient of the toroidal angular velocity drives a heat flux.

Method

expansion in small gyroradius of the Fokker Planck equation in the system of reference moving with the plasma.

The electron parallel momentum

$$\nabla_{\parallel}\left(\ln n_e - \frac{e\Phi_0}{T_e}\right) = 0$$

and the density of electrons

$$n_e = N_e(\psi)\exp\left(\frac{e\widetilde{\Phi}_0}{T_e}\right)$$

the poloidally varying part of the potential is determined from neutrality.

The ion parallel momentum

$$\nabla_{\parallel}\left(\ln n_i + \frac{e\Phi_0}{T_i} - \frac{m_i\omega^2 R^2}{2T_i}\right) = 0$$

(gradient of pressure, electric field, centrifugal force) This is the usual parallel balance : $-\nabla_{\parallel}p_i$ balanced by the electrif field $eE_{\parallel}$, witout collisional friction, BUT, in addition, the centrifugal force $v^2/v_{th,i}^2$ where $v = \omega R$.

Integrated, gives

$$n_i = N(\psi)\exp\left(-\frac{\Xi}{T_i}\right)$$

where

$$\Xi = e\widetilde{\Phi} - \frac{m_i\omega^2 R^2}{2}$$

The centrifugal force is derived from a potential, $m_i v^2/2$, for $v = \omega R$, and this potential is combined with the electrostatic potential.

Using charge neutrality

$$N(\psi) = N_e(\psi) \exp\left[-\frac{m_i \omega^2 \langle R^2 \rangle}{2T_i}\right]$$

and the potential variation on the surface

$$\frac{e\widetilde{\Phi}_0}{T_e} = \frac{m_i \omega^2 \left(R^2 - \langle R^2 \rangle\right)}{2\left(T_e + T_i\right)}$$

the ion density is shifted outward in major radius by the centrifugal forces

then, for neutrality, the plasma must develop an electrostatic field that imposes to electrons to shift and ensure neutrality

The linear operator that occurs at the expansion of the Fokker Planck equation is the convective derivative of the distribution function plus the modified convection in velocity space

$$\begin{aligned}
\Lambda f &= (\mathbf{v}' + \mathbf{u}_0) \cdot \boldsymbol{\nabla}' f \\
&\quad - \left(\frac{e}{m}\boldsymbol{\nabla}\phi_0 + \frac{\partial \mathbf{u}_0}{\partial t_0} + \left[(\mathbf{v}' + \mathbf{u}_0) \cdot \boldsymbol{\nabla}\right]\mathbf{u}_0\right) \cdot \frac{\partial f}{\partial \mathbf{v}'}
\end{aligned}$$

where the velocity is in the *rotating frame* is

$$\mathbf{v}' = \mathbf{v} - \mathbf{u}_0(\mathbf{x}, t)$$

The first term is the convection of the distribution function by the velocity $\mathbf{v}$ which now is written by its composition: the rotation support, $\mathbf{u}_0$ and the *relative* velocity $\mathbf{v}'$.

We note that in the *acceleration* term we have

$$\begin{aligned}
-\frac{e}{m}\boldsymbol{\nabla}\phi_0 &= \mathbf{E} \ \ \text{the electric field acceleration} \\
-\left(\frac{\partial \mathbf{u}_0}{\partial t_0} + \left[(\mathbf{v}' + \mathbf{u}_0) \cdot \boldsymbol{\nabla}\right]\mathbf{u}_0\right) &= \\
-\left(\frac{\partial \mathbf{u}_0}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\,\mathbf{u}_0\right) &= -\frac{d\mathbf{u}_0}{dt} \ \ \text{acceleration of the support rotation}
\end{aligned}$$

One must include here the *gyration* part of the velocity

$$\begin{aligned}
\mathbf{v}' &= \widehat{\mathbf{n}} v_{\|} + \mathbf{v}_{\perp} \\
&= \widehat{\mathbf{n}} v_{\|} + v_{\perp}\left(\widehat{\mathbf{e}}_1 \cos\zeta + \widehat{\mathbf{e}}_2 \sin\zeta\right)
\end{aligned}$$

which implies the transformations

$$\frac{\partial f}{\partial \mathbf{v}'} = \widehat{\mathbf{n}}\frac{\partial f}{\partial v_\parallel} + \frac{\mathbf{v}_\perp}{v_\perp}\frac{\partial f}{\partial v_\perp} + \frac{\widehat{\mathbf{n}} \times \mathbf{v}_\perp}{v_\perp^2}\frac{\partial f}{\partial \zeta}$$

and

$$\begin{aligned}\boldsymbol{\nabla}' f &= \boldsymbol{\nabla} f + (\boldsymbol{\nabla} \cdot \mathbf{b}) \cdot \mathbf{v}_\perp \left(\frac{\partial f}{\partial v_\parallel} - \frac{v_\parallel}{v_\perp}\frac{\partial f}{\partial v_\perp}\right) \\ &\quad + [(\boldsymbol{\nabla} \cdot \widehat{\mathbf{e}}_1)\cos\zeta + (\boldsymbol{\nabla} \cdot \widehat{\mathbf{e}}_2)\sin\zeta] \cdot \frac{\mathbf{v}'}{v_\perp}\frac{\partial f}{\partial \zeta}\end{aligned}$$

Start again.
This is also in *plasma general derivation drift kinetic eq.*

in the *rotating frame*

$$\mathbf{v}' = \mathbf{v} - \mathbf{u}_0 (\mathbf{x}, t)$$

where

$$\begin{aligned}\mathbf{u}_0 &= \omega R \widehat{\mathbf{e}}_\varphi \quad \text{(toroidal)} \\ &\quad + F\mathbf{B} \quad \text{(rigid body)}\end{aligned}$$

the velocity is toroidal and includes a term which is the rotation of a surface as a rigid object, $F\mathbf{B}$.

$$\omega = -\frac{\partial \Phi_{-1}}{\partial \psi}$$

This is the frequency of rotation, $\omega \sim v/R$ where $v \sim E/B_\theta$ (since v is toroidal, the magnetic field is the poloidal component) and $E \equiv$ radial, $E \sim -\partial\Phi/\partial r = (-\partial\Phi/\partial\psi) \times (\partial\psi/\partial r)$. We have

$$|\boldsymbol{\nabla}\psi| = RB_\theta$$

then

$$\begin{aligned}E &\sim -\frac{\partial \Phi}{\partial \psi}|\boldsymbol{\nabla}\psi| = -\frac{\partial \Phi}{\partial \psi} RB_\theta \\ v &\sim \frac{E}{B_\theta} = -\frac{\partial \Phi}{\partial \psi} R\end{aligned}$$

from where

$$\omega \sim \frac{v}{R} = -\frac{\partial \Phi}{\partial \psi}$$

We **note** that the rotation is considered sustained by an electrostatic potential Φ which occurs in order -1. Higher orders correspond to variations on surfaces.

$$n\mathbf{u}_0 = \int d^3v\ \mathbf{v}f$$

and

$$\mathbf{u}_0 \cdot \boldsymbol{\nabla}\psi = 0 \quad \text{(in surface)}$$

$$\begin{aligned}
\mathbf{v}' &= \text{relative to rotating frame} \\
&= \widehat{\mathbf{n}}v_\parallel + \mathbf{v}_\perp \\
&= \widehat{\mathbf{n}}v_\parallel + v_\perp\left(\widehat{\mathbf{e}}_1\cos\zeta + \widehat{\mathbf{e}}_2\sin\zeta\right) \\
\zeta &\equiv \text{gyroangle}
\end{aligned}$$

which implies the transformations

$$\frac{\partial f}{\partial \mathbf{v}'} = \widehat{\mathbf{n}}\frac{\partial f}{\partial v_\parallel} + \frac{\mathbf{v}_\perp}{v_\perp}\frac{\partial f}{\partial v_\perp} + \frac{\widehat{\mathbf{n}} \times \mathbf{v}_\perp}{v_\perp^2}\frac{\partial f}{\partial \zeta}$$

and

$$\begin{aligned}
\boldsymbol{\nabla}' f &= \boldsymbol{\nabla} f + (\boldsymbol{\nabla}\cdot\mathbf{b})\cdot\mathbf{v}_\perp\left(\frac{\partial f}{\partial v_\parallel} - \frac{v_\parallel}{v_\perp}\frac{\partial f}{\partial v_\perp}\right) \\
&\quad + \left[(\boldsymbol{\nabla}\cdot\widehat{\mathbf{e}}_1)\cos\zeta + (\boldsymbol{\nabla}\cdot\widehat{\mathbf{e}}_2)\sin\zeta\right]\cdot\frac{\mathbf{v}'}{v_\perp}\frac{\partial f}{\partial\zeta}
\end{aligned}$$

The invariants, redefined in the rotating frame

$$\begin{aligned}
\mu &= \frac{v_\perp^2}{2B} \\
\epsilon &= \frac{1}{2}\left(v_\parallel^2 + v_\perp^2\right) + \frac{e\widetilde{\Phi}}{m} - \frac{\omega^2R^2}{2}
\end{aligned}$$

The part of the distribution function that depends on the *gyroangle* ζ is

$$\begin{aligned}
\widetilde{f}_1 &= \frac{\mathbf{v}'\times\widehat{\mathbf{n}}}{\Omega}\cdot\boldsymbol{\nabla}\psi\left[\frac{N'}{N} + \frac{e}{T}\left\langle\Phi_0\right\rangle'\right. \\
&\quad \left. + \left(\frac{m\varepsilon}{T} - \frac{3}{2}\right)\frac{T'}{T} + \frac{m}{T}\left(\frac{Iv_\parallel}{B} + \omega R^2\right)\omega'\right] \\
&\quad + \frac{\mathbf{v}'\ \mathbf{v}' : \left(\widehat{\mathbf{e}}_1\widehat{\mathbf{e}}_1 - \widehat{\mathbf{e}}_2\widehat{\mathbf{e}}_2\right)\left|\boldsymbol{\nabla}\psi\right|^2}{2\Omega B v_{th,i}^2}\omega'\ f_0
\end{aligned}$$

The detailed form

$$(\mathbf{v}' \times \widehat{\mathbf{n}}) \cdot \boldsymbol{\nabla}\psi = v_{\parallel} \sin\zeta \ \ |\boldsymbol{\nabla}\psi|$$

$$\mathbf{v}' \ \mathbf{v}' : (\widehat{\mathbf{e}}_1\widehat{\mathbf{e}}_1 - \widehat{\mathbf{e}}_2\widehat{\mathbf{e}}_2) = v_{\perp}^2 \cos(2\zeta)$$

The drift velocity is

$$\begin{aligned}\mathbf{v}_D \ = \ & \frac{1}{\Omega}\widehat{\mathbf{n}} \times \Big(\mu \boldsymbol{\nabla} B + v_{\parallel}^2 (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}} + \frac{e}{m}\boldsymbol{\nabla}\Phi_0 \\ & \qquad -\omega^2 \mathbf{R} \quad \text{centrifugal drift } v^2/R \\ & \qquad +2\omega\widehat{\mathbf{e}}_z \times \widehat{\mathbf{n}} v_{\parallel}\Big) \quad \text{Coriolis}\end{aligned}$$

NOTE
Recall the meaning:
effect Coriolis is the *drift* that a particle suffers when it moves from the North pole toward the Equator, on a planet that has Ω rotation around the axis.

$$\mathbf{v}_D^{Coriolis} = 2\boldsymbol{\Omega} \times \mathbf{v}$$

The drift is felt in the system of reference that rotates with planet, where the initially unperturbed trajectory North-Equator (a meridian) is static.
END

NOTE
A decision is taken
To consider that the rotation velocity

$$\mathbf{u}_0$$

is associated to a vectro of angular rotation

$$\omega \widehat{\mathbf{e}}_z \quad \equiv \text{vertical on the equatorial plane}$$

This is a major restriction, the rotation is toroidal.
END

The new terms

- the centrifugal force (including the mass factor m)

$$m\omega^2 \mathbf{R}$$

The drift resulting from this term is

$$\frac{1}{\Omega}\widehat{\mathbf{n}} \times \left(-\omega^2\right) \mathbf{R} = \text{vertical}$$

- the Coriolis term (including the mass factor m)

$$2mv_{\|}\widehat{\mathbf{n}} \times \omega \widehat{\mathbf{e}}_z$$

The drift resulting from this term is

$$\begin{aligned}
\frac{1}{\Omega}\widehat{\mathbf{n}} \times \left(2v_{\|}\omega\right)\left(\widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_z\right) &= \frac{2v_{\|}\omega}{\Omega}\left[\widehat{\mathbf{n}}\left(\widehat{\mathbf{n}} \cdot \widehat{\mathbf{e}}_z\right) - \widehat{\mathbf{e}}_z\right] \\
&\approx -\frac{2v_{\|}\omega}{\Omega}\widehat{\mathbf{e}}_z \\
&= \text{vertical}
\end{aligned}$$

in the *rotating reference system.*
Comparing the two new terms

$$\frac{\text{centrifugal}}{\text{Coriolis}} \sim \frac{\omega^2 R}{2v_{\|}\omega} \sim \frac{1}{2}\frac{\omega}{v_{\|}/R}$$

where $v_{\|}$ = velocity of the particle. The ratio can be put in connection with the definition of a Mach number which compares the speed of rotation with the speed of sound.

the formulas

$$\widehat{\mathbf{n}} \times \boldsymbol{\nabla}\psi = I\widehat{\mathbf{n}} - BR\widehat{\mathbf{e}}_\varphi$$

$$\left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{n}} \cdot \widehat{\mathbf{e}}_\varphi R = I\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\left(\frac{1}{B}\right)$$

$$\omega\widehat{\mathbf{e}}_z \times \widehat{\mathbf{n}} \cdot \widehat{\mathbf{e}}_\varphi R = \omega\widehat{\mathbf{n}} \cdot \mathbf{R}$$

lead to

$$\mathbf{v}_D \cdot \boldsymbol{\nabla}\psi = \frac{m}{e}v_{\|}\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\left(\frac{Iv_{\|}}{B} + \omega R^2\right)$$

where

$$\boldsymbol{\nabla} \equiv \boldsymbol{\nabla}_{\epsilon=ct,\mu=ct}$$

and

$$v_{\|} = \left\{2\left[\epsilon - \mu B - \frac{e}{m}\widetilde{\Phi}_0 + \frac{\omega^2 R^2}{2}\right]\right\}^{1/2}$$

Identities that involve the versors

$$-\widehat{\mathbf{e}}_2\widehat{\mathbf{e}}_2 : \left(\boldsymbol{\nabla}\ \widehat{\mathbf{n}}\right) = \widehat{\mathbf{e}}_1\widehat{\mathbf{e}}_1 : \left(\boldsymbol{\nabla}\ \widehat{\mathbf{n}}\right) + \frac{\nabla_{\|}B}{B}$$

$$\begin{aligned}\widehat{\mathbf{e}}_1 \cdot (\boldsymbol{\nabla}\ \widehat{\mathbf{n}}) \cdot \widehat{\mathbf{e}}_1 &= \widehat{\mathbf{e}}_2 \cdot (\boldsymbol{\nabla}\times\widehat{\mathbf{e}}_1) \\ &= (\widehat{\mathbf{e}}_2 \cdot \widehat{\mathbf{e}}_\varphi)\, \widehat{\mathbf{e}}_\varphi \cdot (\boldsymbol{\nabla}\times\widehat{\mathbf{e}}_1)\end{aligned}$$

$$\widehat{\mathbf{e}}_\varphi \cdot (\boldsymbol{\nabla}\times\widehat{\mathbf{e}}_1) = \frac{R\mathbf{B}\cdot\boldsymbol{\nabla}\left|\boldsymbol{\nabla}\psi\right|}{\left|\boldsymbol{\nabla}\psi\right|^2}$$

$$\widehat{\mathbf{e}}_2 \cdot \widehat{\mathbf{e}}_\varphi = -\frac{\left|\boldsymbol{\nabla}\psi\right|}{BR}$$

and

$$\begin{aligned}(\widehat{\mathbf{e}}_1\widehat{\mathbf{e}}_1 - \widehat{\mathbf{e}}_2\widehat{\mathbf{e}}_2) &: (\boldsymbol{\nabla}\widehat{\mathbf{n}}) \\ &= -\frac{B}{\left|\boldsymbol{\nabla}\psi\right|^2}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\left(\frac{\left|\boldsymbol{\nabla}\psi\right|^2}{B}\right)\end{aligned}$$

The linearized drift kinetic equation for $\overline{f}_1$ the averaged function of distribution in order 1, dependent on

$$(\mathbf{x}, t, \epsilon, \mu, \sigma = \pm 1)$$

$$\begin{aligned}& v_\parallel \nabla_\parallel \overline{f}_1 - C^{lin}\overline{f}_1 \\ = & -\frac{e}{T} v_\parallel \nabla_\parallel \Phi_1\ f_0 \\ & -v_\parallel \left(\nabla_\parallel \alpha_1\right) \left[\frac{N'}{N} + \frac{e}{T}\left\langle \Phi_0 \right\rangle' + \frac{T'}{T}\right] f_0 \\ & -v_\parallel \left(\nabla_\parallel \alpha_2\right) \left[\frac{T'}{T}\right] f_0 \\ & -v_\parallel \left(\nabla_\parallel \alpha_3\right) \left[\frac{\omega'}{\omega}\right] f_0\end{aligned}$$

where the *fluxes* are

$$\alpha_1 = \frac{m}{e}\left(\frac{I v_\parallel}{B} + \omega R^2\right)$$

$$\alpha_2 = \left(\frac{m\epsilon}{T} - \frac{5}{2}\right)\alpha_1$$

$$\alpha_3 = \frac{m\omega}{e v_{th,i}^2}\left[\left(\frac{I v_\parallel}{B} + \omega R^2\right)^2 + \mu\frac{\left|\boldsymbol{\nabla}\psi\right|^2}{B}\right]$$

with

$$I = RB_\varphi$$

It is introduced the set of *thermodynamical forces*

$$\begin{aligned} A_1 &= \frac{N'}{N} + \frac{e}{T} \langle \Phi_0 \rangle' + \frac{T'}{T} \\ A_2 &= \frac{T'}{T} \\ A_3 &= \frac{\omega'}{\omega} \end{aligned}$$

we note that the operator

$$\begin{aligned} & v_\parallel \nabla_\parallel \\ = & \frac{v_\parallel}{qR} \frac{\partial}{\partial \theta} \end{aligned}$$

occurs in both sides of the drift kinetic equation and this suggests a substitution

$$\begin{aligned} \overline{f}_1 &= f - \frac{e\Phi_1}{T} f_0 \\ & - \left[\left(\frac{I v_\parallel}{B} + \omega R^2 \right) A_1 \right. \\ & + \omega R^2 \left(\frac{m\varepsilon}{T} - \frac{5}{2} \right) A_2 \\ & \left. + \frac{\omega}{v_{th,i}^2} \left(\frac{2 I v_\parallel}{B} \omega R^2 + \omega^2 R^4 \right) A_3 \right] f_0 \end{aligned}$$

The term depending on the electric potential Φ_1 is the *adiabatic* (Boltzmann) response.

Hinton Wong make the observation that the linearized ion-ion collision operator anihilates terms like

$$\left(a + b v_\parallel + c v^2 \right) f_0$$

(due to the conservation in collisions of - respectively, number, momentum and energy).

The drift kinetic equation becomes

$$\begin{aligned} & v_\parallel \nabla_\parallel f - C^{lin} f \\ = & -v_\parallel \left(\nabla_\parallel \beta_2 \right) A_2 \ f_0 \\ & -v_\parallel \left(\nabla_\parallel \beta_3 \right) A_3 \ f_0 \end{aligned}$$

where

$$
\begin{aligned}
\beta_2 &= \frac{Iv_{\parallel}}{\Omega_i}\frac{m_i\epsilon}{T} - \frac{5}{2} \\
\beta_3 &= \frac{\omega}{\Omega_i v_{th,i}^2}\left(\frac{I^2 v_{\parallel}^2}{B} + \mu\left|\boldsymbol{\nabla}\psi\right|^2\right)
\end{aligned}
$$

Note that this approach will be used later by **Helander** and by **Fulop Helander**.

13.6 Transport in toroidally rotating plasma Catto Bernstein Tessarotto

the toroidal rotation can be large

Then

The poloidal variation of the density must be retained

The basic equation

$$
\frac{\partial f}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} f + \mathbf{a}\cdot\frac{\partial f}{\partial \mathbf{v}} = C
$$

$$
\begin{aligned}
\mathbf{a} =& \frac{Ze}{m_i}\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right) \quad \text{acceleration} \\
\mathbf{B} =& I\boldsymbol{\nabla}\varphi + \boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\varphi \\
I =& RB_T
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\nabla}\psi &= -R^2\mathbf{B}\times\boldsymbol{\nabla}\varphi \\
&= R\widehat{\mathbf{e}}_{\varphi}\times\mathbf{B} \\
&\quad (\text{ or } \quad \left|\boldsymbol{\nabla}\psi\right| = RB_{\theta})
\end{aligned}
$$

The time variation

$$
\frac{\partial\psi}{\partial t} = -R\,\widehat{\mathbf{e}}_{\varphi}\cdot\mathbf{E}
$$

and

$$
\begin{aligned}
\left|\boldsymbol{\nabla}\psi\right| &= RB_{\theta} \\
\widehat{\mathbf{e}}_{\psi} &= \frac{\boldsymbol{\nabla}\psi}{RB_{\theta}}
\end{aligned}
$$

$$\begin{aligned}\mathbf{B}_\theta &= \boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\varphi \\ &= B_\theta \widehat{\mathbf{e}}_\psi \times \widehat{\mathbf{e}}_\varphi\end{aligned}$$

The toroidal rotation

$$\omega = \omega(\psi, t)$$

and

$$f_M = \frac{\overline{N}}{(\pi\ 2T/m_i)^{3/2}} \exp\left[-m_i \frac{(\mathbf{v}-\mathbf{V})^2}{2T}\right]$$

$$\mathbf{V} = \omega R\ \widehat{\mathbf{e}}_\varphi$$

and the constraint

$$\nabla_\parallel \left[\ln \overline{N} + \frac{Ze\Phi}{T} - \frac{\omega^2 R^2}{2T/m_i}\right] = 0$$

Other expressions

$$\omega = \frac{\partial \Phi}{\partial \psi}$$

$$\mathbf{E} = -\omega\ \boldsymbol{\nabla}\psi$$

three invariants

$$\begin{aligned}\mathcal{E} &= \frac{m_i v^2}{2} + Ze\Phi \\ P &= Ze\psi + m_i R\ \widehat{\mathbf{e}}_\varphi \cdot \mathbf{v} \qquad \text{(the azimuthal - canonical momentum - invariant)} \\ H &= \mathcal{E} - \omega P + Ze\left(\omega\psi - \int_{\psi_0}^{\psi} d\psi\ \omega\right)\end{aligned}$$

and the invariant H in the *rotating frame*

$$\begin{aligned}H &= \frac{1}{2} m_i \left[(\mathbf{v}-\mathbf{V})^2 - \mathbf{V}^2\right] \\ &\quad + Ze\left(\Phi - \int_{\psi_0}^{\psi} d\psi\ \omega\right)\end{aligned}$$

The distribution function is

$$f_M = \frac{N_0}{(\pi\ 2T/m_i)^{3/2}} \exp\left(-\frac{H}{T}\right)$$

where

$$\begin{aligned}N_0 &= N_0(\psi, t) \\ &= \overline{N}\ \exp\left[\frac{Ze}{T}\left(\Phi - \int_{\psi_0}^{\psi} d\psi\ \omega\right) - \frac{\omega^2 R^2}{2T/m_i}\right]\end{aligned}$$

One has

$$Ze\left(\Phi-\int_{\psi_0}^{\psi}d\psi\ \omega\right) \equiv \text{effective potential energy associated to the electrostatic field in the rotating frame}$$

The change of referential leads to

$$\mu=\frac{m_i\left|\widehat{\mathbf{n}}\times(\mathbf{v}-\mathbf{V})\right|}{2B}=\frac{m_i w}{2B}$$

$$\mathbf{v}=\mathbf{V}+u\widehat{\mathbf{n}}+\mathbf{w}$$

$$\mathbf{w}=\frac{w}{|\boldsymbol{\nabla}\psi|}\left(\cos\zeta\ \boldsymbol{\nabla}\psi+\sin\zeta\ \widehat{\mathbf{n}}\times\boldsymbol{\nabla}\psi\right)$$

with the volume in the new variables

$$d^3v=\frac{B}{u}\frac{1}{m_i^2}dH\ d\mu\ d\zeta$$

The distribution function

$$f=f_M+\overline{g}+\widetilde{g}$$

where

$$\widetilde{g} \equiv \text{part that depends on the gyroangle } \zeta$$
$$\overline{g} \equiv \text{part which is gyrophase averaged}$$

$$\begin{aligned}\widetilde{g} \;=\; & \frac{m_i}{Ze}\left[\left.\left(\frac{\partial f_M}{\partial\psi}\right)\right|_H\right.\\ & \left.+f_M\frac{m_i}{T}\left(\frac{\partial\omega}{\partial\psi}\right)\left(\omega R^2+uR\ \widehat{\mathbf{e}}_\varphi\cdot\widehat{\mathbf{n}}\right)\right]\ R\widehat{\mathbf{e}}_\varphi\cdot\widehat{\mathbf{e}}_w\\ & +\frac{m_i}{Ze}R^2\frac{1}{2T/m_i}\left(f_M\frac{\partial\omega}{\partial\psi}\right)\widehat{\mathbf{e}}_\varphi\cdot\left(\widehat{\mathbf{e}}_w\ \widehat{\mathbf{e}}_w-\frac{1}{2}w^2\left(\mathbf{I}-\widehat{\mathbf{n}}\ \widehat{\mathbf{n}}\right)\right)\cdot\widehat{\mathbf{e}}_\varphi\end{aligned}$$

$$\begin{aligned}\overline{g} \;=\; & h+\frac{m_i}{Ze}\left\{\left(\omega^2R^2+uR\ \widehat{\mathbf{e}}_\varphi\cdot\widehat{\mathbf{n}}\right)\left.\left(\frac{\partial f_M}{\partial\psi}\right)\right|_H\right.\\ & \left.+\left(f_M\frac{1}{2T/m_i}\right)\left[\left(\omega^2R^2+uR\ \widehat{\mathbf{e}}_\varphi\cdot\widehat{\mathbf{n}}\right)^2+\left(\frac{\mu R^2B_\theta^2}{m_iB}\right)\right]\right\}\end{aligned}$$

where

$$u = \widehat{\mathbf{n}} \cdot (\mathbf{v} - \mathbf{V})$$

$$u^2 = v^2 + \frac{2}{m_i}\left[H - \mu B - Ze\left(\Phi - \int_{\psi_0}^{\psi} d\psi\ \omega\right)\right]$$

The equation for the gyrophase averaged part

$$\begin{aligned}
u\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\overline{g} &= f_M \frac{Ze}{T}\widehat{\mathbf{n}} \cdot (\mathbf{E} + \boldsymbol{\nabla}\Phi) \\
&+ \left(\frac{\partial f_M}{\partial \psi}\right)\bigg|_H \frac{m_i}{Ze}\ u\nabla_{\|}\left(\omega R^2 + uR\ \widehat{\mathbf{e}}_{\varphi} \cdot \widehat{\mathbf{n}}\right) \\
&+ \frac{m_i}{Ze}\frac{1}{2T/m_i}\left(f_M \frac{\partial \omega}{\partial \psi}\right) \\
&+ \left[u\ \nabla_{\|}\left(\omega R^2 + uR\ \widehat{\mathbf{e}}_{\varphi} \cdot \widehat{\mathbf{n}}\right) + \frac{u\mu}{m_i}\nabla_{\|}\left(\frac{R^2 B_{\theta}^2}{B}\right)\right]
\end{aligned}$$

The term in the LHS is

$$u\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\overline{g} \sim v_{\|}\nabla_{\|}\overline{g}$$

similar with usual neoclassical equation.

The first term in the RHS is *adiabatic* response

$$\sim v_{\|}\frac{1}{T}Ze\ \nabla_{\|}\Phi$$

the second is Diamagnetic, corrected by the velocity of rotation,

$$\left(\frac{\partial f_M}{\partial \psi}\right)\bigg|_H \frac{m_i}{Ze}\ u\nabla_{\|}\left(uR\ \widehat{\mathbf{e}}_{\varphi} \cdot \widehat{\mathbf{n}}\right)$$

where

$$\widehat{\mathbf{e}}_{\varphi} \cdot \widehat{\mathbf{n}} = \cos\left(\text{angle}\left[\mathbf{B}, \mathbf{B}_T\right]\right) = \frac{B_{\theta}}{B}$$

$$\frac{m_i}{Ze}v_{\|}\frac{RB_{\theta}}{B} = v_{\|}\frac{|\boldsymbol{\nabla}\psi|}{ZeB/m_i} = |\boldsymbol{\nabla}\psi|\ \frac{v_{\|}}{\Omega}$$

and

$$\begin{aligned}
\left(\frac{\partial f_M}{\partial \psi}\right)\bigg|_H \frac{m_i}{Ze}\ u\nabla_{\|}\left(uR\ \widehat{\mathbf{e}}_{\varphi} \cdot \widehat{\mathbf{n}}\right) &= v_{\|}\nabla_{\|}\left(\frac{v_{\|}}{\Omega}\right)\left[|\boldsymbol{\nabla}\psi|\ \left(\frac{\partial f_M}{\partial \psi}\right)\bigg|_H\right] \\
&= v_{\|}\nabla_{\|}\left(\frac{v_{\|}}{\Omega}\right)|\boldsymbol{\nabla} f_M|_r \\
&\sim v_{D,r}\frac{\partial f_M}{\partial r}
\end{aligned}$$

It is due to

$$\mathbf{v}_D \cdot \boldsymbol{\nabla}\psi = -\frac{1}{Ze/m_i} u \nabla_{\|} \left(\omega R^2 + uR\, \widehat{\mathbf{e}}_{\varphi} \cdot \widehat{\mathbf{n}} \right)$$

with the definition

$$\begin{aligned} \mathbf{v}_D &= \frac{1}{B} \left(-\boldsymbol{\nabla}\Phi \times \widehat{\mathbf{n}} \right) \\ &\quad + \frac{1}{\Omega} \widehat{\mathbf{n}} \times \left[\frac{\mu \boldsymbol{\nabla} B}{m_i} + (\mathbf{V}+u\widehat{\mathbf{n}}) \cdot (\boldsymbol{\nabla}\ \mathbf{V}+u\boldsymbol{\nabla}\ \widehat{\mathbf{n}}) \right] \end{aligned}$$

The last term is $\sim v_{\|} \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}}$ the curvature drift.

13.7 Drift-kinetic equation with rotation Hazeltine Ware

In the neoclassical theory (see **Ware Wiley)** it is found the velocity of plasma poloidal rotation which gives the *equilibrium*. An estimate of this mass velocity is

$$V = \frac{\rho_{i\theta}}{L} v_{thi}$$

where L is a typical scale length, like L_n. To obtain this it was supposed that $V \ll v_{thi}$.

To calculate the mass velocity of plasma at equilibrium V with better precision or for **higher values of** V, it is necessary to solve the drift-kinetic equation to second order in the small parameter ρ_θ/L. In [2], it is developed a model of drift-kinetic equation taking:

- first order in ρ_θ/L
- keeping V as a zeroth-order quantity (*i.e.* not very small compared to the thermal velocity)
- taking the zeroth-order expression of the distribution function as a **shifted Maxwellian**.

The computation is performed in the *frame moving with the velocity* V.

$$\text{absolute particle velocity} = \mathbf{V} + \mathbf{u} + \mathbf{s}$$

where $u\widehat{\mathbf{n}}$ is the **parallel velocity relative to the moving frame** and $\mathbf{s}$ is the perpendicular velocity in the moving frame

$$\mathbf{s} = s\widehat{\mathbf{e}}_n \cos\zeta - s\widehat{\mathbf{e}}_{\perp} \sin\zeta$$

where the versors

$$(\widehat{\mathbf{n}}, \widehat{\mathbf{e}}_n, \widehat{\mathbf{e}}_\perp)$$

correspond to the directions: parallel to the magnetic field, perpendicular to the magnetic surface and perpendicular to the vectorial product of these two ones.

Definitions

$$\mathbf{V} \equiv \text{center of mass velocity}$$

The particle velocity relative to the center of mass is

$$\mathbf{v} = \mathbf{u} + \mathbf{s}$$

where

$$\begin{aligned} \mathbf{u} &\equiv \widehat{\mathbf{n}} \ (\widehat{\mathbf{n}} \cdot \mathbf{v}) \\ &= v_\parallel \widehat{\mathbf{n}} \end{aligned}$$

$$\mathbf{s} \equiv \mathbf{v}_\perp$$

The particle **equation of motion**

$$\frac{d}{dt} (\mathbf{V} + \mathbf{u} + \mathbf{s}) = \frac{e\mathbf{E}}{m} + (\mathbf{V} + \mathbf{s}) \times \Omega \widehat{\mathbf{n}}$$

(which is the usual type $md\mathbf{v}/dt = e\mathbf{E} + e\mathbf{v} \times \mathbf{B}$). This equation contains explicitely the *gyration*.

From here the equation takes a *fluid* form, there is divergence of the velocity as if it were a vector field

$$\frac{\partial}{\partial t} (\mathbf{V} + \mathbf{u} + \mathbf{s}) + (\mathbf{V} + \mathbf{u} + \mathbf{s}) \cdot \boldsymbol{\nabla} (\mathbf{V} + \mathbf{u} + \mathbf{s}) = \frac{e\mathbf{E}}{m} + \frac{e}{m} \mathbf{V} \times \mathbf{B} + \mathbf{s} \times \Omega \widehat{\mathbf{n}}$$

or

$$\frac{\partial}{\partial t} (\mathbf{u} + \mathbf{s}) + (\mathbf{V} + \mathbf{u} + \mathbf{s}) \cdot \boldsymbol{\nabla} (\mathbf{u} + \mathbf{s}) = \mathbf{F} - (\mathbf{u} + \mathbf{s}) \cdot \boldsymbol{\nabla} \mathbf{V} + \mathbf{s} \times \Omega \widehat{\mathbf{n}}$$

where the force on the unit mass is

$$\mathbf{F} = \frac{e}{m} (\mathbf{E} + \mathbf{V} \times \mathbf{B}) - \frac{\partial \mathbf{V}}{\partial t} - (\mathbf{V} \cdot \boldsymbol{\nabla}) \mathbf{V}$$

Nota. It is not clear if the change to the moving frame system should be reflected in a change of the electric field (see **Peeters**). Instead of $\mathbf{E}$ we should put the transformed electric field.

Conclusion: there is no change of the coordinate system. Only some convenient quantities are used in the velocity space. The space coordinates remain unchanged.

NOTA on the simultaneous appearence in the equation of motion of the

- particle velocity $\mathbf{u}$ (**parallel**) and $\mathbf{s}$ (**perpendicular on** $\mathbf{B}$) and
- plasma rotation velocity $\mathbf{V}$.

This presence of two velocities of different nature generates particular effects. In the expression of the particle drift velocity (which normally should have to consist of only: gradB, curvature and electric ExB drifts) now appears the *force which is exerted on the plasma mass* and which is related with the plasma rotation velocity $\mathbf{V}$ by the equation of plasma momentum conservation. Certain effects of the **force** appearing in the plasma momentum equation will be transfered in the formula for the particle drift velocity, $\mathbf{v}_D$. In particular the *diamagnetic velocity* will be present in $\mathbf{v}_D$, which is rather unusual.

This change, not only in the formula of $\mathbf{v}_D$ but also in the nature of its composition is obviously related to the change of coordinates in the velocity space:

$\mathbf{v}$ is the ion (particle) velocity in the REST frame

END OF THE NOTE

NOTE on the energetic term. When the term of convection of $\boldsymbol{\nabla} f$ is dominated by the velocity parallel arising from a plasma poloidal rotation:

$$v_{\parallel}\widehat{\mathbf{n}} + \mathbf{V} \text{ where}$$

$$\mathbf{V} = \frac{K(\psi)}{n}\mathbf{B}+R\left(-\frac{\partial\phi}{\partial\psi}\right)\widehat{\mathbf{e}}_{\varphi}$$

(**note** this is the form adopted also by **Helander 3999**) the energetic term is

$$v_{\parallel}\left(\frac{\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\cdot\mathbf{P}}{mn} - \nabla_{\parallel}\frac{v_{\parallel}KB}{n}\right)\frac{\partial\overline{f}}{\partial w}$$

This part also arises from the consideration of the force exerted on the particle by the fluid in motion. The forces arising in the fluid in motion are affecting the particle. This appears automatically since the equation drift-kinetic is written in the **REST FRAME** of the rotation of the plasma.

This is why we obtain the **presence of the (Pressure) viscous force $\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \cdot \mathbf{P}$ appears in the drift-kinetic equation, which is not a fluid equation for balance of forces.**

GENERAL CONCLUSION: when the drift-kinetic equation is written in the rest frame of the rotating plasma,

the fluid motion = plasma rotation
and the particle motion = drift motion and energy change are mixed

This is reflected by the drift-kinetic equation.

END of the NOTE.

The velocity space coordinates are

$$w = \frac{v^2}{2} \text{ (total energy, but relative to the moving frame)}$$

$$\mu = \frac{s^2}{2B} \text{ (magnetic moment, relative to the moving frame)}$$

ζ the gyroangle

The calculation is similar to the gyroaveraging performed in neoclassical theory to obtain the drift-kinetic equation.

$$\frac{dw}{dt} = \mathbf{F} \cdot \mathbf{v} - \mathbf{v} \cdot (\mathbf{v} \cdot \boldsymbol{\nabla}) \mathbf{V}$$

$$B \frac{d\mu}{dt} = \mu \frac{dB}{dt} - u\mathbf{s} \cdot \frac{d\widehat{\mathbf{n}}}{dt} + \mathbf{F} \cdot \mathbf{s} - \mathbf{s} \cdot (\mathbf{v} \cdot \boldsymbol{\nabla}) \mathbf{V}$$

$$\frac{d\zeta}{dt} = \Omega + \widehat{\mathbf{e}}_\perp \cdot \frac{d\widehat{\mathbf{e}}_n}{dt} + \frac{\widehat{\boldsymbol{\rho}}}{s} \cdot \left[u \frac{d\widehat{\mathbf{e}}_\perp}{dt} - \mathbf{F} + (\mathbf{v} \cdot \boldsymbol{\nabla}) \mathbf{V} \right]$$

where

$$\widehat{\boldsymbol{\rho}} = \widehat{\mathbf{e}}_n \sin\zeta + \widehat{\mathbf{e}}_\perp \cos\zeta$$

The *outward* normal to the magnetic surface

$\widehat{\mathbf{e}}_\psi$
is noted $\mathbf{i}_n$ in that work

The perpendicular component of the velocity of the particle $\mathbf{v}$ is $\mathbf{s} \equiv \mathbf{v}_\perp$ as

$$\mathbf{s} = \widehat{\mathbf{e}}_\psi \; v_\perp \cos\zeta - \widehat{\mathbf{e}}_\perp \sin\zeta$$

with the system of versors constrained by

$$\widehat{\mathbf{e}}_\perp = \widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_\psi$$

In the equation dw/dt we note the first term which is *power*: $\mathbf{F} \cdot \mathbf{v}$. The second term is connected with the divergence of the velocity field, which also involves power.

NOTE the equation for $v_\perp^2/2$ in **Novakovskii Sagdeev** is

$$\frac{d}{dt}\left(\frac{v_\perp^2}{2}\right) = \frac{v_\perp^2}{2}\left(\mathbf{v}_E + v_\parallel \widehat{\mathbf{b}}\right) \cdot \boldsymbol{\nabla} \ln B$$

where

$$\mathbf{v}_E \cdot \boldsymbol{\nabla} \ln B = \frac{1}{B}\phi_0' \frac{\sin\theta}{R}$$

This expresses the fact that the change in *perpendicular* energy $v_\perp^2/2$ is due to the *work* done during the motion of the particle against the gradient of the magnetic field. The last term is

$$\begin{aligned} & ... + v_\parallel \frac{v_\perp^2}{2B}\nabla_\parallel B \quad \text{or} \\ v_\parallel \ \mu \nabla_\parallel B &= \text{work against grad } B \end{aligned}$$

This is actually one of the equations of motion of the particle, extended set (including the dynamical changes of the velocities) see **Burrell, Wong, Galeev**, etc.

END

Now the gyroaveraging is performed

$$\langle g \rangle = \oint \frac{d\zeta}{2\pi} g$$

and the notation

$$g = \langle g \rangle + \widetilde{g}$$

The part of g that has dependence on the gyroangle $\widetilde{g}$ has two components: one in $(\sin\xi, \cos\xi)$ and the other in $(\sin(2\zeta), \cos(2\zeta))$.

The averages

$$\left\langle \frac{dw}{dt} \right\rangle = \mathbf{F} \cdot \mathbf{u} - \mu B (\boldsymbol{\nabla} \cdot \mathbf{V}) - \left(u^2 - \mu B\right) \ [\widehat{\mathbf{n}} \cdot (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \mathbf{V}]$$

$$B \left\langle \frac{d\mu}{dt} \right\rangle = -\mu \left[\frac{\partial B}{\partial t} + (\mathbf{V} \cdot \boldsymbol{\nabla}) B\right] - \mu B [(\boldsymbol{\nabla} \cdot \mathbf{V}) - \widehat{\mathbf{n}} \cdot (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \mathbf{V}]$$

The $\cos\zeta$ component

$$\widetilde{\left(\frac{dw}{dt}\right)}_{\cos} = F_\psi \ s - 2 \ e_{\|\psi} \ su$$

$$\widetilde{\left(\frac{dw}{dt}\right)}_{\sin} = -F_\perp \ s + 2 \ e_{\|\perp} \ su$$

$$\widetilde{\left(\frac{dw}{dt}\right)}_{\cos 2} = -\frac{s^2}{2}\left(e_{\psi\psi} - e_{\perp\perp}\right)$$

$$\widetilde{\left(\frac{dw}{dt}\right)}_{\sin 2} = -s^2 \ e_{\psi\perp}$$

For the magnetic momentum

$$B\widetilde{\left(\frac{d\mu}{dt}\right)}_{\cos} = -\left(\mathbf{v}_D\right)_\perp \ \Omega \ s$$

$$B\widetilde{\left(\frac{d\mu}{dt}\right)}_{\sin} = -\left(\mathbf{v}_D\right)_\psi \ \Omega \ s$$

$$B\widetilde{\left(\frac{d\mu}{dt}\right)}_{\cos 2} = -\frac{u \ s^2}{2}\left[\frac{1}{\left(\mathbf{R}_\psi\right)_\|} - \frac{1}{\left(\mathbf{R}_\perp\right)_\|}\right] + \frac{s^2}{2}\left(e_{\psi\psi} - e_{\perp\perp}\right)$$

$$B\widetilde{\left(\frac{d\mu}{dt}\right)}_{\sin 2} = \frac{u \ s^2}{2} \ S - s^2 \ e_{\psi\perp}$$

For the gyro-angle

$$\widetilde{\left(\frac{d\zeta}{dt}\right)}_{\cos} = -\left(\mathbf{v}_D\right)_\psi \frac{\Omega}{s} - \frac{s}{\left(\mathbf{R}_\psi\right)_\perp} - \frac{\mu}{s} \ \frac{\partial B}{\partial x_\perp}$$

$$\widetilde{\left(\frac{d\zeta}{dt}\right)}_{\sin} = \left(\mathbf{v}_D\right)_\perp \frac{\Omega}{s} - \frac{s}{\left(\mathbf{R}_\perp\right)_\psi} - \frac{\mu}{s}\frac{\partial B}{\partial x_\psi}$$

$$\widetilde{\left(\frac{d\zeta}{dt}\right)}_{\cos 2} = \frac{u}{2}S + \left(\widehat{\mathbf{e}}_\psi\right)_\perp$$

$$\widetilde{\left(\frac{d\zeta}{dt}\right)}_{\sin 2} = \frac{u}{2}\left[\frac{1}{\left(\mathbf{R}_\psi\right)_\|} - \frac{1}{\left(\mathbf{R}_\perp\right)_\|}\right] - \frac{1}{2}\left(e_{\psi\psi} - e_{\perp\perp}\right)$$

The notations

$$
\begin{aligned}
\mathbf{v}_D &= -\frac{1}{\Omega}\widehat{\mathbf{n}} \times \mathbf{F} \\
&+\frac{1}{\Omega}\widehat{\mathbf{n}}\ \mu B\ \left(\frac{j_\parallel}{B}\right) \\
&+\frac{1}{\Omega}\widehat{\mathbf{n}} \times \left[\mu \boldsymbol{\nabla} B + u^2 \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{n}} + \left(\mathbf{u} \cdot \boldsymbol{\nabla}\right)\mathbf{V} + \left(\mathbf{V} \cdot \boldsymbol{\nabla}\right)\mathbf{u}\right]
\end{aligned}
$$

The velocity stress tensor

$$
e_{ij} = \frac{1}{2}\left[\widehat{\mathbf{e}}_i \cdot \left(\widehat{\mathbf{e}}_j \cdot \boldsymbol{\nabla}\right)\mathbf{V} + \widehat{\mathbf{e}}_j \cdot \left(\widehat{\mathbf{e}}_i \cdot \boldsymbol{\nabla}\right)\mathbf{V} - \frac{2}{3}\delta_{ij}\left(\boldsymbol{\nabla} \cdot \mathbf{V}\right)\right]
$$

The curvature coefficients

$$
\frac{1}{R_{jk}} = \widehat{\mathbf{e}}_j \cdot \left(\widehat{\mathbf{e}}_j \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{e}}_k
$$

The torsion

$$
\frac{1}{T_{jk}} = \widehat{\mathbf{e}}_j \cdot \left(\widehat{\mathbf{e}}_k \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{e}}_k = -\frac{1}{T_{kj}}
$$

and

$$
S \equiv \frac{1}{T_{\psi\parallel}} + \frac{1}{T_{\perp\parallel}}
$$

The results:

$$
\begin{aligned}
\mathbf{v}_d &= \frac{\mathbf{F} \times \widehat{\mathbf{n}}}{\Omega} + \\
&+\widehat{\mathbf{n}}\frac{\mu B}{\Omega}\left(\frac{j_\parallel}{B}\right) \\
&+\frac{1}{\Omega}\widehat{\mathbf{n}} \times \left(\mu \boldsymbol{\nabla} B + u^2 \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right. \\
&\qquad \left. + \mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{V} + \mathbf{V} \cdot \boldsymbol{\nabla}\mathbf{u}\right)
\end{aligned}
$$

NOTE an alternative formula

$$
\begin{aligned}
\mathbf{v}_D &= \frac{e}{m}\rho_\parallel \boldsymbol{\nabla} \times \left(\rho_\parallel \mathbf{B}\right) \\
&= \frac{e}{m}\rho_\parallel^2 \left(\mu_0 \mathbf{j}\right) \\
&\quad -\frac{eB}{m}\rho_\parallel \left(\widehat{\mathbf{n}} \times \boldsymbol{\nabla}\rho_\parallel\right)
\end{aligned}
$$

END

The change in the particle energy

$$
\begin{aligned}
\dot{w} \;=\;& \mathbf{F}\cdot\left(\mathbf{u}+\mathbf{v}_d\right) \\
& -\frac{\mu B}{\Omega}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\times\mathbf{F} \\
& -\mu B\boldsymbol{\nabla}\cdot\mathbf{V} \\
& -\left(u^2-\mu B\right)\widehat{\mathbf{n}}\cdot\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\mathbf{V} \\
& -\mu B u\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{\pi}}{p}\right) \\
& +\frac{2u}{\Omega}\left(\widehat{\mathbf{e}}_n e_{n\|}+\widehat{\mathbf{e}}_\perp e_{n\perp}\right)\cdot\{-\mathbf{F}\times\widehat{\mathbf{n}} \\
& \qquad\qquad \left(3\mu B-u^2\right)\left(\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\widehat{\mathbf{n}}\right) \\
& \qquad\qquad \frac{\left(\mu B-u^2\right)}{u}\frac{\widehat{\mathbf{n}}\times\left(\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{V}+\mathbf{V}\cdot\boldsymbol{\nabla}\mathbf{u}\right)}{u}\Bigg\}
\end{aligned}
$$

where $\boldsymbol{\pi}$ is the **magnetic viscosity part of the pressure tensor**

$$\pi_{nn}=-\pi_{\perp\perp}=-\frac{p}{\Omega}e_{n\perp}$$

$$\pi_{\|\|}=0$$

$$\pi_{n\perp}=\pi_{\perp n}=\frac{p}{2\Omega}\left(e_{nn}-e_{\perp\perp}\right)$$

where $\mathbf{e}$ is **the velocity stress tensor**

$$(\mathbf{e})_{\alpha\beta}=\frac{1}{2}\left(\widehat{\mathbf{e}}_\alpha\cdot\widehat{\mathbf{e}}_\beta\cdot\boldsymbol{\nabla}\mathbf{V}+\widehat{\mathbf{e}}_\beta\cdot\widehat{\mathbf{e}}_\alpha\cdot\boldsymbol{\nabla}\mathbf{V}-\frac{2}{3}\delta_{\alpha\beta}\boldsymbol{\nabla}\cdot\mathbf{V}\right)$$

The drift-kinetic equation, in its most general form:

$$
\begin{aligned}
\frac{\partial\overline{f}}{\partial t}+ & \\
& +\left(\mathbf{u}+\mathbf{v}_d+\mathbf{V}\right)\cdot\boldsymbol{\nabla}\overline{f} \\
& +\left\langle\frac{d\mu}{dt}\right\rangle\frac{\partial\overline{f}}{\partial\mu} \\
& +\dot{w}\frac{\partial\overline{f}}{\partial w} \\
=\; & 0
\end{aligned}
$$

The ion drift-kinetic equation

$$\begin{aligned}
& \frac{\partial \overline{f}}{\partial t} + \left(v_\parallel \widehat{\mathbf{n}} + \mathbf{V}\right) \cdot \boldsymbol{\nabla} \overline{f} - \\
& - \left[\mathbf{V} \cdot \boldsymbol{\nabla} \mu B + \mu B \left(\boldsymbol{\nabla} \cdot \mathbf{V} - \widehat{\mathbf{n}}\widehat{\mathbf{n}} \colon \boldsymbol{\nabla}\mathbf{V}\right)\right] \frac{\partial \overline{f}}{\partial \mu} + \\
& + \left(v_\parallel \frac{\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \cdot \mathbf{P}}{n m_i} - \mu B \boldsymbol{\nabla} \cdot \mathbf{V} - \left(v_\parallel^2 - \mu B\right) \widehat{\mathbf{n}}\widehat{\mathbf{n}} \colon \boldsymbol{\nabla}\mathbf{V}\right) \frac{\partial \overline{f}}{\partial w} \\
= \; & C\left(\overline{f}\right)
\end{aligned}$$

where the **ion STRESS tensor** is

$$\begin{aligned}
\mathbf{P} & \equiv \\
& = nT\,\mathbf{I} + \mathbf{\Pi} \\
& \simeq nT\mathbf{I} + \frac{3}{2}\pi_\parallel \left(\widehat{\mathbf{n}}\widehat{\mathbf{n}} - \frac{1}{3}\mathbf{I}\right)
\end{aligned}$$

where $\mathbf{\Pi}$ is the **ion VISCOSITY tensor**.

14 Hirshman ambipolarity paradox

Inertial factor.

The damping time of poloidal rotation

$$\begin{aligned}
\tau_p & = \frac{1}{3}\left(1 + 2\widehat{q}^2\right) \frac{\left\langle B_\theta^2 \right\rangle}{\left\langle \left(\nabla_\parallel B\right)^2 \right\rangle} \frac{\sum_j n_j m_j}{\sum_j \mu_j} \\
& \times \frac{1 + \frac{1}{\kappa_p} \frac{\left\langle B^2 \right\rangle}{\left\langle B_\theta^2 \right\rangle} \frac{1}{1+2\widehat{q}^2}}{1 + \frac{1}{\kappa_p}}
\end{aligned}$$

where

$$\rho \equiv \sum_j m_j n_j$$

$$\begin{aligned}
\kappa_p & = 4\pi \sum_j m_j n_j \; c^2 \; \frac{\left\langle R^2 \right\rangle}{\left\langle R^2 B_\theta^2 \right\rangle} \\
& \quad \text{dielectric constant in the poloidal magnetic field} \\
& \gg 1
\end{aligned}$$

[compare **Stringer 1969**, the denominator D which is $D(\omega, k)$ for $\omega = v_\theta/r$ the frequency of the "perturbation" of the magnetic field = actually its variation on poloidal direction and $k_\| = 1/(qR)$. Here it results the factor of ρ_{eff} plus a correction which is $c_{s\theta}^2/v_\theta^2$.]

The following approximation has been used

$$\sum_j \mu_j \left\langle R^2 \boldsymbol{\nabla}\varphi \cdot \mathbf{u}_j \right\rangle \approx \frac{\sum_j \mu_j}{\rho} P_\varphi$$

15 The drift motion of particles according to Novakovskii Galeev Liu Sagdeev Hassam

This is from the old text *Neoclassics2*.

Now it is also in *polarization*.

The paper discusses the poloidal damping due to magnetic pumping in the *plateau* regime.

This is also in **Notes density enhanced confinement**.

See also *plasma, general, viscosity*.

It is considered a fast temporal variation of the radial electric field.

This is accompanied by a change in the neoclassical prolarization.

For the barely circulating ions, it is possible to calculate the radial polarization current (**Novakovski**). It is assumed that the radial electric field has a time variation which can be linearized

$$E_r = E_{r0} + \left(\frac{\partial E_r}{\partial t}\right) t$$

which means that the velocity due to the radial electric field has time variation

$$v_E = v_{E0} + \left(\frac{\partial v_E}{\partial t}\right) t \quad \text{(in the poloidal direction)}$$

The particles that are considered are

$$\begin{array}{rcl} & & \text{trapped ions} \\ \text{with } v_\| & \ll & v_\perp \end{array}$$

The equation for the poloidal motion

$$\frac{rd\theta}{dt} = v_E + v_\parallel \frac{B_\theta}{B_T}$$

where

$$\frac{B_\theta}{B_T} = \frac{\varepsilon}{q} \equiv \Theta$$

is the factor that projects the parallel direction on the poloidal direction. Integrating to find $\theta(t)$

$$r\theta(t) = r\theta_0 + \left(v_{E0} + v_\parallel \frac{B_\theta}{B_T}\right) t + \frac{1}{2}\left(\frac{\partial v_E}{\partial t}\right) t^2$$

The radial velocity is the radial component of the *drift* of the guiding center

$$v_r = \mathbf{v}_D \cdot \widehat{\mathbf{e}}_r = v_D \sin\theta$$

$$v_D = \frac{1}{\Omega} \frac{v_\perp^2/2 + v_\parallel^2}{R}$$

and since we assume very small *parallel* velocity,

$$v_\parallel \ll v_\perp$$

, we approximate

$$v_D \approx \frac{1}{\Omega}\frac{1}{R}\frac{v_\perp^2}{2}\sin\theta = \frac{1}{\Omega}\frac{1}{R}\frac{mv_\perp^2}{2B}\frac{B}{m}\sin\theta = \frac{1}{\Omega R}\frac{\mu B}{m}\sin\theta$$

and

$$v_r(t) = \frac{1}{\Omega R}\frac{\mu B}{m}\sin\left[\theta(t)\right]$$

and the average over a *transit* is

$$\begin{aligned}
\langle v_r \rangle_{transit} &= \frac{1}{2}\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} dt \ v_r(t) \\
&= \frac{1}{\Omega R}\frac{\mu B}{m}\frac{1}{2}\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} dt \ \sin\left[\theta(t)\right]
\end{aligned}$$

Now we expand the function $\sin\theta$ for small argument

$$\begin{aligned}
\sin\left[\theta(t)\right] &= \sin\left[\theta_0 + \frac{1}{r}\left(v_{E0} + v_\parallel \frac{B_\theta}{B_T}\right) t + \frac{1}{2}\left(\frac{\partial v_E}{\partial t}\right) t^2\right] \\
&\approx \sin\theta_0 \\
&\quad + \cos\theta_0 \left[\frac{1}{r}\left(v_{E0} + v_\parallel \frac{B_\theta}{B_T}\right) t + \frac{1}{r}\frac{1}{2}\left(\frac{\partial v_E}{\partial t}\right) t^2\right]
\end{aligned}$$

and the integrations over the bounce period gives

$$\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} dt \;\sin\theta_0 = 0$$

$$\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} dt \;\cos\theta_0 \frac{1}{r}\left(v_{E0} + v_{\parallel}\frac{B_\theta}{B_T}\right) t = \cos\theta_0 \frac{1}{r}\left(v_{E0} + v_{\parallel}\frac{B_\theta}{B_T}\right)\frac{1}{\tau}\left[\frac{\tau^2}{2}\right]_{-\tau/2}^{\tau/2} = 0$$

$$\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} dt \;\cos\theta_0 \frac{1}{r}\frac{1}{2}\left(\frac{\partial v_E}{\partial t}\right) t^2 = \frac{1}{24}\cos\theta_0 \frac{1}{r}\left(\frac{\partial v_E}{\partial t}\right)\tau^2$$

The radial velocity averaged over a bounce is at this moment

$$\begin{aligned} \langle v_r \rangle_{transit} &= \frac{1}{\Omega R}\frac{\mu B}{m}\frac{1}{2}\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} dt \;\sin\left[\theta\left(t\right)\right] \\ &= \frac{1}{\Omega R}\frac{\mu B}{m}\frac{1}{48}\cos\theta_0 \frac{1}{r}\left(\frac{\partial v_E}{\partial t}\right)\tau^2 \end{aligned}$$

Now we take

$$\begin{aligned} \tau &\equiv \text{transit time} = \frac{\text{connection length } 2\pi qR}{\text{parallel velocity } v_{\parallel}} \\ &= \frac{2\pi qR}{\left|v_{\parallel}\right|} \end{aligned}$$

It is necessary to define the regime by few parameters.

$$\begin{aligned} \widehat{\nu} &= \frac{r\;\nu_{ii}}{\Theta v_{th,i}} \\ &= \frac{\text{freq. of ion collisions}}{\text{freq. of ion transit with poloidal velocity } \Theta v_{th,i} \text{ on poloidal circle}} \end{aligned}$$

the same formula is written

$$\begin{aligned} \widehat{\nu} &= \frac{r\;\nu_{ii}}{\Theta v_{th,i}} \\ &= \frac{r\;\nu_{ii}}{\frac{\varepsilon}{q} v_{th,i}} \\ &= \frac{\nu_{ii}}{v_{th,i}/\left(qR\right)} \end{aligned}$$

The parallel velocity of the ions is taken at the limit where the effective ion collision frequency is equal with the parallel transit frequency

$$\nu_{eff} = \frac{v_{\parallel}}{qR}$$

where by definition

$$\nu_{eff} \overset{def}{=} \nu_{ii} \frac{v_{i,th}^2}{v_\parallel^2}$$

We combine the two expressions of ν_{eff} and further use the expression for $\widehat{\nu}$

$$\begin{aligned}
\nu_{ii} \frac{v_{i,th}^2}{v_\parallel^2} &= \frac{v_\parallel}{qR} \quad \text{or} \\
\frac{\nu_{ii}}{v_{i,th}/\left(qR\right)} &= \frac{v_\parallel^3}{v_{i,th}^3} \\
\widehat{\nu} &= \frac{v_\parallel^3}{v_{i,th}^3}
\end{aligned}$$

from where we derive

$$\frac{v_\parallel}{v_{i,th}} = \widehat{\nu}^{1/3}$$

and we replace $v_\parallel$ with the expression in terms of thermal ion velocity and the effective collision parameter $\widehat{\nu}$

$$v_\parallel = v_{i,th} \widehat{\nu}^{1/3}$$

Then the square of the bounce time τ is

$$\begin{aligned}
\tau^2 &= \frac{\left(2\pi\right)^2 q^2 R^2}{v_\parallel^2} \\
&= \frac{\left(2\pi\right)^2 q^2 R^2}{v_{i,th}^2} \widehat{\nu}^{-2/3}
\end{aligned}$$

and

$$\begin{aligned}
\langle v_r \rangle_{transit} &= \frac{1}{\Omega R} \frac{\mu B}{m} \frac{1}{48} \cos\theta_0 \frac{1}{r} \left(\frac{\partial v_E}{\partial t}\right) \tau^2 \\
&= \frac{1}{\Omega R} \frac{\mu B}{m} \frac{1}{48} \cos\theta_0 \frac{1}{r} \left(\frac{\partial v_E}{\partial t}\right) \frac{\left(2\pi\right)^2 q^2 R^2}{v_{i,th}^2} \widehat{\nu}^{-2/3} \\
&= \frac{\left(2\pi\right)^2}{48} \frac{q^2}{\varepsilon \Omega} \frac{\mu B}{m} \cos\theta_0 \left(\frac{\partial v_E}{\partial t}\right) \widehat{\nu}^{-2/3}
\end{aligned}$$

To calculate the radial current two steps are necessary:

- take the fraction of the particles that have this regime

- integrate over the positions θ_0. Actually, the parameter θ_0 appears in the magnitude of the magnetic field $B = B_0 \left(1 - \varepsilon \cos\theta_0\right)$ and this, in turn, appears in the expression of the magnetic momentum $\mu = v_\perp^2 / (2B)$. Then to integrate over the Maxwell distribution of the variable $v_\perp$ we can equivalently integrate over the variable θ_0 for fixed μ.

The fraction of particles is

$$\frac{v_\parallel}{v_{i,th}}$$

and this is

$$\text{fraction of particles} \sim \widehat{\nu}^{1/3}$$

When we multiply the average radial velocity by this factor

$$\begin{aligned} & \widehat{\nu}^{1/3} \times \widehat{\nu}^{-2/3} \\ = & \ \widehat{\nu}^{-1/3} \end{aligned}$$

we get a dependence of the effective collisional parameter as $\widehat{\nu}^{-1/3}$ which will be found in the final expression.

