1 Viscosity

2 Introduction

In **Hirshman Sigmar.**

- the fluid momentum balance and *friction-flow* relations
- *viscosity-flow* relations, obtained by solving the drift kinetic equation.

From **Connor 1973.** (see *impurities* in *plasma general*).
The electron diffusion is small.
For neutrality the radial fluxes of all ions must be approximately zero since the electrons cannot compensate.
If the background ions are diffusing radially outward, then the impurity ions must diffuse inward.

2.1 Notes on viscosity

In **Shaing PoP 3 1996 965**
Distinction and comparison between

1. friction, which is given as the difference of the *parallel* momentum (*mass x velocity*) of ions and electrons multiplied by *collision frequency.* The change of momentum in an interval of time is a *force.* Friction will therefore be the *force* which contributes to the balance of forces in the "momentm" equation.
2. viscosity, whose definition involves the transfer of momentum of one direction to some direction. It is then a tensorial object. It relies on the difference between the two pressures: $p_\perp$ and $p_\parallel$, which are calculated using the solution of the *drift-kinetic equation.* The transfer of momentum directed along some direction to another direction (ex. momentum along x transferred along y) is possible only if there are collisions. The viscosity expresses the periodic transfer between perpendicular and parallel energies, in the local modulations of the magnetic field on helical lines. BUT, there is a collision frequency that multiply the viscosity, otherwise there would be no damping effect. **Stix 1973**. **Rosenbluth Hinton 1997** show that there is no collisionless process that can stop the zonal flows.

Expressions for the components of the viscosity **Staebler Dominguez Anomalous Momentum transport**. Drift waves.

The neoclassical radial transport of the toroidal momentum is negligible (**Ernst, notch** and **Honda**, etc.). It is only carried away by circulating particles, with

$$\left\langle R^2 \nabla\varphi \cdot \nabla \cdot \boldsymbol{\pi} \right\rangle \sim \frac{\rho_i^2}{a^2} \ll 1$$
(Honda)

Or, the circulating particles have small deviation from the magnetic surface (by contrast, trapped particles have large deviations, but they do not have, on the average, a toroidal momentum - except for a small toroidal precession of bananas).

Remember this is

$$\begin{aligned}
\left\langle R^2 \nabla\varphi \cdot \nabla \cdot \boldsymbol{\pi} \right\rangle &= \boldsymbol{\pi}^{(1)} + \boldsymbol{\pi}^{(2)} \sim \frac{\rho_i^2}{a^2} \ll 1 \\
\boldsymbol{\pi}^{(1)} &\approx 0 \\
\boldsymbol{\pi}^{(2)} &\sim \text{very small, the only way} \\
& \quad \text{to transport toroidal momentum} \\
& \quad \text{(Honda)}
\end{aligned}$$

In **Rozhansky Tendler 1992** the equations of motion of particles (see above at the *Section Particle Motion*) are used to write the drift-kinetic equation for the distribution function f_j for species j.

The expansion is done in the inverse aspect ratio

$$\varepsilon = \frac{r}{R}$$

as

$$f_j = f_{0j}(r) + f_{1j}(r,\theta)$$
$$\phi = \phi_0(r) + \phi_1(r,\theta)$$

and we note that in the neoclassical calculation the variation of plasma parameters over the magnetic surface is represented as harmonic components, the first being of 2π periodicity, $\cos\theta$ and $\sin\theta$,

$$\begin{aligned}
f_{1j} &\sim \exp(i\theta) \\
\phi_1 &\sim \exp(i\theta)
\end{aligned}$$

first order variations on the surface. [note that **Novakovskii Sagdeev** PRL use an expansion in harmonics $\exp(i\theta)$ to calculate the function $f^{(2)}$ with which they obtain TTMP].

The solution, *i.e.* the part of the distribution function that has variations on the poloidal direction, on the magnetic surface

$$f_{1j} = \left[\boldsymbol{PV} \frac{1}{\frac{1}{B}\frac{d\phi_0}{dr} + \frac{\varepsilon}{q} V_{\parallel j}} + i\pi\delta\left(\frac{1}{B}\frac{d\phi_0}{dr} + \frac{\varepsilon}{q} V_{\parallel j} \right) \frac{\partial}{\partial\theta} \right] \widehat{A} f_{0j}$$

where

$$\boldsymbol{PV} = \text{principal value}$$

which corresponds to the limit of very low collisions

$$\nu_i \rightarrow 0$$

but still without trapped particles.

See **Galeev**.

For the vanishing of the poloidal velocity by the combination of the poloidal electric flow $\frac{1}{B}\frac{d\phi_0}{dr}$ and the poloidal projection of the parallel flow $\frac{\varepsilon}{q}V_{\parallel j}$ see **Kleva Hassam**:

when the displacement of an element of plasma along the line can produce a displacement on poloidal direction, the tube of plasma also moves poloidally due to E_r such that an element of plasma does NOT move effectively on θ.

It only moves toroidally.

This means compensation of the following two terms

$$\frac{1}{B}\frac{d\phi_0}{dr} + \frac{\varepsilon}{q} V_{\parallel j} \rightarrow 0$$

and this singular factor multiplying the operator acting on the first order distribution function is solved as a $1/x$ singularity.

The factor

$$\delta\left(\frac{1}{B}\frac{d\phi_0}{dr} + \frac{\varepsilon}{q} V_{\parallel j} \right)$$

will select from the next part of the expression exactly the part that corresponds to the balance (compensation) of the two motions.

This state corresponds to the balance discussed by **Stringer**. There is a parallel flow that generates a radial electric field just to preserve neutrality even if the drifts of different species are different.

NOTE

The mutual cancellation of the poloidal electric flow and the poloidal projection of the *parallel* velocity are meant to ensure the exclusive toroidal rotation, due to the strong poloidal damping by magnetic pumping.

If we consider that the electrostatic potential ϕ_0 is uniform on the surface and its radial gradient too, then the electric velocity is constant, $V_E(\psi)$.

In contrast, the parallel velocity is modulated. As seen in **Novakovskii, Sagdeev Liu Rosenbluth** and **Galeev**, the parallel velocity has variation along the magnetic field line, and its amplitude depends on θ.

END

We recognize

$$\frac{\varepsilon}{q} V_{\parallel j} = \frac{B_\theta}{B_T} V_{\parallel j}$$

the projection of the parallel velocity on the poloidal direction. And

$$\frac{1}{B}\frac{d\phi_0}{dr} = \frac{-E_r}{B}$$

the poloidal velocity generated by the radial electric field.

$$V_E + \frac{B_\theta}{B} v_\parallel \approx 0$$

the difference exists however and is

$$\sim \sqrt{\kappa^2 - \sin^2\frac{\theta}{2}}$$

see **Galeev Sagdeev Liu Novakovskii**.

NOTE. The formula that is applied above is

$$\frac{1}{x} = \mathbf{PV}\left(\frac{1}{x}\right) + i\pi\delta(x)$$

where

$$\begin{aligned} x &\equiv \frac{1}{B}\frac{d\phi_0}{dr} + \frac{\varepsilon}{q} V_{\parallel j} \\ &= \text{poloidal velocity resulting from the radial electric field} \\ &\quad\ \text{plus projection of parallel velocities} \end{aligned}$$

This actually is

$$V_E + \Theta V_\parallel$$

which is the combination that occurs usually in the russian literature. This - being the poloidal velocity, must be almost zero

$$V_E + \Theta V_{\|} \approx 0$$

This is the reason of the singularity.

See **Hassam Kulsrud** on the motion of a plasma "tube" simultaneous to the motion of an element of plasma on the tube.

END

In the expression of the *first-order* correction to the distribution function f_{1j} the first factor

$$\left[\boldsymbol{PV} \frac{1}{\frac{1}{B}\frac{d\phi_0}{dr} + \frac{\varepsilon}{q} V_{\|j}} + i\pi\delta\left(\frac{1}{B}\frac{d\phi_0}{dr} + \frac{\varepsilon}{q} V_{\|j}\right)\frac{\partial}{\partial\theta}\right]$$

actually solves the problem of the *propagator*

$$\begin{aligned} G^{-1} &= \frac{1}{G} \\ &= \text{inverse of the derivative to time along a trajectory} \\ &= \text{inverse of a velocity} \\ &= \frac{1}{\frac{1}{B}\frac{d\phi_0}{dr} + \frac{\varepsilon}{q} V_{\|j}} \end{aligned}$$

The Operator $\widehat{A}$ acts on the *equilibrium distribution function* as is usual in a first order expansion of the drift-kinetic equation

$$\begin{aligned} \widehat{A} \equiv & \frac{e_j \phi_1}{m_j}\left(\frac{B_\theta}{B_T}\frac{\partial}{\partial V_{\|j}} \quad \begin{array}{c} \text{energetic,} \\ \text{electrostatic } \phi_1 \sim \theta \text{ on surface} \\ \rightarrow \text{modulation of parallel velocity} \end{array} \right. \\ & \left. \qquad\qquad\qquad + \frac{1}{\Omega_j}\frac{\partial}{\partial r}\right) \\ & -\frac{1}{\Omega_j}\left(\frac{V_{\perp j}^2}{2} + V_{\|j}^2\right)\varepsilon\cos\theta\frac{\partial}{\partial r} \\ & -\frac{1}{T_j}\left(\frac{V_{\perp j}^2}{2} + V_{\|j}^2\right)\varepsilon\cos\theta \end{aligned}$$

(last term to be checked). We find in the first term in the first line an *energetic* effect: the derivation with respect to the parallel velocity of the shifted-Maxwell distribution will produce a parallel velocity

$$\begin{aligned}
& \frac{\partial}{\partial V_{\|j}} \exp\left(-\frac{\left(V_{\|j}-u_{\varphi}\right)^{2}}{v_{i,th}^{2}}\right) \\
= & -2\left(V_{\|j}-u_{\varphi}\right)\frac{1}{2T_{j}/m_{j}}\times\exp\left(...\right)
\end{aligned}$$

and this velocity $\left(V_{\|j}-u_{\varphi}\right)$ is further projected onto the poloidal direction by the factor B_{θ}/B_{T}. The poloidal velocity that results implies a *work* to be done in the electric potential that has variation over the surface, $e\phi_{1}/m_{j}\rightarrow E_{\theta}^{(1)}$. This is a term of the kind *velocity of a particle* $\times$ *energy of the particle in the electric potential.*

This is an energetic term.

This operator is applied on the zeroth order distribution function which is Maxwellian in the frame of the moving ions

$$f_{0i}\left(V_{\perp},V_{\|}\right)=\frac{n_{0}}{\left(2\pi\right)^{3/2}v_{Th,i}^{3}}\exp\left[-\frac{m_{i}}{2T_{i}}\left(V_{\|i}-u_{\varphi}\right)^{2}-\frac{m_{i}V_{\perp i}^{2}}{2T_{i}}\right]$$

(see **Rewoldt Tang Frieman trapped electron mode).**

With this, we dispose of a first order in $\varepsilon=r/R$ expression of the distribution function, which reflects the modification with respect to the Maxwellian due to the *neoclassical* motion of the particles. The collisions have not yet been included. This should correspond to the first function $f_{0}+f_{1}+f_{2}+...$ in **Rutherford**, before adding g which considers the collisions.

With this distribution function one can calculate the components of the *ANISOTROPIC* pressure tensor.

The component $p_{\|i}$ is produced by carrying in the parallel direction with velocity $V_{\|i}$ of the parallel velocity itself $V_{\|i}$.

$$\begin{aligned}
p_{\|i} &= \int d^{3}V\ \left(V_{\|i}^{2}\ \ f_{i}\right) \\
&= \int\left(2\pi\right)dV_{\|i}V_{\perp i}dV_{\perp i}\ \ \ \left(V_{\|i}^{2}\ \ f_{i}\right)
\end{aligned}$$

The component $p_{\perp i}$ is produced by carrying in the perpendicular direction with velocity $V_{\perp i}$ of the perpendicular velocity itself $V_{\perp i}$.

$$\begin{aligned}
p_{\perp i} &= \int d^{3}V\ \left(\frac{1}{2}V_{\perp i}^{2}\ f_{i}\right) \\
&= \int\left(2\pi\right)dV_{\|i}V_{\perp i}dV_{\perp i}\ \ \ \left(\frac{1}{2}V_{\perp i}^{2}\ \ f_{i}\right)
\end{aligned}$$

and with the expressions obtained after introducing the distribution function to first order we can calculate

$$\boldsymbol{\pi}_i^{NEO} = \left(p_{\|i} - p_{\perp i}\right)\left(\widehat{\mathbf{n}}\widehat{\mathbf{n}} - \frac{1}{3}\mathbf{I}\right)$$

Now we have to calculate the *divergence* of the pressure tensor (**Landau Lifshitz**)

$$(\nabla \cdot \boldsymbol{\pi})_l = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^k}\left(\sqrt{|g|}\pi_l^k\right) - \frac{1}{2}\frac{\partial g_{km}}{\partial x^l}\pi^{km}$$

where the metric tensor has the components

$$\begin{aligned} g_{11} &= 1 \\ g_{22} &= r^2 \\ g_{33} &= (1 + \varepsilon\cos\theta)^2 = h^2 \end{aligned}$$

By the way

$$\begin{aligned} g &= \det\left(g_{ij}\right) \\ &= 1 \times r^2 \times (1 + \varepsilon\cos\theta)^2 \\ &= r^2 h^2 \end{aligned}$$

and

$$\sqrt{g} = rh$$

NOTE. In a different system,

$$\begin{aligned} g &= \frac{1}{|\nabla\varphi \cdot (\nabla\psi \times \nabla\theta)|^2} \\ &= \frac{1}{\left[\frac{1}{R} \times 2\pi R B_\theta \times \frac{1}{r}\right]^2} \end{aligned}$$

and it results

$$\sqrt{g} = \frac{1}{2\pi}\frac{r}{B_\theta}$$

which is different by a factor $B_{\theta 0}$ which is a kind of average of the poloidal megnetic field over the surface.

$$\frac{1}{B_\theta} = \frac{1}{B_{\theta 0}}h$$

If the operations that are made afterward are restricted to the surface then the two definitions of the metric tensor are technically equivalent $\sqrt{g} \sim rh$.

END.

It is then possible to find

$$\begin{aligned}
&\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_i \rangle^{NEO} \\
= \;& \sqrt{\pi} n \varepsilon^2 \frac{\sqrt{m_i T_i}}{\sqrt{2}} B_0 \frac{1}{r} \left(V_\theta - V_\theta^{NEO} \right)
\end{aligned}$$

where

$$V_\theta^{NEO} = \left(-\frac{1}{2} \right) \frac{1}{e B_0} \frac{dT_i}{dr} \quad \text{(Hazeltine plateau)}$$

This is the basic neoclassical poloidal rotation derived by **Hazeltine**. Three regimes are identified by the numerical coefficients in front of the gradient of the temperature. For *plateau* the coefficient of the gradient of temperature is 1/2.

Drake Hassam Kleva.

The general equation for the *parallel* momentum averaged over the magnetic surface is

$$\left\langle \mathbf{B} \cdot n m_i \frac{d\mathbf{u}_i}{dt} \right\rangle = - \langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_i \rangle^{NEO}$$

One introduces the notation

$$\begin{aligned}
\overline{u}_\varphi &= \left\langle \frac{B_0}{B_\varphi} u_\varphi \right\rangle \\
&= \langle h u_\varphi \rangle \\
& \quad \text{surface averaged toroidal flow}
\end{aligned}$$

This is similar with the definition, starting from flows, of quantities that are normalized to $B_\theta = \frac{b(r)}{h}$ such that, dividing by B_θ we have, on every magnetic surface, multiplication with h.

This is similar to

$$\frac{\boldsymbol{\nabla}\theta}{\mathbf{B} \cdot \boldsymbol{\nabla}\theta} \cdot () = \frac{1}{B_\theta} ()_\theta = \frac{h}{b(r)} ()_\theta$$

which multiplies with h (that carries the toroidal geometry) the object which is projected on θ.

The flow has *poloidal* θ and *toroidal* φ components, no radial projection. Taking the vector

$$\mathbf{u}_i = (0, u_{i\theta}, u_{i\varphi})$$

and the equation of continuity

$$\boldsymbol{\nabla}\cdot\left(n\mathbf{u}_i\right)=0$$

with the approximation that the density does NOT vary on the magnetic surface $n=$const(θ).

$$\begin{aligned}u_{i\theta} &= V_\theta\left(1+\varepsilon\cos\theta\right)\\ u_{i\varphi} &= \overline{u}_\varphi-2qV_\theta\cos\theta\\ &\quad+\varepsilon\overline{u}_\varphi\cos\theta+2\varepsilon qV_\theta\end{aligned}$$

We recognize in the first line

$$\begin{aligned}V_\theta\left(r,\theta\right) &= \frac{u_{ip}\left(r\right)}{h}\\ \text{where }\ u_{ip} &= u_{ip}\left(r\right)\ \text{ only depends on surface } \psi \text{ or } r\end{aligned}$$

and

$$u_{\|i}\approx u_{i\varphi}+\frac{B_\theta}{B_T}u_{i\theta}$$

The term

$2qV_\theta\cos\theta$ is the Pfirsch-Schluter "flow"

The Pfirsch-Schluter flow (analogous to the PS current) is a harmonic (*i.e.* $\cos\theta$) flow associated to a poloidal flow $V_\theta\left(r,\theta\right)$ due to assumed approximative *incompressibility* of the flow on the magnetic surface

$$\boldsymbol{\nabla}\cdot\mathbf{u}_i\approx 0$$

If we consider the effect of the turbulence as similar to the one of the parallel viscous stress, then instead of

$$\left\langle\mathbf{B}\cdot nm_i\frac{d\mathbf{u}_i}{dt}\right\rangle=-\left\langle\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_i\right\rangle^{NEO}$$

we will have to add an anomalous term

$$\left\langle\mathbf{B}\cdot nm_i\frac{d\mathbf{u}_i}{dt}\right\rangle=-\left\langle\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_i\right\rangle^{NEO}-\left\langle\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_i\right\rangle^{AN}$$

(**Note** that the multiplication with $\mathbf{B}$ has eliminated $\boldsymbol{\nabla}p$ since the gradient of the scalar pressure is perpendicular on $\mathbf{B}$). The equation of balance of the parallel momentum, which is obtained from the equation of motion multiplied by the magnetic field $\mathbf{B}$ and averaged over the surface, with the replacement of the ion-fluid velocity will become

$$
\begin{aligned}
& \left(1+2q^2\right) nm_i \frac{B_\theta}{B_T} B_0 u_r \frac{dV_\theta\left(r,\theta\right)}{dr} + nm_i B_0 [u_r] \frac{d\overline{u}_\varphi\left(r\right)}{dr} \text{ (radial convection } v_r \frac{\partial}{\partial r} \text{)} \\
& - \left(1+2q^2\right) \frac{B_\theta}{B_T} B_0 \frac{1}{r}\frac{d}{dr}\left(r\eta \frac{dV_\theta\left(r,\theta\right)}{dr}\right) - B_0 \frac{1}{r}\frac{d}{dr}\left(r\eta\frac{d\overline{u}_\varphi\left(r\right)}{dr}\right) \\
= & - \left\langle \mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_i\right\rangle^{NEO}
\end{aligned}
$$

This is the equation for the radial distribution of the poloidal rotation velocity.

The parameter that appears here η is the anomalous viscosity coefficient, estimated as

$$
\begin{aligned}
& & \text{anomalous viscosity coefficient} \\
\eta & \sim & D/\left(nm_i\right)
\end{aligned}
$$

with D the particle diffusion coefficient.

$$
\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_i^{AN} = -\frac{1}{r}\frac{d}{dr}\left(r\eta\frac{du_{\|i}}{dr}\right)
$$

In **Hazeltine 1974** the definitions are

$$
\begin{aligned}
\mathbf{P} & = & \widehat{\mathbf{n}}\widehat{\mathbf{n}}P_\| + \left(\mathbf{I}-\widehat{\mathbf{n}}\widehat{\mathbf{n}}\right)P_\perp \\
& & +O\left(\delta^2\right) \quad (\text{where } \delta = r/L)
\end{aligned}
$$

(**note**

$$
\mathbf{P}^{hazel} = \left(P_\| - P_\perp\right)\ \widehat{\mathbf{n}}\ \widehat{\mathbf{n}} + \mathbf{I}\ P_\perp
$$

comparison with

$$
\boldsymbol{\pi}_i^{NEO} = \left(p_{\|i} - p_{\perp i}\right)\left(\widehat{\mathbf{n}}\widehat{\mathbf{n}} - \frac{1}{3}\mathbf{I}\right)
$$

of **Drake, and many others**) **END.**

$$
P_\| \equiv \int d^3v\ mu^2\ f
$$

$$
P_\perp \equiv \int d^3v\ \frac{1}{2}mv_\perp^2\ f
$$

2.2 General expressions

The parallel projection of the momentum conservation equation is the usual instrument to study the connection between the *friction* forces and the divergence of the pressure anisotropy $\nabla \cdot \mathbf{\Pi}$. The flows are involved. The connection, presented by **Hirshman Sigmar review** shows how the parallel friction forces induce radial fluxes.

The following expressions of the parallel plasma viscosity are proposed by **Shaing** et al.

They correspond to the expansion of the kinetic distribution function in series of *flows* u_a . The first is the flow of momentum and the second is the flow of heat normalized to pressure (density of energy). The coefficients of the expansion is in terms of Laguerre polynomials See **Shaing Dominguez resonance viscosity**.

$$\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\Pi} \rangle = nm \left\langle B^2 \right\rangle \left(\mu_1 \boxed{\mathrm{U}_\theta} + \frac{2}{5} \mu_2 \boxed{\frac{q_\theta}{P}} \right)$$

$$\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\Pi} \rangle = nm \left\langle B^2 \right\rangle \left(\mu_2 \boxed{\mathrm{U}_\theta} + \frac{2}{5} \mu_3 \boxed{\frac{q_\theta}{P}} \right)$$

where the angular brakets denote the *flux surface* average. P is the plasma pressure, n is the plasma density (ion's) m is the ion mass. The notations are like in **Hirshman Sigmar**

$$\begin{aligned} U_\theta &\equiv \frac{\nabla \theta}{\mathbf{B} \cdot \nabla \theta} \cdot \mathbf{U} \\ &= \frac{\mathbf{U}/_\theta}{B_\theta} = \text{function of only } \psi \end{aligned}$$

and

$$\begin{aligned} q_\theta &\equiv \frac{\nabla \theta}{\mathbf{B} \cdot \nabla \theta} \cdot \mathbf{q} \\ &= \frac{\mathbf{q}/_\theta}{B_\theta} = \text{function of only } \psi \end{aligned}$$

where $\mathbf{U}$ is the mass flow and $\mathbf{q}$ is the heat flow.

By this definition, the two *new* variables U_θ and $\frac{q_\theta}{P}$ are functions of surface label, r or ψ. No variation on θ.

By this re-definitions, the new symbols U_θ and q_θ are NOT fluxes of mass and heat.

NOTE

The definitions have a particular meaning:

the poloidal velocity divided by the poloidal magnetic field is a function of ONLY the magnetic surface, ψ.

$$\begin{aligned}\frac{u_\theta(r,\theta)}{B_\theta(r,\theta)} &= \frac{1}{b(r)}\; u_\theta(r,\theta)\, h(r,\theta) \quad \text{function of only } \psi \\ &= \text{constant of magnetic surface } \psi\end{aligned}$$

This results from the equation of continuity, or, equivalently, the zero-divergence condition for the flux of fluid and of heat.

For this reason

$$U_\theta \;\text{ and }\; q_\theta$$

are better variables than the physical ones $u_\theta(r,\theta)$.

See below. **END**

See also **Kim Diamond Groebner 1991**.

NOTE. Analytic calculation of Shaing bootstrap pedestal Balance of forces acting in the surface:

parallel viscosity $\boldsymbol{\pi}_i$
parallel electric field $E_\parallel^{(A)}$ induced by transformer
friction forces $\mathbf{F}_{1,2}$ (collisional momentum exchange)

(just like for drift waves). The equations, for electrons and ions.

The momentum equation, projected on the parallel direction and averaged over surface

$$0 = -\left\langle \mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_i\right\rangle + N_i e_i \left\langle B E_\parallel^{(A)}\right\rangle + \left\langle B F_{1i}\right\rangle$$

$$0 = -\left\langle \mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_e\right\rangle + N_e e_e \left\langle B E_\parallel^{(A)}\right\rangle + \left\langle B F_{1e}\right\rangle$$

because $\mathbf{F}_{1i,e}$ is along $\mathbf{B}$, so the scalar product is simply the product of magnitudes. The presence of B means a factor

$$\frac{1}{h}$$

in every surface average $\langle\rangle$.

Note

These equations describe what is the balance of the momentum along $\mathbf{B}$ at stationarity, for electrons and for ions. The transfer of the momenta which exist along different directions to the $\mathbf{B}$ direction is $\boldsymbol{\pi}_{i,e}$. The parallel momentum can get some contribution from such a source. It is balanced by *friction* along the magnetic line.

End.

The heat

$$0 = -\langle \mathbf{B} \cdot \nabla \cdot \boldsymbol{\Theta}_i \rangle + \langle BF_{2i} \rangle$$
$$0 = -\langle \mathbf{B} \cdot \nabla \cdot \boldsymbol{\Theta}_e \rangle + \langle BF_{2e} \rangle$$

where

$$\begin{aligned}
\langle\rangle &\equiv \text{surface average} \\
\boldsymbol{\pi}_i \ , \ \boldsymbol{\pi}_e &\equiv \text{viscous tensors} \\
\boldsymbol{\Theta}_i \ , \ \boldsymbol{\Theta}_e &\equiv \text{heat viscous tensor} \\
F_{1i,e} &\equiv \text{momentum flow friction forces} \\
F_{2i,e} &\equiv \text{heat flow friction forces}
\end{aligned}$$

$$E^{(A)} \equiv \text{externally induced electric field}$$

The forces of friction

$$F_{1i} = -F_{1e}$$

are determined by the collisions between ions, electrons.

Note. In this case the momentum exchange balance is simple,

$$F_{1e} + F_{1i} = 0$$

because only the basic two components exchange momentum. In **Rosenbluth Hinton NBI** the equilibrium must include the *fast ions* from NBI

$$\mathbf{F}_{fe} + \mathbf{F}_{fi} = -\mathbf{F}_{ef} - \mathbf{F}_{if}$$

End.

It is at this moment that the expression of the friction forces is introduced, in terms of *flow velocities.*

This is a result of the *definition* of the friction forces,

the *friction forces* are integrals over the *linearized* collisional operator

In the linearized collision operator one uses the distribution function $f^{(1)}$ resulting from solving the drift-kinetic equation the distribution function is expressed as series of *flows* (momentum and heat)

since the integrand in the definition of the friction forces is the momentum (or heat) flow (projected on $\mathbf{B}$).

$$F_{1e} = l_{11}^{e}\left(V_{\parallel i} - V_{\parallel e}\right) + l_{12}^{e}\,\frac{2}{5}\frac{q_{\parallel,e}}{p_e}$$

friction inducing momentum flow

$$F_{2e} = -l_{21}^{e}\left(V_{\parallel i} - V_{\parallel e}\right) - l_{22}^{e}\,\frac{2}{5}\frac{q_{\parallel,e}}{p_e}$$

friction inducing heat flow

$$F_{2i} = -l_{22}^{i}\,\frac{2}{5}\frac{q_{\parallel,i}}{p_i}$$

The connection between the *friction forces* and the velocities of the flow is *collisional.*

The frequency of collision ν_{ei} must be present in every coefficient.

The numerical coefficients $l_{ij}^{e,i}$ are derived from the <u>collisional</u> operator *linearized* close to Maxwellian

$$\begin{aligned} l_{11}^{e} &= N_e M_e \nu_{ei} \\ l_{12}^{e} &= \frac{3}{2} l_{11}^{e} \\ l_{22}^{e} &= \left(\frac{13}{4} + \frac{\sqrt{2}}{Z}\right) l_{11}^{e} \end{aligned}$$

and

$$l_{22}^{i} = \sqrt{2} N_i M_i \nu_{ii}$$

with the collision frequency

$$\nu_{ab} = \frac{4}{3\sqrt{\pi}} 4\pi \frac{\left(e_a e_b\right)^2}{M_a^2} \ln\Lambda \times \frac{N_b}{v_{th}^3}$$

The parallel projection of the divergence of the pressure anisotropy tensor (the viscous forces) are determined by the parallel friction forces.

The parallel components of the viscous forces, as from $0 = -\left\langle \mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_i\right\rangle + N_i e_i \left\langle B E_{\parallel}^{(A)}\right\rangle + \left\langle B F_{1i}\right\rangle$ (forget the electric field) are

$$\begin{pmatrix} \left\langle \mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}\right\rangle \\ \left\langle \mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Theta}\right\rangle \end{pmatrix} = NMB^2 \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_2 & \mu_3 \end{pmatrix} \begin{pmatrix} V^{\theta} \\ \frac{2}{5} q^{\theta}/p \end{pmatrix}$$

The poloidal components are defined

$$V^\theta = \frac{\boldsymbol{\nabla}\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}\cdot\mathbf{V}$$

$$q^\theta = \frac{\boldsymbol{\nabla}\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}\cdot\mathbf{q}$$

The normalization to poloidal component of the magnetic field $\mathbf{B}\cdot\boldsymbol{\nabla}\theta$ is desired in these expressions: it makes that the ratios are *functions of only the magnetic surface* ψ.

END of NOTE

By mass flow we have to understand the effective displacement of the plasma, not the diamagnetic flow. We have then

$$\begin{aligned} U_\theta &= (\mathbf{U}\cdot\widehat{\mathbf{e}}_\theta)\,\frac{1/r}{B/(qR)} = (\mathbf{U}\cdot\widehat{\mathbf{e}}_\theta)\,\frac{qR}{Br} = (\mathbf{U}\cdot\widehat{\mathbf{e}}_\theta)\,\frac{1}{B_\theta} \\ &= \frac{V_\theta}{B_\theta} \end{aligned}$$

The equilibrium particle distribution function f_0 in the edge region is not a Maxwellian because of the existence of a direct loss region in velocity space. Shaing uses however **a shifted Maxwellian** since in the region of the velocity space which is outside the direct loss region, the distribution function is a shifted Maxwellian. The *viscosity* will be calculated as resulting from the **finite-size orbits of the particles** in the shifted Maxwellian.

3 Parallel viscosity and poloidal rotation damping (Hsu Shaing Gormley 1994)

The equations are the same as those used in next Section.

4 Resonance parallel viscosity Shaing Hsu Dominguez

The *resonance viscosity*:

the plasma viscosity in banana regime is mainly due to the barely circulating and barely trapped particles, whose parallel $v_\parallel$ velocity is in resonance with the poloidal $\mathbf{E}\times\mathbf{B}$ speed.