The Maxwellian in velocity space is

$$\begin{aligned} f_M &= N \exp\left(-\frac{mv^2}{2T}\right) \\ &= N \exp\left(-\frac{m\left(v_\parallel^2 + v_\perp^2\right)}{2T}\right) \sim N \exp\left(-\frac{mv_\perp^2}{2T}\right) \\ &= N \exp\left(-\frac{\mu B}{T}\right) = N \exp\left(-\frac{\mu B_0}{T(1+\varepsilon\cos\theta)}\right) \\ &\approx N \exp\left[-\frac{\mu B_0}{T}(1-\varepsilon\cos\theta)\right] \end{aligned}$$

We use this velocity integration to suppress the indeterminancy given by the presence of θ_0 in the radial current.

$$\frac{\partial v_E}{\partial t} = \frac{1}{B}\left(\frac{\partial E}{\partial t}\right)$$

The radial electric current induced by the time variation of the radial electric field is

$$\langle j_r \rangle \approx \left(1 + q^2 + \widehat{\nu}^{-1/3} q^2\right) \frac{m}{B^2}\left(\frac{\partial E}{\partial t}\right)$$

In this formula, 1 is the standard polarization term. The second term is the neoclassical polarization term due to ions with comparable parallel and perpendicular velocities, $v_\parallel \approx v_\perp$.

The radial current averaged over surface must be zero, $\langle j_r \rangle = 0$.

In the banana regime

$$\widehat{\nu} \approx \varepsilon^{-3/2}$$

The neoclassical polarization radial current due to radial excursions of the banana (*trapped* particles) is

$$j_r^{bananas} \approx \varepsilon^{3/2} \frac{c^2}{v_{A\theta}^2} \left(\frac{\partial E_r}{\partial t} \right)$$

NOTE

This is a usual *neoclassical* polarization (**Robertson Hinton)** in which the dielectric coefficient of $\partial E_r / \partial t$ is replaced

$$\left(1 + \frac{c^2}{v_A^2}\right) \rightarrow \left(1 + \frac{c^2}{v_{A\theta}^2}\right)$$

and, in addition, the banana factor

$$\varepsilon^{3/2}$$

is included.

END

The equations are

$$\frac{d\mathbf{r}}{dt} = v_{\parallel} \widehat{\mathbf{n}} + \mathbf{v}_E + \mathbf{v}_D$$

$$\begin{aligned} \frac{dv_{\parallel}}{dt} &= \left(-\frac{v_{\perp}^2}{2} \widehat{\mathbf{n}} + v_{\parallel} \mathbf{v}_E \right) \cdot \boldsymbol{\nabla} \ln B \\ \frac{d}{dt} \left(\frac{v_{\perp}^2}{2} \right) &= \frac{v_{\perp}^2}{2} \left(v_{\parallel} \widehat{\mathbf{n}} + \mathbf{v}_E \right) \cdot \boldsymbol{\nabla} \ln B \end{aligned}$$

The *drift velocity* now includes the polarization drift $\sim \partial V_E / \partial t$,

$$\mathbf{v}_D = \frac{1}{\Omega_{ci}} \left(\frac{v_{\perp}^2}{2} + v_{\parallel}^2 \right) \widehat{\mathbf{n}} \times \boldsymbol{\nabla} \ln B + \frac{1}{\Omega_{ci}} \widehat{\mathbf{n}} \times \frac{\partial \mathbf{v}_E}{\partial t}$$

Comparing with previous expressions of the drift velocity v_D we note that there is an additional term, which gives the effect of the fast time variation of the radial electric field, like in transitions with rapid change of toroidal and/or poloidal rotation velocity.

We note however that the time variation of the electric drift velocity has the following effect on the drift:

we suppose that

$$\frac{\partial E_r}{\partial t} \sim \widehat{\mathbf{e}}_r$$

exists due to the polarization effect related to the separation of charges which results as forced increase of the poloidal velocity

$$\frac{\partial v_\theta}{\partial t} \rightarrow \frac{\partial E_r}{\partial t}$$

Then $\mathbf{v}_E$ will increase in two possible directions

$$\frac{\partial \mathbf{v}_E}{\partial t} \sim \frac{1}{B}\left(\frac{\partial E_r}{\partial t}\widehat{\mathbf{e}}_r \times \mathbf{B}_\theta + \frac{\partial E_r}{\partial t}\widehat{\mathbf{e}}_r \times \mathbf{B}_T\right)$$

Then the terms mentioned by **Novakovskii** is

$$\begin{aligned} \frac{1}{\Omega_{ci}}\widehat{\mathbf{n}} \times \frac{\partial \mathbf{v}_E}{\partial t} \quad &\sim \quad \frac{1}{\Omega_{ci}}\widehat{\mathbf{n}} \times \left(\frac{\partial E_r}{\partial t}\widehat{\mathbf{e}}_r \times \mathbf{B}_\theta\right) \quad \text{almost zero} \\ &\quad + \frac{1}{\Omega_{ci}}\widehat{\mathbf{n}} \times \left(\frac{\partial E_r}{\partial t}\widehat{\mathbf{e}}_r \times \mathbf{B}_T\right) \quad \text{radial} \end{aligned}$$

Therefore *none* of these contributions is aligned along the toroidal direction, giving a *drift* of the particle population in the toroidal direction.

It seems that a treatment based on the equations of motion of the particles governed by the invariants

$$E, \mu$$

cannot give us a drift of the bananas in the toroidal direction.

Here again the drift-kinetic equation is

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} f + \frac{dv_\parallel}{dt}\frac{\partial f}{\partial v_\parallel} + \frac{d\left(v_\perp^2/2\right)}{dt}\frac{\partial f}{\partial\left(v_\perp^2/2\right)} = St\left(f\right)$$

The paper of **Novakovskii** wants to solve the problem of decay of poloidal rotation in the *plateau* regime.

Then the Drift-Kinetic equation is solved by perturbations.

Zero + order 1 + order 2 are necessary.

We must **note** that the zeroth order is NOT Maxwellian but it is the equilibrium distribution function, f_0.

It takes into account the variation *in the surface*, $\sim \theta$, which comes from the balance netween the parallel advection with $v_\parallel$ (this is *modulated* by the mirror force) with collisions.

In the zeroth order

$$\left(v_{\parallel}\frac{B_{\theta}}{B_T}+v_E\right)\frac{\partial f_0}{r\partial\theta}-St\left(f_0\right)=0$$

which gives a Maxwellian function *possibly* shifted in the parallel direction by a velocity U_0.

$$f_0=\left(1-\frac{mv_{\parallel}U_0}{T}\right)f_M$$

for

$$f_M=\frac{n}{\left(2\pi T/m\right)^{3/2}}\exp\left[-\frac{m\left(v_{\parallel}^2+v_{\perp}^2\right)}{2T}\right]$$

NOTE

We remark the combination

$$\begin{aligned}&v_{\parallel}\frac{B_{\theta}}{B_T}+v_E\\ \sim\ &\Theta\left(v_{\parallel}+\frac{v_E}{\Theta}\right)\\ \approx\ &0 \qquad \text{(since the paranthesis is } \sim 0)\end{aligned}$$

The combination $v_{\parallel}\frac{B_{\theta}}{B_T}+v_E$ is the *poloidal velocity.*

It is composed of the *projection* of the parallel velocity on θ, using Θ, plus the poloidal velocity due to the radial electric field.

The first term $\left(v_{\parallel}\frac{B_{\theta}}{B_T}+v_E\right)\frac{\partial f_0}{r\partial\theta}$ is the convection of the distribution function f_0 in the poloidal direction.

It is either very small or zero.

If it exists, it is balanced by collisions.

END

Nothing at this moment suggests there can be a velocity U in the *parallel* direction, *i.e.* along the magnetic field lines. But the equation for f_0 allows it and since we know it can exist, it is introduced at this point.

Note that the velocity along the magnetic field lines comes from a shift in the parallel *particle* velocity, as

$$\begin{aligned}&-\frac{\left(v_{\parallel}-U_0\right)^2}{2T/m}\\ =\ &-\frac{v_{\parallel}^2}{2T/m}-\frac{2v_{\parallel}U_0}{2T/m}-\frac{U_0^2}{2T/m}\end{aligned}$$

and the last term is much less than 1 since the flow with velocity U_0 is slower than the *thermal* velocity.

Then a substitution is made for the distribution function to extract a rigid body rotation

$$\begin{aligned} f &= f_0 + \varepsilon \left(\frac{m v_\parallel U_0}{T} \right) f_M \\ &\quad + \widetilde{f} \end{aligned}$$

Then the Drift-kinetic equation to order ε^2 gives

$$\begin{aligned} &\frac{\partial \widetilde{f}}{\partial t} + \left(v_E + v_\parallel \frac{B_\theta}{B_T} \right) \frac{\partial \widetilde{f}}{r \partial \theta} - St\left(\widetilde{f}\right) \\ &= \frac{\sin\theta}{R} \frac{m\left(v_\parallel^2 + \frac{v_\perp^2}{2}\right)}{T} W f_M \end{aligned}$$

where W is a velocity in the *poloidal* direction

$$\begin{aligned} W &\equiv v_E + v_{*n} + U_0 \frac{B_\theta}{B_T} \\ &\quad + v_{*T} \left[\frac{m\left(v_\parallel^2 + v_\perp^2\right)}{2T} - \frac{3}{2} \right] \\ &\quad \text{poloidal} \end{aligned}$$

$$\begin{aligned} v_{*n} &\equiv \frac{T}{eB} \frac{d}{dr} \ln n \quad \text{(dia)} \\ v_{*T} &\equiv \frac{T}{eB} \frac{d}{dr} \ln T \quad \text{(temperature-dia)} \end{aligned}$$

COMMENT

The RHS term is

$$\frac{\sin\theta}{R} \frac{m\left(v_\parallel^2 + \frac{v_\perp^2}{2}\right)}{T} W f_M$$

and represents the advection by the *neoclassical drift* in the RADIAL direction, of the equilibrium distribution function, with its radial gradients.

Then W is

$$\begin{aligned} W = \; & \text{poloidal velocity due to the radial electric field} \\ & + \text{diamagnetic-density velocity (poloidal)} \\ & + \text{diamagnetic-temperature velocity (poloidal)} \\ & + \text{poloidal projection of the flow velocity } U_0 \end{aligned}$$

The term $\frac{\sin\theta}{R}\frac{m\left(v_\parallel^2+\frac{v_\perp^2}{2}\right)}{T}Wf_M$ comes from

$$\mathbf{v}\cdot\boldsymbol{\nabla}f_M$$

and reflects the *radial* convective variation of the Maxwellian with the flow velocity that exists in the plasma. The space variation of the Maxwellian f_M is *RADIAL* and the operator $\boldsymbol{\nabla}$ will be reduced to radial derivative

$$\boldsymbol{\nabla}\rightarrow\frac{\partial}{\partial r}=\frac{d}{dr}$$

Then *which is the plasma velocity that will take advantage of this radial variation of the equilibrium distribution function*? It is the particle's drift velocty $\mathbf{v}_D$ which will act along the minor radius v_{Dr}.

The term is actually

$$v_{Dr}\frac{df_M}{dr}$$

This term is dependent on angle θ , the poloidal variable, since the radial projection of the *neoclassical drift* dependes on θ.

This will become the inhomogeneous term that drives a variation of the distribution function, asking therefore for the existence of a f_1.

But what can the correction do to balance this *radial convective variation of the equilibrium distribution Maxwellian function* ?

The correction f_1 actually has variation in the magnetic surface, $\sim\theta$.

It will again be question of a *convective variation*, which means that there is a poloidal velocity that will advect the function f_1 along its variation.

This poloidal advection of the correction $f_1(\theta)$ will compensate for the *radial* variation of the equilibrium distribution function.

To **comment** further, we note in **Rosenbluth Hazeltine Hinton 1972** the equation

$$v_{Dr}\frac{\partial f_0}{\partial r}+v_\parallel\frac{B_\theta}{B}f_0\frac{\partial\widehat{f}}{r\partial\theta}+|e|\,E_\parallel v_\parallel\frac{\partial f_0}{\partial\epsilon}=C(f)$$

where

$$f=f_0\left(1+\widehat{f}\right)$$

We recognize the same picture:

- the zero-order distribution function f_0
- the perturbation of the distribution function is advected by the parallel velocity, which is the maximum it can get.

- the perturbation to the distribution function $\widehat{f}$ has only variation on the poloidal angle θ.
- the balance is ensured by the radial advection of the equilibrium distribution function f_0 by the neoclassical drift of the particles. This drift acts only in the radial direction because the equilibrium f_0 only depends on r.
- there is another term, which is energetic. It is the work done by a parallel electric field
$$v_{\parallel} \ \times \ |e| \, E_{\parallel}$$
and is affecting the space of velocities. It shows that our interest here is on *instabilities*.

END COMMENT

The collision operator is adopted as

$$St\,(f) = -\nu_{eff} f$$

$$\nu_{eff} = \frac{v_{th}^2}{v_{\parallel}^2} \nu_c$$

NOTE
In the paper about the damping of poloidal rotation, which includes a discussion of GAM, **Novakovskii Liu Sagdeev Rosenbluth**, the operator of collision is

$$\begin{aligned} C &= \nu\,\mathcal{L} \quad \text{(Lorentz, pitch angle)} \\ & \quad +\nu\,\frac{2v_{\parallel}}{v_{th}^2} w_a f_M \quad \text{(Hirshman Sigmar Clarke)} \\ \nu &\equiv \text{frequency of collisions} \end{aligned}$$

Similar, in **Rutherford**, etc.
END

Regarding the application of this analysis to the case of a **fast time variation** of the radial electric field (for fast transients of the poloidal or toroidal rotation) the range of validity is established by **Novakovskii et al** by choosing

$$\frac{v_{Th}}{qR} \approx \nu_{eff} \gg \frac{\partial}{\partial t}$$

which means: the frequency of the *bounce* of the trapped particle, $v_{Th}/(qR)$, comparable with the frequency of collisions ν_{eff} is much higher than the frequency associated to the variation of the radial electric field, $\partial/\partial t$. Then during the variation of the radial electric field, which is slow, the trapped particle makes many bounces.

Then a new small parameter has been identified and the distribution function can be expanded in a series. The distribution function is only the *correction* to the shifted Maxwellian, *i.e.* the function $\widetilde{f}$ and the series is

$$\widetilde{f} = f_1 + f_2 + ...$$

Separately and related this time to the spatial variation of the distribution function, it is *considered* the variation in the magnetic surface, *i.e.* the dependence of the distribution functions f_i of the *poloidal angle* θ:

$$f_i = \sum_{\sigma=\pm 1} f_{i\sigma} \exp(i\sigma\theta)$$

Then we get the solution for the first order correction $f_{1\sigma}$ as

$$f_{1\sigma} = -\frac{\varepsilon\frac{\left(v_\perp^2/2+v_\parallel^2\right)}{2T/m}W}{v_\parallel\left(B_\theta/B_T\right)+v_E-\iota\sigma\nu_{eff}}f_M$$

where

$$\iota = -\frac{1}{q}$$

note that usually is $-\frac{2\pi}{\iota} = q$.

$$\sigma = \pm 1$$

and

$$W \equiv v_E + V_n + \frac{B_\theta}{B_T}U_0 + \left[\frac{m\left(v_\parallel^2+v_\perp^2\right)}{2T} - \frac{3}{2}\right]V_T$$

(compare **Hinton Waltz**)

NOTE that the denominator

$$\frac{1}{v_\parallel\left(B_\theta/B_T\right)+V_E-\iota\sigma\nu_{eff}}$$

is not singular only due to collisions. The collisions prevent the resonance.

END.

NOTE

The function $f_{1\sigma}$ contains the factor W.

This W is a velocity and includes the electric velocity V_E.

The electric velocity has *time variation* since the problem consists of finding the rate of damping of the poloidal velocity.

Therefore the expected damping rate $V_E(t)$ means $W(t)$ and further $f_{1\sigma}(t)$.

We will need an equation where $f_{1\sigma}$ has time variation, balanced by other terms in the dynamics.

END

Using the first order in the small parameter

$$\frac{\partial/\partial t}{v_{Th}/(qR)} \ll 1$$

and the ordering

$$\begin{aligned} v_E &\ll v_{Th}\frac{B_\theta}{B_T} \\ v_E &\ll v_{th}\Theta \end{aligned}$$

the second order contribution to the distribution function $\widetilde{f}$ is obtained from the differential equation

$$\frac{\partial f_1}{\partial t} + v_\parallel \frac{B_\theta}{B_T}\frac{\partial f_2}{r\partial\theta} = -\nu_{eff}\ f_2$$

(we **note** that $v_\parallel \frac{B_\theta}{B_T} = v_\theta$) from which a solution is obtained

$$f_{2\sigma} = -\iota \frac{\varepsilon\sigma r \frac{\left(v_\perp^2/2+v_\parallel^2\right)}{2T/m}}{\left[v_\parallel\left(B_\theta/B_T\right) - \iota\sigma r\nu_{eff}\right]^2} f_M\ \frac{\partial v_E}{\partial t}$$

NOTE

The condition

$$\frac{\partial/\partial t}{v_{Th}/(qR)} \ll 1$$

means that the events (damping of v_E) discussed here happen on a scale of time that is *faster* than the transit time.

END

COMMENT

The second order correction is obtained from the balance with the *time variation* of the first order variation, which means

$$f_2 \sim \frac{\partial f_1}{\partial t}$$

This is because the expression for the first oorder correction f_1 contains the factor W which was derived from the radial variation of the equilibrium distribution function $f_0 \sim f_M$.

The factor W contains the *electric potential* ϕ that has radial variation

$$\phi = \phi(r)$$

BUT it also has a time variation

$$\phi = \phi(r, t)$$

since the decay of poloidal rotation consists of the chage of the radial electric field that produces the torque.

$$\begin{array}{rcl} & & \text{torque to stop poloidal rotation} \\ & \downarrow & \\ E_r & = & E_r(t) \\ & \sim & \text{decay of } v_\theta = \dfrac{E_r}{B_T} \end{array}$$

Here it is substituted

$$\nu_{eff} \approx \frac{v_{Th}^2}{v_\parallel^2} \nu_i$$

and the second order contribution to the distribution function, after reversing

$$f_i = \sum_{\sigma = \pm 1} f_{i\sigma} \exp(i\sigma\theta)$$

becomes (omiting the term with $\cos\theta$)

$$f_2 = \frac{\left(\frac{v_\parallel}{v_{Th}}\right)^6 - \widehat{\nu}^2}{\left[\left(\frac{v_\parallel}{v_{Th}}\right)^6 + \widehat{\nu}^2\right]^2} \sin\theta \; \frac{\varepsilon r}{\left(v_{Th} \frac{B_\theta}{B_T}\right)^2} \frac{\left(v_\perp^2/2 + v_\parallel^2\right)}{T/m} \left(\frac{v_\parallel}{v_{Th}}\right)^4 f_M \frac{\partial v_E}{\partial t}$$

where it is noted later

$$\frac{v_\parallel}{v_{Th}} \equiv x$$

and

$$\widehat{\nu} \equiv \frac{r}{v_{Th}\frac{B_\theta}{B_T}}\nu_i$$

which is related to the standard neoclassical collisional parameter ν_* by

$$\begin{aligned}\widehat{\nu} &= \varepsilon^{3/2}\nu_* \\ &= \text{plateau collisionality parameter}\end{aligned}$$

It is extremely important that the expression of f_2 contains a factor $\partial V_E/\partial t$.

This will contribute to the *inertia* to poloidal rotation $1+q^2$.

In order to calculate the magnetic damping of the poloidal rotation it is necessary to start from the radial electric current which on a magnetic surface must have the average equal to zero

$$\langle j_r \rangle = 0$$

The radial fluxes are considered

$$\langle nV_r \rangle = \frac{1}{2\pi}\int_0^{2\pi} d^3v \; d\theta \; v_r \left(1+\varepsilon\cos\theta\right) \; f$$

where the variables in velocity space are

$$\left(v_\parallel, \frac{v_\perp^2}{2}\right)$$

and

$$d^3v = 2\pi dv_\parallel d\left(\frac{v_\perp^2}{2}\right)$$

The radial component of the *particle drift* velocity is

$$v_r = -\frac{1}{\Omega_c}\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)\frac{\sin\theta}{R}+\frac{1}{\Omega_c}\frac{\partial v_E}{\partial t}$$

In the integral for the radial particle fluxes one substitutes the *ion* distribution function

$$f = f_0 + f_1 + f_2 + ...$$

the expansion in the small parameter representing the ratio between the characteristic frequency of the variation of the radial electric field and the bounce frequency.

$$
\begin{aligned}
\langle j_r \rangle &= 0 \\
\text{or} \quad \langle nV_r \rangle &= \frac{1}{2\pi}\int_0^{2\pi} d^3v \ d\theta \ v_r \left(1+\varepsilon\cos\theta\right) \ f \\
& \text{or} \ \frac{1}{2\pi}\int d^3v d\theta \left\{ \ \left[f_0\frac{\partial v_E}{\partial t} - f_2\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)\frac{\sin\theta}{R}\right] - \ f_1\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)\frac{\sin\theta}{R}\right\}
\end{aligned}
$$

or

$$
\begin{aligned}
& \int d^3v d\theta \left[f_0\frac{\partial v_E}{\partial t} - f_2\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)\frac{\sin\theta}{R}\right] \\
= & \int d^3v d\theta \ f_1\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)\frac{\sin\theta}{R}
\end{aligned}
$$

If the plateau collision parameter is small

$$\widehat{\nu} \ll 1$$

then the distribution function in order 1 can be approximated

$$f_1 \approx -\pi q \frac{\frac{v_\perp^2}{2}+v_\parallel^2}{T/m} W \ f_M \ \delta\left(v_\parallel\right)\sin\theta$$

NOTE that the most important contribution to the distribution function, *i.e.* f_1 comes from the *barely trapped ions*, for which $v_\parallel \approx 0$. **End**.

Note similarity with **Rozhansky Tendler** in instabilities. **End.**

In the equation $\langle j_r \rangle = 0$ written above, the term f_2 also contains $\partial V_E/\partial t$. For this reason we have to group the two terms of $\partial V_E/\partial t$ that occur in the integrand $f_0\frac{\partial v_E}{\partial t} - f_2\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)\frac{\sin\theta}{R}$. We now understand why it has been ignored the term that contains $\cos\theta$ in the expression of f_2 reconsituted by inverse Fourier transform: the $\cos\theta$ would give $\sin\theta\cos\theta$ and the integral of this on θ is zero. We retained $\sin\theta$ since this gives $\sin^2\theta$ with finite integral on θ.

The two occurence of $\partial V_E/\partial t$ in the expression $f_0\frac{\partial v_E}{\partial t} - f_2\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)\frac{\sin\theta}{R}$, after integration over θ and over the velocity space d^3v, lead to

$$\left(1+q^2\Lambda\right)\frac{\partial V_E}{\partial t}$$

and this shows the poloidal plasma *inertia*.

Note see alternative derivation in **Hirshman paradox ambipolarity** in *references plasma.* **End.**

Then the equation for the poloidal velocity becomes

$$\left(1+q^2\Lambda\right)\frac{\partial v_E}{\partial t}=-\nu_{MP}\left(v_E+v_{*n}+U_0\frac{B_\theta}{B_T}+\frac{3}{2}v_{*T}\right)$$

where the rate of magnetic pumping damping is

$$\nu_{MP}=\sqrt{\frac{\pi}{2}}\frac{qv_{Th}}{R}$$

and

$$\Lambda\equiv\frac{3}{2}+\Xi\widehat{\nu}^{-1/3}$$

$$\Xi\equiv\frac{1}{2\pi}\int_{-\infty}^{\infty}dx\ x^4\frac{x^6-1}{\left(x^6+1\right)^2}\exp\left(-\widehat{\nu}^{2/3}\frac{x^2}{2}\right)$$

NOTE compare with **Hassam Drake**. **End.**

We remark

- the *inertia* in the poloidal rotation is indissolubly related to the θ variation, the poloidal variation of the distribution function
- this $\sim\theta$ variation comes from the *RADIAL neoclassical drift* of particles, that advects the equilibrium distribution function, with radial gradients
- this $\sim\theta$ advection imposes balance *parallel* to the line, which involves the modulated (magnetic mirror) parallel velocity, collisions;
- the connection is imposed by the overall condition $\langle j_r\rangle=0$
- grouping terms with $\partial V_E/\partial t$ in this constraint we transform it into an equation for the *rate of damping.*

There is an asymptotic poloidal velocity which is sustained by the gradients density+temperature and, if exists, a parallel flow, simply projected on the poloidal direction

$$v_{E\infty}=-V_n-\frac{3}{2}V_T-\frac{B_\theta}{B_T}U_0$$

The equation of variation in time of the poloidal velocity is

$$mn\left(1+2q^2\right)\frac{\partial v_E}{\partial t} = \underset{\text{average parallel viscous force}}{-\frac{1}{2\pi R}\int_0^{2\pi}\left(p_\perp - p_\parallel\right)\sin\theta\ d\theta}$$

where

$$p_\perp = \int d^3v\ \frac{mv_\perp^2}{2}\ f$$
$$p_\parallel = \int d^3v\ mv_\parallel^2\ f$$

16 The dynamics of the radial electric field Novakovskii Liu Sagdeev Rosenbluth

This subject is only mentioned in *polarization.tex* although it is relevant to that chapter too.

16.1 Introduction

This paper is about the fast time scale variation of the radial electric field, therefore includes decay of poloidal rotation and GAM.

Formulation:
a plasma with Maxwellian distribution.
There are radial gradients of temperature and of the density.
The initial radial electric field is zero.
How long does it take for the system to reach the next level of stationarity, after the *relaxation* ?
And what are the electric field and the radial electric field at the end ?
[We can extend and ask: what are the final distribution of parameters (density, temperature, flows) on the magnetic surfaces, with θ dependence].

quilibrium poloidal rotation velocity

$$U_\theta = \underset{\text{poloidal}}{\frac{B_\theta}{B_T}\overline{U}_\parallel} + V_E + V_n + V_T$$

where $\overline{U}$ is an uniform flow along the line.

[**NOTE**
compare with the previous work

$$\begin{aligned} W &\equiv V_E + v_{*n} + U_0 \frac{B_\theta}{B_T} \\ &+ v_{*T} \left[\frac{m\left(v_\parallel^2 + v_\perp^2\right)}{2T} - \frac{3}{2} \right] \\ &\text{poloidal} \end{aligned}$$

END]

where

$$\overline{U}_\parallel \equiv \text{parallel flow of plasma, averaged over surface}$$

$$\begin{aligned} V_n &= \frac{T}{eB}\frac{1}{L_n} \\ V_T &= \frac{T}{eB}\frac{1}{L_T} \\ V_E &= -\frac{E_r}{B} \end{aligned}$$

The relaxation of the poloidal rotation, symbolic eq.

$$\frac{\partial U_\theta}{\partial t} = -\nu_{MP}\left(U_\theta - kV_T\right)$$

with

$$k = \begin{cases} -2.1 & \text{Pfirsch Schluter} \\ -0.5 & \text{plateau} \\ 1.17 & \text{banana} \end{cases}$$

and

$$\begin{aligned} \nu_{MP} &= \text{magnetic pumping} \\ &\quad \text{in Pfirsch Schluter} \\ &= \nu_{PS} = \frac{\nu_b^2}{\nu_{ii}} = \frac{(\text{bounce freq.})^2}{\text{collision freq.}} \end{aligned}$$

Fast time scales are

$$\frac{\partial}{\partial t} \sim \omega_{bounce}$$

In this case there are *Geodesic Acoustic Modes*: oscillations of the plasma column in vertical direction

$$\omega_{GAM} \approx \frac{V_T}{R}$$

Asymptotic

$$\begin{aligned} V_E &= V_{E\infty} \\ &\quad +A\exp\left(-\gamma_{MP}t\right) \\ &\quad +B\cos\left(\omega_{GAM}t+\phi\right)\ \exp\left(-\gamma_{GAM}t\right) \end{aligned}$$

16.2 The drift-kinetic equation

The drift kinetic equation

$$\begin{aligned} &\frac{\partial f}{\partial t}+\left(\frac{B_\theta}{B_T}v_\parallel+V_E\right)\frac{\partial f}{r\partial\theta}+V_r\frac{\partial f}{\partial r} \\ &+\frac{dv_\parallel}{dt}\frac{\partial f}{\partial v_\parallel}+\frac{d\left(v^2/2\right)}{dt}\frac{\partial f}{\partial\left(v^2/2\right)} \\ =\ & C\left[f\right] \end{aligned}$$

where the radial velocity of a particle (kinetic treatment) is the *neoclassical drift*, with $\sim\theta$, and includes the *polarization drift*,

$$\begin{aligned} V_r &= \frac{1}{\Omega_c}\frac{v_\perp^2/2+v_\parallel^2}{R}\sin\theta \quad \text{(drift projected on } r\text{)} \\ &\quad -\frac{1}{\Omega_c}\frac{\partial V_E}{\partial t} \qquad \left(\text{polarization, from } \frac{\partial E_r}{\partial t}\right) \end{aligned}$$

with the observation that the first term should be written

$$\frac{1}{\Omega_c}\frac{1}{R}\frac{v^2+v_\parallel^2}{2}\sin\theta$$

because we will need variables $v^2/2$ and $v_\parallel/v$ later.

The second term in the expression of V_r assumes that there is a *time variation* of the radial electric field. This can only exist if there is a relative *radial* displacement of charges, represented for each by the a velocity $v_r^{e,i}$.

The term is from

$$
\begin{aligned}
\varepsilon_0 \left(1+\frac{c^2}{v_A^2}\right)\frac{\partial E_r}{\partial t} &= e\Gamma_r \\
\text{approx} \quad \varepsilon_0 \times \frac{c^2}{\frac{B^2}{\mu_0 \rho}}\frac{\partial E_r}{\partial t} &= enV_r \\
\varepsilon_0 \times \mu_0 c^2 \frac{1}{eB} nm \frac{\partial}{\partial t}\left(\frac{E_r}{B}\right) &= nV_r \\
\varepsilon_0 \times \frac{1}{\varepsilon_0}\frac{1}{eB/m}\frac{\partial \left(V_E\right)_\theta}{\partial t} &= V_r \\
\frac{1}{\Omega_c}\frac{\partial V_E}{\partial t} &= V_r
\end{aligned}
$$

The dimensional equation is Ampere's law where the current is compensated by the induction of time-varying electric field

$$
\begin{aligned}
0 &= \mu_0 j_r + \frac{1}{c^2}\frac{\partial E_r}{\partial t} \\
0 &= j_r + \varepsilon_0 \frac{\partial E_r}{\partial t}
\end{aligned}
$$

which means that in the equation for polarization one needs a ε_0.

We conclude that in the kinetic equation the term

$$
V_r \frac{\partial f}{\partial r}
$$

is a convection of the distribution function f due to (partly) the radial *current* that is associated to the time variation of the radial electric field (damping, polarization). This results from the Ampere equation, where the radial current j_r (implicitely the radial velocity V_r) is balanced by the time variation of the radial electric field $\frac{1}{c^2}\frac{\partial E_r}{\partial t}$, as the *induction* term in the Ampere's law.

But which is the radial flow velocity V_r ?. It is of the ions, the electrons are too connected with the magnetic surface.

On the other hand this term is the essential source of *time variation of the radial electric field* , equivalently the poloidal rotation velocity, $\frac{\partial V_E}{\partial t}$, which will result from a balance of various mechanisms (damping by magnetic pumping, or acceleration, etc.).

NOTE

The time variation of the poloidal velocity

$$
\frac{\partial}{\partial t}\left(\frac{E_r}{B}\right)
$$

necessarily results from the torque $\mathbf{j} \times \mathbf{B}$ in the *iterative* use of the momentum equation, with zero order velocity being E_r/B, in the term $mn\frac{d}{dt}\mathbf{v}$, while $j = nmv_r$.

This occurs for example in the situation where the plasma has a Boltzmannian response to a potential perturbation and the convection of the Boltzmann response by the $E \times B$ velocity is zero: $[(-\boldsymbol{\nabla}_\perp \phi \times \widehat{\mathbf{n}}) \cdot \boldsymbol{\nabla}_\perp] \left(\frac{e\phi}{T}\right) = 0$. The the radial flow that results is a current of polarization and its magnitude varies with $1/B^2$. Hasegawa Mima regime. It is mainly for small scale perturbations, instabilities (since the nonlinear term has k^4.

The balance along the *poloidal* angle θ,

$$nm\frac{\partial (V_E)_\theta}{\partial r} + (...) = -\nabla_\theta p + enE_\theta + J_r B_T$$

with neglect of the poloidal projected electric field $E_\theta \approx 0$ (too small variation of the electrostatic potential along θ, $\partial\phi/\partial\theta$, a neoclassical effect like Pfirsch Schluter or Stringer), with $\nabla_\theta p \approx 0$ (again weak stationary neoclassical variation of parameters along θ, in the surface) and

$$\begin{aligned} \frac{\partial V_E}{\partial t} &= \frac{eB_T}{m} V_r \\ &= \Omega V_r \end{aligned}$$

valid when there is a radial current $J_r = enV_r$.

There is a difference between the two explanations

In the first one, $\varepsilon_0 \left(1 + \frac{c^2}{v_A^2}\right) \frac{\partial E_r}{\partial t} = e\Gamma_r$, the aspect of polarization is strong.

In the second one the momentum balance along the poloidal direction is dominated by the acceleration (torque) produced by the radial current, $(\mathbf{j}_r \times \mathbf{B})_\theta$ because $V_r \sim J_r$

The first explanation requires from us to accept that the magnetic field is insensitive to the fast changes

$$\boldsymbol{\nabla} \times \mathbf{B}|_r = \mathbf{0}$$

and the radial current is compensated by the induction term, a time-variation of the radial electric field $\partial E_r/\partial t \sim \partial V_\theta/\partial t$.

The presence in the first argument of ε_0 may leave the impression that there is induction effect. Actually, the μ_0 and $\frac{\varepsilon_0}{c^2}$ will remove from the Ampere's law the "induction" aspect.

In the second derivation the balance is directly given by momenta: poloidal torque $(j_r \times B)_\theta$ and acceleration $\partial V_{\theta=E}/\partial t$.

END

NOTE

polarization: a radial displacement of charged particles belonging to different species (electrons and ions) is non-balanced, so there is a radial current.

This produces a charge separation (polarization) and an electric field, which generates rotation by $E \times B$.

END

The other contribution to the radial convection V_r is the neoclassical particle drift.

The equations for velocity

$$\frac{dv_\|}{dt} = -\frac{\varepsilon}{q}\frac{1}{R}\frac{v^2 - v_\|^2}{2}\sin\theta + v_\| V_E \frac{1}{R}\sin\theta$$

$$\frac{d\left(v^2/2\right)}{dt} = -\frac{v^2 + v_\|^2}{2}\frac{1}{R}\sin\theta \ V_E$$

NOTE

In **Wong Burrell** the equations are

$$\begin{aligned}
\frac{dv_\|}{dt} &= -\frac{v_\perp^2}{2}\nabla_\| \ln B + v_\| \frac{-\boldsymbol{\nabla}\phi \times \widehat{\mathbf{n}}}{B} \cdot \boldsymbol{\nabla} \ln B - \frac{e}{m}\nabla_\| \phi \\
\frac{d}{dt}\left(\frac{v_\perp^2}{2}\right) &= \frac{v_\perp^2}{2} v_\| \boldsymbol{\nabla}_\| \ln B + \frac{v_\perp^2}{2}\frac{-\boldsymbol{\nabla}\phi \times \widehat{\mathbf{n}}}{B} \cdot \boldsymbol{\nabla} \ln B
\end{aligned}$$

For comparison, we find in *derivation drift kinetic equation.tex*,

$$\boldsymbol{\nabla}\cdot\widehat{\mathbf{n}} = -\nabla_\| \ln B$$

where

$$\nabla_\| \ln B = \frac{\varepsilon}{q}\frac{\sin\theta}{R}$$

Then for example

$$-\frac{v_\perp^2}{2}\nabla_\| \ln B \quad \text{(Wong Burrell)}$$

$$-\frac{\varepsilon}{q}\frac{1}{R}\frac{v^2 - v_\|^2}{2}\sin\theta \quad \text{(Novakovskii)}$$

are identical.

END

We remind that
$$\frac{\varepsilon}{q} = \Theta = \frac{B_\theta}{B_T} = \frac{r}{qR}$$

The next order introduces the correction to the Maxwellian,
$$\widetilde{f}$$
which verifies the equation
$$\begin{aligned} & \frac{\partial \widetilde{f}}{\partial t} + \frac{v_\parallel}{qR}\frac{\partial \widetilde{f}}{\partial \theta} - \frac{\varepsilon}{q}\frac{v^2 - v_\parallel^2}{2}\frac{1}{R}\sin\theta \frac{\partial \widetilde{f}}{\partial v_\parallel} - C\left[\widetilde{f}\right] \\ = \; & \frac{1}{T/m}\frac{v^2 + v_\parallel^2}{2}\frac{1}{R}\sin\theta \left[V_E + V_n + \left(\frac{v^2}{2T/m} - \frac{3}{2}\right) V_T\right] f_M \end{aligned}$$

We **note** that the RHS consists of *diamagnetic terms*, V_n and V_T and of energy term associated to the time variation of $\frac{d\left(v_\perp^2/2\right)}{dt}$, caused by the modulation of the magnetic field along the line (mirror effect).

The first term in the RHS
$$\sin\theta \frac{1}{T/m}\frac{v^2 + v_\parallel^2}{2}\frac{1}{R} V_E \; \times f_M$$
comes from
$$\frac{d\left(v^2/2\right)}{dt} = -\frac{v^2 + v_\parallel^2}{2}\frac{1}{R}\sin\theta \; V_E$$
and
$$\frac{\partial f_M}{\partial\left(v_\perp^2/2\right)} = -\frac{1}{T/m} f_M$$

The second and third terms come from the *neoclassical drift* part in V_r,
$$\frac{1}{\Omega_c}\frac{v_\perp^2/2 + v_\parallel^2}{R}\sin\theta$$
and the radial derivation of the *advected* Maxwellian
$$\frac{\partial f_M}{\partial r} = \left[V_n + \left(\frac{v^2}{2T/m} - \frac{3}{2}\right) V_T\right] f_M$$

This means that the radial *current* j_r which is V_r advects the Maxwellian distribution f_M, by $(\mathbf{v}\cdot\boldsymbol{\nabla}) f_M \sim V_r \frac{\partial f_M}{\partial r}$.

NOTE
The term

$$-\frac{\varepsilon}{q}\frac{v^2-v_\parallel^2}{2}\frac{1}{R}\sin\theta\frac{\partial \widetilde{f}}{\partial v_\parallel}$$

is energetic and can be explained

- using the equation for the variation of the parallel velocity (*mirror force*), and
- the θ variation of the magnetic field $B(r,\theta)$.

the coefficient of $\partial\widetilde{f}/\partial v_\parallel$ in the equation is

$$-\frac{\varepsilon}{q}\frac{v_\perp^2}{2}\frac{1}{R}\sin\theta$$

should be compared with

$$\frac{dv_\parallel}{dt}=-\frac{\varepsilon}{q}\frac{1}{R}\frac{v^2-v_\parallel^2}{2}\sin\theta$$

Indeed this is the energetic effect of the "mirror force"

$$\frac{dv_\parallel}{dt}\frac{\partial f}{\partial v_\parallel}$$

and of course only the *correction* to the zero-order (Maxwellian) can have variation with θ. However it is NOT stationary neoclassical variation in *surface*, of n, T, (and pressure p) and of electrostatic potential, coming from the same source as Pfirsch Schluter. There is now a dynamical variation $\partial\widetilde{f}/\partial v_\parallel$.
END

The conditions

- the polarization drift has been neglected, $\frac{\partial V_E}{\partial t}$. This was a part of the *radial advection*, $V_r\frac{\partial \widetilde{f}}{\partial r}$ where part of the radial velocity $V_r=(...)-\frac{1}{\Omega_i}\frac{\partial V_E}{\partial t}$ is generated by the time variation of the radial electric field - equivalently, the time variation of the poloidal rotation velocity. It is therefore neglected the effect of advection by the polarization term $\partial V_E/\partial t$ of the equilibrium distribution function

- neglect of V_E terms in the *drift-convective* part in the LHS. It is question of the term $v_{\|} V_E \frac{1}{R} \sin\theta$ from $\frac{dv_{\|}}{dt}$ and of the term $-\frac{v^2+v_{\|}^2}{2} \frac{1}{R} \sin\theta \, V_E$ which is $\frac{d\left(v_{\perp}^2/2\right)}{dt}$. They are neglected.

We conclude from the structure of this equation that the perturbed distribution function $\widetilde{f}$ that will describe the time variation of the poloidal rotation results from a balance of

- variation with θ of this distribution function $\widetilde{f}(\theta)$ (advected by $v_{\|}$)
- mirror force, *i.e.* the modulation of $v_{\|}$ on θ due to the magnetic field $B(\theta)$ which is reflected in energetic effect on $\widetilde{f}\left(v_{\|}\right)$
- collisions

The RHS is the source, $\sim f_M$, with $\sin\theta$ variation from the *neoclassical drifts* that advects RADIALLY f_M, with its radial gradients

Change of variables

$$\left(v^2/2, v_{\|}\right) \rightarrow \left(v, \xi = \frac{v_{\|}}{v}\right)$$

The equation becomes

$$\begin{aligned} & \frac{\partial \widetilde{f}}{\partial t} + \xi v \frac{1}{qR} \frac{\partial \widetilde{f}}{\partial \theta} - \frac{\varepsilon}{q} \frac{v\left(1-\xi^2\right)}{2} \frac{1}{R} \sin\theta \frac{\partial \widetilde{f}}{\partial \xi} - C\left[\widetilde{f}\right] \\ = \ & \sin\theta \frac{1}{T/m} \frac{v^2\left(1+\xi^2\right)}{2} \frac{1}{R} \left[V_E + V_n + \left(\frac{v^2}{2T/m} - \frac{3}{2}\right) V_T\right] f_M \end{aligned}$$

NOTE
In **Sugama Nishimura** the parallel velocity *operator* is

$$V_{\|}\left[\right] = v\xi \nabla_{\|} - \frac{1}{2} v \left(1-\xi^2\right) \nabla_{\|} \ln B \frac{\partial}{\partial \xi}$$

without including the $E \times B$ magnetic drifts.

This is because the coefficients of $\partial \widetilde{f} / \partial \xi$ are

$$
\begin{aligned}
\frac{\varepsilon}{q} \frac{\sin\theta}{R} \text{ (Novakovskii)} &= \nabla_{\|} \ln B \quad \text{(Sugama)} \\
&= \nabla_{\|} \ln \frac{B_0}{1+\varepsilon\cos\theta} = \frac{1}{qR} \frac{\partial}{\partial\theta} \left[\ln B_0 - \ln\left(1+\varepsilon\cos\theta\right)\right] \\
&= -\frac{1}{qR} \frac{-\varepsilon\sin\theta}{1+\varepsilon\cos\theta} = \frac{\varepsilon}{q} \frac{\sin\theta}{R} \left(1-\varepsilon\cos\theta\right) \\
&= \frac{\varepsilon}{q} \frac{\sin\theta}{R} \text{ ok}
\end{aligned}
$$

The two expressions are identical.

END

NOTE

About the term

$$
-\frac{\varepsilon}{q} \frac{v\left(1-\xi^2\right)}{2} \frac{1}{R} \sin\theta \frac{\partial \widetilde{f}}{\partial \xi}
$$

It is in velocity space.

In *Stringer.tex*, it is derived (Hassam Antonsen)

$$
\widehat{u}_{\|} = -2qR\varepsilon \frac{\cos\theta}{r} \widehat{V}_E
$$

but this $\widehat{u}_{\|}$ is not used for energy changes. Its time variation comes from parallel gradient of pressure.

END

16.3 Collision operator

The collision operator must contain *pitch angle scattering* and *slowing down*, the usual structure.

$$
\begin{aligned}
C\left[f\right] &= \nu_c\left(x\right) \frac{\partial}{\partial \xi} \left(1-\xi^2\right) \frac{\partial f}{\partial \xi} \\
&\quad + \xi \widehat{S}_1 \left[f\right]
\end{aligned}
$$

where

$$
x = \frac{v^2}{2T/m}
$$

$$\nu_c(x) = \frac{3\sqrt{2\pi}}{4}\nu_{ii}\left[\left(1-\frac{1}{2x^2}\right)\mathrm{erf}(x)+\frac{\exp(-x^2)}{\sqrt{\pi}x}\right]$$
$$\nu_{ii} = 4\pi\frac{e^4}{m^2}\lambda\frac{n}{v_{th}^3}$$

and

$$\widehat{S}_1[f] = 3[\nu_c(x)-\nu_{slowing}(x)]\int_{-1}^{1}d\xi\ \xi\ f$$
$$+3x\nu_{slowing}(x)f_M\frac{\int dx'\ \nu_{slowing}(x')x'^3\left(\int_{-1}^{1}d\xi'\ \xi'\ f\right)}{\int dx'\ \nu_{slowing}(x')\ x'^4\ f_M}$$

The *slowing down* frequency

$$\nu_{slowing}(x) = \nu_{ii}\frac{2}{x^3}\left[\mathrm{erf}(x)-\frac{2x\exp(-x^2)}{\sqrt{\pi}}\right]$$

NOTE

In **Novakovskii Liu Sagdeev Rosenbluth** the collision operator is

$$St(f) = \nu_c(x)\frac{\partial}{\partial\xi}\left(1-\xi^2\right)\frac{\partial f}{\partial\xi}$$
$$+\xi\ \widehat{S}[f_1]$$

where

$$x^2 \equiv \frac{v^2}{(2T/m)} = \frac{v^2}{v_{th}^2}$$

and

$$\nu_c = \frac{3\sqrt{2\pi}}{4}\nu_{ii}\ \frac{1}{x^3}\left[\left(1-\frac{1}{2x^2}\right)\mathrm{erf}(x)+\frac{1}{\sqrt{\pi}}\frac{\exp(-x^2)}{x}\right]$$

$$\nu_{ii} = \frac{4\pi e^4}{m^2}\frac{1}{\Lambda}\frac{n}{v_{th}^3}$$
$$\text{typical} \sim \frac{n}{T^{3/2}}$$

$$\widehat{S}[f] \equiv \text{Hirshman Sigmar Clarke - type}$$
$$= [\nu_c(x)-\nu_{slowing}(x)]\ 3\int_{-1}^{1}d\xi\ \xi\ f$$
$$+\nu_{slowing}(x)\ 3x\ f_M\ \frac{\int dx'\ x'^3\nu_{slowing}(x')\ \left(\int_{-1}^{1}d\xi'\ \xi'\ f\right)}{\int dx'\ x'^4\ \nu_{slowing}(x')\ f_M}$$

The slowing down frequency is

$$\nu_{slowing}(x) = \nu_{ii}\ \frac{2}{x^3}\left[\mathrm{erf}(x) - \frac{1}{\sqrt{\pi}}2x\ \exp\left(-x^2\right)\right]$$

This operator is again of the form

$$\begin{aligned} C &= \mathcal{L}\ \text{(Lorentz)} + \text{[Hirshman Sigmar Clarke]} \\ & \quad \text{[pitch angle]} + \text{[slowing]} \end{aligned}$$

END

NOTE
In **Helander stellarator** the operator of collision is

$$\begin{aligned} C_a(f_{a1}) &\rightarrow \nu_a\ \mathcal{L}(f_{a1}) \\ & \quad +\nu_a\ v_{\|}\ w_a\ \frac{1}{T_a/m_a} f_{a0} \end{aligned}$$

where

$$\mathcal{L} = \frac{1}{2}\frac{\partial}{\partial\xi}\left(1-\xi^2\right)\frac{\partial}{\partial\xi}\ \text{Lorentz}$$

$$w_a \equiv \text{function of speed } v \text{ determined by integral conditions}$$

Therefore we have the same structure

$$C = \mathcal{L}\ \text{(Lorentz)} + \text{(Hirshman Sigman Clarke)}$$

END

Normalization of the quantities in the equation

$$\widehat{\nu}_c = \nu_c(x)\frac{v_{th}}{qR}$$

$$\widehat{t} = t\frac{v_{th}}{qR}$$

$$\begin{aligned} \widetilde{f} &\rightarrow \widehat{f} \\ \widetilde{f} &= \widehat{f}\ f_M \end{aligned}$$

Then

$$\begin{aligned} & \frac{\partial\widehat{f}}{\partial\widehat{t}} + \xi x\frac{\partial\widehat{f}}{\partial\theta} - \varepsilon\frac{x\left(1-\xi^2\right)}{2}\sin\theta\frac{\partial\widehat{f}}{\partial\xi} - \widehat{C}\left[\widehat{f}\right] \\ = & \sin\theta\ x^2\left(1+\xi^2\right)\left[\widehat{V}_E + \widehat{V}_n + \left(x^2-\frac{3}{2}\right)\widehat{V}_T\right] \end{aligned}$$

where the drift velocities have been normalized

$$\begin{aligned}
\widehat{V}_E &= V_E \frac{q}{v_{th}} \quad \text{(electric)} \\
\widehat{V}_n &= V_n \frac{q}{v_{th}} \quad \text{(dia - n)} \\
\widehat{V}_T &= V_T \frac{q}{v_{th}} \quad \text{(dia - T)}
\end{aligned}$$

The collision operator becomes $\widehat{C}$ after taking

$$\widehat{\nu}_{ii} = \nu_{ii} \frac{v_{th}}{qR}$$

A very important condition: the radial current across a surface must average to zero. This is the quasineutrality.

$$\langle j_r \rangle = \int d^3v \ V_r f \ R d\theta = 0$$

[**NOTE**
in *stringer.tex* this condition is essential in establishing a relation

$$\frac{R}{B_\varphi} \mathbf{B}_{pol} \cdot \left[n m_i \frac{d\mathbf{u}}{dt} + T \boldsymbol{\nabla} n \right] = \mathbf{j} \cdot \boldsymbol{\nabla} \psi$$

between the time variation of the poloidal electric velocity V_E and the $\theta-$dependent density perturbation in surface, $n_1(\theta)$.

This leads to

$$\frac{\partial V_E}{\partial t} = -\frac{1}{r} \varepsilon \frac{c_s^2}{n_0} \int \frac{d\theta}{2\pi} 2 \sin\theta \ n_1$$

END].

The volume in the velocity space is

$$\begin{aligned}
d^3v &= 2\pi v^2 dv d\xi \\
\int d^3v ... &= \int_0^\infty dv \int_{-1}^1 d\xi \ 2\pi v^2 ...
\end{aligned}$$

and

$$R \sim h = 1 + \varepsilon \cos\theta$$

In the expression of the (averaged) radial current $\langle j_r \rangle$ one inserts the radial velocity composed of *drift* and *polarization*

$$V_r = \frac{1}{\Omega_c}\frac{v_\perp^2/2+v_\parallel^2}{R}\sin\theta - \frac{1}{\Omega_c}\frac{\partial V_E}{\partial t}$$

and separate (i.e. focus on) the $\partial V_E/\partial t$,

$$\frac{\partial \widehat{V}_E}{\partial t} + \frac{q^2}{2\pi^{3/2}}\int dx\ d\xi\ d\theta\ \left(1+\xi^2\right)x^4 \exp\left(-x^2\right)\widehat{f}\ \sin\theta = 0$$

NOTE

This step is essential in determination of the rate of change of the poloidal velocity , $\partial V_E/\partial t$.

Here the condition of zero radial current averaged over surface is converted in the equation for time evolution of the electric velocity $V_E(t)$. This arises because the *radial velocity* V_r contains a term that depends of $\partial V_E/\partial t$.

END

Further one takes the surface average of the macroscopic *parallel flow velocity*

$$\overline{U}_\parallel = \frac{\int d^3v\ d\theta\ v_\parallel f}{\int d^3v\ d\theta\ f}$$

after normalization

$$\overline{U}_\parallel \rightarrow \widehat{\overline{U}}_\parallel = \varepsilon\frac{\overline{U}_\parallel}{v_{th}}$$

and for it the surface average is

$$\widehat{\overline{U}}_\parallel = \varepsilon\frac{1}{\pi^{3/2}}\int dx\ d\xi\ d\theta\ x^3\xi\ \widehat{f}\exp\left(-x^2\right)$$

16.4 Solution by expansion with separation of variables v and ξ

The problem of solving the drift-kinetic equation comes from the structure of the collision operator.

The solution si to use series with terms where it is adopted the separation of variables.

The solution of the normalized drift kinetic equation is obtained after adopting the usual separation of variables

$$\widehat{f}(\theta,\xi,x)=\sum F_n(\theta,x)\,P_n(\xi)$$

which must deal with the collision operator.

This is an important separation of variables

$$\begin{array}{lll} x & \equiv & v/v_{th} \text{ the magnitude of the velocity (normalized)} \\ \xi & \equiv & v_{\parallel}/v \text{ the pitch angle variable,} \\ & & \qquad\text{adapted to the structure of } f \text{ in velocity space} \end{array}$$

This expansion is inserted in the equation for $\widehat{f}$. Using the orthogonality of the Legendre equation one obtains a system. The collision operator has the property

$$\widehat{C}\left[\sum B_n(x)\,P_n(\xi)\right]=-\sum\left(\widehat{\nu}_n\;B_n\right)P_n$$

$$\begin{aligned}\widehat{\nu}_n(x)\,B_n &= n(n+1)\left[\widehat{\nu}_c-\delta_{1,n}\left(\widehat{\nu}_c-\widehat{\nu}_S\right)\right]B_n \\ &\quad -x\widehat{\nu}_S\delta_{1,n}\frac{\int dx\;x^3\widehat{\nu}_S B_n}{\int dx\;\widehat{\nu}_S\;\exp\left(-x^2\right)x^4}\end{aligned}$$

The system of equations for F_n is

$$\begin{aligned}&\frac{\partial F_n}{\partial t}+x\left(\frac{n}{2n-1}\frac{\partial F_{n-1}}{\partial\theta}+\frac{n+1}{2n+3}\frac{\partial F_{n+1}}{\partial\theta}\right) \\ &+\varepsilon\frac{\sin\theta}{2}x\;\left[\frac{n(n-1)}{2n-1}F_{n-1}-\frac{(n+1)(n+2)}{2n+3}F_{n+1}\right] \\ &+\widehat{\nu}_n F_n \\ =\;& x^2\;\sin\theta\frac{4\delta_{0,n}+2\delta_{2,n}}{3}\left[V_E+V_n+\left(x^2-\frac{3}{2}\right)V_T\right]\end{aligned}$$

The quasineutrality condition (surface average of radial current is zero $\langle j_r\rangle=0$)

$$\frac{\partial V_E}{\partial t}+\frac{4q^2}{3\pi^{3/2}}\int dx\;d\theta\;\left(F_0+0.1\;F_2\right)\sin\theta\;x^4\exp\left(-x^2\right)$$

Conclusion on the role of parameters

$$\begin{array}{lll}\varepsilon & \rightarrow & \text{control of mirror force} \\ & & \text{which means the NUMBER of banana}\end{array}$$

$$\nu^* \rightarrow \text{collisionality}$$

$$q \rightarrow \text{controls the Landau resonance}$$

Conclusion about the origin of the magnetic damping of poloidal rotation.

It arises a discontinuity at the boundary in the velocity space between trapped and circulating particles.

The different behavior of trapped versus circulating particles is manifested at an *adiabatic change of the temperature gradient.*

The trapped particles are only very weakly collisional.

NOTE

In *collision.tex*

At the limit of the *plateau* to *banana* regime

$$\nu_* = \varepsilon^{3/2} \frac{v}{qR}$$

END

Therefore the trapped particles reach very fast stationarity in the new conditions of the gradient of T.

The circulating particles react slowly.

This produces a discontinuity at the boundary trapped/circulating.

The discontinuity at the boundary acts like a friction, a damping mechanism.

The smoothing of the discontinuity is realized with a rate

$$\frac{\nu_{ii}}{\varepsilon}$$

The damping of the poloidal rotation is due to the collisional friction between the circulating ions (that have poloidal rotation) and the ions on bananas (that cannot have rotation).

The trapping condition for a particle at the poloidal position θ is

$$|\xi| \equiv \left| \frac{v_{\parallel}}{v} \right| \leq \sqrt{\varepsilon \left(1 + \cos\theta\right)}$$

16.5 The static distribution of bananas

Starting from the equation for the first order distribution function (see above)

$$\begin{aligned} & \frac{\partial \widehat{f}}{\partial \widehat{t}} + \xi x \frac{\partial \widehat{f}}{\partial \theta} - \varepsilon \frac{x\left(1-\xi^2\right)}{2} \sin\theta \frac{\partial \widehat{f}}{\partial z} - \widehat{C}\left[\widehat{f}\right] \\ = & \sin\theta \; x^2 \left(1+\xi^2\right) \left[\widehat{V}_E + \widehat{V}_n + \left(x^2 - \frac{3}{2}\right) \widehat{V}_T\right] \end{aligned}$$

Here

- one neglects the collisions $\widehat{C}\left[\widehat{f}\right] \rightarrow 0$ and
- takes stationarity, $\frac{\partial}{\partial t} = 0$,

obtaining

$$\begin{aligned} & \xi x \frac{\partial \widehat{f}}{\partial \theta} - \varepsilon \frac{x\left(1-\xi^2\right)}{2} \sin\theta \frac{\partial \widehat{f}}{\partial z} \\ = & \sin\theta \; x^2 \left(1+\xi^2\right) \left[\widehat{V}_E + \widehat{V}_n + \left(x^2 - \frac{3}{2}\right) \widehat{V}_T\right] \exp\left(-x^2\right) \end{aligned}$$

The solution is the static distribution of bananas

$$\begin{aligned} \widehat{f} &= -\xi \; x \; \left(1 + \varepsilon \cos\theta\right) \frac{2}{\varepsilon} \left[V_E + V_n + \left(x^2 - \frac{3}{2}\right) V_T\right] \exp\left(-x^2\right) \\ & +C \end{aligned}$$

Here C is a function

$$\begin{aligned} C &\equiv C\left(x, \mu, \epsilon\right) \\ & \quad C \;\; \text{is zero for trapped particles} \end{aligned}$$

that is constant along the lines and can be obtained from the surface average of the equation written for the next order in

$$\frac{\widehat{\nu}}{\omega_{bounce}}$$

(the bounce motion is much faster than collisions).

It is introduced a function of distribution

$$\begin{aligned} F\left(\xi, \theta\right) &= \int dx \; \widehat{f} \; x^2 \\ & \text{integration over } v \text{ after which only remaining} \\ & \text{dependence on } v_{\|}/v \end{aligned}$$

and after replacing the static distribution of the bananas, one has

$$\begin{aligned} F \quad &\sim \quad \xi\left(1+\varepsilon\cos\theta\right)\left(V_E+V_n+0.5V_T\right) \\ &\qquad \text{for bananas} \end{aligned}$$

17 Direction of poloidal flows

There is a file *Direction of flows.tex*.

17.1 First choice of system of versors attached to the slab model

The system of reference is :

$\widehat{\mathbf{e}}_x$ directed along r toward the center of plasma
$\widehat{\mathbf{e}}_y$ directed along θ with increasing θ (from equator to major axis)
$\widehat{\mathbf{n}}$ along $\mathbf{B}$

This gives

$$\widehat{\mathbf{e}}_x \times \widehat{\mathbf{e}}_y = \widehat{\mathbf{n}}$$

Here it is assumed that θ is measured from the equatorial plane and increases up in the semispace of positive Z coordinate (this is the major axis of symmetry), toward the major axis of the torus.

The characteristics of this choice is that the gradients are

$$\boldsymbol{\nabla} n \sim \boldsymbol{\nabla} T \sim \widehat{\mathbf{e}}_x$$

Looking in the same direction as the magnetic field, the versor $\widehat{\mathbf{e}}_x$ is directed toward the magnetic axis (the center of plasma), $\widehat{\mathbf{e}}_y$ is tangent to the circle r and points to the left, $\widehat{\mathbf{n}}$ points as $\mathbf{B}$.

Apparently, the paper of **Kishimoto Poloidal shear flow effect** shows a geometry of rotation compatible with this option.

Figure 1: From the paper Poloidal shear effect, Kim, Kishimoto, Wakatani, Tajima PoP3 (1996) 3689.

In this paper the *ion diamagnetic* flow is to the left. The coordinate angle θ is measured as in our option, from the equatorial plane toward left, up, to the major symmetry axis.

In the paper **Laedke Spatschek** it is made an option: the gradient of the density is along the versor $\widehat{\mathbf{e}}_x$, *i.e.* it is positive when it points toward the magnetic axis, which is compatible with the present choice. They find that

$$\begin{aligned} 1+\frac{\rho_s^2\Omega_i\kappa_n}{u} &= 1+\frac{|\rho_s c_s/L_n|}{-|u|} \\ &= 1-\frac{|v_{dia}|}{|u|}>0 \end{aligned}$$

when u is a propagation in the negative y direction (in our system that would be to the right)

$$u=-|u|\,\widehat{\mathbf{e}}_y$$

The diamagnetic velocity

$$\widehat{\mathbf{n}}\times\boldsymbol{\nabla}p\sim\widehat{\mathbf{e}}_y$$

is to the left, also shown by **Laedke Spatschek**

$$\begin{aligned} v_{dia} &= \kappa_n\rho_s^2\Omega_i>0 \\ \mathbf{v}_{dia} &= |\rho_s c_s/L_n|\,\widehat{\mathbf{e}}_y \end{aligned}$$

They also find the conditions for the existence of the nonlinear vortex solutions

$$\frac{1}{\rho_s^2}+\frac{\kappa_n\Omega_i}{u}>0$$

in the two forms.

Form I:

$$u>0$$

(which means that the plasma rotates *in the same direction as the ION diamagnetic drift* v_{dia}, to the left, in the direction of the versor $\widehat{\mathbf{e}}_y$) or

Form II:

$$u<-v_{dia}$$

(which means that the plasma rotates in the direction *opposite to the ION diamagnetic drift* , which means in the direction of the *electron diamagnetic drift* and faster than the ion diamagnetic velocity).

This is from **Scott habillitation** and is connected with the options that will lead to the drift wave theory.

The lowest order flows, for ions

$$\mathbf{v}_{\perp i}^{(0)} = \frac{-\nabla\phi \times \widehat{\mathbf{n}}}{B_0} + \frac{1}{n\,|e|}\frac{\widehat{\mathbf{n}} \times \nabla p_i}{B_0}$$

and for electrons

$$\mathbf{v}_{\perp e}^{(0)} = \mathbf{v}_{\perp e} = \frac{-\nabla\phi \times \widehat{\mathbf{n}}}{B_0} - \frac{1}{n\,|e|}\frac{\widehat{\mathbf{n}} \times \nabla p_e}{B_0}$$

There is no other component in the perpendicular electron velocity since there is no *electron inertia.*

The *electron diamagnetic* velocity (the minus sign $-$ comes from the charge of the electron) is

$$\mathbf{v}_{dia,e} = -\frac{1}{n_e\,|e|}\frac{\widehat{\mathbf{n}} \times \nabla p_e}{B_0}$$

If the pressure p_e increases toward the center then

$$\nabla p_e \sim \widehat{\mathbf{e}}_x$$

and

$$\widehat{\mathbf{n}} \times \nabla p_e \sim \widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_x = \widehat{\mathbf{e}}_y$$

Then

$$\begin{aligned} \mathbf{v}_{dia,e} &= -\frac{1}{n_e\,|e|}\frac{\widehat{\mathbf{n}} \times \nabla p_e}{B_0} \\ &\sim -\widehat{\mathbf{e}}_y \end{aligned}$$

The diamagnetic electrons are rotating in the poloidal direction in the sense of *decreasing* θ (this can be said: in the right or <u>*clockwise*</u> direction).

The *ion diamagnetic* velocity is

$$\mathbf{v}_{dia,i} = \frac{1}{n\,|e|}\frac{\widehat{\mathbf{n}} \times \nabla p_i}{B_0}$$

Since the ion pressure increases toward the center of plasma, the gradient ∇p_i has the direction

$$\nabla p_i \sim \widehat{\mathbf{e}}_x$$

and

$$\widehat{\mathbf{n}} \times \nabla p_i \sim \widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_x = \widehat{\mathbf{e}}_y$$

Then

$$\begin{aligned}\mathbf{v}_{dia,i} &= \frac{1}{n_e\,|e|}\frac{\widehat{\mathbf{n}}\times\boldsymbol{\nabla}p_i}{B_0}\\ &\sim \widehat{\mathbf{e}}_y\end{aligned}$$

The ions have a diamagnetic velocity on the poloidal direction in the sense of *increasing* the angle θ. This can be said : toward left, or <u>*counter-clockwise*</u> direction).

In conclusion, if we look in the direction of the magnetic field, in a point that is outer equatorial, the ion diamagnetic rotation is to the left and up and the electron diamagnetic rotation is to the right and down.

Let us consider an electric potential ϕ that

1. is negative,
$$\phi < 0$$

2. it is increasing toward zero value toward the edge (starting from a deep negative value in the plasma center). The gradient points toward th exterior of the plasma
$$\boldsymbol{\nabla}\phi \sim -\widehat{\mathbf{e}}_x$$
where $\widehat{\mathbf{e}}_x$ is defined as in the *slab* geometry, with y (poloidal) direction toward the left, the z direction from the current origin toward the far point. The x direction is then negative on the minor radius direction, *i.e.* toward the magnetic axis.

Then we have

$$\begin{aligned}-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}} &\sim -\left(-\widehat{\mathbf{e}}_x\right)\times\widehat{\mathbf{n}}\\ &= \widehat{\mathbf{e}}_x\times\widehat{\mathbf{n}}\\ &= -\widehat{\mathbf{e}}_y \text{ (to the right)}\end{aligned}$$

Then the electric velocity is in the <u>**clockwise**</u> direction, exactly as the *electron diamagnetic* velocity and opposite to the *ion diamagnetic* velocity

$$\begin{aligned}\mathbf{v}_E &\sim \mathbf{v}_{dia,e}\sim -\widehat{\mathbf{e}}_y \text{ (clockwise, as the electrons)}\\ \mathbf{v}_E &\sim -\mathbf{v}_{dia,i}\sim -\widehat{\mathbf{e}}_y \text{ (clockwise, opposite than ions)}\end{aligned}$$

Now we can look at the "first order" velocities for electrons and ions.

For ions

$$\begin{aligned}
\mathbf{v}_{\perp i}^{(0)} &= \frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B_0}+\frac{1}{n\left|e\right|}\frac{\widehat{\mathbf{n}}\times\boldsymbol{\nabla}p_i}{B_0} \\
&\sim (\text{clockwise}) + (\text{counter-clockwise}) \\
&\sim \left(-\left|\mathbf{v}_E\right|+\left|\mathbf{v}_{dia,i}\right|\right)\widehat{\mathbf{e}}_y \\
&\sim \text{small, can have any sign}
\end{aligned}$$

hence the velocities are *subtracted* one from the other. The zeroth order velocity for the ions is small when the electric rotation velocity (in the clockwise direction) is almost equal but slightly larger than the *ion*'s diamagnetic velocity (in the anti-clockwise direction).

For electrons

$$\begin{aligned}
\mathbf{v}_{\perp e}^{(0)} &= \mathbf{v}_{\perp e}=\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B_0}-\frac{1}{n\left|e\right|}\frac{\widehat{\mathbf{n}}\times\boldsymbol{\nabla}p_e}{B_0} \\
&\sim (\text{clockwise}) + (\text{clockwise}) \\
&\sim \left(-\left|\mathbf{v}_E\right|-\left|\mathbf{v}_{dia,e}\right|\right)\widehat{\mathbf{e}}_y \\
&\sim \text{large, clockwise}
\end{aligned}$$

The two velocities are added together.

The diamagnetic cancellation is useful only when we consider full balance equations, and look for the fluxes (of particles or heat).

When we calculate the velocities there is no reason to neglect any of these velocities. All must be retained.

In **Su Yushmanov** the velocity $u_\perp$ is defined such that positive E_r gives positive $u_\perp$ which is rotation in the *ion diamagnetic* direction.

In **Bortolon Duval Scarabosio reversal** of toroidal rotation due to density ramp up, the direction in L mode:

plasma rotates toroidally in the counter direction with respect to the plasma current [like in L-mode] *i.e.* electron diamagnetic drift rotation, negative values.

17.2 Second choice of the system of versors

This is used by **Brunner**.

The system of reference is :

$\widehat{\mathbf{e}}_x$ directed along r toward the *exterior* of plasma

$\widehat{\mathbf{e}}_y$ directed along θ with increasing θ (from vertical axis toward the equator)

$\widehat{\mathbf{n}}$ along $\mathbf{B}$

This gives

$$\widehat{\mathbf{e}}_x \times \widehat{\mathbf{e}}_y = \widehat{\mathbf{n}}$$

Here it is assumed that θ is measured from a vertical axis passing through the magnetic axis (parallel with the major symmetry axis of the torus) and increases when going toward the equator.

The characteristics of this choice is that the gradients are

$$\mathbf{\nabla} n \sim \mathbf{\nabla} T \sim -\widehat{\mathbf{e}}_x$$

Looking in the same direction as the magnetic field, the versor $\widehat{\mathbf{e}}_x$ is directed toward the exterior of plasma, $\widehat{\mathbf{e}}_y$ is tangent to the circle r and points to the right, $\widehat{\mathbf{n}}$ points as $\mathbf{B}$.

The Review Modern Phys. **Horton**, uses this system. The gradient of $n(x)$ and $T(x)$ are in a direction opposite to the versor on the radius. The versor on the radius $\widehat{\mathbf{e}}_x$ points to exterior of plasma. The poloidal versor $\widehat{\mathbf{e}}_y$ points to the right.

In the paper **Tajima Horton Morrison Mima** it is said:

the positive potential plasmas (more ions inside) rotate in the *ion diamagnetic* direction [up from equator]

the negative potential plasmas (more electrons inside) rotate in the *electron diamagnetic* direction [down]

18 System for plasma spin-up

18.1 Hassam: system for poloidal asymmetry of diffusion and/or sources

Model of **Hassam**.

The equations governing the *spontaneous poloidal spin-up*:

Continuity with possible source and radial flux depending on θ, θ-asymmetric,

$$\frac{\partial n}{\partial t} + \mathbf{\nabla}\cdot(n\mathbf{u}_\perp) + \mathbf{B}\cdot\mathbf{\nabla}\left(\frac{nu_\parallel}{B}\right) = S - \frac{1}{r}\frac{\partial}{\partial r}(r\Gamma_r)$$

The momentum equation

$$nm_i\left(\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\boldsymbol{\nabla})\,\mathbf{u}\right)=-T\boldsymbol{\nabla}n+\mathbf{j}\times\mathbf{B}-m_i\mathbf{u}S\ ,\ \ (T \text{ is constant})$$

$$\boldsymbol{\nabla}\cdot\mathbf{j}=0$$

$$-\boldsymbol{\nabla}\phi+\mathbf{u}\times\mathbf{B}=0 \text{ (no resistivity)}$$

Note
The amount of momentum that is due to the source S is

$$-m_i\mathbf{u}S$$

and is negative. The source provides particles that have very low velocity and must be rised to $\mathbf{u}$. This means that the fluid *loses* momentum $m_i\mathbf{u}$ for every new particle of the source. It is collisional.
End

The equations used by **Hassam** for studying the spin-up contain a term (we note here $\mathbf{v}_\perp=\mathbf{u}_\perp$)

$$\boldsymbol{\nabla}\cdot(n\mathbf{v}_\perp)$$

In the lowest approximation, the perpendicular velocity is that given by the radial electric field

$$\mathbf{v}_\perp=\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}$$

This is a *perpendicular* velocity. Let us consider the approximation of the velocity along the *poloidal* direction, which is very close to the perpendicular one

$$\begin{aligned}\mathbf{v}_\perp &\simeq \frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B_0}\frac{B_0}{B}=\left|\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B_0}\right|\widehat{\mathbf{e}}_\theta\frac{B_0}{B}\\ &= u_E\widehat{\mathbf{e}}_\theta\frac{B_0}{B}\end{aligned}$$

where u_E is constant on the magnetic surface. Then in the equation of continuity

$$\begin{aligned}\boldsymbol{\nabla}\cdot(n\mathbf{v}_\perp) &= \boldsymbol{\nabla}\cdot\left(nu_E\widehat{\mathbf{e}}_\theta\frac{B_0}{B}\right)\\ &= n_0\boldsymbol{\nabla}\cdot\left(u_E\widehat{\mathbf{e}}_\theta\frac{B_0}{B}\right)+\frac{1}{r}\frac{\partial n_1}{\partial\theta}u_E\end{aligned}$$

In the last term, the correction to the density n_0 (constant on the surface) is due to the variation of the density on the surface and is of the order ε. Then the difference between B and B_0 being of order ε as well, has been neglected for that term. The first term is made more explicit writting the dependence of B of the angle θ:

$$B = \frac{B_0}{1+\varepsilon\cos\theta}$$

$$\boldsymbol{\nabla}\cdot\left(u_E\widehat{\mathbf{e}}_\theta\frac{B_0}{B}\right) = \boldsymbol{\nabla}\cdot\left[u_E\widehat{\mathbf{e}}_\theta\left(1+\varepsilon\cos\theta\right)\right]$$

In the orthogonal coordinates (r,θ,φ) we have the element of distance:

$$dl^2 = (dr)^2 + r^2(d\theta)^2 + (R_0 + r\cos\theta)^2 d\varphi^2$$

which gives the Lamé metric coefficients

$$\begin{aligned} h_1 &= 1 \\ h_2 &= r \\ h_3 &= R_0 + r\cos\theta \end{aligned}$$

Then the divergence of a vector $\mathbf{a}$ is written

$$\boldsymbol{\nabla}\cdot\mathbf{a} = \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial r}(h_2h_3a_1) + \frac{\partial}{\partial\theta}(h_1h_3a_2) + \frac{\partial}{\partial\varphi}(h_1h_2a_3)\right)$$

which gives

$$\begin{aligned} \boldsymbol{\nabla}\cdot\left[u_E\widehat{\mathbf{e}}_\theta\left(1+\varepsilon\cos\theta\right)\right] &= u_E\frac{1}{r\left(R_0+r\cos\theta\right)}\frac{\partial}{\partial\theta}\left(\left(R_0+r\cos\theta\right)\left(1+\varepsilon\cos\theta\right)\right) \\ &= u_E\frac{1}{r\left(R_0+r\cos\theta\right)}R_0\frac{\partial}{\partial\theta}\left[\left(1+\varepsilon\cos\theta\right)^2\right] \\ &= u_E\varepsilon\frac{(-2\sin\theta)}{r} \end{aligned}$$

With this, we get

$$\boldsymbol{\nabla}\cdot(n\mathbf{v}_\perp) = n_0\; u_E\varepsilon\frac{(-2\sin\theta)}{r} + \frac{\partial n_1}{r\partial\theta}u_E$$

and is of order ε.

In **PRL66(1991)309 Hassam** uses the equations

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla}\cdot(n\mathbf{v}_\perp) + \mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{nv_\parallel}{B}\right) = 0$$

(no source, no diffusion, i.e. $S - \frac{1}{r}\frac{\partial}{\partial r}(r\Gamma_r)$ as present in the treatment of the spontaneous poloidal rotation).

The momentum conservation equation

$$nM\frac{\partial(\mathbf{B}\cdot\mathbf{v})}{\partial t} + nM\mathbf{B}\mathbf{v} : \nabla\mathbf{v} = -T\mathbf{B}\cdot\nabla n - \mathbf{B}\nabla : \mathbf{\Pi}$$

We **note** that in this equation
- the term $\mathbf{j}\times\mathbf{B}$ has been suppressed by the projection (scalar multiplication with $\mathbf{B}$) and
- there is explicit presence of the anisotropy of the pressure tensor.

The equation of current conservation $\nabla\cdot\mathbf{j} = 0$, i.e.

$$\mathbf{B}\cdot\nabla\left(\frac{j_\parallel}{B}\right) = -\nabla_\perp\cdot\mathbf{j}_\perp$$

where

$$\mathbf{v}_\perp = \frac{-\nabla\phi\times\mathbf{B}}{B^2} + \frac{\mathbf{R}_\perp\times\mathbf{B}}{B^2}$$

$$\mathbf{j}_\perp = \widehat{\mathbf{n}}\times\left(nM\frac{d\mathbf{v}}{dt} + T\nabla n + \nabla\cdot\mathbf{\Pi}\right)$$

Here $\mathbf{R}_\perp$ is a generalized electron-ion momentum transfer term. (`is a` B_0 `missing at the denominator of this formula?`). This term generates *drift* motion of particles by the interaction between the *force* $\mathbf{R}$ and the magnetic field (the usual $\frac{\mathbf{F}\times\mathbf{B}}{B^2}$).

The parallel viscous stress $\mathbf{\Pi}$ is the origin of *magnetic pumping*.

Note that **Su Yushmanov Dong Horton** express the neoclassical *force* in the equation for the perpendicular momentum

$$nm\left(1+2\widehat{q}^2\right)\frac{\partial u_\perp}{\partial t} = -F^{nc} - F^R - F^R_\sim - F^a_\perp$$

as

$$F^{nc} = -\frac{B_T}{B_\theta}\frac{1}{B_0}\left\langle B^2(\mathbf{B}\cdot\nabla)\frac{P_\parallel - P_\perp}{2B^2}\right\rangle$$

Therefore the effect of magnetic pumping on the poloidal rotation is expressed as a parallel divergence of the anisotropy of the pressure.

Remember the factor $(1+2\widehat{q}^2)$ which enhances the *inertia* of plasma at poloidal acceleration/decay is calculated by **Novakovski**.

End.

Note in **Beer thesis** the Boltzmann equation is for

$$\frac{\partial\left(FB\right)}{\partial t}...$$

where the presence of B is explained by the Jacobian for the change of variables.

End.

There are new velocities that are defined on the basis of the poloidal and toroidal velocities:

$$\begin{array}{l}\left(V_{pol},V_{tor}\right) \text{ and}\\ \left(v_\theta,v_\varphi\right)\end{array}$$

The definitions of the poloidal and toroidal velocities are

$$\begin{aligned}
V_{pol} &\equiv \left\langle v_\theta\left(1+\frac{r}{R}\cos\theta\right)\right\rangle \\
&\equiv \text{function of only the surface, } r \\
V_{tor} &\equiv \left\langle v_\varphi\left(1+\frac{r}{R}\cos\theta\right)\right\rangle \\
&\equiv \text{function of only the surface, } r
\end{aligned}$$

and the surface averaging operator is

$$\langle f\rangle \equiv \oint \frac{d\theta}{2\pi} f\left(1+\frac{r}{R}\cos\theta\right)$$

The velocities are

$$v_\theta = \frac{V_{pol}\left(r\right)}{1+\frac{r}{R}\cos\theta}$$

$$v_\varphi \simeq V_{tor} - 2qV_{pol}\cos\theta + \varepsilon\left[V_{tor}\cos\theta + 2qV_{pol}\left(1+\frac{1}{4}\cos 2\theta\right)\right]$$

NOTE

The formula 1.22 of **Rozhansky Tendler Rev Plasma** is

$$\begin{aligned}
u_{i\varphi} &= \overline{u}_\varphi\left(1-\varepsilon\cos\theta\right) - 2qV_\theta\cos\theta + 2\varepsilon qV_\theta \\
&\qquad -1.5\varepsilon^2\overline{u}_\varphi
\end{aligned}$$

END.

The flux surface average of the toroidal angular momentum

$$\langle nRv_{\varphi}\rangle \simeq nR_0 V_{tor}$$

As is typical in fluid studies on vorticity, one can define the *circulation*.

This quantity is the integral of the velocity along a closed loop, in particular a streamfunction

$$\text{circulation} = \oint \mathbf{dl}\cdot\mathbf{v} = \oint dl_{\parallel}\ v_{\parallel}$$

where $\parallel$ refers to the tangent to the loop curve.

The average (calculated as $\langle f\rangle \equiv \oint \frac{d\theta}{2\pi} f\left(1+\frac{r}{R}\cos\theta\right)$) of the quantity defined by $v_{\parallel}$ (to the magnetic line) multiplied by the factor $\frac{1}{h}$ is an integral

$$\begin{aligned}
\left\langle v_{\parallel}\frac{B}{B_0}\right\rangle &= \left\langle v_{\parallel}\frac{1}{h}\right\rangle \\
&= \oint \frac{d\theta}{2\pi}\ v_{\parallel}\frac{1}{h}\ h \\
&= \oint \frac{d\theta}{2\pi}\ v_{\parallel} \\
&= \frac{1}{r}\times\left(\text{circulation of } v_{\parallel}\right)
\end{aligned}$$

circulation

$$\left\langle \frac{v_{\parallel}B}{B_0}\right\rangle \simeq V_{tor} + \Theta\left(1+2q^2\right)V_{pol}$$

where

$$\begin{aligned}
\Theta(r) &= \frac{r}{qR} = \frac{\varepsilon}{q} \\
&= \frac{B_{\theta}}{B_T}
\end{aligned}$$

or

$$\begin{aligned}
\left\langle \frac{v_{\parallel}}{h}\right\rangle &\simeq V_{tor}(r) \\
&\quad + \frac{B_{\theta}}{B_{tor}}\left(1+2q^2\right)V_{pol}(r)
\end{aligned}$$

The flux of transport of various quantities transversal on the magnetic surfaces is given by

$$nv_r = \frac{R_{\perp}}{eB}$$

Note this is essentially the radial velocity which is calculated here as *drift velocity* produced by the force $\mathbf{R}_\perp$ in $\frac{\mathbf{R}_\perp \times \mathbf{B}}{B^2}$. The factors ne were already in $\mathbf{R}_\perp$.

End.

NOTE for comparison, the radial velocity v_r in *hose-like* non-resistive ($\eta = 0$) is due to J_r and this in turn results from balancing $J_r \times B_\theta$ with the toroidal flow $(\mathbf{v} \cdot \boldsymbol{\nabla})\, v_\varphi$ with $\mathbf{v} \cdot \boldsymbol{\nabla} = \mathbf{v}_\theta \partial / (r\partial\theta)$.

[**super-Note** remember this static equilibrium can possibly be invoked to saturate - in the ideal $\eta = 0$ case) the polarization generated by the continuous input of fast ions NBI]

END

And, when $\eta \neq 0$ as in Stringer (Pfirsch Schluter)

$$v_r B_\theta = \eta J_z$$

with the explanation: *the simple existence of the diamagnetic flow and of toroidal geometry leads to the toroidal (harmonic) Pfirsch Schluter current; and this one, from Ohm's law is due to the* $\mathbf{v} \times \mathbf{B}$ *with an obligation to have a radial velocity.*

18.1.1 Simplified form of this system of equations

In **IAEA-CN-53 Hassam** uses a system of equations for three quantities:

1. averaged density

$$\langle n \rangle$$

2. the **toroidal angular momentum**

$$\langle n R v_\varphi \rangle$$

the **circulation**

$$\left\langle v_\parallel B \right\rangle$$

In the absence of sources the density and the toroidal angular momentum are **conserved**. The *circulation* is only **convected**.

They obtain the following system, the first two representing the two conservation laws:

$$\frac{\partial n}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r n \overline{v}_r\right) = 0$$

$$\frac{\partial}{\partial t}\left(nV_{\varphi}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left[rn\left(V_{\varphi}\overline{v}_r-qV_{\theta}\widetilde{v}_r\right)\right]=0$$

$$\frac{\partial}{\partial t}\left[V_{\varphi}+\Theta\left(1+2q^2\right)V_{\theta}\right]+\overline{v}_r\frac{\partial V_{\varphi}}{\partial r}-\widetilde{v}_r\frac{\partial}{\partial r}\left(qV_{\theta}\right)+\text{magnetic pumping}=0$$

where

$$\langle n\rangle = n\left(r\right)$$

$$nV_{\varphi}\left(r\right)\equiv\frac{1}{R_0}\left\langle nv_{\varphi}R\right\rangle$$

$$V_{\theta}\left(r\right)\equiv\frac{1}{R_0}\left\langle v_{\theta}R\right\rangle$$

with the **magnetic field**

$$\begin{aligned}
\mathbf{B} &\equiv \left[0,\Theta\left(r\right),1\right]\frac{B_0R_0}{R}\\
&= \left[0,\ \frac{r}{qR_0}\frac{B_0R_0}{R},\ \frac{B_0R_0}{R}\right]\\
&= \left[0,\frac{B_{\theta}}{B_T}B_0,\ \frac{B_0}{h}\right]
\end{aligned}$$

$$R=R_0+r\cos\theta$$

$$\begin{aligned}
1+\varepsilon\cos\theta &= \frac{R}{R_0}\\
&\equiv h
\end{aligned}$$

$$\begin{aligned}
\Theta\left(r\right) &= \frac{r}{qR_0}\\
&= \frac{B_{\theta}}{B_T}\ll 1
\end{aligned}$$

The *surface averaging* operation is

$$\begin{aligned}
\langle f\rangle &\equiv \oint\frac{d\theta}{2\pi}\left(1+\varepsilon\cos\theta\right)\ f\\
&= \oint\frac{d\theta}{2\pi}\ h\ f
\end{aligned}$$

which leads to

$$nV_{\varphi}\left(r\right)\equiv\frac{1}{R_0}\left\langle nv_{\varphi}R\right\rangle=\left\langle nv_{\varphi}\ h\right\rangle=\oint\frac{d\theta}{2\pi}h^2\ nv_{\varphi}$$

and

$$V_{\theta}\left(r\right) \equiv \frac{1}{R_0}\left\langle v_{\theta}R\right\rangle = \left\langle v_{\theta}\ h\right\rangle = \oint \frac{d\theta}{2\pi}h^2\ v_{\theta}$$

These should be compared with

$$\left\langle f\right\rangle = \frac{\oint \frac{d\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}f}{\oint \frac{d\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}}$$

$$\begin{aligned}
\frac{d\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta} &= \frac{1}{B}\frac{d\theta}{\nabla_{\|}\theta} = \frac{1}{B}qRd\theta = \frac{1}{B}\frac{rB_T}{RB_{\theta}}Rd\theta \\
&= rd\theta\ \frac{B_T}{B}\frac{1}{B_{\theta}} \approx \frac{rd\theta}{B_{\theta}}
\end{aligned}$$

The quantities $\overline{v}_r$ and $\widetilde{v}_r$ represent **radial diffusion velocities** arising from *electron-ion momentum transfer*. The force associated to this momentum transfer is

$$\mathbf{R}_{\perp} \text{ with the direction of } \mathbf{B}\times\boldsymbol{\nabla}n$$

Then the radial diffusion flux is

$$nv_r = \frac{R_{\perp}}{eB}$$

from which the two definitions are obtained:

$$\overline{v}_r \equiv \left\langle v_r\right\rangle$$

$$\widetilde{v}_r \equiv \left\langle 2\cos\theta\ v_r\right\rangle$$

The resulting **equation for the plasma poloidal velocity** is

$$\Theta\left(1+2q^2\right)\left(\frac{\partial V_{\theta}}{\partial t}+\gamma_{MP}V_{\theta}\right)+qV_{\theta}\frac{1}{nr}\frac{\partial}{\partial r}\left(nr\widetilde{v}_r\right)=0$$

18.2 hose-like: System of equation for spontaneous poloidal rotation and shock formation (Rosenbluth Lee Hazeltine 2)

Important NOTE

This calculation is very general: it is simply *assumed* that there is a poloidal rotation and the variations on the magnetic surface of the poloidal velocity, density, toroidal velocity, radial current, parallel current, are calculated (first in the absence of the resistivity $\eta = 0$ then with small resistivity $\eta \neq 0$). The variations on surfaces are determined by the toroidal geometry.

The Pfirsch Schluter flow and the Stringer mechanism exist within this treatment and they can be obtianed, in particular, by choosing the diamagnetic flow as poloidal rotation.

END

This part is also in **Stringer**.

This part is also in **equilibrium flows notes.tex.**

The paper **Resistive plasma rotation shock formation Rosenbluth Lee Hazeltine 1971**. Hose.

The physical picture: in zeroth order in the dissipative mechanism (here the resistivity η) the quantities that are θ-averaged over the magnetic surface

$$\begin{aligned}
&\overline{\rho} \text{ density} \\
&\overline{u} \text{ poloidal rotation speed} \\
&\overline{v} \text{ toroidal velocity}
\end{aligned}$$

can be prescribed independently from surface to surface, by arbitrary functions.

The zeroth-order (in η) equations uniquely determine the *azimuthally-varying* parts of the full functions

$$\begin{aligned}
&\overline{\rho} + \delta\rho\,(\theta) \\
&\overline{u} + \delta u\,(\theta) \\
&\overline{v} + \delta v\,(\theta)
\end{aligned}$$

The poloidal variation occurs due to the toroidal geometry, reflected in the expressions of the operators of the MHD equations.

Note the variation of the poloidal rotation is also discussed in **Hassam Kulsrud** as bycicle effect. **End.**

In this *zero order* different possible steady states are identified. [**Ware** finds a large class of poloidal rotation equilibria]

When there is non-zero *resistivity* η it exists *interaction*. The effect is a slow

$$\tau \sim \frac{1}{\eta}$$

transition between the steady states found at the zeroth-order. The time-dependent equations resulted from the inclusion of the small η show that the rotations with small poloidal speeds have the tendency to accelerate.

The MHD equations + small resistivity

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla}) \mathbf{v} = -\frac{c_s^2}{\rho} \boldsymbol{\nabla} \rho + \frac{1}{\rho} \mathbf{J} \times \mathbf{B} \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0 \tag{2}$$

$$\boldsymbol{\nabla} \cdot \mathbf{J} = 0 \tag{3}$$

$$-\boldsymbol{\nabla} \phi + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J} \tag{4}$$

Note the absence of anisotropy $\left(\Pi_{\|},\ \Pi_{\perp}\right)$ of the pressure tensor in $\|$ relative to $\perp$ directions, that are usually invoked to represent the magnetic pumping damping of poloidal rotation (see **Hassam**).
End.

Note the absence of the temperature variation (in the gad-pressure term).
End.

Note the absence of the Ampere's law

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}$$

from the system of equations. They assume that the fields are produced by external coils and are given and with no change.
There is no magnetic fluctuations induced by the waves.
End.

The magnetic field

$$\begin{aligned} \mathbf{B} &= \left(0, \frac{\varepsilon}{q} \frac{B_0}{h}, \frac{B_0}{h}\right) \\ h &\equiv 1 + \varepsilon \cos \theta \end{aligned}$$

We note that the toroidal component B_0/h is used in the definition of the poloidal component. It is just projected with the Russian factor

$$
\begin{aligned}
B_\theta &= B_\varphi \times \Theta \\
&= B_\varphi \times \left(\frac{B_\theta}{B_\varphi}\right) = B_\varphi \times \frac{RB_\theta}{rB_\varphi}\frac{r}{R} \\
&= B_\varphi \times \frac{\varepsilon}{q} \quad \text{where} \quad \Theta = \frac{\varepsilon}{q} \ll 1
\end{aligned}
$$

Note that looks like **Novakovskii**. **End.**

Axial symmetry

$$\frac{\partial}{\partial z} = 0$$

The zeroth-order equations are obtained

$$\text{zero-th order} \begin{cases} \eta \simeq 0 \\ \frac{\partial}{\partial t} \to 0 \end{cases}$$

NOTE. The fact that the zero-order is obtained taking the *time-variation* as zero means that this result cannot be applied to the fast time increase of the poloidal flow at the onset of the cells of convection in **Inverse RH** problem.

Also there is no decay of an initial poloidal rotation, at relaxation, like in **Taguchi**, etc.

No fast variation of kinetic f as decay **Novakovski** of the poloidal rotation by TTMP.

End.

It results that the average radial velocity is zero and there is only θ-dependent radial velocity

$$v_r = 0$$

and the Ohm's law (4) gives

$$\frac{\partial \phi}{\partial \theta} = 0$$

In the zeroth order the potential does not vary on the magnetic surface, there is no poloidal electric field. We draw conclusion that **the variation of the electrostatic potential on a magnetic surface is connected with a dissipative mechanism: diffusion, resistivity.**

This is explained in **Stringer, Rosenbluth**, etc.

The information that $v_r = 0$ can now be used to extend our knowledge on the poloidal velocity, via the *continuity equation.*

This is the very important assumption, usually taken as *zero radial current*

$$j_r = 0$$

It also results from the continuity equation

$$\nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial}{\partial \theta} (h \rho v_\theta) = 0$$

(**Note** that this comes from, see **Morse Feshbach I pages 25-35** and **Geometry.tex**

$$\nabla \cdot \mathbf{a} = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial r} (h_2 h_3 a_1) + \frac{\partial}{\partial \theta} (h_1 h_3 a_2) + \frac{\partial}{\partial \varphi} (h_1 h_2 a_3) \right)$$

with

$$\begin{aligned} h_r &= 1 \\ h_\theta &= r \\ h_\varphi &= R + r \cos \theta \end{aligned}$$

).

Recall:

$$\begin{aligned} \rho &\equiv \text{mass density} \\ &= n m_i \end{aligned}$$

The consequence of this equation is

$$\begin{aligned} h \rho v_\theta &= L(r) \\ &= \text{function of ONLY the surface} \end{aligned}$$

We multiply the equation of conservation of momentum $\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{c_s^2}{\rho} \nabla \rho + \frac{1}{\rho} \mathbf{J} \times \mathbf{B}$ (1) with $h\mathbf{B}$, at stationarity $\partial / \partial t = 0$,

$$h\mathbf{B} \cdot [(\mathbf{v} \cdot \nabla) \mathbf{v}] = -\frac{c_s^2}{\rho} h\mathbf{B} \cdot \nabla \rho + \frac{1}{\rho} h\mathbf{B} \cdot (\mathbf{J} \times \mathbf{B})$$

The last term is identically zero.

The first term in RHS is the gradient of pressure for constant temperature

$$\mathbf{B}\cdot\left(-\frac{1}{\rho}\boldsymbol{\nabla}p\right)=-\frac{c_s^2}{\rho}h\mathbf{B}\cdot\boldsymbol{\nabla}\rho$$

(hence there is variation of the density ρ along the magnetic lines).

The resulting equation will become a *Bernoulli-like law*

$$h\mathbf{B}\cdot\left[(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}\right]=-\frac{c_s^2}{\rho}h\mathbf{B}\cdot\boldsymbol{\nabla}\rho$$

It is a balance of forces along the magnetic field and must reflect the variation on the surface $f(\theta)$ of the physical parameters like density n (or ρ).

The static advection of the plasma along the field is balanced by the variation of the density (pressure) along the field.

In the paper **poloidal rotation growth Rosenbluth Hazeltine Lee** it is written

$$\mathbf{B}\cdot\left[(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}\right]=\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{v^2}{2}\right)+(\boldsymbol{\nabla}\times\mathbf{v})\cdot(\mathbf{v}\times\mathbf{B})$$

Note that the last term is for small resistivity

$$-\boldsymbol{\omega}\cdot\mathbf{E}$$

or

$$\eta\boldsymbol{\omega}\cdot\mathbf{J}$$

To calculate $(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}$ we use the formulas from **geometry.tex** notes, or from **Morse and Feshbach**.

At this stage there will be no approximation yet.

The velocity is introduced with the non-zero components which correspond to the lowest order, derived from $\eta=0$ condition

$$\begin{aligned}(v_r,v_\theta,v_\varphi)&=(0,v_\theta,v_\varphi)\\ \text{for }\ \eta&=0\\ &\quad\text{and lowest order in }\varepsilon\end{aligned}$$

Digression.

Consider two vectors

$$\mathbf{A} \text{ and } \mathbf{B}$$

$$\begin{aligned}
& \text{component of } (\mathbf{B}\cdot\boldsymbol{\nabla})\,\mathbf{A} \text{ along } \widehat{\mathbf{e}}_1 \\
= & \left(B_1 \frac{1}{h_1}\frac{\partial}{\partial\xi_1} + B_2 \frac{1}{h_2}\frac{\partial}{\partial\xi_2} + B_3 \frac{1}{h_3}\frac{\partial}{\partial\xi_3}\right) A_1 \\
& + \frac{A_2}{h_1 h_2}\left(B_1 \frac{\partial h_1}{\partial\xi_2} - B_2 \frac{\partial h_2}{\partial\xi_1}\right) + \frac{A_3}{h_1 h_3}\left(B_1 \frac{\partial h_1}{\partial\xi_3} - B_3 \frac{\partial h_3}{\partial\xi_1}\right)
\end{aligned}$$

$$\begin{aligned}
& \text{component of } (\mathbf{B}\cdot\boldsymbol{\nabla})\,\mathbf{A} \text{ along } \widehat{\mathbf{e}}_2 \\
= & \left(B_1 \frac{1}{h_1}\frac{\partial}{\partial\xi_1} + B_2 \frac{1}{h_2}\frac{\partial}{\partial\xi_2} + B_3 \frac{1}{h_3}\frac{\partial}{\partial\xi_3}\right) A_2 \\
& + \frac{A_3}{h_2 h_3}\left(B_2 \frac{\partial h_2}{\partial\xi_3} - B_3 \frac{\partial h_3}{\partial\xi_2}\right) + \frac{A_1}{h_2 h_1}\left(B_2 \frac{\partial h_2}{\partial\xi_1} - B_1 \frac{\partial h_1}{\partial\xi_3}\right)
\end{aligned}$$

$$\begin{aligned}
& \text{component of } (\mathbf{B}\cdot\boldsymbol{\nabla})\,\mathbf{A} \text{ along } \widehat{\mathbf{e}}_3 \\
= & \left(B_1 \frac{1}{h_1}\frac{\partial}{\partial\xi_1} + B_2 \frac{1}{h_2}\frac{\partial}{\partial\xi_2} + B_3 \frac{1}{h_3}\frac{\partial}{\partial\xi_3}\right) A_3 \\
& + \frac{A_1}{h_3 h_1}\left(B_3 \frac{\partial h_3}{\partial\xi_1} - B_1 \frac{\partial h_1}{\partial\xi_3}\right) + \frac{A_2}{h_3 h_2}\left(B_3 \frac{\partial h_3}{\partial\xi_2} - B_2 \frac{\partial h_2}{\partial\xi_3}\right)
\end{aligned}$$

Then for our operators the identifications are

$$\begin{aligned}
\mathbf{B} & \equiv \mathbf{v} \\
\mathbf{A} & \equiv \mathbf{v}
\end{aligned}$$

and from the general expression we obtain the components

$$\begin{aligned}
& \text{component of } (\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v} \text{ along } \widehat{\mathbf{e}}_r \\
= & v_r \frac{\partial v_r}{\partial r} + v_\theta \frac{\partial v_r}{r\partial\theta} - \frac{v_\theta^2}{r} - \frac{v_\varphi^2}{R}\cos\theta
\end{aligned}$$

$$\begin{aligned}
& \text{component of } (\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v} \text{ along } \widehat{\mathbf{e}}_\theta \\
= & \left(v_\theta \frac{1}{h_\theta}\frac{\partial}{\partial\theta} + v_\varphi \frac{1}{h_\varphi}\frac{\partial}{\partial\varphi}\right) v_\theta \\
& + \frac{v_\varphi}{h_\theta h_\varphi}\left(v_\theta \frac{\partial h_\theta}{\partial\varphi} - v_\varphi \frac{\partial h_\varphi}{\partial\theta}\right)
\end{aligned}$$

We assume that there is no variation along φ for the functions involved here. We insert

$$\frac{\partial h_\varphi}{\partial \theta} = \frac{\partial}{\partial \theta}(R + r\cos\theta) = -r\sin\theta$$

or

$$-\sin\theta = \frac{1}{\varepsilon}\frac{\partial h}{\partial \theta}$$

The components.
On r,

$$\begin{aligned} & \text{component of } (\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v} \text{ along } \widehat{\mathbf{e}}_r \\ = \; & v_r\frac{\partial v_r}{\partial r} + v_\theta\frac{\partial v_r}{r\partial\theta} - \frac{v_\theta^2}{r} - \frac{v_\varphi^2}{R}\cos\theta \end{aligned}$$

On θ,

$$\begin{aligned} & \text{component of } (\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v} \text{ along } \widehat{\mathbf{e}}_\theta \\ = \; & v_\theta\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} - \frac{v_\varphi^2}{r\,(R + r\cos\theta)}(-r\sin\theta) \end{aligned}$$

Further, on φ,

$$\begin{aligned} & \text{component of } (\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v} \text{ along } \widehat{\mathbf{e}}_\varphi \\ = \; & \left(v_\theta\frac{1}{h_\theta}\frac{\partial}{\partial\theta} + v_\varphi\frac{1}{h_\varphi}\frac{\partial}{\partial\varphi}\right)v_\varphi \\ & + \frac{v_\theta}{h_\varphi h_\theta}\left(v_\varphi\frac{\partial h_\varphi}{\partial\theta} - v_\theta\frac{\partial h_\theta}{\partial\varphi}\right) \\ = \; & v_\theta\frac{1}{r}\frac{\partial v_\varphi}{\partial\theta} + \frac{v_\theta v_\varphi}{r(R + r\cos\theta)}(-r\sin\theta) \end{aligned}$$

At this moment we have

$$\begin{aligned} & (\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v} \\ = \; & \widehat{\mathbf{e}}_r\left[v_r\frac{\partial v_r}{\partial r} + v_\theta\frac{\partial v_r}{r\partial\theta} - \frac{v_\theta^2}{r} - \frac{v_\varphi^2}{R}\cos\theta\right] \\ & + \widehat{\mathbf{e}}_\theta\left[v_\theta\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} - \frac{v_\varphi^2}{r\,(R + r\cos\theta)}(-r\sin\theta)\right] \\ & + \widehat{\mathbf{e}}_\varphi\left[v_\theta\frac{1}{r}\frac{\partial v_\varphi}{\partial\theta} + \frac{v_\theta v_\varphi}{r(R + r\cos\theta)}(-r\sin\theta)\right] \end{aligned}$$

We will be interested in the *parallel* projection of the static inertial term,

$$\mathbf{B}\cdot\left[\left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{v}\right]$$

and we must calculate every term.

Returning to the equation of momentum conservation multiplied by $h\mathbf{B}$, which is a static inertia parallel balance.

We have

$$h\mathbf{B}\cdot\left[\left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{v}\right]=-\frac{c_s^2}{\rho}h\mathbf{B}\cdot\boldsymbol{\nabla}\rho$$

and replacing the detailed form of $\left[\left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{v}\right]$, where only the θ and φ terms contribute, we have

$$B_\theta h\left[v_\theta\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}-\frac{v_\varphi^2}{Rh}\left(-\sin\theta\right)\right]+B_\varphi h\left[v_\theta\frac{1}{r}\frac{\partial v_\varphi}{\partial\theta}+\frac{v_\theta v_\varphi}{Rh}\left(-\sin\theta\right)\right]=-\frac{c_s^2}{\rho}hB_\theta\frac{\partial\rho}{r\partial\theta}$$

NOTE that it is *here* that the diamagnetic velocity cannot be included since there was a projection along the magnetic field line, - by multiplying with $\mathbf{B}$. Then this equation is a balance of forces which just describes the variation of parameters like velocity and density on the magnetic surface. **END.**

The radial component of the Ohm's law

$$-\frac{\partial\phi}{\partial r}+\left.\begin{pmatrix}\widehat{\mathbf{e}}_r & \widehat{\mathbf{e}}_\theta & \widehat{\mathbf{e}}_\varphi\\ v_r & v_\theta & v_\varphi\\ 0 & B_\theta & B_\varphi\end{pmatrix}\right|_r=0$$

$$-\frac{\partial\phi}{\partial r}+v_\theta B_\varphi-v_\varphi B_\theta=0$$

NOTE that it is the classical formula to determine the radial electric field

$$E_r=B\frac{1}{n}\frac{1}{m\Omega_c}\frac{dp}{dr}+\left(\mathbf{v}\times\mathbf{B}\right)_\theta$$

and we note that the diamagnetic velocity is neglected.

END

We can eliminate v_φ by expressing it through v_θ and the radial derivative of the potential

$$v_\varphi=\frac{1}{B_\theta}\left(-\frac{\partial\phi}{\partial r}\right)+\frac{B_\varphi}{B_\theta}v_\theta$$

we also have

$$
\begin{aligned}
\frac{B_\varphi}{B_\theta} &= \frac{q}{\varepsilon} \\
&= \frac{1}{\Theta} \gg 1
\end{aligned}
$$

From the first square bracket we take separately the second term. We have

$$
\begin{aligned}
& B_\theta h \left[-\frac{v_\varphi^2}{Rh} (-\sin\theta) \right] \\
= \; & \frac{\varepsilon}{q} \frac{B_0}{h} \left(-\frac{1}{R} \right) (-\sin\theta) \left[\frac{1}{B_\theta} \left(-\frac{\partial \phi}{\partial r} \right) + \frac{B_\varphi}{B_\theta} v_\theta \right]^2 \\
= \; & \frac{\varepsilon}{q} \frac{B_0}{h} \left(-\frac{1}{R} \right) (-\sin\theta) \left[\frac{1}{B_\theta^2} \left(\frac{\partial \phi}{\partial r} \right)^2 + \frac{q^2}{\varepsilon^2} v_\theta^2 - 2 \frac{1}{B_\theta} \left(\frac{\partial \phi}{\partial r} \right) \frac{q}{\varepsilon} v_\theta \right]
\end{aligned}
$$

The *first* term now becomes

$$
\begin{aligned}
& B_\theta h \left[v_\theta \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} - \frac{v_\varphi^2}{Rh} (-\sin\theta) \right] \\
= \; & B_\theta h \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\theta^2}{2} \right) \\
& + \frac{\varepsilon}{q} \frac{B_0}{h} \left(-\frac{1}{R} \right) (-\sin\theta) \frac{1}{B_\theta^2} \left(\frac{\partial \phi}{\partial r} \right)^2 \\
& + \frac{\varepsilon}{q} \frac{B_0}{h} \left(-\frac{1}{R} \right) (-\sin\theta) \frac{q^2}{\varepsilon^2} v_\theta^2 \\
& + \frac{\varepsilon}{q} \frac{B_0}{h} \left(-\frac{1}{R} \right) (-\sin\theta) \left[-2 \frac{1}{B_\theta} \left(\frac{\partial \phi}{\partial r} \right) \frac{q}{\varepsilon} v_\theta \right]
\end{aligned}
$$

Few re-formulations for this *first* term,

$$
\begin{aligned}
& B_\theta h \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\theta^2}{2} \right) + \frac{qh}{B_0 \varepsilon} \left(-\frac{1}{R} \right) \frac{1}{\varepsilon} \frac{\partial h}{\partial \theta} \left(\frac{\partial \phi}{\partial r} \right)^2 \\
& + \frac{B_0}{h} \left(-\frac{1}{R} \right) \frac{1}{\varepsilon} \frac{\partial h}{\partial \theta} \frac{q}{\varepsilon} v_\theta^2 + \left(-\frac{1}{R} \right) \frac{1}{\varepsilon} \frac{\partial h}{\partial \theta} \frac{q}{\varepsilon} v_\theta \left[-2 \left(\frac{\partial \phi}{\partial r} \right) \right]
\end{aligned}
$$

and taking into account that

$$
\left(-\frac{1}{R} \right) \frac{1}{\varepsilon} = -\frac{1}{r}
$$

$$
\begin{aligned}
\text{first term} &= B_\theta h \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\theta^2}{2}\right) - \frac{h}{B_0}\frac{q}{\varepsilon}\frac{1}{r}\frac{\partial h}{\partial\theta}\left(\frac{\partial\phi}{\partial r}\right)^2 \\
&\quad -\frac{B_0}{h}\frac{q}{\varepsilon}\frac{1}{r}\frac{\partial h}{\partial\theta}v_\theta^2 + 2\frac{q}{\varepsilon}v_\theta\frac{1}{r}\frac{\partial h}{\partial\theta}\left(\frac{\partial\phi}{\partial r}\right)
\end{aligned}
$$

NOTE

We draw attention on the occurence of the term $\frac{\partial}{\partial\theta}\left(\frac{v_\theta^2}{2}\right)$. This will be involved in the hose-like "Bernoulli law" applied to the flow in the poloidal direction.

It is the result of the "inertial" static part $[(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}]$ in the momentum equation.

This is not similar with the term u^2 from the **transient amplification Alfven Lau**.

That factor, $1+u^2$ is a spectral (Fourier) representation of the Laplacian Δ that acts on ξ_{1x}, the radial component of the displacement due to Alfven waves.

END

The next (*second*) term where we will substitute the expression of v_φ is

$$
\begin{aligned}
& B_\varphi h\left[v_\theta\frac{1}{r}\frac{\partial v_\varphi}{\partial\theta} + \frac{v_\theta v_\varphi}{Rh}(-\sin\theta)\right] \\
=\ & \frac{B_0}{h}hv_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{B_\theta}\left(-\frac{\partial\phi}{\partial r}\right)+\frac{q}{\varepsilon}v_\theta\right] \\
& +\frac{B_0}{h}h\frac{v_\theta}{Rh}(-\sin\theta)\left[\frac{1}{B_\theta}\left(-\frac{\partial\phi}{\partial r}\right)+\frac{q}{\varepsilon}v_\theta\right]
\end{aligned}
$$

which we write

$$
\begin{aligned}
& \frac{B_0}{h}hv_\theta\frac{1}{r}\left(-\frac{\partial\phi}{\partial r}\right)\frac{\partial}{\partial\theta}\left(\frac{1}{\frac{\varepsilon}{q}\frac{B_0}{h}}\right) + \frac{B_0}{h}hv_\theta\frac{1}{r}\frac{1}{B_\theta}\left(-\frac{\partial^2\phi}{\partial\theta\partial r}\right) + \frac{B_0}{h}hv_\theta\frac{q}{\varepsilon}\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} \\
& +\frac{B_0}{h}h\frac{v_\theta}{Rh}(-\sin\theta)\frac{1}{B_\theta}\left(-\frac{\partial\phi}{\partial r}\right) \\
& +\frac{B_0}{h}h\frac{v_\theta}{Rh}(-\sin\theta)\frac{q}{\varepsilon}v_\theta
\end{aligned}
$$

The first contribution to this *second* term is

$$
\begin{aligned}
\frac{B_0}{h}hv_\theta\frac{1}{r}\left(-\frac{\partial\phi}{\partial r}\right)\frac{\partial}{\partial\theta}\left(\frac{1}{\frac{\varepsilon}{q}\frac{B_0}{h}}\right) &= \frac{B_0}{h}hv_\theta\frac{1}{r}\left(-\frac{\partial\phi}{\partial r}\right)\frac{q}{\varepsilon}\frac{1}{B_0}\frac{\partial h}{\partial\theta} \\
&= v_\theta\frac{q}{\varepsilon}\frac{1}{r}\frac{\partial h}{\partial\theta}\left(-\frac{\partial\phi}{\partial r}\right)
\end{aligned}
$$

The second contribution to this *second* term is

$$\begin{aligned}\frac{B_0}{h} h v_\theta \frac{1}{r}\frac{1}{B_\theta}\left(-\frac{\partial^2\phi}{\partial\theta\partial r}\right) &= B_0 v_\theta \frac{qh}{B_0\varepsilon}\left(-\frac{1}{r}\frac{\partial^2\phi}{\partial\theta\partial r}\right)\\ &= v_\theta\frac{q}{\varepsilon}h\left(-\frac{1}{r}\frac{\partial^2\phi}{\partial\theta\partial r}\right)\end{aligned}$$

The third contribution to this *second* term is

$$\frac{B_0}{h} h v_\theta \frac{q}{\varepsilon}\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} = B_0\frac{q}{\varepsilon}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\theta^2}{2}\right)$$

The fourth contribution to this *second* term is

$$\begin{aligned}\frac{B_0}{h} h \frac{v_\theta}{Rh}(-\sin\theta)\frac{1}{B_\theta}\left(-\frac{\partial\phi}{\partial r}\right) &= B_0\frac{v_\theta}{Rh}\frac{1}{\varepsilon}\frac{\partial h}{\partial\theta}\frac{qh}{\varepsilon B_0}\left(-\frac{\partial\phi}{\partial r}\right)\\ &= \frac{v_\theta}{R\varepsilon}\frac{q}{\varepsilon}\frac{\partial h}{\partial\theta}\left(-\frac{\partial\phi}{\partial r}\right)\\ &= v_\theta\frac{q}{\varepsilon}\frac{1}{r}\frac{\partial h}{\partial\theta}\left(-\frac{\partial\phi}{\partial r}\right)\end{aligned}$$

The fifth contribution to this *second* term is

$$\begin{aligned}\frac{B_0}{h} h \frac{v_\theta}{Rh}(-\sin\theta)\frac{q}{\varepsilon}v_\theta &= B_0 v_\theta^2\frac{q}{\varepsilon}\frac{1}{Rh}\frac{1}{\varepsilon}\frac{\partial h}{\partial\theta}\\ &= B_0 v_\theta^2\frac{q}{\varepsilon}\frac{1}{h}\frac{1}{r}\frac{\partial h}{\partial\theta}\end{aligned}$$

Then the full *second* term is

$$\begin{aligned}&v_\theta\frac{q}{\varepsilon}\frac{1}{r}\frac{\partial h}{\partial\theta}\left(-\frac{\partial\phi}{\partial r}\right)\\ &+B_0\frac{q}{\varepsilon}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\theta^2}{2}\right)\\ &+v_\theta\frac{q}{\varepsilon}h\left(-\frac{1}{r}\frac{\partial^2\phi}{\partial\theta\partial r}\right)\\ &+v_\theta\frac{q}{\varepsilon}\frac{1}{r}\frac{\partial h}{\partial\theta}\left(-\frac{\partial\phi}{\partial r}\right)\\ &+B_0 v_\theta^2\frac{q}{\varepsilon}\frac{1}{h}\frac{1}{r}\frac{\partial h}{\partial\theta}\end{aligned}$$

the first and the fourth contributions are added together and the full *second* term is

$$2v_\theta\frac{q}{\varepsilon}\frac{1}{r}\frac{\partial h}{\partial\theta}\left(-\frac{\partial\phi}{\partial r}\right)+B_0\frac{q}{\varepsilon}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\theta^2}{2}\right)+v_\theta\frac{q}{\varepsilon}h\left(-\frac{1}{r}\frac{\partial^2\phi}{\partial\theta\partial r}\right)+B_0 v_\theta^2\frac{q}{\varepsilon}\frac{1}{h}\frac{1}{r}\frac{\partial h}{\partial\theta}$$

Now we put together the *first* term and the *second* term

$$B_\theta h \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\theta^2}{2}\right) - \frac{h}{B_0}\frac{q}{\varepsilon}\frac{1}{r}\frac{\partial h}{\partial\theta}\left(\frac{\partial\phi}{\partial r}\right)^2$$

$$\underline{-\frac{B_0}{h}\frac{q}{\varepsilon}\frac{1}{r}\frac{\partial h}{\partial\theta}v_\theta^2} + \underline{\underline{2\frac{q}{\varepsilon}v_\theta\frac{1}{r}\frac{\partial h}{\partial\theta}\left(\frac{\partial\phi}{\partial r}\right)}}$$

$$+\underline{\underline{2v_\theta\frac{q}{\varepsilon}\frac{1}{r}\frac{\partial h}{\partial\theta}\left(-\frac{\partial\phi}{\partial r}\right)}} + B_0\frac{q}{\varepsilon}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\theta^2}{2}\right) + v_\theta\frac{q}{\varepsilon}h\left(-\frac{1}{r}\frac{\partial^2\phi}{\partial\theta\partial r}\right) + \underline{B_0 v_\theta^2\frac{q}{\varepsilon}\frac{1}{h}\frac{1}{r}\frac{\partial h}{\partial\theta}}$$

We note that the underlined terms cancel. Then the left hand side of the equation of momentum conservation is

$$B_\theta h \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\theta^2}{2}\right) - \frac{h}{B_0}\frac{q}{\varepsilon}\frac{1}{r}\frac{\partial h}{\partial\theta}\left(\frac{\partial\phi}{\partial r}\right)^2 + v_\theta\frac{q}{\varepsilon}h\left(-\frac{1}{r}\frac{\partial^2\phi}{\partial\theta\partial r}\right)$$

the LHS of momentum equation

A last modification, we replace

$$B_\theta h = \frac{\varepsilon}{q}B_0$$

The right hand side of the equation of momentum conservation takes the form

$$-\frac{c_s^2}{\rho}hB_\theta\frac{\partial\rho}{r\partial\theta} = -c_s^2\frac{\varepsilon}{q}B_0\frac{\partial}{r\partial\theta}\ln\rho$$

the RHS of momentum equation

These two terms are equal

$$\begin{aligned}
&\frac{\varepsilon}{q}B_0\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\theta^2}{2}\right) + B_0\frac{q}{\varepsilon}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\theta^2}{2}\right) \\
&-\frac{h}{B_0}\frac{q}{\varepsilon}\frac{1}{r}\frac{\partial h}{\partial\theta}\left(\frac{\partial\phi}{\partial r}\right)^2 + v_\theta\frac{q}{\varepsilon}h\left(-\frac{1}{r}\frac{\partial^2\phi}{\partial\theta\partial r}\right) \\
= &-c_s^2\frac{\varepsilon}{q}B_0\frac{\partial}{r\partial\theta}\ln\rho
\end{aligned}$$

We multiply everything by

$$\frac{\varepsilon}{q}$$

and divide everything by B_0.

$$\begin{aligned}
& \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\theta^2}{2}\right)\left(1+\frac{\varepsilon^2}{q^2}\right) \\
& -\frac{1}{B_0^2}\left[\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{h^2}{2}\right)\right]\left(\frac{\partial\phi}{\partial r}\right)^2 - \frac{v_\theta}{B_0 h}h^2\left(\frac{1}{r}\frac{\partial^2\phi}{\partial\theta\partial r}\right) \\
= \quad & -c_s^2\left(\frac{\varepsilon}{q}\right)^2\frac{\partial}{r\partial\theta}\ln\rho
\end{aligned}$$

Note. We need to substitute

$$v_\theta \to \frac{1}{B_0/h}\left(\frac{\partial\phi}{\partial r}\right) = \frac{1}{B_T}\left(\frac{\partial\phi}{\partial r}\right)$$

such that the last term on the left becoms

$$-\frac{v_\theta}{B_0 h}h^2\left(\frac{1}{r}\frac{\partial^2\phi}{\partial\theta\partial r}\right) \to -\frac{1}{B_0^2}\left(\frac{\partial\phi}{\partial r}\right)h^2\left(\frac{1}{r}\frac{\partial^2\phi}{\partial\theta\partial r}\right)$$

and then we get

$$\begin{aligned}
& \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\theta^2}{2}\right)\left(1+\frac{\varepsilon^2}{q^2}\right) - \frac{1}{B_0^2}\left[\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{h^2}{2}\right)\right]\left(\frac{\partial\phi}{\partial r}\right)^2 - \frac{1}{B_0^2}\frac{h^2}{2}\frac{1}{r}\frac{\partial}{\partial\theta}\left[\left(\frac{\partial\phi}{\partial r}\right)^2\right] \\
= \quad & -c_s^2\left(\frac{\varepsilon}{q}\right)^2\frac{\partial}{r\partial\theta}\ln\rho
\end{aligned}$$

or

$$\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\theta^2}{2}\right)\left(1+\frac{\varepsilon^2}{q^2}\right) - \frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{B_0^2}\frac{h^2}{2}\left(\frac{\partial\phi}{\partial r}\right)^2\right] + \frac{1}{r}\frac{\partial}{\partial\theta}\left[c_s^2\left(\frac{\varepsilon}{q}\right)^2\ln\rho\right] = 0$$

$$\frac{1}{r}\frac{\partial}{\partial\theta}\left[\left(1+\frac{\varepsilon^2}{q^2}\right)\frac{v_\theta^2}{2} + c_s^2\left(\frac{\varepsilon}{q}\right)^2\ln\rho - \frac{1}{B_0^2}\frac{h^2}{2}\left(\frac{\partial\phi}{\partial r}\right)^2\right] = 0$$

The second order term ε^2/q^2 from the paranthesis multiplying $v_\theta^2/2$ will be neglected.

A Bernoulli law

$$\frac{\partial}{\partial\theta}\left[\frac{v_\theta^2}{2} + \left(\frac{\varepsilon}{q}\right)^2 c_s^2\ln\rho - \left(\frac{\partial\phi}{\partial r}\right)^2\frac{h^2}{2B_0^2}\right] = 0$$

Comment
The flow is "static", there is no time variation.
Then we only describe the spatial variations.
End

Normalization

$$\begin{aligned}
v_\theta \quad &\rightarrow \quad u = \frac{q}{\varepsilon}\frac{v_\theta}{c_s} \qquad \text{poloidal velocity} \\
v_z \quad &\rightarrow \quad v = \frac{v_z}{c_s} \qquad \text{toroidal velocity} \\
E \quad &= \quad \frac{q}{\varepsilon}\frac{1}{c_s}\frac{h}{B_0}\frac{\partial \phi}{\partial r} \qquad \text{radial electric field}
\end{aligned}$$

NOTE that the first normalization introduces the poloidal projection of the sound speed

$$c_s \frac{\varepsilon}{q} = c_s \frac{B_\theta}{B_T} = c_s^\theta$$

In the papers of **Friedberg** on the shock solutions and their evolution from radially directed to circles in the poloidal plane (as magnetic surfaces) it is reminded that the poloidal projection of the parallel sound speed is zero in the center and is zero at the edge of the plasma.
note also that this is in **Stringer1969 PRL** see *025_stringer*.
END.

NOTE that the last normalization introduces a new poloidal velocity

$$\begin{aligned}
E \quad &\equiv \quad \text{normalized poloidal velocity} \\
&= \quad \frac{\frac{1}{B_0/h}\frac{\partial \phi}{\partial r}}{c_s^\theta} = \frac{v_E^\theta}{c_s^\theta}
\end{aligned}$$

END.
Defining

$$\overline{E} \equiv \frac{q}{\varepsilon}\frac{1}{c_s}\frac{1}{B_0}\frac{\partial \phi}{\partial r}$$

we separate in this way the factor which does not contain any poloidal θ variation and we have

$$E = \overline{E}\ h = \overline{E}\ \left(1 + \varepsilon \cos\theta\right)$$

The *zeroth-order* equations, *i.e.* $\eta = 0$ are

$$\begin{aligned}
h\rho u \quad &= \quad L\left(r\right) \ \text{(arbitrary function of } r\text{)} \\
E \quad &= \quad \overline{E}h = \overline{E}\left(1 + \varepsilon \cos\theta\right) \\
\frac{u^2}{2} - \frac{E^2}{2} + \ln\rho \quad &= \quad K\left(r\right) \ \text{(arbitrary function of } r\text{)}
\end{aligned}$$

the last equation ("Bernoulli law") means that $K(r)$ does not depend on the poloidal variable θ.

$$\frac{\partial}{\partial\theta}K(r)=0$$

Also **remember** that the first equation comes from the *continuity*.

$$\boldsymbol{\nabla}\cdot(\rho\mathbf{v})=0$$

$$\frac{\partial}{\partial\theta}(h\rho v_\theta)=0$$

after normalization of v_θ.

The system is reduced by extracting ρ from the first equation and replacing it in the "Bernoulli "-like equation

$$\rho=\frac{L}{hu}$$

$$\frac{u^2}{2}-\frac{E^2}{2}+\ln\left(\frac{L}{hu}\right)=K(r)$$

$$\begin{aligned}\frac{u^2}{2}-\frac{E^2}{2}-\ln(hu) &= K-\ln L\\ \frac{u^2}{2}-\frac{E^2}{2}-\ln(u)-\ln(1+\varepsilon\cos\theta) &= K-\ln L\\ \frac{u^2}{2}-\frac{\left(\overline{E}h\right)^2}{2}-\ln(u)-\varepsilon\cos\theta &\approx K-\ln L\end{aligned}$$

It is defined the quantity

$$f(u)\equiv\frac{u^2}{2}-\ln|u|$$

and its average over the surface

$$\overline{f}=\frac{1}{2\pi}\int_0^{2\pi}d\theta f$$

we have

$$f(u)=\overline{f}(u)+\left(1+\overline{E}^2\right)\varepsilon\cos\theta$$

This is an already interesting result.