The poloidal rotation

$$\mathbf{V}_E = \frac{\mathbf{E}\times\mathbf{B}}{B^2}$$

with Mach number

$$M_p = \frac{|V_E|}{v_{th}\frac{B_\theta}{B}} \sim 1$$

The equation

$$\left(v_\parallel \widehat{\mathbf{n}} + \mathbf{v}_D + \mathbf{V}\right)\cdot\boldsymbol{\nabla} f + \dot{w}\frac{\partial f}{\partial w} = 0$$

where

$$\begin{aligned} w &\equiv \frac{v^2}{2} \\ \mu &= \frac{v_\perp^2}{2B} \end{aligned}$$

$$\mathbf{V} \equiv \text{center of mass flow velocity}$$

The drift velocity

$$\begin{aligned} \mathbf{v}_D &= \frac{\mathbf{F}\times\widehat{\mathbf{n}}}{\Omega} + \frac{\mu B}{\Omega}\widehat{\mathbf{n}}\left(\frac{j_\parallel}{B}\right) \\ &\quad + \frac{1}{\Omega}\widehat{\mathbf{n}}\times\left[\mu\boldsymbol{\nabla} B + v_\parallel^2\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right. \\ &\qquad\qquad \left. + v_\parallel\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\mathbf{V} + \left(\mathbf{V}\cdot\boldsymbol{\nabla}\right)\left(v_\parallel\widehat{\mathbf{n}}\right)\right] \end{aligned}$$

Dimensionally, the second term is

$$\frac{1}{\Omega}\mu B \sim \frac{v_D}{\nabla}$$

and the rest is

$$\frac{j_\parallel}{B} \sim \frac{\nabla\times B}{B} \sim \nabla$$

$$\frac{\mu B}{\Omega}\widehat{\mathbf{n}}\left(\frac{j_\parallel}{B}\right) \sim \frac{v_D}{\nabla}\ \nabla \sim v_D$$

and this is OK.

NOTE

Remember

$$
\begin{aligned}
v_{drift} &= \frac{m}{e} v_{\parallel} \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right) \left(\frac{v_{\parallel}}{B_{\theta}}\right) \\
&= \frac{m}{e} v_{\parallel} \nabla_{\parallel} \left(\frac{v_{\parallel}}{B_{\theta}}\right)
\end{aligned}
$$

From **Rutherford 1970.**

And

$$
\mathbf{v}_D = \frac{e\rho_{\parallel}}{mD} \left\{ \boldsymbol{\nabla}\times \left(\rho_{\parallel}\mathbf{B}\right) - \widehat{\mathbf{n}} \left[\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\times \left(\rho_{\parallel}\mathbf{B}\right)\right] \right\}
$$

See and compare

$$
\frac{\mu B}{\Omega} \widehat{\mathbf{n}} \left(\frac{j_{\parallel}}{B}\right) \quad ?
$$

with

$$
\mathbf{v}_D = \frac{e}{m} \rho_{\parallel}^2 \left(\mu_0 \mathbf{j}\right) - \frac{eB}{m} \rho_{\parallel} \left(\widehat{\mathbf{n}} \times \boldsymbol{\nabla}\rho_{\parallel}\right)
$$

END

The *force* $\mathbf{F}$ is derived from the *fluid force balance*, static

$$
\mathbf{F} = \frac{e}{m} \left(\mathbf{E} + \mathbf{V} \times \mathbf{B}\right) - \frac{\partial \mathbf{V}}{\partial t} - \left(\mathbf{V} \cdot \boldsymbol{\nabla}\right) \mathbf{V}
$$

The variation of the energy

$$
\begin{aligned}
\dot{w} = &\ \mathbf{F} \cdot \mathbf{v}_{\parallel} - \mu B \ \left(\boldsymbol{\nabla} \cdot \mathbf{V}\right) \\
&- \left(v_{\parallel}^2 - \mu B\right) \left(\widehat{\mathbf{n}} \cdot \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\mathbf{V}\right) \\
&+ \mathbf{v}_D \cdot \mathbf{F} \\
&- \frac{\mu B}{\Omega} \widehat{\mathbf{n}} \cdot \left(\boldsymbol{\nabla} \times \mathbf{F}\right)
\end{aligned}
$$

The velocity stress tensor

$$
\begin{aligned}
&\boldsymbol{\nabla} \cdot \mathbf{V} \quad \text{and} \\
&\widehat{\mathbf{n}} \cdot \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\mathbf{V}
\end{aligned}
$$

The distribution

$$
\begin{aligned}
f = &\ f_M - \frac{2u}{v_{th}^2} \left(\frac{2}{5} L_1^{(3/2)}\right) \frac{q_{\parallel}}{p} \ f_M \\
&+ g
\end{aligned}
$$

The text is **Shaing Hsu Dominguez resonance**.

The first part is very similar to **poloidal parallel viscosities Shaing 1989**.

The drift-kinetic equation with **a mass flow** is

$$\left(u\widehat{\mathbf{n}}+\mathbf{v}_D+\mathbf{V}\right)\cdot\boldsymbol{\nabla}f+\dot{w}\frac{\partial f}{\partial w}=C\left(f\right)$$

Note that u is the *parallel* velocity of the particle in the frame of the *center-of-mass* velocity.

The neoclassical drift velocity is

$$\begin{aligned}\mathbf{v}_D &= \frac{\mathbf{F}\times\widehat{\mathbf{n}}}{\Omega}+\frac{\mu B\widehat{\mathbf{n}}}{\Omega}\left(\frac{j_{\parallel}}{B}\right) \qquad (1)\\ &+\frac{\widehat{\mathbf{n}}}{\Omega}\times\left[\mu\boldsymbol{\nabla}B+u_{\parallel}^{2}\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}+u_{\parallel}\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\mathbf{V}+\left(\mathbf{V}\cdot\boldsymbol{\nabla}\right)\left(u_{\parallel}\widehat{\mathbf{n}}\right)\right]\end{aligned}$$

where $j_{\parallel}$ is the parallel current density, a term that comes from Ampere's law

$$\boldsymbol{\nabla}\times\mathbf{B}=\mu_0\mathbf{J}$$

projected along $\mathbf{B}$, with

$$\left(\mathbf{J}\cdot\widehat{\mathbf{n}}\right)\widehat{\mathbf{n}}=\frac{1}{\mu_0}\left[\widehat{\mathbf{n}}\cdot\left(\boldsymbol{\nabla}\times\mathbf{B}\right)\right]\widehat{\mathbf{n}}$$

$$\Omega=\frac{eB}{m}$$

$$\mu=\frac{s^2}{2B}\quad,\quad s\equiv v_{\perp}$$

The force is derived from the *fluid* momentum balance

$$\begin{aligned}\mathbf{F} &= \frac{e}{m}\left(\mathbf{E}+\mathbf{V}\times\mathbf{B}\right)-\frac{\partial\mathbf{V}}{\partial t}-\left(\mathbf{V}\cdot\boldsymbol{\nabla}\right)\mathbf{V}\\ &= \frac{\boldsymbol{\nabla}p}{nm}-\frac{\mathbf{R}}{nm}\end{aligned}$$

with $\mathbf{R}$ the friction force and the viscous (anisotropy, $\boldsymbol{\pi}$) force is neglected.

The velocity s (perpendicular) and the energy

$$w=v^2/2$$

are in the frame of the center of mass velocity.

(we keep the notation ϵ for the *total* energy, including the electrostatic potential energy).

The **energetic contribution in the drift-kinetic equation** is:

$$\begin{aligned}
\dot{w} &= \text{time variation of the particle's kinetic energy} \\
&= \mathbf{F}\cdot\mathbf{u} - \mu B\left(\boldsymbol{\nabla}\cdot\mathbf{V}\right) - \left(u^2-\mu B\right)\left(\widehat{\mathbf{n}}\cdot\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\mathbf{V}\right) \\
&\quad +\mathbf{F}\cdot\mathbf{v}_D - \left(\frac{\mu B}{\Omega}\right)\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\times\mathbf{F}
\end{aligned}$$

We see that the time variation of the energy is due to

- the effect of the force $\cdot$ velocity :

  $$\mathbf{F}\cdot\mathbf{u} \text{ and } \mathbf{F}\cdot\mathbf{v}_D$$

  which means that the plasma must do *work* when moving with velocity $\mathbf{u}$ and with the drift velocity $\mathbf{v}_D$ against a force $\mathbf{F}$;

- compression of the flow, by a nonzero compressibility $\boldsymbol{\nabla}\cdot\mathbf{V}\neq 0$. *This is IN the velocity stress tensor.* This is combined with the perpendicular energy ;

  $$\begin{aligned}
  & -\mu B\left(\boldsymbol{\nabla}\cdot\mathbf{V}\right) - \left(u^2-\mu B\right)\left(\widehat{\mathbf{n}}\cdot\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\mathbf{V}\right) \\
  = & -u^2\left(\widehat{\mathbf{n}}\cdot\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\mathbf{V}\right) \\
  & -\mu B\left[\boldsymbol{\nabla}\cdot\mathbf{V} - \left(\widehat{\mathbf{n}}\cdot\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\mathbf{V}\right)\right]
  \end{aligned}$$

- a sort of viscosity stress: difference in the **parallel** and **perpendicular** energies, $(u^2-\mu B)$ combined with the parallel drift of the parallel divergence of the velocity $(\widehat{\mathbf{n}}\cdot\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\mathbf{V})$ which is also in the velocity stress tensor.

- a direct action of the force, by its nonzero rotational, $\boldsymbol{\nabla}\times\mathbf{F}\neq\mathbf{0}$.

The solution of the drift-kinetic equation can be written

$$\begin{aligned}
f &= f_{MS} \qquad \text{(Maxwellian)} \\
&\quad -\left(\frac{2u}{v_{th}^2}\right)\frac{2}{5}L_1^{(3/2)}\frac{q_\|}{p}f_{MS} \quad \text{(shifted Maxwellian)} \\
&\quad +g \qquad \text{(neoclassical)}
\end{aligned}$$

Note. The part which differs from the *shifted maxwellian* is in the rotating system (with the velocity $\mathbf{V}$). This part must be the same as that in other solutions of the drift-kinetic equation. However here is proportional with the parallel *heat* flux $q_\|$.

But, what is the neoclassical correction ?

Here

$$L_1^{(3/2)} = \frac{5}{2} - \frac{2w^2}{v_{th}^2}$$

is the Laguerre polynomial, and

$$\begin{aligned} \frac{2}{5} L_1^{(3/2)} &= 1 - 4w^2/\left(5v_{th}^2\right) \\ &= 1 - \frac{2}{5}\frac{v^2}{v_{th}^2} \end{aligned}$$

For particle velocities which are much higher than the thermal velocity, the value of the $2/5 \times L$ can be negative, which gives a negative correction to the shifted maxwellian in the first order.

$$p = nT$$

is the pressure.

The drfit-kinetic equation will become an equation for g,

$$\begin{aligned} & \left(u\widehat{\mathbf{n}} + \mathbf{v}_D + \mathbf{V}\right) \cdot \boldsymbol{\nabla} g \quad \text{(parallel, drift and flow - advection terms)} \\ & +\dot{w}\frac{\partial g}{\partial w} \quad \text{(energy gain)} \\ & -C\left(g\right) \quad \text{(collisions)} \\ = \; & 2\frac{v^2}{v_{th}^2}\left(\frac{1}{2} - \frac{3}{2}\frac{u^2}{v^2}\right)\left(\widehat{\mathbf{n}} \cdot \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\mathbf{V} - \frac{1}{3}\left(\boldsymbol{\nabla} \cdot \mathbf{V}\right) - \frac{2}{5}L_1^{(3/2)}\,\frac{\mathbf{q}}{p}\cdot\boldsymbol{\nabla}\ln B\right) f_{MS} \end{aligned}$$

where

$$\mathbf{q} = q_{\|}\widehat{\mathbf{n}} + \mathbf{q}_{\perp}$$

and

$$\mathbf{q}_{\perp} = \frac{5}{2}\frac{p}{m\Omega}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} T$$

(this is the *diamagnetic flow of the heat*).

The particle rotation velocity $\mathbf{V}$ is obtained from:

- particle parallel velocity, $V_{\|}\widehat{\mathbf{n}}$,
- electric field-driven ($\mathbf{E} \times \mathbf{B}$) and the
- *diamagnetic flux divided to the particle density.*

$$\mathbf{V} = V_{\|}\widehat{\mathbf{n}} + \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{1}{n}\frac{1}{m\Omega}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p$$

Here it is more than the **mass flow**. We must compose the total velocity which multiplies ∇f in the drift-kinetic equation.

$$\begin{aligned} u\widehat{\mathbf{n}}+\mathbf{v}_{dia}+\mathbf{V} &= \\ &= u\widehat{\mathbf{n}}-\frac{1}{n}\frac{1}{m\Omega}\widehat{\mathbf{n}}\times\nabla p+\mathbf{V} \\ &= \left(u+V_{\|}\right)\widehat{\mathbf{n}}+\frac{\mathbf{E}\times\mathbf{B}}{B^2} \end{aligned}$$

In this formula we have made use of an expression for the **particle drift velocity** which is the first term in (**??**), coming from the the "external force" $\mathbf{F}$ which generates a drift by acting on the *Larmor gyration*. Any force which is perpendicular to the magnetic field generates a drift of the form $\mathbf{F}\times\widehat{\mathbf{n}}/\Omega$ (**Galeev Sagdeev**). Now, replacing here $\mathbf{F}$ by the gradient of the pressure divided by the density, we get a drift velocity of the particle which looks identical but with opposite sign to the diamagnetic flow velocity coming from the total flow $\mathbf{V}$.

Acest amestec de cinetic si fluid in care parti din viteza (care este solutia ecuatiei de miscare in campurile $\mathbf{E}$ si $\mathbf{B}$ date) sunt inlocuite prin expresii care provin de la fluid (consideratii de bilant de impuls) sunt dificil de urmarit.

It results that the **diamagnetic contributions** (one from $\mathbf{v}_{dia}$ one from the general flow $\mathbf{V}$) <u>*cancel each other*</u> and there remains only the $\mathbf{E}\times\mathbf{B}$ flow:

$$\begin{aligned} &\left[\left(u+V_{\|}\right)\widehat{\mathbf{n}}+\frac{\mathbf{E}\times\mathbf{B}}{B^2}\right]\cdot\nabla g-C(g) \\ =\ & 2\frac{v^2}{v_{th}^2}\left(\frac{1}{2}-\frac{3}{2}\frac{u^2}{v^2}\right) \\ &\times\left[\frac{1}{B}\left(\mathbf{V}\cdot\nabla\right)B-\frac{2}{3}\frac{1}{n}\left(\mathbf{V}\cdot\nabla\right)n-\frac{2}{5}L_1^{(3/2)}\frac{\mathbf{q}}{p}\cdot\nabla\ln B\right]f_{MS} \end{aligned}$$

NOTE on the cancelling between the diamagnetic contributions.

It is interesting to remark that in the *mass flow*, it was originally accepted to be introduced only the effective displacement of plasma, not the diamagnetic flow.

In the work of Hazeltine on the drift-kinetic equation in the presence of large plasma rotation the velocity which multiplies the gradient of the distribution function in the Boltzmann equation is only composed of

$$v_{\|}\widehat{\mathbf{n}}+\mathbf{V}$$

and the **particle drift velocity** $\mathbf{v}_D$ is absent. We conclude:

When the *particle* drift *velocity* $\mathbf{v}_D$ is not included in the total velocity which convects ∇f , then in the *mass velocity* $\mathbf{V}$ we do not include the diamagnetic velocity.

When we include explicitely in the expression of the plasma rotation velocity $\mathbf{V}$ the diamagnetic flow then it is necessary to form the total particle velocity which convects ∇f by including the particle drift $\mathbf{v}_D$.

This shows that between the particle drift velocity (gradB, curvature) and the diamagnetic flow (gradP) it is a connection. How this can appear? The explanation is given by the form of the equation (1). We see that in this approach the "particle drift velocity" $\mathbf{v}_D$ contains much more than the *gradB* and curvature motions. It also contains the effect of a force which is exerted on the plasma, and this force if $\mathbf{F}$, obtained from the momentum equation divided by the density of particles. *This is not exactly what we used to do starting from the equation of gyromotion of the particle and expanding in the small Larmor radius and averaging over the gyration.* This was the typical approach and the **gradient of the pressure could not appear** in the drift velocity formula.

In the early Russian literature the *drift velocity* is called *diamagnetic* velocity.

This is because the diamagnetic velocity is proportional with ∇p and in the drift velocity there is the term ∇B which multiplied by B becomes ∇B^2 and the density of magnetic energy occurs. The force balance (*i.e.* momentum equation leads separately to

$$\begin{aligned} 0 &= -\nabla p + \mathbf{j} \times \mathbf{B} \\ &\downarrow \\ \nabla \left(p + \frac{B^2}{2\mu_0} \right) &= 0 \end{aligned}$$

and in this way we have a connection between ∇p and ∇B.

Then the diamagnetic flow $\sim \nabla p$ is connected with the drift velocity $\sim \mu \nabla B$.

With the remark that the diamagnetic velocity is *fluid*, while the drift velocity is *particle-like*.

See **Galeev Sagdeev book**.

In the article by **Galeev Sagdeev 1968** the names are

- diamagnetic, for $\mu \nabla B / m$ part of the drift v_D velocity
- centrifugal, for $v_\parallel^2 (\widehat{\mathbf{n}} \cdot \nabla) \widehat{\mathbf{n}}$, *i.e.* for curvature drift

- electric $V_E = -\frac{\partial \phi}{B\partial r}$.
- projected of the parallel velocity, $\Theta v_{\|}$ to take into account the motion along the lines projected on the poloidal plane.

END OF THE NOTE

Following the paper **shaing parallel viscosity banana.**
The kinetic equation

$$\left(u\widehat{\mathbf{n}} + \mathbf{v}_D + \mathbf{V}\right) \cdot \boldsymbol{\nabla} f + \dot{w}\frac{\partial f}{\partial w} = C\left(f\right)$$

The force is identified in the *fluid* description

$$\mathbf{F} = \frac{1}{NM}\boldsymbol{\nabla} p + \frac{1}{NM}\boldsymbol{\nabla}\cdot\boldsymbol{\Pi} - \frac{1}{NM}\mathbf{R}$$

where $\mathbf{R}$ is the friction force.

The equation for g is

$$\begin{aligned}
&\left[\left(u + V_{\|}\right)\widehat{\mathbf{n}} + \mathbf{V}_E + \mathbf{v}_{D,r}\right]\cdot\boldsymbol{\nabla} g \\
&+\dot{w}\frac{\partial g}{\partial w} \\
&-C\left(g\right) \\
= \; & 2\frac{v^2}{v_{th}^2}\left(\frac{1}{2} - \frac{3}{2}\frac{u^2}{v^2}\right) f_M\,\mathcal{D} - \frac{8}{15} f_M L_2^{(1/2)}\left[-\frac{5}{2}\frac{1}{NeB}\left(\widehat{\mathbf{n}}\times\boldsymbol{\nabla} T\right)\cdot\boldsymbol{\nabla} N + \frac{1}{p}\boldsymbol{\nabla}\cdot\mathbf{q}\right]
\end{aligned}$$

$$L_2^{(1/2)} = \frac{1}{2}\left(\frac{v}{v_{th}}\right)^4 - \frac{5}{2}\left(\frac{v}{v_{th}}\right)^2 + \frac{15}{8}$$

It is defined the factor

$$\begin{aligned}
\mathcal{D} \; = \; & \left(\widehat{\mathbf{n}}\cdot\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\mathbf{V} - \frac{1}{3}\boldsymbol{\nabla}\cdot\mathbf{V}\right) \\
& + \left(\frac{v^2}{v_{th}^2} - \frac{5}{2}\right)\frac{2}{5}\frac{1}{p}\left(\widehat{\mathbf{n}}\cdot\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\mathbf{q} - \frac{1}{3}\boldsymbol{\nabla}\cdot\mathbf{q}\right) \\
& + \frac{1}{3}\left(\frac{v^2}{v_{th}^2} - \frac{7}{2}\right)\frac{1}{NeB}\left(\widehat{\mathbf{n}}\times\boldsymbol{\nabla} T\right)\cdot\boldsymbol{\nabla} N
\end{aligned}$$

Here the variations of T and of N in the surface ($\sim \theta$) are assumed to be of the order

$$\begin{aligned}
\frac{\partial \ln T}{\partial \theta} &\sim \sqrt{\varepsilon} \\
\frac{\partial \ln N}{\partial \theta} &\sim \sqrt{\varepsilon}
\end{aligned}$$

NOTE

The equation for the distribution function used to calculate the *average parallel viscosity* $\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\Pi} \rangle$ in the work **Hsu Shaing Gromley poloidal damping** is

$$\begin{aligned}
& \frac{\partial \overline{f}}{\partial t} + \left(v_{\parallel} \widehat{\mathbf{n}} + \mathbf{V} \right) \cdot \nabla \overline{f} \\
& - \left[\mathbf{V} \cdot \nabla \left(\mu B \right) + \mu B \left(\nabla \cdot \mathbf{V} - \widehat{\mathbf{n}} \cdot \widehat{\mathbf{n}} \cdot \nabla \ \mathbf{V} \right) \right] \frac{\partial \overline{f}}{\partial \mu} \\
& + \left[v_{\parallel} \frac{1}{nm} \widehat{\mathbf{n}} \cdot \nabla \cdot \mathbf{P} - \mu B \left(\nabla \cdot \mathbf{V} \right) - \left(v_{\parallel}^2 - \mu B \right) \widehat{\mathbf{n}} \cdot \widehat{\mathbf{n}} \cdot \nabla \ \mathbf{V} \right] \frac{\partial \overline{f}}{\partial w} \\
= \ & C \left(\overline{f} \right)
\end{aligned}$$

Note that to re-derive this equation we must return to the *change of variables* that we have done in *neoclassics.*

The ion stress tensor

$$\begin{aligned}
\mathbf{P} &= nT \ \mathbf{I} + \mathbf{\Pi} \\
&= nT \ \mathbf{I} + \frac{3}{2} \pi_{\parallel} \left(\widehat{\mathbf{n}} \widehat{\mathbf{n}} - \frac{1}{3} \mathbf{I} \right)
\end{aligned}$$

and $\mathbf{\Pi}$ is the *viscosity.*

The velocity

$$\begin{aligned}
\mathbf{V} &= \frac{K(\psi)}{n} \mathbf{B} \quad \text{poloidal flow} \\
& \quad + \omega_{\varphi} R^2 \nabla \varphi \quad \text{electric flow not parallel to } \mathbf{B} \text{ IN surface}
\end{aligned}$$

$$\begin{aligned}
\omega_{\varphi}(\psi) &= -\frac{\partial \phi}{\partial \psi} \quad (\text{comes from } E_r \times B_{\theta}) \\
\text{the potential } \phi \text{ has radial } &\sim \psi \text{ variation. } E_r \text{ is radial}
\end{aligned}$$

The second term in $\mathbf{V}$ is projected on toroidal φ direction.

END

The heat balance

$$\nabla \cdot \mathbf{q} = p \left[\left(\mathbf{V} \cdot \nabla \right) \left(\ln N - \frac{3}{2} \ln T \right) \right]$$

note the combination

$$\frac{N}{T^{3/2}}$$

occurs in any collision frequency. Here we have advection of this quantity by the velocity $\mathbf{V}$.

and continuity

$$\boldsymbol{\nabla}\cdot(N\mathbf{V})=0$$

lead to

$$\begin{aligned}\mathcal{D} &= (\mathbf{V}\cdot\boldsymbol{\nabla})\left(\ln B-\frac{2}{3}\ln N\right)\\ &+\left(\frac{v^2}{v_{th}^2}-\frac{5}{2}\right)\frac{2}{5}\left[\frac{1}{p}\mathbf{q}\cdot\boldsymbol{\nabla}\ln B+\mathbf{V}\cdot\boldsymbol{\nabla}\left(\frac{N^{2/3}}{T}\right)\right]\\ &+\frac{2}{3}\left(\frac{v^2}{v_{th}^2}-\frac{7}{2}\right)\frac{1}{NeB}\left(\widehat{\mathbf{n}}\times\boldsymbol{\nabla}T\right)\cdot\boldsymbol{\nabla}N\end{aligned}$$

4.1 The poloidal angular velocity of a particle

Define

$$I=R^2\mathbf{B}\cdot\boldsymbol{\nabla}\varphi$$

(this is RB_T) and introduce the variable *poloidal angular frequency*

$$\begin{aligned}\omega &= \left(u\widehat{\mathbf{n}}+\mathbf{V}_E\right)\cdot\boldsymbol{\nabla}\theta\\ &= \left(u+I\frac{1}{B}\frac{\partial\phi}{\partial\psi}\right)\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta\end{aligned}$$

(**note** $I\frac{1}{B}\frac{\partial\phi}{\partial\psi}=RB_T\frac{1}{B}\frac{1}{|\boldsymbol{\nabla}\psi|}\frac{\partial\phi}{\partial r}=RB_T\frac{1}{B}\frac{1}{RB_\theta}\frac{\partial\phi}{\partial r}=\frac{B_T}{B_\theta}\left(\frac{1}{B}\frac{\partial\phi}{\partial r}\right)=\frac{B_T}{B_\theta}\left(V_\perp\right)$ =projection of $V_\perp$ on the *parallel* $\widehat{\mathbf{n}}$ direction, **end**). Next one takes a projection on the poloidal direction, *i.e.* multiplication by

$$\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta=\nabla_\|\theta=\frac{1}{r}\frac{rd\theta}{dl_\|}=\frac{1}{r}\frac{dl_\theta}{dl_\|}=\frac{1}{qR}$$

The ratio

$$\omega=\left(u+I\frac{1}{B}\frac{\partial\phi}{\partial\psi}\right)\frac{1}{qR}\sim\text{frequency}$$

Two invariants

$$E = \frac{u^2}{2}+\mu B(\theta)+\frac{e}{M}\phi$$

$$(u \text{ is a parallel velocity})$$

$$P_\varphi = -\frac{Iu}{\Omega} + \psi$$
$$\text{longitudinal invariant}$$

since

$$\begin{aligned} \frac{Iu}{\Omega} &= R^2 \mathbf{v} \cdot \boldsymbol{\nabla} \varphi \\ &= R \, v_{tor} \end{aligned}$$

and

$$\psi = A_\varphi$$

It is adopted a reference surface

$$\psi_0$$

and the electric potential is considered constant on surfaces.

The potential is expanded to second order

$$\phi(\psi) = \phi(\psi_0) + (\psi - \psi_0) \frac{d\phi}{d\psi} + \frac{1}{2} (\psi - \psi_0)^2 \frac{d^2\phi}{d\psi^2}$$

NOTE

This procedure is similar to the usual expansion of the *longitudinal invariant* to second order in the *radial* deviation of a particle from the reference radius

$$(r - r_0)$$

A procedure which leads to a second-degree algebraic equation for this deviation and introduces κ in **Galeev Sagdeev** etc.

END

NOTE

that this equation is an algebraic equation of second degree with the *given* left hand side member.

This function $\phi(\psi)$ will enter the expression of the parallel velocity ω through the "radial distance" $\Delta\psi = \psi - \psi_0$, solution of this second-degree equation.

END

It is adopted a reference *parallel velocity*

$$u_0$$

with the property that

$$u_0 = 2\left[E - \mu B\left(\theta_0, \psi_0\right) - \frac{e}{M}\phi\left(\psi_0\right)\right]^{1/2}$$

and

$$\theta_0 = \text{ angle of origin of the orbit}$$

like in **Galeev Sagdeev**.

The orbit is

$$\Delta\psi = \psi\left(\theta, P_\varphi, E, \mu\right) - \psi_0$$

We note that

$$B = \frac{B_0}{h\left(\psi, \theta\right)}$$

$$\begin{aligned}
V_{E0} &= I\frac{1}{B_0}\frac{d\phi_0}{d\psi} \\
&= RB_T\frac{1}{RB_\theta}\left(\frac{1}{B_0}\frac{d\phi_0}{dr}\right) = \frac{B_T}{B_\theta}\left(V_\perp\right) \\
&= \text{projection of } V_\perp \text{on the parallel } \widehat{\mathbf{n}}
\end{aligned}$$

Note also that

$$V_E = I\frac{1}{B_0}\frac{d\phi_0}{d\psi} \sim RB_\varphi\frac{1}{B_0}\frac{d\phi}{dr}\frac{1}{RB_\theta} \sim E_r \times B_\theta \sim \widehat{\mathbf{e}}_\varphi$$

For the poloidal angular velocity

$$\begin{aligned}
\omega &= \left(u\widehat{\mathbf{n}} + \mathbf{V}_E\right)\cdot\boldsymbol{\nabla}\theta \\
&= \widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta\left[\frac{h_0}{h}u_0 + hV_{E0} + \left(\frac{1 + h^2(S-1)}{h}\right)\frac{\Omega_0}{I}\Delta\psi\right]
\end{aligned}$$

Note that S - squeezing factor, occurs due to the second order derivative of the potential close to the reference ψ_0, $\frac{\partial^2\phi}{\partial\psi^2}$. **End.**

$$\begin{aligned}
\omega = \; & \widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta\left\{\left(hV_{E0} + \frac{h_0}{h}u_0\right)^2\right. \\
& \left. - \left[1 + h^2\left(S-1\right)\right]\left[2\left(E - \frac{e}{M}\phi\right)\left(\frac{h_0^2}{h^2} - 1\right) - \frac{2\mu B_0}{h}\left(\frac{h_0}{h} - 1\right)\right]\right\}^{1/2}
\end{aligned}$$

In the limit

$$\begin{aligned}
S &\to 1 \\
V_{E0} &\to 0
\end{aligned}$$

one has for the poloidal angular frequency

$$\omega \approx u \, \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \theta$$

This is

$$\begin{aligned} \omega &\approx u \frac{1}{r} \widehat{\mathbf{n}} \cdot \widehat{\mathbf{e}}_{\theta} = u \frac{1}{r} \frac{B_{\theta}}{B_{\varphi}} = u \frac{1}{qR} \\ &= u_{\|} \frac{1}{qR} \end{aligned}$$

define

$$\widehat{\omega} \equiv \left| \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \theta \right| \left(u_0 + V_{E0} \right)$$

or

$$\widehat{\omega} = \frac{1}{qR} \left(u_0 + V_{E0} \right)$$

then

$$\omega = \widehat{\omega} \left[1 - k \sin^2 \frac{\theta}{2} \right]^{1/2}$$

and

$$k = 4 S \varepsilon \frac{\left| u_0^2 + \mu B_0 \right|}{\left(\frac{\widehat{\omega}}{\left| \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \theta \right|} \right)^2}$$

The trapped particles

$$1 < k < \infty$$

the circulating particles

$$0 < k < 1$$

Therefore

$$k = 1$$

is the boundary between trapped and circulating.