The function $u^2/2 - \ln|u|$ has a correction that is *harmonic* $\varepsilon\cos\theta$ on the poloidal direction, like the Pfirsch Schluter toroidal current. This result does not have any connection with the Pfirsch Schluter current since the damagnetic flow (implicitely $\boldsymbol{\nabla}p$) has not been involved.

The *geometry* constraints the expression $u^2/2 - \ln|u|$ to have a variation on the poloidal direction θ. Since this combination involves poloidal velocity it results that the way plasma rotates on the magnetic surfaces in the poloidal direction must be variable with θ, even at stationarity of the flow.

This is the *hose effect.*

The function $u^2/2 - \ln|u|$ has two minima

$$u = \pm 1$$

For given $\overline{f}$ and angle θ the Bernoulli equation for u gives *four* solutions.

The minima of the function corresponds to the *critical speed*

$$|u| = 1$$

where the poloidal velocity is equal to the sound velocity projected along the poloidal direction

$$\frac{\varepsilon}{q}c_s$$

The *acoustic* waves propagate along the magnetic field lines with c_s . And the plasma rotates in the poloidal direction with u . The result is that the velocity of the waves on the poloidal direction is zero, since there is compensation.

NOTE

See also the explanation of **Peeters** on bootstrap.

END.

The quantity

$$\frac{u^2}{2} - \frac{\left(\overline{E}h\right)^2}{2} - \ln(u) - \varepsilon\cos\theta \approx \text{const on } \theta$$

can be used to see how are correlated the θ variations in the surface.

There is a function

$$\begin{aligned}
-\frac{\left(\overline{E}h\right)^2}{2} - \varepsilon\cos\theta &\approx -\left[\overline{E}^2\left(\frac{1}{2} + \varepsilon\cos\theta\right) + \varepsilon\cos\theta\right] \\
&= \text{ct} - \varepsilon\left(\overline{E}^2 + 1\right)\cos\theta \\
&\sim \alpha - \beta\cos\theta
\end{aligned}$$

and

$$f\left(u\right)+\alpha-\beta\cos\theta\approx\text{const on }\theta$$

When $|\cos\theta|$ is small we have $f\left(u\right)$ large.

When θ grows from $\theta=0$ to $\pi/2$ the term $|\beta\cos\theta|$ decreases from $|\beta|$ to 0 therefore the function $f\left(u\right)$ must increase to keep the whole expression constant =ct-α.

When $\theta=\pi/2$ the function $|\beta\cos\theta|$ is zero and the function $f\left(u\right)$ is maximum.

When θ grows from $\pi/2$ to π the function $-\beta\cos\theta$ increases from $|\beta|$ and the function $f\left(u\right)$ must decrease to keep the sum constant.

The minimum value of the combination $u^2/2-\ln|u|$ is 1/2 and corresponds to the critical poloidal velocity $|u|=1$.

Then the function must be larger than

$$\overline{f}\geq\frac{1}{2}+a\varepsilon$$

The function $\overline{f}$ close to the minimum 1/2 corresponds to a poloidal velocity u that has certain particularities since it is close to the singularity given by the cancellation of the velocity of the acoustic waves, $|u|=1$.

For velocities u such that the function $u^2/2-\ln|u|$ is far from the minimum, the flow has a different pattern.

18.2.1 The regime of poloidal rotation which is far from the critical value

Consider the poloidal velocity expanded around the equilibrium value

$$u=\overline{u}+\delta u$$

and

$$\begin{aligned}f\left(u\right) &= \frac{\overline{u}^2}{2}-\ln\overline{u}\\ &\quad+\left(\overline{u}-\frac{1}{\overline{u}}\right)\delta u+\frac{1}{2}\left(1+\frac{1}{\overline{u}^2}\right)\left(\delta u\right)^2\end{aligned}$$

For velocities u that are far from the critical value $|u|=1$, which means

$$\overline{u}-1\gg\frac{\delta u}{2}$$

the term $(\delta u)^2$ is neglected and the equation for the calculation of δu is

$$\begin{aligned}
f(u) &= \frac{\overline{u}^2}{2} - \ln \overline{u} + \left(\overline{u} - \frac{1}{\overline{u}}\right)\delta u \\
\overline{f}(u) + a\varepsilon\cos\theta &= \frac{\overline{u}^2}{2} - \ln \overline{u} + \left(\overline{u} - \frac{1}{\overline{u}}\right)\delta u \\
\frac{\overline{u}^2}{2} - \ln \overline{u} + a\varepsilon\cos\theta &= \frac{\overline{u}^2}{2} - \ln \overline{u} + \left(\overline{u} - \frac{1}{\overline{u}}\right)\delta u \\
\frac{a\varepsilon\cos\theta}{\overline{u} - \frac{1}{\overline{u}}} &= \delta u
\end{aligned}$$

or

$$\begin{aligned}
\delta u &= \varepsilon\overline{u}\frac{\overline{E}^2 + 1}{\overline{u}^2 - 1}\cos\theta \\
&\quad + O\left(\varepsilon^2\right)
\end{aligned}$$

With this expression for the poloidal velocity $\overline{u} + \delta u$ we return to the *continuity equation*

$$h\rho u = L(r)$$

where we now admit the variation of the density on the magnetic surface

$$\rho = \overline{\rho} + \delta\rho$$

Using the expression of the poloidal velocity u with harmonic correction, we obtain

$$\delta\rho = -\overline{\rho}\left(\frac{\delta u}{\overline{u}} + \varepsilon\cos\theta\right) + O\left(\varepsilon^2\right)$$

or

$$\delta\rho \approx -\overline{\rho}\ \varepsilon\ \cos\theta\ \left(1 + \frac{\overline{E}^2 + 1}{\overline{u}^2 - 1}\right)$$

We have derived the variation of the density on the magnetic surface, and found that it is harmonic like that of the poloidal rotation u.

There is also the θ-variation in the magnetic surface of the *toroidal velocity.*

The absence of resistivity $\eta = 0$ specifies the zeroth order Ohm's law

$$-\frac{\partial\phi}{\partial r} + (\mathbf{v} \times \mathbf{B})_r = 0$$

which in normalized variables is

$$v = u - E$$

Now we assume a small θ-dependent departure from the average value

$$\delta v = v - \overline{v}$$

$$\delta v = \delta u - \overline{E}\varepsilon \cos\theta + O\left(\varepsilon^2\right)$$

$$\delta v \approx \left(-\overline{E} + \overline{u}\frac{\overline{E}^2 + 1}{\overline{u}^2 - 1}\right)\varepsilon \cos\theta$$

We have derived the θ dependent velocities of the flows in the magnetic surface, when these are far from the critical one v_c.

18.2.2 The regime of the poloidal rotation that is close to the critical velocity

The regime is characterized by

$$\overline{u} - 1 \lesssim \frac{\delta u}{2}$$

The order $(\delta u)^2$ cannot be neglected.

Then

$$\begin{aligned} & -2\left(1-\overline{u}\right)\delta u + \left(\delta u\right)^2 \\ = \; & \left(\delta u\right)^2 + a\varepsilon \cos\theta \end{aligned}$$

The authors make the observation that

$$u\left(\pi\right) \approx 1$$

where

$$\frac{\partial f}{\partial \theta} = 0 \;\; \text{at} \;\; u = 1$$

The reason for this is the physical acceleration of the fluid when going to a narrower zone of the duct. The velocity must increase to keep the flux constant but the limit is, as choosen by us, the critical velocity

$$|u| \approx 1$$

after traversing the narrowest section, at $\theta = \pi$ the fluid can deccelerate.

Then

$$\delta u \approx 1 - \overline{u} \pm \sqrt{\varepsilon \left(1 + \overline{E}^2\right)(1 + \cos\theta)}$$

The other variables have the following corrections

$$\delta \rho \approx -\frac{\overline{\rho}}{\overline{u}} \left\{ 1 - \overline{u} \pm \sqrt{\varepsilon \left(1 + \overline{E}^2\right)(1 + \cos\theta)} \right\}$$

$$\delta v \approx 1 - \overline{u} \pm \sqrt{\varepsilon \left(1 + \overline{E}^2\right)(1 + \cos\theta)}$$

18.2.3 The current.

From the equation of conservation of the momentum.

There is toroidal flow

$$v = v_{\varphi}$$

and its velocity v_{φ} has θ variation.

The toroidal flow is connected with the *radial* current $\sim v_r$ and with the *poloidal* magnetic field B_θ through the usual $j_r \times B_\theta =$ acceleration on φ.

The radial component

$$J_r = c_s^2 \frac{1}{B_0} \rho u \frac{\partial}{r \partial \theta} (hv)$$

This radial current has 0 average on the magnetic surface, via the periodicity in θ of the factors.

Here a comment is needed: the existence of a radial current appears to be in contradiction with $v_r = 0$ the first derived result in zeroth order ($\eta = 0$ and no θ dependence, *i.e.* $\varepsilon = 0$). But since there is a *toroidal flow*, $v \equiv v_\varphi \neq 0$, then the product

$$\begin{aligned} \mathbf{J}_r \times \mathbf{B}\big|_{\varphi} &= en\mathbf{v}_r \times \mathbf{B}\big|_{\varphi} = J_r B_\theta \\ &= \rho \left(\mathbf{v} \cdot \boldsymbol{\nabla}\right) \mathbf{v}\big|_{\varphi} \ , \end{aligned}$$

where

$$\rho = nm_i$$

is calculated using the formula derived before

$$
\begin{aligned}
& (\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v} \\
= \; & \widehat{\mathbf{e}}_r\left[v_r\frac{\partial v_r}{\partial r}+v_\theta\frac{\partial v_r}{r\partial\theta}-\frac{v_\theta^2}{r}-\frac{v_\varphi^2}{R}\cos\theta\right] \\
& +\widehat{\mathbf{e}}_\theta\left[v_\theta\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}-\frac{v_\varphi^2}{r\,(R+r\cos\theta)}\,(-r\sin\theta)\right] \\
& +\widehat{\mathbf{e}}_\varphi\left[v_\theta\frac{1}{r}\frac{\partial v_\varphi}{\partial\theta}+\frac{v_\theta v_\varphi}{r(R+r\cos\theta)}\,(-r\sin\theta)\right]
\end{aligned}
$$

$$
\begin{aligned}
\left.(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}\right|_\varphi &= v_\theta\frac{1}{r}\frac{\partial v_\varphi}{\partial\theta}+\frac{v_\theta v_\varphi}{r(R+r\cos\theta)}\,(-r\sin\theta) \\
&= v_\theta\left[\frac{1}{r}\frac{\partial v_\varphi}{\partial\theta}+v_\varphi\frac{1}{r}\frac{1}{h}\frac{\partial h}{\partial\theta}\right] \\
&= v_\theta\frac{1}{h}\left[h\frac{\partial v_\varphi}{r\partial\theta}+v_\varphi\frac{\partial h}{r\partial\theta}\right] \\
&= \frac{v_\theta}{h}\frac{\partial}{r\partial\theta}\,(hv_\varphi)
\end{aligned}
$$

From this we obtain

$$
\begin{aligned}
J_rB_\theta &= \rho\frac{1}{h}\left(v_\theta\frac{\partial}{r\partial\theta}\right)(hv_\varphi) \\
J_r &= \rho\frac{1}{B_\theta}\frac{1}{h}\left(v_\theta\frac{\partial}{r\partial\theta}\right)(hv_\varphi)
\end{aligned}
$$

At this moment we must take into account the normalization of the velocities

$$
\begin{aligned}
v_\theta &\rightarrow u=\frac{q}{\varepsilon}\frac{v_\theta}{c_s} \\
v_\varphi &\rightarrow v=\frac{v_\varphi}{c_s}
\end{aligned}
$$

Then

$$
\begin{aligned}
J_r &= \rho\frac{1}{B_\theta}\frac{1}{h}\left[\frac{\varepsilon}{q}c_su\;\frac{\partial}{r\partial\theta}\,(hc_sv)\right] \\
&= c_s^2\rho\frac{1}{B_\theta}\frac{1}{h}\frac{\varepsilon}{q}\left[u\frac{\partial}{r\partial\theta}\,(hv)\right] \\
&= c_s^2\rho\frac{1}{B_T}\frac{1}{h}\left[u\frac{\partial}{r\partial\theta}\,(hv)\right]
\end{aligned}
$$

and since

$$B_T = \frac{B_0}{h}$$

it results

$$J_r = c_s^2 \rho \frac{1}{B_0} \left[u \frac{\partial}{r \partial \theta} (hv) \right]$$

The poloidal component is derived from the same *static* momentum balance equation

$$(\mathbf{v} \cdot \boldsymbol{\nabla}) \, \mathbf{v} = -c_s^2 \frac{1}{\rho} \boldsymbol{\nabla} \rho + \frac{1}{\rho} \mathbf{J} \times \mathbf{B}$$

We **note** that there is only one possibility to involve the diamagnetic flow, which is to include the gradient of pressure. The force balance is based on the gradient of the pressure (density) and the term $\mathbf{J} \times \mathbf{B}$. This is balanced by the stationary inertial term $(\mathbf{v} \cdot \boldsymbol{\nabla}) \, \mathbf{v}$.

To evaluate the LHS contribution we project along r (since this involves $\mathbf{J}_{\perp}$) and we use the formula

$$\begin{aligned} & \text{component of } (\mathbf{v} \cdot \boldsymbol{\nabla}) \, \mathbf{v} \text{ along } r \\ = \quad & v_r \frac{\partial v_r}{\partial r} + v_\theta \frac{\partial v_r}{r \partial \theta} - \frac{v_\theta^2}{r} - \frac{v_\varphi^2}{R_0 h} \cos \theta \end{aligned}$$

We must use the equation of continuity

$$\boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0$$

with the formula

$$\boldsymbol{\nabla} \cdot \mathbf{a} = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial r} (h_2 h_3 a_1) + \frac{\partial}{\partial \theta} (h_1 h_3 a_2) + \frac{\partial}{\partial \varphi} (h_1 h_2 a_3) \right)$$

where

$$\begin{aligned} h_r &= 1 \\ h_\theta &= r \\ h_\varphi &= R_0 + r \cos \theta \end{aligned}$$

This gives

$$
\begin{aligned}
& \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) \\
= \; & \frac{1}{rR_0 h}\left[\frac{\partial}{\partial r}\left(rR_0 h\ \rho v_r\right)+\frac{\partial}{\partial\theta}\left(R_0 h\ \rho v_\theta\right)+\frac{\partial}{\partial\varphi}\left(r\rho v_\varphi\right)\right] \\
= \; & \frac{1}{rh}\left(h\rho v_r + r\frac{\cos\theta}{R_0}\rho v_r + rhv_r\frac{\partial\rho}{\partial r}+rh\rho\frac{\partial v_r}{\partial r}\right) \\
& +\frac{1}{rh}\rho\left(\frac{-r\sin\theta}{R_0}v_\theta + h\frac{\partial v_\theta}{\partial\theta}\right) \\
= \; & \rho\frac{v_r}{r}+\frac{\cos\theta}{hR_0}\rho v_r + v_r\frac{\partial\rho}{\partial r}+\rho\frac{\partial v_r}{\partial r}-\rho\frac{\sin\theta}{hR_0}v_\theta+\rho\frac{\partial v_\theta}{r\partial\theta} \\
= \; & 0
\end{aligned}
$$

This expression may serve to extract some of the terms from this equation and to replace them in $\rho\left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{v}\big|_r$. This should allow to to calculate $J_\perp$ and after it, the projection $J_{\perp\theta}$.

$$
J_{\perp\theta} = c_s^2\frac{1}{B_0}\rho h\left(\frac{1}{\rho}\frac{\partial\rho}{\partial r}-\frac{v^2}{h}\frac{\partial h}{\partial r}\right)
$$

The first term is diamagnetic

$$
\begin{aligned}
env_{\perp\theta} &= \frac{T_e}{m_i}\frac{1}{\left(\frac{B_0}{h}\right)}\frac{d\left(nm_i\right)}{dr} \\
v_{\perp\theta} &= \frac{T_e}{eB}\frac{d}{dr}\ln n
\end{aligned}
$$

The second term is

$$
-c_s^2\frac{1}{B_0}\rho v_z^2\left(\frac{\cos\theta}{R_0}\right)
$$

and looks like an approximation of the *neoclassical particle drift* flow, like in Stringer.

The poloidal (θ) average of the poloidal component of the parallel current is zero

$$
\overline{J_{\parallel\theta}}
$$

and

$$
J_{\parallel\theta}\approx -2c_s^2\frac{1}{B_0}\varepsilon\cos\theta\frac{\partial}{\partial r}\left[\overline{\rho}\left(1+\frac{\overline{v}^2}{2}\right)\right]
$$

Using the θ part of the perpendicular and of the parallel current we obtain

$$
J_z = \frac{q}{\varepsilon}J_{\parallel\theta}-\frac{\varepsilon}{q}J_{\perp\theta}
$$

18.2.4 The radial velocity and the poloidal electric field at zeroth order (for low η)

The resistivity gives the Ohm's law

$$v_r = \eta \frac{J_z}{B_\theta}$$

$$v_r \approx -\eta c_s^2 \left(\frac{1}{B_0^2}\right) \left\{ 2\left(\frac{q}{\varepsilon}\right)^2 \frac{\partial}{\partial r}\left[\overline{\rho}\left(1+\frac{\overline{v}^2}{2}\right)\right] h\varepsilon\cos\theta \right\}$$

Here it is noted that v_r has a factor of order

$$\varepsilon^{-1}$$

and has zero poloidal average.

The *poloidal* electric field

$$\begin{aligned} -B_\theta \frac{\partial \phi}{r\partial\theta} &= \eta \mathbf{B}\cdot\mathbf{J} \\ &= \eta \frac{B_0}{h}\frac{q}{\varepsilon} J_{\parallel\theta} \end{aligned}$$

or

$$\frac{\partial\phi}{r\partial\theta} \approx 2\eta\left(\frac{q}{\varepsilon}\right)^2 \frac{1}{B_0} c_s^2 \frac{\partial}{\partial r}\left[\overline{\rho}\left(1+\frac{\overline{v}^2}{2}\right)\right]\varepsilon\cos\theta$$

These are solutions: uniform on the magnetic surface plus corrections depending on θ, stationary, generated by the

- geometry
- the presence of a small resistivity

Now there will be time variation.

18.2.5 Time variation of the plasma variables between equilibrium states

The equation of momentum is multiplied by

$$h\mathbf{B}$$

resulting

$$\begin{aligned} & \frac{\partial}{\partial t}\left(h\mathbf{B}\cdot\mathbf{v}\right)+h\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{\left|\mathbf{v}\right|^{2}}{2}+c_{s}^{2}\ln\rho\right) \\ = & \; h\mathbf{B}\cdot\left[\mathbf{v}\times\left(\boldsymbol{\nabla}\times\mathbf{v}\right)\right] \end{aligned}$$

We note that the first term in the paranthesis and the right hand side term represent $(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}$ part of the convective derivative $d\mathbf{v}/dt$. The term $\ln\rho$ results from the assumed expression for the pressure, with $T=$const.

Using the Ohm's law the RHS is written

$$h\mathbf{B}\cdot\left[\mathbf{v}\times\left(\boldsymbol{\nabla}\times\mathbf{v}\right)\right]=-h\left[\boldsymbol{\nabla}\times\mathbf{v}\cdot\eta\mathbf{J}+\boldsymbol{\nabla}\cdot\left(\mathbf{v}\times\boldsymbol{\nabla}\phi\right)\right]$$

After averaging over θ,

$$\frac{\partial}{\partial t}\left(h\mathbf{B}\cdot\mathbf{v}\right)=\frac{1}{r}\frac{\partial}{\partial r}\left[r\;\overline{\left(hv_{z}\frac{\partial\phi}{r\partial\theta}\right)}\right]-\eta\;\overline{\left(h\boldsymbol{\nabla}\times\mathbf{v}\cdot\mathbf{J}\right)}$$

It is remarked that the variation of the electrostatic potential on the magnetic surface θ is of the order of the resistivity η, which means that the time variation induced by it is on the resistive time scale.

In the present problem $\boldsymbol{\nabla}\phi$ is the variation of the electrostatic potential on the magnetic surface,

$$\frac{\partial\phi}{r\partial\theta}$$

[and here we ma recall the expression that has already been derived

$$\frac{\partial\phi}{r\partial\theta}\approx2\eta\left(\frac{q}{\varepsilon}\right)^{2}\frac{1}{B_{0}}c_{s}^{2}\frac{\partial}{\partial r}\left[\overline{\rho}\left(1+\frac{\overline{v}^{2}}{2}\right)\right]\varepsilon\cos\theta$$

In applying the Ohm's law the last term is

$$\boldsymbol{\nabla}\cdot\left(\mathbf{v}\times\boldsymbol{\nabla}\phi\right)\rightarrow\frac{1}{r}\frac{\partial}{\partial r}\left[rh\;v_{z}\frac{\partial\phi}{r\partial\theta}\right]$$

which is further averaged over θ.

Note that the averaging over θ of terms that are periodic on θ gives zero and this is the reason for the disappearence of the second term of the LHS.

The first term

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\overline{\left(hv_z\frac{\partial\phi}{r\partial\theta}\right)}\right]$$

arises from the parallel resistivity is related to the divergence of a mass flux and is positive. *This is the term that causes the rotational speed-up.*

The scond term

$$\begin{aligned}&\eta\ \overline{(h\boldsymbol{\nabla}\times\mathbf{v}\ \cdot\ \mathbf{J})}\\ \approx\ &\eta\frac{1}{B_0}c_s^3\left[\overline{\rho}\ \overline{u}\ \overline{\left(\frac{\partial v}{r\partial\theta}\right)^2}-\frac{\partial\overline{\rho}}{\partial r}\frac{\partial\overline{v}}{\partial r}\right]\end{aligned}$$

The time variation of the zeroth order parameters.

There are conservation equations.

Conservation of mass

$$\frac{\partial}{\partial t}\overline{(h\rho)}+\frac{1}{r}\frac{\partial}{\partial r}\left[r\ \overline{(h\rho v_r)}\right]=0$$

Conservation of momentum

$$\frac{\partial}{\partial t}\overline{(h^2\rho v)}+\frac{1}{r}\frac{\partial}{\partial r}\left[r\ \overline{(h^2\rho v\ v_r)}\right]=0$$

Using these cnservation equations together with the equation (above)

$$\frac{\partial}{\partial t}\left(h\mathbf{B}\cdot\mathbf{v}\right)=\frac{1}{r}\frac{\partial}{\partial r}\left[r\ \overline{\left(hv_z\frac{\partial\phi}{r\partial\theta}\right)}\right]-\eta\ \overline{(h\boldsymbol{\nabla}\times\mathbf{v}\cdot\mathbf{J})}$$

one obtains

$$\begin{aligned}&\frac{\partial}{\partial t}\overline{v}+\left(\frac{\varepsilon}{q}\right)^2\frac{\partial}{\partial t}\overline{u}\\ =\ &\frac{1}{r}\frac{\partial}{\partial r}\left[r\ A_{\parallel}\left(\overline{u}-2\overline{v}-\frac{2}{\varepsilon}\overline{\cos\theta\ \delta u}\right)\right]\\ &-A_{\perp}\left[\frac{\partial\overline{v}}{\partial r}-\overline{\rho}\overline{u}\frac{1}{\frac{\partial\overline{\rho}}{\partial r}}\overline{\left(\frac{\partial v}{r\partial\theta}\right)^2}\right]\end{aligned}$$

where

$$\begin{aligned} A_{\parallel} &= -\eta_{\parallel} c_s^2 \frac{1}{B_0^2} q^2 \frac{\partial}{\partial r} \left[\overline{\rho} \left(1 + \frac{\overline{v}^2}{2} \right) \right] \\ A_{\perp} &= -\eta_{\perp} c_s^2 \frac{1}{B_0^2} \frac{\partial \overline{\rho}}{\partial r} \end{aligned}$$

if the resistivity would be different $\eta_{\parallel} \neq \eta_{\perp}$.

The mass flux

$$F \equiv \overline{(h \rho v_r)}$$

$$F = L(r) \times \overline{\left(\frac{v_r}{u} \right)}$$

and this is expanded to lowest order in ε

$$F \approx \overline{\rho} \left[\overline{v_r} - \overline{\left(v_r \frac{\delta u}{\overline{u}} \right)} + \overline{v_r \left(\frac{\delta u}{\overline{u}} \right)^2} \right]$$

The flux of angular momentum is

$$\begin{aligned} \overline{(h^2 \rho v \; v_r)} &= L \frac{\overline{(hv \; v_r)}}{u} \\ &\approx \overline{\rho} \; \overline{u} \; \overline{(h v_r)} - \overline{\rho} \; \overline{E} \; \overline{\left(\frac{h^2 v_r}{1 + \frac{\delta u}{u}} \right)} \end{aligned}$$

where it has been used

$$v = u - E$$

the radial projection of the Ohm's law.

The lowest order expression is

$$\begin{aligned} \overline{(h^2 \rho v \; v_r)} &= (\overline{v} - \overline{u}) F \\ & \quad + \overline{\rho u} \; A_{\perp} \\ & \quad + 2 \overline{\rho v} \; A_{\parallel} \end{aligned}$$

where

$$\overline{v_r} = A_{\parallel} + A_{\perp}$$

When the velocity on poloidal direction $\overline{u}$ is far from the critical velocity (sound speed projected on poloidal direction)

$$\overline{u} \ll 1$$

the second order term, with $(\delta u)^2$ can be neglected, and u has the θ variation that has been derived in the first case. Next, oit is possible to calculate the average over θ of the second term in the expansion of F since we use the expression for δu derived before.

The mass flux becomes

$$F = \left[A_{\parallel} \left(1 + \frac{a}{1 - \overline{u}^2} \right) + A_{\perp} \right] \overline{\rho}$$

or

$$F = -\eta c_s^2 \frac{1}{B_0^2} \overline{\rho} \left\{ \frac{\partial \overline{\rho}}{\partial r} + q^2 \left(1 + \frac{1 + (\overline{u} - \overline{v})^2}{1 - \overline{u}^2} \right) \frac{\partial}{\partial r} \left[\overline{\rho} \left(1 + \frac{\overline{v}^2}{2} \right) \right] \right\}$$

This expression generalizes the Pfirsch Schluter flux for the case of the presence of a velocity of a flow $\mathbf{v} \neq \mathbf{0}$.

The Pfirsch Schluter mass flux is

$$F_{PS} = -\eta c_s^2 \frac{1}{B_0^2} \left(1 + 2q^2 \right) \overline{\rho} \, \frac{\partial \overline{\rho}}{\partial r}$$

obtained from above for $\overline{v} = 0$.

18.2.6 Comment on comparison with Transient Alfven Amplification

Transient Alfven Lau Davidson Hui.

This work is commented upon in *plasma, general, theory, Alfven.*

This image is stationary.

There is flow in the surface and the velocity has spatial variation on surface.

The density has spatial variation - so there is gradient of pressure (= force).

These forces $nm_i \left[(\mathbf{v} \cdot \boldsymbol{\nabla}) \, \mathbf{v} \right]$ (inertial part of the momentum conservation) and $-\boldsymbol{\nabla} p$ must compensate each other, along a magnetic field line (to exclude $\mathbf{j} \times \mathbf{B}$).

The inertial term will produce

$$\frac{v_\theta^2}{2}$$

under the θ-derivation operator (plus others).

When we compare with

$$\frac{\partial^2}{\partial t^2}\boldsymbol{\nabla}^2\xi_{1x}$$

of **Lau transient Alfven amplification** we cannot find the factor $1+u^2$ which, in that case, produces an effect like the *hose*. A narrow profile (in spectral space) that produces for a wave packet an amplification when it traverses that zone.

One of the operators $\partial/\partial t$ comes from the connection between the displacement $\boldsymbol{\xi}_1$ and the velocity $\mathbf{v}_1$ both of order 1. The other operator $\partial/\partial t$ comes from $nm_i\frac{\partial \mathbf{v}_1}{\partial t}$, the momentum conservation. The Laplacian $\boldsymbol{\nabla}^2$ comes from applying $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times...)$ operator on the momentum conservation. The result will consist of Laplacian (due to the zero-divergence of $\boldsymbol{\xi}_1$ and of $\mathbf{B}_1$).

Therefore the inertial aspect consists of $\partial/\partial t$ part of d/dt, not of the $(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}$.

However the second time derivative (explicit time variation from inertial $d\mathbf{v}/dt$) is not at the origin of the factor that becomes narrow $1+u^2$. This factor is produced by the Laplacian, after Fourier transform.

18.3 Multiple equilibria and poloidal rotation instabilities (Ware Wiley)

The paper by **Ware Wiley** mentions that the plasma is unstable to poloidal rotation.

the rate of instability is small, proportional with the *resistivity*.

[see **Stringer** in *impurity accumulation Notes*, or in *stringer*.]

Usual neoclassical assumption

any mass motion of the plasma is at most first order in

$$\frac{\rho_{i\theta}}{L}\quad\text{for motion}\quad\parallel\mathbf{B}$$

for motion parallel to $\mathbf{B}$,

and is at most first order in

$$\frac{\rho_i}{L}\quad\text{for motion}\quad\perp\mathbf{B}$$

for any motion in perpendicular direction $\perp\mathbf{B}$.

The condition of equilibrium of the poloidal rotation is

$$\begin{aligned} mn\frac{\partial\overline{V}_\theta}{\partial t} &= -\frac{\left\langle\left(\widetilde{P}_\parallel-\widetilde{P}_\perp\right)\sin\theta\right\rangle}{R} \\ &= 0 \end{aligned}$$

NOTE that later we will have **Shaing Crume** to say that

$$m_i n \frac{\partial v_\theta}{\partial t} \approx \langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \mathbf{\Pi} \rangle$$

with the parallel viscosity calculated from the drift-kinetic equation. However this is for *relaxation* or damping of the poloidal rotation. Not for *driving.* **END**.

where

$$mn \equiv \rho \ \ \text{plasma density}$$

To first order in

$$\frac{\rho_{i\theta}}{L}$$

we have

$$\widetilde{P}_{e\|} = \widetilde{P}_{e\perp} = -\widetilde{P}_{i\|} = \overline{n} e \widetilde{\Phi}$$

where $\overline{n} \equiv \langle n \rangle$.

Then the condition of equilibrium in order

$$\frac{\rho_{i\theta}}{L_n}$$

$$\begin{aligned} \rho \frac{\partial \overline{V}_\theta}{\partial t} &= \frac{\left\langle \left(\widetilde{P}_{i\|} - \widetilde{P}_{i\perp} \right) \sin\theta \right\rangle}{R} \\ &= 0 \end{aligned}$$

Now it is used the expression for the difference $\widetilde{P}_{i\|} - \widetilde{P}_{i\perp}$ for the *plateau* regime

$$\rho \frac{\partial \overline{V}_\theta}{\partial t} = -\frac{\sqrt{\pi}}{2} n m_i v_{thi} \frac{r}{R^2} \left[\overline{V}_\| - \frac{E_r}{B_\theta} + \frac{T_i}{eB_\theta} \left(\frac{n'}{n} + \frac{3}{2} \frac{T_i'}{T_i} \right) \right]$$

To first order

$$\overline{V}_\theta = \frac{B_\theta}{B} \overline{V}_\| - \frac{E_r}{B} + \frac{p_i'}{neB}$$

(which is simply the very popular equation for the determination of E_r in experiments). Then introducing $\frac{B_\theta}{B}\overline{V}_\| - \frac{E_r}{B}$ from this expression in the square bracket above it results the slow time variation (higher than lowest order) of the poloidal rotation velocity

$$\begin{aligned} \rho \frac{\partial \overline{V}_\theta}{\partial t} &= -\frac{\sqrt{\pi}}{2} n m_i v_{thi} \frac{q}{R} \left(\overline{V}_\theta + \frac{T_i'}{2eB} \right) \\ &= 0 \end{aligned}$$

Then the equilibrium is

$$\overline{V}_{\parallel} - \frac{E_r}{B_\theta} = -\frac{T_i}{eB_\theta}\left(\frac{n'}{n} + \frac{3}{2}\frac{T_i'}{T_i}\right)$$

or

$$\overline{V}_\theta = \left[-\frac{1}{2}\right] \times \frac{1}{eB}\frac{dT_i}{dr}$$

This is the well-known result about the equilibrium poloidal rotation in tokamak. (Hazeltine and Hinton).

Here it is the plateau regime.

In the banana regime, the coefficient $-1/2$ is replaced by $+1.17$.

In the Pfirsch-Schluter regime the coefficient $-1/2$ is replaced by -2.1.

See also **Novakovskii Liu Sagdeev Rosenbluth** for polarization, very fast changes, GAM.

Orders of magnitude, after taking

$$v_{th,j} \sim 1$$

$$\begin{aligned}
\overline{V}_{e,i\ \parallel} &\sim \frac{\rho_{e,i\ \theta}}{L} \\
\widetilde{V}_{e,i\ \parallel} &\sim \left(\frac{r}{R}\right)\frac{\rho_{e,i\ \theta}}{L} = \varepsilon\ \frac{\rho_{e,i\ \theta}}{L} \\
\overline{V}_{i\perp} &\sim \left(\frac{r}{R}\right)\frac{\rho_{e,i\ \theta}}{L} = \varepsilon\ \frac{\rho_{e,i\ \theta}}{L} \\
\widetilde{V}_{e,i\ \perp} &\sim \left(\frac{r}{R}\right)^2\frac{\rho_{e,i\ \theta}}{L} = \varepsilon^2\ \frac{\rho_{e,i\ \theta}}{L} \\
V_{e,i\ r} &\sim \left(\frac{r}{R}\right)^2\left(\frac{\rho_{e,i\ \theta}}{L}\right)^2 = \varepsilon^2\ \left(\frac{\rho_{e,i\ \theta}}{L}\right)^2
\end{aligned}$$

The equation of continuity to the first order in $\rho_{i\theta}/L$ for the species $j \equiv e, i, Z$

$$\begin{aligned}
&\frac{\partial\widetilde{n}_j}{\partial t} - \frac{E_r}{B}\frac{\partial\widetilde{n}_j}{r\partial\theta} + B_\theta\frac{\partial}{r\partial\theta}\left(\frac{n_j V_{j\parallel}}{B}\right) \\
&-\frac{2\sin\theta}{Z_j eBR}\left(\overline{p}_j' - \overline{n}_j Z_j e\overline{E}_r\right) \\
= \ & 0
\end{aligned}$$

18.4 System of equations taking into account the SOL rotation (McCarthy et al)

This system is derived by **McCarthy, Drake, Guzdar and Hassam PF B5 (1993) 1188.**

In the paper **McCarthy, Drake, Guzdar and Hassam PF B5 (1993) 1188** it is reviewed the variety of regimes of relation between the *diamagnetic* velocity and *poloidal rotation* velocity.

In the H-mode the poloidal velocity can exceed the diamagnetic velocity

$$v_\theta > \frac{c_s \rho_s}{L_n}$$

It has been found that

$$\begin{aligned} E_r &> 0 \text{ outward in SOL, changes sharply to} \\ E_r &< 0 \text{ inward inside the LCFS} \end{aligned}$$

In TEXT, inside relative to the separatrix

$$\begin{aligned} E_r &\simeq -30 \ \ (V/cm) \text{, which means} \\ v_\theta &\simeq 3 \times v_{dia} \end{aligned}$$

In SOL, in TEXT, the electric field scales typically for a *plasma sheath* as derived from the condition

$$\frac{e\phi}{T_e} \sim 1$$

$$E_r \sim -\frac{\partial T_e}{\partial r}$$

and the poloidal rotation velocity is about fourtimes the diamagnetic velocity

$$v_\theta^{SOL} \sim 4 \times v_{dia}$$

In this paper it is explained that the *magnetic pumping* damps out the poloidal rotation and the pressure gradient (*i.e.* the diamagnetic flow) is balanced by a radial electric field

$$\mathbf{E} \times \mathbf{B} \sim \text{ion diamagnetic velocity}$$

However, in experiments it is found

$$v_\theta = 3...5 \ \ \times v_{dia}$$

A simplified form of the equation for the poloidal rotation

$$\frac{\partial \overline{v}_\theta}{\partial t} = -\frac{\partial}{\partial r}\left(\overline{\widetilde{v}_r \widetilde{v}_\theta}\right) + \frac{c_s^2}{R\overline{n}} 2\overline{(n \sin\theta)}$$

from where it is concluded that the Reynolds stress term is much less than the Stringer term.

NOTE

The convective cell that can be generated according to **Kuo, Shapiro**, at the plasma edge, leading to the H -mode layer, is a source of variation of the density over the surface. This is because the flow should be initiated at the hot surface, which is in the interior of the plasma and will be directed to the edge, where it follows the poloidal direction and closes at some distance (poloidally) from the equator. The flow is initiated in a region with higher density,

$$n\,(r < a) > n\,(r = a)$$

and this density is carried over the last closed magnetic surface. On this surface the enhanced density will not be uniform in the first stage of the formation of the convective cell. This non-uniformity leads to an enhancement of the Stringer effect which accelerates the flow in the poloidal direction.

See also **Rozhansky** for the angular momentum one-dimensional equation, based on transport from SOL.

18.5 The Pfirsch-Schluter flow and the basics of spin-up

For a poloidally rotating plasma, there exist flows in the toroidal direction which are necessarily associated to the poloidal rotation. **There cannot be a purely poloidally rotating plasma**.

The origin of these harmonic toroidal flows is the (approximate) *incompressibility of plasma*

$$\nabla \cdot \mathbf{v} \approx 0$$

The flux tubes rotating poloidally compress and decompress alternatively, thus driving flows along the tube in order to maintain the zero-divergence of the total flow velocity.

NOTE however that we should use the equation of continuity

$$\nabla \cdot (n\mathbf{v}) = 0$$

END.

It can be shown that there is the following connection between the two (poloidal and toroidal) averaged velocities defined above and the local velocities varying on the magnetic surface

$$v_\theta = V_{pol} \frac{R_0}{R}$$

or,

$$v_\theta (r, \theta) = \frac{V_{pol}(r)}{1 + \varepsilon \cos \theta}$$

and

$$v_\varphi \simeq V_{tor} - 2q V_{pol} \cos \theta$$

Thus, if there is a poloidal velocity $v_\theta \neq 0$ there is a harmonic component of the toroidal flow, even if there is no average toroidal flow.

NOTE an extension of this argument for the "*diamagnetic* " flow generated by the gradient of density interacting with the banana trajectories. The banana trajectories, projected on the poloidal plane, have a density that is modulated from the center $r = 0$ to the edge $r = a$ by the fraction of trapping $\sqrt{\varepsilon}$. This is an equivalent *diamagnetic* flow, poloidal, with the velocity

$$v_\theta^{banana} = \frac{\rho_s^{banana} c_s}{L_n} = \frac{(width)_{ion}^{banana} c_s}{L_n}$$

The electrons, however, have smaller flow since the for them the width of the banana is smaller. This diamagnetic-type flow must generate an equivalent Pfirsch-Schluter flow, harmonic on the meridional section.

The paper by **Nycander Yankov** shows that when the parallel flow is strong, the high value of the parallel velocity will alter the condition of trapping for many ions, some of the trapped ions will get untrapped. Then we will have a substantial change in the "equivalent Pfirsch-Schluter" harmonic current.

Actually, we should examine what happens when there is a strong toroidal rotation, in general: the ions may get a ordered motion along the bananas, much more would go in one direction on the trapping trajectories than in the opposite direction. In this case what happens is a substantial increase (actually from quasi-zero) of the banana-diamagnetic flow and correspondingly the generation of the associated Pfirsch-Schluter current.

END

18.6 Spontaneous poloidal spin-up (Hassam Drake)

See *impurity accumulation, Notes.*

It should be collected in *plasma, general, studies, Stringer.*

See [16].

When the particle diffusion is poloidally asymmetric and the local particle confinement time is shorter than the damping time of the poloidal rotation, the poloidal rotation is unstable. The plasma can spontaneously spin-up.

We should study the the connection which can exist between the plasma rotation induced by the asymmetric diffusion and the localization in poloidal direction of the perturbation of the envelope of the ITG turbulence, verifying the Nonlinear Schrodinger Equation. It might be an interplay:

- the localization induced by the NSE in the poloidal direction exerts an influence on the rate of diffusion and supports an asymmetric flux of particles;
- the plasma rotation arising from the asymmetric diffusion exerts an influence on the localization of the envelope governed by NSE.

The equations describing the plasma rotation are:

$$\frac{\partial n}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(rn\overline{v}_r\right) = 0$$

$$\frac{\partial}{\partial t}\left(nV_\varphi\right) + \frac{1}{r}\frac{\partial}{\partial r}\left[rn\left(V_\varphi\overline{v}_r - qV_\theta\widetilde{v}_r\right)\right] = 0$$

$$\frac{\partial}{\partial t}\left[V_\varphi + \Theta\left(1+2q^2\right)V_\theta\right] + \overline{v}_r\frac{\partial V_\varphi}{\partial r} - \widetilde{v}_r\frac{\partial}{\partial r}\left(qV_\varphi\right) + \text{magnetic pumping} = 0$$

The definition of the variables is

$$n = \left\langle n\left(r\right)\right\rangle$$

$$nV_\varphi\left(r\right) \equiv \left\langle nv_\varphi\frac{R}{R_0}\right\rangle$$

$$V_\theta\left(r\right) = \left\langle v_\theta\frac{R}{R_0}\right\rangle$$

and the magnetic field is

$$\mathbf{B} = \left(0,\Theta\left(r\right),1\right)\frac{B_0\left(r\right)}{R/R_0}$$

$$R = R_0 + r\cos\theta$$

$$q = \frac{r}{R_0}\Theta$$

The surface average is made according to the formula

$$\langle f \rangle = \oint \frac{d\theta}{2\pi}\frac{R}{R_0}\, f$$

The quantities $\widetilde{v}_r$ and $\overline{v}_r$ are velocities which arise from diffusive fluxes in the radial direction. They are defined from the expressions of the radial fluxes. It is known that any friction force which is exerted *in* the magnetic surface and is not parallel to the magnetic field generates a particle flux in the radial direction. The equation of motion at equilibrium

$$0 = -\boldsymbol{\nabla} p_\alpha + e_\alpha n_\alpha \mathbf{E} + e_\alpha n_\alpha \mathbf{V}_\alpha \times \mathbf{B} + \mathbf{R}_{in,\alpha}$$

which is dot-multiplied by the versor of the poloidal direction, $\widehat{\mathbf{e}}_\theta$:

$$0 = 0 + 0 + e_\alpha n_\alpha B V_{\alpha,r} + R_{in,\alpha,\theta}$$

from this formula it results

$$n_\alpha V_{\alpha,r} = -\frac{1}{e_\alpha B} R_{in,\alpha,\theta}$$

This is the **radial diffusive flux** induced by the friction force $\mathbf{R}_{in,\alpha}$ which act in the poloidal direction. This is the usual effect of the combination between a force $\mathbf{R}_{in,\alpha}$ and the magnetic field, generating a drift of the particles as $\mathbf{R}\times\mathbf{B}/B^2$.

From this formula we define the two velocities

$$\overline{v}_r = \langle V_{\alpha,r} \rangle$$

Note the fact that an average radial velocity exists is a result of the assumption that there is a non-zero averaged radial diffusion flux. **End.**

The part that has poloidal θ variation

$$\widetilde{v}_r = \langle 2\cos\theta\; V_{\alpha,r} \rangle$$

The second velocity has a trigonometric variation on the poloidal direction. This asymmetry can be induced by the poloidal variation of the diffusive flux (related to a poloidal variation of the friction force $\mathbf{R}_{in}$) or, simply because the radial diffusive flux depends on the poloidal coordinate (has poloidal asymmetry):

$$\widetilde{v}_r \sim \frac{\delta D}{L_n}$$

where D is the diffusion coefficient. Even the Pfirsch-Schluter coefficient is sufficient to induce such a poloidal variation.

Poloidal spin-up occurs when the solution of the equation for the poloidal velocity is **growing**. The equation is:

$$\Theta\left(1+2q^2\right)\left(\frac{\partial V_\theta}{\partial t}+\gamma_{MP}V_\theta\right)+qV_\theta\frac{1}{n}\frac{1}{r}\frac{\partial}{\partial r}\left(nr\widetilde{v}_r\right)=0$$

Here γ_{MP} is the **magnetic pumping damping rate of the poloidal rotation**.

18.7 Poloidal rotation induced by variation of parameters over the magnetic surface

The paper **Multiple Equilibria poloidal rotation Ware Wiley**.

The order of the time derivative

$$\omega \sim \frac{\rho_{i\theta}}{L}\Omega_{c,i}$$

is the frequency associated to adjustments between different equilibria.

The time derivative of of the toroidal velocity is very small

$$\frac{\partial \overline{V}_\varphi}{\partial t} \sim \frac{r}{R}\frac{\partial V_\theta}{\partial t}$$

and implies the same order for the time variation of the *parallel* velocity

$$\frac{\partial V_\parallel}{\partial t} \sim \frac{r}{R}\frac{\partial V_\theta}{\partial t}$$

since

$$\frac{B_\theta}{B} \sim \frac{r}{R}$$

It results

$$\frac{\partial \overline{V}_\theta}{\partial t} \simeq -\frac{1}{B}\frac{\partial E_r}{\partial t}$$

This is actually the usual equation for E_r, time-derivated and with time vatiations of the other components, v_φ and *diamagnetic*, taken zero. (From here the neoclassical polarization explanation of poloidal-toroidal connection can start).

The momentum balance on *perpendicular* on $\mathbf{B}$ direction

$$\begin{aligned} & \sum_j n_j m_j \left(\frac{\partial V_{j\perp}}{\partial t} + \left(\mathbf{V}_j \cdot \boldsymbol{\nabla} \right) V_{j\perp} \right) \\ = & -\frac{B_\varphi}{B} \frac{\partial P_\perp}{r\partial\theta} \\ & -\frac{B_\varphi}{B} \frac{\left(P_\parallel - P_\perp \right) \sin\theta}{R} \\ & -\widehat{\mathbf{e}}_\perp \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi} \\ & -J_r B \end{aligned}$$

The parallel projection of the momentum equation

$$\begin{aligned} & \sum_j n_j m_j \left(\frac{\partial V_{j\parallel}}{\partial t} + \left(\mathbf{V}_j \cdot \boldsymbol{\nabla} \right) V_{j\parallel} \right) \\ = & -\frac{B_\theta}{B} \frac{\partial P_\parallel}{r\partial\theta} \\ & +\frac{B_\theta}{B} \frac{\left(P_\parallel - P_\perp \right) \sin\theta}{R} \\ & -\widehat{\mathbf{e}}_\parallel \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi} \end{aligned}$$

From these two equations we can write a new equation, this time for the *poloidal* component of the plasma rotation

$$\begin{aligned} & n_j m_j \left(\frac{\partial V_{j\theta}}{\partial t} + \left(\mathbf{V}_j \cdot \boldsymbol{\nabla} \right) V_{j\theta} \right) \\ = & -\frac{\partial}{r\partial\theta} P_\parallel \\ & +B_\varphi^2 \frac{\partial}{r\partial\theta} \left(\frac{P_\parallel - P_\perp}{B^2} \right) \\ & +\frac{\left(P_\parallel - P_\perp \right) \sin\theta}{R} \\ & -J_r B_\varphi \\ & -\widehat{\mathbf{e}}_\theta \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi} \end{aligned}$$

See also **Zhu Horton Sugama**.

The equation of continuity for the part of the density that is varying over the magnetic surface.

The equation of continuity for the species j to first order in $\rho_{i\theta}/L$ (**Ware Wiley**)

$$\begin{aligned} & \frac{\partial \widetilde{n}_j}{\partial t} - \frac{E_r}{B}\frac{\partial \widetilde{n}_j}{r\partial\theta} + B_\theta \frac{\partial}{r\partial\theta}\left(\frac{\widetilde{n}_j \overline{V}_{j\parallel}}{B}\right) \\ & -\frac{2\sin\theta}{R}\frac{1}{Z_j e B}\left(\overline{p}_j - \overline{n}_j Z_j e \overline{E}_r\right) \\ = \; & 0 \end{aligned}$$

To first order the follwoing relationship exists between the poloidal and parallel velocities

$$\overline{V}_\theta = \frac{B_\theta}{B}\overline{V}_\parallel - \frac{E_r}{B} + \frac{p_i'}{neB}$$

Then the equation of continuity, this time for the part of the density of impurity Z that varies over the magnetic surface, similar to the eqaution written above for a general species j is

$$\frac{\partial \widetilde{n}_z}{\partial t} + B_\theta \frac{\partial}{r\partial\theta}\left(\frac{\widetilde{n}_z \overline{V}_{z\theta}}{B_\theta}\right) = 0$$

18.8 Radial current in tokamak

this is from **Novakovskii** where the formula is derived

$$\begin{aligned} \langle j_r \rangle &= \langle n v_{i,r} \rangle \\ &= \left(1 + q^2 + \frac{q^2}{\left(\varepsilon^{3/2}\nu_*\right)^{1/3}}\right)\frac{m^2c^2}{B^2}\frac{\partial E_r}{\partial t} \end{aligned}$$

This is a *polarization current*, induced by the fast time variation of the radial electric field.

A discussion about the radial current generated by a fast rise of E_r is in **Chang White icrh rotation**.

Essentially, the phases are

- E_r is created on a very short time scale, by charge separation, due for example to the banana width change at ICRH one has a displacement of the center of the banana, which means a radial current.

- The fast rise of E_r produces a *polarization* current j_r^p which is proportional with $\partial E_r / \partial t$ and the *dielectric constant* $\varepsilon_\perp$ which is very large. There is *classical* and *neoclassical* polarization. The latter is much higher since it involves the banana width.

- then the bulk plasma responds by creating a counter-current which will become equal to the polarization one

$$j^{ret} = j_r^p$$

  and will produce a *torque* on the bulk ions.

See the discussion in **Honda**.

18.9 Radial electric field in tokamak

The paper **Density Clumping** by **Shaing, Houlberg, Crume** in Comments 12 (1988) 69 discusses the effect of the *radial electric field* becoming more *negative* as a source for particle improved confinement in tokamak.

It has been observed that a more negative E_r in tokamak leads to:

1. particle confinement τ_p increases with negative E_r and

2. the impurities are accumulating in the center

This regime looks close to the H-mode. The experiments with Impurity Study Experiment (ISX-B) with **NBI injection**.

1. For **co-injection**, the electric field was least negative. Particle confinement was poor: this is so-called *density clamping*;

2. **balanced injection**: more negative E_r than above

3. **counter-injection**: most negative. Improved density confinement. (See also paper on NBI co and counter, with direct loss).

The impurities are accumulated in the center in the H mode (since E_r is more negative, see **McCarthy Drake Hassam Guzdar**). DIII-D: 48% of NBI ions are lost to the limiter.

19 Viscosity

There is a special text on *viscosity* in *plasma, general, theory.*

19.1 Introduction

The sources for the expression of neoclassical viscosity.

Shaing.

Stacey with expansions in $\sin\theta$ and $\cos\theta$ components on a magnetic surface, asymmetries connected with *Inverse Stringer effect* in my definition.

Kim Diamond.

Yushmanov Su Horton.

Hazeltine and Hinton for neoclassic.

Hirshman Sigmar.

Spontaneous Poloidal Galeev.

Viscosity Stacey.

19.2 The parallel viscosity and the radial electric field

There is a file on *VISCOSITY in General, Theory.*

The paper by **Zhu, Horton Sugama**.

The model is the $13M$ of **Grad**.

The distribution function is expanded in terms of the flows

$$\begin{aligned}\mathbf{u} &\equiv \text{flow of the fluid}\\ \mathbf{q} &\equiv \text{flow of heat}\end{aligned}$$

$$f^{(1)} = f_M \left(1 + \frac{2\mathbf{v}}{v_{thi}^2} \cdot \left[\mathbf{u}_i + \frac{2\mathbf{q}_i}{5p_i}\left(\frac{v^2}{v_{thi}^2} - \frac{5}{2}\right)\right]\right)$$

where

$$\mathbf{q}_i \rightarrow \frac{\mathbf{q}_i}{5p_i/2}$$

to have the same dimension as $\mathbf{u}_i$.

Using this distribution function the set of closed equations for momenta is obtained.

The *parallel momentum balance* equations involve the poloidal flows $\mathbf{u}_\theta$ and $\mathbf{q}_\theta$.

$$\begin{aligned}0 &= -\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\Pi}_i \rangle + \langle \mathbf{B} \cdot \mathbf{F}_{1i} \rangle\\ 0 &= -\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\Theta}_i \rangle + \langle \mathbf{B} \cdot \mathbf{F}_{2i} \rangle\end{aligned}$$

the surface averaged *stresses* balance the *friction forces* along the magnetic field direction.

Using the distribution function $f^{(1)}$ one obtains the parallel components of the *stress* expressed in terms of *poloidal* and *parallel* flows

$$\begin{pmatrix} \langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Pi}_i \rangle \\ \langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Theta}_i \rangle \end{pmatrix} = \frac{n_i m_i}{\tau_{ii}} \begin{pmatrix} \widehat{\mu}_{i1} & \widehat{\mu}_{i2} \\ \widehat{\mu}_{i2} & \widehat{\mu}_{i3} \end{pmatrix} \begin{pmatrix} \widehat{u}_{\theta i} \\ \widehat{q}_{\theta i} \end{pmatrix} \langle B^2 \rangle$$

$$\begin{pmatrix} \langle \mathbf{B} \cdot \mathbf{F}_{1i} \rangle \\ \langle \mathbf{B} \cdot \mathbf{F}_{2i} \rangle \end{pmatrix} = \frac{n_i m_i}{\tau_{ii}} \begin{pmatrix} \widehat{l}_{11}^{ij} & -\widehat{l}_{12}^{ij} \\ -\widehat{l}_{21}^{ij} & \widehat{l}_{22}^{ij} \end{pmatrix} \begin{pmatrix} \langle u_{\parallel j} B \rangle \\ \langle q_{\parallel j} B \rangle \end{pmatrix}$$

where

$$\widehat{\mu}_{aj} \text{ and } \widehat{l}_{ij}^{ab}$$

are normalized neoclassical transport coefficients.

The

parallel, and
poloidal

components

$$\widehat{u}_{\theta i} \ , \ \widehat{q}_{\theta i}$$
$$u_{\parallel j} \ , \ q_{\parallel j}$$

are connected through the equations

$$\begin{aligned} u_{\parallel} &= -\frac{T}{ZeB_\theta} \left(\frac{1}{p} \frac{dp}{dr} + \frac{Ze}{T} \frac{d\Phi}{dr} \right) + \widehat{u}_\theta (\psi) B \\ q_{\parallel} &= -\frac{1}{ZeB_\theta} \frac{dT}{dr} + \widehat{q}_\theta (\psi) B \\ u_\theta &= \widehat{u}_\theta B_\theta \\ q_\theta &= \widehat{q}_\theta B_\theta \end{aligned}$$

The *driving forces* are

$$V_1 \equiv -\frac{T}{ZeB_\theta} \left(\frac{1}{p} \frac{dp}{dr} + \frac{Ze}{T} \frac{d\Phi}{dr} \right)$$

and

$$V_2 \equiv -\frac{1}{ZeB_\theta} \frac{dT}{dr}$$

The equations for the parallel flows are

$$\begin{aligned} &\sum_{j(species)} \left[\begin{pmatrix} \widehat{\mu}_{i1} & \widehat{\mu}_{i2} \\ \widehat{\mu}_{i2} & \widehat{\mu}_{i3} \end{pmatrix} \delta_{ij} - \begin{pmatrix} \widehat{l}_{11}^{ij} & -\widehat{l}_{12}^{ij} \\ -\widehat{l}_{21}^{ij} & \widehat{l}_{22}^{ij} \end{pmatrix} \right] \begin{pmatrix} \widehat{u}_{\theta i} \\ \widehat{q}_{\theta i} \end{pmatrix} \\ = &\sum_{j(species)} \begin{pmatrix} \widehat{l}_{11}^{ij} & -\widehat{l}_{12}^{ij} \\ -\widehat{l}_{21}^{ij} & \widehat{l}_{22}^{ij} \end{pmatrix} \begin{pmatrix} \widehat{V}_{1j} \\ \widehat{V}_{2j} \end{pmatrix} \end{aligned}$$

where the new notations are

$$\widehat{V}_{1j} \equiv \frac{\langle V_{1j} B \rangle}{\langle B^2 \rangle}$$

representing the averaged *driving forces*.

19.3 General expressions

The following expressions of the parallel plasma viscosity are proposed by **Shaing** et al.

$$\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Pi} \rangle = nm \left\langle B^2 \right\rangle \left(\mu_1 U_\theta + \frac{2}{5} \mu_2 \frac{q_\theta}{P} \right)$$

$$\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Pi} \rangle = nm \left\langle B^2 \right\rangle \left(\mu_2 U_\theta + \frac{2}{5} \mu_3 \frac{q_\theta}{P} \right)$$

where the angular brakets denote the *flux surface* average. P is the plasma pressure, n is the plasma density (ion's) m is the ion mass. The notations are

$$U_\theta \equiv \frac{\mathbf{U} \cdot \boldsymbol{\nabla} \theta}{\mathbf{B} \cdot \boldsymbol{\nabla} \theta}$$

and

$$q_\theta \equiv \frac{\mathbf{q} \cdot \boldsymbol{\nabla} \theta}{\mathbf{B} \cdot \boldsymbol{\nabla} \theta}$$

where $\mathbf{U}$ is the mass flow and $\mathbf{q}$ is the heat flow. By mass flow it seems that we have to understand the effective displacement of the plasma, not the diamagnetic flow. We have then

$$\begin{aligned} U_\theta &= \left(\mathbf{U} \cdot \widehat{\mathbf{e}}_\theta \right) \frac{1/r}{B_\theta / r} \\ &= \left(\mathbf{U} \cdot \widehat{\mathbf{e}}_\theta \right) \frac{1/r}{B / (qR)} = \left(\mathbf{U} \cdot \widehat{\mathbf{e}}_\theta \right) \frac{qR}{Br} = \left(\mathbf{U} \cdot \widehat{\mathbf{e}}_\theta \right) \frac{1}{B_\theta} \\ &= \frac{V_\theta}{B_\theta} \end{aligned}$$

The equilibrium particle distribution function f_0 in the edge region is not a Maxwellian because of the existence of a direct loss region in velocity space. Shaing uses however **a shifted Maxwellian** since in the region of the velocity space which is outside the direct loss region, the distribution function is a shifted Maxwellian. The *viscosity* will be calculated as resulting from the **finite-size orbits of the particles** in the shifted Maxwellian.

The drift-kinetic equation with **a mass flow** is

$$\left(u\widehat{\mathbf{n}}+\mathbf{v}_d+\mathbf{V}\right)\cdot\boldsymbol{\nabla}f+\dot{w}\frac{\partial f}{\partial w}=C\left(f\right)$$

The drift velocity is

$$\begin{aligned}
\mathbf{v}_d &= \frac{\mathbf{F}\times\widehat{\mathbf{n}}}{\Omega}+\frac{\mu B\widehat{\mathbf{n}}}{\Omega}\left(\frac{j_{\|}}{B}\right) \\
&\quad +\frac{\widehat{\mathbf{n}}}{\Omega}\times\left[\mu\boldsymbol{\nabla}B+u_{\|}^2\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}+u_{\|}\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\mathbf{V}+\left(\mathbf{V}\cdot\boldsymbol{\nabla}\right)\left(u_{\|}\widehat{\mathbf{n}}\right)\right]
\end{aligned} \tag{5}$$

where $j_{\|}$ is the parallel current density

$$\Omega=\frac{eB}{m}$$

$$\mu=\frac{s^2}{2B}$$

The force is

$$\begin{aligned}
\mathbf{F} &= \frac{e}{m}\left(\mathbf{E}+\mathbf{V}\times\mathbf{B}\right)-\frac{\partial\mathbf{V}}{\partial t}-\left(\mathbf{V}\cdot\boldsymbol{\nabla}\right)\mathbf{V} \\
&= \frac{\boldsymbol{\nabla}p}{nm}-\frac{\mathbf{R}}{nm}
\end{aligned}$$

with $\mathbf{R}$ the friction force and the viscous force is neglected.

NOTE

The first term has the meaning of a *universal* reason for drift of particles transversal to a magnetic surface: the existence of a force $\mathbf{F}$ acting perpendicular to the magnetic field $\widehat{\mathbf{n}}$ leads to a drift $\sim\mathbf{F}\times\widehat{\mathbf{n}}$. But after that we look for the force $\mathbf{F}$ that acts on the fluid and find that the momentum equation gives a force that comes from the gradient of the pressure. Or, with this $\boldsymbol{\nabla}p$ the drift is actually the *diamagnetic* velocity.

END

The velocity s (perpendicular) and the energy $w=v^2/2$ are in the frame of the center of mass velocity.

The **energetic contribution in the drift-kinetic equation** is:

$$\begin{aligned}
\dot{w} &= \mathbf{F}\cdot\mathbf{u}-\mu B\left(\boldsymbol{\nabla}\cdot\mathbf{V}\right)-\left(u^2-\mu B\right)\left(\widehat{\mathbf{n}}\cdot\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\mathbf{V}\right) \\
&\quad +\mathbf{F}\cdot\mathbf{v}_d-\left(\frac{\mu B}{\Omega}\right)\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\times\mathbf{F}
\end{aligned}$$

We see that the time variation of the energy is due to

- the effect of the force $\times$ velocity : $\mathbf{F} \cdot \mathbf{u}$ and $\mathbf{F} \cdot \mathbf{v}_d$.
- compression of the flow, by a nonzero compressibility $\boldsymbol{\nabla} \cdot \mathbf{V} \neq 0$. *This is IN the velocity stress tensor.* This is combined with the perpendicular energy ;
- a sort of viscosity stress: difference in the **parallel** and **perpendicular** energies, $(u^2 - \mu B)$ combined with the parallel drift of the parallel divergence of the velocity $(\widehat{\mathbf{n}} \cdot \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\mathbf{V})$ which is also in the velocity stress tensor.
- a direct action of the force, by its nonzero rotational, $\boldsymbol{\nabla} \times \mathbf{F} \neq \mathbf{0}$.