Note

$$\widehat{\omega} = \frac{1}{qR} \left(u_0 + V_{E0} \right)$$

NOTE in **turbulence driven bootstrap McdeWitt** the "pitch angle variable"

$$\kappa = \sqrt{\frac{1 - \lambda \left(1 - \varepsilon \right)}{2 \varepsilon \lambda}}$$

where

$$\begin{aligned}
\lambda &= \frac{2\mu B_0}{v^2} = \frac{v_\perp^2}{v^2}\frac{B_0}{B} \\
&= \frac{v_\perp^2}{v^2} h
\end{aligned}$$

the change of variables

$$\left(\mathbf{x}, v^2, \mu\right) \to (\mathbf{x}, v, \kappa)$$

takes place with

$$\left.\frac{\partial}{\partial \mathbf{x}}\right|_{\mu, v^2} = \left.\frac{\partial}{\partial \mathbf{x}}\right|_{v,\kappa} + \frac{\frac{1}{2} - \kappa^2}{2\kappa} \frac{\partial \ln \varepsilon}{\partial \mathbf{x}} \left.\frac{\partial}{\partial \kappa}\right|_{\mathbf{x}, v}$$

$$\left.\frac{\partial}{\partial v^2}\right|_{\mathbf{x},\mu} = \frac{1}{2v}\left.\frac{\partial}{\partial v}\right|_{\mathbf{x},\kappa} + \frac{1 - \varepsilon + 2\varepsilon\kappa^2}{4\varepsilon\kappa v^2}\left.\frac{\partial}{\partial \kappa}\right|_{\mathbf{x}, v}$$

The Jacobian

$$\mathcal{I} = 2\pi\varepsilon\kappa \left(\frac{v}{v_{th,e}}\right)^3 \lambda^2 \left(\frac{B}{B_0}\right) \frac{v_{th,e}}{v_\parallel}$$

The domain of the variable κ,

$$1 < \kappa \quad \text{passing particles}$$

$$0 < \kappa < 1 \quad \text{trapped particles}$$

$$\kappa = 1 \quad \text{boundary trapped/circulating}$$

END

NOTE
The formulas are similar to **Shaing Hazeltine (SH) orbit squeezing 1992**. But in SH

$$\begin{aligned}
\widehat{\omega} &= \left|\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta\right| \sqrt{\left(V_{E0} + v_{\parallel 0}\right)^2 + 4S\varepsilon\left(v_{\parallel 0}^2 + \mu B_0\right)} \\
\text{compared with } \widehat{\omega} &\equiv \left|\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta\right| \left(u_0 + V_{E0}\right) \quad \text{(HSD)}
\end{aligned}$$

and

$$
\begin{aligned}
k &= 4\left|S\right|\varepsilon\frac{\left(v_{\parallel 0}^2+\mu B_0\right)}{\left(V_{E0}+v_{\parallel 0}\right)^2+4S\varepsilon\left(v_{\parallel 0}^2+\mu B_0\right)} \\
\text{compared with } \kappa &= \sqrt{\frac{1-\lambda\left(1-\varepsilon\right)}{2\varepsilon\lambda}} \quad \text{(SHD)}
\end{aligned}
$$

The boundary trapped/passing is

$$k=1$$

$$\left(V_{E0}+v_{\parallel 0}\right)^2 = 8\left|S\right|\varepsilon\left(v_{\parallel 0}^2+\mu B_0\right)$$

The fraction of trapped particles is

$$f_t \sim \sqrt{\varepsilon\left|S\right|}$$

END

To solve the equation for the function g one replaces the term with f_M by a new function, extracting explicitely the expected trigonometric variation with θ. The new function, just a notation

$$\mathcal{N}$$

$$
\begin{aligned}
& 2\frac{v^2}{v_{th}^2}\left(\frac{1}{2}-\frac{3}{2}\frac{u^2}{v^2}\right)f_M\,\mathcal{D}-\frac{8}{15}f_M L_2^{(1/2)}\left[-\frac{5}{2}\frac{1}{NeB}\left(\widehat{\mathbf{n}}\times\boldsymbol{\nabla}T\right)\cdot\boldsymbol{\nabla}N+\frac{1}{p}\boldsymbol{\nabla}\cdot\mathbf{q}\right] \\
= {} & \mathcal{N}f_M\ \sin\theta
\end{aligned}
$$

where it was assumed that

$$
\begin{aligned}
T &\sim \cos\theta \\
N &\sim \cos\theta
\end{aligned}
$$

Later it will be explained that the trigonometric functions showing variation *in the surface* for the density and temperature are imposed by the *heat* flow.

See *drift kinetic derivation, expressions...*

To this one should add the Pfirsch-Schluter modulation of the toroidal flow.

Approximations

$$\frac{v^2}{2} \text{ does NOT depend on } \theta$$
it is an invariant of the particle motion

$$\begin{aligned} u^2 \quad &\sim \quad V_{E0}^2 \\ & \qquad \text{no } \theta \text{ dependence} \end{aligned}$$

Change the variables
$$(E,\mu) \rightarrow (E,\omega)$$

and take
$$\frac{dE}{dt}\frac{\partial g}{\partial E} = 0$$

To the lowest order
$$\omega \frac{\partial g_0}{\partial \theta} = \mathcal{N} f_M \sin\theta$$
next order is dominated by *collisions*

$$\omega \frac{\partial g_1}{\partial \theta} = C\left(g_0\right)$$

Solution of the zeroth order

$$g_0 = -\mathcal{N} f_M \frac{4}{\widehat{\omega}^2 k}\, \omega + h_i$$

The integration "constant" is determined by *bounce-averaging* the equation in the next order
$$\left\langle \frac{1}{\omega} C\left(g_0\right) \right\rangle_{bounce} = 0$$

The authors adopt for the collision operator a model of *test particle collision.*

It is assumed that the collisional flow across the boundary defined by the exact compensation of the flow velocities along the *parallel* direction.

$$\frac{u}{v} + \frac{1}{v} I \frac{1}{B} \frac{d\phi}{d\psi} = 0$$

has the main contribution.

Then the form of the adopted collision operator is

$$C\left(g_0\right)=\frac{1}{2}\nu_D\ F\ \frac{v^2}{q^2R^2}\frac{\partial^2 g_0}{\partial\omega^2}$$

where the factor

$$F=\left(1-\frac{V_{E0}^2}{v^2}\right)+\frac{\nu_\parallel}{\nu_D}\frac{V_{E0}^2}{v^2}$$

and

$$\begin{aligned}\nu_D &\equiv \text{deflection frequency}\\ &= \sum_b \nu_b\frac{\overline{\Phi}\left(v/v_{th,b}\right)-G\left(v/v_{th,b}\right)}{\left(\frac{v}{v_{th}}\right)^3}\end{aligned}$$

$$\begin{aligned}\nu_\parallel &\equiv \text{energy scattering frequency}\\ &= 2\sum_b \nu_b\frac{G\left(v/v_{th,b}\right)}{\left(\frac{v}{v_{th}}\right)^3}\end{aligned}$$

The Chandrasekhar function is G.

The frequancy

$$\nu_b=\frac{4\pi\left(ee_b\right)^2}{M^2}\ln\Lambda\frac{N_b}{\left(2T/M\right)^{3/2}}$$

(see **Hirshman Sigmar approx FP 1976**).

NOTE

Regarding the collision operator

In **Shaing Hazeltine orbit squeezing 1992, the collisions are** *pitch angle*

$$C\left(f\right)=\nu_D\frac{v_\parallel}{B}\frac{\partial}{\partial\mu}\mu v_\parallel\frac{\partial f}{\partial\mu}$$

where

$$\begin{aligned}\nu_D &\equiv \text{deflection frequency}\\ \mu &= \frac{v_\perp^2}{2B}\end{aligned}$$

After changing variables

$$\left(E,\mu\right)\rightarrow\left(E,\omega\right)$$

AND neglecting

$$\frac{\partial f}{\partial \omega}$$

the operator is

$$C(f) = \nu_D \frac{v^2}{2R^2 q^2} \frac{\partial^2 f}{\partial \omega^2}$$

END

Return to the definitions

$$\widehat{\omega} = \frac{1}{qR} (u_0 + V_{E0})$$

the frequency of the rotation on parallel direction, and

$$\omega = \widehat{\omega} \left[1 - k \sin^2 \frac{\theta}{2} \right]^{1/2}$$

where

$$k = 2S\varepsilon \frac{\left| u_0^2 + \mu B_0 - \frac{e}{M} \overline{\Phi}_1 \right|}{\widehat{\omega} / (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \theta)}$$

The potential

$$\overline{\Phi}_1$$

is the coefficient of the *cosinus* term in the expansion in harmonics of Φ_1,

$$\Phi_1 = \overline{\Phi}_1 \ \varepsilon \ \cos \theta$$

We must solve the drift kinetic equation to find the function

$$g$$

For this one re-wries

$$\begin{aligned} & \left[\left(u + V_{\|} \right) \widehat{\mathbf{n}} + \mathbf{V}_E + \mathbf{v}_{D,r} \right] \cdot \boldsymbol{\nabla} g \\ & + \dot{w} \ \frac{\partial g}{\partial w} \\ & - C(g) \\ = \ & \mathcal{W} \end{aligned}$$

where $\mathcal{W}$ is composed of factors of the equilibrium Maxwellian function, f_M.

As mentioned above, it is now taken an interesting decision.

This term will receive a form that exhibits the essential harmonic content of it.

$$
\begin{aligned}
& 2\frac{v^2}{v_{th}^2}\left(\frac{1}{2}-\frac{3}{2}\frac{u^2}{v^2}\right) f_M \;\; \mathcal{D} \\
& -\frac{8}{15} f_M \; L_2^{(1/2)} \; \left[\frac{\boldsymbol{\nabla}\cdot\mathbf{q}}{p} - \frac{5}{2}\frac{1}{NeB}\left(\widehat{\mathbf{n}}\times\boldsymbol{\nabla}T\right)\cdot\boldsymbol{\nabla}N\right] \\
= \;\; & \mathcal{N} \; f_M \;\; \sin\theta
\end{aligned}
$$

Approximations

- the poloidal variations of T and of N are
$$T,\; N \sim \sqrt{\varepsilon}$$
- the kinetic energy
$$\frac{1}{2}v^2$$
is considered constant in θ, although there is energetic effect on θ due to the electrostatic potential $\phi \sim \theta$ associated to the pressure *parallel* balance $\sim \theta$.
- the velocity u is assumed constant on θ, and is considered very close to
$$u^2 \approx V_{E0}^2$$

Use
$$\frac{\partial \omega}{\partial k} \approx -\frac{\widehat{\omega}^2}{2k\omega}$$
and the constraint of zero-bounce-average (after replacement of g_0) leads to the equation for h_i,
$$k\frac{\partial}{\partial k}\left(\frac{k}{\widehat{\omega}^2}\left\langle\omega\right\rangle_\theta \frac{\partial h_i}{\partial k}\right) = 0$$
Here ω is averaged over the orbits

$$
\begin{aligned}
\text{circulating } 0 & < k < 1 \\
\left\langle\omega\right\rangle_\theta & = \frac{1}{2\pi}\int_0^{2\pi} d\theta \; \omega\left(\theta\right)
\end{aligned}
$$

$$
\begin{aligned}
\text{trapped } 1 & < k < \infty \\
\left\langle\omega\right\rangle_\theta & = \frac{1}{2\pi}\int_{-\theta_t}^{\theta_t} d\theta \;\; \left|\omega\right|
\end{aligned}
$$

The solution

$$\frac{\partial h_i}{\partial k} = C \frac{1}{k} \frac{\widehat{\omega}^2}{\langle \omega \rangle_\theta}$$

Returning to g_0,

$$\frac{\partial g_0}{\partial \omega} = -\mathcal{N} f_M \frac{4}{\widehat{\omega}^2 k} \; - 2C \frac{\omega}{\langle \omega \rangle_\theta}$$

This function is localized in the *trapped part of the velocity space* and

$$\lim_{k \to 0} \langle \omega \rangle_\theta = \omega$$

The limit $\lim\limits_{k \to 0} \langle \omega \rangle_\theta = \omega$ means the state of *strongly circulating* , $v_\perp^2 \to 0$. One choses the constant C as

$$C = -\mathcal{N} f_M \frac{2}{\widehat{\omega}^2 k}$$

Then, for

$$0 < k < 1 \quad \text{circulating}$$

the solution is

$$\begin{aligned} \frac{\partial g_0}{\partial \omega} &= -\mathcal{N} f_M \frac{4}{\widehat{\omega}^2 k} \left(1 - \frac{\omega}{\langle \omega \rangle_\theta} \right) \\ & \quad \text{circulating} \end{aligned}$$

The constant C must be chosen according to the constraints resulting from the equation for $\partial h_i / \partial k$.

For trapped particles, $\partial h_i / \partial k$ is even in ω

$$\frac{\partial h_i}{\partial k} (-\omega) = \frac{\partial h_i}{\partial k} (+\omega)$$

For

$$1 < k < \infty \;\; \text{trapped}$$

the choice is

$$C = 0$$

The *even part* of $\partial h_i / \partial k$ for circulating particles is zero. Then the continuity of $\partial h_i / \partial k$ at the boundary $k = 1$ requires $C = 0$.

Then the solution for

$$1 < k < \infty \;\; \text{trapped}$$

is

$$\frac{\partial g_0}{\partial \omega} = -\mathcal{N} f_M \frac{4}{\widehat{\omega}^2 k}$$
trapped

There is therefore a discontinuity at the boundary TRAPPED/CIRCULATING, $k = 1$.

4.2 Resonance plasma viscosity

One has to calculate (**Shaing Hazeltine Sanuki**)

$$\left\langle \frac{\mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Pi}}{N} \right\rangle_{\theta,\psi}$$

average over poloidal angle θ

and average over a radial extension $\int \frac{d\psi}{\Delta\psi} (...)$

which is

$$\left\langle \frac{\mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Pi}}{N} \right\rangle = \left\langle \frac{1}{N} \int d^3v \ M v^2 \left(\frac{1}{2} - \frac{3}{2} \frac{u^2}{v^2} \right) \ g \ B \left(\nabla_\| \ln B - \frac{2}{3} \nabla_\| \ln N \right) \right\rangle$$

It is assumed that the variation in surface of the density is

$$N \sim N_0 + N_1 \cos\theta$$

(See *drift kinetic derivation, Expressions...*)

Introduce a new variable

$$\mathcal{M} \equiv \frac{1}{\sin\theta} B \left(\nabla_\| \ln B - \frac{2}{3} \nabla_\| \ln N \right)$$

Then

$$\left\langle \frac{\mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Pi}}{N} \right\rangle = \left\langle \frac{1}{N} \int d^3v \ M v^2 \left(\frac{1}{2} - \frac{3}{2} \frac{u^2}{v^2} \right) \ g \ \mathcal{M} \sin\theta \right\rangle$$

the average $\langle\rangle$ is on magnetic surface (θ) and also on a radial extension

$$\int \frac{d\psi}{\delta\psi}$$

Assumption

the transport is dominated by particles close to the resonance

$$u^2 \approx V_{E0}^2$$

The following equations should be considered for the next step

the kinetic equation for g

$$\begin{aligned}
& \left[\left(u+V_{\|}\right)\widehat{\mathbf{n}}+\mathbf{V}_E+\mathbf{v}_{D,r}\right]\cdot\boldsymbol{\nabla} g \\
& +\dot{w}\frac{\partial g}{\partial w} \\
& -C\left(g\right) \\
= \; & 2\frac{v^2}{v_{th}^2}\left(\frac{1}{2}-\frac{3}{2}\frac{u^2}{v^2}\right) f_M \; \mathcal{D}-\frac{8}{15}f_M L_2^{(1/2)}\left[-\frac{5}{2}\frac{1}{NeB}\left(\widehat{\mathbf{n}}\times\boldsymbol{\nabla}T\right)\cdot\boldsymbol{\nabla}N+\frac{1}{p}\boldsymbol{\nabla}\cdot\mathbf{q}\right]
\end{aligned}$$

in which we introduce

the definition

$$\omega=\left(u\widehat{\mathbf{n}}+\mathbf{V}_E\right)\cdot\boldsymbol{\nabla}\theta$$

the notation (it is question of $\mathcal{N}$)

$$\begin{aligned}
& 2\frac{v^2}{v_{th}^2}\left(\frac{1}{2}-\frac{3}{2}\frac{u^2}{v^2}\right) f_M \; \mathcal{D}-\frac{8}{15}f_M L_2^{(1/2)}\left[-\frac{5}{2}\frac{1}{NeB}\left(\widehat{\mathbf{n}}\times\boldsymbol{\nabla}T\right)\cdot\boldsymbol{\nabla}N+\frac{1}{p}\boldsymbol{\nabla}\cdot\mathbf{q}\right] \\
= \; & \mathcal{N} f_M \; \sin\theta
\end{aligned}$$

the two orders: zero

$$\omega\frac{\partial g_0}{\partial\theta}=\mathcal{N} f_M\sin\theta$$

and next

$$\omega\frac{\partial g_1}{\partial\theta}=C\left(g_0\right)$$

that together are

$$\omega\frac{\partial g}{\partial\theta}-C\left(g\right)=\mathcal{N} f_M\sin\theta$$

From the last equation we extract the factor

$$\sin\theta=\left(\omega\frac{\partial g}{\partial\theta}-C\left(g\right)\right)\frac{1}{\mathcal{N} f_M}$$

and replace in the viscosity

$$\begin{aligned}
&\left\langle \frac{\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\mathbf{\Pi}}{N}\right\rangle \\
&= \left\langle \frac{1}{N}\int d^3v\ Mv^2\left(\frac{1}{2}-\frac{3}{2}\frac{u^2}{v^2}\right)\ g\ \mathcal{M}\sin\theta\right\rangle \\
&= \left\langle \frac{1}{N}\int d^3v\ Mv^2\left(\frac{1}{2}-\frac{3}{2}\frac{u^2}{v^2}\right)\ g\ \frac{1}{\mathcal{N}f_M}\left(\omega\frac{\partial g}{\partial\theta}-C\left(g\right)\right)\right\rangle
\end{aligned}$$

In this formula the integrations

$$\oint d\theta\ \int d\psi\ \int d^3v\left(...\right)$$

will eliminate the term

$$\omega\frac{\partial g}{\partial\theta}$$

so the result is

$$\begin{aligned}
&\left\langle \frac{\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\mathbf{\Pi}}{N}\right\rangle \\
&= -\left\langle \frac{1}{N}\int d^3v\ Mv^2\left(\frac{1}{2}-\frac{3}{2}\frac{u^2}{v^2}\right)\ g\ \mathcal{M}\frac{1}{\mathcal{N}f_M}C\left(g\right)\right\rangle
\end{aligned}$$

It is now possible to insert the expression for the collision operator (see above)

$$C\left(g_0\right)=\frac{1}{2}\nu_D\ F\ \frac{v^2}{q^2R^2}\frac{\partial^2 g_0}{\partial\omega^2}$$

after which an integration by part of $g\frac{\partial^2 g_0}{\partial\omega^2}$ produces a factor

$$\left(\frac{\partial g}{\partial\omega}\right)^2$$

and it is

$$\begin{aligned}
&\left\langle \frac{\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\mathbf{\Pi}}{N}\right\rangle \\
&= \left\langle \frac{1}{N}\int d^3v\ Mv^2\left(\frac{1}{2}-\frac{3}{2}\frac{u^2}{v^2}\right)\ \frac{\mathcal{M}}{\mathcal{N}f_M}\frac{1}{2}\nu_D\ F\ \frac{v^2}{q^2R^2}\left(\frac{\partial g}{\partial\omega}\right)^2\right\rangle
\end{aligned}$$

IMPORTANT NOTE

In **shaing hsu dominguez resonance** it is examined a simpler form of the *viscosity* and it is taken $\partial N/\partial\theta = 0$ and $\partial T/\partial\theta = 0$ and this is attributed to the situation where there is no *heat flow.*

The variation in surface, with θ, of N and T comes from the flow of the heat.

END

5 Viscosity diffusion and current Sugama Nishimura

Prepared to be applicable to stellarator too.

Usually pitch angle scattering collision operators are used (Lorentz) instead of full Landau.

The distribution function for

$$e_a \ , \ m_a$$

is

$$\begin{aligned} f_a &= f_{aM}\left[1+\frac{e_a}{T_a}\int^l \frac{dl}{B}\left(BE_\parallel - \frac{B^2}{\langle B^2\rangle}\langle BE_\parallel\rangle\right)\right] \\ &\quad +f_{a1} \end{aligned}$$

where

$$\begin{aligned} f_{aM} &= \frac{n_a}{\left(\sqrt{\pi}v_{th,a}\right)^3}\exp\left(-x_a^2\right) \\ v_{th,a} &= \sqrt{\frac{2T_a}{m_a}} \\ x_a &= \frac{v}{v_{th,a}} \end{aligned}$$

The integral is taken along the field line.

The averaging operator is

$$\langle ...\rangle = \frac{1}{\frac{dV}{d\psi}}\oint d\theta \oint d\varphi \ \sqrt{g}\,(...)$$

which is the surface average,

$$\begin{aligned} \frac{dV}{d\psi} &= \oint d\theta \oint d\varphi \ \sqrt{g} \\ &= \text{derivative of the volume to the streamfunction } \psi \end{aligned}$$

The neoclassical transport is caused by the *perturbation* function, f_{a1}. This is a gyrophase averaged function.

The linearized drift kinetic equation

$$\begin{aligned} & V_{\parallel}\left[f_{a1}\right]+\mathbf{v}_D\cdot\boldsymbol{\nabla}f_{1M}-\frac{e_a}{T_a}v_{\parallel}B\frac{\left\langle BE_{\parallel}\right\rangle}{\left\langle B^2\right\rangle}f \\ = & \; C_a^{lin}\left[f_{a1}\right] \end{aligned}$$

where

$$\begin{aligned} V_{\parallel}\left[f_{a1}\right] &= \text{collisionless orbit operator} \\ &= v_{\parallel}\nabla_{\parallel}f_{a1} \end{aligned}$$

$$\mathbf{v}_D=\frac{1}{\Omega_a}\widehat{\mathbf{n}}\times\left[\frac{1}{m_a}\mu\boldsymbol{\nabla}B+v_{\parallel}^2\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}+\frac{e_a}{m_a}\boldsymbol{\nabla}\phi\right]$$

since the definition is

$$\mu=\frac{m_a v_{\perp}^2}{2B}$$

The energy

$$\epsilon=\frac{1}{2}m_a v^2+e_a\phi$$

The variables are

$$\left(\mathbf{x},\epsilon,\mu\right)$$

the operator of collision is Landau

$$C_a^{lin}\left[f_{a1}\right]=\sum_{species\ b}\left\{C_{ab}\left[f_{a1},f_{bM}\right]+C_{ab}\left[f_{aM},f_{1b}\right]\right\}$$

Transformation of variables

$$\begin{aligned} \left(\mathbf{x},\epsilon,\mu\right) &\rightarrow \left(\mathbf{x},v,\xi\right) \\ \xi &\equiv \frac{v_{\parallel}}{v} \end{aligned}$$

The collisionless orbit operator

$$V_{\parallel}\left[f_{a1}\right]=v\xi\ \widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}-\frac{1}{2}v\left(1-\xi^2\right)\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\ln B\right)\frac{\partial}{\partial\xi}$$

which is

$$v_{\parallel}\nabla_{\parallel}+\left[\text{magnetic mirror force}\right]$$

Here there are no

- curvature drift $\frac{1}{\Omega_a}\widehat{\mathbf{n}} \times \left[v_\parallel^2 \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} \right]$,
- electric drift $E \times B$.

For the interest of future use of the explicit form of the collision operator, one has to consider the *series expansion in Legendre polynomials*

For a function $F(\mathbf{x}, v, \xi)$,

$$F(\mathbf{x}, v, \xi) = \sum_{l=0}^{\infty} F^{(l)}(\mathbf{x}, v, \xi)$$

with the components

$$F^{(l)}(\mathbf{x}, v, \xi) = P_l(\xi) \frac{2l+1}{2} \int_{-1}^{1} d\eta \; P_l(\eta) \;\; F(\mathbf{x}, v, \xi)$$

The polynomials of the set are

$$\begin{aligned} P_0(\xi) &= 1 \\ P_1(\xi) &= \xi \\ P_2(\xi) &= \frac{3}{2}\xi^2 - \frac{1}{2} \\ &\ldots \end{aligned}$$

The suggestion to project a distribution function $F(\mathbf{x}, v, \xi)$ on the set of Legendre function comes from the ξ dependence of the collision operator. This is an essential variable of the velocity space since is involved in the separation trapped/circulating. Or, the collisions acting in this boundary layer produce conversion between these two types of behaviors.

the set of Legendre polynomials is orthonormal on the interval

$$[-1, 1]$$

which is precisely what ξ can have: parallel and anti-parallel motion.

The Legendre expansion only facilitates the work with the variable ξ.

The dependence of the distribution function on the full kinetic energy $\frac{mv^2}{2}$ normalized to the thermal energy

$$x_a^2 = \frac{v^2}{v_{th,a}^2}$$

is another essential variable in the problem.

Then one uses a new expansion, in Laguerre polynomials, functions of x_a^2.

Every term in the expansion in the Legendre polynomials $F^{(l)}(\mathbf{x}, v, \xi)$ will be projected on the set of Laguerre plynomials.

$$\begin{aligned} L_0^{(3/2)}\left(x_a^2\right) &= 1 \\ L_1^{(3/2)}\left(x_a^2\right) &= \frac{5}{2} - x_a^2 \\ &\ldots \end{aligned}$$

Consider the function

$$f_{a1}(\mathbf{x}, v, \xi)$$

which is the correction to the equilibrium Maxwellian distribution function that produces the *neoclassical transport*.

This will be first projected on the Legendre set, and the first term is

$$f_{a1}^{(l=1)}(\mathbf{x}, v, \xi)$$

This first term of the Legendre series

$$\begin{aligned} f_{a1}^{(l=1)}(\mathbf{x}, v, \xi) &= P_{l=1}(\xi)\ \frac{2l+1}{2} \int_{-1}^{1} d\eta\ P_{l=1}(\eta)\ \ f_{a1}(\mathbf{x}, v, \eta) \\ &= \xi\ \frac{3}{2} \int_{-1}^{1} d\left(\frac{v_\parallel}{v}\right)\ \left(\frac{v_\parallel}{v}\right)\ f_{a1}\left(\mathbf{x}, v, \frac{v_\parallel}{v}\right) \end{aligned}$$

will be now projected on the set or Laguerre polynomials

$$\begin{aligned} f_{a1}^{(l=1)}(\mathbf{x}, v, \xi) &= \frac{2}{v_{th,a}} \xi\ x_a u_{\parallel a}\ f_{aM} \\ &\quad + \frac{2}{v_{th,a}} \xi\ x_a\ \left(x_a^2 - \frac{5}{2}\right) \frac{2}{5} \frac{q_{\parallel a}}{p_a}\ f_{aM} \\ &\quad + f_{a1}^{(l=1,\ j\geq 2)} \end{aligned}$$

The three indices of the last term are

$$\begin{aligned} a1 &\rightarrow \text{from Maxwell + neoclassic} \\ l &= 1 \text{ from Legendre } (\xi) \text{ expansion} \\ j &\geq 2 \text{ from Laguerre } \left(x_a^2\right) \text{ expansion} \end{aligned}$$

and the definition

$$u_{\parallel a} = \frac{1}{n_a} \int d^3v\ f_{a1}\ v_\parallel$$

$$q_a = T_a \int d^3v \; f_{a1} \; v_{\|} \left(x_a^2 - \frac{5}{2} \right)$$

The conservation laws for the flow of particles and for the flow of heat are

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{u}_a &= 0 \\ \mathbf{u}_a &= u_{\|a} \widehat{\mathbf{n}} + \mathbf{u}_{\perp a} \end{aligned}$$

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{q}_a &= 0 \\ \mathbf{q}_a &= q_{\|a} \widehat{\mathbf{n}} + \mathbf{q}_{\perp a} \end{aligned}$$

where

$$\mathbf{u}_{\perp a} = \frac{1}{e_a B} X_{a1} \; \boldsymbol{\nabla} \psi \times \widehat{\mathbf{n}}$$

$$\frac{\mathbf{q}_{\perp a}}{p_a} = \frac{5}{2} \frac{1}{e_a B} \; X_{a2} \; \boldsymbol{\nabla} \psi \times \widehat{\mathbf{n}}$$

with the *thermodynamic forces*

$$X_{a1} = -\frac{1}{n_a} \frac{\partial p_a}{\partial \psi} - e_a \frac{\partial \phi}{\partial \psi}$$

$$X_{a1} = -\frac{\partial T_a}{\partial \psi}$$

The pressure and the temperature are only functions of surface, ψ.

The incompressibility condition

$$\boldsymbol{\nabla} \cdot \mathbf{u}_a = 0$$

leads to

$$\begin{aligned} u_{\|a} &= \frac{\langle u_{\|a} B \rangle}{\langle B^2 \rangle} B \\ &\quad + \frac{1}{e_a} \; X_{a1} \; \widetilde{U} \end{aligned}$$

and

$$\begin{aligned} \frac{2}{5} \frac{q_{\|a}}{p_a} &= \frac{2}{5} \frac{\left\langle \frac{q_{\|a}}{p_a} B \right\rangle}{\langle B^2 \rangle} B \\ &\quad + \frac{1}{e_a} X_{a2} \; \widetilde{U} \end{aligned}$$

where the new function

$$\widetilde{U}$$

can be obtained from

$$\begin{aligned}
\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{\widetilde{U}}{B}\right) &= \mathbf{B}\times\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\left(\frac{1}{B^2}\right)\\
\left\langle B\widetilde{U}\right\rangle &= 0
\end{aligned}$$

Note In tokamak the second equation is

$$\left\langle \frac{B_0}{R_0 h}\widetilde{U}\right\rangle = 0 \rightsquigarrow \left\langle \widetilde{U}\left(1-\frac{r}{R_0}\cos\theta\right)\right\rangle = 0$$

and for a circular magnetic surface

$$\left\langle \widetilde{U}\cos\theta\right\rangle = 0$$

[if we assume that $\left\langle \widetilde{U}\right\rangle = 0$], which means

$$\widetilde{U} \sim \sin\theta$$

The LHS of the first equation

$$\begin{aligned}
B\ \nabla_\parallel\left(\frac{\widetilde{U}}{\frac{B_0}{R_0 h}}\right) &= \frac{B_0}{R_0 h}\frac{R_0}{B_0}\nabla_\parallel\left(\widetilde{U}h\right)\\
&= \frac{1}{h}\nabla_\parallel\left(\widetilde{U}+\frac{r}{R_0}\cos\theta\ \widetilde{U}\right)\\
&= \frac{1}{h}\frac{1}{qR}\frac{\partial}{\partial\theta}\left(\widetilde{U}+\frac{r}{R_0}\cos\theta\ \widetilde{U}\right)
\end{aligned}$$

The RHS is

$$\begin{aligned}
\mathbf{B}\times\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\left(\frac{1}{B^2}\right) &= B\left(\widehat{\mathbf{n}}\times\widehat{\mathbf{e}}_r\right)RB_\theta\cdot\boldsymbol{\nabla}\left(\frac{1}{\left(\frac{B_0}{R}\right)^2}\right)\\
&= \frac{B_0}{R}RB_\theta\frac{1}{B_0^2}\widehat{\mathbf{e}}_\theta\cdot\boldsymbol{\nabla}\left(R^2\right) = \frac{B_\theta}{B_0}\nabla_\theta\left(R^2\right)
\end{aligned}$$

where

$$\nabla_\theta = \frac{\partial}{\partial l_\theta} = \frac{1}{r}\frac{\partial}{\partial\theta}$$

then

$$\frac{B_\theta}{B_0}\frac{1}{r}\frac{\partial}{\partial\theta}\left(R^2\right)=\frac{B_\theta R}{r\left(\frac{B_0}{R}\right)}\frac{1}{R^2}\frac{\partial}{\partial\theta}\left(R^2\right)=\frac{1}{qR^2}2R\frac{\partial}{\partial\theta}\left(R\right)=2\frac{1}{qR}\frac{\partial}{\partial\theta}\left(R\right)$$

and the equation becomes

$$\frac{1}{h}\frac{1}{qR}\frac{\partial}{\partial\theta}\left(\widetilde{U}+\frac{r}{R_0}\cos\theta\ \widetilde{U}\right)=2\frac{1}{qR}\frac{\partial}{\partial\theta}\left(R\right)$$

an approximation would be

$$\begin{aligned}\widetilde{U}h-2R &\approx \text{function of surface label}\\ \widetilde{U} &\sim \frac{\text{const}}{h}\end{aligned}$$

End.

Further it is separated the part that comes from higher than $l=1$ order Legendre polynomials

$$g_a=f_{a1}-f_{a1}^{(l=1)}$$

in particular

$$l=2$$

For g_a the equation is

$$V_\| g_a-C_a^{lin}\left[g_a\right]=H_a^{(l=1)}+H_a^{(l=2)}$$

where

$$\begin{aligned}H_a^{(l=1)} &= \frac{e_a}{T_a}v_\| B\frac{\left\langle BE_\|\right\rangle}{\left\langle B^2\right\rangle}f_{aM}\\ &\quad +C_a^{lin}\left[f_{a1}^{(l=1)}\right]\end{aligned}$$

and

$$\begin{aligned}H_a^{(l=2)} &= \frac{f_{aM}}{T_a}\left\{\sigma_{Ua}\left[\frac{\left\langle u_{\|a}B\right\rangle}{\left\langle B^2\right\rangle}+\frac{2}{5}\frac{\left\langle\frac{q_{\|a}}{p_a}B\right\rangle}{\left\langle B^2\right\rangle}\left(x_a^2-\frac{5}{2}\right)\right]\right.\\ &\quad \left.+\sigma_{Xa}\left[X_{a1}+X_{a2}\left(x_a^2-\frac{5}{2}\right)\right]\right\}\end{aligned}$$

This term can be rewritten

$$H_a^{(l=2)} = \frac{f_{aM}}{T_a}\left(\frac{dV}{d\psi}\right)\frac{1}{4\pi^2}\left\{\sigma_{p_a}\frac{1}{\left(\frac{d\chi}{d\psi}\right)}\left[\left\langle u_a^\theta\right\rangle+\frac{2}{5}\left\langle\frac{q_a^\theta}{p_a}\right\rangle\left(x_a^2-\frac{5}{2}\right)\right]\right.$$
$$\left.+\sigma_{T_a}\frac{1}{\left(\frac{d\psi}{d\psi}\right)}\left[\left\langle u_a^\varphi\right\rangle+\frac{2}{5}\left\langle\frac{q_a^\varphi}{p_a}\right\rangle\left(x_a^2-\frac{5}{2}\right)\right]\right\}$$

This is the equation for the distribution function g_a.
Solution g_a will allow to calculate f_{a1}.
In the composition of f_{a1} there is also the term $f_{a1}^{(l=1)}$.
Therefore $f_{a1}^{(l=1)}$ in terms of $u_{\|a}$ and of $\frac{q_{\|a}}{p_a}$ PLUS the function g_a also in terms of u_a and q_a/p_a (poloidal toroidal) will give finally f_{a1} .
In the expressions of the *viscosity* only the function g_a is present.

The objective is to find a matrix (linear relationship) between the viscosity components and the flows.

The coefficients

$$\begin{aligned}\sigma_{Ua} &= -m_a v^2\ P_2(\xi)\ \mathbf{B}\cdot\boldsymbol{\nabla}\ln B \\ &= -V_{\|}\ [m_a v\xi B]\end{aligned}$$

Then replace

$$\begin{aligned}P_2(\xi) &= \frac{3}{2}\xi^2-\frac{1}{2} \\ &= \frac{1}{2}\left(3\frac{v_\|^2}{v^2}-1\right) = \frac{1}{2v^2}\left(3v_\|^2-v_\|^2-v_\perp^2\right) \\ &= \frac{v_\|^2-\frac{v_\perp^2}{2}}{v^2}\end{aligned}$$

this combination of squared velocities is - of course - distinct of the combination occuring in the approximative expression of the neoclassical drift velocity $\mathbf{v}_D$. And

$$\mathbf{B}\cdot\boldsymbol{\nabla}\ln B = \nabla_\| B$$

such that

$$\begin{aligned}&-m_a v^2\ P_2(\xi)\ \mathbf{B}\cdot\boldsymbol{\nabla}\ln B \\ &= -m_a v^2\frac{v_\|^2-\frac{v_\perp^2}{2}}{v^2}\nabla_\| B\end{aligned}$$

Note that the operator $V_{\parallel}$ has the explicit form

$$V_{\parallel}\left[... \right] = v_{\parallel} \nabla_{\parallel} \left[... \right]$$

in the coordinates $(\mathbf{x}, \epsilon, \mu)$ and the expression

$$V_{\parallel}\left[... \right] = \left(v\xi\nabla_{\parallel} - \frac{1}{2} v \left(1-\xi^2\right) \nabla_{\parallel} \ln B \frac{\partial}{\partial \xi} \right) \left[... \right]$$

in the coordinates $(\mathbf{x}, v, \xi)$.

Applying this operator $V_{\parallel}\ \left[m_a v \xi B\right]$,

$$\begin{aligned}
& V_{\parallel}\ \left[m_a v \xi B\right] \\
= & \left(v\xi\nabla_{\parallel} - \frac{1}{2} v \left(1-\xi^2\right) \nabla_{\parallel} \ln B \frac{\partial}{\partial \xi} \right) \left[m_a v \xi B\right] \\
= & m_a v^2 \xi^2 \nabla_{\parallel} B - m_a \frac{1}{2} v \left(1-\xi^2\right) v B \nabla_{\parallel} \ln B\ \frac{\partial \xi}{\partial \xi} \\
= & m_a v^2 \nabla_{\parallel} B \left[\xi^2 - \frac{1}{2} + \frac{\xi^2}{2} \right] = m_a v^2 \nabla_{\parallel} B \left(\frac{3}{2} \xi^2 - \frac{1}{2} \right) \\
= & m_a v^2 \nabla_{\parallel} B\ P_2(\xi)
\end{aligned}$$

This verifies the formula, we have obtained the original form of $\sigma_{Ua} = -m_a v^2\ P_2(\xi)\ \mathbf{B} \cdot \boldsymbol{\nabla} \ln B$.

Next

$$\begin{aligned}
\sigma_{Xa} &= -v^2\ P_2(\xi)\ \frac{B}{\Omega_a} \left(\widetilde{U}\ \widehat{\mathbf{n}} + \frac{\boldsymbol{\nabla}\psi \times \widehat{\mathbf{n}}}{B} \right) \cdot \boldsymbol{\nabla} \ln B \\
&= -v^2 P_2(\xi)\ \frac{\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \left(B \widetilde{U} \right)}{2\Omega_a}
\end{aligned}$$

The last term contains the product

$$\begin{aligned}
& \frac{B}{\Omega_a} \frac{\boldsymbol{\nabla}\psi \times \widehat{\mathbf{n}}}{B} \cdot \boldsymbol{\nabla} \ln B \\
= & \frac{1}{\Omega_a} (-) \frac{1}{B} \left[\widehat{\mathbf{n}} \times \boldsymbol{\nabla} B \right] \cdot \boldsymbol{\nabla}\psi \\
\sim & \mathbf{v}_D \cdot \boldsymbol{\nabla}\psi
\end{aligned}$$

which is part (no factor μ) of the projection on the radial direction ($\boldsymbol{\nabla}\psi$) of the particle drift due to magnetic mirror.

NOTE

In **Hirshman Sigmar PF 20, 1977, 418**
See comparison with **Galeev Sagdeev Liu Novakovskii**.
It is mentioned the *mirror force*

$$\mu \nabla_{\|} B$$

which, if small, does not change substantially the parallel velocity

$$v_{\|}$$

It is a weak modulation of the parallel velocity of a circulating particle.
The equation for the distribution function is

$$\begin{aligned}
& v_{\|} \nabla_{\|} f_{a1} - \frac{1}{2} \left(v^2 - v_{\|}^2 \right) \nabla_{\|} \left(\ln B \right) \ \frac{\partial f_{a1}}{\partial v_{\|}} \\
& - \frac{e_a v_{\|} E_{\|}}{T_a} f_{a0} \\
= & \sum_{species\ \ b} C_{ab} \\
& + \nabla_{\|} \left(\ln B \right) \frac{\frac{v_{\perp}^2}{2} + v_{\|}^2}{\Omega_a} \ I \left(\psi \right) \ \frac{df_{a0}}{dr}
\end{aligned}$$

where

$$\begin{aligned}
\frac{df_{a0}}{dr} &= \left(A_{1a} + \frac{e_a}{T_a} \frac{d\phi}{dr} + x_a^2 A_{2a} \right) \ f_{a0} \\
A_{1a} &= \frac{1}{n_a} \frac{dn_a}{dr} - \frac{3}{2} \frac{1}{T_a} \frac{dT_a}{dr} \\
A_{2a} &= \frac{1}{T_a} \frac{dT_a}{dr}
\end{aligned}$$

END

One can see

$$\begin{aligned}
& - \frac{1}{2} \left(v^2 - v_{\|}^2 \right) \nabla_{\|} \left(\ln B \right) \ \frac{\partial f_{a1}}{\partial v_{\|}} \\
= \quad & - \frac{1}{2} v_{\perp}^2 \frac{1}{B} \nabla_{\|} B \ \frac{\partial f_{a1}}{\partial v_{\|}} = - \mu \nabla_{\|} B \ \frac{\partial f_{a1}}{\partial v_{\|}}
\end{aligned}$$

Next

$$\sigma_{P_a} = - m_a v^2 \ P_2 \left(\xi \right) \ \mathbf{B}_{pol} \cdot \boldsymbol{\nabla} \ln B$$

$$\sigma_{T_a} = -m_a v^2 \; P_2(\xi) \; \mathbf{B}_{tor} \cdot \boldsymbol{\nabla} \ln B$$

The *parallel* neoclassical viscosity

$$\langle \mathbf{B} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a) \rangle = \left\langle \int d^3 v \; g_a \; \sigma_{Ua} \right\rangle$$

$$\langle \mathbf{B} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{\Theta}_a) \rangle = \left\langle \int d^3 v \; g_a \; \sigma_{Ua} \; \left(x_a^2 - \frac{5}{2} \right) \right\rangle$$

The *poloidal* neoclassical viscosity

6 Ambipolarity Hirshman

The toroidal component of the momentum balance

$$\begin{aligned} m_j n_j \frac{\partial \mathbf{u}_j}{\partial t} &= -\boldsymbol{\nabla} p_j - \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_j \\ & \quad + e_j n_j (\mathbf{E} + \mathbf{u}_j \times \mathbf{B}) \\ & \quad + \mathbf{R}_j \end{aligned}$$

where

$$\mathbf{R}_j = \int d^3 v \; m_j \mathbf{v} \; \sum_k C_{jk}$$

The relation between the friction force $\mathbf{R}_j$ and the cross-field particle flux

$$\Gamma_j = \langle n_j \mathbf{u}_j \cdot \boldsymbol{\nabla} \psi \rangle$$

is determined from this equation.

The inertial part

$$(\mathbf{u} \cdot \boldsymbol{\nabla}) \, \mathbf{u}$$

is neglected.

A flux friction relation is obtained by multiplying with

$$R \widehat{\mathbf{e}}_\varphi$$

and averaging over the surface ψ.

$$\begin{aligned} & e_j \Gamma_j \quad \text{(radial current)} \\ = \; & e_j (\Gamma_R)_j \quad \text{(resistive part)} \\ & + e_j (\Gamma_p)_j \quad \text{(polarization current)} \\ & + e_j (\Gamma_{orth-cond})_j \quad \text{(orthogonal conduction current)} \end{aligned}$$

with

$$e_j \left(\Gamma_R\right)_j = -\left\langle R\widehat{\mathbf{e}}_\varphi \cdot \left(\mathbf{R}_j^{Coulomb} + e_j n_j \mathbf{E}\right)\right\rangle$$

the *resistive* component of the radial current is determined by the *toroidal component* of the Coulomb friction force.

The *polarization* current

$$e_j \left(\Gamma_p\right)_j = \left\langle m_j n_j \ R\widehat{\mathbf{e}}_\varphi \cdot \frac{\partial \mathbf{u}_j}{\partial t} \right\rangle$$

The orthogonal conduction current is driven by perpendicular viscous stress and non-Coulomb friction

$$e_j \left(\Gamma_{ortho-cond}\right)_j = \left\langle R\widehat{\mathbf{e}}_\varphi \cdot \left[\boldsymbol{\nabla} \cdot \boldsymbol{\pi}_j - \left(\mathbf{R}_j - \mathbf{R}_j^{Coulomb}\right)\right]\right\rangle$$

This can be written as

$$\begin{aligned} e_j \left(\Gamma_{orth-cond}\right)_j &= \frac{1}{V'}\frac{\partial}{\partial \psi} V' \left\langle R\widehat{\mathbf{e}}_\varphi \cdot \boldsymbol{\pi}_j \cdot \boldsymbol{\nabla}\psi \right\rangle \quad \text{(toroidal-radial)} \\ & -\left\langle R\widehat{\mathbf{e}}_\varphi \cdot \left(\mathbf{R}_j - \mathbf{R}_j^{Coulomb}\right)\right\rangle \ \text{(residual friction)} \end{aligned}$$

Here

$$V' = \oint \frac{dl_\theta}{B_\theta}$$

The property

$$\boldsymbol{\pi}_j : \boldsymbol{\nabla}\left(R\widehat{\mathbf{e}}_\varphi\right) = 0$$

for symmetric $\boldsymbol{\pi}_j$. And

$$\begin{aligned} \left\langle R\widehat{\mathbf{e}}_\varphi \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_j^{(1)} \right\rangle &= 0 \\ & \text{in first order in } \rho/L \end{aligned}$$

If there is a radial current in the initial state

$$\left\langle \mathbf{j} \cdot \boldsymbol{\nabla}\psi \right\rangle \neq 0$$

then this radial current combines with the poloidal magnetic field and produces a torque in toroidal direction

$$\mathbf{j} \times \mathbf{B}_\theta$$

The motion of plasma in toroidal direction will produce a radial electric field that will cancel the radial current

$$\langle \mathbf{j} \cdot \boldsymbol{\nabla} \psi \rangle + \left\langle \frac{\partial \mathbf{E}}{\partial t} \cdot \boldsymbol{\nabla} \psi \right\rangle = 0$$

The toroidal momentum evolution is

$$\frac{\partial P_{\varphi}}{\partial t} = \frac{\partial}{\partial t} \left\langle \sum_j m_j n_j \ R \widehat{\mathbf{e}}_{\varphi} \cdot \mathbf{u}_j + R \ E_r B_{\theta} \right\rangle$$

where

$$R \ E_r B_{\theta} = \mathbf{E} \cdot \boldsymbol{\nabla} \psi$$

comes from the charge separation leading to polarization.

$$\begin{aligned} \mathbf{u}_j &= \left(\frac{\mathbf{u}_j \cdot \mathbf{B}_{\theta}}{B_{\theta}^2} \right) \mathbf{B} \\ &+ \left(\left\langle R \widehat{\mathbf{e}}_{\varphi} \cdot \mathbf{u}_j \right\rangle - \frac{\mathbf{u}_j \cdot \mathbf{B}_{\theta}}{B_{\theta}^2} \left\langle R B_{\varphi} \right\rangle \right) \frac{1}{\langle R^2 \rangle} R \widehat{\mathbf{e}}_{\varphi} \end{aligned}$$

The first term is the known notation $K(\psi)$,

$$\frac{\mathbf{u}_j \cdot \mathbf{B}_{\theta}}{B_{\theta}^2} = K_j(\psi)$$

The equation for the damping of the poloidal rotation must introduce in the balance the parallel component of the divergence of the stress. For this purpose one starts with the equation of motion (for $\mathbf{u}_j$) and projects it on the parallel direction, by scalar multiplication with $\mathbf{B}$ followed by surface averaging.

$$\sum_j m_j n_j \left\langle \mathbf{B} \cdot \frac{\partial \mathbf{u}_j}{\partial t} \right\rangle = - \sum_j \left\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_j^{(1)} \right\rangle$$

(One remembers that NOT along parallel direction but along the *toroidal* direction we have for the first order

$$\left\langle R \widehat{\mathbf{e}}_{\varphi} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_j^{(1)} \right\rangle = 0$$

This means that for the damping of the surface averaged toroidal momentum one needs higher order terms of viscosity, off-diagonal).

The equation for the damping of the poloidal rotation is derived from the sum of j contributions projected on parallel direction, averaged over surface. It is

$$\begin{aligned}
&\sum_j m_j n_j \frac{\partial}{\partial t}\left[\left(1+2\widehat{q}^2\right)\langle \mathbf{u}_j\cdot\mathbf{B}_\theta\rangle + \frac{\langle RB_\varphi\rangle}{\langle R^2\rangle}\langle R\widehat{\mathbf{e}}_\varphi\cdot\mathbf{u}_j\rangle\right] \\
= & -\sum_j \langle \mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_j\rangle
\end{aligned}$$

where

$$\begin{aligned}
\widehat{q}^2 &= \frac{1}{2}\frac{1}{\langle B_\theta^2\rangle}\left(\left\langle B_\varphi^2\right\rangle - \frac{1}{\left\langle \frac{1}{B_\varphi^2}\right\rangle}\right) \\
&\approx q^2
\end{aligned}$$

Hirshman proposes an approximation that allows to close the equation for the poloidal velocity

$$\langle \mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_j\rangle = 3\mu_j\ \left\langle \left(\nabla_\parallel B\right)^2\right\rangle \frac{\mathbf{u}_j\cdot\mathbf{B}_\theta}{B_\theta^2}$$

where μ_j is the neolcassical viscosity coefficient.

7 Fluxes, flows and friction forces Hirshman Sigmar review

7.1 Definitions

In **Hirshman Sigmar review**.

For the specis a

$$n_a = \int d^3v\ f_a$$

Scalar pressure (density of energy) isotrop

$$p_a = \int d^3v\ m_a \frac{\left|\mathbf{v}-\mathbf{u}_a\right|^2}{3}\ f_a$$

There is a *directed flow* of the species a

$$\underset{\text{flow}}{\mathbf{u}_a = \frac{1}{n_a}\int d^3v\ \mathbf{v}\ f_a}$$

$$\mathbf{q}_a = \int d^3v \ (\mathbf{v}-\mathbf{u}_a) \ \frac{m_a \left|\mathbf{v}-\mathbf{u}_a\right|^2}{2} \ f_a$$

conductive heat flow

in the framework of the flow

The viscosity tensor

$$\boldsymbol{\pi}_a = \int d^3v \ m_a \left[(\mathbf{v}-\mathbf{u}_a) \ (\mathbf{v}-\mathbf{u}_a) - \frac{\left|\mathbf{v}-\mathbf{u}_a\right|^2}{3} \mathbf{I} \right] \ f_a$$

The isotropic scalar pressure p_a is subtracted from the direct tensorial product. This leaves only anisotropic part.

The collisional momentum source *rate* for the species a

$$\mathbf{F}_{a1} = \int d^3v \ m_a \mathbf{v} \ C_a(f_a)$$

friction force, rate of MOMENTUM change

The collisional heat source *rate*

$$Q_a = \int d^3v \ \frac{m_a \left|\mathbf{v}-\mathbf{u}_a\right|^2}{2} \ C_a(f_a)$$

Total energy flux, includes the flow $\mathbf{u}_a$,

$$\begin{aligned}\mathbf{Q}_a &= \int d^3v \ \mathbf{v} \ \frac{m_a v^2}{2} \ f_a \\ &= \mathbf{q}_a + \mathbf{u}_a \frac{5}{2} p_a + \mathbf{u}_a n_a \frac{m_a u_a^2}{2} + \mathbf{u}_a \cdot \boldsymbol{\pi}_a\end{aligned}$$

Later in **Hirshman Sigmar** this is written

$$\mathbf{Q}_{a\perp} = \mathbf{q}_{a\perp} + \frac{5}{2} T_a \boldsymbol{\Gamma}_{a\perp}$$

since it is introduced

$$\boldsymbol{\Gamma}_a = n_a \mathbf{u}_{a\perp}$$

The energy-weighted stress tensor

$$\mathbf{R}_a = \int d^3v \ \mathbf{v} \ \mathbf{v} \ \frac{m_a v^2}{2} \ f_a$$

The *rate* of collisional heat flux generation

$$\begin{aligned}\mathbf{G}_a &= \int d^3v\ \mathbf{v}\ \frac{m_a v^2}{2}\ C\left(f_a\right)\\ &= \frac{T_a}{m_a}\left(\frac{5}{2}\mathbf{F}_{a1}+\mathbf{F}_{a2}\right)\end{aligned}$$

where $\mathbf{F}_{a1}=\int d^3v\ m_a\mathbf{v}\ C_a\left(f_a\right)$ and the second term is

$$\begin{aligned}\mathbf{F}_{a2} &= \int d^3v\ m_a\mathbf{v}\ \left(\frac{\mathbf{v}^2}{2T_a/m_a}-\frac{5}{2}\right)\ C_a\left(f_a\right)\\ & \quad \text{heat friction, rate of coll. HEAT change}\end{aligned}$$

(we note that the last term in the paranthesis, $-5/2\times()$ cancels $5/2\times\mathbf{F}_{a1}$. Then why writting $\mathbf{G}_a$ as a sum of two terms ? We **note** that the round paranthesis is actually a Laguerre polynomial,

$$\frac{\mathbf{v}^2}{2T_a/m_a}-\frac{5}{2}=-L_1^{(3/2)}\left(x_a^2\right)$$

End).

Note that later it will result that the friction force $\mathbf{F}_{a1}$ leads to the *momentum* flux from the dependence on gyration.

The friction force $\mathbf{F}_{a2}$ will produce the *heat* flux that results from the dependence on the gyration.

End.

In **Hirshman Sigmar**

$$I=R^2\mathbf{B}\cdot\boldsymbol{\nabla}\varphi$$

and

$$\left|\boldsymbol{\nabla}\varphi\right|=\frac{1}{R}\ ,\ I=RB_T$$

and

$$\begin{aligned}\mathbf{B} &= \frac{1}{2\pi}\frac{\partial\chi}{\partial\psi}\boldsymbol{\nabla}\varphi\times\boldsymbol{\nabla}\psi \quad \text{(poloidal)}\\ &\quad +I\boldsymbol{\nabla}\varphi \qquad\qquad\qquad \text{(toroidal)}\end{aligned}$$

where

$$\begin{aligned}I &= R^2\mathbf{B}\cdot\boldsymbol{\nabla}\varphi\\ &\sim RB_\varphi\end{aligned}$$

$$\begin{aligned}\chi' &= \frac{\partial \chi}{\partial \psi} = 2\pi \sqrt{g}\, \mathbf{B} \cdot \boldsymbol{\nabla} \theta \\ &= \text{poloidal magnetic flux density}\end{aligned}$$

$$\begin{aligned}\sqrt{g} &= \frac{1}{(\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta) \cdot \boldsymbol{\nabla}\varphi} \\ &\sim \frac{1}{\left(2\pi R B_\theta \times \frac{1}{r}\right) \times \frac{1}{R}} = \frac{r}{2\pi B_\theta}\end{aligned}$$

The function χ is introduced in relation with the poloidal magnetic field. Due to

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0$$

one can introduce a *streamfunction* χ for B_θ,

$$\mathbf{B}_\theta = \frac{1}{2\pi} \boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\chi$$

When the surfaces of constant χ are nested toroidal there is a mapping

$$\psi \leftrightarrow \chi$$

With the usual geometry

$$\sqrt{g} = \frac{1}{2\pi} \frac{r}{B_\theta}$$

and it results

$$\chi' = \frac{\partial \chi}{\partial \psi} = 2\pi \sqrt{g}\, \mathbf{B} \cdot \boldsymbol{\nabla}\theta = 2\pi \frac{1}{2\pi} \frac{r}{B_\theta} B_\theta \frac{1}{r} = 1$$

Then it is confirmed that in vacuum and in circular magnetic surfaces we have

$$\begin{aligned}\chi' &= \frac{\partial \chi}{\partial \psi} = 1 \\ &\rightarrow \chi = \psi\end{aligned}$$

The definitions. The tensor of pressure

$$\begin{aligned}\mathbf{P}_a = {} & p_a\, \mathbf{I} \qquad \text{(scalar pressure)} \\ & + \left(p_{\|a} - p_{\perp a}\right) \left(\widehat{\mathbf{n}}\widehat{\mathbf{n}} - \frac{1}{3}\mathbf{I}\right) \qquad \text{(anisotropy, stress tensor)}\end{aligned}$$

and

$$\begin{aligned}
\mathbf{R}_a &= R_a\,\mathbf{I} \\
&\quad + \left(R_{\|a} - R_{\perp a}\right)\ \left(\widehat{\mathbf{n}}\,\widehat{\mathbf{n}} - \frac{1}{3}\mathbf{I}\right)
\end{aligned}$$

where

$$\begin{aligned}
R_a &= \mathrm{Tr}\ \left(\frac{1}{3}\mathbf{R}_a\right) \\
&= \frac{5}{2} p_a \frac{T_a}{m_a}
\end{aligned}$$

The magnitude of anisotropy

$$\frac{p_{\|a} - p_{\perp a}}{p_a} \sim \frac{R_{\|a} - R_{\perp a}}{R_a} \sim \varepsilon$$

A new definition of a tensor has been introduced

$$\begin{aligned}
\mathbf{\Theta}_a &= \left[\frac{m_a}{T_a}\left(R_{\|a} - R_{\perp a}\right) - \frac{5}{2}\left(P_{\|a} - P_{\perp a}\right)\right]\left(\widehat{\mathbf{n}}\,\widehat{\mathbf{n}} - \frac{1}{3}\mathbf{I}\right) \\
&= \left(\Theta_{\|a} - \Theta_{\perp a}\right)\left(\widehat{\mathbf{n}}\,\widehat{\mathbf{n}} - \frac{1}{3}\mathbf{I}\right)
\end{aligned}$$

The kinetic content of the two components of the tensor $\mathbf{\Theta}$ are

$$\Theta_{\|a} = \int d^3v\ m_a\left(v_{\|} - u_{\|a}\right)\left(x_a^2 - \frac{5}{2}\right)\overline{f}_a$$

$$\Theta_{\perp a} = \int d^3v\ m_a\ \mu B\ \left(x_a^2 - \frac{5}{2}\right)\ \overline{f}_a$$

where

$$x_a^2 = \frac{v^2/2}{T_a/m_a} = \frac{v^2}{v_{th,a}^2}$$

7.2 Radial fluxes

In **Hirshamn Sigmar review**.

From the momentum conservation equation, multiplied vectorially with $\mathbf{B}$, one extracts the fluxes that are perpendicular on $\mathbf{B}$. The contributions come from *drifts*

$$
\begin{aligned}
\mathbf{\Gamma}_{a\perp} &= n_a \frac{\mathbf{E}\times\mathbf{B}}{B^2} \qquad \text{(electric)} \\
&+ \frac{1}{m_a\Omega_a}\widehat{\mathbf{n}}\times\mathbf{\nabla}\cdot\mathbf{P}_a \qquad \text{(diamagnetic)} \\
&- \frac{1}{m_a\Omega_a}\widehat{\mathbf{n}}\times\mathbf{F}_{a1} \;\; \text{(force-induced)}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{q}_{a\perp} &= \mathbf{Q}_{a\perp} - \frac{5}{2}T_a\mathbf{\Gamma}_{a\perp} \\
&= \frac{1}{\Omega_a}\widehat{\mathbf{n}}\times\left(\mathbf{\nabla}\cdot\mathbf{R}_a - \frac{5}{2}\frac{T_a}{m_a}\mathbf{\nabla}\cdot\mathbf{P}_a\right) \\
&\quad - \frac{1}{m_a\Omega_a}T_a\widehat{\mathbf{n}}\times\mathbf{F}_{a2}
\end{aligned}
$$

(absence of terms with $\mathbf{E}$). In the first line, we neglect the terms arising from a directed flow $\mathbf{u}_a$.

The *radial fluxes* are composed of a part that depends on the *gyrophase* ζ and a part that results from averaging over the gyration

$$
\begin{aligned}
\text{full } \Gamma_a^\psi &= \langle \Gamma_a\cdot\mathbf{\nabla}\psi\rangle \\
&= \overline{\Gamma}_a^\psi + \widetilde{\Gamma}_a^\psi
\end{aligned}
$$

$$
\text{full } \; q_a^\psi = \overline{q}_a^\psi + \widetilde{q}_a^\psi
$$

The part that depends on the *gyrophase* is driven by the perpendicular friction forces $\mathbf{F}_{a1}$ and $\mathbf{F}_{a2}$

$$
\begin{aligned}
\widetilde{\Gamma}_a^\psi &= -\left\langle \frac{1}{m_a\Omega_a}\left(\widehat{\mathbf{n}}\times\mathbf{F}_{a1}\right)\cdot\mathbf{\nabla}\psi\right\rangle \\
&\qquad \text{(surface average)}
\end{aligned}
$$

$$
\frac{\widetilde{q}_a^\psi}{T_a} = \left\langle \frac{1}{m_a\Omega_a}\left(\widehat{\mathbf{n}}\times\mathbf{F}_{a2}\right)\cdot\mathbf{\nabla}\psi\right\rangle
$$

The part that does NOT depend on the gyrophase, is the *neoclassical* part.