The solution of the drift-kinetic equation can be written

$$f = f_{MS} + g - \left(\frac{2u}{v_{th}^2}\right) \frac{2}{5} L_1^{(3/2)} \left(\frac{q_\parallel}{p}\right) f_{MS}$$

Note. The part which differs from the *shifted maxwellian* is in the rotating system (with the velocity $\mathbf{V}$). This part must be the same as that in other solutions of the drift-kinetic equation.

Here

$$L_1^{(3/2)} = \frac{5}{2} - \frac{2w^2}{v_{th}^2}$$

is the Legendre polynomial, and

$$2/5\, L_1^{(3/2)} = 1 - 4w^2/\left(5v_{th}^2\right) = 1 - 2v^2/\left(5v_{th}^2\right)$$

For particle velocities which are much higher than the thermal velocity, the value of the $2/5 \times L$ can be negative, which gives a negative correction to the shifted maxwellian in the first order.

$$p = nT$$

is the pressure.

The drfit-kinetic equation will become an equation for g,

$$\begin{aligned}
& (u\widehat{\mathbf{n}} + \mathbf{v}_d + \mathbf{V}) \cdot \boldsymbol{\nabla} g + \dot{w}\frac{\partial g}{\partial w} - C(g) \\
= \; & 2\frac{v^2}{v_{th}^2}\left(\frac{1}{2} - \frac{3}{2}\frac{u^2}{v^2}\right)\left(\widehat{\mathbf{n}} \cdot \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\mathbf{V} - \frac{1}{3}(\boldsymbol{\nabla} \cdot \mathbf{V}) - \frac{2}{5} L_1^{(3/2)} \frac{\mathbf{q} \cdot \boldsymbol{\nabla} B}{pB}\right) f_{MS}
\end{aligned}$$

where

$$\mathbf{q} = q_\parallel \widehat{\mathbf{n}} + \mathbf{q}_\perp$$

and

$$\mathbf{q}_{\perp} = \frac{5}{2} \frac{p\widehat{\mathbf{n}} \times \boldsymbol{\nabla} T}{m\Omega}$$

(this seems to be essentially the *diamagnetic flow of the heat*).

The particle rotation velocity $\mathbf{V}$ is obtaind from:

- particle parallel velocity,
- electric field-driven ($\mathbf{E} \times \mathbf{B}$) and the
- *diamagnetic velocity divided at the particle density.*

$$\mathbf{V} = V_{\parallel}\widehat{\mathbf{n}} + \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{1}{n}\frac{1}{m\Omega}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p$$

(asta trebuie clarificata, desi este asa). Here it is more than the **mass flow**. We must compose the total velocity which multiplies $\boldsymbol{\nabla} f$ in the drift-kinetic equation.

$$\begin{aligned} u\widehat{\mathbf{n}} + \mathbf{v}_d + \mathbf{V} &= \\ &= u\widehat{\mathbf{n}} - \frac{\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p}{nm\Omega} + \mathbf{V} \\ &= \left(u + V_{\parallel}\right)\widehat{\mathbf{n}} + \frac{\mathbf{E} \times \mathbf{B}}{B^2} \end{aligned}$$

In this formula we have made use of an expression for the **particle drift velocity** which is the first term in (**??**), coming from the the "external force" $\mathbf{F}$ which generates a drift acting on the *Larmor gyration*. Any force which is perpendicular to the magnetic field generates a drift of the form $\mathbf{F} \times \widehat{\mathbf{n}}/\Omega$. Now, replacing here $\mathbf{F}$ by the gradient of the pressure divided by the density, we get a drift velocity of the particle which looks identical but with opposite sign to the diamagnetic flow velocity comming from the total flow $\mathbf{V}$.

Acest amestec de cinetic si fluid in care parti din viteza (care este solutia ecuatiei de miscare in campurile $\mathbf{E}$ si $\mathbf{B}$ date) sunt inlocuite prin expresii care provin de la fluid (consideratii de bilant de impuls) este dificil de urmarit.

It results that the **diamagnetic contributions** (one from $\mathbf{v}_d$ one from the general flow $\mathbf{V}$) cancel each other and there remains only the $\mathbf{E} \times \mathbf{B}$ flow:

$$\begin{aligned} &\left[\left(u + V_{\parallel}\right)\widehat{\mathbf{n}} + \frac{\mathbf{E} \times \mathbf{B}}{B^2}\right] \cdot \boldsymbol{\nabla} g - C(g) \\ = \;& 2\frac{v^2}{v_{th}^2}\left(\frac{1}{2} - \frac{3}{2}\frac{u^2}{v^2}\right)\left(\frac{1}{B}\left(\mathbf{V} \cdot \boldsymbol{\nabla}\right) B - \frac{2}{3}\frac{1}{n}\left(\mathbf{V} \cdot \boldsymbol{\nabla}\right) n - \frac{2}{5} L_1^{(3/2)} \frac{\mathbf{q} \cdot \boldsymbol{\nabla} B}{pB}\right) f_{MS} \end{aligned}$$

NOTE on the cancelling between the diamagnetic contributions.

It is interesting to remark that in the *mass flow*, it was originally accepted to be introduced only the effective displacement of plasma, not the diamagnetic flow. This is why in the work of **Hazeltine** on the drift-kinetic equation in the presence of large plasma rotation the velocity which multiplies the gradient of the distribution function in the Boltzmann equation is only composed of

$$v_{\parallel}\widehat{\mathbf{n}}+\mathbf{V}$$

and the **particle drift velocity** $\mathbf{v}_D$ is absent. We conclude:

When the *particle drift velocity* $\mathbf{v}_D$ is not included in the total velocity which convects $\boldsymbol{\nabla} f$, then in the *mass velocity* $\mathbf{V}$ we do not include the diamagnetic velocity.

When we include explicitely in the form of the plasma rotation velocity $\mathbf{V}$ the diamagnetic flow then it is necessary to form the total particle velocity which convects $\boldsymbol{\nabla} f$ by including the particle drift $\mathbf{v}_D$.

This shows that between the particle drift velocity (gradB, curvature) and the diamagnetic flow (gradP) it is a connection. How this can appear? The explanation is given by the form of the equation (5). We see that in this approach the "particle drift velocity" $\mathbf{v}_D$ contains much more than the gradB and curvature motions. It also contains the effect of a force which is exerted on the plasma, and this force if $\mathbf{F}$, obtained from the momentum equation divided by the density of particles. *This is not exactly what we used to do starting from the equation of gyromotion of the particle and expanding in the small Larmor radius and averaging over the gyration.* This was the typical approach and the **gradient of the pressure could not appear** in the drift velocity formula.

About diamagnetic cancellation see **Carreras,** but also **Horton 1990**.

END OF THE NOTE

NOTE on the form of the velocity which convects $\boldsymbol{\nabla} f$, in the Boltzmann equation. The following expressions are used in different contexts

$$v_{\parallel}\widehat{\mathbf{n}}+\mathbf{V}$$

$$u\widehat{\mathbf{n}}+\mathbf{v}_D+\mathbf{V}$$

etc. In fact, we have to be careful to compose correctly the formulas for the velocities. For the first form

$$v_{\parallel}\widehat{\mathbf{n}}+\mathbf{V} \text{ and we should use}$$

$$\mathbf{V} = \frac{K(\psi)}{n}\mathbf{B}+R\left(-\frac{\partial \phi}{\partial \psi}\right)\widehat{\mathbf{e}}_{\varphi}$$

Here $K(\psi) B$ is a general form of the *flux in the magnetic surface, parallel to the magnetic lines.* Divided by the density, n we have a general form of the *PLASMA* parallel velocity. This parallel flow arises from a poloidal plasma flow. In this formula we can see that the *diamagnetic contributions have been cancelled* each other.

The term $K(\psi) B$ is the famous *poloidal* part in the expression of the rotation. See **Helander 3999** and many others. The determination of $K(\psi)$ from condition of periodicity in the lower order distrbution function equation.

The part $R\left(-\frac{\partial\phi}{\partial\psi}\right)\widehat{\mathbf{e}}_{\varphi}$ is due to the **perpendicular drifts which take place in the surface**. It is NOT toroidal electric ExB flow.

(Later. Or, maybe it is exactly the $E \times B$ velocity, but within the magnetic surface?)

END OF THE NOTE

NOTE. This way to put the problem, starting with a general plasma flow, $K(\psi)$, allows to put into evidence the fact that the **plasma flow** is the *origin of the difference between the actual distribution function and the Maxwellian.*

One can say however that the drift kinetic equation starts from a different premise, compared with the standard approach, used when there is no significant plasma rotation:

- The case without plasma rotation evidences the convection of the ∇f by the **parallel** and the **drift** (gradB and curvature) particle velocities

$$\left(v_{\parallel}\widehat{\mathbf{n}}+\mathbf{v}_D\right)\cdot\nabla f$$

- The case with a general plasma flow

$$\left(v_{\parallel}\widehat{\mathbf{n}}+\frac{K}{n}\mathbf{B}\right)\cdot\nabla f$$

  (see **Shaing, Hazeltine Hinton 1976**) We remark the absence of the **drift velocity** and only the parallel components are retained as significative. *So it is assumed that the driving force for the distribution function to deviate from the Maxwellian is the plasma flow,* not the drift of the particles which convects the **radial variation of the Maxwellian.**

As a result, the expressions for the distribution function in the two cases are:

- When the deviation of $\overline{f}$ from the Maxwellian is due to the **drift of particles** $\mathbf{v}_D$,

$$f_1^{(0)} = -\left(\frac{Iv_\parallel}{\Omega}\right)\frac{\partial f_0}{\partial \psi} + g\left(\psi, \lambda, w, \sigma\right)$$

  (here the superscript (0) to the first order distribution function is related to series expansion in the small neocalssical **second** parameter, related to the time scales: bounce frequency and collision frequency).

- When the drive of the deviation from the Maxwellian is the **poloidal plasma rotation** (projected on parallel direction)

$$\overline{f} = f_M - \frac{2KBv_\parallel}{nv_{th}^2} f_M + \begin{cases} +g\left(v^2, v_\perp^2, r\right) & \text{for } v_\parallel > 0 \quad \text{passing particles} \\ -g\left(v^2, v_\perp^2, r\right) & \text{for } v_\parallel < 0 \quad \text{passing particles} \end{cases}$$

  (and g is zero for the trapped particles).

END of the NOTE.

In this formula the term has been neglected

$$\dot{w}\frac{\partial g}{\partial w} \approx 0$$

Also **the mirror force term** has been neglected, in the Pfirsch-Schluter plateau regime.

To solve the equation:

- Take a simple **Krook collision operator**

$$C\left(g\right) = -\nu_k g$$

- Group the terms on the right hand and **perform a development in Fourier series in the coordinate** θ

$$\frac{1}{B}\left(\mathbf{V}\cdot\boldsymbol{\nabla}\right)B - \frac{2}{3}\frac{1}{n}\left(\mathbf{V}\cdot\boldsymbol{\nabla}\right)n = \sum_{m\neq 0}\left(A_m \cos m\theta + B_m \sin m\theta\right)$$

$$\frac{\mathbf{q}\cdot\boldsymbol{\nabla}B}{pB} = \sum_{m\neq 0}\left(C_m \cos m\theta + D_m \sin m\theta\right)$$

- In the left hand side we must make explicit the velocity

$$\left(u + V_{\parallel}\right) \widehat{\mathbf{n}} + \frac{\mathbf{E} \times \mathbf{B}}{B^2} \tag{6}$$

here we only need the poloidal component since the gradient of g is in the poloidal direction (*absence of ripple, etc*).

$$\begin{aligned} \boldsymbol{\nabla} g &= \frac{1}{qR} \frac{\partial}{\partial \theta} g \\ &= \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \theta \frac{\partial}{\partial \theta} g \end{aligned}$$

When the velocity (6) is projected on the **poloidal direction** the first term will be simply

$$u \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \theta$$

To calculate the rest of the velocity prejected on θ, we write the total flow velocity $\mathbf{V}$ projected

$$\begin{aligned} V_p &\equiv \mathbf{V} \cdot \widehat{\mathbf{e}}_\theta = \left(V_{\parallel} \widehat{\mathbf{n}} + \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p}{nm\Omega} \right) \cdot \widehat{\mathbf{e}}_\theta \\ &= \left(V_{\parallel} \widehat{\mathbf{n}} + \frac{\mathbf{E} \times \mathbf{B}}{B^2} \right) \cdot \widehat{\mathbf{e}}_\theta + \frac{\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p}{neB} \cdot \widehat{\mathbf{e}}_\theta \\ &= (\text{projection of the parallel velocity and of the electric } E \times B \text{ velocity}) \\ &\quad + (\text{diamagnetic velocity}) \end{aligned}$$

then

$$\left(V_{\parallel} \widehat{\mathbf{n}} + \frac{\mathbf{E} \times \mathbf{B}}{B^2} \right) \cdot \widehat{\mathbf{e}}_\theta = V_p - \frac{\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p}{neB} \cdot \widehat{\mathbf{e}}_\theta$$

We obtain

$$\begin{aligned} & \left(u \widehat{\mathbf{n}} + V_{\parallel} \widehat{\mathbf{n}} + \frac{\mathbf{E} \times \mathbf{B}}{B^2} \right) \cdot \widehat{\mathbf{e}}_\theta \\ = \; & u \widehat{\mathbf{n}} \cdot \widehat{\mathbf{e}}_\theta + V_p - \frac{\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p}{neB} \cdot \widehat{\mathbf{e}}_\theta \end{aligned}$$

Next we introduce the notations

$$u = vU$$

and recompose the left hand side

$$\begin{aligned}
&\left[\left(u+V_{\|}\right)\widehat{\mathbf{n}}+\frac{\mathbf{E}\times\mathbf{B}}{B^{2}}\right]\cdot\boldsymbol{\nabla}g-C\left(g\right)\\
=&\left[\left(u+V_{\|}\right)\widehat{\mathbf{n}}+\frac{\mathbf{E}\times\mathbf{B}}{B^{2}}\right]\cdot\boldsymbol{\nabla}\theta\frac{\partial}{\partial\theta}g-\nu_{k}g\\
=&\left[u\widehat{\mathbf{n}}+V_{\|}\widehat{\mathbf{n}}+\frac{\mathbf{E}\times\mathbf{B}}{B^{2}}\right]\cdot\widehat{\mathbf{e}}_{\theta}\left|\boldsymbol{\nabla}\theta\right|\frac{\partial}{\partial\theta}g-\nu_{k}g\\
=&\left(u\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta+\left(V_{p}-\frac{\widehat{\mathbf{n}}\times\boldsymbol{\nabla}p}{neB}\cdot\widehat{\mathbf{e}}_{\theta}\right)\left|\boldsymbol{\nabla}\theta\right|\right)\frac{\partial}{\partial\theta}g-\nu_{k}g
\end{aligned}$$

- represent in Fourier series the function g

See also **Stacey**

The article **Time dependent parallel viscosity Hsu Shaing Gormley**.

The distribution function

$$\overline{f}\equiv\text{ gyro-averaged dist. function}$$

$$\begin{aligned}
\mu&\equiv\frac{v_{\perp}^{2}}{2B}\\
w&\equiv\frac{v^{2}}{2}
\end{aligned}$$

The drift-kinetic eq. for *ions* in the presence of strong rotation that can be toroidal and/or poloidal $\mathbf{V}$. The equation is derived after **Hazeltine and Ware Plasma Physics**.

$$\begin{aligned}
&\frac{\partial\overline{f}}{\partial t}+\left(v_{\|}\widehat{\mathbf{b}}+\mathbf{V}\right)\cdot\boldsymbol{\nabla}\overline{f}\\
&-\left[\mathbf{V}\cdot\boldsymbol{\nabla}\left(\mu B\right)+\mu B\left(\boldsymbol{\nabla}\cdot\mathbf{V}-\widehat{\mathbf{b}}\widehat{\mathbf{b}}\colon\boldsymbol{\nabla}\mathbf{V}\right)\right]\frac{\partial\overline{f}}{\partial\mu}\\
&+\left[v_{\|}\frac{\widehat{\mathbf{b}}\cdot\boldsymbol{\nabla}\cdot\mathbf{P}}{mn}-\mu B\left(\boldsymbol{\nabla}\cdot\mathbf{V}\right)-\left(v_{\|}^{2}-\mu B\right)\widehat{\mathbf{b}}\widehat{\mathbf{b}}\colon\boldsymbol{\nabla}\mathbf{V}\right]\frac{\partial\overline{f}}{\partial w}\\
=&\ C\left(\overline{f}\right)
\end{aligned}$$

The equation for the flux due to the strong rotation $\mathbf{V}$

$$\boldsymbol{\nabla}\cdot(n\mathbf{V})=0$$

has a general solution

$$\mathbf{V}=\frac{K(\psi)}{n}\mathbf{B}+\omega_\varphi R^2\boldsymbol{\nabla}\varphi$$

$$\begin{aligned}
\frac{K(\psi)}{n}\mathbf{B} &\equiv \text{ part of the velocity } \mathbf{V} \text{ which is in the surface along } \mathbf{B}\\
\omega_\varphi R^2\boldsymbol{\nabla}\varphi &= -\frac{\partial\phi(\psi)}{\partial\psi}R^2\boldsymbol{\nabla}\varphi=-\frac{\partial\phi}{\partial r}\frac{1}{RB}R^2\frac{1}{R}\widehat{\mathbf{e}}_\varphi\\
&= \frac{1}{B}\left(-\frac{\partial\phi}{\partial r}\right)\widehat{\mathbf{e}}_\varphi\equiv \ (E\times B)/B^2 \text{ velocity perp. drift in the surface}
\end{aligned}$$

The ion stress tensor

$$\begin{aligned}
\mathbf{P} &\equiv \int d^3v\ m\mathbf{v}\ \mathbf{v}\ \overline{f}\\
\mathbf{P}_{ij} &= nT\delta_{ij}+\frac{3}{2}\pi_\parallel\left(\widehat{b}_i\widehat{b}_j-\frac{1}{3}\delta_{ij}\right)
\end{aligned}$$

The equation in which we insert this quantities

$$\begin{aligned}
&\frac{\partial\overline{f}}{\partial t}+\left(v_\parallel+\frac{K}{n}B\right)\nabla_\parallel\overline{f}\\
&+v_\parallel\left[\frac{\widehat{\mathbf{b}}\cdot\boldsymbol{\nabla}\cdot\mathbf{P}}{mn}-\nabla_\parallel\frac{v_\parallel KB}{n}\right]\frac{\partial\overline{f}}{\partial w}\\
=\ & C\left(\overline{f}\right)
\end{aligned}$$

Shaing Hsu Gromley assert that K is the *driving force* that induces a difference between $\overline{f}$ and a Maxwellian.

The scales

$$\begin{aligned}
1 &\gg \frac{\frac{K}{n}B}{v_{i,th}}\gg\Delta\\
&\widetilde{T}(\theta,t)\ \text{ and } \widetilde{n}(\theta,t)\ \text{ are of order } \Delta
\end{aligned}$$

It is defined

$$\begin{aligned}
\nu_p &= -\frac{\partial}{\partial t}\ln\overline{f}\\
&\sim \left[s^{-1}\right]
\end{aligned}$$

to describe the rotation damping due to magnetic pumping

$$\overline{f} = f_M + \widetilde{f}$$

$$f_M = \frac{n(\psi)}{\pi^{3/2} v_{i,th}^3} \exp\left[-\frac{mw}{T(\psi)}\right]$$

The following observations

$$\begin{aligned} \omega_\varphi \equiv\ & E \times B \text{ velocity in the surface} \\ & \text{coming from the velocity perpendicular on } \mathbf{B} \\ & \text{but directed along } \widehat{\mathbf{e}}_\varphi \end{aligned}$$

This velocity can be absorbed in the Maxwellian distribution in an axisymmetric system.

Only K is the driving of a difference between $\overline{f}$ and f_M.

The equation for this difference $\widetilde{f}$ is

$$\begin{aligned} & -\nu_p \widetilde{f} + v_\parallel \nabla_\parallel \left(\widetilde{f} + \frac{2KBv_\parallel}{n v_{th}^2} f_M\right) \\ = \ & \frac{\widehat{\mathbf{b}} \cdot \boldsymbol{\nabla} \cdot \mathbf{P}}{\overline{p}} v_\parallel f_M \\ & + C\left(\widetilde{f}\right) \end{aligned}$$

The solution.

To the zeroth order

$$\begin{aligned} \widetilde{f}_0 &= -\frac{2KBv_\parallel}{n v_{th}^2} f_M \\ & \quad + \sigma g(w, \mu, \psi) \\ \text{where } \sigma &\equiv sign\left(v_\parallel\right) \end{aligned}$$

To the first order in Δ, $O(\Delta)$

$$\begin{aligned} v_\parallel \nabla_\parallel \widetilde{f}_1 &= \frac{\widehat{\mathbf{b}} \cdot \boldsymbol{\nabla} \cdot \mathbf{P}}{\overline{p}} v_\parallel f_M \\ & + \nu_p \widetilde{f}_0 \quad \left(\text{this is } \frac{\partial f}{\partial t} \text{ due to damping}\right) \\ & + C\left(\widetilde{f}_0\right) \end{aligned}$$

Here we will replace the expression of the distribution function at zeroth order, $\widetilde{f}_0$ which contains the function g.

We will use the *periodicity* in θ to obtain a constraint from where we get g.

To calculate the function g we must take the average of the above equation over the bounce of ions. This will eliminate the sign of the parallel velocity.

First we devide by $v_{\parallel}$ and multiply with B:

$$\frac{B}{v_{\parallel}} \times \text{[Equation]}$$

and then integrate over the magnetic surface. *This is the average over the bounce.*

$$\begin{aligned}
& \frac{\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \mathbf{P} \rangle}{\overline{p}} f_M \\
& + \nu_p \left(-\frac{2K \langle B^2 \rangle}{n v_{th}^2} f_M + \left\langle \frac{B}{|v_{\parallel}|} \right\rangle g \right) \\
& + \left\langle \frac{B}{|v_{\parallel}|} C(g) \right\rangle \\
= & \ 0
\end{aligned}$$

$$g(w, \mu, \lambda) \equiv 0 \text{ in the trapping region}$$

A detailed treatment of the collision operator.

$$C[P_l(\xi) \phi(v)] = P_l(\xi) C^l [\phi(v)]$$

This can be found also in **Taguchi**.

The collisional operator is *linearized* and *expanded* as

$$\begin{aligned}
C(f) &= 2\nu(v) h |\xi| \frac{\partial}{\partial \lambda} \lambda |\xi| \frac{\partial}{\partial \lambda} f \\
& + \sum_{l=0}^{\infty} P_l(\xi) \left[C^l \left(\widehat{f_l} \right) + \frac{l(l+1)}{2} \nu(v) \widehat{f_l} \right]
\end{aligned}$$

where

$$\lambda \equiv \frac{\mu B}{w} = \frac{v_{\perp}^2}{v^2}$$

$$\xi = \frac{v_{\parallel}}{v}$$

$$h \equiv \frac{B_0}{B} = 1 + \frac{r}{R}\cos\theta$$
$$P_l(\xi) \equiv \text{Legendre polynomials}$$
$$\widehat{f}_l \equiv \frac{2l+1}{2}\int_{-1}^{1} d\xi P_l(\xi)$$
$$x \equiv \frac{v}{v_{th}}$$
$$\nu(v) \equiv \frac{3\sqrt{2\pi}}{4}\nu_{ii}\frac{\phi(x)}{x^3}$$
$$\phi(x) \equiv \left(1 - \frac{1}{2x^2}\right)\text{erf}(x) - \frac{1}{\sqrt{\pi}}\frac{\exp(-x^2)}{x}$$

(see **Novakovskii Liu Sagdeev Rosenbluth** where the collision operator is in the equation for the distribution function with FAST time dependence $\frac{\partial}{\partial t} \sim \omega_{bounce}$, for polarization, GAM, magnetic pumping).

In the paper **Shaing Callen PF26 (1983) 3315** the viscosity is calculated

$$\begin{aligned} \langle \mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_a \rangle &= \left\langle \left(p_{\perp a} - p_{\| a}\right)\widehat{\mathbf{n}}\cdot\nabla B \right\rangle \\ &= \left\langle \widehat{\mathbf{n}}\cdot\nabla B \int d^3 v m_a \left(\frac{v_\perp^2}{2} - v_\|^2\right) f_b \right\rangle \end{aligned}$$

(See also **Zhu Horton Sugama**).

To calculate this integral one introduces two space-coordinate variables

$$\begin{aligned} V &\equiv \text{volume inside a magnetic flux surface} \\ \beta &\equiv \frac{d\psi}{dV}\theta - \frac{d\chi}{dV}\zeta \end{aligned}$$

where

$$\begin{aligned} \frac{d\psi}{dV} &\equiv \text{density of toroidal flux} \\ &= \mathbf{B}\cdot\nabla\zeta \\ \frac{d\chi}{dV} &\equiv \text{density of poloidal flux} \\ &= \mathbf{B}\cdot\nabla\theta \end{aligned}$$

with (θ, ζ) angle variables, roughly poloidal and toroidal angles. The magnetic field is

$$\mathbf{B} = \nabla V \times \nabla\beta$$

when β is taken as the variable that measures the distance along the magnetic field line.

The differential volume in the *velocity space* is expressed in terms of the new variables: (E,μ) (note that without ζ on which we have integrated out for *gyroaveraging*).

$$d^3v = \frac{dE\ d\mu}{\varepsilon_0 m_a^2}\frac{B}{\left|v_\parallel\right|}$$

or

$$d^3v = \frac{1}{\varepsilon_0 m_a^2}\frac{B}{\left|v_\parallel\right|}dEd\mu$$

Then the integral becomes

$$\begin{aligned}\langle \mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_a\rangle &= \\ &= \oint d\beta d\zeta \int \frac{1}{\varepsilon_0 m_a^2} dE\ d\mu\ m_a \left[\frac{1}{\left|v_\parallel\right|}\frac{\partial B}{\partial \zeta}\left(\frac{v_\perp^2}{2}-v_\parallel^2\right)\right] f_b\end{aligned}$$

The factor in the square brackets is a total derivative

$$\frac{1}{\left|v_\parallel\right|}\frac{\partial B}{\partial \zeta}\left(\frac{v_\perp^2}{2}-v_\parallel^2\right) = -\frac{\partial}{\partial \zeta}\left(\left|v_\parallel\right| B\right)$$

Then, since the distribution function does not depend on ζ in the surface, the integration can be carried out and gives

$$\langle \mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_a\rangle = 0$$

In the same paper

$$p_\parallel - p_\perp = 3n\tau\left[\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\left(\widehat{\mathbf{n}}\cdot\mathbf{u}\right) - \left(\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right)\cdot\mathbf{u}\right]$$

or

$$p_\parallel - p_\perp = 3n\tau\left[\nabla_\parallel u_\parallel - \left(\nabla_\parallel \widehat{\mathbf{n}}\right)\cdot\mathbf{u}\right]$$

and if we take

$$\begin{aligned}\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}} &\approx -\frac{1}{R}\widehat{\mathbf{e}}_R \\ &\quad\text{(directed toward the major axis))}\end{aligned}$$

then

$$p_\parallel - p_\perp = 3n\tau\left[\nabla_\parallel u_\parallel + \frac{1}{R}u^{axis}\right]$$

where τ is the ion-ion collision time.

$$p_{\|} - p_{\perp} = 3n\tau \left[\nabla_{\|} u_{\|} - \frac{\widehat{\mathbf{e}}}{R} \cdot \mathbf{u} \right]$$

where $\widehat{\mathbf{e}}$ is the versor of the curvature direction, which is directed toward the axis of symmetry of the torus.

Note The inertia of poloidal rotation is calculated in terms of an effective mass by **Galeev Sagdeev** 1996. And in **Hassam Drake Kleva** on Stringer, in *impurity accumulation, Notes.*

The *viscosity* leading to damping of the poloidal rotation in **Stacey Neoclassical Poloidal Rotation** is

$$\mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_j = 3 \left\langle (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} B)^2 \right\rangle \left(qRn_j m_j v_{thj} \frac{\nu_{*j}}{(1+\nu_{*j})(1+\varepsilon^{3/2}\nu_{*j})} \right) \frac{\mathbf{v} \cdot \mathbf{B}_\theta}{B_\theta^2}$$

where

$$\left\langle (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} B)^2 \right\rangle = \frac{1}{2} \left(\frac{\varepsilon}{qR} \right)^2 B^2$$

where

$$\varepsilon \equiv \frac{r}{R}$$

The result is geometric

$$\begin{aligned} \left\langle \left(\nabla_{\|} B \right)^2 \right\rangle &= \frac{1}{2} \left(\frac{\varepsilon}{qR} \right)^2 B^2 \\ &= \frac{1}{2} \left(\frac{1}{R} \frac{B_\theta}{B_T} \right)^2 B^2 \\ &\approx \frac{1}{2} \left(\frac{B_\theta}{R} \right)^2 \end{aligned}$$

(see also *geometry magnetic field.tex*).

The paper **Poloidal Rotation Stacey 1992** presents the density asymmetries on magnetic surfaces together with neoclassical viscosity.

19.4 Variation of plasma parameters on a magnetic surface

There is a file in *General, Theory.*

This subject is closely related to the viscosity and plasma rotation, toroidal and poloidal.

It has a *neoclassic* component, as Inverse Stringer Effect in my definition.

Paper **Role Flow Shear**.

Stacey makes detailed calculations of such asymmetries on the surface, based on an expansion in $\sin\theta$ and $\cos\theta$.

19.5 About plasma rotation velocity and viscosity (Shaing, Crume, Houlberg)

Let us consider the plasma velocity

$$\mathbf{U} = U_{\parallel}\widehat{\mathbf{n}} + \mathbf{U}_{\perp}$$

The perpendicular plasma velocity has two components:

- the radial electric field x B, *i.e.* $\mathbf{E} \times \mathbf{B}$ flow;
- the **diamagnetic** flow.

Then

$$\mathbf{U}_{\perp} = E_r \frac{\widehat{\mathbf{e}}_r \times \widehat{\mathbf{n}}}{B} + \frac{1}{nm\Omega}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p$$

Multiplying the equation of definition of $\mathbf{U}$ by versors in the poloidal and in the toroidal direction, we get

$$U_p = U_t\left(\frac{B_p}{B}\right) - \frac{E_r}{B} + \frac{1}{nm\Omega}\frac{dP_i}{dr}$$

From this formula it results that **the poloidal rotation velocity** becomes more positive when **the radial electric field** becomes more positive.

The contribution of U_t to the radial electric field E_r is a fraction of B_p/B from the contribution of B_p when the toroidal U_t and poloidal U_p velocities are equal.

This shows that it is the poloidal flow which can strongly influence the value of the radial electric field.

The determination of the poloidal plasma velocity is done by solving the momentum equation.

In this momentum equation the poloidal momentum is damped by the poloidal (parallel) viscosity

$$\langle \mathbf{B}_p \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Pi} \rangle$$

In the neoclassical theory this is all we need, since there is no source or sink of momentum. In practial situation of the tokamak there are *orbit losses* which provide additional sources in the equation close to the edge.

In order to calculate $\langle \mathbf{B}_p \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Pi} \rangle$ we need to solve the *drift-kinetic* equation.

$$\langle \mathbf{B}_p \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Pi} \rangle = \frac{\sqrt{\pi}}{4} \frac{\varepsilon^2}{r} nm\, v_{thi} B \left(I_p U_p + I_T U_{p0} \right)$$

where

$$U_{p0} = -\frac{\rho_i v_{thi}}{2} \frac{1}{T_i} \frac{dT_i}{dr}$$

and the integrals are

$$\begin{aligned} \left. \begin{array}{c} I_p \\ I_T \end{array} \right\} &= \frac{1}{\pi} \int_0^{\nu_{*i}^{1/2}} dx \left\{ \begin{array}{c} 1 \\ \left(\frac{5}{2} - x \right) \end{array} \right\} x^2 \exp\left(-x\right) \\ &\times \int_{-1}^{1} d\left(\frac{v_\parallel}{v}\right) \left[1 - 3 \left(\frac{v_\parallel}{v} \right)^2 \right]^2 \frac{\nu_{*i} \varepsilon^{3/2} \left(\frac{\nu_T}{\nu \sqrt{x}} \right)}{\left(\frac{v_\parallel}{v} + U_{p,m}/\sqrt{x} \right)^2 + \left[\nu_{*i} \varepsilon^{3/2} \left(\frac{\nu_T}{\nu\sqrt{x}} \right) \right]^2} \end{aligned}$$

The other notations

$$U_{p,m} = U_p \frac{B}{v_{thi} B_p} + \lambda_p / 2$$

$$\lambda_p = -\rho_{pi} \left(\frac{1}{P} \frac{dP}{dr} \right)$$

$$\begin{aligned} \nu_{*i} &= \nu \frac{Rq}{v_{thi} \varepsilon^{3/2}} \\ &= \frac{\nu}{\left(v_{th,i} / Rq \right)} \varepsilon^{-3/2} \end{aligned}$$

$$\frac{v_\parallel}{v} = \cos(\text{pitch angle})$$

$$\nu_T = \text{the collisional rate of anisotropy relaxation}$$

The term $\lambda_p/2$ will cancel the diamagnetic flow, such that only the **radial electric field** $\times$ **magnetic field** and the **parallel flow** will remain.

20 Rotation decay through ion bulk viscosity

20.1 Introduction

In the text **Soliton NSE Self Modulation** I find a reference :

The decay rate of the poloidal rotation by this mechanism is [**?**]:

$$\gamma_{MP} \simeq \frac{3}{4}\left(1+\frac{1}{2q^2}\right)^{-1}\left(\frac{l}{qR}\right)^2 \nu_{ii} \tag{7}$$

where l is the mean-free path.

And this is later calculated giving an estimate of

$$\gamma_{MP} = 12 \ \left[s^{-1}\right]$$

(*Later* Not correct, much higher 10^4).

There is another reference, mentioned by **Taguchi**, it is: **Hirschman 1978**.

The paper **Neutral H Mode Peeters** gives the following decay coefficient

$$\gamma_{MP} = \frac{\langle B_\theta \rangle \langle \mathbf{B}_\theta \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_{NC} \rangle}{\langle B_\theta^2 \rangle m_i n_i \langle V_\theta \rangle}$$

20.2 The general schema of the problem (Stix 1973)

One starts with an initial rotation of the plasma or, equivalently, with a radial electric field. No need to specify the origin of these.

initial E_r and Plasma Rotation

This rotation is executed in a modulated magnetic field, which generates dissipation through two mechanisms:

- collisions
- transit time magnetic pumping

These mechanisms of dissipation induce a *radial current.*

NOTES

The ("average") plasma rotation induces *a radial current.* This is generated by the standard mechanism of the difference between the motions of ions and electrons sustained by the continuous alimentation of the density in a point due to the average motion of particles. This is necessary since otherwise the number of pairs ion/electron arriving at a certain point from one direction would be equal to the number of pairs of ion/electron arriving from the other direction. Now, at the current point, the pair comming from one direction separates (: in opposite direction on the local radial direction, due to the guiding centre neoclassical drift) with different velocities. The pair comming from the other direction will also separate in opposite radial direction, and in opposition relative to the first pair **NO**. If the distribution function would be symmetric, the number of ions passing through a surface at a particular radius in one sense on the radius would be equal to the number passing in the opposite sense. In order to have a current we need these numbers to be unequal, *i.e.* we need a distribution function which is not symmetric in $v_{\|}$. This is achived if the plasma particles move along the magnetic line with in one direction, so that an average parallel velocity exists. It is thus related to the existence of an average *parallel* motion of both particles. But also the poloidal motion (due to the initial rotation = initial electric field) and the *diamagnetic* rotation can sustain the mechanism of radial current generation. **Note**. The continuous flow of particles arriving at a point has the same effect as

- the ionization of the neutral *beam-injected* at that point
- the local heating of the ion by ICRF heating (put energy in the perpendicular ion motion means to increase its radial deviation from the magnetic surface due to the drift);

This radial current interacts with the magnetic field and produces a force which acts on the plasma. The plasma slows down and the rotation decays.

END OF THE NOTES

It is studied : **The momentum decay of a low-β plasma moving through a spatially periodic magnetic field**. Due to the motion the plasma sees in its rest frame *a time-varying magnetic field* and is therefore subject to the dissipative process of **magnetic pumping**.

The dissipation arises from the **bulk ion viscosity** and has components

- collisional
- non-collisional

which means that there is also dissipation of the nature of *Landau damping* (not transforming the motion into heat but in incoherent motions).

The viscous drag slows down the plasma motion across the field irregularities and **the electric field originally responsible for the motion is wiped out by the sum of the viscosity-induced and polarization-induced cross-B currents**.

A list of possible sources of *radial electric field*:

1. plasma heating expands the orbits of ions more than those of the electrons; the width of an orbit is proportional with the drift velocity, which is $\sim v_{\perp}^2$;

2. turbulent transport may be non-ambipolar

3. imperfect magnetic surfaces may allow electrons to escape

4. α particles released from nuclear reactions near the edge of a reactor plasma can be carried out of the plasma by their **large Larmor radius** or by their **banana orbits**

5. the orbits of the **fast ions** deposited in a plasma by NBI deviate from the magnetic surface (where they are produced by ionization) by displacements as large as the banana width.

The picture is as following:

- there is an *initial radial electric field* E_r

- due to this electric field there is a *plasma flow*; this flow is in the magnetic surface;

- due to this plasma flow it arises a *radial current*; this current will be calculated as *space averaged*; **this space-averaged radial current is second order in the amplitude of the magnetic field perturbation**.(Nota: it is question of the **variation** of the magnetic field intensity along a magnetic line, wiewed as a **perturbation**). The amplitude of this variation is

$$\lambda \sim \frac{\Delta_{\parallel} B}{B} \sim \frac{r}{R}$$

- the total current is composed of **viscous** and **polarization** parts.

NOTA. The most important part of the work is to calculate the *radial current* density, j_r. Its value arises from the difference of the radial drifts of the electrons and the ions. The generation of this current and the interaction of this current with the magnetic field (leading to a force which stops the plasma rotation) is characterized with the name: **viscosity**. Using this name is probably justified because there is transfer of momentum in transversal and parallel directions, convected by the motion of the plasma.

The plasma undergoing *magnetic pumping* dissipates via **ion viscosity** the kinetic energy of the rotation and heat will be produced. This is way the time for decay of the initial rotation is of the order of the **ion-ion collision time**. But, in the less collisional regimes (so-called long mean-free path regimes) the trapped particles operates the **decay** of the initial motion, through a mechanism similar to Landau damping.

There is a Lagrangian for the particles in inhomogeneous magnetic field:

$$\begin{aligned} L &= \frac{1}{2}\sum_i m_i \mathbf{v}_i^2 + \sum_i q_i \mathbf{v}_i \cdot \mathbf{A}_{ext}\left(\mathbf{r}_i, t\right) - \sum_i q_i \phi_{ext}\left(\mathbf{r}_i, t\right) \\ &\quad + L_{int} \end{aligned}$$

where L_{int} is the part of the Lagrangian due to the interaction between the particles

$$L_{int} = \frac{1}{2}\sum_{i,j\ i\neq j} \frac{q_i q_j}{r}\left[-1 + \frac{1}{2}\left(\mathbf{v}_i \cdot \mathbf{v}_j + \frac{\left(\mathbf{v}_i \cdot \mathbf{r}\right)\left(\mathbf{v}_j \cdot \mathbf{r}\right)}{r^2}\right)\right]$$

and

$$r \equiv \left|\mathbf{r}_i - \mathbf{r}_j\right|$$

From this coordinates $\mathbf{r}_i$ one can go to generalized coordinates appropriate for the geometry under investigation

$$\left(\xi_i, \dot{\xi}_i\right)$$

and write the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\xi}_i} - \frac{\partial L}{\partial \xi_i} = 0$$

To take into account the geometry, one introduces the Lamé coefficients (from the elementery differential form of distance ds^2): h_ξ as in

$$v_\xi = h_\xi \dot{\xi}$$

The interaction term makes no contribution to the second term in the Euler-Lagrange equation:

$$\sum_k \frac{\partial L}{\partial \xi_k} = 0$$

and it results

$$\begin{aligned} \frac{d}{dt}\sum_k \left[m_k h_\xi^2 \dot{\xi}_k + q_k h_\xi A_\xi \left(\mathbf{r}_k, t \right) \right] &= \\ &= \sum_k \frac{\partial}{\partial \xi_k} q_k \mathbf{v}_k \cdot \mathbf{A}_{ext} - \sum_k \frac{\partial}{\partial \xi_k} q_k \phi_{ext} \end{aligned}$$

where A_ξ is the magnetic potential due to the **external** sources *and* to all the particles except the k-th particle.

The **momentum conservation law** is connected with the case where at leats one of the coordinates of the system is ignorable. We can consider the external potential independent of the coordinate ξ, *i.e.*:

$$h_\xi A_{ext\xi} \ , \ h_\eta A_{ext\eta} \ , \ h_\zeta A_{ext\zeta} \ \text{ are independent of } \xi$$

this gives

$$\frac{d}{dt}\sum_k \left[m_k h_\xi^2 \dot{\xi}_k + q_k h_\xi A_{ext\xi} \left(\mathbf{r}_k, t \right) \right] = 0$$

NOTE. The last term of this formula has not an obvious physical interpretation: it is the product of the particle's charges with the magnetic potential.

From this one gets

$$\begin{aligned} \frac{d}{dt}\sum_k m_k h_\xi^2 \dot{\xi}_k &= -\frac{\partial}{\partial t}\int \rho h_\xi A_\xi \left(\mathbf{r}, t \right) d^3 r \\ &= \int \left(\boldsymbol{\nabla} \cdot \mathbf{j} \right) h_\xi A_\xi \left(\mathbf{r}, t \right) d^3 r + \int \rho h_\xi E_{induction\xi} d^3 r \end{aligned}$$

NOTE. It appears that only a nonvanishing *divergence of the radial current* can sustain a time variation of the poloidal plasma momentum. This calls for a dynamic phenomenon, where the charges are accumulating at each point, as in a transient process of polarization. However, at the end of the calculation, one gets a simple $\mathbf{j} \times \mathbf{B}$ force acting of plasma, and the **non-vanishing divergence of the current density** is no more visible.

Actually, what is more important in this formula is the non-vanishing of the **time derivative of the charge density**. This is correlated with one of the following processes:

- creation of charges from: polarization (very small relative displacement of positive and negative charges); or from the difference in the motion of the particles just created in some spatial point;

- regular displacement = flow of charges (of only one sign is sufficient) but variation of the flow due to geometry. If plasma rotates there is a periodic concentration and dilation of charges in the magnetic surface.

NOTE. It is also possible that the **divergence of the current density arises simply from the toroidal geometry**. But in a slab, there is no other possibility than accepting a transient process where charges are created at every point in the plasma, or they move one relative to the other, as in polarization.

If the coordinate of symmetry, ξ is the toroidal angle, φ then

$$m_k h_k^2 \dot{\xi}_k \rightarrow mR^2 \dot{\varphi} = mRv_\varphi$$

which is the particle's **kinetic momentum** in the toroidal direction, a vector which is along the main axis of the torus, oriented upward. The conservation equation is

$$\frac{d}{dt} \sum \left(mR^2 \dot{\varphi} \right) = \text{torque on the plasma, from a force in the } \varphi \text{ direction}$$

The last term contains the *charge density* which is allways zero on the scales of interest.

$$\frac{d}{dt} \sum_k m_k h_\xi^2 \dot{\xi}_k = - \int \mathbf{j} \cdot \boldsymbol{\nabla} \left(h_\xi A_\xi \right) d^3 r \tag{8}$$

after an integration by parts. **This is a relation between the NON-AMBIPOLAR flow of particles j and the plasma acceleration in the direction of the ignorable coordinate**.

There is a relation between the time-derivative of the radial electric field and the radial current (across the magnetic surfaces).

$$\frac{1}{\mu_0} \left. \left(\boldsymbol{\nabla} \times \mathbf{B} \right) \right|_r = j_r + \varepsilon_0 \frac{\partial E_r}{\partial t}$$

If this relation is averaged over the magnetic surface,

$$\langle j_r \rangle = \frac{\text{total current } J_r}{\text{surface}} = \frac{J_r}{(2\pi R)(2\pi r)}$$

$$0 = \frac{J_r}{(2\pi R)(2\pi r)} + \varepsilon_0 \frac{\partial \langle E_r \rangle}{\partial t}$$

Taking

$$h_\phi = R + r\cos\theta$$

$$A_\phi = -\frac{R}{R + r\cos\theta}\int^r b(r)\,dr$$

$$B_\theta = \frac{R}{R + r\cos\theta} b(r)$$

$$d^3r = rdr\,d\theta\,(R + r\cos\theta)\,d\phi$$

The total current crossing a magnetic surface is

$$J_r(r) = \iint j_r(r,\theta,\phi)\,rd\theta\,(R + r\cos\theta)\,d\phi$$

returning to the equation (8) we obtain

$$\frac{d}{dt}\sum_k m_k (R + r\cos\theta)^2 \dot{\phi} = R\int J_r(r)\,b(r)\,dr$$

This relation expresses the connection between the change of the *toroidal angular kinetic momentum* of plasma to the *total torque* around the major axis due to the force $\mathbf{j}\times\mathbf{B}_\theta$. (Este vorba de momentul cinetic, mvr sau $m\omega r^2$).

This a **force relation**: $\mathbf{j}\times\mathbf{B}$ is a *force*: the plasma *is accelerated*, there is no stationarity at this moment.

20.3 The connection between the toroidal and poloidal total momenta

The definitions

$$L_{tor} = \sum_k m_k (R + r\cos\theta)^2 \dot{\phi}$$

$$L_{pol} = \sum_k m_k r_k^2 \dot{\theta}$$

We note

$$\sin\alpha = \frac{b}{B}$$

Taking the particle drift as the electric drift

$$\mathbf{E} + \mathbf{v}\times\mathbf{B} = \mathbf{0}$$

we obtain the relation which connects the *time variations* of the **angular kinetic momenta** is

$$\left(1+\sin\alpha\frac{bB}{4\pi n}\right)\frac{d}{dt}L_{tor}=\frac{R}{r}\frac{bB}{4\pi n}\cos\alpha\frac{d}{dt}L_{pol}$$

which shows that: **the change in the poloidal angular kinetic momentum P_{pol} has a very weak influence on the toroidal angular kinetic momentum.**

(What means that? This suggests that a change in the poloidal rotation cannot be transfered to the toroidal one? Then we will have a difficulty to explain the important toroidal rotation when the vorticity is pinching).

20.4 The drift-kinetic equation

The general form of the drift-Boltzmann equation

$$\frac{\partial f}{\partial t}+\mathbf{v}_D\cdot\boldsymbol{\nabla}f+q\frac{\partial\phi}{\partial t}\frac{\partial f}{\partial\varepsilon}=C\left(f\right)$$

where the variables of the distribution function imposed by the operations of *gyro-phase averaging* of the Boltzmann equation are

$$f=f\left(\varepsilon,\mu,\mathbf{r},t\right)$$

with

$$\begin{aligned}\varepsilon &= \frac{mv^2}{2}+q\phi\\ \mu &= \frac{mv_\perp^2}{2B}\end{aligned}$$

NOTE
The drift velocity v_D actually contains the parallel velocity $v_\parallel\widehat{\mathbf{n}}$.
END OF NOTE

NOTE. It is necessary to observe that the *neoclassical* study which is developed by Stix relays on: convection of the distribution function by the drift of the particles and **energetic effect associatied to the variation of the electric potential**. So, the fact that the particle moves in a spatially nonuniform magnetic field will have as consequence the appearence of an energy change due to the variation of the electric potential in which the particle evolves. The electric potential is changing in time. By the change of variables, the potential variation in time will formally disappear from the equation for the distribution function.

It is interesting to note that this term does not appear in the treatments based on the **drift-kinetic equation with a strong plasma rotation** (Hinton and Chang, Hazeltine and Ware, Shaing). When we keep the time variation of ϕ, the only way to interact with this time variation of ϕ is the motion of the particle, $Wk_x + Vk_z$ in Stix. When the velocity is already present, there is no need to mention that.

END of NOTE.

In the equation, the partial derivatives are performed *holding five of the six variables* $(\varepsilon, \mu, \mathbf{r}, t)$ constant. The velocities are

$$\mathbf{v}_D = \widehat{\mathbf{n}} v_\parallel + \frac{1}{\Omega} \widehat{\mathbf{n}} \times \left[\frac{1}{m} \mu \boldsymbol{\nabla} B + v_\parallel^2 \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} + \frac{q}{m} \boldsymbol{\nabla} \phi \right]$$

where

$$v_\parallel = \left[\frac{2}{m} \left(\varepsilon - \mu B - q\phi \right) \right]^{1/2}$$

Changing to the system of coordinates

$$(\varepsilon, \mu, \mathbf{r}, t) \rightarrow \left(\frac{v_\perp^2}{2}, v_\parallel, \mathbf{r}, t \right)$$

the operators becomes

$$\left. \frac{\partial}{\partial t} \right|_{\varepsilon, \mu, \mathbf{r}} = \left. \frac{\partial}{\partial t} \right|_{v_\perp, v_\parallel, \mathbf{r}} - \frac{q}{m} \frac{\partial \phi}{\partial t} \left. \frac{\partial}{\partial \left(v_\parallel^2 / 2 \right)} \right|_{v_\perp, v_\parallel, \mathbf{r}}$$

and

$$\begin{aligned} \left. \boldsymbol{\nabla} \right|_{\varepsilon, \mu, t} &= \left. \boldsymbol{\nabla} \right|_{v_\perp, v_\parallel, t} + \\ & \frac{\mu \boldsymbol{\nabla} B}{m} \left. \frac{\partial}{\partial \left(v_\perp^2 / 2 \right)} \right|_{v_\perp, v_\parallel, t} - \left(\frac{\mu \boldsymbol{\nabla} B + q \boldsymbol{\nabla} \phi}{m} \right) \left. \frac{\partial}{\partial \left(v_\parallel^2 / 2 \right)} \right|_{v_\perp, v_\parallel, t} \end{aligned}$$

and

$$q \frac{\partial \phi}{\partial t} \left. \frac{\partial}{\partial \varepsilon} \right|_{\mu, \mathbf{r}, t} = \frac{q}{m} \frac{\partial \phi}{\partial t} \left. \frac{\partial}{\partial \left(v_\parallel^2 / 2 \right)} \right|_{v_\perp, v_\parallel, t}$$

NOTE

This calculation is very instructive. It shows that we have to consider the space operators acting on the distribution function as being connected with the dependence of the distribution function with respect to the energetic variables, the parallel energy and the perpendicular energy. It is normal, since the variation of the magnetic field induces a variation of energy for every particle moving in this geometry. The way to take this into account is to make a change of variables in the **space**-differential operator.

END OF NOTE

(**NOTA**: this change of variable is different of that which is performed just before the gyro-phase averaging. There the variables naturally contains also the gyro-phase ζ. Possibly after averaging, the operators may be identical, but this is not of interest to the neoclassical calculations of gyro-phase averaging).

NOTE By the change of variables, the term with the *energy change by the time variation of the electric potential* disappears. It is a cancelling of the two terms that contain the time derivative of the electric potential.

The **Drift-Boltzmann equation** is

$$\begin{aligned}
&\left\{ \frac{\partial}{\partial t} + \right. \\
&+ \left[\widehat{\mathbf{n}} v_{\|} + \frac{1}{\Omega} \widehat{\mathbf{n}} \times \left(\frac{1}{m} \mu \boldsymbol{\nabla} B + v_{\|}^2 \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} + \frac{q}{m} \boldsymbol{\nabla} \phi \right) \right] \times \\
&\qquad \times \left. \left[\boldsymbol{\nabla} + \frac{\mu \boldsymbol{\nabla} B}{m} \frac{\partial}{\partial \left(v_{\perp}^2 / 2 \right)} - \left(\frac{\mu \boldsymbol{\nabla} B + q \boldsymbol{\nabla} \phi}{m} \right) \frac{\partial}{\partial \left(v_{\|}^2 / 2 \right)} \right] \right\} f \\
= & \quad -\nu \left(f - f_0 \right)
\end{aligned}$$

NOTA It is clear that this treatment is developed before or independent of the *neoclassical* standard methods of expansion in the two parameters typical for neoclassical orbits: **banana width/plasma radius** and **bounce frequency/collision frequency**.

20.4.1 The magnetic field

It is chosen a magnetic field which is in a slab geometry but which essentially keeps the periodic spatial variation of the tokamak field.

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$$

$$\mathbf{B}_1 \left(x, z \right) = \lambda B_0 \operatorname{Re} \left[\widehat{\mathbf{e}}_z - \widehat{\mathbf{e}}_x \frac{k_z}{k_x} \right] \exp \left(i k_x x + k_z z \right)$$

where λ measures the fractional modulation of the magnetci field along the line B. It is actually of the order

$$\lambda \sim \frac{r}{R}$$

20.4.2 Solution of the drift-kinetic equation

The distribution function and the electric potential are expanded in the small parameter λ and are Fourrier transformed

$$f = f_0 + \mathrm{Re}\, f_1 \exp\left(ik_x x + ik_z z - i\omega t\right) + \cdots$$

$$\phi = \phi_0 + \mathrm{Re}\, \phi_1 \exp\left(ik_x x + ik_z z - i\omega t\right) + \cdots$$

In the first order we have

$$\begin{aligned}
\left(-\omega + k_x W + k_z v_\parallel - i\nu\right) f_1 \;\; = \;\; & \frac{q\phi_1}{m}\left(k_z \frac{\partial}{\partial v_\parallel} - \frac{k_x}{\Omega}\frac{\partial}{\partial y}\right) f_0 \\
& -\lambda \frac{\delta(\omega)}{m}\left(\mu k_z B_0 v_\parallel \frac{\partial}{\partial\left(v_\perp^2/2\right)} - \mu k_z B_0 v_\parallel \frac{\partial}{\partial\left(v_\parallel^2/2\right)}\right. \\
& +\mu k_x B_0 W \frac{\partial}{\partial\left(v_\perp^2/2\right)} - \frac{m v_\parallel^2 k_z^2}{k_x} W \frac{\partial}{\partial\left(v_\parallel^2/2\right)} + \\
& \left. +\frac{\mu B_0}{\Omega} k_x \frac{\partial}{\partial y} - \frac{m v_\parallel^2}{\Omega}\frac{k_z^2}{k_x}\frac{\partial}{\partial y}\right) f_0
\end{aligned}$$

where the poloidal velocity has been introduced

$$W = \frac{1}{B_0}\frac{\partial \phi_0}{\partial y}$$

NOTE. Here k_z plays the role of the inverse connection length

$$k_z \sim \frac{1}{qR}$$

and appears in the position of the **radius of curvature**, $1/R$ arising from the $v_\parallel^2 \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right) \widehat{\mathbf{n}}$. We also remark that the wavevector k_x describes the variation in the direction perpendicular to the magnetic field line.

20.4.3 The radial electric current

It is obtained from the combination of the

- radial drift of the particles and
- the distribution function in the first order of approximation in powers of λ, the variation of the magnetic field modulus along the line.

$$\begin{aligned}\widehat{\mathbf{e}}_y \cdot \mathbf{v}_D &= i\frac{k_x \lambda}{q}\left[\mu - \frac{k_z^2 m v_\parallel^2}{k_x^2 B_0}\right] \\ &\approx i\frac{k_x \lambda}{q}\frac{v_\perp^2}{2\Omega}\end{aligned}$$

since

$$k_z^2 \ll k_x^2$$

Here

$$k_x \lambda \sim \frac{1}{r}\frac{r}{R} \sim \frac{1}{R} \text{ the curvature}$$

and the drift is

$$\sim \frac{1}{\Omega}\frac{v_\perp^2/2 + v_\parallel^2}{R}$$

The projection of the drift velocity on the *radial* versor is a periodic quantity. Since $\mathbf{v}_D$ is a vector oriented approximately in vertical direction, the projection of $\mathbf{v}_D$ on $\widehat{\mathbf{e}}_r$ is

$$\mathbf{v}_D \cdot \widehat{\mathbf{e}}_r \sim \sin\theta$$

It is zero at $\theta = 0$ (on the equatorial plane) and maximum at the highest top point on the surface, at $\theta = \pi/2$; it decreases afterward and changes sign on the side of the surface which is below the equatorial plane. So it is a $\sin\theta$.

On the other hand, we see that the first order distribution function is also periodic in space on the x and z directions, due to the presence of the Fourier exponential.

Since the radial current is the product of the radially-projected drift velocity with the first order distribution function **the result is essentially periodic with the same periodicity**, k_x on the x coordinate (perpendicular on the magnetic field line) and k_z on the direction of the magnetic line. Then the spacial average "on the magnetic surface" (*i.e.* on (x, z)) is **zero** and there would be **no radial average electric field**.

Actually the average of the radial electric field is not zero and **this is due to the shift in phase between the**

- radially projected drift velocity $\mathbf{v}_D \cdot \widehat{\mathbf{e}}_r \sim \sin\theta$; and the
- first order distribution function $f_1 \sim \exp(ik_x x + ik_z z)$.

SHIFT in phase which is due to collisions:

$$\nu > 0$$

The **average on the magnetic surface is equivalent to the average on** (x, z):

$$\langle AB \rangle = \frac{1}{4}\left(\widetilde{A}^* \widetilde{B} + \widetilde{A}\widetilde{B}^*\right)$$

where $\widetilde{A}$ and $\widetilde{B}$ are the complex coefficients Fourier transform of A and B. Then

$$\left(\widetilde{\mathbf{v}_D \cdot \widehat{\mathbf{e}}_r}\right)^* = -i\frac{k_x \lambda^*}{q}\frac{v_\perp^2}{2\Omega}$$

and

$$\widetilde{(f_1)} = f_1 = \text{solution of the drift-kinetic equation}$$

Then

$$\begin{aligned}
\langle j_r \rangle &= \int 2\pi v_\perp dv_\perp dv_\parallel (qm)\left(\frac{1}{4}\sum_{\text{ions, elect.}}\left(-i\frac{k_x\lambda^*}{q}\frac{v_\perp^2}{2\Omega}\right) f_1 + \text{complex conjugate}\right) \\
&= \frac{1}{4}\left(2J_I + J_{II}\right) + \text{complex conjugate}
\end{aligned}$$

with the definitions

$$J_I = i\left|\lambda\right|^2 \frac{k_x}{B_0}\sum_{i,e}\int dv_\parallel \left(\frac{N_I T_\perp}{-k_z v_\parallel - k_x W + i\nu}\right)$$

and

$$\begin{aligned}
N_I \equiv & \left(k_x W + k_z v_\parallel\right) f_0 + \\
& + k_z \frac{T_\perp}{m}\frac{\partial f_0}{\partial v_\parallel} - \frac{k_x}{B_0}\frac{\partial f_0}{\partial y}
\end{aligned}$$

where

$$f_0 = f_0\left(v_\parallel, y\right)$$

which is what remains from the Maxwellian after the integration over the perpendicular velocity, $v_\perp$.

We can recognize the last term: the diamagnetic term.

We can see that if $\nu = 0$ the quantity N_I is pure real and J_I is pure complex this would simply mean that

$$\begin{aligned}
\langle j_r \rangle &\sim \frac{1}{2}\left(J_I + \text{complex conjugate}\right) \\
&= \frac{1}{2}\left(J_I - J_I\right) \\
&= 0
\end{aligned}$$

and there is no averaged radial current.

This **is not so due to the collisions**

$$\nu \neq 0$$

20.4.4 Conclusion at this perspective on the poloidal velocity damping

From this treatment we can get the same conclusion, but using eventually other words.

The drift-kinetic equation gives the distribution function reflecting the motion of particles in the geometry of the tokamak. This motion enters the drift-kinetic equation by the expression of the neoclassical particle drift velocity $\mathbf{v}_D$.

The neoclassical calculus stops at the time derivative of f, the convection $\left(v_{\parallel}\widehat{\mathbf{n}} + \mathbf{v}_D\right)$ applied on the gradient of f and the Collisions. The electric potential ϕ terms cancel.

The **drift of the particle** is oscillatory on the magnetic surface and has **zero average** on the magnetic surface, for passing particles in particular. At Stix, has zero average on (x, z).

The distribution function in the first order has **zero average** since it passes over the maxima and minima of the spatially periodic distribution function.

There is a decalage between the **drift of the particle** and the **distributiuon function** which in principle can provide non-zero average of the product $(\langle j_y \rangle \neq 0)$. But, essential is the presence of the **COLLISIONS** ν. They produce and sustain the current across the magnetic surfaces which, when combined with the magnetic field gives the **decay of the poloidal rotation**.

With the solution of the drift-kinetic equation (in certain approximations given by the usual expansion in neoclassical small parameters) it is calculated the **radial particle fluxes**, for electrons and ions. Due to the difference in the drift velocities of the alectrons and ions, there is a **intrinsic difference between the ion and electron radial fluxes leading to a radial current**.

This radial current interacts with the main magnetic field and produces a force which is opposed to the initial poloidal plasma flow.

20.5 Decay by viscosity of the rotation generated by an initial electric field

In the banana regime the continuity equation has as solution the general form of the plasma velocity

$$\mathbf{V} = \frac{K(\psi)}{n}\mathbf{B} + \omega_\varphi R^2 \boldsymbol{\nabla}\varphi$$

to the lowest order in the Larmor radius.

$$\omega_\varphi = -\frac{\partial \phi(\psi)}{\partial \psi}$$

corresponds to the $\mathbf{E} \times \mathbf{B}$ rotation and has to be a flux function. Then

$$\mathbf{V} = \frac{K(\psi)}{n}\mathbf{B} + R\left(-\frac{\partial \phi(\psi)}{\partial \psi}\right)\widehat{\mathbf{e}}_\varphi \tag{9}$$

or

$$\mathbf{V} = \frac{K(\psi)}{n}\mathbf{B} + \left(-\frac{\partial \phi(\psi)}{\partial r}\frac{1}{B}\right)\widehat{\mathbf{e}}_\varphi$$

$K(\psi)$ corresponds to **the parallel flow**, in the surface. It contains the poloidal flow.

This is a general solution of the **continuity equation**

$$\boldsymbol{\nabla}\cdot(n\mathbf{V}) = 0$$

after neglecting the radial flow. See **Hazeltine Hinton**.