For momentum

$$\overline{\Gamma}_a^{\psi} = -\left\langle \frac{1}{m_a \Omega_a} \left[\widehat{\mathbf{n}} \times \left(-\boldsymbol{\nabla} p_a - \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a + e_a n_a \boldsymbol{\nabla} \Phi \right) \right] \cdot \boldsymbol{\nabla} \psi \right\rangle$$

this comes from the term $j \times B$ in the *fluid* momentum conservation, at stationarity.

This can be written, in particle kinetic form

$$\overline{\Gamma}_a^{\psi} = \left\langle \int d^3 v \ \left(\mathbf{v}_{Da} \cdot \boldsymbol{\nabla} \psi \right) \overline{f}_a \right\rangle$$

For heat

$$\frac{\overline{q}_a^{\psi}}{T_a} = -\left\langle \frac{1}{m_a \Omega_a} \left[\widehat{\mathbf{n}} \times \left(-\frac{5}{2} n_a \boldsymbol{\nabla} T_a - \boldsymbol{\nabla} \cdot \boldsymbol{\Theta}_a \right) \right] \cdot \boldsymbol{\nabla} \psi \right\rangle$$

in fluid terms.

This can be written

$$\frac{\overline{q}_a^{\psi}}{T_a} = \left\langle \int d^3 v \ \left(\frac{v^2}{2T_a/m_a} - \frac{5}{2} \right) \mathbf{v}_{Da} \cdot \boldsymbol{\nabla} \psi \right\rangle$$

This means that the *neoclassical* fluxes (not dependent on the gyrophase) are due to the neoclassical drifts of the particles.

In conclusion the part of the radial fluxes $\boldsymbol{\Gamma}_a^{\psi}$ and $\frac{\mathbf{q}_a^{\psi}}{T_a}$

- that depends on gyrophase ζ is connected with the *friction* forces $\mathbf{F}_{a1}$ (momentum) and $\mathbf{F}_{a2}$ (heat), and
- that DOES NOT depend on the gyrophase is connected with the *neoclassical particle drift* $\mathbf{v}_D$.

The drift velocity is

$$\begin{aligned} \mathbf{v}_{Da} &= \frac{1}{\Omega_a} \widehat{\mathbf{n}} \times \left(\mu \boldsymbol{\nabla} B + v_{\parallel}^2 \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} \right) \\ &\quad + \frac{1}{B} \widehat{\mathbf{n}} \times \boldsymbol{\nabla} \Phi \end{aligned}$$

The sources of the neoclassical "radial" flux $\overline{\Gamma}_a^{\psi}$.

The formula for the *neoclassical* part is

$$
\begin{aligned}
& \text{neoclassical } \overline{\Gamma}_a^{\psi} \\
= & -\left\langle \frac{1}{m_a \Omega_a} \left[\widehat{\mathbf{n}} \times \left(-\boldsymbol{\nabla} p_a - \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a + e_a n_a \boldsymbol{\nabla} \Phi \right) \right] \cdot \boldsymbol{\nabla} \psi \right\rangle
\end{aligned}
$$

1. variation of $p_a - \langle n_a \rangle e_a \Phi$ and of the temperature T_a within the magnetic surface; in collisional regimes. Normally $[\widehat{\mathbf{n}} \times (-\boldsymbol{\nabla} p_a)] \cdot \boldsymbol{\nabla} \psi \approx 0$ since the gradient of pressure is perpendicular on surface, as it is $\boldsymbol{\nabla} \psi$. The pressure anisotropy $\boldsymbol{\pi}$ is small (the collisions almost make equal the parallel and perpendicular components) $p_{\parallel a} - p_{\perp a} \approx 0$. There is a variation of the pressure with the poloidal angle θ. The gradient $\frac{\partial p}{r \partial \theta}$ along the poloidal direction vector-multiplied by $\widehat{\mathbf{n}}$ gives radial flux, $\sim \boldsymbol{\nabla} \psi$. And there is an electric field along the magnetic field line, due to the finite resistivity (see **Stringer**)

$$
\frac{\partial}{\partial \theta} \left(\frac{e \Phi}{T} \right) \sim \varepsilon \frac{L}{\lambda_{\parallel a}}
$$

   This variation of the electric potential along the line (*i.e.* the parallel electric field $E_{\parallel}$) combines with the toroidal magnetic field as included in $\mathbf{v}_D$ in the term : $\widehat{\mathbf{n}} \times \boldsymbol{\nabla} \Phi$ to produce *radial flux* $\sim \boldsymbol{\nabla} \psi$.

2. *stress anisotropy*; in the long-free-path regime. The anisotropy of the pressure is due to the *poloidal flow* of particles, whose nonuniformity is caused by the nonuniformity of the strength of the magnetic field along the line. The parallel $v_{\parallel}$ and the perpendicular $v_{\perp}$ components of the particle velocity are "exchanging magnitude" between them during the motion along the line, to keep conserved ϵ and μ. Too few collisions to isotropize the pressure tensor. This means *bananas* are visible.

3. the variation of the electrostatic potential $\widetilde{\Phi}$, and density $\widetilde{n}$ within a magnetic surface, $\widetilde{\Phi}(\psi, \theta)$. The variation of the density enters through the distribution function $\overline{f}_a$. The distribution function will reflect the *neoclassical* processes.

"*in an axisymmetric toroidal system the* <u>*neoclassical*</u> *(radial) fluxes can be related to the parallel component of the friction forces, [* $F_{a1}^{\parallel}$ *and* $F_{a2}^{\parallel}$*]*

just as the <u>*classical*</u> *fluxes are related to the perpendicular components of* $F_{a1}^{\perp}$ *and* $F_{a2}^{\perp}$.

The neoclassical flux-friction relations arise as a consequence of the toroidal angular momentum conservation by the viscous forces."

In **Honda** it is demonstrated the equation

$$\left\langle R^2 \boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}^{(1)} \right\rangle = 0$$

where the first order viscosity tensor is Chow Goldberger Low

$$\boldsymbol{\pi}^{(1)} = \left(p_\parallel - p_\perp\right) \left(\widehat{\mathbf{n}}\,\widehat{\mathbf{n}} - \frac{1}{3}\mathbf{I}\right)$$

Honda takes a general formula

$$\boldsymbol{\xi} \cdot \boldsymbol{\nabla} \cdot \mathbf{T} = \boldsymbol{\nabla} \cdot (\boldsymbol{\xi} \cdot \mathbf{T}) - \boldsymbol{\nabla}\, \boldsymbol{\xi} : \mathbf{T}$$

and write

$$\begin{aligned} & R^2 \boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}^{(1)} \\ = \; & \boldsymbol{\nabla} \cdot \left(R^2 \boldsymbol{\nabla}\varphi \cdot \boldsymbol{\pi}^{(1)}\right) - \boldsymbol{\nabla}\left(R^2 \boldsymbol{\nabla}\varphi\right) : \boldsymbol{\pi}^{(1)} \end{aligned}$$

The tensor $\boldsymbol{\pi}^{(1)}$ is symmetric.
Then the second term is zero

$$\boldsymbol{\nabla}\left(R^2 \boldsymbol{\nabla}\varphi\right) : \boldsymbol{\pi}^{(1)} = 0$$

In detail, the equations are as follows. We note the action of the first $\boldsymbol{\nabla}$ operator which is the first factor in the full contraction

$$\begin{aligned} \boldsymbol{\nabla}\ \left(R^2 \boldsymbol{\nabla}\varphi\right) &= (\boldsymbol{\nabla} R)\ R \boldsymbol{\nabla}\varphi + R \boldsymbol{\nabla}\ (R \boldsymbol{\nabla}\varphi) \\ &= R \left[\boldsymbol{\nabla} R\ \boldsymbol{\nabla}\varphi + \boldsymbol{\nabla}\ (R \boldsymbol{\nabla}\varphi)\right] \end{aligned}$$

equivalently

$$\begin{aligned} \partial_\alpha\ R^2 \partial_\beta \varphi &= (\partial_\alpha R)\ R \partial_\beta \varphi + R \partial_\alpha\ (R \partial_\beta \varphi) \\ &= R \left[(\partial_\alpha R)\ (\partial_\beta \varphi) + \partial_\alpha\ (R \partial_\beta \varphi)\right] \\ & \quad \text{tensor with two indices} \end{aligned}$$

This will be contracted with $\pi_{\alpha\beta}$.

Honda recalls that

$$\boldsymbol{\nabla}\ (R \boldsymbol{\nabla}\varphi) = -\boldsymbol{\nabla}\varphi\ \boldsymbol{\nabla} R$$

This can be understood from

$$\boldsymbol{\nabla}\varphi = \frac{1}{R}\widehat{\mathbf{e}}_{\varphi}$$

Then

$$\begin{aligned}
\boldsymbol{\nabla}\left(R^2\boldsymbol{\nabla}\varphi\right) &: \boldsymbol{\pi}^{(1)} = R\left[\boldsymbol{\nabla}R\ \boldsymbol{\nabla}\varphi + \boldsymbol{\nabla}\ \left(R\boldsymbol{\nabla}\varphi\right)\right] : \boldsymbol{\pi}^{(1)} \\
&= R\left[\boldsymbol{\nabla}R\ \boldsymbol{\nabla}\varphi - \boldsymbol{\nabla}\varphi\ \boldsymbol{\nabla}R\right] : \boldsymbol{\pi}^{(1)}
\end{aligned}$$

But $\boldsymbol{\pi}^{(1)}$ is symmetric, and the square paranthesis is anti-symmetric, therefore the contraction is zero.

Returning to the problem we find that the local (not-averaged) viscosity tensor is a divergence.

$$R^2\boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}^{(1)} = \boldsymbol{\nabla}\cdot \left(R^2\boldsymbol{\nabla}\varphi \cdot \boldsymbol{\pi}^{(1)}\right)$$

After averagng, this is zero.

In **Hirshman Sigmar review** it is demonstrated the conservation property.

It starts from the *geometric* relation

$$R^2\boldsymbol{\nabla}\varphi = I\frac{\widehat{\mathbf{n}}}{B} - \frac{1}{B}\widehat{\mathbf{n}} \times \boldsymbol{\nabla}\psi\ \frac{1}{2\pi}\frac{\partial\chi}{\partial\psi}$$

We see the geometrical aspect

$$\begin{aligned}
R^2\boldsymbol{\nabla}\varphi &\equiv \text{toroidal vector} \\
&= R\widehat{\mathbf{e}}_{\varphi}
\end{aligned}$$

is expressed through a combination of

- parallel
$$\frac{I}{B}\widehat{\mathbf{n}} = R\frac{B_T}{B}\widehat{\mathbf{n}}$$
and the factor B_T/B projects the versor $\widehat{\mathbf{n}}$ to the toroidal direction

- perpendicular
$$\begin{aligned}
&\frac{1}{B}\widehat{\mathbf{n}} \times \boldsymbol{\nabla}\psi\ \frac{1}{2\pi}\frac{\partial\chi}{\partial\psi} \\
&= \frac{1}{B}\widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_r \left|\boldsymbol{\nabla}\psi\right| \\
&= R\frac{B_\theta}{B}\widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_r
\end{aligned}$$

where $\widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_r$ is a vector perpendicular on $\mathbf{B}$, and contained in the surface $\widehat{\mathbf{e}}_\perp$. With the factor B_θ/B it is projected along the toroidal direction

components

Note details

The toroidal projection of the divergence of the *anisotropic* pressure (the viscous stress) need to calculate

$$
\begin{aligned}
& R^2 \boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a \\
= & \left[I \frac{\widehat{\mathbf{n}}}{B} - \frac{1}{B} \widehat{\mathbf{n}} \times \boldsymbol{\nabla}\psi \, \frac{1}{2\pi} \frac{\partial \chi}{\partial \psi} \right] \cdot \boldsymbol{\nabla} \cdot \left[\left(p_{\|a} - p_{\perp a} \right) \left(\widehat{\mathbf{n}}\widehat{\mathbf{n}} - \frac{1}{3} \mathbf{I} \right) \right]
\end{aligned}
$$

Here one should start by taking out the operator $\boldsymbol{\nabla}$ and finding a supplementary term that is shown to be zero as contraction of a symmetric and antisymmetric tensor (**Honda**).

We expand. The first term is

$$
\begin{aligned}
& I \frac{\widehat{\mathbf{n}}}{B} \cdot \boldsymbol{\nabla} \cdot \left[\left(p_{\|a} - p_{\perp a} \right) \left(\widehat{\mathbf{n}}\widehat{\mathbf{n}} - \frac{1}{3} \mathbf{I} \right) \right] \\
= & \frac{I}{B} n_\alpha \partial_\beta \left[\left(p_{\|a} - p_{\perp a} \right) \left(n_\beta n_\alpha - \frac{1}{3} \delta_{\beta\alpha} \right) \right]
\end{aligned}
$$

The tensor

$$
n_\beta n_\alpha - \frac{1}{3} \delta_{\beta\alpha}
$$

is symmetric.

$$
n_\alpha \partial_\beta \left(n_\beta n_\alpha - \frac{1}{3} \delta_{\beta\alpha} \right) = n_\alpha n_\alpha \left(\partial_\beta n_\beta \right) + n_\alpha n_\beta \left(\partial_\beta n_\alpha \right)
$$

for the scond term in this line

$$
n_\alpha n_\beta \left(\partial_\beta n_\alpha \right)
$$

we consider

$$
\begin{aligned}
n_\alpha n_\alpha &= \left(\widehat{\mathbf{n}} \right)^2 = 1 \\
\partial_\beta \left(n_\alpha^2 \right) &= 2 n_\alpha \left(\partial_\beta n_\alpha \right) = 0 \quad \text{for any } \beta \\
\text{then } n_\alpha n_\beta \left(\partial_\beta n_\alpha \right) &= n_\beta \left[n_\alpha \left(\partial_\beta n_\alpha \right) \right] = 0
\end{aligned}
$$

and it remains

$$n_\alpha \partial_\beta \left(n_\beta n_\alpha - \frac{1}{3}\delta_{\beta\alpha} \right) = \partial_\beta n_\beta$$

Returning

$$\begin{aligned}
& \frac{I}{B} n_\alpha \partial_\beta \left[\left(p_{\|a} - p_{\perp a} \right) \left(n_\beta n_\alpha - \frac{1}{3}\delta_{\beta\alpha} \right) \right] \\
= & \left[\frac{I}{B} n_\alpha \partial_\beta \left(p_{\|a} - p_{\perp a} \right) \right] \left(n_\beta n_\alpha - \frac{1}{3}\delta_{\beta\alpha} \right) \\
& + \frac{I}{B} \left(p_{\|a} - p_{\perp a} \right) \left[n_\alpha \partial_\beta \left(n_\beta n_\alpha - \frac{1}{3}\delta_{\beta\alpha} \right) \right]
\end{aligned}$$

consider now from the first term

$$\sum_\alpha \sum_\beta n_\alpha \left(n_\beta n_\alpha - \frac{1}{3}\delta_{\beta\alpha} \right) = \frac{2}{3} \sum_\beta n_\beta$$

or

$$\begin{aligned}
& \frac{I}{B} n_\alpha \partial_\beta \left[\left(p_{\|a} - p_{\perp a} \right) \left(n_\beta n_\alpha - \frac{1}{3}\delta_{\beta\alpha} \right) \right] \\
= & \frac{I}{B} \partial_\beta \left(p_{\|a} - p_{\perp a} \right) \times \frac{2}{3} n_\beta \\
& + \frac{I}{B} \left(p_{\|a} - p_{\perp a} \right) \partial_\beta n_\beta
\end{aligned}$$

The second term is related with

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\
\boldsymbol{\nabla} \cdot \left(B \widehat{\mathbf{n}} \right) &= \nabla_\| B + B \boldsymbol{\nabla} \cdot \left(\widehat{\mathbf{n}} \right) = \nabla_\| B + B \boldsymbol{\nabla} \cdot \left(\frac{\mathbf{B}}{B} \right) \\
&= \nabla_\| B + B \left(-\frac{1}{B} \nabla_\| B \right) + \frac{1}{B} \boldsymbol{\nabla} \cdot \mathbf{B} = 0 \\
\text{then} \quad \boldsymbol{\nabla} \cdot \widehat{\mathbf{n}} &= -\frac{1}{B} \nabla_\| B = -\nabla_\| \ln B
\end{aligned}$$

$$\partial_\beta n_\beta = -\nabla_\| \ln B$$

There are two contributions

$$\begin{aligned}
& \frac{I}{B} \partial_\beta \left(p_{\|a} - p_{\perp a} \right) \times \frac{2}{3} n_\beta \\
= & \frac{2I}{3B} \nabla_\| \left(p_{\|a} - p_{\perp a} \right)
\end{aligned}$$

and

$$
\begin{aligned}
& \frac{I}{B}\left(p_{\|a}-p_{\perp a}\right)\partial_\beta n_\beta \\
= \quad & -\frac{I}{B}\left(p_{\|a}-p_{\perp a}\right)\nabla_{\|}\ln B
\end{aligned}
$$

Summarizing the first term is

$$
\begin{aligned}
& I\frac{\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}\cdot\left[\left(p_{\|a}-p_{\perp a}\right)\left(\widehat{\mathbf{n}}\widehat{\mathbf{n}}-\frac{1}{3}\mathbf{I}\right)\right] \\
= \quad & \frac{2I}{3B}\nabla_{\|}\left(p_{\|a}-p_{\perp a}\right) \\
& -\frac{I}{B}\left(p_{\|a}-p_{\perp a}\right)\nabla_{\|}\ln B
\end{aligned}
$$

Putting together

$$
\begin{aligned}
& R^2\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_a \\
= \quad & \left[I\frac{\widehat{\mathbf{n}}}{B}-\frac{1}{B}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\psi\;\frac{1}{2\pi}\frac{\partial\chi}{\partial\psi}\right]\cdot\boldsymbol{\nabla}\cdot\left[\left(p_{\|a}-p_{\perp a}\right)\left(\widehat{\mathbf{n}}\widehat{\mathbf{n}}-\frac{1}{3}\mathbf{I}\right)\right] \\
= \quad & \frac{2I}{3B}\nabla_{\|}\left(p_{\|a}-p_{\perp a}\right)-\frac{I}{B}\left(p_{\|a}-p_{\perp a}\right)\nabla_{\|}\ln B \\
& +\boldsymbol{\nabla}\cdot\left[\frac{1}{2\pi}\frac{\partial\chi}{\partial\psi}\left(p_{\|a}-p_{\perp a}\right)\frac{1}{B}\boldsymbol{\nabla}\psi\times\widehat{\mathbf{n}}\;\right]
\end{aligned}
$$

Taking the first term to be small to order ε^2, we have

$$
\begin{aligned}
& R^2\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_a \qquad \text{(toroidal)} \\
= \quad & \boldsymbol{\nabla}\cdot\left[\frac{1}{2\pi}\frac{\partial\chi}{\partial\psi}\left(p_{\|a}-p_{\perp a}\right)\frac{1}{B}\boldsymbol{\nabla}\psi\times\widehat{\mathbf{n}}\;\right]
\end{aligned}
$$

The content of the vector field on which the *divergence* is applied

$$
\left(p_{\|a}-p_{\perp a}\right)\frac{1}{B}\boldsymbol{\nabla}\psi\times\widehat{\mathbf{n}}\approx\frac{RB_\theta}{B}\left(p_{\|a}-p_{\perp a}\right)\widehat{\mathbf{e}}_\perp
$$

End details

The result

$$
R^2\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_a=\boldsymbol{\nabla}\cdot\left(\frac{1}{2\pi}\frac{\partial\chi}{\partial\psi}\frac{p_{\|a}-p_{\perp a}}{B}\;\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\psi\right)
$$

It is very important that this is a *divergence*. Because parforming surface average the divergence will give *zero*, as integrating an exact differential on a periodic domain

$$\left\langle R^2 \boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a \right\rangle = 0$$

Recall that the tensor of pressure

$$\begin{aligned} \mathbf{P}_a &= p_{\| a}\, \widehat{\mathbf{n}}\, \widehat{\mathbf{n}} + p_{\perp a} \left(\mathbf{I} - \widehat{\mathbf{n}}\, \widehat{\mathbf{n}} \right) \\ &\quad + O\left(\varepsilon^2\right) \end{aligned}$$

or

$$\mathbf{P}_a = p_a \mathbf{I} + \boldsymbol{\pi}_a$$

$$\begin{aligned} p_a &= \frac{1}{3} \mathrm{Tr}\left(\mathbf{P}_a\right) \\ &= \frac{1}{3}\left(p_{\| a} + 2 p_{\perp a}\right) \\ &\quad \text{scalar pressure} \end{aligned}$$

$$\begin{aligned} \boldsymbol{\pi}_a &= \left(p_{\| a} - p_{\perp a}\right)\left(\widehat{\mathbf{n}}\, \widehat{\mathbf{n}} - \frac{1}{3}\mathbf{I}\right) \\ &\quad \text{pressure anisotropy} \end{aligned}$$

From the toroidal component of the Faraday law ($\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$)

$$\frac{1}{q(\psi)} \frac{1}{2\pi} \frac{dV}{d\psi} \left\langle \frac{I}{R^2} \right\rangle = \frac{\partial \chi}{\partial \psi}$$

since

$$\frac{d\phi}{d\psi} = \frac{1}{2\pi} \frac{dV}{d\psi} \left\langle \frac{I}{R^2} \right\rangle$$

where ϕ is the toroidal flux function and

$$\frac{d\phi}{d\chi} = q(\psi)$$

Now we *average* over the flux surface

$$\left\langle R^2 \boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a \right\rangle = 0$$

to order $O\left(\varepsilon^2\right)$. This results from the equations (above) where the toroidal projection of the divergence of the pressure anisotropic part is a divergence of a vector. Then the surface average acts like an *annihilator* and suppresses this term.

NOTE

This is derived also in **Honda** where the residual anisotropy is

$$\begin{aligned} \boldsymbol{\pi}_a &= \boldsymbol{\pi}_a^{(1)} \\ &\quad + \boldsymbol{\pi}_a^{(2)} \ \text{(higher order off-diagonal)} \\ &\qquad\qquad \text{(perpendicular viscosity)} \end{aligned}$$

and

$$\begin{aligned} \left\langle R^2 \boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a^{(1)} \right\rangle &= 0 \\ &\quad \text{toroidal} \\ &\quad \text{for any species } a \end{aligned}$$

What remains is the residual viscosity $\boldsymbol{\pi}_a^{(2)}$ which leads to the *damping* of the surface-averaged toroidal momentum.

This will remain in the expression of the radially projected *current*

$$\begin{aligned} \langle \mathbf{j} \cdot \boldsymbol{\nabla}\psi \rangle &= \sum_{a=e,i} \left[m_a n_a \frac{\partial}{\partial t} \langle R u_{a\varphi} \rangle \ \text{polarization} \right. \\ &\quad + \left\langle R^2 \boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a^{(2)} \right\rangle \ \text{perp. viscosity} \\ &\quad \left. - \left\langle R^2 \boldsymbol{\nabla}\varphi \cdot \mathbf{R}_a^{non-Coulomb} \right\rangle \right] \ \text{non-Coulomb friction} \end{aligned}$$

This level of analysis requires to examine the balance between

- polarization, since a radial current leads to charge separation and it is given by the time variation of the poloidal rotation
- perpendicular viscosity, transfer of momentum from other directions to the parallel direction
- frictio, of turbulence origin

END

The meaning of this vanishing is: The angular momentum is conserved in each surface.

Transfer of angular momentum exists at higher than ε^2 order, due to new terms in the structure of $\boldsymbol{\pi}_a$. [see **Honda** term $\boldsymbol{\pi}_a^{(2)}$].

In order to use this conservation equation the equations of momentum and heat balance are multiplied by

$$R^2 \boldsymbol{\nabla}\varphi = R\widehat{\mathbf{e}}_\varphi$$

and averaged over surface $\langle\rangle$.

Take the momentum conservation equation

$$\begin{aligned} n_a m_a \left(\frac{\partial \mathbf{u}_a}{\partial t} + (\mathbf{u}_a \cdot \boldsymbol{\nabla}) \mathbf{u}_a \right) &= -\boldsymbol{\nabla} p_a - \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a \\ &\quad + n_a e_a \mathbf{E} \\ &\quad + n_a e_a \mathbf{u}_a \times \mathbf{B} \\ &\quad + \mathbf{F}_{a1} \end{aligned}$$

This equation is projected on the toroidal direction by scalar multiplication with

$$R^2 \boldsymbol{\nabla}\varphi \cdot$$

For the static case all LHS is zero

$$\begin{aligned} 0 &= R^2 \boldsymbol{\nabla}\varphi \cdot (-\boldsymbol{\nabla} p_a) - R^2 \boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a \\ &\quad + n_a e_a \; R^2 \boldsymbol{\nabla}\varphi \cdot \mathbf{E} \\ &\quad + n_a e_a R^2 \boldsymbol{\nabla}\varphi \cdot (\mathbf{u}_a \times \mathbf{B}) \\ &\quad + R^2 \boldsymbol{\nabla}\varphi \cdot \mathbf{F}_{a1} \end{aligned}$$

Now we take the surface average.

The first term

$$\begin{aligned} \left\langle R^2 \boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla} p_a \right\rangle &= 0 \\ & \quad \text{axisymmetry} \end{aligned}$$

The second term is known

$$\begin{aligned} \left\langle R^2 \boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a \right\rangle &= 0 \\ \left\langle R\widehat{\mathbf{e}}_\varphi \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a \right\rangle &= 0 \\ & \quad \text{average of a divergence} \end{aligned}$$

this comes from this term being a exact divergence (in lowest order in $\delta = \rho/L$. In the second order $\boldsymbol{\pi}^{(2)}$ it is the "perpendicular viscosity" the factor that produces damping of the flux surface averaged toroidal momentum **Honda**)

The third term is $\mathbf{j}\times\mathbf{B}$ projected on the toroidal direction;
This - we expect, will contain $\mathbf{j}_r\times\mathbf{B}_\theta$ which is along toroidal direction.
Then we multiply by $R^2\boldsymbol{\nabla}\varphi$ and surface-average.

$$\begin{aligned}
\left\langle n_a e_a R^2\boldsymbol{\nabla}\varphi\cdot(\mathbf{u}_a\times\mathbf{B})\right\rangle &= n_a e_a\left\langle R^2\boldsymbol{\nabla}\varphi\cdot\left(\mathbf{u}_a\times\left[\frac{\chi'}{2\pi}\boldsymbol{\nabla}\varphi\times\boldsymbol{\nabla}\psi+I\boldsymbol{\nabla}\varphi\right]\right)\right\rangle\\
&= n_a e_a\frac{\chi'}{2\pi}\left\langle R^2\boldsymbol{\nabla}\varphi\cdot\left[\boldsymbol{\nabla}\varphi\ \left(\mathbf{u}_a\cdot\boldsymbol{\nabla}\psi\right)-\boldsymbol{\nabla}\psi\ \left(\mathbf{u}_a\cdot\boldsymbol{\nabla}\varphi\right)\right]\right\rangle\\
&= n_a e_a\frac{\chi'}{2\pi}\left\langle R^2\left|\boldsymbol{\nabla}\varphi\right|^2\ u_a^\psi\right\rangle
\end{aligned}$$

since the last term from the paranthesis in the second line is zero $\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\psi=0$.
We reintroduce the notation as radial (ψ) flux

$$\begin{aligned}
\left\langle n_a e_a R^2\boldsymbol{\nabla}\varphi\cdot(\mathbf{u}_a\times\mathbf{B})\right\rangle &= n_a e_a\frac{\chi'}{2\pi}\left\langle R^2\left|\boldsymbol{\nabla}\varphi\right|^2\ u_a^\psi\right\rangle\\
&= e_a n_a\frac{\chi'}{2\pi}\left\langle u_a^\psi\right\rangle\\
&= e_a\frac{\chi'}{2\pi}\Gamma_a^\psi
\end{aligned}$$

and we will remember that it is *averged on surface*.

Returning to the equation of static force balance, we add this term to the *electric* and the *friction* terms

$$0=\left\langle n_a e_a\ R^2\boldsymbol{\nabla}\varphi\cdot\mathbf{E}+R^2\boldsymbol{\nabla}\varphi\cdot\mathbf{F}_{a1}\right\rangle+e_a\frac{\chi'}{2\pi}\Gamma_a^\psi$$

or

$$\Gamma_a^\psi=-\frac{1}{e_a}\frac{2\pi}{\chi'}\left\langle R^2\boldsymbol{\nabla}\varphi\cdot\left(\mathbf{F}_{a1}+e_a n_a\mathbf{E}\right)\right\rangle$$

The results for the FULL radial fluxes [classical, $\sim$(gyroangle) ζ plus neoclassical $\sim\mathbf{v}_D$]

$$\text{full (radial)}\quad \Gamma_a^\psi=-\frac{2\pi}{\chi'}\frac{1}{e_a}\left\langle R^2\boldsymbol{\nabla}\varphi\cdot\left(\mathbf{F}_{a1}+e_a n_a\mathbf{E}\right)\right\rangle$$

$$\text{full (radial)}\ \ \frac{q_a^\psi}{T_a}=-\frac{2\pi}{\chi'}\frac{1}{e_a}\left\langle R^2\boldsymbol{\nabla}\varphi\cdot\mathbf{F}_{a2}\right\rangle$$

which express the *radial flows* through the *toroidal friction forces* $\mathbf{F}_{1,2}$.

These equations for the radial fluxes Γ_a^ψ and q_a^ψ/T_a contain the *classical* parts, that are $\widetilde{\Gamma}_a^\psi$ and $\widetilde{q}_a^\psi/T_a$ dependent of the *gyroangle*. They can be subtracted, leaving the *neoclassical* flux-friction relations.

$$\begin{aligned}
\text{neoclassical } \overline{\Gamma}_a^\psi &= \Gamma_a^\psi - \widetilde{\Gamma}_a^\psi \\
&= -\frac{2\pi}{\chi'} \left\langle \frac{1}{m_a \Omega_a} \, I \, \left(F_{a1}^{\parallel} + e_a n_a E_{\parallel}^A \right) \right\rangle
\end{aligned}$$

$$\begin{aligned}
\text{neoclassical } \frac{\overline{q}_a^\psi}{T_a} &= \frac{q_a^\psi}{T_a} - \frac{\widetilde{q}_a^\psi}{T_a} \\
&= -\frac{2\pi}{\chi'} \left\langle \frac{1}{m_a \Omega_a} \, I \, F_{a2}^{\parallel} \right\rangle
\end{aligned}$$

Now it is very clear that the *neoclassical radial fluxes* $\overline{\Gamma}_a^\psi$ are determined by the *parallel friction forces*, $F_{1,2}^{\parallel}$.

Explanation of **HS**

The *conservation of the angular momentum* is

$$\left\langle R^2 \boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a \right\rangle = 0$$

is proved using the *geometric relation*

$$R^2 \boldsymbol{\nabla}\varphi = I \frac{\widehat{\mathbf{n}}}{B} - \frac{\widehat{\mathbf{n}} \times \boldsymbol{\nabla}\psi}{B} \frac{1}{2\pi} \frac{d\chi}{d\psi}$$

which expresses the toroidal direction through parallel and perpendicular directions.

Note again that this is true in the first order in $\delta = \rho/L$. Higher order leads to damping of toroidal rotation, $\boldsymbol{\pi}^{(2)}$ **Honda**. **End.**

Replacing the factor $R^2\boldsymbol{\nabla}\varphi$ in the conservation equation by the geometrical expression above,

$$\left\langle \left(I \frac{\widehat{\mathbf{n}}}{B} - \frac{\widehat{\mathbf{n}} \times \boldsymbol{\nabla}\psi}{B} \frac{1}{2\pi} \frac{d\chi}{d\psi} \right) \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a \right\rangle = 0$$

This shows a connection between *parallel viscous stress* and the *perpendicular viscous stress*

$$\begin{aligned}
\left\langle I \, \frac{\widehat{\mathbf{n}}}{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a \right\rangle &= \frac{1}{2\pi} \frac{d\chi}{d\psi} \left\langle \frac{\widehat{\mathbf{n}} \times \boldsymbol{\nabla}\psi}{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a \right\rangle \\
\text{(parallel)} &\sim \text{(perpendicular)}
\end{aligned}$$

Or, this provides an expression for the *perpendicular* part

$$\left\langle \frac{1}{m_a \Omega_a} \frac{\widehat{\mathbf{n}} \times \boldsymbol{\nabla} \psi}{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a \right\rangle$$
$$\left(\text{part of } \overline{\Gamma}_a^{\psi}\right)$$

which occurs in the equation of balance of momentum on *radial* ψ direction, $\overline{\Gamma}_a^{\psi}$, - in terms of the *parallel* viscous stress

$$\left\langle I \ \frac{\widehat{\mathbf{n}}}{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a \right\rangle$$

In other words, we have expressed the perpendicular NEOCLASSICAL flux $\overline{\Gamma}_a^{\psi}$ *by the parallel viscous stresses*, $F_{a1}^{\|}$ and $F_{a2}^{\|}$.

We must correctly appreciate this.

The *friction forces* $\mathbf{F}_{1a}$ and $\mathbf{F}_{a2}$ are normally present in the *gyro-phase* part of the radial fluxes.

By contrast, the *neoclassical* radial fluxes are expressed in terms of *drift velocity* $\mathbf{v}_{Da}$.

Now we find the *neoclassical* radial fluxes expressed in terms of the *friction forces*, as if they were gyro-phase dependent.

The reason is - of course, - the connection between the components of the flux, connection that is provided by the conservation of the canonical angular momentum.

In conclusion, we have expressed the *radial* particle and heat diffusions Γ_a^{ψ} and q_a^{ψ}/T_a in terms of *parallel friction forces*, $\mathbf{F}_{1a} \cdot \mathbf{B}$ and $\mathbf{F}_{2a} \cdot \mathbf{B}$.

In order to find concrete expressions for the radial fluxes we need precise formulas for the friction forces.

This requires the distribution function.

Returning

$$\text{neoclassical } \ \overline{\Gamma}_a^{\psi} = -\frac{2\pi}{\chi'} \left\langle \frac{1}{m_a \Omega_a} \ I \ \left(F_{a1}^{\|} + e_a n_a E_{\|}^{A} \right) \right\rangle$$

and further we know that the *friction forces* can be expressed by the pressure anisotropy

$$\left\langle \left[F_{a1}^{\|} + e_a n_a E_{\|}^{A} \right] B \right\rangle = \left\langle \left(p_{\perp a} - p_{\| a} \right) \ \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} B \right\rangle + \left\langle \widetilde{n}_a e_a \nabla_{\|} \Phi \right\rangle$$

and

$$\left\langle F^{\|}_{a2}\ B\right\rangle = \left\langle \left(\Theta_{\perp a} - \Theta_{\|a}\right)\ \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} B\right\rangle + \frac{5}{2}\left\langle \widetilde{n}_a\ \mathbf{B} \cdot \boldsymbol{\nabla} T_a\right\rangle$$

Now the *neoclassical radial fluxes*, initially expressed through the *parallel friction forces* are now expressed through the surface average of the pressure anisotropy.

7.3 Flows

Consider the particle flow and the heat flow.

Define for the species a,

$$\begin{aligned}\mathbf{V}^a &= V_{\|}^a \widehat{\mathbf{n}} + \mathbf{V}_{\perp}^a \\ &\quad \text{flow velocity}\end{aligned}$$

$$\begin{aligned}\mathbf{Q}^a &= -\frac{2}{5}\frac{\mathbf{q}^a}{p_a} \\ &\quad \text{equivalent heat flow velocity}\end{aligned}$$

The perpendicular components of these flows are

$$\begin{aligned}\mathbf{V}_{\perp}^a &= \frac{1}{n_a}\frac{1}{e_a B}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p_a + \frac{\mathbf{E}\times\mathbf{B}}{B^2} \quad \text{(diamagnetic + electric)} \\ \mathbf{Q}_{\perp}^a &= \frac{1}{e_a B}\widehat{\mathbf{n}} \times p_a \boldsymbol{\nabla} T_a\end{aligned}$$

Note that in **Hirschman Sigmar (HS)** the formulas are

$$\mathbf{V}_{\perp}^a \rightarrow \mathbf{u}_{\perp}^{a(1)} = \frac{1}{n_a}\frac{1}{m_a \Omega_{ca}}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p_a + \frac{\mathbf{E}\times\widehat{\mathbf{n}}}{B}$$

$$\mathbf{q}_{\perp}^{a(1)} = \frac{1}{m_a \Omega_{ca}}\widehat{\mathbf{n}} \times \frac{5}{2} p_a \boldsymbol{\nabla} T_a$$

and the flux of heat $\mathbf{q}_{\perp,\|}^{a(1)}$ are used instead of $\mathbf{Q}_{\perp}^a$ in **KimDiamondGroebner**.

The upperscript (1) is retained since there is an expansion in the neoclassical small parameter ρ_θ / L_n. The order 0 is the Maxwellian and the order 1 (f_{1a} in **Sugama**) is the *neoclassical* correction.

End

The poloidal components are obtained by dot-multiplying with

$$\frac{\boldsymbol{\nabla}\theta}{\boldsymbol{\nabla}\theta\cdot\mathbf{B}}.$$

Then

$$\frac{\boldsymbol{\nabla}\theta}{\boldsymbol{\nabla}\theta\cdot\mathbf{B}}\cdot\left(\mathbf{V}^a=V_{\parallel}^a\widehat{\mathbf{n}}+\mathbf{V}_{\perp}^a\right)$$

and define

$$\begin{aligned} U_\theta^a &= \frac{\boldsymbol{\nabla}\theta}{\boldsymbol{\nabla}\theta\cdot\mathbf{B}}\cdot\mathbf{V}^a \\ &\to \frac{V_\theta^a}{B_\theta} \end{aligned}$$

and

$$\begin{aligned} H_\theta^a &= \frac{\boldsymbol{\nabla}\theta}{\boldsymbol{\nabla}\theta\cdot\mathbf{B}}\cdot\mathbf{Q}^a \\ &\to \frac{Q_\theta^a}{B_\theta} \end{aligned}$$

NOTE
In Hirshman Sigmar the two variables on θ are

$$u_{a\theta}(\psi)=\frac{\boldsymbol{\nabla}\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}\cdot\mathbf{u}_a^{(1)}$$

$$q_{a\theta}(\psi)=\frac{\boldsymbol{\nabla}\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}\cdot\mathbf{q}_a^{(1)}$$

END.

For the perpendicular term, $\mathbf{V}_\perp^a$, after multiplication we have

$$\frac{\boldsymbol{\nabla}\theta}{\boldsymbol{\nabla}\theta\cdot\mathbf{B}}\cdot\left(\mathbf{V}_\perp^a=\frac{1}{n_a}\frac{1}{e_aB}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}p_a+\frac{\mathbf{E}\times\mathbf{B}}{B^2}\right)$$

$$\frac{\boldsymbol{\nabla}\theta}{\boldsymbol{\nabla}\theta\cdot\mathbf{B}}\cdot\mathbf{V}_\perp^a = -\frac{1}{B^2}I\frac{T_a}{e_a}\left(\frac{\partial}{\partial\psi}\ln p_a+\frac{e_a}{T_a}\frac{\partial}{\partial\psi}\phi\right)$$

In **Kim et al 1991** it is introduced a new function

$$\overline{U}_\theta^a\equiv -I\frac{T_a}{e_a}\left(\frac{\partial}{\partial\psi}\ln p_a+\frac{e_a}{T_a}\frac{\partial}{\partial\psi}\phi\right)$$

such that

$$\frac{\boldsymbol{\nabla}\theta}{\boldsymbol{\nabla}\theta\cdot\mathbf{B}}\cdot\mathbf{V}_\perp^a=\frac{\overline{U}_\theta^a}{B^2}$$

The last term to be calculated is the poloidal projection of the parallel component

$$\begin{aligned}
\frac{\nabla\theta}{\nabla\theta\cdot\mathbf{B}}\cdot V_{\parallel}^{a}\widehat{\mathbf{n}} &= V_{\parallel}^{a}\frac{1}{B_{\theta}}\widehat{\mathbf{e}}_{\theta}\cdot\widehat{\mathbf{n}} \\
&= -V_{\parallel}^{a}\frac{1}{B_{\theta}}\frac{B_{\theta}}{B} \\
&= -V_{\parallel}^{a}\frac{1}{B}
\end{aligned}$$

Now we have calculated all three terms

$$\begin{aligned}
& \frac{\nabla\theta}{\nabla\theta\cdot\mathbf{B}}\cdot\left(\mathbf{V}^{a}=V_{\parallel}^{a}\widehat{\mathbf{n}}+\mathbf{V}_{\perp}^{a}\right) \\
\frac{\nabla\theta}{\nabla\theta\cdot\mathbf{B}}\cdot\mathbf{V}^{a} &= \frac{\nabla\theta}{\nabla\theta\cdot\mathbf{B}}\cdot V_{\parallel}^{a}\widehat{\mathbf{n}}+\frac{\nabla\theta}{\nabla\theta\cdot\mathbf{B}}\cdot\mathbf{V}_{\perp}^{a} \\
-U_{\theta}^{a} &= -V_{\parallel}^{a}\frac{1}{B}+\frac{\overline{U}_{\theta}^{a}}{B^{2}}
\end{aligned}$$

The final form is

$$V_{\parallel}^{a}\frac{1}{B}=U_{\theta}^{a}+\frac{\overline{U}_{\theta}^{a}}{B^{2}}$$

NOTE
that in **HS** Eq.(3.43) is

$$u_{a\parallel}=u_{a\theta}\left(\psi\right)B+V_{a1}$$

END

Analogously, for the heat flow, we start from the relations

$$\mathbf{Q}^{a}=-\frac{2}{5}\frac{\mathbf{q}^{a}}{p_{a}}$$

and the perpendicular projection

$$\mathbf{Q}_{\perp}^{a}=\frac{1}{e_{a}B}\widehat{\mathbf{n}}\times\nabla T_{a}$$

There is also the *parallel* component of the flow

$$Q_{\parallel}^{a}=-\frac{2}{5}\frac{q_{\parallel}^{a}}{p_{a}}$$

for

$$Q_{\theta}^{a}=\frac{B_{\theta}}{B}Q_{\parallel}^{a}$$

and the full expression of the heat velocity is

$$\mathbf{Q}^a = \frac{1}{e_a B}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} T_a - \frac{2}{5}\frac{q_\parallel^a}{p_a}\widehat{\mathbf{n}}$$

Now we want to determine the poloidal projection by dot-multiplying as above

$$\frac{\boldsymbol{\nabla}\theta}{\boldsymbol{\nabla}\theta \cdot \mathbf{B}} \cdot \left(\mathbf{Q}^a = \frac{1}{e_a B}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} T_a - \frac{2}{5}\frac{q_\parallel^a}{p_a}\widehat{\mathbf{n}} \right)$$

The left hand side is a definition

$$\frac{\boldsymbol{\nabla}\theta}{\boldsymbol{\nabla}\theta \cdot \mathbf{B}} \cdot \mathbf{Q}^a \equiv -H_\theta^a$$

The right hand side

$$\frac{\boldsymbol{\nabla}\theta}{\boldsymbol{\nabla}\theta \cdot \mathbf{B}} \cdot \left(\frac{1}{e_a B}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} T_a \right) = \frac{1}{B^2}\frac{T_a}{e_a} I \frac{\partial}{\partial \psi} \ln T_a$$

It is introduced as above the notation

$$\frac{\boldsymbol{\nabla}\theta}{\boldsymbol{\nabla}\theta \cdot \mathbf{B}} \cdot \mathbf{Q}_\perp^a \overset{def}{=} \frac{\overline{Q}^a}{B^2}$$

or

$$\overline{Q}^a = \frac{T_a}{e_a} I \frac{\partial}{\partial \psi} \ln T_a$$

Now we have all terms and write the dot-product of the equation for $\mathbf{Q}^a$ as

$$\begin{aligned} -H_\theta^a &= \frac{\boldsymbol{\nabla}\theta}{\boldsymbol{\nabla}\theta \cdot \mathbf{B}} \cdot \mathbf{Q}^a \\ &= -\frac{2}{5}\frac{q_\parallel^a}{p_a} + \frac{T_a}{e_a} I \frac{\partial}{\partial \psi} \ln T_a \\ &= -\frac{Q_\parallel^a}{B} + \frac{\overline{Q}^a}{B^2} \end{aligned}$$

$$\frac{Q_\parallel^a}{B} = H_\theta^a + \frac{\overline{Q}^a}{B^2}$$

The *equation of continuity* for incompressibility

$$\boldsymbol{\nabla} \cdot (n_a \mathbf{V}^a) = 0$$

and

$$\boldsymbol{\nabla} \cdot \mathbf{q}^a = 0$$

where

$$\nabla \cdot \mathbf{A} = \frac{1}{\sqrt{g}} \left[\frac{\partial}{\partial \psi} \left(\sqrt{g}\ \mathbf{A} \cdot \nabla \psi \right) + \frac{\partial}{\partial \theta} \left(\sqrt{g}\ \mathbf{A} \cdot \nabla \theta \right) \right]$$

are used to show that

$$U_{\theta}^{a} = \frac{V_{\theta}^{a}}{B_{\theta}}$$

and

$$H_{\theta}^{a} = -\frac{Q_{\theta}^{a}}{B_{\theta}}$$

are only functions of surface ψ.

The poloidal components of the flow $[\sim (\psi, \theta)]$ *divided by the poloidal magnetic field are functions of only the surface label (coordinate)* ψ.

From **Hirshamn Sigmar review**.

In the following (1) means order 1 in ε, as we will find it from the order ε perturbation to the Maxwellian distribution function, solution of the Fokker Planck kinetic equation.

Order (1) is for the neoclassical effects.

The neoclassical flows

$$\begin{aligned} \mathbf{u}_{a}^{(1)} &= \mathbf{u}_{\perp a}^{(1)} + u_{\| a}\ \widehat{\mathbf{n}} \\ \mathbf{q}_{a}^{(1)} &= \mathbf{q}_{\perp a}^{(1)} + q_{\| a}\ \widehat{\mathbf{n}} \end{aligned}$$

The neoclassical perpendicular fluid velocity consists of *diamagnetic* part and *electric* part

$$n\mathbf{u}_{\perp a}^{(1)} = \frac{1}{m_a \Omega_a} \widehat{\mathbf{n}} \times \nabla p_a(\psi) + n_a \frac{\mathbf{E} \times \widehat{\mathbf{n}}}{B}$$

$$\mathbf{q}_{\perp a}^{(1)} = \frac{5}{2} p_a \frac{1}{m_a \Omega_a} \widehat{\mathbf{n}} \times \nabla T_a(\psi)$$

At this level ($\sim \varepsilon$) the flows are tied to surfaces.

This means that

$$\mathbf{u}_{a}^{(1)} \cdot \nabla \psi = 0 \quad \text{no flow transversal on surfaces (radial)}$$

$$\mathbf{q}_{a}^{(1)} \cdot \nabla \psi = 0$$

But the NBI ions have anisotropic pressure and traverse the surfaces.

Next, one takes into account the *continuity*

$$\begin{aligned}\boldsymbol{\nabla}\cdot\left(n_a\mathbf{u}_a^{(1)}\right) &= 0\\ \boldsymbol{\nabla}\cdot\mathbf{q}_a^{(1)} &= 0\end{aligned}$$

The operator of divergence

$$\boldsymbol{\nabla}\cdot\mathbf{A}=\frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial\psi}\left(\sqrt{g}\mathbf{A}\cdot\boldsymbol{\nabla}\psi\right)+\frac{\partial}{\partial\theta}\left(\sqrt{g}\mathbf{A}\cdot\boldsymbol{\nabla}\theta\right)\right]$$

We can express

$$\boldsymbol{\nabla}\cdot\left(n_a\mathbf{u}_a^{(1)}\right)=\frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial\psi}\left(\sqrt{g}\;n_a\mathbf{u}_a^{(1)}\cdot\boldsymbol{\nabla}\psi\right)+\frac{\partial}{\partial\theta}\left(\sqrt{g}\;n_a\mathbf{u}_a^{(1)}\cdot\boldsymbol{\nabla}\theta\right)\right]$$

The first term is 0. Then

$$\boldsymbol{\nabla}\cdot\left(n_a\mathbf{u}_a^{(1)}\right)=\frac{1}{\sqrt{g}}\frac{\partial}{\partial\theta}\left(\sqrt{g}\;n_a\mathbf{u}_a^{(1)}\cdot\boldsymbol{\nabla}\theta\right)=0$$

Neglecting the poloidal variation of n_a, one has

$$\frac{\partial}{\partial\theta}\left(\frac{r}{B_\theta}\mathbf{u}_a^{(1)}\cdot\boldsymbol{\nabla}\theta\right)=0$$

which means that inside the paranthesis does NOT depend on θ, only depends on ψ, it is constant on surfaces

$$\frac{\mathbf{u}_a^{(1)}\cdot\boldsymbol{\nabla}\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}\equiv u_{a\theta}\left(\psi\right)$$

$$\frac{\mathbf{q}_a^{(1)}\cdot\boldsymbol{\nabla}\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}=q_{a\theta}\left(\psi\right)$$

These are two NEW fields, flows in poloidal direction that do not depend on θ but only on the surface.

New variables are defined instead of the poloidal projections

$$u_{a\theta}\left(\psi\right)=\frac{\boldsymbol{\nabla}\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}\cdot\mathbf{u}_a^{(1)}$$

$$q_{a\theta}\left(\psi\right)=\frac{\boldsymbol{\nabla}\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}\cdot\mathbf{q}_a^{(1)}$$

Using these new variables for the flow, one can express the *parallel* velocities

$$
\begin{aligned}
u_{\|a} &= V_{1a} + u_{a\theta}(\psi)\ B \\
q_{\|a} &= \frac{5}{2} p_a V_{2a} + q_{a\theta}(\psi)\ B
\end{aligned}
$$

The first term in these expressions are the parallel projection of the *diamagnetic* velocity

$$
\begin{aligned}
V_{1a} &= -\frac{\boldsymbol{\nabla}\theta}{\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta} \cdot \mathbf{u}_{\perp a}^{(1)} \\
&= -\frac{2\pi}{\chi'} I\ \frac{T_a/m_a}{e_a B/m_a} \left(\frac{p_a'}{p_a} + \frac{e_a \Phi'}{T_a} \right) \\
&\quad \text{function of only } \psi
\end{aligned}
$$

$$
\begin{aligned}
V_{2a} &= -\frac{2}{5} \frac{\boldsymbol{\nabla}\theta}{\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta} \cdot \mathbf{q}_{\perp a}^{(1)} \\
&= -\frac{2\pi}{\chi'} I\ \frac{T_a'}{e_a B} \\
&\quad \text{function of only } \psi
\end{aligned}
$$

where

$$
{}' \equiv \frac{\partial}{\partial \psi}
$$

After finding the distribution function we can calculate the friction forces by integrations of the *collision operator*.

The choice consists of adopting a particular expression for the distribution function, which places emphasis on flow velocities, for momentum and for heat.

Then the friction forces will result as expressions in terms of these velocities.

In these formulas there are *coefficients* that are essentially separated formulas.

8 Friction forces

NOTE Hirschman Sigmar review. See also **Hirschman 1977**.

These relations *friction-force* are called in **HS** *constitutive relations*.

From HS, the distribution function

$$f_a = f_a^{(0)} + f_a^{(1)}$$

and the zero order is maxwellian

$$f_a^{(0)} = \frac{n_a(\psi)}{\pi^{3/2} v_{th,a}^3} \exp\left(-\frac{v^2}{v_{th,a}^2}\right)$$

$$v_{th,a} = \left(\frac{2T_a}{m_a}\right)^{1/2}$$

Now it is necessary to find an expression for the distribution function $f_a^{(1)}$ to be used in the velocity-space integrations.

Normally we must write the drift kinetic equation, expand in neoclassical small terms $\delta = \rho/L$ and find the solution as a series. The first term must be the ρ_θ/L correction to the Maxwellian distribution. The next term should be the function (denoted in general) g, which has an equation for circulating particles and is zero for trapped ones. The equation for g includes the collision operator in *pitch-angle*. Then, use this solution to calculate the radial fluxes.(see **Rutherford 1970** and **Connor1973** and **Galeev**).

Alternatively, one assumes that the distribution function is a series in terms of *flows* of momentum and of heat.

How this arises?

The reason is the invariant which results from the *canonical* angular momentum

$$\frac{d}{dt}\left[Rmv_\varphi + R\ eA_\varphi\right] = 0$$

Now we have the connection

$$\psi = R\ A_\varphi$$

and from this it results that

$$\begin{aligned} \psi' &= \psi + \frac{mRv_\varphi}{e} \\ &= \text{dynamical constant} \end{aligned}$$

Another dynamical constant is the energy

$$\epsilon = \frac{v^2}{2} + \frac{e\phi}{m}$$

The distribution function f_j will be function of these invariants.

Returning to **Hirshman Sigmar review** to evaluate the classical cross-field fluxes.

They are due to the part of the distribution function that depends on the gyroangle

$$\widetilde{f}_a$$

in first order in ε.

This is solution of the drift kinetic equation without collisions.

Then $\widetilde{f}_a$ is a function of the *invariants*.

The distribution function is a Maxwellian calculated in the position that results from the conservation

$$\mathbf{x}'_{\perp} = \mathbf{x}_{\perp} + \frac{1}{\Omega}\widehat{\mathbf{n}} \times \mathbf{v}$$

and the difference

$$x'_{\perp} - x_{\perp} \sim \frac{v_{\perp}}{\Omega} \sim \varepsilon$$

is small. The *gyrophase-dependent* part of the distribution function is

$$\widetilde{f}_a = f_a^{(0)}\left(\mathbf{x}'_{\perp}, \epsilon\right) - f_a^{(0)}\left(\mathbf{x}_{\perp}, \varepsilon\right)$$

Then

$$\widetilde{f}_a = \frac{2\mathbf{v}}{v_{th,a}^2} \cdot \left[\mathbf{u}_{\perp a}^{(1)} + \frac{2}{5}\left(x_a^2 - \frac{5}{2}\right)\frac{\mathbf{q}_{\perp a}^{(1)}}{p_a}\right] f_a^{(0)}$$

where

$$\begin{aligned} \mathbf{u}_{\perp a}^{(1)} &= \frac{1}{n_a}\frac{1}{m_a\Omega_a}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p_a(\psi) \qquad \text{(diamagnetic)} \\ &\quad -\widehat{\mathbf{n}} \times \frac{\mathbf{E}}{B} \qquad (E \times B \quad \text{velocity}) \end{aligned}$$

$$\mathbf{q}_{\perp a}^{(1)} = \frac{1}{m_a\Omega_a}\widehat{\mathbf{n}} \times \frac{5}{2} p_a \boldsymbol{\nabla} T_a$$

These expressions appear by expansion of the variables in the Maxwellian that depends on space $\mathbf{x}$.

NOTE

This is what **Rutherford 1970** does:

$$\psi' = \psi + \frac{mRv_{\varphi}}{e}$$

and the Maxwellian is expanded in the small difference

$$\psi' - \psi = \frac{v_{\varphi}}{\Omega} \sim \text{small}$$

First it is adopted a separation of the situations

- without collisions f
- with collisions g

The distribution function is

$$F = f\left(\epsilon, \psi'\right) + g$$

The part which does not include collisions, f, is expanded since the argument is shifted spatially by a small quantity

$$f = f^{(0)} + f^{(1)} + ...$$

where

$$\begin{aligned} f^{(0)} &= f_M\left(\epsilon, \psi\right) \quad \text{Maxwellian} \\ &= \frac{N\left(\psi\right)}{\left(\sqrt{\pi} v_{th}\right)^3} \exp\left(-\frac{\epsilon}{v_{th}^2}\right) \end{aligned}$$

but

$$v_{th} = \sqrt{\frac{T}{m}}$$

$$\begin{aligned} f^{(1)} &= \frac{mRv_{\varphi}}{e} \frac{\partial f^{(0)}\left(\epsilon, \psi\right)}{\partial \psi} \\ & \quad \text{precisely a shift on } \psi \\ &\sim \frac{v_{\varphi}}{B_{\theta}} \frac{\partial f^{(0)}}{\partial r} = \rho_{\theta} \frac{\partial f^{(0)}}{\partial r} \end{aligned}$$

Further, the correction to the distribution function that is due to <u>collisions</u>, g, is expanded in powers of the small ε

$$g = g^{(1)} + g^{(2)} + ...$$

END

NOTE
In **Galeev JETP1971** the zeroth order distribution function, in the presence of an electric field is

$$\begin{aligned} f_a^{(0)} &= \frac{n_a}{\left(\sqrt{\pi} v_{th,a}\right)^3} \exp\left[-\frac{\left(v_\parallel - u_a^{(0)}\right)^2}{2T_a/m_a} - \frac{\mu B_0}{T_a}\right] \\ &\quad \times \left\{1 - \frac{2 v_\parallel u_a^{(0)}}{(2T_a/m_a)} \sum_{j=1}^{\infty} a_j \; L_j^{(3/2)}\left(x_a^2\right)\right\} \end{aligned}$$

The zeroth order flows are

$$\begin{aligned} u_i^{(0)} &= 0 \\ u_e^{(0)} &= u^{(0)} \end{aligned}$$

and further

$$\begin{aligned} f_a^{(1)} &= -\frac{\frac{\mu B_0}{m_a} + v_\parallel^2}{R}\left[\Theta \frac{\partial}{\partial v_\parallel} + \frac{1}{\Omega_a}\frac{\partial}{\partial r}\right] f_a^{(0)} \\ &\quad \times \left\{\mathbf{P}\frac{r\cos\theta}{\Theta v_\parallel + v_E} - \pi r \sin\theta \; \delta\left(\Theta v_\parallel + v_E\right)\right\} \end{aligned}$$

END

REMARK
Briefly, the first order correction to the Maxwellian for a *non-collisional* plasma is due to the shift of the spatial variable

$$\mathbf{x} = \mathbf{x}' = \mathbf{x} + \frac{\rho_\theta}{\Omega}$$

caused by the need to obey the canonical invariant.

Expansion of the Maxwellian around the laboratory spatial position $\mathbf{x}$ leads to first order correction to the Maxwellian and this contains the derivatives of the basic parameters, p_a and T_a from the density $N(\psi)$ and from the thermal velocity. These are actually the *flows*.

Therefore the first correction to the Maxwellian, for the simplest *non-collisional* case, is expressed in terms of *flows.*

Naturally, after disposing of the distribution function

$$f_M + f^{(1)}$$

we can calculate the *friction forces* since they are defined as integrals over the velocity space of collision operator, with the two interesting kernels: momentum $m\mathbf{v}$ and heat $\frac{mv^2}{2}\mathbf{v}$ (see the definitions).

Integrating the *linearized* collisional operator we obtain a linear relationship that defines the *forces*

$$F_{1a} \quad \text{and} \quad F_{2a}$$

as linear sum of flows.

We need the distribution function $f = f^{(0)} + f^{(1)} + ... = f_{Ma} + f_a^{(1)}$.

We should solve a kinetic Fokker Planck equation. Since the collision operator contains the *pitch angle* scatterin part, an expansion of $f_a^{(1)}$ in series of Legendre polynomials allows separation of the variables (v, ξ). Every term in the expansion consists of a product of two factors, one is funvtion of v and the other is $P_l\left(\frac{v_\parallel}{v}\right)$, function of ξ.

Only the first "harmonic" is retained, $l = 1$.

They use the $l = 1$ *velocity space harmonic component* (Legendre) of the distribution function $f_a^{(1)}$.

There is separation of variables v from ξ. It is made an expansion in the set of order-(3/2) Laguerre (Sonin) polynomials $L_k^{(3/2)}\left(x_a^2\right)$ where

$$x_a^2 = \frac{v^2}{v_{th,a}^2}$$

$$\begin{aligned}
f_a^{(1)} &= \frac{2\mathbf{v}}{v_{th,a}^2} \cdot \left[\sum_{j=0}^{\infty} \mathbf{u}_{a,j}\ L_j^{(3/2)}\left(x_a^2\right)\right] f_a^{(0)} \\
&= \frac{2\mathbf{v}}{v_{th,a}^2} \cdot \left[\mathbf{u}_a \right. \\
&\qquad\qquad -\frac{2}{5}\left(\frac{5}{2} - x_a^2\right)\frac{\mathbf{q}_a}{p_a} \\
&\qquad\qquad\quad \left. +L_2^{(3/2)}\ \mathbf{u}_{a,2} + ...\right] f_a^{(0)}
\end{aligned}$$

In this expansion one just introduces *notations* for the coefficients,

$$\mathbf{v} \cdot \mathbf{u}_a \quad , \quad \mathbf{v} \cdot \frac{\mathbf{q}_a}{p_a} \quad , \quad \mathbf{v} \cdot \mathbf{u}_{a,2} \quad , \quad ...$$

They do not have a clear physical meaning at this moment.