$K(\psi)$ is the **driving force** of the deviation of the averaged distribution function.

The drift-kinetic equation becomes

$$\frac{\partial \overline{f}}{\partial t} + \left(v_\parallel + \frac{K}{n}B\right)\nabla_\parallel \overline{f} + v_\parallel\left(\frac{\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\cdot\mathbf{P}}{nm_i} - \nabla_\parallel \frac{v_\parallel K B}{n}\right)\frac{\partial \overline{f}}{\partial w} = C\left(\overline{f}\right)$$

NOTE on the origin of this equation. As said before, this equation comes from the drift-kinetic equation in a plasma with strong rotation. The general form of the mass velocity for this rotation is given by the general formula of the solution of the of the *continuity* equation, Eq.([**?**]). The drift-kinetic equation has an energetic term, arising from the combination of the effects of the **mass flow** and the **particle velocity**. There are two components which are retained, when it is calculated the *time-dependent parallel viscosity and relaxation rate of poloidal rotation*:

$$v_{\parallel}\frac{\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\cdot\mathbf{P}}{nm} \qquad \begin{array}{c}\text{time rate of energy exchange}\\ \text{against the parallel } \textit{Pressure gradient}\end{array}$$

This term is exactly that of the full equation. And

$$-v_{\parallel}\nabla_{\parallel}\frac{v_{\parallel}KB}{n}$$

This term has the following source

$$-\left(v_{\parallel}^{2}-\mu B\right)\widehat{\mathbf{n}}\cdot\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\mathbf{V}\ \frac{\partial\overline{f}}{\partial w}$$

with $w=v^2/2$. The first part

$$-v_{\parallel}^{2}\widehat{n}_{\alpha}\widehat{n}_{\beta}\nabla_{\alpha}V_{\beta}=-v_{\parallel}\left(v_{\parallel}\nabla_{\parallel}\frac{KB}{n}\right)$$

and the second term seems to be neglected. This is rather bizarre: the force here arises from the parallel divergence of the **flow velocity** of the rotation. More precisely, in the form

$$-v_{\parallel}\nabla_{\parallel}\frac{v_{\parallel}KB}{n}$$

the energy change is connected with the work done against the force which appears from the parallel divergence of the flux of parallel momentum. Or, the parallel flux of parallel momentum has a space variation in the parallel direction. This can be in principle related to the magnetic field variation but all the difficulty of calculation is transfered to the mass flow KB/n.

PROBLEMS. It is necessary to evaluate the

- divergence of the flow velocity (the compressibility of the poloidal flow)

  $$\boldsymbol{\nabla}\cdot\mathbf{V}$$

  which is multiplied by μB;

- the term

$$\mu B \,\widehat{\mathbf{n}} \cdot \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \mathbf{V}$$

Here, the parallel variation of the distribution function $\overline{f}$ is convected by the parallel velocity, composed of:

- particle parallel velocity $v_{\|}$;
- fluid parallel velocity, (which is also a velocity for each particle) $\frac{K}{n} B$; this is found in other treatments as $\mathbf{V}$;

There is a parallel variation of the distribution function along the magnetic lines, *i.e.* in the magnetic surface. There is a variation of the density and the temperature in the magnetic surface: $\widetilde{n}(\theta, t)$ and $\widetilde{T}(\theta, t)$. It can be seen that the variation in the surface of the density and temperature are conected with MAGNETIC PUMPING.

The **Energetic term** in the drift-kinetic equation.

- The first term: the gradient of the ion stress tensor is projected in the parallel direction and then is convected by the parallel velocity. The main variation of the ion pressure is radial, so has zero projection along the field line. But, there is a variation of the pressure "in the magnetic surface" and thus there is a parallel gradient of the pressure, *i.e.* a force. The particle works against this force when it moves along the line.
- The second term $-\nabla_{\|} \frac{v_{\|} K B}{n}$ is connected with the **parallel variation of the parallel flux of parallel momentum**: It is question of the parallel momentum from the flow, $\frac{KB}{n}$, convected by the parallel velocity $v_{\|}$; this is a *parallel flux of parallel momentum.* Its variation in the parallel direction is connected with the geometry (toroidal: the lines of flow agglomerate when we go to the interior of the torus) and to the variation of the magnitude of the magnetic field. The flux has non-zero divergence. This divergence is a force in the parallel direction. Then, the particle works against this force, since it moves with velocity $v_{\|}$. It seems that actually it is question of the *magnetic pumping force*, which is an energy sink for the direct motion.

In the lowest approximation, the distribution function is Maxwellian

$$f_M = \frac{\overline{n}(\psi)}{\pi^{3/2} v_{th}^3(\psi)} \exp\left(-\frac{mw}{T(\psi)}\right)$$

and the gyro-averaged distribution function differs from the Maxwellian by

$$\overline{f} = f_M + \widetilde{f}$$

and in the function $\widetilde{f}$ tehre is a part which is the first order of the neoclassical distribution function:

$$\widetilde{f} = -\frac{2KBv_\parallel}{nv_{th}^2} f_M + \delta\widetilde{f}$$

This part can be attributed to the effect of the MAGNETIC PUMPING associated to the poloidal motion of the plasma with the poloidal velocity which results from the *parallel velocity* $\frac{K(\psi)}{n}\mathbf{B}$. This "modulation" of the distribution function $\overline{f} - f_M$ is related to the neoclassical effects: (1) geometry of the torus, and (2) variation of the magnetic field along the line, *i.e.* the magnetic pumping.

(Observatie personala: 2-ul este interesant, apare misterios si la poloidal spin-up de Drake si Hassam).

Equation for this part $\delta\widetilde{f}$ of the distribution function:

$$\begin{aligned} v_\parallel \nabla_\parallel \left(\widetilde{f} + \frac{2KBv_\parallel}{nv_{th}^2} f_M \right) &= \frac{\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \cdot \mathbf{P}}{\overline{p}} v_\parallel f_M + \\ &+ \left(\nu_p \widetilde{f} \right) \\ &+ C\left(\widetilde{f} \right) \end{aligned}$$

Here, since the objective were to calculate the **rotation damping** by ion viscosity, the explicit time derivative has been replaced with the rate of damping.

This equation

$$\begin{aligned} v_\parallel \nabla_\parallel \delta\widetilde{f} &= \frac{\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \cdot \mathbf{P}}{\overline{p}} v_\parallel f_M + \\ &\quad \nu_p \widetilde{f} + C\left(\widetilde{f} \right) \end{aligned} \tag{10}$$

shows that the parallel variation of this "supplement" of the distribution function is convected due to a force which arises from the parallel variation of the **pressure**. (It is already here the origin of the spontaneous poloidal spin-up: the differneces in the density and temperature on the surface drive the distribution function, which further can enhance the differences. However, here it is said that only the *poloidal rotation* is the cause of the $\delta\widetilde{f}$ on the surface. And, in the analysis of Hassam and Drake, it is said that the variation on the surface of the *fluxes from diffusion* or of the *sources* can drive the spin-up, according to Stringer).

The **zeroth order** of the function $\delta\widetilde{f}$ is denoted $g(w,\mu,\psi)$

$$\delta\widetilde{f}_0=\widetilde{f}_0+\frac{2KBv_\parallel}{nv_{th}^2}f_M=\sigma g$$

where σ is the sign of the parallel velocity. The next order:

$$\delta\widetilde{f}_1=\left(\widetilde{f}+\frac{2KBv_\parallel}{nv_{th}^2}f_M\right)\Big|_1=\sigma g_1$$

is obtained after sunstituting $\delta\widetilde{f}_1$ in the left hand side of eq.(10) and $\widetilde{f}_0$ in the last two terms in the right hand side:

$$\begin{aligned} v_\parallel\nabla_\parallel\widetilde{f}_1 &= \frac{\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\cdot\mathbf{P}}{\overline{p}}v_\parallel f_M+ \\ &\nu_p\widetilde{f}_0+C\left(\widetilde{f}_0\right) \end{aligned} \tag{11}$$

20.6 The transit time magnetic pumping induced in the MHD relaxation (Hasegawa, Yoshida)

The paper of **Hasegawa Yoshida**.

The helicity dissipation is

$$\mathbf{E}\cdot\mathbf{B}d^3x$$

where

$$\mathbf{E}=\eta\mathbf{J}-\mathbf{v}\times\mathbf{B}$$

The *viscosity* does not change the helicity. Viscosity is just a transfer of momentum.

Only the resistive dissipation.

But it dissipates the energy of the fluctuations to heat ions.

The force experienced by a fluid that makes a motion along the direction of magnetic field lines (parallel) is

$$\mathbf{F}_\parallel=-\mu\boldsymbol{\nabla}_\parallel B$$

(magnetic mirror effect) where

$$\mu=\frac{mv_\perp^2}{2B_0}$$

This force acts on particle energy, i.e. it has effect in the velocity space, on the parallel velocity.

For the *ions* distribution function we have

$$\frac{\partial f}{\partial t}+v_{\parallel}\nabla_{\parallel}f+\boldsymbol{\nabla}_{\perp}\cdot\left(\mathbf{v}_{drift}\ f\right)-\frac{\mu}{m}\boldsymbol{\nabla}_{\parallel}B_{\parallel}\frac{\partial f}{\partial v_{\parallel}}=0$$

To solve this equation it is made a Laplace transform after firstly taking a Fourier representation of the parallel magnetic field.

The imaginary part of the distribution function is

$$f_k=i\pi\delta\left(\omega-k_{\parallel}v_{\parallel}\right)\frac{\mu}{m}k_{\parallel}B_{\parallel,k}\frac{\partial f}{\partial v_{\parallel}}$$

(we **Note** that the propagator is reduced to a simple resonance in the parallel direction, no drift taken into account).

The average parallel fluid velocity

$$\begin{aligned}u_{\parallel,k} &= \frac{1}{n_0}\int dv_{\parallel}\ v_{\parallel}f_k \\ &= -i\sqrt{\frac{\pi}{2}}\frac{\overline{v}_{\perp}^2}{v_{th,i}}\frac{B_{\parallel,k}}{B_0}\frac{k_{\parallel}}{\left|k_{\parallel}\right|}\left(\frac{v_{\varphi}}{v_{th,i}}\right)^2\exp\left[-\left(\frac{v_{\varphi}}{v_{th,i}}\right)^2\right]\end{aligned}$$

the phase velocity is

$$v_{\varphi}=\frac{\omega}{k_{\parallel}}$$

The rate of dissipation into ions is

$$P_k=n_0\left\langle F_{\parallel,k}u_{\parallel,k}\right\rangle$$

where the operator is defined as

$$\left\langle A_kB_k\right\rangle\equiv\frac{1}{2}\operatorname{Re}\left\langle A_kB_k^*\right\rangle$$

20.7 Kinetic damping of the poloidal rotation (Haines)

Equation for the parallel component of the **ion** momentum equation [7]

$$m_in_i\frac{\partial}{\partial t}\left\langle\mathbf{B}\cdot\mathbf{u}_i\right\rangle=-\left\langle\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_i\right\rangle \tag{12}$$

where $\boldsymbol{\Pi}_i$ is the *traceless* stress tensor. See also **Shaing Crume Houlberg**.

Note Rosenbluth Hinton mention that $\left\langle\left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{v}\right\rangle=0$) **End.**

The *evolving parallel component for of the total poloidal flow.* The pressure balance on a flux surface

$$0 = -\frac{\boldsymbol{\nabla} P_i}{n_i e} + \mathbf{E} + \mathbf{u}_i \times \mathbf{B}$$

where P_i is the scalar pressure of ions. The **perpendicular flow** is

$$\mathbf{u}_\perp = \frac{-\boldsymbol{\nabla} P_i \times \mathbf{B}}{e n_i B^2} + \frac{-\boldsymbol{\nabla}\Phi\times\mathbf{B}}{B^2}$$

it is actually composed of : *diamagnetic flow* and ExB flow. One has to add the parallel flow, $\mathbf{u}_\|$.

The **toroidal** (ϕ) component of the total flow

$$\begin{aligned} u_{\phi\alpha} &= \frac{\mathbf{u}_\alpha \cdot \boldsymbol{\nabla}\phi}{|\boldsymbol{\nabla}\phi|} \\ &= u_{\|\alpha}\frac{B_\phi}{B} - \frac{T_\alpha B_\theta}{\Omega_\alpha m_\alpha B}\left(\frac{e\Phi'}{T_\alpha} + \frac{P'_\alpha}{P_\alpha}\right) \end{aligned}$$

The **poloidal** (θ) component

$$\begin{aligned} u_{\theta\alpha} &= \frac{\mathbf{u}_\alpha \cdot \boldsymbol{\nabla}\theta}{|\boldsymbol{\nabla}\theta|} \\ &= u_{\|\alpha}\frac{B_\theta}{B} + \frac{T_\alpha B_\phi}{\Omega_\alpha m_\alpha B}\left(\frac{e\Phi'}{T_\alpha} + \frac{P'_\alpha}{P_\alpha}\right) \end{aligned}$$

Using these relations, Hirschman showed that the equation (12) can be written in terms of the poloidal flow within a surface

$$U_\theta(\psi) \equiv \frac{\mathbf{u} \cdot \mathbf{B}_\theta}{B_\theta^2}$$

$$m_i n_i \left(1 + 2q^2\right) \left\langle B_\theta^2 \right\rangle \frac{\partial}{\partial t} U_\theta(\psi) \approx -\left\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Pi}_i \right\rangle$$

The term in the right hand side can be written

$$\left\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Pi}_i \right\rangle = \left\langle Bm \int d^3v\, v_\|^2 \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \overline{f}_i \right\rangle$$

where $\overline{f}_i$ is the gyro-averaged distribution function. This is the solution of the equation

$$\frac{\partial}{\partial t}\overline{f}_i + \left(v_\|\widehat{\mathbf{n}} + \mathbf{v}_d\right) \cdot \boldsymbol{\nabla}\overline{f}_i + e\left(v_\|\widehat{\mathbf{n}} + \mathbf{v}_d\right) \cdot \mathbf{E}\frac{\partial}{\partial \varepsilon}\overline{f}_i = C\left\{\overline{f}_i\right\} \qquad (13)$$

where ε is the particle energy.

See also **Zhu Horton Sugama**.

Note : it is taken into account the **energetic effect from the *drift* motion of the charged particle in the electric field**. It is normal that a motion imposed by the geometry and effected *against* the electric field to be a source of energy change.

The solution of the neoclassical equation for the distribution function $\overline{f}_i$ The *bar* means gyro-phase averaging, which is a basic assumption and we shall not write the bar in the following. The solution is obtained by perturbation based on the smallness of the two neoclassical parameters:

- the **drift parameter** δ

$$f_i = f_0 + f_1 + ... \text{ where } f_1 \sim \delta f_0$$

- the **bounce parameter** η

$$f_1 = f_1^{(0)} + f_1^{(1)} + ... \text{ where } f_1^{(1)} \sim \eta f_1^{(0)}$$

First the expansion in δ. The **zeroth order** in δ:

$$\frac{\partial}{\partial t} f_0 + v_{\|} \widehat{\mathbf{b}} \cdot \boldsymbol{\nabla} f_0 = C\left(f_0\right)$$

The solution of this equation is a Maxwellian:

$$f_0 = n\left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\frac{w}{T}\right)$$

Nota. Here at the exponent we have to include also the energy of the particle in the electric potential.

The first order in in δ:

$$\frac{\partial}{\partial t} f_1 + v_{\|} \widehat{\mathbf{b}} \cdot \boldsymbol{\nabla} f_1 + \mathbf{v}_d \cdot \boldsymbol{\nabla} f_0 - \mathbf{v}_d \cdot e\mathbf{E} \frac{\partial}{\partial w} f_0 = C\left(f_1\right)$$

Now we go to the **expansion in the parameter** η. The **zeroth order** in η of the above equation:

$$v_{\|} \widehat{\mathbf{b}} \cdot \boldsymbol{\nabla} f_1^{(0)} + \mathbf{v}_d \cdot \boldsymbol{\nabla} f_0 = 0$$

We can use the relation

$$\mathbf{v}_d \cdot \boldsymbol{\nabla} f_0 = -v_{\|} \widehat{\mathbf{b}} \cdot \boldsymbol{\nabla} \left(\frac{I v_{\|}}{\Omega}\right) \frac{\partial f_0}{\partial \psi}$$

and the solution for the function $f_1^{(0)}$ is of the form

$$f_1^{(0)} = -\left(\frac{Iv_\parallel}{\Omega}\right)\frac{\partial f_0}{\partial \psi} + g\left(\psi, \lambda, w, \sigma\right)$$

The integration "constant" g is independent of θ.

The first order in η of the drift kinetic equation

$$v_\parallel \widehat{\mathbf{b}} \cdot \boldsymbol{\nabla} f_1^{(1)} = -\frac{\partial}{\partial t} f_1^{(0)} + C\left(f_1^{(0)}\right)$$

One can use the *periodicity* on θ to obtain a constraint from which to determine $f_1^{(0)}$.

Taking the **bounce average** of this equation we eliminate the left hand side and obtain an equation for the **zeroth order function**, $f_1^{(0)}$. The **bounce average** is taken using the operator

$$\left\langle \frac{B}{v_\parallel} \cdot \right\rangle$$

with the integral in the poloidal coordinate taken between $-\pi$ and π for untrapped particles and betwrrn $-\theta^*$ and $+\theta^*$ for trapped particles. This equation is called : **banana constraint equation**.

$$\left\langle \frac{B}{v_\parallel} \left[-\frac{\partial}{\partial t} f_1^{(0)} + C\left(f_1^{(0)}\right)\right]\right\rangle = 0$$

The lowest order non-zero contribution to the parallel stress comes from the function $f_1^{(1)}$ and since the parallel component of the stress is bounce-averaged

$$\left\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Pi} \right\rangle = \left\langle mB \int d^3v \; v_\parallel \left[-\frac{\partial}{\partial t} f_1^{(0)} + C\left(f_1^{(0)}\right)\right]\right\rangle$$

which shows that only $f_1^{(0)}$ must be determined.

21 The ambipolarity Hirshman

This text is from **NF18, (1978) 917.**

It is here just to remind how the "neoclassical inertia factor" arises.

The equation of momentum conservation projected on the parallel direction

$$\sum_j m_j n_j \left\langle \mathbf{B} \cdot \frac{\partial \mathbf{u}_j}{\partial t} \right\rangle = -\sum_j \left\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Pi}_j \right\rangle$$

Now we need an expression for the fluid velocity u_j,

$$\begin{aligned}\mathbf{u}_j &= \left(\frac{\mathbf{u}_j \cdot \mathbf{B}_\theta}{B_\theta^2}\right)\mathbf{B} \\ &\quad + \frac{R^2\boldsymbol{\nabla}\varphi}{\langle R^2\rangle}\left(\left\langle R^2\boldsymbol{\nabla}\varphi \cdot \mathbf{u}_j\right\rangle - \frac{\mathbf{u}_j \cdot \mathbf{B}_\theta}{B_\theta^2}\langle R B_T\rangle\right)\end{aligned}$$

It has been introduced the polodal projection of the velocity, which after projection is scaled by B_θ to remove the θ dependence and produce a function (velocity) that depends on only ψ.

$$\frac{\mathbf{u}_j \cdot \mathbf{B}_\theta}{B_\theta^2} = K_j(\psi)$$

(usual notation for the poloidal part of the velocity. **Hsu Gromley Shaing bootstrap** α)

Then

$$\begin{aligned}&\sum_j m_j n_j \frac{\partial}{\partial t}\left[\left(1+2\widehat{q}^2\right)\ \langle \mathbf{u}_j \cdot \mathbf{B}_\theta\rangle + \langle R\ B_T\rangle \frac{\langle R^2\boldsymbol{\nabla}\varphi \cdot \mathbf{u}_j\rangle}{\langle R^2\rangle}\right] \\ &= -\sum_j \langle \mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_j\rangle\end{aligned}$$

where

$$\widehat{q}^2 \equiv \frac{1}{2\langle B_\theta^2\rangle}\left(\left\langle B_T^2\right\rangle - \frac{1}{\left\langle \frac{1}{B_T^2}\right\rangle}\right)$$

This reduces to

$$\widehat{q}^2 \to q^2$$

for small ε.

The static form of the neoclassical parallel stress tensor

$$\begin{aligned}&\langle \mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_j\rangle \\ &= 3\mu_j\ \left\langle \left(\nabla_\parallel B\right)^2\right\rangle\ \frac{\mathbf{u}_j \cdot \mathbf{B}_\theta}{B_\theta^2}\end{aligned}$$

22 Inverse Stringer-effect and turbulence modulation

There is a text on *Stringer*.

The Stringer effect is the generation of plasma rotation due to a poloidal non-uniformity of the rate of transport due to plasma turbulence. This transport modulation will couple with the Pfirsch-Schluter toroidal current and produce a rotation of plasma in the poloidal direction.

The *inverse Stringer* effect is the opposite form of this relationship.

We assume that there is poloidal rotation and that this is coupled with Pfirsch-Schluter toroidal current. Then it should induce a non-uniformity of the poloidal distribution of transport. This is equivalent to a modulation of the amplitude of the turbulence.

Possibly observations like the coherent modes may be connected with this.

Or, the strong importance of the direction of ∇B drift relative to the X-point.

Check again, I remember to have found an objection.

23 Inverse Ranque-Hilsch effect

It is question of the possibility that the effect leading to temperature separation in the case of the ranque-Hilsch vortex tube can have an opposite manifestation.

The gradient of temperature and a poloidal rotation, together with an initial toroidal (axial) movement of plasma = toroidal rotation represent the initial state. The poloidal rotation is enhanced as a result of an increase of the radial gradient of the temperature during an external heating applied to plasma. Then *by an inverse Ranque-Hilsch effect* the plasma enhances its poloidal rotation, evolving to a state where the gradient of temperature (equivalent to a separation of temperature, in the terminology of the Ranque-Hilsch vortex tube) becomes compatible with the radial distribution of the angular velocity in the vortex (*i.e.* the poloidal velocity).

The problem of the possible role of the drift waves in generating and sustaining the *direct* Ranque-Hislch effect. This is suggested by several approaches to the classical, direct, R-H effect, by invoking the structures arising in the vortex by an instability which breakes the azimuthal symmetry. This instability is further invoked by **Colgate Lovelace** in accreating disks of protoplanetary formations, and named *Rossby instabilities.* This practically means drift wave instabilities in plasma physics. They are acting as semi-rigid pedals and possibly are the agent that transports the angular momentum from the center toward the periphery.

The difference between the plasma and the R-H vortex tube: *there is no Larmor radius* in the R-H case. Since there is no magnetic field. If Rossby waves are going to be excited (as propose **Colgate Lovelace**) they will be drift waves.

We **NOTE** an important difference between the convective cells of Rayleigh-Benard problem and the rolls of the Rossby waves:

1. in the Rayleigh-Benard problem the convective cells have alternating direction of rotation

2. in the Rossby rolls the direction of rotation are the same for all the rolls. There is a picture in **Vortex Nucleation**, from atmosphere paper.

What determines the radial and poloidal extension of the rolls? It must be an optimum, related to the efficiency of thermal transfer via convection. And the limitation comes intrinsically since the poloidal rotation which is generated by the rolls acts as a limitation of the radial extension of the rolls.

In Fig.1 of **Bortolon Duval Pochelon TCV** it can be seen that there is a small decrease of the temperature profile for electrons and carbon, which must be explained by the presence of convective cells, more efficient thermal transporters.

The generation of convective rolls from drift waves can be seen as a sequence of Eckhaus instabilities, leading to suppression of high-k eddies and generation of larger convective structures. The paper **pR1735Eckhaus**.

the work by **Riley Davies** treats the situation where the curve of marginal stability of the convective solution of the Rayleigh-Benard system is *flat* in its dependence on the *wavenumber* k, like $(k - k_{crit})^{2n}$, the system is unstable to Eckhaus instabilities (sideband instabilities) of the secondary type, with rapid succession of them. This means that it is possible that the number of rolls that is created evolves rapidly from the envelope of the drift waves to a poloidal wavenumber which is optimum. In the paper **Roll like patterns Proctor** it is found a condition for instability.

This may be invoked, if confirmed in the case of convective cells of drift waves in tokamak, to explain the fast change of the roll geometry until it stabilizes at a sequance of few rotating rolls located with their centers on a magnetic surface. The fact that the poloidal wavenumber is decreasing is invoked to support the idea that the poloidal velocity that the rolls generate at their periphery is higher and higher during the sequence of Eckhaus transitions.

In **Shakura Sunyaev** the angular momentum propagation is due to turbulent Reynolds stress.

The paper **Coherent structures zonal flows Smolyakov Diamond Malkov** discusses the transformation of the zonal flows into coherent structures. They say: for generation of zonal flows the underlying mechanism is inverse cascade.

Comment: inverse cascade is a stochastic process, it takes time to generate a coherent flow (zonal) out of turbulence. Then the process

turbulence →	inverse cascade →	zonal flow →	suppression of turbulence

follwed by generation of large scale coherent structures, needs long time (Busse says: *two days*) and has low efficiency. In addition, they obtain an equation for the poloidal flow that exhibits solutions of the *kink* type, here domain-walls separating regions with different poloidal rotation velocities and propagating radially.

Possibly not all solutions have been considered. Harmonic perturbations in the poloidal direction are possible and can trigger KH instabilities leading to breaking of the zonal flow into large scale coherent convective cells.

Instead, the process

turbulence →	coherent structures (convective cells) →	wind →	zonal flow or poloidal r

can be more easily accessible to the plasma under a large temperature gradient.

The paper **Instability Conv Cells Weiland Phys Plasma** shows that the ITG large scale convective cells are subject to instability of shear flow.

24 The ambipolarity

This is our work.

See also **Novakovskii**.

We assume that there is a difference between the radial fluxes of electrons and ions. thismay be due to intrinsic diffusion mechanisms, ion-orbit loss to the limiter, electron drain through an external metallic pin (external plasma polarization), etc. Then there is a radial **electric current**. Suppose for simplicity that the coordinates are

$$\begin{aligned}
x &\equiv r \text{ with } \widehat{\mathbf{e}}_x \equiv -\widehat{\mathbf{e}}_r \text{ (toward the plasma interior)} \\
y &\equiv r\theta \text{ with } \widehat{\mathbf{e}}_y \equiv \widehat{\mathbf{e}}_\theta \\
z &\equiv R\varphi \text{ with } \widehat{\mathbf{e}}_z \equiv \widehat{\mathbf{e}}_\varphi
\end{aligned}$$

The radial electric field is necessarly accompagned by an inductive electric field, as results from the radial component of the $\nabla \times \mathbf{B}$ equation

$$\begin{aligned}
\nabla \times \mathbf{B} &= 0 = \mu_0 \mathbf{j} + \varepsilon_0 \mu_0 \left.\frac{\partial \mathbf{E}}{\partial t}\right|_x \\
0 &= \mu_0 j_x + \varepsilon_0 \mu_0 \frac{\partial E_x}{\partial t}
\end{aligned}$$

The conservation of the ion momentum in the radial direction is

$$n_i m_i \left(\frac{\partial v_x}{\partial t} + v_y \frac{\partial v_x}{\partial y} \right) = -\frac{\partial p}{\partial x} + e n_i E_x + e n_i v_y B_z + R_x$$

or, after evaluating the order of magnitude and neglecting some terms,

$$v_y = -\frac{E_x}{B_z}$$

The conservation of the ion momentum in the poloidal direction is

$$n_i m_i \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} \right) = -\frac{\partial p}{\partial y} + e n_i E_y - e n_i v_x B_z + R_y$$

or

$$v_x = -\frac{m_i}{e B_z} \frac{\partial v_y}{\partial t}$$

It results that the ion poloidal velocity is

$$v_y^i = \frac{1}{B_z \Omega_i} \frac{\partial E_x}{\partial t}$$

This velocity is actually the poloidal **polarization velocity**.

The radial current is

$$\begin{aligned}
j_x &= e n_i v_x^i - e n_e v_x^e \\
&= -\varepsilon_0 \frac{\partial E_x}{\partial t}
\end{aligned}$$

which can be rewritten

$$e n_i v_x^i - e n_e v_x^e = \varepsilon_0 B_z \frac{\partial v_y^i}{\partial t}$$

Replacing the formula for the poloidal ion velocity, it reads

$$-n_i \frac{m_i}{B_z} \frac{\partial v_y^i}{\partial t} - e n_e v_x^e = \varepsilon_0 B_z \frac{\partial v_y^i}{\partial t}$$

or

$$-e n_e v_x^e = \varepsilon_0 B_z \left(1 + \frac{c^2}{v_A^2} \right) \frac{\partial v_y^i}{\partial t}$$

Expressing in quantitaive terms, we have

$$\begin{aligned} v_x^e &= -\frac{\varepsilon_0 B_z}{en} \left(1 + \frac{c^2}{v_A^2} \right) \frac{\partial v_y^i}{\partial t} \\ &= -2 \cdot 10^{-12} \left(1 + 2 \cdot 10^3 \right) \left(\frac{\partial v_y^i}{\partial t} \right) \end{aligned}$$

The same relation can be expressed as

$$j_x \left(1 + \frac{n_i m_i}{\varepsilon_0 B_z^2} \right) = -e n_e v_x^e$$

The physical image can be as follows: suppose we generate a different radial flux of electrons and ions. This generates a radial electric current which, constrained by the $\nabla \times \mathbf{B}$ equation, produces the time variation of a radial electric field. Since there is a connection between the radial electric field and the poloidal velocity, the time variation of the radial field is equivalent to a time variation of the poloidal velocity: plasma begins to rotate without any delay. But the time variation of the poloidal velocity (momentum) is related to the other forces, in particular (and it is essentially this one) to the $\mathbf{v} \times \mathbf{B}$ force. The time variation of the poloidal velocity imposes the generation of a radial ion flux (such as this one, combined with the magnetic field, to keep the momentum balance in poloidal direction). This new radial ion flux follows the radial *electron* flux in order to reduce the radial **electric current** and reinstore the ambipolarity.

24.1 Sheared velocity rotation Horton Dong. Kinetic treatment for ITG

The equation of motion for the particles

$$\begin{aligned} \frac{d^2 \mathbf{r}}{dt^2} &= \frac{e}{m} E(x) \widehat{\mathbf{e}}_x + \Omega \mathbf{v} \times \widehat{\mathbf{n}} \\ \mathbf{v} &= \frac{d\mathbf{r}}{dt} \\ \Omega &= \frac{|e| B}{m} \end{aligned}$$

the constants of motion

$$\begin{aligned}\alpha &\equiv \text{total energy of the perpendicular motion}\\ &= \frac{1}{2}\left(v_x^2+v_y^2\right)+\frac{|e|\,\Phi_0}{m}\end{aligned}$$

$$\begin{aligned}\beta &\equiv \text{thermal energy of the parallel motion}\\ &= \frac{1}{2}\left[v_{\parallel}-v_{0\parallel}(x)\right]^2\end{aligned}$$

$$\begin{aligned}X_g &\equiv \text{position of the guiding center}\\ &= x+\frac{v_y-v_E(X_g)}{\Omega}\\ \text{where}\quad v_E(X_g) &= \frac{-E(X_g)}{B}\end{aligned}$$

Note the system of coordinates that makes the electric velocity to be $-E/B$. **End.**

25 Torque due to the ICRH

25.1 The physical explanation of the ICRH - induced rotation (Chang White)

The papers by **Chang, White, Bonoli** about ICRH rotation.

The idea is the loss of *omnigenity* of the bananas of ions after transversal heating.

The width of the banana of the heated ion is larger

$$width \sim v_{\perp}^2$$

and the *virtual center* of positions of the ion on banana moves slightly, which is a radial current.

Later, numerical simulation where the collisional transfer from the energetized ions to the bulk ions and electrons are calculated. Radial profiles of torque for rotation in the toroidal direction.

Eriksson measurements on JET of ICRH rotation.

The ICRH torque is due to space variation of the density of torque transferred to each resonant ion. The ions move on bananas and the conservation of J leads to displacement of the tips of the bananas.

It results that the ICRH torque is *opposite* to the rotation that is seen in experiments.

Their conclusion is that

$$\text{ion rotation} \sim \boldsymbol{\nabla} p_i$$

is related to the gradient of the pressure of the ions.

Comment on this result

We must consider the change in the local spatial distribution of the pressure due to heating. Then baroclinic term arises.

This would produce vorticity

$$\frac{\partial \omega}{\partial t} \sim \frac{1}{\rho^3} \boldsymbol{\nabla} \rho \times \boldsymbol{\nabla} p_i$$

but this would mean poloidal rotation, which, being weak, cannot exist. The other term from this equation balances the baroclinic term

$$0 = \nabla_{\parallel} j_{\parallel} + \frac{1}{\rho^3} \boldsymbol{\nabla} \rho \times \boldsymbol{\nabla} p_i$$

and this gives a chance to toroidal rotation born from baroclinic term, related with the gradient of the ion pressure.

Their measurement has shown good correlation

$$\text{momentum density} \sim \nabla p_i$$

END

The radial current

$$j_r^{rf} > 0$$

which means from the interior to the exterior of the plasma, produces

$$E_r < 0$$

This means that the potential increases as we go to larger radii (toward the edge). Normally the potential is *negative* inside plasma and, being negative, rises slowly toward the plasma edge $r = a$.

The condition of equilibrium is that the polarization radial current with which the plasma responds to the rf current is

$$j_r^{plasma} = -j_r^{rf}$$

and the force due to this radial return current is balanced by the viscosity force

$$0 = -\overline{\boldsymbol{\nabla} \cdot \boldsymbol{\Pi}} - \overline{j_r^p B_\varphi}$$

where Π is derived from the perpendicular component of the viscous tensor

$$\Pi_\perp = 3\varepsilon\mu P_i \left(\frac{V_\theta}{v_i}\right)\left(\frac{B}{B_\theta}\right)\sin\theta$$

The viscosity comes from the collisional friction between the *trapped* and *untrapped* particles.

The net torque is zero:

1. the high energy particles, the tail of the distribution function, with the current j_r^{rf} have a torque, and
2. the bulk plasma, due to the return current j_r^p.

We have

$$j_r^{rf} \times B + j_r^p \times B = 0$$

which means that when the tail particles are accelerated in one direction the bulk plasma particles are accelerated in the opposite direction.

Observation in Alcator C by **Rice** the toroidal velocity due to ICRH *central* heating in H-mode is

$$\begin{aligned} v_{tor} &= 1.2 \times 10^5 \ \ (m/s) \\ &= 120 \ \ (km/s) \end{aligned}$$

in *co-current* direction. The toroidal rotation profile decreases with increasing r.

The radial electric field

$$\begin{aligned} E_r &= 300 \ \ (V/cm) \\ &= 30 \ kV/m \end{aligned}$$

When the plasma current is reversed, the direction of toroidal rotation changes accordingly such as to remain co-current.

This means that there is a connection with the current distribution or with the direction of the poloidal magnetic field B_θ. This may come from the *compatibility of the helicities* (v_θ, v_{tor}) and (B_θ, B_{tor}).

26 Rotation induced by *alpha* particles

The paper by **Rosenbluth Hinton**.

The alpha particles creation leads to a current that is radial and induces a torque.

After a reaction of fusion in which a α particle is created the α particle will move to the trajectory banana which is characteristic for the magnetic surface where the particle was born. This leads to a radial current.

It is because there is a spatial (radial) profile of the rate of creation of α particles, *i.e.* a nonuniformity of the rate of born, that there is a radial current of alpha particles.

$$\frac{\partial f_\alpha}{\partial t} + \left(v_\parallel \widehat{\mathbf{n}} + \mathbf{v}_d\right) \cdot \boldsymbol{\nabla} f_\alpha = C_{\alpha e} f_\alpha + S_\alpha$$

The guiding centre drift velocity of the α particle is

$$\mathbf{v}_d = \frac{1}{\Omega} \widehat{\mathbf{n}} \times \left(\mu \boldsymbol{\nabla} B + v_\parallel^2 \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} + \frac{e}{m} \boldsymbol{\nabla} \phi \right)$$

which can be written as

$$\mathbf{v}_d = -v_\parallel \widehat{\mathbf{n}} \times \boldsymbol{\nabla} \left(\frac{v_\parallel}{\Omega} \right)$$

$$\Omega_\alpha = \frac{e_\alpha B}{m_\alpha}$$

$$v_\parallel = v \sqrt{1 - \lambda B}$$

where

$$\begin{aligned} \lambda &= \frac{\mu}{\epsilon} = \frac{v_\perp^2 / (2B)}{v^2/2} = \frac{1}{B} \frac{v_\perp^2}{v^2} \\ &= \text{velocity pitch angle} \end{aligned}$$

The collision operator

$$C_{\alpha e} f = \nu_s \frac{1}{v^2} \frac{\partial}{\partial v} \left(v^3 f \right)$$

where

$$\begin{aligned} \nu_s &\equiv \text{slowing down on the electrons} \\ \nu_s &= \frac{m_e}{m_p} \frac{1}{\tau_e} \end{aligned}$$

$$\tau_e \equiv \text{electron-ion collision time}$$

The source of alphas

$$S_\alpha = \dot{n} \left(\psi, t \right) \frac{1}{4 \pi v_0^2} \delta \left(v - v_0 \right)$$

26.1 Characteristic parameters

The ratio between the slowing down frequency and the bounce frequency

$$\frac{\nu_s}{v_{th}/\left(qR\right)} \ll 1$$

which means that the particle makes many bounces before being slowed down collisionally by the electrons.

The ratio between the frequency associated to the guiding centre *drift* and the frequency of bounce

$$\frac{v_d/L_\alpha}{v_{th}/\left(qR\right)} \ll 1$$

Here L_α is the gradient length of the distribution of alpha particles. This means that the particle makes many bounces before the trajectory moves due to the guiding centre drift on a distance L_α. The radial motion is much slower than the bounce motion.

It is defined a single range and a singe small parameter

$$\begin{aligned} \delta &\equiv \frac{\nu_s}{v_{th}/\left(qR\right)} \sim \frac{v_d/L_\alpha}{v_{th}/\left(qR\right)} \\ \delta &\ll 1 \end{aligned}$$

26.2 Expansion of the distribution function of the α particles

The expansion is made in the parameter δ.

$$f_\alpha = f_\alpha^{-1} + f_\alpha^0 + f_\alpha^1 + ...$$

The expansion must be replaced in

$$\frac{\partial f_\alpha}{\partial t} + \left(v_\parallel \widehat{\mathbf{n}} + \mathbf{v}_d\right) \cdot \boldsymbol{\nabla} f_\alpha = C_{\alpha e} f + S_\alpha$$

26.2.1 The order f_{-1}

The first order is -1:

$$\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} f_{-1} = 0$$

which means that the function f_{-1} is constant on the surfaces.

26.2.2 The order of f_0

The next order is *zero-th* order

$$\frac{\partial f_{-1}}{\partial t}+\left(v_{\parallel}\widehat{\mathbf{n}}+\mathbf{v}_d\right)\cdot\boldsymbol{\nabla}\left(f_{-1}+f_0\right)=C_{\alpha e}f_{-1}+S_{\alpha}$$

The time variation of the function f_{-1} is slow and is comparable with the variation of the function f_0 in the magnetic surface, *i.e.* the term $v_{\parallel}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}f_0$. The collision operator is applied on the function f_{-1} since the resulting term is small. The deviation from due to guiding centre drift motion $\mathbf{v}_d$ are small when combined with f_0-gradient. When they combine withthe gradient of f_{-1} the deviation must be retained

$$\frac{\partial f_{-1}}{\partial t}+v_{\parallel}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}f_0+\mathbf{v}_d\cdot\boldsymbol{\nabla}f_{-1}=C_{\alpha e}f_{-1}+S_{\alpha}$$

or

$$\begin{aligned}v_{\parallel}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}f_0 &= C_{\alpha e}f_{-1}+S_{\alpha}\\ &\quad-\frac{\partial f_{-1}}{\partial t}\\ &\quad-\mathbf{v}_d\cdot\boldsymbol{\nabla}f_{-1}\end{aligned}$$

Here we replace the gradient with derivation to the function ψ, which means almost radial

$$\begin{aligned}v_{\parallel}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}f_0 &= C_{\alpha e}f_{-1}+S_{\alpha}\\ &\quad-\frac{\partial f_{-1}}{\partial t}\\ &\quad-\mathbf{v}_d\cdot\boldsymbol{\nabla}\psi\frac{\partial f_{-1}}{\partial\psi}\end{aligned}$$

The variation of the function f_0 *in* the magnetic surface can be calculated if we know f_{-1}.

The equation is now averaged on the *bouncing motion*, of either trapped or passing particles

$$\overline{A}\equiv\frac{\oint\frac{d\theta}{v_{\parallel}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta}A}{\oint\frac{d\theta}{v_{\parallel}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta}}$$

The denominator is replaced by the notation

$$T\equiv\oint\frac{d\theta}{v_{\parallel}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta}$$

We notice that

$$v_{\parallel}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta=\frac{v_{\parallel}}{qR}=\frac{1}{\text{time of bouncing}}$$

The average on bounce motion

$$\begin{aligned}\overline{v_{\parallel}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}f_{0}} &= \overline{C_{\alpha e}}f_{-1}+\overline{S}_{\alpha} \\ & -\frac{\partial f_{-1}}{\partial t} \\ & -\overline{\mathbf{v}_{d}\cdot\boldsymbol{\nabla}\psi}\frac{\partial f_{-1}}{\partial\psi}\end{aligned}$$

and we take into account the periodicity of the left-hand side term, which means that it will disappear by bounce averaging

$$\overline{v_{\parallel}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}f_{0}}=0$$

We have the equation

$$\frac{\partial f_{-1}}{\partial t}=\overline{C_{\alpha e}}f_{-1}+\overline{S}_{\alpha}-\overline{\mathbf{v}_{d}\cdot\boldsymbol{\nabla}\psi}\frac{\partial f_{-1}}{\partial\psi}$$

We have to calculate

$$\begin{aligned}\mathbf{v}_{d}\cdot\boldsymbol{\nabla}\psi &= -v_{\parallel}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\left(\frac{v_{\parallel}}{\Omega}\right)\cdot\boldsymbol{\nabla}\psi \\ &= -v_{\parallel}\left[\boldsymbol{\nabla}\psi\times\widehat{\mathbf{n}}\right]\cdot\boldsymbol{\nabla}\left(\frac{v_{\parallel}}{\Omega}\right)\end{aligned}$$

and, assuming that

1. the magnetic potential ψ grows toward the exterior of the torus such that its gradient is positive directed to the exterior
2. the radial coordinate is directed toward the magnetic axis, it is then opposite to the direction of $\boldsymbol{\nabla}\psi$.

Then

$$\boldsymbol{\nabla}\psi=-\widehat{\mathbf{e}}_{r}\left(RB_{\varphi}\right)$$

$$\begin{aligned}\boldsymbol{\nabla}\psi\times\widehat{\mathbf{n}} &= -I\widehat{\mathbf{e}}_{r}\times\widehat{\mathbf{n}} \\ &= -I\left(-\widehat{\mathbf{e}}_{\perp}\right) \\ &\simeq I\widehat{\mathbf{e}}_{\theta}\end{aligned}$$

We have

$$\boldsymbol{\nabla}Q\left(r,\theta,\varphi\right)=\left(\boldsymbol{\nabla}r\frac{\partial}{\partial r}+\boldsymbol{\nabla}\theta\frac{\partial}{\partial\theta}+\boldsymbol{\nabla}\varphi\frac{\partial}{\partial\varphi}\right)Q$$

We will take another way. From the text *plasma.tex* we use

$$\mathbf{v}_D \cdot \boldsymbol{\nabla}\psi = v_{\parallel} \frac{\mathbf{B} \cdot \boldsymbol{\nabla}\theta}{B} \frac{\partial}{\partial\theta} \left(\rho_{\parallel} \frac{\mathbf{B} \cdot (\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta)}{\mathbf{B} \cdot \boldsymbol{\nabla}\theta} \right)$$

and we take into account that in the case of circular surface

$$\begin{aligned} \frac{\mathbf{B} \cdot \boldsymbol{\nabla}\theta}{B} \frac{\partial}{\partial\theta} &= \frac{B_\theta}{B_T} \frac{1}{r} \frac{\partial}{\partial\theta} \\ &= \frac{1}{qR} \frac{\partial}{\partial\theta} \\ &\simeq \frac{\partial}{\partial z} \\ &\sim \nabla_{\parallel} \end{aligned}$$

which makes these factors to combine in

$$v_{\parallel} \frac{\mathbf{B} \cdot \boldsymbol{\nabla}\theta}{B} \frac{\partial}{\partial\theta} = v_{\parallel} \frac{1}{qR} \frac{\partial}{\partial\theta}$$

from here we retain

$$\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta = \frac{1}{qR}$$

The other factor of the full expression requires the calculation of

$$\begin{aligned} \frac{\mathbf{B} \cdot (\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta)}{\mathbf{B} \cdot \boldsymbol{\nabla}\theta} &= \frac{\widehat{\mathbf{n}} \cdot (RB_\varphi) (-\widehat{\mathbf{e}}_r) \times (1/r) \widehat{\mathbf{e}}_\theta}{\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta} \\ &= \frac{1}{1/(qR)} RB_\varphi \frac{1}{r} \widehat{\mathbf{n}} \cdot \widehat{\mathbf{e}}_\varphi \\ &= (RB_\varphi)(qR) \frac{1}{r} \left(\frac{B_\theta}{B_\varphi} \right) \\ &= RB_\varphi \end{aligned}$$

(check however the signs) and

$$\rho_{\parallel} \frac{\mathbf{B} \cdot (\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta)}{\mathbf{B} \cdot \boldsymbol{\nabla}\theta} = \rho_{\parallel} (RB_\varphi)$$

Now we return to the initial expression of the projection of the drift velocity on the radial direction and we obtain

$$\begin{aligned} \mathbf{v}_D \cdot \boldsymbol{\nabla}\psi &= v_{\parallel} \frac{1}{qR} \frac{\partial}{\partial\theta} \\ &\quad \times \left(\rho_{\parallel} RB_\varphi \right) \end{aligned}$$

since

$$RB_{\varphi} = R_0 B_{0\varphi} = const$$

$$\mathbf{v}_D \cdot \boldsymbol{\nabla}\psi = v_{\parallel} \frac{1}{qR} RB_{\varphi} \frac{\partial}{\partial\theta}\left(\frac{v_{\parallel}}{\Omega}\right)$$

Introducing the expressions

$$I \equiv RB_{\varphi}$$

$$\frac{1}{qR}\frac{\partial}{\partial\theta} = \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta$$

The expression can be re-written

$$\mathbf{v}_D \cdot \boldsymbol{\nabla}\psi = I v_{\parallel} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta \frac{\partial}{\partial\theta}\left(\frac{v_{\parallel}}{\Omega}\right)$$

and this is the expression used by **Rosenbluth Hinton** for alphas.

Taking into account that $\boldsymbol{\nabla}\psi = RB_{\varphi}\left(\widehat{\mathbf{e}}_r\right)$ the factor RB_{φ} simplifies from the left and right, leaving

$$v_{D,r} = \frac{v_{\parallel}}{Rq}\frac{\partial}{\partial\theta}\left(\rho_{\parallel}\right)$$

which may be a useful approximation for the radial component of the drift velocity.

Now, separately, the term with the guiding centre drift is

$$\begin{aligned}
\mathbf{v}_d \cdot \boldsymbol{\nabla}\psi &= I v_{\parallel} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta \frac{\partial}{\partial\theta}\left(\frac{v_{\parallel}}{\Omega}\right) \\
&= I \frac{v_{\parallel}}{qR}\frac{\partial}{\partial\theta}\left(\frac{v_{\parallel}}{\Omega}\right) \\
&= I v_{\parallel} \frac{\partial}{\partial l_{\parallel}}\left(\frac{v_{\parallel}}{\Omega}\right)
\end{aligned}$$

The average on bouncing is

$$\begin{aligned}
\overline{\mathbf{v}_d \cdot \boldsymbol{\nabla}\psi} &= \frac{1}{T}\oint \frac{d\theta}{v_{\parallel}/\left(qR\right)} I \frac{v_{\parallel}}{qR}\frac{\partial}{\partial\theta}\left(\frac{v_{\parallel}}{\Omega}\right) \\
&= \frac{1}{T} I \oint d\theta \frac{\partial}{\partial\theta}\left(\frac{v_{\parallel}}{\Omega}\right) \\
&= 0
\end{aligned}$$

by periodicity, for both trapped and passing particles.

Then the equation obtained from the *zeroth* order, after bounce-averaging, is

$$\frac{\partial f_{-1}}{\partial t} = C_{\alpha e} f_{-1} + S_\alpha$$

and does not imply at all f_0, but exclusively f_{-1}. **Rosenbluth and Hinton** solve this equation

$$f_{-1} = \frac{1}{\nu_s} \dot{n}_\alpha \left[\psi, t - \tau(v)\right] \frac{1}{4\pi v^3} \left[\Theta\left(v - v_0 \exp\left(-\nu_s t\right)\right) - \Theta\left(v - v_0\right)\right]$$

with the initial condition

$$f_{-1}(t = 0) = 0$$

where

$$\tau(v) \equiv \frac{1}{\nu_s} \ln\left(\frac{v_0}{v}\right)$$

The factor containing the Heaviside functions Θ works in the velocity space. In the velcoity space this is a finite interval on the v-axis: it starts to be different of zero (actually equal to 1) after the value $v_0 \exp(-\nu_s \tau)$ and becomes again zero at $v = v_0$ due to the combination of the θ functions. The interval on the v-axis is dependent on time and becomes at the limit $\tau \to \infty$ the interval $[0, v_0]$.

After obtaining a solution for the distribution function f_{-1} we can return to the equation written for the order f_0 but now without the bounce averaging.

$$\begin{aligned} v_\parallel \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} f_0 &= C_{\alpha e} f_{-1} + S_\alpha \\ &\quad - \frac{\partial f_{-1}}{\partial t} \\ &\quad - \mathbf{v}_d \cdot \boldsymbol{\nabla} \psi \frac{\partial f_{-1}}{\partial \psi} \end{aligned}$$

but the three terms in the Right Hand side give zero since they have balanced in the *averaged* version of the equation. It remains

$$\begin{aligned} v_\parallel \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} f_0 &= I v_\parallel \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \theta \frac{\partial}{\partial \theta} \left(\frac{v_\parallel}{\Omega}\right) \frac{\partial f_{-1}}{\partial \psi} \\ &= v_\parallel \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \left(\frac{I v_\parallel}{\Omega}\right) \frac{\partial f_{-1}}{\partial \psi} \end{aligned}$$

The solution is

$$\begin{aligned} f_0 &= -I \left(\frac{v_\parallel}{\Omega}\right) \frac{\partial f_{-1}}{\partial \psi} \\ &\quad + g \end{aligned}$$

where g is the homogeneous solution. To determine g we will write the equation for the next order f_1 and average over bounce to find a constraint for g.

26.2.3 The order of f_1

The next order is f_1 and the drift-kinetic equation gives

$$v_{\|}\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} f_1 = C_{\alpha e} f_0 - \frac{\partial f_0}{\partial t} - \mathbf{v}_d \cdot \boldsymbol{\nabla} f_0$$

Now we need the derivative of the zeroth order function mf$_0$ in the magnetic surface, along the direction of θ. This is possible since f_0 has a variation on the poloidal direction

$$\mathbf{v}_d \cdot \boldsymbol{\nabla}\theta = -I v_{\|}\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta \frac{\partial}{\partial \psi}\left(\frac{v_{\|}}{\Omega}\right)$$

The bounce average of the gradient of the function f_0 advected by the guiding centre drift velocity is zero

$$\overline{\mathbf{v}_d \cdot \boldsymbol{\nabla} f_0} = 0$$

This will be used in the bounce average of the equation for f_1, and allows to determine g:

$$\begin{aligned} & \frac{\partial g}{\partial t} - \nu_s \frac{1}{v^2}\frac{\partial}{\partial v}\left(v^3 g\right) \\ = \; & I\overline{\left(\frac{v_{\|}}{\Omega}\right)}\left[\frac{\partial}{\partial \psi}\frac{\partial f_{-1}}{\partial t} - \nu_s \frac{1}{v^3}\frac{\partial}{\partial v}\left(v^4 \frac{\partial f_{-1}}{\partial \psi}\right)\right] \end{aligned}$$

To calculate the bounce average of the ratio $v_{\|}/\Omega$ we use

$$\begin{aligned} \overline{\left(\frac{v_{\|}}{\Omega}\right)} &= \frac{1}{T}\oint \frac{dl}{\Omega} \\ &= \frac{1}{T}\oint \frac{qR d\theta}{\Omega} \end{aligned}$$

The solution for g is

$$g = I\overline{\left(\frac{v_{\|}}{\Omega}\right)}\frac{\partial f_{-1}}{\partial \psi}$$

Clearly it is non-zero only for passing, untrapped, particles.

Returning to the expression of the function f_0 we get

$$\begin{aligned} f_0 &= -I\left(\frac{v_\parallel}{\Omega}\right)\frac{\partial f_{-1}}{\partial\psi}+g \\ &= -I\left(\frac{v_\parallel}{\Omega}\right)\frac{\partial f_{-1}}{\partial\psi}+I\overline{\left(\frac{v_\parallel}{\Omega}\right)}\frac{\partial f_{-1}}{\partial\psi} \end{aligned}$$

or

$$f_0 = -I\left[\left(\frac{v_\parallel}{\Omega}\right)-\overline{\left(\frac{v_\parallel}{\Omega}\right)}\right]\frac{\partial f_{-1}}{\partial\psi}$$

26.3 Calculation of the radial current from alpha particles

The torque exists only if there is a radial electric current which combines with the magnetic field to induce that torque and plasma rotation.

The radial current will be determined as average over the magnetic surface of the guiding centre drift velocity projected on the direction perpendicular on the surface and averaged over the surface

$$\langle \mathbf{j}_\alpha\cdot\boldsymbol{\nabla}\psi\rangle = e_\alpha\left\langle\int d^3v\ \mathbf{v}_d\cdot\boldsymbol{\nabla}\psi\ f_\alpha\right\rangle$$

where the operator of averaging over the surface si

$$\langle A\rangle = \frac{\oint\frac{d\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}A}{\oint\frac{d\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}}$$

The expression of the guiding centre drift velocity projected on $\boldsymbol{\nabla}\psi$ is

$$\begin{aligned} \mathbf{v}_d\cdot\boldsymbol{\nabla}\psi &= Iv_\parallel\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta\frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{\Omega}\right) \\ &= Iv_\parallel\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\left(\frac{v_\parallel}{\Omega}\right) \end{aligned}$$

and suggests an integration by parts in the expression of the current

$$\langle \mathbf{j}_\alpha\cdot\boldsymbol{\nabla}\psi\rangle = -e_\alpha I\left\langle\int d^3v\ \left(\frac{v_\parallel}{\Omega}\right)v_\parallel\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}f_\alpha\right\rangle$$

Now we note that f_{-1} does NOT contribute to the integral due to the condition of remaining constant over the magnetic surface, the first equation in the expansion, $\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}f_{-1}=0$.

The function f_0 also does NOT contribute to the integration-average.

The first to contribute is f_1. For f_1 we do not have yet an eplicit solution but we have precisel

$$v_{\|}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}f_1 = C_{\alpha e}f_0 - \frac{\partial f_0}{\partial t} - \mathbf{v}_d\cdot\boldsymbol{\nabla}f_0$$

The last term, the drift term $\mathbf{v}_d\cdot\boldsymbol{\nabla}f_0$ does NOT contribute

$$\left\langle \int d^3v \left(\frac{v_{\|}}{\Omega}\right)\mathbf{v}_d\cdot\boldsymbol{\nabla}f_0\right\rangle = 0$$

We remain with only two terms

$$\left\langle \mathbf{j}_\alpha\cdot\boldsymbol{\nabla}\psi\right\rangle = -e_\alpha I\left\langle \int d^3v\ \left(\frac{v_{\|}}{\Omega}\right)\left(C_{\alpha e}f_0 - \frac{\partial f_0}{\partial t}\right)\right\rangle$$

The transitory term

$$\left\langle \mathbf{j}_\alpha^{tr}\cdot\boldsymbol{\nabla}\psi\right\rangle = e_\alpha I\left\langle \frac{1}{\Omega}\int d^3v\ v_{\|}\frac{\partial f_0}{\partial t}\right\rangle$$

The frictional term

$$\left\langle \mathbf{j}_\alpha^{frict}\cdot\boldsymbol{\nabla}\psi\right\rangle = -m_\alpha I\left\langle \frac{1}{B}\int d^3v\ v_{\|}C_{\alpha e}f_0\right\rangle$$

26.3.1 The transitory term

The surface average of the radial current in the transitory regime is

$$\left\langle \mathbf{j}_\alpha^{tr}\cdot\boldsymbol{\nabla}\psi\right\rangle = e_\alpha I\left\langle \frac{1}{\Omega}\int d^3v\ v_{\|}\frac{\partial f_0}{\partial t}\right\rangle$$

The function f_0 depends on the first order, f_{-1}.

$$f_0 = -I\left[\left(\frac{v_{\|}}{\Omega}\right) - \overline{\left(\frac{v_{\|}}{\Omega}\right)}\right]\frac{\partial f_{-1}}{\partial\psi}$$

and

$$\frac{\partial f_0}{\partial t} = -I\left[\left(\frac{v_{\|}}{\Omega}\right) - \overline{\left(\frac{v_{\|}}{\Omega}\right)}\right]\frac{\partial^2 f_{-1}}{\partial\psi\partial t}$$

and we need the time derivative of the distribution function in order -1.

$$\begin{aligned}\frac{\partial f_{-1}}{\partial t} &= \dot{n}_\alpha\frac{1}{4\pi v^2}\delta\left(v - v_0\exp\left(-\nu_s t\right)\right)\\ &+\frac{\partial\dot{n}_\alpha}{\partial t}\frac{1}{\nu_s}\frac{1}{4\pi v^3}\left[\Theta\left(v - v_0\exp\left(-\nu_s t\right)\right) - \Theta\left(v - v_0\right)\right]\end{aligned}$$

With this term we return to the expression of the function $\partial f_0/\partial t$ which we use further to calculate the transitory part of the radial alpha current

$$\begin{aligned}\left\langle \mathbf{j}_\alpha^{tr} \cdot \boldsymbol{\nabla}\psi \right\rangle &= -e_\alpha I^2 \frac{v_0^2}{2} \mathcal{I} \frac{\partial}{\partial \psi} \left[\overset{\cdot}{n}_\alpha (\psi, 0) \exp(-2\nu_s t) \right. \\ &\left. + \int_0^t d\tau \exp(-2\nu_s \tau) \frac{\partial}{\partial t} \overset{\cdot}{n}_\alpha (\psi, t-\tau) \right]\end{aligned}$$

In this expression it is further assumed that the source is constant. This makes disappear the second term in the bracket. The result is

$$\left\langle \mathbf{j}_\alpha^{tr} \cdot \boldsymbol{\nabla}\psi \right\rangle = -e_\alpha I^2 \frac{v_0^2}{2} \mathcal{I} \frac{\partial}{\partial \psi} \left[\overset{\cdot}{n}_\alpha (\psi, 0) \exp(-2\nu_s t) \right]$$

and shows that the torque lasts only on the duration of the *slowing down* time, τ_s.

Note. This result may seem strange: we have continuous creation of the alpha particles, which results from taking

$$\overset{\cdot}{n}_a (\psi, t) = const$$

and we know that every new alpha particle will make a motion toward its final position on the banana trajectory, with an associated radial current. Therefore we have continuous events of radial currents and torques, at the limit a continuous torque applied on plasma. It is true that every individual act of alpha-motion and radial current is finite and has no further effect. The alpha particle and its associate electrons become usual particles of a population that has been Maxwellianized by slowing down due to collisions with electrons. Everything that they will do further will be symmetric, etc. similar to what happens for protons of the background gas.
End.

The quantity

$$\mathcal{I} = \sum_\sigma \left\langle \frac{1}{2\Omega} \int B d\lambda \left[\frac{\xi}{\Omega} - \overline{\frac{\xi}{\Omega}} \right] \right\rangle$$

where

$$\xi \equiv \left| \sqrt{1 - \lambda B} \right|$$

$$\begin{aligned}\mathcal{I} &= \frac{(2\varepsilon)^{3/2}}{\Omega_{\alpha 0}^2} \left\{ \frac{8}{9\pi} + \int_0^1 \frac{dk}{k^{5/3}} \left[\frac{2}{\pi} E\left(k^{1/2}\right) - \frac{\pi}{2K\left(k^{1/2}\right)} \right] \right\} \\ &\simeq \frac{(2\varepsilon)^{3/2}}{\Omega_{\alpha 0}^2} \times 0.38\end{aligned}$$

26.3.2 The collisional (frictional) term

The collisional term

$$\begin{aligned}\left\langle \mathbf{j}_{\alpha}^{frict} \cdot \boldsymbol{\nabla}\psi \right\rangle &= -m_{\alpha} I \left\langle \frac{1}{B} \int d^3v \ v_{\parallel} C_{\alpha e} f_0 \right\rangle \\ &= -e_{\alpha} I \left\langle \frac{1}{\Omega} \int d^3v \ v_{\parallel} \ \nu_s \frac{1}{v^2} \frac{\partial}{\partial v} \left(v^3 f_0 \right) \right\rangle\end{aligned}$$

Here we replace

$$f_0 = -I \left[\left(\frac{v_{\parallel}}{\Omega} \right) - \overline{\left(\frac{v_{\parallel}}{\Omega} \right)} \right] \frac{\partial f_{-1}}{\partial \psi}$$

The loss condition

$$2I \frac{v_{\parallel}}{\Omega} > |\psi - \psi_s|$$

where ψ_s is the value of the magnetic potential at the plasma surface.

26.4 The equation of the toroidal momentum

The baisc equation is the momentum conservation

$$\begin{aligned}&m_i n_i \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla}) \mathbf{u} \right] \\ =\ & -\boldsymbol{\nabla} (p_e + p_i) - \boldsymbol{\nabla} \cdot \boldsymbol{\Pi}_i \\ & + \frac{1}{c} \mathbf{j} \times \mathbf{B} \\ & + \mathbf{F}\end{aligned}$$

where the forces are

$$\begin{aligned}&\mathbf{F} \\ =\ & \mathbf{F}_{i\alpha} \quad \text{(collisional momentum transfer to ions)} \\ & + \mathbf{F}_{e\alpha} \quad \text{(collisional momentum transfer to electrons)} \\ & + \mathbf{F}_{if} \quad \text{(collisional momentum transfer to fast ions)} \\ & + \mathbf{F}_{ef} \quad \text{(collisional momentum transfer to fast electrons)}\end{aligned}$$

The approximation is

$$\mathbf{F} \simeq \mathbf{F}_{i\alpha}$$

i.e. the NBI ions are ommitted. Also loss on ions.

The equation is multiplied with

$$\times R \widehat{\mathbf{e}}_{\varphi}$$

and averaged over the magnetic surface

$$\begin{aligned} & m_i n_i \frac{\partial}{\partial t} \langle u_\varphi R \rangle + m_i n_i \langle R \widehat{\mathbf{e}}_\varphi \cdot (\mathbf{u} \cdot \boldsymbol{\nabla}) \mathbf{u} \rangle \\ = & - \langle R \widehat{\mathbf{e}}_\varphi \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{\Pi}_i) \rangle \\ & + \frac{1}{c} \langle \mathbf{j} \cdot \boldsymbol{\nabla} \psi \rangle \\ & + \langle R \widehat{\mathbf{e}}_\varphi \cdot \mathbf{F}_{e\alpha} \rangle \end{aligned}$$

The surface average of the term containing the convective momentum derivative is zero

$$\langle R \widehat{\mathbf{e}}_\varphi \cdot (\mathbf{u} \cdot \boldsymbol{\nabla}) \mathbf{u} \rangle = 0$$

The reason for which we need a current radial in plasma opposite to the current of the alphas, results from the Ampere equation

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 (\mathbf{j}_\alpha + \mathbf{j}) + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

the left hand side term is zero. The equation is multiplied with $\boldsymbol{\nabla}\psi$ to obtain the radial component and averaged over the surface

$$0 = \langle (\mathbf{j}_\alpha + \mathbf{j}) \cdot \boldsymbol{\nabla} \psi \rangle + \varepsilon_0 \left\langle \frac{\partial \mathbf{E}}{\partial t} \cdot \boldsymbol{\nabla} \psi \right\rangle$$

The term with the electric field will aquire the coefficient

$$\varepsilon_0 \left(1 + \frac{c^2}{v_{A^2}} \right) \gg 1$$

and it will be neglected. Then

$$\langle \mathbf{j}_\alpha \cdot \boldsymbol{\nabla} \psi \rangle = - \langle \mathbf{j} \cdot \boldsymbol{\nabla} \psi \rangle$$

Then one obtains

$$\begin{aligned} m_i n_i \left\langle R \frac{\partial u_\varphi}{\partial t} \right\rangle &= - \langle R \widehat{\mathbf{e}}_\varphi \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{\Pi}_i) \rangle \\ & \quad - \langle \mathbf{j} \cdot \boldsymbol{\nabla} \psi \rangle - \langle R \widehat{\mathbf{e}}_\varphi \cdot \mathbf{F}_{\alpha e} \rangle \end{aligned}$$

The last two terms in the right will cancel each other.

27 Ion orbit loss rate in tokamak (Shaing [11])

This part (rotation due to ion-loss) is also in *plasma, general, ion_ loss notes.*

To explain the H-mode in tokamaks. The loss of ions is localized in the high energy part of the distribution function, since here the ions are less collisional. Being less collisional, they have rather clear banana trajetories and in their motion they can hit the limiter.

But the expulsion of a "hot" ion from the plasma is simultaneously compensated (electrically) by the entrance of an ion from the exterior of the plasma toward the interior. This influx is driven by the ion viscosity which is essentially determined by a non-zero rotation of the plasma.

Note

It must be clarified in a physical picture how the viscosity acts to sustain the return current which compensates the ion-loss.

The ion viscosity is a transport of momentum. It is probably collisional. It seems that the rotation of plasma is essential.

Why the electrostatic charge created in the entire plasma column by the loss of a cherged ion is not a sufficient force to attract and swallow a cold ion from SOL just to maintain neutrality.

End.

It concerns the lower temperature ions, *i.e.* the ions of the Pfirsch-Schluter-plateau, more collisional. This again rises the problem of Stix : it is necessary to have the poloidal rotation + collisions in order to get a radial electric current. This current will participate in the radial component of the $\nabla \times \mathbf{B}$ equation, to compensate (together with the polarization electric current) the outflux of directly lost hot ions. Also in Roenbluth it is the current of compensation (return current) of the lost very-hot-ions at NBI.

This problem of balance of ion radial currents can be re-stated: the loss of ions is the primary process; but the plasma establishes an electric radial field andthe corresponding rotation in order to obtain, through the ion-viscosity, a radial electric current of ions which balances the outgoing ion flux.

- the *outgoing* ion orbit loss flux (in **banana regime**) , is balanced by
- incoming *viscosity driven* ion flux (plateau-Pfirsch-Schluter)

to mentain ambipolarity at the steady state.

Nota. It must be understood that the *disappearence of an ion (* whose banana orbit hits the wall or the limitator) from the plasma **should not be seen as a current directed to the wall**. On the contrary, one expects that the ion which will replace the lost ion, will come from the plasma border toward the centre. So, there is a current of response, the so-called *incoming current* or *return current.*

The effect of viscosity is separated in two in order to emphasize the two different effects:

1. "viscosity-driven flux" is the flux driven by the viscosity contributed by the particles in the regime **plateau-Pfirsch-Schluter**. Actually is the flux of ions of replacement, the return current.

2. "ion orbit loss flux" is the flux driven by the viscosity of the particles in the banana regime. (This is because the loss of ions is due to the difference in the radial drifts of the electrons and the ions, when the drifts are not too much perturbed by the collisions , *i.e.* in the banana regime). But it is not clear how the viscosity is involved in the **direct loss of banana ions to the limiter**. It is first said that the high energy ions are likely to leav plasma since (1) they have large bananas, and (2) they are less collisional.

When the two fluxes (outcoming and incoming) are integrated over the velocity space, they *approximately cancel* each other. The **net** ion flux cancels to order $\sqrt{m_e/m_i}$ *when E_r is determined proprely from the momentum balance equation.*

NOTE. In the paper Shaing insists on the difference between the *ion loss current* and *the plasma current density.* The latter is the current formed by ions which replace the ions lost by the intersection of their banana with the wall.

The torque associated to the ion orbit loss flux is counterbalanced by the torque associated with the viscosity. At steady state there is no net radial current across flux surfaces and there is no net torque applied on the plasma.

Various currents which constitutes the radial plasma current:

- the **ion orbit loss current** $e\Gamma_{orbit}$
- the **viscosity-driven** current;
- the **polarization** current;

At *steady state*:

- the radial current density j_r which is proportional with $\partial E_r/\partial t$ **and** the polarization current vanish

- the ion orbit loss current $e\Gamma_{orbit}$ is balanced by the viscosity-driven flux

The equation used by Shaing

$$\left(v_{\|}\widehat{\mathbf{n}}+\mathbf{V}_E\right)\cdot\boldsymbol{\nabla}\theta\frac{\partial f}{\partial\theta}+\mathbf{v}_d\cdot\boldsymbol{\nabla}\psi\frac{\partial f}{\partial\psi}=C\left(f\right)$$

where $\mathbf{V}_E$ is the electric velocity. The equation is simply $\left(v_{\|}\widehat{\mathbf{n}}+\mathbf{v}_d+\mathbf{V}_E\right)\cdot\boldsymbol{\nabla}f=C\left(f\right)$: *without explicit time derivative* (since the process here is not dynamic, as for example when we study the decay of plasma rotation by the torque generated from the radial electric current of ions (compared to electrons) in the presence of **rotation** and **collisions**, see Stix; also, without *energetic term* in the drift kinetic equation; this should be accounted for in other cases, as for example the **transit time magnetic pumping**, where the viscosity is noncollisional. This appears in [12] where the equation is $\left(u\widehat{\mathbf{n}}+\mathbf{v}_d+\mathbf{V}\right)\cdot\boldsymbol{\nabla}f+\dot{w}\partial f/\partial w=0$; and the equation is so because there is a strong plasma rotation composed of : *diamagnetic*, *electric* and *parallel.*

The $\mathbf{v}\cdot\boldsymbol{\nabla}f$ part of the equation (left hand side) The first part in the equation is of this form because the drift kinetic distribution function f is function of only the **poloidal** θ coordinate and **radial** r coordinate. The variables are (the ion charge is e, *i.e.* $e=|e|$):

$$\begin{aligned} & \psi,\theta \\ E &= \frac{v^2}{2}-\frac{|e|}{m_i}\Phi \\ & \mu \end{aligned}$$

The effects of orbit squeezing can be taken into account by employing a new coordinate ψ_* istead of ψ.

$$\psi_*=\psi-\frac{I}{S\Omega}\left(v_{\|}+\frac{I}{\Omega}\frac{B^2}{B_0^2}\frac{e}{m_i}\frac{\partial\Phi}{\partial\psi}\right)$$

where the **squeezing factor** is

$$S=1+\left(\frac{I}{\Omega_0}\right)^2\frac{e}{m_i}\frac{\partial^2\Phi}{\partial\psi^2}$$

NOTE. For comparison, the variable ψ_* representing the *drift surface* is given in [13] as

$$\psi_*\equiv\psi-\frac{v_{\|}}{\Omega}=\text{const}$$

(we could say $\psi_* = \psi - \rho_\|$ but take the *variable* poloidal Larmor radius, with the parallel velocity variable up to zero and change of direction, and with B also variable). Here the function ψ is

$$\psi = \frac{\Psi}{RB_\varphi}$$

where Ψ is the poloidal magnetic flux function. A reasonable approximation for the magnetic field variation in space

$$RB_\varphi = \text{const}$$

If we assume *a constant current density* we obtain

$$\psi = \frac{1}{qR}\frac{r^2}{2}$$

END of the NOTE

NOTE
The second term in

$$v_\| + \frac{I}{\Omega}\frac{B^2}{B_0^2}\frac{e}{m_i}\frac{\partial\Phi}{\partial\psi}$$

represents a correction to the parallel velocity when a radial electric field is present

$$\begin{aligned}
& \frac{RB_\varphi}{\frac{eB}{m_i}}\frac{B^2}{B_0^2}\frac{e}{m_i}\frac{1}{2\pi RB_\theta}\frac{\partial\Phi}{\partial r} \\
\approx & \frac{1}{2\pi}\frac{B_\varphi}{B_\theta}\frac{B}{B_0^2}(-E_r) = \frac{1}{2\pi}\frac{1}{hB_0}\frac{B_\varphi}{B_\theta}(-E_r) \\
= & \frac{1}{2\pi}\frac{B_\varphi}{hB_0}V_r \\
\approx & \frac{1}{2\pi}\frac{1}{h^2}V_r
\end{aligned}$$

Then approximate h.
End.

The quantity Ω_0 is Ω calculated at the magnetic axis. The shear of the electric field Φ'' is considered constant over the width of the banana orbit.

It can be shown that

$$\omega = (v_\|\widehat{\mathbf{n}} + \mathbf{V}_E)\cdot\boldsymbol{\nabla}\theta \simeq -\frac{I}{\Omega}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta\frac{\partial\psi_*/\partial\psi}{\partial\psi_*/\partial E}$$

and

$$\mathbf{v}_d \cdot \boldsymbol{\nabla}\psi = \frac{I}{\Omega}\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta \frac{\partial \psi_* / \partial \theta}{\partial \psi_* / \partial E}$$

We **note**

$$\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta = \frac{\partial \theta}{\partial l_{\|}}$$

with

$$\begin{aligned} dl_\theta &= dl_{\|} \times \cos\left(\widehat{\mathbf{n}}, \widehat{\mathbf{e}}_\theta\right) = dl_{\|} \sin\left(\widehat{\mathbf{n}}, \widehat{\mathbf{e}}_\varphi\right) \\ &= dl_{\|} \frac{B_\theta}{B} \end{aligned}$$

$$\begin{aligned} dl_{\|} &= dl_\theta \frac{B}{B_\theta} = \frac{rB}{B_\theta} d\theta = qRd\theta \\ \frac{\partial \theta}{\partial l_{\|}} &= \frac{1}{qR}, \quad \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta = \frac{1}{qR} \end{aligned}$$

The meaning of the notation is

$$I = R^2 \mathbf{B} \cdot \boldsymbol{\nabla}\varphi = RB_\varphi$$

The drift associated with the poloidal field variation $\partial B / \partial \theta$ have been neglected. Taking this relation into the drift-kinetic equation we get

$$\left(v_{\|}\widehat{\mathbf{n}} + \mathbf{V}_E\right) \cdot \boldsymbol{\nabla}\theta \frac{\partial f}{\partial \theta} + \mathbf{v}_d \cdot \boldsymbol{\nabla}\psi \frac{\partial f}{\partial \psi} \simeq \omega \left. \frac{\partial f}{\partial \theta} \right|_{\psi_*, E, \mu}$$

We note that ω is a frequency which has the role of a coordinate which combines the parallel velocity and electric $\mathbf{E} \times \mathbf{B}$ particle velocities **projected on the poloidal direction**, divided at the local small radius r. **Note** We must check the flux variable ψ_* with the "drift surface" variable, as introduced in the review of Hazeltine and Hinton.

The collision operator Consider only the **pitch-angle scattering** operator

$$C(f) = \frac{\nu_D}{2} \frac{\partial}{\partial \xi} \left(1 - \xi^2\right) \frac{\partial f}{\partial \xi}$$

where ν_D is the deflection collision frequency and

$$\xi = \frac{v_{\|}}{v}$$

is the **pitch angle**.

Change of variables

$$(\psi, \xi) \rightarrow (\psi_*, \omega)$$

Keeping the highest order derivatives, one obtains

$$C(f) \simeq \frac{\nu_D}{2}\left[(v\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta)^2\frac{\partial^2 f}{\partial\omega^2} - 2\frac{Iv^2\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta}{\Omega S}\frac{\partial^2 f}{\partial\omega\partial\psi_*} + \left(\frac{Iv}{\Omega S}\right)^2\frac{\partial^2 f}{\partial\psi_*^2}\right]$$

Here it has been approximated $1-\xi^2 \approx 1$. This is valid since **it is the barely circulating and the barely trapped particles which contribute to the ion orbit loss.**

A new variable is introduced, $\widehat{\omega}$ which makes possible to connect these expressions which will finally give f with the *neoclassical distribution function for ions*:

$$\omega = \sigma\widehat{\omega}\ \widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta\ \left(1 - k\sin^2\frac{\theta}{2}\right)^{1/2}$$

Then

- $k < 1$ corresponds to *poloidally circulating particles,* and
- $1 < k < \infty$ corresponds to the *poloidally trapped particles.*

Here the **direction** of the variable ω is $\sigma = \pm 1$.

NOTE

that we have now a factor that gives the variation of population of trapped particles on magnetic surface

$$\sqrt{1 - \kappa\sin^2(\theta/2)}$$

as in **Galeev Sagdeev**.

End.

Another change of variables is suggested by the form of the **drift flux function**. We take

$$\psi_* = \psi_0 - \frac{I}{\Omega S}\widetilde{\omega}_0$$

assuming

$$\varepsilon \ll 1 \ \text{ but } \ |S|\,\varepsilon < 1$$

then

$$
\begin{aligned}
\widehat{\omega}^2 &= \overline{\omega}_0^2 \frac{B_0^2}{B_x^2} \\
&+2SE+ \\
&+2\left|S\right|\varepsilon\left[\mu B_0+\left(\overline{\omega}_0\frac{B_0}{B_x}-I\frac{1}{B_0}\frac{\partial\Phi}{\partial\psi}\right)^2\right]
\end{aligned}
$$

where

$$
E = E-\mu B_0-\frac{e\Phi}{m_i}-\frac{1}{2}\left(\overline{\omega}_0\frac{B_0}{B_x}-I\frac{1}{B_0}\frac{\partial\Phi}{\partial\psi}\right)^2
$$

The notation $\widetilde{\omega}_0$ represents $\overline{\omega}_0 B/B_x$ where

$$
\overline{\omega}_0 = v_\| + I\frac{1}{B_0}\frac{\partial\Phi}{\partial\psi}
$$

evaluated at $\psi=\psi_0$ and $\theta=0$ or π (depending on where the particles are trapped: $\theta=0$ for inside of a tokamak, $\theta=\pi$ if the particle is trapped outside of the tokamak). B_x is the value of B evaluated at $\theta=0$ or $\theta=\pi$.

27.0.1 Possibility of reprezentation of the effect of the ion orbit loss flux as a force in the balance equations

The effect of the ion orbit loss flux can be *modelled* by a force $\mathbf{\Sigma}$. The basis for this modellization is the fact that **every particle loss mechanism has a force which corresponds to it**. The example is the intrinsically nonambipolar flux of particles which is driven by the **viscous force**

$$
\left\langle \mathbf{\Gamma}_\pi\cdot\mathbf{\nabla}\psi\right\rangle = -\left(\frac{\left\langle R^2\right\rangle}{Ie}\right)\left\langle \mathbf{B}\cdot\mathbf{\nabla}\cdot\mathbf{\Pi}\right\rangle
$$

(this is also in **Shaing Crume Houlberg**) . Similar to this example, *the flux that exists only in the presence of the* **ion orbit loss region** *in the phase space* is considered associated to a force $\mathbf{\Sigma}$. This method, of separating the ion orbit loss flux from the viscous flux is useful, since the viscosity can be calculated in the neoclassical approach but only if we ignore such processes ion orbit loss, which limits the integration on velocity.

Introducing this force $\mathbf{\Sigma}$, one can repeat the calculation of Hirschman to obtain

$$
\begin{aligned}
&\frac{1}{\left\langle B_p^2\right\rangle}\left(\left\langle B^2\right\rangle-\frac{I^2}{\left\langle R^2\right\rangle}\right)\frac{\partial\left\langle \mathbf{V}\cdot\mathbf{B}_p\right\rangle}{\partial t}+\frac{I}{\left\langle R^2\right\rangle nm_i}\left\langle \mathbf{J}\cdot\mathbf{\nabla}\psi\right\rangle \\
=\;&\frac{I}{\left\langle R^2\right\rangle nm_i}\left\langle \mathbf{J}_{orb}\cdot\mathbf{\nabla}\psi\right\rangle-\frac{1}{nm_i}\left\langle \mathbf{B}\cdot\mathbf{\nabla}\cdot\mathbf{\Pi}\right\rangle+\left\langle\left(\frac{I^2}{\left\langle R^2\right\rangle B^2}-1\right)\frac{\mathbf{B}\cdot\mathbf{\Sigma}}{nm_i}\right\rangle
\end{aligned}
$$

where the *radial current due to the orbit ion loss* is related to the force $\mathbf{\Sigma}$

$$\langle \mathbf{J}_{orb} \cdot \nabla \psi \rangle = \left\langle \nabla \psi \cdot \frac{\mathbf{B} \times \mathbf{\Sigma}}{B^2} \right\rangle$$

At steady state, the radial component of the $\nabla \times \mathbf{B}$ equation which is

$$0 = \left(\mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \right)_r$$

will give

$$\begin{aligned} \mu_0 J_r &= 0 \\ \frac{\partial E_r}{\partial t} &= 0 \end{aligned}$$

However, the orbit loss curent $\mathbf{J}_{orb}$ is not zero. It is compensated by the viscosity driven current, which is seen from the equation at stationarity

$$\frac{I}{\langle R^2 \rangle n m_i} \langle \mathbf{J}_{orb} \cdot \nabla \psi \rangle + \left\langle \left(\frac{I^2}{\langle R^2 \rangle B^2} - 1 \right) \frac{\mathbf{B} \cdot \mathbf{\Sigma}}{n m_i} \right\rangle = \frac{1}{n m_i} \langle \mathbf{B} \cdot \nabla \cdot \mathbf{\Pi} \rangle$$

if we neglect the second term in the left hand side.

The meaning of this expression is that: *high energy ions are lost* and *the low energy ions respond by a current driven by their viscosity.*

28 Plasma flows associated with ion orbit loss

28.1 Notes

Effects to be accounted for:

- neoclassical viscosity effect
- torque due to the prompt loss of ions

The physical reason for the formation of the electrostatic potential at the plasma edge is *the difference of the orbit sizes of electrons and ions.* High energy ions (suprathermal component of a tail of a Maxwellian distribution) have the largest orbit width and are lost from the largest distance from the wall : this means that the ions which are lost in a point very close to the the border belong actually to a region deeper into the plasma. Those which have greater parallel velocity but still smaller than that necessary to become transiting particles, are stoped (i.e. have banana tips) in the inner part of the tokamak plasma. They have the larger radial drift on their bananas, which also means that they belong to magnetic surfaces (in the sense that the average of the trajectory is there) deeper into the plasma. When such ion is lost through collision, a positively charged particle from a point deep in the plasma disappears.

29 Toroidal momentum input to Tokamak plasma from neutral beams and generation of rotation (Rosenbluth Hinton)

This is the paper by **Rosenbluth and Hinton**. Similar with another paper by the two authors in which the rotation is generated by the alpha particles.

29.0.1 General

Neutral beam injection will be made **transversal** to plasma, due to problem of accessibility.

Most of the ions will be trapped.

One usually works with a separation of populations:

- *fast ions*
- *plasma*

The mechanism of momentum transfer to the plasma, in the case of injection into trapped ion orbits, is (A) **a $\mathbf{j} \times \mathbf{B}$ torque which results from** *fast ion radial current*. The birth of fast ions and their subsequent orbital motion cause a **radial current**. This current must be cancelled by a **radial current flowing in the rest of the plasma**. This *radial current* causes a torque on the plasma .

In addition, (B) there is a torque from due to **collisional friction with the fast ions**.

$$T = -\left\langle \mathbf{j}_{fast} \cdot \boldsymbol{\nabla}\psi \right\rangle - I \left\langle \frac{1}{B} \left(F_{\parallel fast-e} + F_{\parallel fast-i} \right) \right\rangle \tag{14}$$

In this paper the equation of motion of plasma is

1. multiplied with R,
2. projected along the toroidal direction

3. averaged over a magnetic surface

The equation is

$$\begin{aligned} m_i n_i \frac{\partial}{\partial t} \langle u_\phi R \rangle &= -\langle R\widehat{\mathbf{e}}_\phi \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Pi}_i \rangle \ \ \text{(toroidal projection of the DIV(viscosity tensor))} \\ &+ \frac{1}{c} \langle \mathbf{j} \cdot \boldsymbol{\nabla} \psi \rangle \quad \text{(radial current)} \\ &+ \langle R\widehat{\mathbf{e}}_\phi \cdot \mathbf{F} \rangle \end{aligned}$$

where it is taken into account that

$$\langle R\widehat{\mathbf{e}}_\phi \cdot (\mathbf{u} \cdot \boldsymbol{\nabla})\, \mathbf{u} \rangle \equiv 0$$

From the equation of balance it results that when there is a change in the *inertia* term due to a change in the toroidal velocity, a radial current is generated. This may induce an effect on the poloidal direction. Normally this should connect the two velocities. The change in the inertial term $\partial/\partial t$ apparently induces a *polarization* effect, expressed by the flowing of a current in the plasma. This current, Hinton and Rosenbluth consider that it is a *return* current and use the Ampere 's law to balance the two currents: one of the particles, the other is the *return* current and the electric field derivative to time is negligible due to the very small value of the perpendicular dielectric.

The *torque* applied on the plasma in the toroidal direction is **the right hand side of the momentum equation after projection and averaging**

$$T \equiv \frac{1}{c} \langle \mathbf{j} \cdot \boldsymbol{\nabla} \psi \rangle + \langle R\widehat{\mathbf{e}}_\phi \cdot (\mathbf{F}_{ef} + F_{if}) \rangle$$

An interesting formula

$$\langle R\widehat{\mathbf{e}}_\phi \cdot (\mathbf{F}_{ef} + F_{if}) \rangle = I \left\langle \frac{1}{B} \left(F_{\| ef} + F_{\| fi} \right) \right\rangle$$

One writes the **drift-kinetic equation** for the *fast ions.*

29.1 NBI-fast ion drift-kinetic equation Hinton Rosenbluth

In the Ref. [8] the following equation is used to describe the fast ions from neutral beam injection:

$$\frac{\partial f}{\partial t} + \left(v_\| \widehat{\mathbf{b}} + \mathbf{v}_d \right) \cdot \boldsymbol{\nabla} f = C_{fast}\, f + S_{fast}$$

where the drift velocity of the guiding centre is

$$\mathbf{v}_d = -v_{\parallel}\widehat{\mathbf{b}} \times \boldsymbol{\nabla}\left(\frac{v_{\parallel}}{\Omega}\right)$$

NOTE

In the paper about the rotation due to alpha particles, the radial component of the drift velocity is

$$\begin{aligned}
\mathbf{v}_d \cdot \boldsymbol{\nabla}\psi &= I v_{\parallel}\left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta\right)\frac{\partial}{\partial\theta}\left(\frac{v_{\parallel}}{\Omega}\right) \\
&= I\frac{v_{\parallel}}{qR}\frac{\partial}{\partial\theta}\left(\frac{v_{\parallel}}{\Omega}\right)
\end{aligned}$$

END

The notations are introduced

$$\begin{aligned}
\xi &= \frac{v_{\parallel}}{v} \quad \text{(pitch angle)} \\
&= (1-\lambda B)^{1/2} \\
E &= \frac{v^2}{2} \quad \text{(kinetic energy)} \\
\mu &= \lambda E \quad \text{(magnetic moment invariant)}
\end{aligned}$$

where

$$\begin{aligned}
\mu &\equiv \frac{v_{\perp}^2}{2B} = \lambda\frac{v^2}{2} \quad \rightarrow \quad \lambda \equiv \frac{1}{B}\frac{v_{\perp}^2}{v^2} \\
B\lambda &= \frac{v_{\perp}^2}{v^2} \\
\text{such that } \ 1 - B\lambda &= \frac{v_{\parallel}^2}{v^2} = \xi^2
\end{aligned}$$

The **spatial variables** are

$$\psi, \theta, \phi$$

and the **velocity space** variables are

$$v, \lambda, \sigma \ (= \text{sign of } v_{\parallel}\,)$$

The **fast ion collision term** is

$$C_{fast}f = C_{fast-e}f + C_{fast-i}f$$

where

$$C_{fast-e}f = \nu_s\frac{1}{v^2}\frac{\partial}{\partial v}\left(v^3 f\right)$$

$$C_{fast-i}f = \frac{2m_i}{m_f}\nu_s \frac{v_c^3}{v^3}\frac{\xi}{B}\frac{\partial}{\partial\lambda}\lambda\xi\frac{\partial f}{\partial\lambda}$$

where the **fast ion slowing down rate** is

$$\nu_s = \frac{m_e}{m_f}\frac{Z_f^2}{\tau_e}$$

and the **electron collision time** is

$$\tau_e = \frac{16\sqrt{\pi}}{3}\frac{n_e e^4 \ln\Lambda}{m_e^2 v_e^3}$$

and the **critical speed,** at which the collisional transfer of energy from fast ions to elecrons equals that from fast ions to background ions (see **NBI**)

$$v_c = \left(\frac{3\sqrt{\pi}}{4}\frac{m_e}{m_i}\right)^{1/3} v_e$$

with the thermal electron velocity

$$v_e = \sqrt{\frac{2T_e}{m_e}}$$

In this treatment it is assumed that:

- The velocity of the fast ions is somewhere between the *critical* and *electron thermal* velocities

  $$v_c \ll v \ll v_e$$

  this assumption is rather strange, since the injected neutrals have high energy. Probably the slowing down process produces a *hump* in the distribution function, which allows the definition of a fast-ion temperature or an average velocity, smaller than that of the electrons. This condition is probably used in the expression of the *collision operators.* What ever they do, the *fast* ions are SLOWER than the thermal electrons.

- The fast ion density is small

  $$n_f \ll n_e$$

- Slowing down to thermal ions can be neglected, the transfer of energy from fast ions is to electrons

- The **pitch angle scattering is kept**. This means that we consider the displacements of the fast ions in the velocity space, due to collisions: trapped, untrapped.

The following parameters are considered small and this permits the perturbative solution of the drift-kinetic equation: **the ratio of the slowing down rate of the fast ions to the transit time frequency**: the fast ion performs many bounces or transits in the typical time of slowing down.

$$\frac{\nu_s}{\left(\frac{v_{th}}{qR}\right)} \ll 1$$

And **the ratio of the guiding centre drift frequency to the bounce frequency**

$$\frac{\frac{v_d}{L_{fast}}}{\frac{v_{th}}{qR}} \ll 1$$

The two parameters are considered of the same order of magnitude δ and both very small, $\delta \ll 1$. **Note**: there is no reference to a space-type small parameter like the ratio of the banana width to the minor radius.

The perturbative expansion of f for the solution of the drift-kinetic equation for the fast ions.

$$f = f_{-1} + f_0 + f_1 + ... \tag{15}$$

The lowest order

$$\widehat{\mathbf{b}} \cdot \boldsymbol{\nabla} f_{-1} = 0$$

This means that the lowest order function f_{-1} is **constant** along the magnetic lines and then on the **magnetic surfaces** but it says nothing about the velocity space dependence.

The **zeroth order equation** is

$$\begin{aligned} v_{\parallel}\widehat{\mathbf{b}} \cdot \boldsymbol{\nabla} f_0 &= -\mathbf{v}_d \cdot \boldsymbol{\nabla}\psi \frac{\partial f_{-1}}{\partial \psi} - \frac{\partial f_{-1}}{\partial t} + \\ &\quad + C_{fast} f_{-1} + S_{fast} \end{aligned} \tag{16}$$

We note that the scalar product with $\mathbf{v}_d$ retains only the **radial** derivative in the gradient of f_{-1} since it does not vary in the surface. The function f_0 should be sensitive to the spacial modifications which are introduced by:

- the radial dependence of the distribution function f_{-1} (due to static plasma gradients, like in density) and

- the space changes due to the source of fast ions and due to collisions

By **bounce averaging** the function f_0 is eliminated. The operator is

$$\overline{A} = \frac{1}{T} \oint \frac{d\theta}{v_\parallel \widehat{\mathbf{b}} \cdot \boldsymbol{\nabla}\theta} A$$

where the **bounce time** is

$$T = \oint \frac{d\theta}{v_\parallel \widehat{\mathbf{b}} \cdot \boldsymbol{\nabla}\theta}$$

The *limits of integrations* are

$$-\pi \ , \ \pi \text{ for untrapped ions}$$

and for trapped ions the integral is defined

$$\oint d\theta = \sum_\sigma \sigma \int_{\theta_1}^{\theta_2} d\theta$$

where θ_1 and θ_2 are the *turning points.*

We write the **radial part of the guiding centre drift velocity**

$$\mathbf{v}_d \cdot \boldsymbol{\nabla}\psi = I v_\parallel \widehat{\mathbf{b}} \cdot \boldsymbol{\nabla}\theta \frac{\partial}{\partial\theta} \left(\frac{v_\parallel}{\Omega} \right) \tag{17}$$

Nota: this formula can be written

$$v_{d,r} = I \frac{v_\parallel}{qR} \frac{\partial}{\partial\theta} \left(\frac{v_\parallel}{\Omega} \right)$$

By performing the bounce averaging the radial displacements averages to zero (**because there is no electric field and no ripple**)

$$\overline{(\mathbf{v}_d \cdot \boldsymbol{\nabla}\psi)} = 0$$

The **bounce averaged zeroth-order** equation

$$\frac{\partial f_{-1}}{\partial t} = \overline{C}_{fast} f_{-1} + \overline{S}_{fast} \tag{18}$$

Note. It results that the lowest order function f_{-1} which is constant on the magnetic surfaces has a variation in the velocity space given by the *source* of fast ions and *the collisions.* It will result that the **source** is essential for a stationary **radial current** of fast ions, even if the collisions and the limitator do not remove ions from plasma.

Before solving the equation, we calculate the **radial current of fast ions**.

The *magnetic surface-averaged radial fast ion current density* is

$$\langle \mathbf{j}_{fast} \cdot \boldsymbol{\nabla}\psi \rangle = e_{fast} \left\langle \int d^3 v \, \mathbf{v}_d \cdot \boldsymbol{\nabla}\psi \, f \right\rangle$$

where we have to replace f by the solution of the *drift-kinetic equation for the fast ions*. Then f here should be seen as $f_{-1} + f_0 + \ldots$.

The following **surface average operator** is used

$$\langle A \rangle = \frac{\oint \frac{d\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta} A}{\oint \frac{d\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}}$$

The equation (17) gives

$$\begin{aligned}
\langle \mathbf{j}_{fast} \cdot \boldsymbol{\nabla}\psi \rangle &= e_{fast} \left\langle \int d^3 v \, \mathbf{v}_d \cdot \boldsymbol{\nabla}\psi \, f \right\rangle \\
&= e_{fast} \left\langle \int d^3 v \; I v_\parallel \widehat{\mathbf{b}} \cdot \boldsymbol{\nabla}\theta \frac{\partial}{\partial\theta} \left(\frac{v_\parallel}{\Omega} \right) f \right\rangle
\end{aligned}$$

and we perform an integration by parts in θ:

$$e_{fast} \left\langle \int d^3 v \, \mathbf{v}_d \cdot \boldsymbol{\nabla}\psi \, f \right\rangle = -e_{fast} I \left\langle \int d^3 v \; \left(\frac{v_\parallel}{\Omega} \right) v_\parallel \widehat{\mathbf{b}} \cdot \boldsymbol{\nabla} f \right\rangle$$

Now we must replace f by its expansion (**??**). The lowest order f_{-1} does not depend on the parallel coordinate (in the surface) so it will not contribute. The first contribution is from the term f_0. This is obtained in equation (16) where we have to insert the equation (18):

$$\begin{aligned}
v_\parallel \widehat{\mathbf{b}} \cdot \boldsymbol{\nabla} f_0 &= -\mathbf{v}_d \cdot \boldsymbol{\nabla}\psi \frac{\partial f_{-1}}{\partial\psi} \\
&\quad + C_{fast} f_{-1} - \overline{C}_{fast} f_{-1} + S_{fast} - \overline{S}_{fast}
\end{aligned}$$

The first term on the right will not contribute after **surface-averageing**

$$\begin{aligned}
e_{fast} I \left\langle \int d^3 v \; \left(\frac{v_\parallel}{\Omega} \right) \mathbf{v}_d \cdot \boldsymbol{\nabla}\psi \frac{\partial f_{-1}}{\partial\psi} \right\rangle &= e_{fast} I \left\langle \int d^3 v \; \left(\frac{v_\parallel}{\Omega} \right) I v_\parallel \widehat{\mathbf{b}} \cdot \boldsymbol{\nabla}\theta \frac{\partial}{\partial\theta} \left(\frac{v_\parallel}{\Omega} \right) \frac{\partial f_{-1}}{\partial\psi} \right\rangle \\
&= \frac{1}{2} e_{fast} I^2 \left\langle \int d^3 v \; v_\parallel \widehat{\mathbf{b}} \cdot \boldsymbol{\nabla}\theta \frac{\partial}{\partial\theta} \left[\left(\frac{v_\parallel}{\Omega} \right)^2 \right] \frac{\partial f_{-1}}{\partial\psi} \right\rangle
\end{aligned}$$

The integrand is the product of a *odd function* of $v_\parallel$ ($v_\parallel$) with an *even function* of $v_\parallel$ (f_{-1}). The integral on the velocity space is **zero**. This result is great importance: it shows that the surface averaged radial current of the ions is zero. If we expected to have an effective current simply due to the difference in radial drift motions between the ions and the electrons, this result shows clearly that it cannot exist. (A particularity of this calculation is that we assume that the current is entirely due to the ions, *i.e.* the electrons are considered tied to the magnetic lines. This changes nothing since actually the electrons really have small radial drifts compared to ions. The total radial current is practically the ions radial current.)

The radial current due to the difference between iond and electron drifts exists, but only locally. When it is integrated over the magnetic surface, it gives zero.

In conclusion we have to find somewhere else to produce an effective radial current (*i.e.* whose surface average is not zero). We find:

- the collisions, and
- the source of fast ions

NOTE We have to compare this with the work of Stix 73 where the surface averaged radial current seems to depend on the plasma rotation. If there is no plasma rotation (in that paper's notation, $U = W = 0$ then it still remains in the expression of the "flow" the *diamagnetic velocity.*

The **surface-averaged** radial current of fast ions is

$$\langle \mathbf{j}_{fast} \cdot \boldsymbol{\nabla}\psi \rangle = -e_{fast} \left\langle \int d^3v \, \left(\frac{v_\parallel}{\Omega}\right) \left[C_{fast} f_{-1} - \overline{C}_{fast} f_{-1} + S_{fast} - \overline{S}_{fast} \right] \right\rangle$$

NOTE ON THE TORQUE

We can continue along this line and use the above expression of the *surface-averaged radial fast ion current* to calculate the torque exessed on plasma at injection. It must be added the **torque due to the frictional forces** in the lowest order of the distribution function, f_{-1}:

$$\mathbf{F}_{fast-e} + \mathbf{F}_{fast-i} = \int d^3v \, m_{fast} \mathbf{v} \, C_{fast} f_{-1}$$

It results from the equation (14)

$$T = -m_{fast} \left\langle \frac{I}{B} \int d^3v \, v_\parallel \left[\overline{C}_{fast} f_{-1} - S_{fast} + \overline{S}_{fast} \right] \right\rangle$$

and using the equation (18)

$$T = -m_{fast} \left\langle \frac{I}{B} \int d^3v \; v_\parallel \left[\frac{\partial f_{-1}}{\partial t} - S_{fast} \right] \right\rangle$$

If the injection is made in **trapped particle region** the function f_{-1} will not depend on the **direction** of the parallel velocity so its integral is zero. The result is then

$$\begin{aligned} T &= m_{fast} \left\langle \frac{I}{B} \int d^3v \; v_\parallel S_{fast} \right\rangle \\ &= \left\langle \dot{M}_\phi \right\rangle \end{aligned} \tag{19}$$

The conclusion is that the torque exists only if there is a source of fast ions.

Also at stationarity, the time-derivative of the distribution function is zero and, for **trapped and untrapped** particles, the torque is given by equation (19) which reflects **the angular momentum conservation in the neutral injection**.

The equation (18) must be solved for

trapped ions
untrapped ions

Solution of the equation for the lowest order distribution function of the fast ions injected into untrapped orbits, f_{-1}^{untr} The equation is (18) and gives the nostationar dist. function from the source and the collisions averaged over the bounce. The notation f_{-1}^{untr} will be simplified to f. The bounce-averaged collision operator is

$$\begin{aligned} \overline{C}_{fast} f &= 2 \frac{m_i}{m_{fast}} \frac{\nu_s v_c^3}{v^3} \frac{1}{I_2(\lambda)} \frac{\partial}{\partial \lambda} \lambda I_1(\lambda) \frac{\partial f}{\partial \lambda} + \\ &+ \frac{\nu_s}{v^2} \frac{\partial}{\partial v} \left(v^3 f \right) \end{aligned}$$

where

$$I_1(\lambda) = \int_{-\pi}^{\pi} \frac{d\theta}{\mathbf{B} \cdot \boldsymbol{\nabla} \theta} \xi = \int_{-\pi}^{\pi} \frac{r d\theta}{B_\theta} \xi$$

$$I_2(\lambda) = \int_{-\pi}^{\pi} \frac{d\theta}{\widehat{\mathbf{b}} \cdot \boldsymbol{\nabla} \theta} \frac{1}{\xi} = \int_{-\pi}^{\pi} r d\theta \frac{1}{\xi}$$

where

$$\begin{aligned}\xi &= \frac{v_{\|}}{v} \\ &= (1-\lambda B)^{1/2}\end{aligned}$$

The *bounce-averaged source term* for injection into trapped orbits is

$$\overline{S}_{fast} = N(\psi)\,\delta_{\sigma,\sigma_0}\Theta(t)\,\delta(\lambda-\lambda_0)\,\frac{\delta(v-v_0)}{\pi v_0^2}$$

where

$$N(\psi) = \frac{\int_{-\pi}^{\pi}\frac{d\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}\dot{n}_{fast}}{\int_{-\pi}^{\pi}\frac{d\theta}{\widehat{\mathbf{b}}\cdot\boldsymbol{\nabla}\theta}\frac{1}{|\xi_0|}}$$

The function step Θ is introduced since the injection begins at a specific moment.

30 Plasma rotation due to alpha particles generated in fusion reactions

The origin of the rotation is the **alpha birth current**, a radially directed current due to the fact that the alpha particles after creation evolveto get their neoclassical trajectories. They have a radial drift.

The equation for the distribution function of the *alpha* particles

$$\frac{\partial f_\alpha}{\partial t} + \left(\mathbf{v}_d + v_{\|}\widehat{\mathbf{n}}\right)\cdot\boldsymbol{\nabla} f_\alpha = C_{\alpha e} f_\alpha + S_\alpha$$

The drift velcoity is

$$\mathbf{v}_d = -v_{\|}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\left(\frac{v_{\|}}{\Omega}\right)$$

where

$$\begin{aligned}\Omega_\alpha &= \frac{e_\alpha B}{m_\alpha} \\ v_{\|} &= v(1-\lambda B)^{1/2}\end{aligned}$$

The energy and magnetic moment are

$$\begin{aligned}E &= \frac{v^2}{2} \\ \mu &= \lambda E\end{aligned}$$

The variables are

$$(v, \lambda, \sigma)$$

where σ is the sign of the parallel velocity.

The alpha particle - electrons collision operator is

$$C_{\alpha e} f = \nu_s \frac{1}{v^2} \frac{\partial}{\partial v} \left(v^3 f \right)$$

NOTE that this is only the *slowing down* on electrons, while in the case of the plasma rotation due to the NBI. **END**.

The source of alphas

$$S_\alpha = \frac{\dot{n}_\alpha (\psi, t)}{4\pi v_0^2} \delta (v - v_0)$$

where v_0 is the velocity of the alpha particle at the birth event. The rate of generation of alphas is $\dot{n}_\alpha (\psi, t)$ and depends on the radius and time.

31 Spontaneous toroidal rotation

31.1 Observations

It is observed in **TCV** as toroidal rotation :

1. counter-current for central (core) plasma in the L-mode
2. co-current for central plasma in the H-mode
3. there is a transition, fast and sharp, 100 $m\,$sec. In **TCV**.

See also **Rice**. This is the paper of **Rice 2003.**

The toroidal speed characterised by the *Alfvenic Mach* number

$$\begin{aligned} M_A &= \frac{v_\varphi}{c_A} \\ &= \frac{B^2}{\mu_0 n_e m_{ave}} \end{aligned}$$

$$M \simeq 0.3$$

In the H mode, there is a scaling of the Alfvenic Mach number with the plasma pressure is

$$M_A \sim \beta_N$$

In H mode the *spontaneous* toroidal rotation is co-current. If the current direction is reversed, the spontaneous toroidal rotation changes too, such as to be again co-current.

Note the electric current I_p is the motion of the ions (as direction). Looking along the axis in the direction where the electric current flows, the magnetic lines are helical in the clockwise direction.

The plasma motion (toroidal rotation) is the motion of ions and is in the same direction as the electric current. Therefore the the toroidal plasma rotation *is compatible with and enhances the current.* **End**.

Note that high toroidal rotation reduces the number of trapped ions and reduces the ITG turbulence.

Strong correlation between toroidal velocity and plasma pressure or stored energy.

$$\begin{aligned} v_\varphi &\sim \frac{W}{I_p} \quad \text{or} \\ &\sim \frac{T_e(0)}{T_i(0)} \frac{W}{I_p} \end{aligned}$$

$$v_\varphi \sim \frac{1}{I_p}$$

No correlation between Mach number with *normalized gyro-radius*

$$\rho_* = 1.02 \times 10^{-4} \sqrt{\mu(AMU) \frac{T(eV)}{B(T)\, a(m)}}$$

$$M_A \;\; \text{not correlated with } \rho_*$$

No correlation of rotation with *collisionality.*

In the paper of **Ware Wiley** it is given the following explanation for the toroidal rotation.

There is a substantial effect of the largeness of

$$\rho_{i\theta}$$

In particular this means that there is an *asymmetry* of the boundary of the region of trapped particles, in the velocity space. The asymmetry is around

$$v_\parallel = 0$$

plane. (Tip of the banana orbit),

For

$$v_{\parallel} > 0$$

only a *small* fraction of particles are trapped because mirroring requires

$$mv_{\parallel} < e\psi$$

where ψ is the small poloidal flux that can be accumulated in the region between $r = 0$ and r.

For

$$v_{\parallel} < 0$$

the fraction of particles that are trapped is larger because they mirror at larger minor radius r.

So, there is an asymmetry in the trapped particle fraction in the region around $v_{\parallel} = 0$.

The large value of the poloidal Larmor radius $\rho_{i\theta}$ is implied in this asymmetry. It is cited **Goldston** thesis 1977.

This asymmetry does not have an important effect on the radial fluxes.

But the asymmetry has a strong effect on the *parallel-radius* component of the ion pressure tensor

$$P_{i\parallel r}$$

The trapped particles with $v_{\parallel} < 0$ conducting heat out also carry momentum

$$-m_i \left| v_{\parallel} \right|$$

which is NOT cancelled by the trapped particles that have $v_{\parallel} > 0$. For a given gradient of ion temperature

$$T_i'$$

the heat radially-transported flux is

$$q_i$$

and the order of magnitude of the parallel-radial ion pressure tensor component is

$$P_{i\parallel r} \approx \left(\frac{r}{R}\right)^{1/2} \frac{q_i}{v_{Th,i}}$$

Relative to the small-Larmor radius result the above result is larger by a factor

$$\left(\frac{R}{r}\right)^{1/2}$$

Ware Wiley conclude that there is a strong parallel viscosity in the center of the plasma.

31.2 Flow generation by turbulence

Suggestion

"intrinsic rotation in tokamaks is an example of a "negative viscosity phenomenon" in which an up-gradient component of the momentum flux organizes a structured mean flow"

The equation

$$\begin{aligned} & \frac{\partial f_0}{\partial t} + \left(v_\parallel \widehat{\mathbf{n}} + \mathbf{v}_D + \mathbf{v}_{E_0}\right) \cdot \boldsymbol{\nabla} f_0 \\ & -\widehat{\mathbf{n}}^* \cdot \boldsymbol{\nabla} \left(\mu B + \frac{e}{m_i} \boldsymbol{\nabla} \phi\right) \frac{\partial f_0}{\partial v_\parallel} \\ = \ & C_i \left(f_0, f_0\right) \end{aligned}$$

where

$$\mathbf{v}_D \equiv \boldsymbol{\nabla} B \text{ drift}$$

$$\begin{aligned} & & c \\ \widehat{\mathbf{n}}^* & = & \widehat{\mathbf{n}} \\ & & +\rho_\parallel \widehat{\mathbf{n}} \times \left[\left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right) \widehat{\mathbf{n}}\right] \end{aligned}$$

$$C_i \equiv \text{coulomb collision operator}$$

Solution
shifted Maxwellian with large toroidal rotation

$$\begin{aligned} f_0 & = f_{SM} \\ & = n\left(r, \theta\right) \frac{1}{\left[\pi \ \left(2 T_i m_i\right)\right]^{3/2}} \exp\left[-\frac{\left(v_\parallel - U_i\right)^2 + 2\mu B}{\left(2 \ T_i / m_i\right)}\right] \end{aligned}$$

$$U_i = I \frac{\omega_\varphi}{B}$$

The ion density has poloidal variation that is modified by the presence of the rotation U_i,

$$n_i\left(r, \theta\right) = N\left(r\right) \ \exp\left[\frac{U_i^2}{2 T_i / m_i} - \frac{e \widetilde{\phi}_0}{T_i}\right]$$

The potential

$$\phi_0 = \langle \phi_0 \rangle + \widetilde{\phi}_0$$

the part that has poloidal variation is due to the centrifugal force, which produces charge separation.

the equation for the turbulence-disturbed kinetic distribution

$$
\begin{aligned}
& \frac{\partial}{\partial t}\delta f_i + \left(v_\parallel \widehat{\mathbf{n}} + \mathbf{v}_D + \mathbf{v}_{E_0} + \mathbf{v}_E\right)\cdot \boldsymbol{\nabla}\delta f_i \\
& -\widehat{\mathbf{n}}^* \cdot \boldsymbol{\nabla}\left(\mu \boldsymbol{\nabla} B + \frac{e}{m_i}\phi_0 + \frac{e}{m_i}\overline{\phi}\right)\frac{\partial}{\partial v_\parallel}\delta f_i \\
= & \left(-\left\{\frac{\left(v_\parallel - U_i\right)^2 + 2\mu B}{2T_i/m_i} - \frac{3}{2}\right\}\mathbf{v}_E \cdot \boldsymbol{\nabla}\ln T\right. \\
& \qquad\qquad\qquad\qquad -\mathbf{v}_E \cdot \boldsymbol{\nabla}\ln n\left(r,\theta\right) \\
& -\frac{v_\parallel - U_i}{T_i/m}\mathbf{v}_E \cdot \boldsymbol{\nabla} U_i\left(r,\theta\right) \\
& +\frac{U_i}{v_\parallel}\frac{1}{T_i/m_i}\mathbf{v}_E \cdot \mu \boldsymbol{\nabla} B \\
& \left.-\frac{v_\parallel \widehat{\mathbf{n}} + \mathbf{v}_D}{T_i/m_i}\cdot \boldsymbol{\nabla}\left(\frac{e}{m_i}\overline{\phi}\right)\left(1 - \frac{U_i}{v_\parallel}\right)\right) f_0 \\
& +C_i^l\left[\delta f_i\right]
\end{aligned}
$$

where

$$\mathbf{v}_E \equiv \text{corresponds to the fluctuation potential}$$

The term propotional with

$$\boldsymbol{\nabla} U_i$$

is the Kelvin-Helmholtz instability.

31.3 The broken spectral $k_\parallel$ symmetry

The brroken symmetry for

$$k_\parallel \rightarrow -k_\parallel$$

is essentially a broken chirlaity of the isntability.

32 Connection between the poloidal and toroidal rotations

The connection should allow understanding the changes that appear simultaneously in the two rotations, at the transition like L to H. **Rice**.

A possibility is in the work **Coherent structures in shear flow-driven plasma microturbulence Jovanovic Shukla Horton de Angelis**.

They study coherent vortices in a plasma with parallel *sheared* flow

$$v_0(x) \ \parallel \ \mathbf{B}_0, \text{ and } x \equiv \text{radial}$$

There is a possibility of a linear instability propagating in the y (poloidal) direction whose energy source is the parallel velocity shear. The modes are purely growing. The flow can support coherent dipolar vortices, obtained as in Larichev Reznik.

To this evolution consisting of coherent vortices sustained by parallel velocity shear, we must add a new physical factor: the inverse transformation from a series of vortices to a sheared poloidal flow. The process can be inverse Kelvin-Helmholtz, or the tilting instability (**Shapiro Rosenbluth**).

There is a paper about inverse KH, it actually is on *tilting* or alignement of flow out of a collection of vortices.

There is another reference **Yushmanov** who cites **Waelbroek** (?).

The work of **Yushmanov Horton** shows that the Reynolds stress produced by one turbulence is able to generate rotation on the other direction.

And **ParallelVelocityShearInstability_DongHorton**.

The paper **conversion poloidal toroidal kasuga diamond**.

Phase space structures drive phase space transport through dissipation.

Bounce average kinetic equation

$$\frac{\partial f}{\partial t} + v_D(E)\frac{\partial f}{\partial y} + \frac{1}{B}\{\phi, f\} = 0$$

Gyrokinetic Poisson equation (neutrality)

$$\frac{e\widetilde{\phi}}{T} - \rho^2 \nabla_\perp^2 \frac{e\widetilde{\phi}}{T} = \frac{1}{n_0}\frac{2}{\sqrt{\pi}}\int_0^\infty dE\ \sqrt{E}\ \delta f$$

where the magnetic precession drift velocity depends on E,

$$v_D(E) = v_{D,0}\frac{E}{T_i}$$

$$\rho = \rho_s \left(1 + 1.6q^2 \frac{1}{\sqrt{\varepsilon}}\right)$$

The simplified equation for the toroidal and poloidal rotation of a plasma produced by a *radial current* are

$$\frac{\partial}{\partial t} \langle v_\varphi \rangle \approx \frac{1}{m_i n} \langle J_r \rangle B_\theta$$

$$\left(1 + 1.6q^2 \frac{1}{\sqrt{\varepsilon}}\right) \frac{\partial}{\partial t} \langle v_\theta \rangle = -\frac{1}{m_i n} \langle J_r \rangle B_\varphi$$

The ratio is

$$\begin{aligned} M_{tor \leftarrow pol} &= \frac{\frac{\partial}{\partial t} \langle v_\varphi \rangle}{\frac{\partial}{\partial t} \langle v_{pol} \rangle} \\ &= -\left(1 + 1.6q^2 \frac{1}{\sqrt{\varepsilon}}\right) \frac{\varepsilon_0}{q} \end{aligned}$$

The *inertia* factor that affect the poloidal rotation, $\left(1 + 1.6q^2 \frac{1}{\sqrt{\varepsilon}}\right)$ is the cause for the two rotation rates to be *comparable*. This is against the intuition that suggests that a common drive of rotation, the radial current J_r, is more efficient in driving poloidal rotation (due to the large ration B_φ / B_θ).

33 LH transition, rotation and radial electric field

33.1 Experimental observations

At transition there is a strong increase in the poloidal rotation.

DIIID, Burrell.
JFT.
JET? Giannella. Andrew.

33.2 Suppression of fluctuations

There are two ways:

1. drastic reduction of radial correlations of the turbulent fluctuations by a *sheared* rotation;

2. linear suppression of the instabilities, by coupling with damped modes due to shift of the eigenmode relative to the resonant surface

3. the rise of T_i over T_e makes $\eta - \eta_c$ to change sign and the growth rate of is suppressed (Horton Kim Kishimoto Tajima JT60)

33.2.1 Notes on linear suppression of instabilities

Pfb92 Carreras Diamond. An effect of the sheared rotation on the dissipative trapped electron drift instability is:

- Doppler shift of the frequency
- asymmetry of the eigenfunction (shift away from the $\mathbf{k} \cdot \mathbf{B}$ surface), which makes the turning point (of ion absorbtion of Pearlstein-Berk-radiated energy) to be inside the mode width

The shear flow stabilizes the drift wave when the shear flow term cancels the difference between the instability drive and the magnetic shear damping

$$\begin{array}{cc} ik_y^2 D_0 & \text{instability drive} \\ \omega_{*e}\frac{\rho_s^2}{W_k^2} & \text{magnetic shear damping} \\ -i\frac{W_k^2 \Omega_s^2}{4\rho_s^2}\omega_{*e} & \text{sheared flow stabilisation} \end{array}$$

34 Impurity fluxes and plasma rotation

This part is also in *plasma general impurities.*

The paper on **Observation of central toroidal rotation in ICRF**, Rice,..., NF 38 (1998) 75. From **Observationtoroidal rotation Alcator-C Rice**:

In the L-mode the impurities rotates toroidally in direction *opposite* to the current. (Counter-current seems a state favorable for confinement of particles, see counter-NBI).

The fact that at the H-mode we have a much better particle confinement extends also over the impurity confinement. In the H-mode there is accumulation of impurities in the center of the plasma.

Regarding the relative direction of impurity radial motion, the text **Density and Omega.tex** presents a hypothesis, in which it is involved the *swirl*.

But the problem is not only *accumulation*, seen as a result of better confinement.

It is also the *flow of impurities* in the radial direction.

From the paper **Varenna ICRHTorque**: there is a pinch of particles during the RF heating, because a toroidal acceleration displaces the trapping ion turning points accross the magnetic surfaces. The turning points are displaced inwardly for *heated* ions. And viceversa.

The origin of displacement of the tips of banana after heating is the conservation of J. See **White Chang ICRH**.

Usually the impurities have *inward* flux in a tokamak. This is due to their radial gradient being created by high density at the edge and low density in the plasma core. Another explanation, *non-diffusive* is given by **Hayashi JT60**.

The paper **NBI influence on impurity Isler** ISX-B tokamak. It is mentioned that theoretical works in neoclassics have revealed that:

1. NBI and electric current in the same direction (*co-injection*) can reduce the inward diffusive flow of impurities.

2. NBI and electric current in opposite direction (*counter-injection*) has the effect of accumulation of impurities in the centre.

In general, in Ohmically heated discharges: classical and neoclassical processes lead to flux of impurities toward the centre.

However in reality the accumulation of impurities is slow or even inexistent. One has to use empirical anomalous diffusion coefficients to explain this.

In the paper of **Isler**. The Argon is concentrated in the centre but only in Deuterium and *not* in Hydrogen.

Conclusion for Argon:

1. the Ar accumulation in the centre is *inhibited* by **co-injection**, and

2. the Ar accumulation in the centre is *enhanced* by **counter-injection**.

The same conclusions for the *iron*.

the influx of Fe toward the centre is very fast in *counter-injection*. Plasma begins to cool in the centre rapidly. (**Note** looks like the H-mode; normally the impurities are better confined. However the effect is not due to new atoms absorbed and confined, but to redistribution of the Fe ions.)

By contrast, the *co-injection* of NBI does not lead to accumulation of Fe in the plasma centre. (**Note** looks like the L-mode).

The evolution of Fe atoms *is not due to supplementary input of Fe in the discharge but to redistribution.*

The change in transport, the only acceptable explanation for this behavior of impurities, cannot be attributed to only the known transport mechanisms (classical and neoclassical) since this would imply a flatter profile of Fe while the experiment shows a peaked profile.

NOTE the possibility to associate this peaking of the impurity (Fe) density in the center with the density pinch induced by the vorticity pinch.

From the article: **BootstrapNeo Hirshman** 1996 :

1. in quiescent H-mode the absolute value of impurity velocity is neoclassic

2. hot-ion mode in JET: neoclassical effect of temperature gradient screening explains the expulsion of the Carbon from the core

3. in PEP (Pellet Enhanced Performance) the same effect (temperature gradient screening) drives impurities inward

When the impurities are introduced in plasma (seeded) as in the experiments on **DIII-D** with Krypton, Argon, Neon, the collisionality for ions increases. As a result there is a higher diffusion of banana in the radial direction. This should be seen in the suplementary *bootstrap* current.

35 Notes

The sheared rotation suppresses the saw-teeth. Either poloidal or toroidal. **Kleva. Strauss.**

The paper on **Observation of central toroidal rotation in ICRF**, Rice,..., NF 38 (1998) 75. From **Observationtoroidal rotation Alcator-C Rice**:

In the $L-$mode the impurities rotates toroidally in direction *opposite* to the current. (Counter-current seems a state favorable for confinement of particles, see counter-NBI).

From the paper **Varenna ICRHTorque**: there is a pinch of particles during the RF heating, because a toroidal acceleration displaces the trapping ion turning points accross the magnetic surfaces. The turning points are displaced inwardly for *heated* ions. And viceversa.

The paper **Generation of pasma rotation ICRH** by Shaing PoP6 (1999) 1969.

The paper **The mechanism for toroidal momentum input neutral beams**, Hinton Rosenbluth PL A 259 (1999) 267-275.

The paper **Plasma rotation driven by alpha particles reactor**, Rosenbluth and Hinton NF 36 (1996) 55.

The paper **Neoclassical Poloidal Toroidal Diamond Kim** says that only the shear of the radial electric field matters, since there is no poloidal rotation. Here the *impurity* velocities are calculated and compared with those of the bulk ions.

The paper **Rotation Ohmic** studies the toroidal and poloidal rotations and the effect of the neutral-ions friction and *charge-exchange* which leads to toroidal rotation damping but not quenching. One observation:

the toroidal velocity is co-current at the periphery and it is counter-current at core. (Remember that NBI counter-injection at **Shaing Houlberg** is the scanario that ensures most density confinement or density clumping). (**Note** looks like the periphery is in the L-mode and the center is in the H-mode).

The *counter-current* toroidal flow is associated to the L-mode and there is a transition to *co-current* toroidal rotation in the centre of plasma (**TCV**).

The LH transition: always inject power. But the power injects also vorticity therefore the injection sustains the poloidal rotation against damping by magnetic pumping and forces the stationary solution (organised) to evolve toward a stable state. But this should be compatible with the length L of the domain, according to our equation, for given Ω_{ci}, ρ_s.

See **Hmode_Groebner**. Beginning of an understanding.

From the paper **Stability ITG sheared poloidal flows Wang Diamond Rosenbluth**.

Reasons for poloidal rotation

1. neoclassical drive

2. orbit loss of high energy ions

3. flow drive by radially propagating waves with rapidly varying wave energy flux
4. poloidally asymmetric radial particle transport (Stringer)

the steep density gradients characteristic of the H-mode confinement will naturally suppress the ITG (since ITG is η_i-mode, where $\eta_i = \frac{L_n}{L_T}$ and the η_i mode is excited when the density is *flat*, $L_n \rightarrow \infty$).

The paper **Galeev Sagdeev Liu Novakovskii Spontaneous poloidal** explains the generation of radial electric field by non-ambipolar diffusion of particles in the ion-trapped regime. Electrons stay close to their magnetic axis and the ions have wide deviations.

It is in *plasma general viscosity and TTMP*.

36 The relationship between the density and the rotation

It is question of density peaking.

The density peaking is always associated to the higher central confinement.

The pinch is higher than the neoclassical one.

There are at least two connections:

1. the ERTEL' s theorem
$$\frac{d}{dt}\left(\frac{\omega + \Omega_{ci}}{n}\right) = 0$$
This must be accompanied by ergodicity which converts the Lagrangian invariants into Eulerian invariants.
2. the rotation of the plasma, in the toroidal direction has effects
   (a) leads to less trapped particles. Then a particle changes from trapped to circulating will have its virtual center of the periodic positions moving a small amount and only once, when the transformation trapped $\rightarrow$ circulating takes place. This is a current and is a source of torque; This returns to Ertel's theorem.
   (b) ICRH makes the tips of the bananas to move towards interior of plasma. This is a motion of the particles.