(see **Galeev 1970**) where definitions have beed introduced, that can be derived by reversing the above relation

$$\mathbf{u}_{a,k} \equiv \frac{3}{2} \frac{\int d^3v \ \mathbf{v} \ L_k^{(3/2)} \ f_a^{(1)}}{\int d^3v \ x_a^2 \ \left[L_k^{(3/2)}\right]^2 \ f_a^{(0)}}$$

This looks like a flow velocity, at least for low k's. They are *fluid* quantities, they do not depend anymore on the particle velocity $\mathbf{v}$.

We see that the perturbed (neoclassical) distribution function is expressed like a series in terms of fluxes (or velocities)

$$\begin{aligned} \mathbf{u}_{a,0} &\equiv \mathbf{u}_a \\ \mathbf{u}_{a,1} &\equiv -\frac{2}{5}\frac{\mathbf{q}_a}{p_a} \\ \mathbf{u}_{a,2} &= ... \end{aligned}$$

This notation is meant to place together, in a single writting, the velocity of particle flow and the heat "velocity"

For these indices, 0 for flow, 1 for heat, the Laguerre polynomials are

$$L_0^{(3/2)}\left(x_a^2\right) = 1$$

$$L_1^{(3/2)}\left(x_a^2\right) = \frac{5}{2} - x_a^2$$

$$L_2^{(3/2)}\left(x_a^2\right) = \frac{35}{8} - \frac{7}{2}x_a^2 + \frac{1}{2}x_a^4$$

Usually the cntributions of physical flows (particle, heat,...) higher than

$$k > 2$$

are neglected.

The *friction forces* at **HSreview**.

The friction forces are integrals of *momentum* or *heat* multiplying the collision operator, as is $\mathbf{F}_V^a \equiv \int d^3v \ m_a\mathbf{v} \ L_0^{(3/2)} \ C_a^{lin}\left(f_a\right)$.

The collision operator depends on the distribution function, solution to the kinetic equation (close to Spitzer problem, according to **Hirshman 1978**).

The distribution function, as formal solution of the Fokker Planck equation (through a series of 3/2 Laguerre polynomials), is expressed as a series of flows $\mathbf{u}_a$ and $\frac{\mathbf{q}_a}{T_a}$, ...

Introducing this expression for the distribution function in the linearized collisional operator, the result will be again a series of these "flows".

Further, returning to the friction forces, *e.g.* $\mathbf{F}_V^a \equiv \int d^3v\, m_a \mathbf{v}\, L_0^{(3/2)}\, C_a^{lin}(f_a)$, one sees that they are series constituted from these *flows.*

$$\begin{aligned}
\mathbf{F}_V^a &= \sum_b \big(l_{VV}^{ab}\ \mathbf{u}_b \\
&\quad -\frac{2}{5} l_{VQ}^{ab} \frac{\mathbf{q}_b}{p_b} \\
&\quad + l_{V3}^{ab}\ \mathbf{u}_{b,2} \big)
\end{aligned}$$

$$\begin{aligned}
\mathbf{F}_Q^a &= \sum_b \left(-l_{QV}^{ab}\ \mathbf{u}_b + \frac{2}{5} l_{QQ}^{ab}\ \frac{\mathbf{q}_b}{p_b} \right. \\
&\quad \left. - l_{Q3}^{ab}\ \mathbf{u}_{b,2} \right)
\end{aligned}$$

where the third term $\mathbf{u}_{b,2}$ is obtained from the general expression of $\mathbf{u}_{a,k}$, next after $k = 0$ (which is $\mathbf{u}_a$) and $k = 1$ (which is $-\frac{2}{5}\frac{\mathbf{q}_a}{p_a}$), ...

END

The **friction forces** are expressed as

friction for velocity of the particle flow

$$\mathbf{F}_V^a \equiv \int d^3v\ m_a \mathbf{v}\ L_0^{(3/2)}\ C_a^{lin}(f_a)$$

friction for the equivalent velocity of the heat flow

$$\mathbf{F}_Q^a \equiv \int d^3v\ m_a\ \mathbf{v}\ L_1^{(3/2)}\ C_a^{lin}(f_a)$$

Summary

The Laguerre polynomials are $L_{0,1}^{(3/2)}$ (k is 0 and 1).

The collision operator is linearized.

When the integral over velocity space d^3v is computed, the distribution function is expanded near the Maxwellian and the collision operator is implicitely expanded.

The integrations over the Maxwellian can be done explicitely.

By linearization of the collision operator, the integrals will contain formal d^3v integrations of the first order distribution function $f^{(1)}$.

But this $f^{(1)}$ is formally expressed as a series of *flow velocities.*

In this way the *friction* forces will be expressed in terms of *flow velocities.*

NOTE
In **Hirschman 1977**

$$\begin{aligned} R_a &= \int d^3v \; m_a v_\parallel \; C_a \\ &= F_{V\parallel}^a \\ &= \text{parallel friction} \end{aligned}$$

$$\begin{aligned} H_a &= \int d^3v \; m_a v_\parallel \left[\left(\frac{v}{v_{th,a}} \right)^2 - \frac{5}{2} \right] C_a \\ &= F_{Q\parallel}^a \\ &= \text{parallel heat friction} \end{aligned}$$

where $C_a = \sum C_{ab}$.
END

The *momentum flow friction* $\mathbf{F}_V^a$ comes from the projection $L_0^{(3/2)}$ of the *collision operator*.

The *heat friction* $\mathbf{F}_Q^a$ comes from the projection $L_1^{(3/2)}$ of the *collision operator*.

The harmonic function in the velocity space are eigenfunctions of the *linearized* collisional operator in velocity space.

The linear equations for the *friction forces* in terms of *flow velocities*

$$\mathbf{F}_V^a = \sum_b \left(l_{VV}^{ab} \mathbf{V}^b + l_{VQ}^{ab} \mathbf{Q}^a \right)$$

$$\mathbf{F}_Q^a = \sum_b \left(l_{QV}^{ab} \mathbf{V}^b + l_{QQ}^{ab} \mathbf{Q}^b \right)$$

We introduce the notations

$$i, j \equiv V, Q$$

The coefficients are

$$\begin{aligned} l_{ij}^{ab} &= \left(\sum_{\text{species } k} m_a \frac{n_a}{\tau_{ak}} M_{ak}^{ij} \right) \delta_{ab} \\ &\quad + m_a \frac{n_a}{\tau_{ab}} N_{ab}^{ij} \end{aligned}$$

and the two new coefficients M_{ab}^{ij} and N_{ab}^{ij} are

$$\frac{n_a}{\tau_{ab}} M_{ab}^{ij} = \int d^3v \; v_\| L_i^{(3/2)} \; C_{ab} \left(\frac{2v_\|}{v_{th,a}^2} L_j^{(3/2)} \; f_a^{(0)} \; , \; f_b^{(0)} \right)$$

$$\frac{n_a}{\tau_{ab}} N_{ab}^{ij} = \int d^3v \; v_\| \; L_i^{(3/2)} \; C_{ab} \left(f_a^{(0)} \; , \; \frac{2v_\|}{v_{th,b}^2} L_j^{(3/2)} f_b^{(0)} \right)$$

The Braginskii Coulomb collision time

$$\frac{1}{\tau_{ab}} = \frac{4}{3\sqrt{\pi}} 4\pi e_a^2 e_b^2 \ln \Lambda \frac{n_a}{m_a^2 v_{th,a}^3}$$

NOTE for comparison, in **Fundamenski Garcia**,

$$\gamma_{ss'} = \frac{1}{8\pi} \frac{e_s^2 e_{s'}^2}{\varepsilon_0^2} \ln \Lambda_{ss'}$$

and the time of decay of a parallel velocity in *slowing down* is

$$\begin{aligned} \frac{1}{\tau_\|} &= 4\frac{1}{m_s^2} \gamma_{ss'} \frac{\Psi(v)}{v} \times \frac{n_{s'}}{v_{th,s'}^3} \\ &= 4\frac{1}{m_s^2} \frac{1}{8\pi} \frac{e_s^2 e_{s'}^2}{\varepsilon_0^2} \ln \Lambda_{ss'} \frac{\Psi(v)}{v} \times \frac{n_{s'}}{v_{th,s'}^3} \end{aligned}$$

If the velocity of the species s notmalized to the thermal velocity of the species s' is much smaller than 1

$$\begin{aligned} v &= \frac{v_s}{v_{th,s'}} \\ &\ll 1 \end{aligned}$$

then

$$\frac{\Psi(v)}{v} \approx \frac{2}{3\sqrt{\pi}}$$

and

$$\begin{aligned} \frac{1}{\tau_\|} &= 4\frac{1}{m_s^2} \frac{1}{8\pi} \frac{e_s^2 e_{s'}^2}{\varepsilon_0^2} \ln \Lambda_{ss'} \frac{2}{3\sqrt{\pi}} \times \frac{n_{s'}}{v_{th,s'}^3} \\ &= \frac{1}{3\pi\sqrt{\pi}} \frac{1}{m_s^2} \frac{e_s^2 e_{s'}^2}{\varepsilon_0^2} \ln \Lambda_{ss'} \times \frac{n_{s'}}{v_{th,s'}^3} \end{aligned}$$

We can now compare, since the transition to SI units is

$$\frac{1}{\varepsilon_0} \rightarrow 4\pi$$

Then

$$\begin{aligned}
\frac{1}{\tau_{\|}} &= \frac{1}{3\pi\sqrt{\pi}} \frac{1}{m_s^2} \frac{e_s^2 e_{s'}^2}{\varepsilon_0^2} \ln \Lambda_{ss'} \times \frac{n_{s'}}{v_{th,s'}^3} \\
&= \frac{4^2\pi^2}{3\pi\sqrt{\pi}} \frac{1}{m_s^2} e_s^2 e_{s'}^2 \ln \Lambda_{ss'} \times \frac{n_{s'}}{v_{th,s'}^3} \\
&= \frac{4}{3\sqrt{\pi}} \frac{1}{m_s^2} 4\pi e_s^2 e_{s'}^2 \ln \Lambda_{ss'} \times \frac{n_{s'}}{v_{th,s'}^3} \\
&= \frac{1}{\tau_{ss'}}
\end{aligned}$$

We conclude that the time of collision τ_{ab} (Coulombian) in Braginskii is the momentum loss time of Fundamenski Garcia.

END

The self-adjoiness property of the Coulomb collision operator leads to the symmetry properties of the coefficients

$$\begin{aligned}
M_{ab}^{ij} &= M_{ab}^{ji} \\
\text{for } i,j &\equiv V,Q
\end{aligned}$$

$$\begin{aligned}
T_b^2 v_{th,b} N_{ab}^{ij} &= T_a^2 v_{th,a} N_{ab}^{ji} \\
\text{for } i,j &= V,Q
\end{aligned}$$

From these it results the symmetry of the friction coefficients

$$\begin{aligned}
l_{ij}^{ab} &= l_{ji}^{ab} \\
\text{for } i,j &\equiv V,Q
\end{aligned}$$

The Table of coefficients

$$\begin{aligned}
M_{ab}^{VV} &= -\left(1+\frac{m_a}{m_b}\right) \frac{1}{\left(1+x_{ab}^2\right)^{3/2}} \\
M_{ab}^{VQ} &= -\frac{3}{2}\left(1+\frac{m_a}{m_b}\right) \frac{1}{\left(1+x_{ab}^2\right)^{5/2}} \qquad = M_{ab}^{QV} \\
M_{ab}^{QQ} &= -\left(\frac{13}{4}+4x_{ab}^4+\frac{15}{2}x_{ab}^4\right) \frac{1}{\left(1+x_{ab}^2\right)^{5/2}}
\end{aligned}$$

and

$$
\begin{aligned}
N_{ab}^{VV} &= \left(1+\frac{m_a}{m_b}\right)\frac{1}{\left(1+x_{ab}^2\right)^{3/2}}\\
N_{ab}^{VQ} &= \frac{3}{2}\frac{T_a}{T_b}\frac{1}{x_{ab}}\left(1+\frac{m_a}{m_b}\right)\frac{1}{\left(1+x_{ab}^2\right)^{5/2}}\\
N_{ab}^{QV} &= \frac{3}{2}\left(1+\frac{m_a}{m_b}\right)\frac{1}{\left(1+x_{ab}^2\right)^{5/2}}\\
N_{ab}^{QQ} &= \frac{27}{4}\frac{T_a}{T_b}x_{ab}^2\frac{1}{\left(1+x_{ab}^2\right)^{5/2}}
\end{aligned}
$$

where

$$x_{ab} \equiv \frac{v_{th,b}}{v_{th,a}}$$

8.1 The parallel viscosity

From **Hirshman Sigmar review 1981**

The dependence of the parallel viscosity tensors

$$
\begin{gathered}
\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Pi}_a \rangle \\
\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Theta}_a \rangle
\end{gathered}
$$

on the FIRST order flows

$$
\begin{gathered}
\mathbf{u}_a^{(1)} \\
\mathbf{q}_a^{(1)}
\end{gathered}
$$

To first order in the Larmor radius it is assumed that the stress tensor depends linraly on the spatial gradients of the fluid flows $\mathbf{u}_a^{(1)}$ and $\mathbf{q}_a^{(1)}$.

There is no first order parallel viscous stress arising from nonuniformities of the flows in the radial $\boldsymbol{\nabla}\psi$ direction.

Parallel viscous tensors of specis (a) *only involve SURFACE components of the gradient that operates on the flows* $\mathbf{u}_a$ and $\mathbf{q}_a$.

Rigid body rotations in the toroidal symmetry direction produce no components in the parallel viscous tensor, even though the angular rotation frequency may be different on each magnetic surface.

For particle flow The components of the viscosity tensor are

$$
\begin{aligned}
& \boldsymbol{\Pi} \\
= \; & \boldsymbol{\Pi}_{\parallel} \\
& +\boldsymbol{\Pi}_{gyro} \\
& +\boldsymbol{\Pi}_{\perp}
\end{aligned}
$$

When the dstribution function is separated in *gyroaveraged* and a part depending on the gyration angle

$$f = \overline{f} + \widetilde{f}$$

the definition of the parallel viscosity tensor is made explicit

$$\mathbf{\Pi}_{\|,a} = \int d^3v \; m_a \; \left(v_\| - \frac{v_\perp^2}{2} \right) \left(\widehat{\mathbf{n}}\,\widehat{\mathbf{n}} - \frac{1}{3}\mathbf{I} \right) \; \overline{f}_a$$

This is

$$\begin{aligned} \mathbf{\Pi}_{\|,a} &= \int d^3v \; m_a \; v^2 \; P_2^0\left(\xi\right) \; \left(\widehat{\mathbf{n}}\,\widehat{\mathbf{n}} - \frac{1}{3}\mathbf{I} \right) \; \overline{f}_a \\ &= \left(p_\| - p_\perp\right) \left(\widehat{\mathbf{n}}\,\widehat{\mathbf{n}} - \frac{1}{3}\mathbf{I} \right) \end{aligned}$$

The viscosity tensor that mixes parallel and perpendicular directions is caused by the gyro-angle dependent part of the distribution function

$$\mathbf{\Pi}_{1a} = \int d^3v \; m_a \; v_\| v_\perp \; \left(\widehat{\mathbf{e}}_\perp \widehat{\mathbf{n}} \; - \; \widehat{\mathbf{n}}\,\widehat{\mathbf{e}}_\perp\right) \; \widetilde{f}_a$$

This can be written as

$$\mathbf{\Pi}_{1a} = \int d^3v \; m_a \; v^2 P_2^1\left(\xi\right) \; \left(\widehat{\mathbf{e}}_\perp \widehat{\mathbf{n}} \; - \; \widehat{\mathbf{n}}\,\widehat{\mathbf{e}}_\perp\right) \; \widetilde{f}_a$$

The viscosity tensor resulting from perpendicular components of the velocity is also due to the gyro-angle dependent distribution function

$$\mathbf{\Pi}_{2a} = \int d^3v \; m_a \; v_\perp^2 \; \left[\widehat{\mathbf{e}}_\perp \; \widehat{\mathbf{e}}_\perp - \left(\mathbf{I} - \widehat{\mathbf{n}}\,\widehat{\mathbf{n}}\right)\right] \; \widetilde{f}_a$$

and can be rewritten

$$\mathbf{\Pi}_{2a} = \int d^3v \; m_a \; v^2 \; P_2^2\left(\xi\right) \; \left[\widehat{\mathbf{e}}_\perp \; \widehat{\mathbf{e}}_\perp - \left(\mathbf{I} - \widehat{\mathbf{n}}\,\widehat{\mathbf{n}}\right)\right] \; \widetilde{f}_a$$

From these tensors, $\mathbf{\Pi}_{1a}$ and $\mathbf{\Pi}_{2a}$ it is composed $\mathbf{\Pi}_{gyro}$.

For heat flow The formulas are similar

$$\mathbf{\Theta}_{\parallel,a}=\int d^3v\ m_a\ v^2\ L_1^{(3/2)}\ P_2^0(\xi)\ \left(\widehat{\mathbf{n}}\,\widehat{\mathbf{n}}-\frac{1}{3}\mathbf{I}\right)$$

or

$$\mathbf{\Theta}_{\parallel,a}=\left(r_\parallel-r_\perp\right)\ \left(\widehat{\mathbf{n}}\,\widehat{\mathbf{n}}-\frac{1}{3}\mathbf{I}\right)$$

The part which mixes parallel and perpendicular movements

$$\mathbf{\Theta}_{1a}=\int d^3v\ m_a\ v^2\ L_1^{(3/2)}\ P_2^1(\xi)\ \left(\widehat{\mathbf{e}}_\perp\widehat{\mathbf{n}}\ -\ \widehat{\mathbf{n}}\,\widehat{\mathbf{e}}_\perp\right)\ \widetilde{f}_a$$

The part involving the perpendicular velocities

$$\mathbf{\Theta}_{2a}=\int d^3v\ m_a\ v^2\ L_1^{(3/2)}\ P_2^2(\xi)\ \left[\widehat{\mathbf{e}}_\perp\ \widehat{\mathbf{e}}_\perp-(\mathbf{I}-\widehat{\mathbf{n}}\,\widehat{\mathbf{n}})\right]\ \widetilde{f}_a$$

8.1.1 The Fokker Planck equation for the distribution function

The equation

$$v_\parallel\nabla_\parallel\left[\overline{f}_a^{(1)(0)}+\frac{2\pi}{\partial\chi/\partial\psi}\frac{I}{\Omega_a}v_\parallel\frac{\partial f_{Ma}}{\partial\psi}\right]=0$$

Here both expansions are present

- expansion in $\delta=\rho_\theta/L$ and it is (1) and
- expansion in ν_*/ω_b (collisional) and it is (0).

This is the equation in order *zero* for the *collisionality.*
The next order in expansion for collisionality

$$v_\parallel\nabla_\parallel\overline{f}_a^{(1)(1)}+e_aE_\parallel v_\parallel\frac{\partial f_{Ma}}{\partial\epsilon}=C_a\left[\overline{f}_a^{(1)(0)},\overline{f}_b^{(1)(0)}\right]$$

This equation offers a *constraint* if it is used the periodicity

$$\oint\frac{dl_\parallel}{v_\parallel}(...)=0$$

$$e_a\oint dl_\parallel\ E_\parallel\frac{\partial f_{Ma}}{\partial\epsilon}-\oint\frac{dl_\parallel}{v_\parallel}C_a\left[-\frac{2\pi}{\partial\chi/\partial\psi}\frac{I}{\Omega_a}v_\parallel\frac{\partial f_{Ma}}{\partial\epsilon}\ ,\ -\frac{2\pi}{\partial\chi/\partial\psi}\frac{I}{\Omega_b}v_\parallel\frac{\partial f_{Mb}}{\partial\epsilon}\right]$$
$$=\oint\frac{dl_\parallel}{v_\parallel}C\left[g_a,g_b\right]$$

where g_a is a perturbation of $f_a^{(1)(0)}$ beyond the neoclassical one, $-\frac{2\pi}{\partial\chi/\partial\psi}\frac{I}{\Omega_a}v_\parallel\frac{\partial f_{Ma}}{\partial\epsilon}$.

One now needs an expression for the collision operator.
It may be taken a Coulomb approximation.

8.1.2 Closure of the hierarchy of momenta

"*It has been shown that in an axisymmetric torus the parallel viscous forces are driven mainly by their own poloidal flows in all collisionality regimes.*"

The *parallel viscosity is much higher than the perpendicular one*

$$\frac{\Pi_{1,2a}}{\Pi_{\parallel a}} \sim \frac{\nu_a}{\Omega_{ca}} \ll 1$$

The equations connecting the forces coming from viscosity tensor with the flows (particles and heat)

$$\begin{aligned}
\left\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Pi}_{\parallel a} \right\rangle &= m_a \frac{n_a}{\tau_{aa}} \\
&\times \left[\mu^a_{VV} \; U^a_\theta \; \left\langle B^2 \right\rangle \; + \; \mu^a_{VQ} \; Q^a_\theta \; \left\langle B^2 \right\rangle \right]
\end{aligned}$$

$$\begin{aligned}
\left\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Theta}_{\parallel a} \right\rangle &= m_a \frac{n_a}{\tau_{aa}} \\
&\times \left[\mu^a_{QV} \; U^a_\theta \; \left\langle B^2 \right\rangle \; + \; \mu^a_{QQ} \; Q^a_\theta \; \left\langle B^2 \right\rangle \right]
\end{aligned}$$

The coefficients are calculated in the banana regime.
For the electrons

$$\begin{aligned}
\mu^{elect}_{VV} &= \frac{f_{trap}}{f_{circ}} \left[Z_{eff} + \sqrt{2} - \ln\left(1+\sqrt{2}\right) \right] \\
\mu^{elect}_{VQ} &= \frac{f_{trap}}{f_{circ}} \left[\frac{3}{2} Z_{eff} + \frac{4}{\sqrt{2}} - \frac{5}{2} \ln\left(1+\sqrt{2}\right) \right] = \mu^{elect}_{QV} \\
\mu^{elect}_{QQ} &= \frac{f_{trap}}{f_{circ}} \left[\frac{13}{4} Z_{eff} + \frac{39}{4\sqrt{2}} - \frac{25}{4} \ln\left(1+\sqrt{2}\right) \right]
\end{aligned}$$

For the ions

$$\begin{aligned}
\mu^{ions}_{VV} &= \frac{f_{trap}}{f_{circ}} \left[\alpha + \sqrt{2} - \ln\left(1+\sqrt{2}\right) \right] \\
\mu^{ions}_{VQ} &= \frac{f_{trap}}{f_{circ}} \left[\frac{3}{2} \alpha + \frac{4}{\sqrt{2}} - \frac{5}{2} \ln\left(1+\sqrt{2}\right) \right] = \mu^{ions}_{QV} \\
\mu^{ions}_{QQ} &= \frac{f_{trap}}{f_{circ}} \left[\frac{13}{4} \alpha + \frac{39}{4\sqrt{2}} - \frac{25}{4} \ln\left(1+\sqrt{2}\right) \right]
\end{aligned}$$

where

$$\alpha \equiv \frac{n_{imp} Z^2_{imp}}{n_{ion} Z^2_{ion}}$$

For the impurities

$$\begin{aligned}
\mu_{VV}^{imp} &= \frac{f_{trap}}{f_{circ}}\left[\frac{1}{\alpha}+\sqrt{2}-\ln\left(1+\sqrt{2}\right)\right] \\
\mu_{VQ}^{imp} &= \frac{f_{trap}}{f_{circ}}\left[\frac{3}{2}\frac{1}{\alpha}+\frac{4}{\sqrt{2}}-\frac{5}{2}\ln\left(1+\sqrt{2}\right)\right] = \mu_{QV}^{imp} \\
\mu_{QQ}^{imp} &= \frac{f_{trap}}{f_{circ}}\left[\frac{13}{4}\frac{1}{\alpha}+\frac{39}{4\sqrt{2}}-\frac{25}{4}\ln\left(1+\sqrt{2}\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
f_{circ} &= \frac{3}{4}\left\langle B^2\right\rangle\int_0^{1/B_{\max}}\frac{\lambda\, d\lambda}{\left\langle\sqrt{1-\lambda B}\right\rangle} \\
&= 1-f_{trap}
\end{aligned}$$

$$f_{circ}\approx 1-1.46\sqrt{\varepsilon}+0.46\varepsilon^{3/2}$$

NOTE

In **Rozhansky Tendler** the parallel projection of the stress. It is then possible to find

$$\begin{aligned}
&\left\langle \mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_i\right\rangle^{NEO} \\
&= \sqrt{\pi}n\varepsilon^2\frac{\sqrt{m_iT_i}}{\sqrt{2}}B_0\frac{1}{r}\left(V_\theta-V_\theta^{NEO}\right)
\end{aligned}$$

where

$$V_\theta^{NEO}=-\frac{1}{2}\frac{1}{eB_0}\frac{dT_i}{dr}\quad\text{(Hazeltine plateau)}$$

This is the basic neoclassical poloidal rotation derived by **Hazeltine**. Three regimes are identified by the numerical coefficients in front of the gradient of the temperature. For *plateau* the coefficient of the gradient of temperature is 1/2.

8.1.3 Application to the case of one impurity in a background of ions and electrons (Kim Diamond Groebner)

Difference between main ions and impurity ions rotation velocities (**Kim, Diamond, Groebner**)

This is **Kim Diamond Groebner 1991**.

Usually

$$n_I \ll n_i$$

and

$$v_{th,I} \ll v_{th,i}$$

$$\begin{array}{ll} & \text{the impurity-impurity friction force} \\ \ll & \\ & \text{impurity-ion friction force} \end{array}$$

It is assumed that

$$\begin{aligned} \frac{\frac{n_I m_I}{\tau_{II}}}{\frac{n_i m_i}{\tau_{ii}}} \mu^I &= \alpha^2 \\ & \times \left(\frac{m_I}{m_i}\right)^{1/2} \left(\frac{T_i}{T_I}\right)^{3/2} \mu^I \\ &\ll \alpha \end{aligned}$$

9 General effects of the viscosity (Hirshman Sigmar)

1. Only the poloidal components of the fluxes are damped,

2. There is a viscous stress generated by the nonuniform heat flux.

3. Contribution to the visocus stress of species a come only from its own nonuniform flow fields.

COMMENT on this.

A problem to be clarified.

A heavy ion is cold when it enters plasma at the edge. The collisions with the background ions - apparently - do not transfer energy from the bulk population to the contaminant with a high rate. The energy of the heavy impurity ion is for a long time in a transient phase. This is important since the equilibrium distribution function of the sub-population of heavy ions now depends strongly on the equilibrium distribution function of the bulk ions.

Then, even if the viscosity tensor of the heavy ions is expressed as velocity-space integral of an expression (Laguerre 3/2 of order 0 or 1, or 2,...) it only contains the average distribution of the heavy impurities, $\overline{f}_a$.

It is so, - but the effect of f_b of other ions on this function is strong.

The surface average of the *parallel* viscosity tensor (**Hirshman Sigmar review** eq.4.18) is

$$
\begin{aligned}
-\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a \rangle &= -3 \left\langle \left(\nabla_{\|} B \right)^2 \right\rangle \\
&\quad \times \left[\mu_{a1} \boxed{u_{a\theta}(\psi)} + \frac{2}{5} \mu_{a2} \boxed{\frac{q_{a\theta}(\psi)}{p_a}} \right]
\end{aligned}
$$

for *momentum* and

$$
\begin{aligned}
-\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Theta}_a \rangle &= -3 \left\langle \left(\nabla_{\|} B \right)^2 \right\rangle \\
&\quad \times \left[\mu_{a2} \boxed{u_{a\theta}(\psi)} + \frac{2}{5} \mu_{a3} \boxed{\frac{q_{a\theta}(\psi)}{p_a}} \right]
\end{aligned}
$$

for *heat*; and

the poloidal flows

$u_{a\theta}$ and $\dfrac{q_{a\theta}}{p_a}$

are obtained from the full flow vectors from which we substract the perpendicular components, which are determined by the *diamagnetic flows* and by the *electric field*.

This is the *moments method* in which it is retained only two of the moements: for momentum (impulse) and heat flow.

The coefficients of the *LINEAR* expression : viscosity - flows, μ or K, in **HS**

$$
\begin{aligned}
\mu_{a1} &= K_{11}^a \\
\mu_{a2} &= K_{12}^a - \frac{5}{2} K_{11}^a \\
\mu_{a3} &= K_{22}^a - 5 K_{12}^a + \frac{25}{4} K_{11}^a
\end{aligned}
$$

Note that we have three physical quantities that are related

viscosity tensors $\boldsymbol{\pi}$ and

flow fluxes (velocities) $u_{a\theta}$ and $\dfrac{q_{a\theta}}{p_a}$

friction forces $\mathbf{R}$ and $\mathbf{H}$

The coefficients K replace the coefficients μ just to simplify the linear relation between flows and frictions: *linear flux-force* relation. **End.**

It will be shown later (HS) that the coefficients K (further μ) are

$$\begin{aligned} K_{ij}^a &= p_a \ \tau_{aa} \ \iota_{ij}^a \\ \iota_{ij}^a &\equiv \text{dimensionless constants} \end{aligned}$$

These constants are sums over expressions that involve other species, in terms of

$$x_{ab} = \frac{v_{th,b}}{v_{th,a}}$$

Then:

indeed the expression $\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a \rangle$ of the surface averaged - parallel projected viscosity stress is only dependent on the flows of the species a, the same.

But, the coefficients of linear relation contain sums over contributions of the other species.

9.1 Neoclassical viscosity coefficients (Hirshman Sigmar)

See also **Horshman Sigmar 1977** in *drift-kinetic derivation.*

We have to see what we have at this moment.

In the equation of forces (balance of momentum) for the species a, it enters the gradient of the scalar pressure, $-\boldsymbol{\nabla} p_a$ and the gradient of the tensor of stress for the species a.

This term shows the force exerced in one direction due to the space variations (gradients) of the stress-tensor in all the other direction. For example, if the stress tensor has a variation in the poloidal direction, $\boldsymbol{\nabla}_\theta \cdot \boldsymbol{\pi}_a$ then this is one contribution to the force in the radial direction, for example (and also contributes to the forces in other directions).

The momentum equation is projected parallel to the magnetic field $\mathbf{B} \cdot$ (this extracts the force along the parallel direction) and a surface averaging is taken.

The balance of momentum, so transformed, shows the role of the surface average of the parallel projection of the gradient of the stress tensor.

Now, we have an expression of the *viscous stress* $\boldsymbol{\pi}_a$ which is *linear* in the flows projected along the poloidal direction, $u_{a\theta}$ and $q_{a\theta}/p_a$.