36.1 Vortex in drift wave turbulence

The condition that a vortex exist and is stable without radiating drift waves

Vortex (condition to exist)	must move (either)	$\rightarrow$	in the electron diamagnetic direction and faster than $v_{dia,e}$ $\quad v > v_{dia,e}$
			or
			in the ion diamagnetic direction $v_{dia,i}$

Short notation

$$\begin{array}{lll} V_{vortex} & \| & V_{*,e} \text{ (clockwise) and } V > V_{*,e} \text{ or} \\ V_{vortex} & \| & V_{*i} \text{ (counter-clockwise)} \end{array}$$

37 Vortex (clump or hole) on a background of gradient of vorticity

The works of **Lin**, **Schecter Dubin** and **Marcus**.

Take Γ_v to denote the *circulation*, i.e. the integral of the velocity along a closed curve around the vortex

clump ($\Gamma_v > 0$)	prograde (shear $\|$ vorticity) retrograde (shear anti-$\|$ vorticity)	move toward the maximum of the background vorticity	
hole ($\Gamma_v < 0$)	prograde (shear $\|$ vorticity) retrograde (shear anti-$\|$ vorticity)	move toward the minimum of the background vorticity	

See also **Mikhailovsky and a lady.**

38 Sheared toroidal rotation (Rosenbluth Liu Catto 1973)

Tentative comments on flows 1

This is also in *instabilities.*

In **Rosenbluth Catto Liu 1973 sheared parallel** it is described the physical mechanism of the drift wave and of the KH instability when there is a radial gradient of the parallel velocity.

The first assumption is the existence of an electric field

$$\mathbf{E} = -(ik_y\phi)\,\widehat{\mathbf{e}}_y - (ik_z\phi)\,\widehat{\mathbf{e}}_z$$

Due to the electric field the *ions* will move alng the magnetic field line $\mathbf{B}_0$ to neutralize the charge. Then they will get a velocity

$$\delta u_z^{ion} = -\frac{|e|}{m_i} ik_z\phi\delta t$$

which is (acceleration $-\frac{|e|}{m_i}ik_z\phi$) x (time interval δt).

(We must understand the definitions

$$\begin{aligned}\mathbf{E} &= -\boldsymbol{\nabla}\phi \text{ and } \phi \sim \exp(ik_y y + ik_z z - i\omega t) \\ E_z &= -\frac{\partial}{\partial z}\phi = -ik_z\phi \text{ and } E_y = -\frac{\partial}{\partial y}\phi = -ik_y\phi\end{aligned}$$

$$m_i\frac{\delta u_z^{ion}}{\delta t} = |e|\,E_z = -|e|\,\frac{\partial\phi}{\partial z} = -|e|\,ik_z\phi$$

Rosenbluth Catto and Liu next discuss the other reason for a change in velocity: the advection, by the field of the wave, of the ions having different parallel velocities, coming from different locations on x, since $U \equiv U(x)$. In the case without shear we do not have such contribution.

Tentative comments on flows 2

This takes into account the fact that the polarization drift leads to a continuous charge separation

$$\mathbf{E}_\perp = -\mathbf{v}\times\mathbf{B}$$

with $\mathbf{v}$ the flow in the layer (like in H-mode layer) assumed *without shear.*

The electric field $\mathbf{E}_\perp$ is transversal on the plasma flow in the layer (rotation) and is generated by the accumulation of charges at *virtual boundaries* of the layer. Since the flow is with fixed velocity $\mathbf{v}$, the drifts of the ions and electrons transversal to the direction of plasma flow, and are produced by $q\mathbf{v}\times\mathbf{B}$, the Lorentz force. This leads to a current that is flowing transversally, in the layer. If the magnetic field pointing away from the page in the direction of sight, the ions are moving *upward* and the electrons are moving *downward.* This makes an accumulation of ions at the upper level and of electrons at the lower level. The electric field that is produced by this accumulation of charge is transversal on the flow and is oriented from *up* toward *down. The transversal electric field varies continuously in time,* $dE_\perp/dt \neq 0$

until the force resulting from the electric field, $q_a\mathbf{E}$ will become equal and will cancel the Lorentz force

$$q_a\mathbf{E} = -q_a\mathbf{v} \times \mathbf{B}$$

Then a stationary state is established and no other flow of charges transversally on the layer can occur. If there are no boundaries at finite distance, the flow must continue since there is no reason to stop the accumulation of charge at the two *virtual* boundaries, up and down the layer.

This electric field should be called *polarization* field since the piece of material now has charges on the two opposite boundaries and is polarised.

Regarding the *sign* of the transversal electric field: since the ions are going in the direction of the electric field, the electric field must be considered as pointing upward.

The polarization flow needs a time-varying electric field to exist. This may happen only if the boundaries are at infinity. If they are at finite distance then the accumulated charges stops when the electric force equals the Lorentz force and the motion of the object (or fluid) is no more accompanied by charge displacement.

$$\mathbf{v}_{pol} = \frac{1}{\Omega_{ci}B}\frac{d\mathbf{E}_\perp}{dt}$$

This polarization flow results from the *guiding centre* motion of the particles (which perform larmor gyromotion) and is of high order having power 2 in the magnetic field in the denominator $\sim \Omega_i B \sim B^2$.

This polarization velocity should *practically* be considered only for *ions* since for electrons the mass is too small.

$$\mathbf{v}_{pol}^{ion} = \frac{1}{\Omega_{ci}B}\frac{d\mathbf{E}_\perp}{dt}$$

Now, the dispacement of ions due to the polarization drift must be compared with the displacement of the ions due to the drift-wave oscillation of the potential in the direction parallel with the flow.

For this we consider

$$k_y \to 0$$

very longwave oscillation in the flow direction (poloidal). The electric field in the poloidal direction is

$$E_y = -ik_y\phi$$

and the x velocity is

$$v_x = \frac{E_y}{B} = -ik_y\phi\frac{1}{B}$$

This velocity acting on an interval of time

$$\delta t$$

will produce a displacement of the ions on a distance

$$\delta x = -ik_y \phi \frac{1}{B} \delta t$$

and the *advection* of the ions in the x direction takes place on a background of a gradient of density. This means that there will be a perturbation of the density of ions

$$\begin{aligned} \delta n^{ion} &= \frac{dn_0}{dr} \delta x \\ &= -ik_y \phi \frac{1}{B} \delta t \frac{dn_0}{dr} \end{aligned}$$

Taking the interval of time δt as the inverse of the frequency of the drift wave

$$\begin{aligned} \delta t &= \omega_{*e}^{-1} \\ &= \left(k_y \frac{T_e}{|e| B} \frac{1}{n_0} \frac{dn_0}{dr} \right)^{-1} \end{aligned}$$

we get

$$\begin{aligned} \delta n^{ion} &= -ik_y \phi \frac{1}{B} \frac{dn_0}{dr} \frac{1}{k_y \frac{T_e}{|e| B} \frac{1}{n_0} \frac{dn_0}{dr}} \\ &= -in_0 \frac{|e| \phi}{T_e} \end{aligned}$$

where $-i = \exp\left(-i\frac{\pi}{2}\right)$ is a phase factor. In **Rosenbluth Catto** the modulus is taken.

We want to examine the situation where the transversal velocity $\mathbf{v}_{pol}$ due to the polarization is equal with the x velocity due to the oscillation of the drift wave. Normally this should never happen, they have different orders of magnitude.

39 The problem of compatibility of the two *helicities*: the magnetic field lines and the swirl of the poloidal/toroidal rotation

This compatibility is the major factor.

Montgomery finds alignment of vorticity and magnetic field.

The association that is suggested by several properties is however

$$\boldsymbol{\omega} \text{ and } \mathbf{j}$$

The reversal of the direction of the toroidal rotation **Rice**.

The increase of the density $n(t)$ leads at a threshold to the reversal of toroidal rotation. The increase of n implies increase of the number of trapped ions. This generates and changes continuously the radial current which is due to the radial drifts of the newly trapped ions. At a certain value the radial current induces a torque which is higher than the magnetic damping and the poloidal rotation is set in. Then the toroidal rotation must adapt to the new helicity.

Rice 1998: the change of the direction of the plasma current I_p relative to the magnetic field, leads to a change of the direction of toroidal rotation, such that

the toroidal rotation remains co-current

This is because

- the radial current of the ions is invariant (geometric) being generated by the radial drift of ions that perform motions along the bananas, and
- the confining toroidal magnetic field, B_T, with fixed direction and magnitude

then,

the poloidal rotation, associated with the torque $\mathbf{J}_r \times \mathbf{B}_T$, is in consequence invariant. When the plasma current is reversed, the helicity of (B_θ, B_{tor}) changes sign and the helicity of (v_θ, v_{tor}) must also change sign to remain compatible. This means that v_{tor} must be reversed too.

Later: however there can be a different cause

Consider the process of ionization that generates the radial current at each event of ionization. This total current has a radial distribution of the magnitude which is established by the profile of the source of ionization (gas puff, pellet, etc.). The total radial current results from a substraction of two components with opposite radial direction:

- one comes from the radial displacement for the bananas that are entirely inside the magnetic surface and
- the other from the radial displacement for the bananas that are entirely outside the magnetic surface.

Only because there is a difference between the dimensions of the two bananas (internal and external) a current exists.

This will be the contribution of the ionizations at that particular surface.

Now, imagine there is a change of direction of I_p. Then the helicity of the magnetic field changes.

However the drift of an ion will remain invariant $\mathbf{v}_D = \frac{1}{\Omega}\widehat{\mathbf{n}} \times \mu \boldsymbol{\nabla} B$ is vertical.

The vertical drift $\mathbf{v}_D$ now will combine with a parallel velocity $\mathbf{v}_{\parallel}$ of the same ion and, since this parallel velocity has changed the helicity, there will be another final result

any banana that previously was exterior to the magnetic surface will now be internal

any banana that previously was interior to the magnetic surface will now be external

If we assume that the radial profile of the rate of ionization is unchanged then the direction of the radial current will be reversed.

If this radial current produced by ionization is combined with the poloidal magnetic field then the direction of toroidal rotation does not change

If the radial current produced by ionization is combined with the toroidal magnetic field then the direction of poloidal rotation will reverse.

Note We must clarify if the change of direction of the current takes place from one experiment to another, or if in the same discharge. If the latter is true then

$$\frac{dv_{tor}}{dt} \rightarrow \frac{dJ_{tor}}{dz}$$

and this means connection with

$$\frac{\partial \omega}{\partial t} = \nabla_{\parallel} j_{\parallel}$$

which is a source of vorticity.

40 Contributions to a model

If we move a dielectric across a magnetic field, with a constant velocity, the charges will separate. The state at one position is different of the state of the state at a different position, after a distance, since more positive charges are in the part of the space which is *up* compared with the quantity of the positive charge in the same part of the space at the previous position. The same for the negative charges in the opposite part of the space.

This means that there is no Lorentz or Galilei transformation that can remove the velocity of the flow, even if constant.

On the other hand there is an electric field in the first position (due to the separation of charges) and there will be another electric field at the next position, where the charge separation has been accentuated. The time variation of the transversal electric field has an effect to oppose the flow of charges that are separating across the flowing plasma. In a *streamer* there is a moment where the two forces are balanced.

The time variation of the transversal electric field induces a polarization velocity.

The current will stop and there will be no additional accumulation of charges at the top and bottom.

The electric field is saturated.

There are similar situations in fluids and atmosphere or oceans, etc. We mention few of them:

- works of **Kuo** in the physics of atmosphere. They actually start with a simplified model consisting of a fluid rotating in the region defined by two cylinders which are kept at constant different temperatures. Kuo looks for the convective cell that can form as a motion of the fluid along the axis of the cylinders, closing heat convective transport flow between the two cylinders. This is for the meridional convection in atmosphere.

- **Kuo Hadley** mentions three regimes of this convective flow, one of them being fast and substantial. Possibly similar to the H-mode state. See the book **Thermal Convection**.

- **Ranque-Hilsch** tube. The separation of temperature and the inverse effect. The five rolling balls.

- **Rosenbluth** formation of shock. The shock at $\theta = \pi$ in the high field part of the equatorial plane. The equation derived in the cylindrical geometry, for $(\mathbf{B} \cdot \boldsymbol{\nabla})\,\mathbf{A}$, with Christoffel symbols. Nozzle flow $u^2/2 - \ln u$.

- **Ware Wiley** multiple equilibria of poloidal rotation. Anisotropy of the pressure tensor, various species. Impurities may be important since they amplify the variation of the potential and density over the surface, leading to stronger Stringer effect.

- **Drake Guzdar Hassam** article on *Stringer poloidal rotation.* Possible positive instability:

1. the poloidal rotation is driven by the same process as in Rayleigh-Benard convection. The geometry is similar to the **Kuo** system.
2. the flow in poloidal direction associated to the convective cell that has been generated between two magnetic surfaces will transport heat along the surface in the poloidal direction, before closing the flow toward the other surface. The heat transported poloidally will produce a variation of the temperature on the magnetic surface.
3. the Stringer effect will be enhanced by this nonuniform distribution of temperature on the magnetic surface
4. the enhanced Stringer effect enhances in turn the flow in the poloidal direction that defines the convection cell.
5. higher flow in the convection cell will transport even more heat from the hot surface and will increase the variation of temperature on the surface

- **Shapiro Rosenbluth** theory of *tilting instability.* Paper **TiltInstAtmosphere**.
- **Shapiro** theory for convective cell generation.
- **Shapiro Diamond** the rate of generation of convective cells is higher than the ITG growth rate.
- **Diamond Malkov** coherent structures in zonal flows, but only radial propagation. No cenvective cell. Negative viscosity, wave kinetic equation.
- **Weiland** convective cells (**Sanuki**)
- **Shakura Sunyaev** and **Colgate Lovelace** local maximum entropy on accreation disks. *Rossby* waves.
- ADAF advection dominated accretion flux, vs. CDAF convection dominated accertion flux, theory of **Igumenschev** and **Lovelace Colgate**: analysis by **Balbus** showing that only the MRI Magneto Rotational instabilit can explain accretion.
- **Howard Krishnamurthi** on Rayleigh Benard system. The wind which occurs after several bifurcations is similar to the convective cell in meridional plane at **Kuo**.
- *Taylor Couette* flow between two cylinders. Bifurcations. Book **Rotating Fluids**. Page 356 shows the physical system.

- velocity of propagation of perturbations on a background of rotating fluid. The paper **0108083 Anti Centrifugal**.
- symmetry breaking in Taylor Couette flows.
- the paper **IntermittencyTransport DIII D** where it is presented the *blobs* transport. Actually they may be convective cells that close short and also have captured current.
- field theory: the barrier that separates two states, solutions of the CHM equation.

The physical image for this, connected with *generation of convection rolls* used in **Reversal Toroidal Rotation (ex Ranque Hilsch**) is as follows:

we have a transient flow as a convection roll, spontaneously generated and, soon after its creation, vanishing by dis-organization of the coherency of the flow. It is connex on part of its circumference with part of the layer of sheared poloidal rotation. The direction of flow on the common region is the same. The convection rolls are sustained by the gradient of pressure, like RB system. They transfer by friction and by Reynolds stress (on the zone of mixing of the two fluids) momentum, sustaining the poloidal flow. There is also the temperature that is transferred to the sheared flow, leading to local perturbation of the temperature on the poloidal circumference and to a Stringer effect. If these are compatible (direction of initial flow and direction of Stringer effect) we have instability.

Later, **2012, in EPS**, we propose another mechanism to generate rotation:

the pre-existing poloidal rotation has the property of gradient of vorticity. This is a background. Then any robust vortex will necessarily evolve on this background:

1. positive vortices (clumps, *i.e.* the circulation $\Gamma = \oint \mathbf{v} \cdot \mathbf{dl}$) are moving toward the maximum of the vorticity
2. negative vortices (holes, *i.e.* negative circulation) are evolving toward the minimum of the vorticity

Then we have drops of vorticity adding themselves to the background vorticity layer, where they melt down and release their vorticity as a contribution to the angular momentum.

The robust vortices are generated in the turbulent plasma, close to the layer of rotation. They separate (positive and negative) and they move in opposite directions.

This process is the reversed form of the generation of vortices in a rotating Bose-Einstein condensate (BEC) See **Aranson**, etc.

At **Padova TTG 2012** we present the effect of the random addition of vortices on a rotating layer. The result is the **Davey Stewartson** system, but in **k** space, whose solution (**Chow**) shows accumulation of the spectral energy close to $k_y \approx 0$, *i.e.* poloidal flow.

41 Importance of the problem for Reactor Scenarios

In **Burrell 1996** it is mentioned that the terms in the equation

$$E_r = \frac{1}{Z_i \left|e\right| n_i} \nabla p_i - v_{i\theta} B_\varphi + v_{i\varphi} B_\theta$$

are important in various regions of the discharge:

- for the H mode, at the edge:

$$\nabla p_i \text{ and } v_{i\theta}$$

- for the VH mode, in the core

$$v_{i\varphi}$$

- for the core plasma, in the Internal Transport Barriers

$$\nabla p_i \quad \text{and} \quad v_{i\varphi}$$

Therefore scenarios of active control can be forseen:

1. NBI to modify $v_{i\varphi}$,
2. Ion Bernstein to modify $v_{i\theta}$.

In **Burrell 1996** it is reminded that there is a problem that cannot be understood on the basis of the $E \times B$ shear suppression of turbulence:

it is the fact that the electron transport is not affected, as shown by experiments. The ion transport is affected. This may be due to *rotation* since the ions are involved.

42 The effective Larmor radius

There is my text **DriftWaves_rho_effective**.

43 L to H mode transition

Wagner : the transition takes place when the *diamagnetic velocity* reaches a certain threshold

$$v_{dia,i} > V_{crit}$$

$$\begin{aligned} n_i \mathbf{v}_{dia,i} &= \frac{1}{m_i \Omega_{ci}} \widehat{\mathbf{n}} \times \boldsymbol{\nabla} p_i \\ v_{dia,i} &= \frac{1}{n_i |e| B_T} \left| \frac{dp_i}{dr} \right| \end{aligned}$$

The threshold of L *to* H transition depends on the **direction of $\boldsymbol{\nabla}B$ drift of ions**:

when the direction of the drift $\boldsymbol{\nabla}B$ is toward the X-point of the separatrix, the power threshold is smaller.

This means that this direction of ion drift helps the transition.

We can make the following supposition.

The change of the direction of the toroidal rotation of plasma from *counter-current* in the L-mode to *co-current* in the H-mode can only be explained by an electric phenomenon.

The only electric process seems to be of the *polarization* type.

[*LATER*: there is also the torque due to ionization.]

When there is a strong poloidal rotation in the H-mode there is also an electric field at the edge. This modifies the banana trajectories, by squeezing them. There is less loss of trapped ions to the edge.

This means that there is a change in the regime that was before, in L-mode.

The equilibrium in L-mode consisted of a certain continuous loss of trapped ions at the edge with an associated flow of charged particles (electrons) toward the center of the plasma to compensate for the loss of charges. This current provides the toroidal rotation in a stationary L-mode regime.

After the transition to H-mode we have less trapped ions lost to the border. Therefore less inflow of charges and a change in the radial current.

NOTE

This is also in **Notes Toroidal Rotation Reversal.**

The change in the direction of *toroidal* plasma rotation from *counter-current* to *co-current* refers almost exclusively to IONS, the massive plasma component. The electrons, that carry most of the electric current, continue to flow in the same direction, the direction (inverse) of the current I_p. There is a change however in the value of the parallel current

$$\begin{array}{ll} \text{from} & |e|\, n_0 \left| v_{i\parallel}^{(1)} \right| - |e|\, n_0 v_{e\parallel} \\ \text{to} & |e|\, n_0 \left(- \left| v_{i\parallel}^{(2)} \right| \right) - |e|\, n_0 v_{e\parallel} \end{array}$$

with the change of sign and of magnitude of directed velocity of the ions. Then there is a short time a high value of

$$\nabla_{\parallel} j_{\parallel}$$

and this means that there is a pinch of vorticity in the meridional section of plasma

$$\frac{d}{dt} \boldsymbol{\nabla}_{\perp}^2 \phi$$

These relationships are local. We have however to go to *integral* forms of these equations, to refer to volumes in the core or edge.

For example, in **Rice PoP19 056106, 2012**, the toroidal velocity of rotation changes from

$$\begin{aligned} & 10 \ (km/s) \ \text{ to } \ -20 \ (km/s) \\ \Delta v_{tor} &= 30 \ (km/s) \end{aligned}$$

at a density of

$$\begin{aligned} n_{thresh} &= 0.6 \times 10^{20} \ \left(m^{-3}\right) \ \text{ at a current } I_p = 0.62 \ (MA) \\ n_{thresh} &= 1 \times 10^{20} \ \left(m^{-3}\right) \ \text{ at a current } I_p = 1 \ (MA) \end{aligned}$$

In the first case the variation of I_p due to reversal of direction of ions is

$$\begin{aligned} \Delta I_p &= |e|\, n_{thresh} \Delta v_{tor} \ \times Area \\ &= 1.6 \times 10^{-19} \,(C) \times 0.6 \times 10^{20} \left(m^{-3}\right) \times 30 \times 10^3 \,(m/s) \times Area \left(m^2\right) \end{aligned}$$

we take

$$\begin{aligned} Area &\approx \pi \left(\frac{a}{2}\right)^2 = 3.14 \times (0.1)^2 \ \left(m^2\right) \\ &= 0.03 \ m^2 \end{aligned}$$

$$\begin{aligned}\Delta I_p &= 1.6\times 0.6\times 30\times 0.03\times 10^{-19+20+3}\\ &= 10^4\ A\end{aligned}$$

Or

$$\Delta j_{\parallel} = \frac{\Delta I_p}{Area} = \frac{10^4}{0.03} = \frac{1}{3}\times 10^6\ \left(A/m^2\right)$$

and the variation is

$$\begin{aligned}\nabla_{\parallel} j_{\parallel} &\sim \frac{0.3\times 10^6\ \left(A/m^2\right)}{2\pi R\ (m)}\\ &= \frac{0.3\times 10^6\ \left(A/m^2\right)}{6\times 0.67\ (m)} \approx 10^5\ \left(A/m^3\right)\end{aligned}$$

Now, the integral of the vorticity over the area gives the circulation

$$\iint dA\ \omega(r) = \iint dA\,(\boldsymbol{\nabla}\times\mathbf{v}) = \oint \mathbf{dl}\cdot\mathbf{v} = 2\pi r\ v_\theta$$

The vorticity is

$$\omega \sim \frac{\boldsymbol{\nabla}_\perp^2\phi}{B}$$

The variation of the vorticity

$$\begin{aligned}\Delta\omega &= \left|\frac{\boldsymbol{\nabla}_\perp^2\phi}{B}\right|_{t_1}^{t_2} = \frac{1}{B}\left|\boldsymbol{\nabla}_\perp^2\phi\right|_{t_1}^{t_2}\\ &= \frac{1}{B}\frac{\Delta\left(\left|\boldsymbol{\nabla}_\perp^2\phi\right|\right)}{\Delta t}\Delta t\end{aligned}$$

where (**Rice**)

$$\Delta t \sim 0.05\ (s)$$

$$\begin{aligned}\Delta\Omega_{tot} &= \Delta\left(\iint dA\ \omega(r)\right) = \left|\iint dA\ \omega(r)\right|_{t_1}^{t_2}\\ &= \left|(2\pi r v_\theta)\right|_{t_1}^{t_2}\end{aligned}$$

or

$$\begin{aligned}&2\pi r\left[v_\theta(t_2) - v_\theta(t_1)\right]\\ =\ &\Delta\Omega_{tot}\\ =\ &\frac{1}{B}\frac{\Delta\left(\left|\boldsymbol{\nabla}_\perp^2\phi\right|\right)}{\Delta t}\Delta t\times\left(\pi r^2\right)\\ =\ &\frac{1}{B}\times\text{coeff}\times\left[\left(\nabla_{\parallel} j_{\parallel}\right)\times\left(\pi r^2\right)\right]\Delta t\\ =\ &\frac{1}{B}\times\text{coeff}\times\left[\left(10^5\ \left(A/m^3\right)\right)\times\left(\pi r^2\right)\right]\Delta t\end{aligned}$$

$$[v_\theta(t_2) - v_\theta(t_1)] = \frac{1}{2r}\frac{1}{B} \times \text{coeff} \times \left[10^5 \ \left(A/m^3\right)\right]$$

$$\begin{aligned} r &= a/2 = 0.1 \ (m) \\ B &= 5 \ (T) \end{aligned}$$

$$\begin{aligned} [v_\theta(t_2) - v_\theta(t_1)] &= \text{coeff} \times \left[10^5 \ \left(A/m^3\right)\right] \frac{1}{(mT)} \Delta t \\ &= \text{coeff} \times \left[10^5 \ \left(A/m^3\right)\right] \frac{1}{(mT)} \times 0.05 \ (s) \end{aligned}$$

END

From **Rice ICRF NF38 1998 p75**.

The direction of the toroidal rotation changes when the direction of the plasma current changes, remaining *co-current* .

What happens when the system makes the transition to the H-mode: the density is better confined and there is an increase of the density in the central region. This means that instead of loosing trapped ions the plasma will get new trapped ions. This means that the radial current may reverse its sign.

To describe this the kinetic treatment for the distribution function as developed by **Rosenbluth Hinton** for α-particles and for **NBI** must be used. The source can be taken as a time variation of the density $n(t)$ which is a coefficient of the Maxwell distribution function. We can assume that due to the effective closure of the loss-channel at the border in H-mode, the density in the core increases and this is simply $\dot{n} > 0$. The radial current should come again from the drift motion of the particles.

In the L-mode, the toroidal plasma rotation in the centre is *counter-current* with a magnitude of

$$u_{tor} = -10 \ km/s$$

If this velocity is due to the radial current then

$$0 = j_r B_\theta + F_{tor}^{\text{coll}}$$

where the collisions have been considered as balance of the electric force (see however **Hinton Robertson** neoclassical polarization). Or, if there is a radial electric field

$$E_r = u_\varphi B_\theta - u_{pol} B_T + \frac{1}{|e| n_i} \frac{\partial p_i}{\partial r}$$

but B_θ is very small, $B_\theta \sim 10^{-2} B_T = 10^{-2}$ (T). $E_r = u_\varphi B_\theta = 10^4\,(m/s) \times 10^{-2} = 10^2$ V/m.

When there is strong poloidal rotation two changes can be seen for the trapped particles.

1. the banana width is reduced by the radial electric field by **squeezing** (Shaing)

2. the number of trapped ions becomes small, because the fast parallel flow makes $\lambda \equiv v_\perp^2/v^2$ to become smaller and the region in $\mathbf{v}$-space will be reduced. **Nycander Yankov**.

The quantity that is used by **Nycander Yankov** to describe the H mode is the Mach-number
$$M = \frac{v_\theta B_T}{v_{th} B_\theta}$$
and it is found that $M > 1$ in the H mode.

The following experimental facts support the idea that the *trapped ions* are the principal actor in the H mode physics

1. the Mach number of the poloidal rotation is greater than 1 in H mode

2. when there is poloidal rotation the trapped ions are fewer since the parallel component of the rotation is high and reduces the chance of ions to be trapped (increase of the parallel energy of ions)

3. the suppression of transport in the edge small region is due to the reduction of the trapped particles, or, the transport was due to trapped particles. It is NOT the shear that destroys the turbulence but it is simply the reduction of the factor on which the instabilities were based: trapped ions.

4. the idea that trapped particles are essential is supported by the revealing their role also in the density pinch. The *inverse magnetic shear* in the central core of plasma makes that the trapped particles are in a minimum of energy and therefore the transport is low: indeed the transport is reduced with a factor of 40 in the central region. (**Note** however that the **Ranque-Hilsch** phenomenon can also claim that the center is hot and the border is cold - which is opposite to the classical Ranque-Hilsch - but connected with the different poloidal rotations).

44 The ambipolar electric field in *stellarators*

The paper **Hastings Houlberg Shaing NF25 (1985) 445**.

It is question of the existence of two roots of the algebraic or differential equation for the radial electric field in stellarators:

1. the ion root
2. the electron root

which exist in various *collisional* regimes.

By comparison there is a double root for the poloidal velocity due to non-linear variation of the parallel viscosity on this parameter. **Shaing, Crume, Houlberg**.

45 Proposed developments

45.1 Front propagation as Ambipolaron

Possibly a propagation of a front of *shear of poloidal velocity*

This is, equivalently, the propagation of a "wave" or a soliton of vorticity.

We should combine the **Ambipolarons Morrison** with the suppression of the η_i modes due to sheared poloidal rotation from **Horton Dong**. One should be able to derive a diffusion coefficient

$$D(E) \sim \frac{1}{a+bE}$$

since the effect of sheared rotation is to reduce: the growth rate of the mode γ and to decrease the mode width δx, then

$$D \sim \frac{(\delta x)^2}{\gamma^{-1}}$$

This should first be expressed in terms of rotation and also of spatial variation of velocities

$$E_r = \frac{1}{|e|\,n}\nabla p + v_\theta B_T - v_\varphi B_\theta$$

Then it may become possible to obtain a dependence of

$$\varepsilon_\perp \frac{\partial E_r}{\partial t}$$

of the fluxes $\Gamma_{e,i}$ which in turn are functions of the electric field

$$\Gamma_{e,i}(E)$$

since they are functions of velocities and their shear; and further on the diffusion coefficient for the electric field, D.

45.2 The convective cell similar to Taylor-Couette experiment

The sheared toroidal rotation combined with the gradient of temperature can be unstable to generation of *streamwise* rolls with axis along the magnetic field. These rolls are actually convective cells that can be transformed into poloidal flow with radial shear, after *tilting* instability.

(See also the effect of *helicity* in **Moiseev Ruthkievich Tur Yanovskii**).

The model is Taylor Cuette. See **Busse conf**.

The generation of rolls (convective cells with axis along magnetic field) is followed by tilting as in **Rosenbluth Shapiro** and generation of the *wind* in poloidal direction as in **Howard Krishnamurthi**.

the possiblity of tunneling directly to the *wind* phase.

See also **Kuo** athmosphere.

46 A text for EFDA WP11-TRA-04-01 (april 2011)

Title: Studies of the relative magnitude of intrinsic organization of vorticity versus neoclassical and/or applied torque

Motivation:

- Confinement:

  1. L-mode: toroidal *counter-current* rotation; E_r is the least negative.
  2. H-mode: toroidal *co-current* rotation. E_r is deep negative. At transition there is a fast transfer of momentum from toroidal rotation to poloidal rotation.

- NBI generates rotation of plasma

1. NBI *co-current injection* leads to a decrease of impurity concentration in the center: their transport from the border toward the center is *reduced*; on 75 KeV ions injected in the core 38% are lost to the wall within the *first* bounce period.
2. NBI *counter–current injection* leads to higher concentration of Argon impurity in the center (ISX). The influx of the impurities to the center is very fast and plasma begins to cool.

- Experimental observation: when input power increases the ITB moves from the core toward the edge.

Hydrodynamic references:

1. a fluid evolves to vorticity organization into coherent large scale structures
2. there is evidence of vorticity concentration in $2D$ fluids

What should be examined:

- Recall: the effect of the rotation (and possibly of the *sheared* rotation) on the instabilities, drift waves and ITG. Linear suppression of high k spectral region, favor excitation of poloidally long-wavelength waves, with low ω.
- A fast, massive, change of the toroidal rotation in the center of plasma (*reversing direction* at the L to H transition) leads to strong E_r change. This increases the poloidal rotation term $v_{i\theta}B_{\varphi}$ since there is a interval when $v_{i\varphi}B_{\theta}$ is very small. The poloidal rotation is only weakly damped in the core, the magnetic pumping is less efficient.
- The radial profile of the poloidal velocity should be seen as a front which advances from the core toward the edge.
- The poloidally elongated eddies, tilted, are easily converted into rotating layer and absorbed by the advancing profile.

First mechanism to be formalised and studied: the poloidal rotation advances toward the edge as a *front propagation*. The model is **van Saarlos**:

$$\frac{\partial \gamma}{\partial t} = \frac{\partial^2 \gamma}{\partial x^2} + F(\gamma)$$

where

$$F(\gamma) = \gamma(\gamma + b)(\gamma - 1)$$

This is one of the objectives.

Elementary vortical elements have the property of self-organization into coherent structures.

Well-known in the physics of fluids, atmosphere, MHD.

Estimated strength: spectral flow of energy in inverse cascade. Two concrete sources of quantitative characterization:

1. conversion of drift wave turbulence into convective structures (two-field fluid treatment)

2. generation of coherent large scale flow from helicity turbulence (statistical methods)

We have developed a field theoretical description of self-organized states in $2D$ plasma. It shows vorticity radial distributions that are smoothly evolving toward a cuasi-singular state on the magnetic axis. This evolution drags the density too.

We have already a large collection of numerical results.

This is the second objective.

We will compare with the driven case (first objective).

Join to this study:

1. torque transferred to plasma by α particle creation

2. torque transferred to plasma by NBI

An important aspect from *neoclassics*: The balance equation for E_r is based on fluid equation where the fluid motion can be attributed to the guiding centers. We actually need a balance equation involving the *neoclassical polarization* current, where the velocities can be attributed to centers resulting from bounce-averaging the ion trajectories. This will allow to discuss the loss from the plasma of the energetic ions coming from NBI. The effective polarization dielectric constant in much larger than the classical one (a factor $(B_\varphi/B_\theta)^2$).

Objectives:

To find the competing mechanisms acting on poloidal rotation of plasma.

To find the way plasma converts toroidal rotation into poloidal rotation (imposed by helicity constraint)

47 Rotation in mirrors (Horton)

This is from **Horton Phys Rep 1990**.

Also in **Horton Liu** PF27, 1984.

The ion hydrodynamic regime with an external potential.

The continuity equation

$$\frac{\partial n_i}{\partial t} + \nabla\cdot (n_i \mathbf{v}_i) = 0$$

and the momentum balance equation

$$\begin{aligned} m_i n_i \left(\frac{\partial \mathbf{v}_i}{\partial t} + (\mathbf{v}_i \cdot \nabla) \mathbf{v}_i \right) &= -T_i \nabla n_i - \nabla \cdot \boldsymbol{\pi}_i \\ &\quad + e_i n_i (\mathbf{E} + \mathbf{v}_i \times \mathbf{B}) \\ &\quad - n_i m_i \mathbf{g} \end{aligned}$$

where the effect of the curvature is represented as

$$\mathbf{g} = \nabla U$$

Observation of **Horton**:

taking the ion temperature to be constant will eliminate the modes η_i and η_e. This is a simplification that the model has assumed.

The momentum equation can be solved by expansion in $1/\Omega_{ci}$. The ion flux is composed of

$$\begin{aligned} n\mathbf{v}_i &= \frac{-\nabla\phi \times \widehat{\mathbf{n}}}{B} \qquad \text{electric drift flow} \\ &\quad + \frac{T_i}{e_i B} \widehat{\mathbf{n}} \times \nabla n_i \qquad \text{diamagnetic flow} \\ &\quad + \frac{1}{e_i B} \widehat{\mathbf{n}} \times \left(m_i n_i \left[\frac{\partial \mathbf{v}_i}{\partial t} + (\mathbf{v}_i \cdot \nabla) \mathbf{v}_i \right] + \nabla \cdot \boldsymbol{\pi} \right) \quad \text{ion polarization drift flow} \\ &\quad - m_i n_i \frac{1}{e_i B} \widehat{\mathbf{n}} \times \mathbf{g} \qquad \text{flow induced by the gravity force } \times B \end{aligned}$$

and now we expand the ion velocity taking as first order

$$\mathbf{v}_i^{(1)} = \frac{1}{B}\left[-\boldsymbol{\nabla}\left(\phi + \frac{T_i}{e_i}\ln n_i\right) \times \widehat{\mathbf{n}}\right]$$

this approximation retains only the *electric* drift plus the *diamagnetic* drift.

This *first order* ion velocity is inserted in the expressions of

$$\begin{array}{l} \dfrac{1}{e_i B}\widehat{\mathbf{n}} \times m_i n_i \dfrac{d}{dt}\mathbf{v}_i^{(1)} \;\; \text{ion polarization drift} \\ \dfrac{1}{e_i B}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} \cdot \boldsymbol{\pi} \;\; \text{Finite Larmor Radius stress tensor} \end{array}$$

Horton Phys Rep 1990: the nonlinear ion continuity equation.

The equation reads

$$\begin{aligned} & \frac{\partial n_i}{\partial t} + \frac{-\boldsymbol{\nabla}\phi \times \widehat{\mathbf{n}}}{B} \cdot \boldsymbol{\nabla} n_i \\ & -\frac{1}{\Omega_{ci} B}\boldsymbol{\nabla}\cdot \left[n_i \frac{\partial}{\partial t}\boldsymbol{\nabla}\phi + n_i \frac{1}{B}\left(\frac{-\boldsymbol{\nabla}\phi \times \widehat{\mathbf{n}}}{B} \cdot \boldsymbol{\nabla}\right)\boldsymbol{\nabla}\phi \right. \\ & \qquad\qquad\qquad\qquad \left. + \frac{T_i}{e_i B}\left(\frac{-\boldsymbol{\nabla} n_i \times \widehat{\mathbf{n}}}{B} \cdot \boldsymbol{\nabla}\right)\boldsymbol{\nabla}\phi\right] \\ & -\frac{1}{\Omega_{ci}}\left(\frac{-\boldsymbol{\nabla} n_i \times \widehat{\mathbf{n}}}{B} \cdot \boldsymbol{\nabla}\right) U \\ = \;\; & 0 \end{aligned}$$

NOTE for comparison the equation from **Petviashvili Pokhotelov** in the form

$$\begin{aligned} & \frac{\partial n}{\partial t} + \frac{-\boldsymbol{\nabla}\phi \times \widehat{\mathbf{n}}}{B_0} \cdot \boldsymbol{\nabla} n \\ & -n_0 \frac{1}{\Omega_{ci} B_0}\left(\frac{\partial}{\partial t} + \frac{-\boldsymbol{\nabla}\phi \times \widehat{\mathbf{n}}}{B_0} \cdot \boldsymbol{\nabla}_\perp\right)\Delta_\perp \phi \\ & -\frac{1}{\Omega_{ci} B_0}\boldsymbol{\nabla}_\perp \cdot \left[\left(\frac{1}{m_i \Omega_{ci}}\left(\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p_i\right) \cdot \boldsymbol{\nabla}_\perp\right)\boldsymbol{\nabla}_\perp \phi\right] \\ = \;\; & 0 \end{aligned}$$

where

$$\begin{aligned} p_i &= \text{ion pressure} \\ \frac{\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p_i}{m_i \Omega_{ci}} &= n_0 v_{*i} \;\; \text{the diamagnetic flux of ions} \end{aligned}$$

and

$$n = n_0(x)\exp\left(\frac{|e|\,\phi}{T_e(x)}\right)$$

Boltzmann distribution.

We **note**:

- the divergence operator applied on the polarization (inertial) velocity $\mathbf{v}_p$ goes directly to the time derivative of the electric field $\nabla_\perp \cdot (d\mathbf{E}_\perp/dt)$ leading to the Laplacean of the potential ϕ, which is the *vorticity*. The divergence is multiplied with n_0 as it should since the density continuity is $\partial n/\partial t + (\mathbf{v}\cdot\nabla)\,n + n_0\,(\nabla_\perp\cdot\mathbf{v}) = 0$.

- the term of divergence of the flux

$$-\frac{1}{\Omega_{ci}B_0}\nabla_\perp\cdot\left[\left(\frac{1}{m_i\Omega_{ci}}\left(\widehat{\mathbf{n}}\times\nabla p_i\right)\cdot\nabla_\perp\right)\nabla_\perp\phi\right]$$

  coming from the *diamagnetic* advection of the electric velocity $\left(\frac{\widehat{\mathbf{n}}\times\nabla p_i}{m_i\Omega_{ci}}\cdot\nabla_\perp\right)\nabla_\perp\phi$ belongs to the same part of the density continuity, as the advection part in the total derivative of the inertial (polarization) velocity. In other words, in the expression of the total time derivative that is applied on the polarization (inertial) velocity $\mathbf{v}_p$, the velocity of advective term is composed of both *electric* and *diamagnetic* parts

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{v}_E + \mathbf{v}_{*i})\cdot\nabla_\perp$$

  is applied on the electric velocity $\dfrac{d\mathbf{E}_\perp}{dt}$ and so $\mathbf{v}_p$ is obtained then one takes the divergence $\operatorname{div}\mathbf{v}_p$

- there is no gravity term at **Petviashvili Pokhotelov** since at **Horton** there were mirrors.

End.
END.

Cylindrical geometry allows Fourier representation

$$\begin{aligned}\phi(r,\theta,t) &= \phi_0(r,t)\\ &\quad+\sum_m \phi_m(r)\exp(im\theta - i\omega t)\end{aligned}$$

$$\begin{aligned} n(r,\theta,t) &= n_0(r,t) \\ &\quad + \sum_m n_m(r)\exp(im\theta - i\omega t) \end{aligned}$$

The averages

$$\frac{\partial}{\partial t}\langle n\rangle(r,t) + \frac{1}{r}\frac{\partial}{\partial r}\langle r v_r n_i\rangle = 0$$

$$\frac{\partial}{\partial t}\langle n_i v_\theta\rangle + \frac{1}{r}\frac{\partial}{\partial r}\left[\langle r n_i v_\theta v_r\rangle\right] + \frac{1}{m_i}\langle \pi_{r\theta}\rangle = r\langle n_i g_\theta\rangle$$

where

$$\begin{aligned} v_r &= \left.\frac{-\boldsymbol{\nabla}\phi \times \widehat{\mathbf{n}}}{B}\right|_r \\ v_\theta &= \left.\frac{-\boldsymbol{\nabla}\phi \times \widehat{\mathbf{n}}}{B}\right|_\theta \end{aligned}$$

$$\langle r\pi_{r\theta}\rangle = \frac{T_i}{2\Omega_{ci}}\left\langle r n_i\left(\frac{\partial v_r}{\partial r} - \frac{\partial v_\theta}{r\partial\theta} - \frac{v_r}{r}\right)\right\rangle$$

The linear (excluding the mode coupling terms) equation for the poloidal components with mode number m is

$$\begin{aligned} & -i(\omega - m\Omega)\,\delta n_{im} - i\frac{m}{r}\frac{1}{B}\frac{\partial n_0}{\partial r}\delta\phi_m \\ & +\boldsymbol{\nabla}\cdot\left(\delta\left(n_i\mathbf{v}_i\right)^{(3)}_m\right) \\ = {} & 0 \end{aligned}$$

We recognize

$$\begin{aligned} -i\omega\;\delta n_{im} &\leftarrow \frac{\partial}{\partial t}\,\delta n_{im} \\ im\Omega\;\delta n_{im} &\leftarrow \frac{-\boldsymbol{\nabla}\phi \times \widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}\delta n_{im}\quad\text{where}\quad \phi = \phi_0 \\ &= \frac{1}{B}\frac{d\phi_0}{dr}\frac{\partial}{\partial y}\delta n_{im} = \frac{1}{B}\frac{d\phi_0}{dr}ik_y\;\delta n_{im} = \frac{1}{B}\frac{d\phi_0}{dr}i\frac{m}{r}\;\delta n_{im} \\ &= \left(\frac{1}{B}\frac{1}{r}\frac{d\phi_0}{dr}\right)im\;\delta n_{im} \overset{\text{def}}{=} im\Omega\;\delta n_{im} \end{aligned}$$

with the definition

$$\Omega \equiv \frac{1}{B}\frac{1}{r}\frac{d\phi_0}{dr}$$

since

$$k_y \equiv k_\theta = \frac{m}{r}$$

The next term is *diamagnetic* and comes from the same term as above, expanded in the two small perturbations δn_{im} and $\delta\phi_m$

$$\frac{-\nabla\delta\phi\times\widehat{\mathbf{n}}}{B}\cdot\nabla n_i \to -\frac{1}{B}\frac{d(\delta\phi_m)}{dy}\frac{dn_i^{(0)}}{dr} = -\frac{1}{B}ik_y(\delta\phi_m)\frac{dn_i^{(0)}}{dr} = -\frac{1}{B}i\frac{m}{r}(\delta\phi_m)\frac{dn_i^{(0)}}{dr}$$

The last term is

$$\begin{aligned}
& \nabla\cdot\left[\delta\left(n_i\mathbf{v}_i\right)_m^{(3)}\right] \\
= \; & i\frac{1}{\Omega_{ci}B}\left[n_0\left(\omega - m\Omega - \omega_{*i}\right)\nabla_\perp^2\delta\phi_m\right. \\
& +\left(\omega - m\Omega\left(r\right)\right)\frac{dn_0}{dr}\frac{d\delta\phi_m}{dr} + \frac{m}{r}2\Omega\frac{dn_0}{dr}\delta\phi_m \\
& +m\frac{d\Omega}{dr}\frac{dn_0}{dr}\delta\phi_m + \frac{m}{r}\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r^2\Omega\right)\right)n_0\delta\phi_m \\
& +m\frac{T_i}{e_iB}r\frac{d}{dr}\left(\frac{1}{r}\frac{dn_0}{dr}\right)\frac{d}{dr}\left(\frac{\delta\phi_m}{r}\right) \\
& -\frac{m}{r}B\left(r\Omega^2 + g\right)\delta n_{im} \\
& +\frac{m}{r}\frac{T_i}{e_i}\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r^2\Omega\right)\right)\delta n_{im} \\
& \left.+m\frac{T_i}{e_i}r\frac{d\Omega}{dr}\frac{d}{dr}\left(\frac{\delta n_{im}}{r}\right)\right]
\end{aligned}$$

where

$$\Omega \equiv \frac{1}{B}\frac{1}{r}\frac{d\phi_0}{dr}$$

$$\omega_{*i} = k_\theta\frac{T_i/m_i}{(e_iB/m_i)}\frac{1}{n_0}\frac{dn_0}{dr} = k_\theta\frac{\rho_s c_s}{L_n}$$

The first line

$$\frac{i}{\Omega_{ci}B}n_0\left(\omega - m\Omega - \omega_{*i}\right)\nabla_\perp^2\delta\phi_m$$

has the following composition

$$n_0 i\omega\nabla_\perp^2\delta\phi_m \leftarrow \nabla\cdot\left(n_i\frac{\partial}{\partial t}\nabla_\perp\delta\phi_m\right) \quad \text{with } n_i \text{ unperturbed } n_i \equiv n_i^{(0)}$$

$$
\begin{aligned}
-im\Omega\boldsymbol{\nabla}_{\perp}^{2}\delta\phi_{m} &\leftarrow \boldsymbol{\nabla}_{\perp}\cdot\left(\frac{1}{B}\frac{d\phi_{0}}{dr}\boldsymbol{\nabla}_{\perp}^{2}\delta\phi_{m}\right)\sim\leftarrow\boldsymbol{\nabla}_{\perp}\cdot\left(n_{i}^{(0)}\frac{-\boldsymbol{\nabla}_{\perp}\phi_{0}\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}_{\perp}^{2}\delta\phi_{m}\right) \\
&\sim \boldsymbol{\nabla}\cdot\left(n_{i}\frac{-\boldsymbol{\nabla}_{\perp}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}_{\perp}^{2}\phi\right)\quad\text{with } n_{i}=n_{i}^{(0)}\text{ unperturbed,} \\
&\sim n_{i}^{(0)}\frac{-\boldsymbol{\nabla}_{\perp}\phi_{0}\times\widehat{\mathbf{n}}}{B}\cdot\widehat{\mathbf{e}}_{y}\frac{\partial}{\partial y}\boldsymbol{\nabla}_{\perp}^{2}\left(\delta\phi_{m}\right) \\
&\sim n_{i}^{(0)}\left(-\right)\frac{1}{B}\frac{d\phi_{0}}{dr}i\frac{m}{r}\boldsymbol{\nabla}_{\perp}^{2}\left(\delta\phi_{m}\right) \\
&= -n_{0}im\Omega\boldsymbol{\nabla}_{\perp}^{2}\left(\delta\phi_{m}\right)
\end{aligned}
$$

Therefore the origin of this term is

$$
\boldsymbol{\nabla}\cdot\left(n_{i}\frac{-\boldsymbol{\nabla}_{\perp}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}_{\perp}^{2}\phi\right)
$$

which is the second term in the parathesis of the polarization drift term in the main equation.

We also note that

$$
im\Omega=i\frac{m}{r}\frac{1}{B}\frac{d\phi_{0}}{dr}\leftarrow ik_{y}\frac{E_{\perp}}{B}\leftarrow ik_{y}v_{y}^{(0)}\leftarrow v_{y}^{(0)}\frac{\partial}{\partial y}\leftarrow\mathbf{v}\cdot\boldsymbol{\nabla}|_{y}
$$

Analogously there is a term of the form (the third in the paranthesis)

$$
\boldsymbol{\nabla}\cdot\left(\frac{T_{i}}{e_{i}}\frac{-\boldsymbol{\nabla}_{\perp}n_{i}\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}_{\perp}^{2}\phi\right)
$$

and this generates

$$
-\omega_{*i}\boldsymbol{\nabla}_{\perp}^{2}\left(\delta\phi_{m}\right)
$$

This comes from the diamagnetic velocity that convects the electric velocity (to get the polarization - inertial velocity) followed by taking the divergence of this flux.

We can see that

$$
\left(\omega-m\Omega-\omega_{*i}\right)
$$

comes from

$$
\frac{\partial}{\partial t}+v_{E}\left(-ik_{\theta}\right)+v_{*i}\left(-ik_{\theta}\right)\sim\frac{\partial}{\partial t}+\left(\mathbf{v}_{E}+\mathbf{v}_{*i}\right)\cdot\boldsymbol{\nabla}_{\perp}=\frac{d}{dt}
$$

which is applied on the *electric velocity* v_E.

This is actually (the inverse of) a *propagator*. The propagator $1/\left(\omega-m\Omega-\omega_{*i}\right)$ can have resonances. The singularity can be associated with the state where the wave and the particles are strongly interacting. If there are no dissipative mechanisms like collisions or drifts that force the non-coincidence of the phases of the particles and the wave, the result is Landau damping. If there are dissipative mechanisms then the wave can grow or decay through the interaction with the populaion of particles: absorbs energy from them or looses the energy toward them.

What happens with the propagator when there is cancelling $v_E + v_{*i} \approx 0$? The propagator becomes

$$\sim \frac{1}{i\omega}$$

like in the Navier-Stokes fluid. They introduce a viscosity $\nu \Delta v$ which leads to

$$\frac{1}{i\omega - \nu k^2}$$

The other terms can also be identified.

The neutrality is

$$n_i = n_e$$

and this takes place at un-perturbed state

$$n_i^{(0)} = n_e^{(0)}$$

From the ion continuity equation

$$-i(\omega - m\Omega)\delta n_m - \frac{1}{B}\frac{dn_0}{dr} i\frac{m}{r}\delta\phi_m + \mathbf{\nabla}_\perp \cdot \left[\delta\left(n_i \mathbf{v}_i\right)_m^{(3)}\right] = 0$$

we extract the perturbed part of the *density*, δn_m, to be used in the neutrality equation

$$\delta n_m = -\frac{1}{(\omega - m\Omega)}\frac{1}{B}\frac{dn_0}{dr}\frac{m}{r}\delta\phi_m + \frac{\mathbf{\nabla}_\perp \cdot \left[\delta\left(n_i \mathbf{v}_i\right)_m^{(3)}\right]}{i(\omega - m\Omega)}$$

We see that

$-i(\omega - m\Omega)\delta n_m$	$-\frac{1}{B}\frac{dn_0}{dr} i\frac{m}{r}\delta\phi_m$	$\mathbf{\nabla}_\perp \cdot \left[\delta\left(n_i \mathbf{v}_i\right)_m^{(3)}\right]$	
$\downarrow$	$\downarrow$ (drive)	$\downarrow$	
$\left(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \mathbf{\nabla}_\perp\right) n$	$(v_{*i}\nabla_\theta)\widetilde{v}_E$	$\mathrm{div}\,(\mathbf{v}_p)$	$\widetilde{v}_E$ is the wave, $\delta\phi_m$

We also **Note** that the denominator

$$\omega - m\Omega$$

is a *resonance* and it is not a drive that can eventually be cancelled. It is the equivalent to a *propagator*, coming from the inverse of a derivative along a trajectory, represented here by d/dt.

This is for the ions.

For electrons, in mirrors there are two kind of populations: trapped and passing. We only take one density n_e which for ITG should be taken adiabatic.

It results, after **Horton**

$$\begin{aligned}
& \frac{m_i}{e_i B^2} \left\{ \boldsymbol{\nabla}_\perp \cdot \left[n_0 \left(\omega - m\Omega - \omega_{*i} \right) \boldsymbol{\nabla}_\perp \left(\delta\phi \right) \right] \right. \\
& + m \frac{d\Omega}{dr} \frac{d}{dr} \left[n_0 \left(1 - \frac{\omega_{*i}}{\omega - m\Omega} \right) \delta\phi \right] \\
& + \left[\left(2m\Omega + \frac{m^2 \left(\omega^2 + g/r \right)}{\omega - m\Omega} - \frac{1}{\omega - m\Omega} \frac{m^2}{r^2} \frac{T_i}{e_i B} \frac{d}{dr} \left(r^2 \frac{d\Omega}{dr} \right) \right) \left(\frac{1}{r} \frac{dn_0}{dr} \right) \right. \\
& \qquad + \frac{m}{r} \frac{d}{dr} \left(\frac{1}{r} \frac{d}{dr} \left(r^2 \Omega \right) \right) n_0 \\
& \qquad \left. \left. + \frac{1}{r} \frac{d}{dr} \left(n_0 \omega_{*i} \right) \right] \delta\phi \right\} \\
= \; & n_e
\end{aligned}$$

NOTE

In the absence of sheared electric velocity

$$\begin{aligned}
\Omega &= \text{const} \\
\frac{d\Omega}{dr} &= 0
\end{aligned}$$

we have

$$\begin{aligned}
& \frac{m_i}{e_i B^2} \left\{ \boldsymbol{\nabla}_\perp \cdot \left[n_0 \left(\omega - m\Omega - \omega_{*i} \right) \boldsymbol{\nabla}_\perp \left(\delta\phi \right) \right] \right. \\
& \left. + \left[\left(2m\Omega + \frac{m^2 \omega^2}{\omega - m\Omega} \right) \left(\frac{1}{r} \frac{dn_0}{dr} \right) + \frac{1}{r} \frac{d}{dr} \left(n_0 \omega_{*i} \right) \right] \delta\phi \right\} \\
= \; & 0
\end{aligned}$$

We see that the first part comes from

$$\begin{aligned} & \frac{\partial}{\partial t} + \mathbf{v}_E \cdot \boldsymbol{\nabla}_\perp + \mathbf{v}_{*i} \cdot \boldsymbol{\nabla}_\perp \\ = & \frac{d}{dt} \end{aligned}$$

applied on the *fluctuating part* of the electric velocity $\widetilde{v}_E$. Therefore it comes from the fluctuating part of the polarization (inertial) velocity $\widetilde{\mathbf{v}}_p$. This part can only be a <u>propagator</u> and describes the trajectory advection of a perturbation.

We must see the *drive* of the perturbations.

And find in this drive the possible *cancellation* due to the equality

$$v_E \approx v_{*i}$$

We can look to the terms

$$\begin{aligned} & (2m\Omega) \left(\frac{dn_0}{dr} \right) + \frac{d}{dr} (n_0 \omega_{*i}) \\ \sim & \; 2m\Omega \left(\frac{dn_0}{dr} \right) + \omega_{*i} \frac{dn_0}{dr} \sim k_\theta (v_E + v_{*i}) \frac{dn_0}{dr} + m\Omega \frac{dn_0}{dr} \end{aligned}$$

which, when

$$\omega - m\Omega \approx 0$$

becomes the combination

$$\omega + \omega_E + \omega_{*i}$$

This is the drive. Then when we have

$$\omega_E + \omega_{*i} \approx 0$$

we have a problem for the drive. But in this case the the frequency of the mode is

$$\omega = m\Omega$$

END.

Horton works for mirrors and for this the electron response is calculated in a special way.

We will simply take adiabatic electrons.

NOTE that we do not retain *two fields*, density perturbations and potential perturbations $\delta\phi$. This would be equivalent to the passage from Hasegawa-Mima to Hasegawa-Wakatani. **END**.

Radial eigenmode eauation

The neutrality

$$n_i = n_e = \frac{|e|\,\varphi}{T_e}$$

leads to

$$\begin{aligned}
& \frac{m_i}{e_i B}\left\{\boldsymbol{\nabla}_\perp \cdot \left[n_0\left(\omega - m\Omega - \omega_{*i}\right)\boldsymbol{\nabla}_\perp \delta\Phi\right]\right. \\
& + m\frac{d\Omega}{dr}\frac{d}{dr}\left[n_0\left(1-\frac{\omega_{*i}}{\omega - m\Omega}\right)\delta\Phi\right] \\
& + \left[\left(2m\Omega + \frac{m^2\left(\omega^2+\frac{g}{r}\right)}{\omega - m\Omega} - \frac{m^2}{r^2}\frac{1}{\omega - m\Omega}\frac{T_i}{e_i B}\frac{d}{dr}\left(r^2\frac{d\Omega}{dr}\right)\right)\left(\frac{1}{r}\frac{dn_0}{dr}\right)\right. \\
& \qquad\qquad + \frac{m}{r}\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r^2\Omega\right)\right)n_0 \\
& \qquad \left.\left. + \frac{1}{r}\frac{d}{dr}\left(n_0\omega_{*i}\right)\right]\delta\Phi\right\} \\
& - n_0\frac{|e|\,\delta\Phi}{T_e} \\
= \; & 0
\end{aligned}$$

In distinction from *mirrors* we take a adiabatic response for the *electron* density.

The equation is

$$\begin{aligned}
& \boldsymbol{\nabla}_\perp \cdot \left[n_0\left(\omega - m\Omega - \omega_{*i}\right)\boldsymbol{\nabla}_\perp \delta\Phi\right] \\
& + m\frac{d\Omega}{dr}\frac{d}{dr}\left[n_0\left(1-\frac{\omega_{*i}}{\omega - m\Omega}\right)\delta\Phi\right] \\
& + \left[\left(2m\Omega + m^2\frac{\left(\omega^2+\frac{g}{r}\right)}{\omega - m\Omega}\right)\frac{1}{r}\frac{dn_0}{dr}\right. \\
& - \frac{m}{r}\frac{\omega_{*i}}{\omega - m\Omega}\frac{d}{dr}\left(r^2\frac{d\Omega}{dr}\right) \\
& + \frac{m}{r}n_0\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r^2\Omega\right)\right) \\
& \qquad\qquad \left. + \frac{1}{r}\frac{d}{dr}\left(n_0\omega_{*i}\right)\right] \\
= \; & \frac{|e|\,\delta\Phi}{T_e}
\end{aligned}$$

Horton (since it was question of mirrors) takes

$$\begin{aligned} \delta\Phi\left(r\rightarrow 0\right) &= r^{m} \\ & \qquad \delta\Phi_{m} \quad \text{or} \quad \frac{\delta\Phi}{dr} \quad (\text{at} \quad r=b) \end{aligned}$$

Horton says that the term

$$\frac{1}{\omega - m\Omega}$$

describes the interchange of vorticity elements.

$$\zeta = \frac{1}{r}\frac{d}{dr}\left(r^{2}\Omega\right) = \frac{1}{r}\frac{d}{dr}\left(rv_{\theta}\right)$$

47.0.1 Solid body rotation

In **Horton** the following particular case is examined

$$\begin{aligned} \Omega &= \text{const} \quad (\text{solid body rotation}) \\ n_{0}\left(r\right) &= n_{0}\exp\left(-\frac{r^{2}}{a^{2}}\right) \\ g\left(r\right) &= g_{0}\frac{r}{a} \end{aligned}$$

The equation becomes

$$\frac{d^{2}}{dr^{2}}\delta\Phi + \left(\frac{1}{r} - \frac{2r}{a^{2}}\right)\frac{d\delta\Phi}{dr} + \left(\frac{2}{a^{2}}\nu - \frac{m^{2}}{r^{2}}\right)\delta\Phi = 0$$

The eigenvalue is introduced by

$$\begin{aligned} & \nu\left(\omega, m, \Omega, \frac{g_{0}}{a}, \frac{T_{i}}{T_{e}}\right) \\ = & -\frac{1}{\omega - m\Omega}\frac{1}{\omega - m\Omega - \omega_{*,i}} \\ & \times\left[m^{2}\left(\Omega^{2} + \frac{g_{0}}{a}\right) + \left(\omega - m\Omega\right)\left(2m\Omega + \omega_{*i}\right) + \frac{|e|}{T_{e}}\left(\omega - m\Omega\right)\right] \end{aligned}$$

The solution is

$$\delta\Phi_{m,n}\left(r\right) = A_{m}\left(\frac{a}{r}\right)\exp\left(\frac{r^{2}}{a^{2}}\right)W_{p,q}\left(\frac{r^{2}}{a^{2}}\right)$$

the Whittaker function of indices

$$
\begin{aligned}
p &\equiv \frac{\nu_{m,n}+1}{2} \\
q &\equiv \frac{m}{2}
\end{aligned}
$$

In the *pedestal* the profiles of the parameters have fast radial variation.

The radial electric field starts from about 0 at the some distance from the plasma edge in SOL, increases in absolute magnitude, being negative, on a distance of about 1 cm then rises to be again close to zero at about $1.5cm$ from the Last Closed Flux Surface. The pictures from **Burrell 1997 in Additional LH** show that the maximum of the radial electric field is approximately at the LCFS. This should be also, in our idea, the profile of the diamagnetic velocity ω_{*i}. This means that there is a strong variation of the gradient of density, with the presence of a maximum of the gradient at a radius inside this radial interval

$$
\begin{aligned}
\frac{d}{dr}\omega_{*i} &= 0 \text{ or} \\
\frac{d}{dr}u_\theta &= 0 \text{ or} \\
&\approx \frac{d}{dr}\Omega = 0
\end{aligned}
$$

This would allow us to simplify the equation for $\delta\phi_m$ by taking approximately $d\Omega/dr \approx 0$.

48 Bibliography

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