This linearization is mediated by coefficients μ (or K) and they must be calculated.

The objective of calculation of the coefficients μ (or K) is to derive, finally, a linear relationship between the flows and the forces, as in thermodynamics

$$
\begin{array}{rcl}
 & & \text{fluxes} \\
 & & \Gamma_a^{BP} \\
Q_a^{BP} & = & q_a^{BP} + \frac{5}{2} T_a \Gamma_a^{BP}
\end{array}
$$

$$
\begin{array}{l}
\text{forces} \\
\frac{1}{p_a} \frac{\partial p_a}{\partial \psi} - \frac{5}{2} \frac{1}{T_a} \frac{\partial T_a}{\partial \psi} \\
\frac{1}{T_a} \frac{\partial T_a}{\partial \psi}
\end{array}
$$

Therefore we have to calculate the *distribution function*, solution of the Fokker Planck.

The F-P eq. is first averaged over the gyromotion. It results the *drift-kinetic* equation for the averged distribution function

$$\overline{f}_a(\mathbf{x}, \mu, \epsilon)$$

This function contains as main part a Maxwellian

$$f_{a0} \equiv \text{Maxwellian}$$

Then it is expanded around this Maxwellian

$$\overline{f}_a = f_{a0} + \overline{f}_{a1}$$

The *drift kinetic* equation (see **Hirshman Sigmar 1977** in *drift-kinetic derivation*) and for comparison **Galeev Sagdeev Liu Novakovskii**)

$$
\begin{array}{rl}
 & v_\parallel \nabla_\parallel \overline{f}_{a1} + \mathbf{v}_{drift} \cdot \boldsymbol{\nabla} \psi \frac{\partial f_{a0}}{\partial \psi} + e_a E_\parallel v_\parallel \frac{\partial f_{a0}}{\partial \epsilon} \\
= & C_a\left(\overline{f}_{a1}, \overline{f}_{b1}\right)
\end{array}
$$

The operator of collision can be re-written, after extracting the Maxwellian parts that cannot contribute

$$C_a\left(\overline{f}_{a1}, \overline{f}_{b1}\right) = \sum_{species\ b} \left[C_{ab}\left(\overline{f}_{a1}, f_{b0}\right) + C_{ab}\left(f_{a0}, \overline{f}_{b1}\right)\right]$$

NOTE

The species a has a Maxwellian distribution function like the other species b.

But we cannot see here the equilibration of the temperatures by collisions: the small number of tungsten ions are collided by the much more numerous deuterons but the collisions are not efficient for energy transfer. Then the Tungsten sub-population is for a long time in a transient state of continuous heating. Even for a Test Tungsten ion, the effect of collisions with deuterons must be preserved, not as a neoclassical effect.

We see that the interest here goes to the departure of the distribution function a from a Maxwellian, $\overline{f}_{a1}$. This is usual neoclassical process: the drift of the basic maxwellian and the parallel change of the perturbation are balanced by collisions: all species are Maxwellians, $\overline{f}_{b0}$, and according to this *restricted* interest, the basic Maxwellian of species a is influenced by the neoclassical perturbation $\overline{f}_{1b}$.

END

The collisional regime.

See the text on *collisions*.

In second order

$$\begin{aligned} & v_\parallel \left[\left(\nabla_\parallel \ln p_a - \frac{e_a}{T_a} E_\parallel \right) \; L_0^{(3/2)} \right. \\ & \qquad \left. + \quad \left(-\nabla_\parallel \ln T_a \right) L_1^{(3/2)} \right] \\ = \quad & C_a \left[\overline{f}_{a1}^{l=1} \; , \; \overline{f}_{b1}^{l=1} \right] \end{aligned}$$

The upperscript

$$l = 1$$

means that the $l = 1$ spherical harmonic (Legendre) in velocity space is retained.

$$\begin{aligned} \overline{f}_{a1}^{l=1} \quad = \quad & \frac{2 v_\parallel}{v_{th,a}^2} \left[u_{\parallel a} \right. \\ & \qquad - \frac{2}{5} L_1^{(3/2)} \; \frac{q_{\parallel a}}{p_a} \\ & \qquad \left. + \; L_2^{(3/2)} \; u_{a2,\parallel} + ... \right] \end{aligned}$$

(see also **Hirshman neoclassic current**).

Introducing this expression in the drift-kinetic equation for $\overline{f}_{a1}$, in the collision operator appear *harmonics* Legendre $l=2$,

$$\begin{aligned} & 2x_a^2 \ \nabla_{\|} B \left[u_{a\theta}(\psi) - \frac{2}{5} \frac{q_{a\theta}(\psi)}{p_a} L_{l=1}^{(3/2)} \right] \times P_2\left(\frac{v_{\|}}{v}\right) f_{a0} \\ = \ & C_a \left[\overline{f}_{a1}^{l=2}, \overline{f}_{b1}^{l=2} \right] \end{aligned}$$

where

$$x_a = \frac{v}{v_{th,a}}$$

and the Legendre polynomial of order 2 is

$$P_2(\xi) = \frac{3}{2}\xi^2 - \frac{1}{2}$$

with

$$\xi \equiv \frac{v_{\|}}{v} \ \ \text{pitch angle}$$

To solve the equation for $\overline{f}_{a1}^{l=2}$ one has to expand this function in Laguerre polynomials of order $5/2$.

$$\overline{f}_{a1}^{l=2} = \frac{2}{3} x_a^2 \ P_2\left(\frac{v_{\|}}{v}\right) \sum_{j=0} \ p_{aj} \ L_j^{(5/2)}\left(x_a^2\right) \ f_{a0}$$

The Laguerre polynomials of order $5/2$ are

$$\begin{aligned} L_0^{(5/2)} &= 1 \\ L_1^{(5/2)} &= \frac{7}{2} - x_a^2 \\ L_2^{(5/2)} &= \frac{68}{8} - \frac{9}{2} x_a^2 + \frac{1}{2} x_a^4 \end{aligned}$$

One introduces an operator

$$\{A(v)\} \equiv \frac{8}{3\sqrt{\pi}} \int_0^{\infty} dx \ \exp\left(-x^2\right) \ x^4 \ A\left(x v_{th,a}\right)$$

This operator is useful and has been met before.

Note. In **Hirshman Sigmar 1977** it is

$$\left\{F^{ab}(v)\right\} = 2 \int d^3 v \ \left(\frac{v_{\|}}{v_{th,a}}\right)^2 \ F^{ab}(v) \ \frac{f_{a0}}{n_a}$$

there it is applied to $\nu^{ab}_{slowing}$ the frequency for slowing down part of the collision operator of species a and b; the operator also contains the *deflection* part. **End.**

Using this operator the coefficients of the expansion in terms of Laguerre polynomials of order 5/2 is

$$p_{aj} \equiv 5 \frac{1}{n_a} \frac{\int d^3v \; x_a^2 \; L_j^{(5/2)} \; \overline{f}_{a1}^{l=2} \; P_2 \left(\frac{v_\parallel}{v} \right)}{\left\{ x_a^2 \left[L_j^{(5/2)} \right]^2 \right\}}$$

For example

$$\frac{p_{a\parallel} - p_{a\perp}}{p_a} = p_{a0}$$

$$\frac{\Theta_{a\parallel} - \Theta_{a\perp}}{p_a} = p_{a0} - \frac{7}{2} p_{a1}$$

This is an expansion of the function $\overline{f}_{a1}^{l=2}$ and it can be used in the drift kinetic equation, to solve it.

Replacing the expansionon the drift kinetic equation and taking moments

$$P_2 L_j^{(5/2)}$$

it is obtained an algebraic system.

10 Connection between radial fluxes and forces/viscosity (Houlberg Shaing Hirshman Zarnstorff)

Parallel and radial force balance

The connections for the *radial fluxes*

classical	$\longleftarrow$	friction forces in surface perp. on B
banana plateau	$\longleftarrow$	flux surface averaged viscous forces
Pfirsch-Schluter	$\longleftarrow$	variation of the friction force on surface

Parallel force balance, averaged on surface and including Sources

$$0 = -\left\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \mathbf{P}_\alpha^{ai} \right\rangle + \left\langle \mathbf{B} \cdot \mathbf{F}_\alpha^{ai} \right\rangle + S_{\|,\alpha}^{ai}$$

where

$$\begin{aligned} \alpha &\equiv \text{order of the moments} \\ \alpha &= 1 \;\; \text{momentum balance} \\ \alpha &= 2 \;\; \text{heat flux balance} \end{aligned}$$

The sources

$$S_{\|,\alpha}^{ai} = S_{E\|,\alpha}^{ai} + S_{NBI\|,\alpha}^{ai} + S_{cx\|,\alpha}^{ai} + S_{anom\|,\alpha}^{ai}$$

The averaging

$$\langle A \rangle = \frac{\int \frac{rd\theta}{B_\theta} A}{\int \frac{rd\theta}{B_\theta}}$$

10.1 Friction forces

The friction force

$$\mathbf{F}_\alpha^{ai} = \int d^3v \; m_a v \; L_{\alpha-1}^{(3/2)}\left(x_a^2\right) \; C_{ai}$$

$$x_a = \frac{v}{v_{th,a}}$$

and the Laguerre polynomials

$$\begin{aligned} L_0^{(3/2)}\left(x^2\right) &= 1 \\ L_1^{(3/2)}\left(x^2\right) &= \frac{5}{2} - x^2 \\ L_2^{(3/2)}\left(x^2\right) &= \frac{35}{8} - \frac{7}{2}x^2 + \frac{1}{2}x^4 \end{aligned}$$

The first two collisional friction forces

$$\begin{aligned} \mathbf{F}_1 &\equiv \mathbf{R} \;\; \text{momentum} \\ \mathbf{F}_2 &\equiv \mathbf{H} \;\; \text{heat} \end{aligned}$$

The construction of the friction forces

parallel friction forces

=

(parallel friction coefficients)

×

(parallel flows)

Then

$$\left\langle \mathbf{B}\cdot\mathbf{F}_{\alpha}^{ai}\right\rangle = \sum_{b,j}\sum_{\beta-\mathrm{moment}} l_{\alpha\beta}^{ai,bj}\ \widehat{u}_{\beta}^{bj}$$

where

$$\begin{aligned}\text{upper index } i &\equiv \text{ charge state } i\\ j &\equiv \text{ charge state } j\end{aligned}$$

$$a \equiv \text{isotope (species) } a$$

$$\widehat{u}_{\alpha}^{ai} \equiv \left\langle \mathbf{B}\cdot\mathbf{u}_{\alpha}^{ai}\right\rangle$$

and

$$\mathbf{u}_{\alpha}^{ai} = (-1)^{\alpha-1}\ \frac{\frac{3}{2}\int d^3v\ \mathbf{v}\ L_{\alpha-1}^{(3/2)}\left(x_a^2\right)\ f_{ai,1}^{(1)}}{\int d^3v\ \left[L_{\alpha-1}^{(3/2)}\left(x_a^2\right)\right]^2\ f_{ai,0}}$$

The most known forms are

$$\begin{aligned}\widehat{u}_{1}^{ai} &\equiv \left\langle \mathbf{B}\cdot\mathbf{u}^{ai}\right\rangle\\ \widehat{u}_{2}^{ai} &\equiv \left\langle \mathbf{B}\cdot\frac{2}{5}\frac{\mathbf{q}^{ai}}{p_{ai}}\right\rangle\end{aligned}$$

The theory that is applied is the *reduced charge state* when there are many impurities (species) and many degrees of charge.

$$l_{\alpha\beta}^{ai,bj} = Z_{ai}\ \left(\delta_{ai,bj}\ M_{\alpha\beta}^{a}\ +\ Z_{bj}\ N_{\alpha\beta}^{ab}\right)$$

where

$$\alpha,\beta \equiv \text{moment, 1 is momentum flow and 2 is heat flow}$$

$$Z_{ai} \equiv \frac{n_{ai}Z_i^2}{\sum_i n_{ai}Z_i^2}$$

10.2 Viscous stress

The most used

$$
\begin{aligned}
\mathbf{P}_1 &\equiv \mathbf{\Pi} \\
\mathbf{P}_2 &\equiv \mathbf{\Theta}
\end{aligned}
$$

with

$$
\mathbf{\Pi}_{ai} = \left(p_{\parallel ai} - p_{\perp ai}\right)\left(\widehat{\mathbf{n}}\,\widehat{\mathbf{n}} - \frac{1}{3}\mathbf{I}\right)
$$

$$
\mathbf{\Theta}_{ai} = \left(\Theta_{\parallel ai} - \Theta_{\perp ai}\right)\left(\widehat{\mathbf{n}}\,\widehat{\mathbf{n}} - \frac{1}{3}\mathbf{I}\right)
$$

The flux surface averaged parallel viscous forces are expressed through *flows* $\widehat{u}_{\alpha}^{ai}$ as

$$
\left\langle \mathbf{B}\cdot\boldsymbol{\nabla}\cdot\mathbf{P}_{\alpha}^{ai}\right\rangle = \left\langle B^2\right\rangle \sum_{\beta} \widehat{\mu}_{\alpha\beta}^{ai}\ \widehat{u}_{\theta,\beta}^{ai}
$$

where

$$
\widehat{\mu}_{\alpha\beta}^{ai} \equiv \text{normalized viscosity coefficients}
$$

Most used

$$
\widehat{u}_{\theta,1}^{ai} \equiv \frac{1}{\left\langle \mathbf{B}\cdot\boldsymbol{\nabla}\theta\right\rangle}\left\langle \boldsymbol{\nabla}\theta\cdot\mathbf{u}^{ai}\right\rangle
$$

$$
\widehat{u}_{\theta,2}^{ai} \equiv \frac{1}{\left\langle \mathbf{B}\cdot\boldsymbol{\nabla}\theta\right\rangle}\left\langle \boldsymbol{\nabla}\theta\cdot\frac{2}{5}\frac{\mathbf{q}^{ai}}{p_{ai}}\right\rangle
$$

These expressions consist of removing the θ dependence by the factor

$$
\frac{\boldsymbol{\nabla}\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}.
$$

to produce functions that are only dependent on the label of the magnetic surface r or ψ.

11 Transit Time Magnetic Pumping (Yoshida Hasegawa) in reconnection

The paper **PF-B 4 (1992) 3013 Yoshida Hasegawa**.

The power lost by TTMP

$$
P_k = n_0 \left\langle F_{\parallel,k}\ u_{\parallel,k}\right\rangle
$$

where

$$\mathbf{F}_{\|} = -\mu \boldsymbol{\nabla}_{\|} B$$

$$u_{\|,k} = \frac{1}{n_0} \int dv_{\|} \; v_{\|} f_k$$

and f_k verifies

$$\frac{\partial f}{\partial t} + v_{\|} \nabla_{\|} f + \boldsymbol{\nabla}_{\perp} \cdot (\mathbf{v}_D f) - \frac{\mu}{m} \nabla_{\|} B_{\|} \frac{\partial f}{\partial v_{\|}} = 0$$

The same drift kinetic is used by **Galeev Sagdeev Liu Novakovskii**.

12 The viscosity associated to the poloidal rotation (Shaing Hazeltine Sanuki)

The equation of motion projected on the parallel $\mathbf{B}$ direction

$$\begin{aligned} & \mathbf{B} \cdot \left[(\mathbf{V} \cdot \boldsymbol{\nabla}) \, \mathbf{V} \right] \\ = \; & -\frac{1}{nm_i} \mathbf{B} \cdot \boldsymbol{\nabla} (p_e + p_i) \\ & -\frac{1}{nm_i} \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Pi}_i \end{aligned}$$

where $\boldsymbol{\Pi}_i \equiv$ ion viscous tensor

The *contribution of electrons to the viscous force is neglected since it is of order*

$$\sqrt{\frac{m_e}{m_i}}$$

It is assumed

- ions are adiabatic due to the high poloidal velocity
- electrons have constant temperature along the magnetic field

Then

$$\frac{1}{n} \mathbf{B} \cdot \boldsymbol{\nabla} (p_e + p_i) = \left(p_e + \frac{5}{3} p_i \right) \mathbf{B} \cdot \boldsymbol{\nabla} (\ln n)$$

The magnetic field is

$$\begin{aligned} \mathbf{B} &= \boldsymbol{\nabla} \varphi \times \boldsymbol{\nabla} \psi + I \boldsymbol{\nabla} \varphi \\ I &= R^2 \mathbf{B} \cdot \boldsymbol{\nabla} \varphi \end{aligned}$$

and

$$\begin{aligned}\mathbf{B}_\theta &\equiv \text{poloidal magnetic field} \\ &= \boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi\end{aligned}$$

The convective term

$$\mathbf{B}\cdot\left[\left(\mathbf{V}\cdot\boldsymbol{\nabla}\right)\mathbf{V}\right] = \mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{1}{2}V^2\right) + \left(\mathbf{V}\times\mathbf{B}\right)\cdot\left(\boldsymbol{\nabla}\times\mathbf{V}\right)$$

The diamagnetic flow can be neglected if

$$\frac{d}{d\psi}\ln\Phi \gg \frac{d}{d\psi}\ln n$$

The equation of continuity

$$\boldsymbol{\nabla}\cdot\left(n\mathbf{V}\right) = 0$$

leads to

$$\mathbf{V} = \frac{K\left(\psi\right)}{n}\mathbf{B} - \frac{d\Phi}{d\psi}R^2\boldsymbol{\nabla}\varphi$$

where

$$\begin{aligned}K\left(\psi\right) &= n\frac{\boldsymbol{\nabla}\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}\cdot\mathbf{V} \\ &= \text{flux function}\end{aligned}$$

(see **ambipolarity Hirshman**).

The Chew Goldberger Low form of the viscous tensor

$$\boldsymbol{\Pi} = \left(P_\parallel - P_\perp\right)\left(\widehat{\mathbf{n}}\,\widehat{\mathbf{n}} - \frac{1}{3}\mathbf{I}\right)$$

and the *parallel viscous force* is

$$\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Pi} = \frac{2}{3}\mathbf{B}\cdot\boldsymbol{\nabla}\left(P_\parallel - P_\perp\right) - \left(P_\parallel - P_\perp\right)\frac{\mathbf{B}\cdot\boldsymbol{\nabla}B}{B}$$

NOTE

In the Section above on Hirshman Sigmar radial fluxes we derive in detail

$$\begin{aligned}& I\frac{\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}\cdot\left[\left(p_{\parallel a} - p_{\perp a}\right)\left(\widehat{\mathbf{n}}\widehat{\mathbf{n}} - \frac{1}{3}\mathbf{I}\right)\right] \\ =\ & \frac{2I}{3B}\nabla_\parallel\left(p_{\parallel a} - p_{\perp a}\right) \\ & -\frac{I}{B}\left(p_{\parallel a} - p_{\perp a}\right)\nabla_\parallel \ln B\end{aligned}$$

END

One needs the expression for the pressure anisotropy

$$P_{\|} - P_{\perp}$$

which can be obtained from a *drift-kinetic* equation which takes into account the flow of the plasma

$$P_{\|} - P_{\perp} = -2\sqrt{\pi} I_{PS} \ K \ m_i v_{th,i} \ B \left(\frac{\partial}{\partial \theta} \ln B - \frac{2}{3} \frac{\partial}{\partial \theta} \ln n \right)$$

where

$$v_{th,i} = \sqrt{\frac{2T_i}{m_i}}$$

$$I_{PS} = \frac{1}{\pi} \int dx \ x^2 \exp\left(-x^2\right) \int_{-1}^{1} dy \left(\frac{1}{2} - \frac{3}{2} y^2 \right) \frac{\nu}{U_t^2 + \nu^2}$$

$$\nu = \frac{v_T}{v_{th,i} \sqrt{x} \nabla_{\|} \theta}$$

$$U_t = G_r \left[y + \frac{\left(\mathbf{V}_E + \widehat{\mathbf{n}} V_{\|} \right) \cdot \boldsymbol{\nabla} \theta}{v_{th,i} \sqrt{x} \nabla_{\|} \theta} \right]$$

and

$$G_r \equiv \text{geometric factor}$$

$$x = \frac{v^2}{v_{th,i}^2}$$

$$\nu_T = 3\nu_D + \nu_E$$

The local parallel balance of force

$$\begin{aligned} \mathbf{B} \cdot \boldsymbol{\nabla} \Bigg(& \frac{1}{2} K^2 \frac{B^2}{n} - \frac{1}{2} \frac{I^2 \left(\frac{d\langle \Phi \rangle}{d\psi} \right)^2}{B^2} + \frac{p_e}{m_i n} \ln n \\ & + \frac{5}{3} \frac{\langle p_i \rangle}{\langle n^{5/3} \rangle m_i} n^{2/3} + \frac{2}{3} \frac{1}{\langle n \rangle m_i} \left(p_{\|} - p_{\perp} \right) \Bigg) \\ = \ & 0 \end{aligned}$$

The balance of forces along $\mathbf{B}$ is integrated on the magnetic field line

$$
\begin{aligned}
& \frac{1}{2}K^2\left(\frac{B^2}{n^2}-\left\langle\frac{B^2}{n^2}\right\rangle\right)-\frac{1}{2}I^2\left(\frac{d\langle\Phi\rangle}{d\psi}\right)^2\left(\frac{1}{B^2}-\left\langle\frac{1}{B^2}\right\rangle\right) \\
& +\frac{p_e}{nm_i}\left(\ln n-\langle\ln n\rangle\right) \\
& +\frac{5}{3}\frac{\langle p_i\rangle}{\langle n^{5/3}\rangle m_i}\left(n^{2/3}-\langle n^{2/3}\rangle\right) \\
& -\frac{2}{3}\frac{1}{\langle n\rangle m_i}\left(p_{\parallel}-p_{\perp}\right) \\
= \; & 0
\end{aligned}
$$

where

$$\langle\rangle \equiv \text{surface average}$$

At the integration it resulted a constant of integration. This has been obtained using the fact that the expression of the anisotropy of the pressure tensor

$$p_{\parallel}-p_{\perp}$$

has zero value when it is averaged over the magnetic surface.

New variables

$$
\begin{aligned}
\chi &= \ln\left(\frac{n}{\overline{n}}\right) \\
&= \frac{e\widetilde{\Phi}}{T_e}
\end{aligned}
$$

where $\widetilde{\Phi}$ is the poloidally varying part of the electric potential

$$\overline{n} \equiv \langle n\rangle\left(1-\frac{\langle\chi^2\rangle}{2}\right)$$

Take

$$\frac{B}{B_0}=1-\varepsilon\cos\theta$$

assume

$$\chi\sim\sqrt{\varepsilon}$$

then

$$
\begin{aligned}
& \frac{2}{3}D\frac{d\chi}{d\theta}+\left(1-m_p^2\right)\chi \\
& -2\frac{dA}{d\psi}\left(\chi^2-\langle\chi^2\rangle\right) \\
= \; & \varepsilon\left[\left(m_p^2+2C\right)\cos\theta+D\sin\theta\right]
\end{aligned}
$$

where

$$D = \frac{8\sqrt{\pi}}{3} I_{PS} K B_0 \frac{1}{2\overline{n} v_{th,i} C_r^2}$$

$$C = \frac{I^2 \left(\frac{d\Phi}{d\psi}\right)^2}{2 v_{th,i}^2 B_0^2 C_r^2}$$

$$\frac{dA}{d\psi} = \frac{m_p^2}{2} + \frac{5}{18} \frac{1}{C_r^2}$$

$$C_r^2 = \frac{1}{2} \left(\frac{5}{3} + \frac{T_e}{T_i} \right)$$

$$M_p = \frac{K \ B_0}{\overline{n} v_{th,i} C_r}$$

12.1 The linear regime

It corresponds to

$$\begin{aligned} M_p &\ll 1 \\ M_p &\gg 1 \end{aligned}$$

The nonlinear term

$$\chi^2 - \langle \chi^2 \rangle \quad \text{is neglected}$$

Then

$$\begin{aligned} & \frac{2}{3} D \frac{d\chi}{d\theta} + \left(1 - M_p^2\right) \chi \\ = \quad & \varepsilon \left[\left(M_p^2 + 2C \right) \cos\theta + D \sin\theta \right] \end{aligned}$$

13 The radial electric field and the rotation damping (Novakovskii Liu Sagdeev Rosenbluth)

The more detailed discussion on this work is in *rotation.tex*.

This is also in *drift kinetic solutions*.

There is a connection with *polarization*.

The equation for the ions

$$\begin{aligned}
& \frac{\partial f}{\partial t} \\
& +\left(\Theta v_{\|}+V_E\right) \frac{\partial f}{r \partial \theta}+V_r \frac{\partial f}{\partial r} \quad \text{(space convection)} \\
& +\frac{d v_{\|}}{d t} \frac{\partial f}{\partial v_{\|}}+\frac{d\left(v_{\perp}^2 / 2\right)}{d t} \frac{\partial f}{\partial\left(v_{\perp}^2 / 2\right)} \quad \text{(energetic)} \\
= \; & C(f)
\end{aligned}$$

The radial velocity

$$\begin{aligned}
V_r \; = \; & -\frac{1}{\Omega} \frac{1}{R} \frac{v_{\|}^2+v^2}{2} \sin \theta \quad \text{(neoclassical drift } v_{Dr}\text{)} \\
& -\frac{1}{\Omega} \frac{\partial V_E}{\partial t} \quad \text{(polarization)}
\end{aligned}$$

$$V_E=-\frac{E_r}{B}$$

and for the radial component of the polarization $-\frac{1}{\Omega} \frac{\partial V_E}{\partial t}$,

$$n m \frac{\partial V_E}{\partial t}+\ldots=\left[-\boldsymbol{\nabla}_{\perp} p \ldots\right]+e n\left.|\mathbf{v} \times \mathbf{B}\right|_\theta$$

The equations of motion for the acceleration of the velocities

$$\frac{d v_{\|}}{d t}=-\varepsilon \frac{v^2-v_{\|}^2}{2} \frac{1}{q R} \sin \theta+v_{\|} \frac{V_E}{R} \sin \theta$$

$$\frac{d\left(v_{\perp}^2 / 2\right)}{d t}=-\frac{1}{R} \frac{v_{\|}^2+v^2}{2} V_E \sin \theta$$

NOTE

In **Novakovskii Liu Sagdeev Rosenbluth** the equations are

$$\begin{aligned}
\frac{d v_{\|}}{d t} & =\left(-\frac{v_{\perp}^2}{2} \widehat{\mathbf{b}}+v_{\|} \mathbf{v}_E\right) \cdot \boldsymbol{\nabla} \ln B \\
\frac{d}{d t}\left(\frac{v_{\perp}^2}{2}\right) & =\frac{v_{\perp}^2}{2}\left(\mathbf{v}_E+v_{\|} \widehat{\mathbf{b}}\right) \cdot \boldsymbol{\nabla} \ln B
\end{aligned}$$

where

$$\mathbf{v}_E \cdot \boldsymbol{\nabla} \ln B=\frac{1}{B} \phi_0^{\prime} \frac{\sin \theta}{R}$$

because

$$\nabla_{\|} \ln B = \frac{\varepsilon}{qR} \sin\theta$$

and

$$\mathbf{v}_E \cdot \boldsymbol{\nabla} \ln B = \left(V_E \widehat{\mathbf{e}}_\theta\right) \cdot \boldsymbol{\nabla} \ln B = -\frac{E_r}{B} \nabla_\theta \ln B$$

but

$$\begin{aligned}
\nabla_\theta &= \frac{\partial}{\partial l_\theta} \text{ and replace } dl_\theta = dl_{\|} \times \frac{B_\theta}{B} \\
\text{from where } \frac{\partial}{\partial l_\theta} &= \frac{\partial}{\partial l_{\|}} \frac{B}{B_\theta} \\
\text{then } \nabla_\theta &= \frac{B}{B_\theta} \nabla_{\|}
\end{aligned}$$

and

$$\nabla_\theta \ln B = \frac{B}{B_\theta} \nabla_{\|} \ln B = \frac{B}{B_\theta} \frac{\varepsilon}{qR} \sin\theta \approx \frac{1}{R} \sin\theta$$

We obtain the desired result

$$\begin{aligned}
\mathbf{v}_E \cdot \boldsymbol{\nabla} \ln B &= -\frac{E_r}{B} \nabla_\theta \ln B \\
&= \frac{1}{B} \phi_0' \frac{\sin\theta}{R}
\end{aligned}$$

In the case of *deep* trapped particles

$$\frac{v_{\|}^2 + v^2}{2} \rightarrow \frac{v_{\perp}^2}{2}$$

END

Notations

$$\Theta = \frac{\varepsilon}{q}$$

The expansion

$$f = f_{Mi} + \widetilde{f}$$

then

$$\begin{aligned}
& \frac{\partial \widetilde{f}}{\partial t} + \frac{v_{\|}}{qR} \frac{\partial \widetilde{f}}{\partial \theta} \\
& -\varepsilon \frac{v^2 - v_{\|}^2}{2} \frac{1}{qR} \sin\theta \ \frac{\partial \widetilde{f}}{\partial v_{\|}} \\
& -C\left(f\right) \\
= \ & \frac{1}{R} \frac{v^2 - v_{\|}^2}{2} \sin\theta \frac{m}{T} \left[V_E + V_n + \left(\frac{v^2}{2T/m} - \frac{3}{2} \right) V_T \right] f_M
\end{aligned}$$

Here we note that the first spatial variation of $\widetilde{f}$ is along the poloidal direction and is exploited by the parallel velocity projected on θ.

The next term is essential: it is the energetic effect on $\widetilde{f}$ by the change of the parallel velocity along the field line $\frac{dv_{\parallel}}{dt}$. All *mirror* effect along the magnetic field line, the trapping.

Condition of neutrality $\langle j_r \rangle = 0$ (on a magnetic surface there is NO net radial current) gives the equation of $\frac{\partial V_E}{\partial t}$.

where

$$\begin{aligned} V_n &= \frac{T}{eB}\frac{1}{L_n} \\ V_T &= \frac{T}{eB}\frac{1}{L_T} \end{aligned}$$

$$V_E = \frac{1}{B}\frac{d\phi}{dr}$$

Notes

- the polarization velocity has been neglected (Hazeltine Robertson)
- moderate electric field
$$\frac{V_E}{\Theta v_{th}} \ll 1$$
(see the comment by **Hassam Drake** on the threshold of this inequality where $v_{th} \rightarrow c_s$)

New variable

$$\xi = \frac{v_{\parallel}}{v}$$

The equation

$$\begin{aligned} & \frac{\partial \widetilde{f}}{\partial t} + \frac{\xi v}{qR}\frac{\partial \widetilde{f}}{\partial \theta} \\ & -\varepsilon \frac{v\left(1-\xi^2\right)}{2}\frac{1}{qR}\sin\theta\,\frac{\partial \widetilde{f}}{\partial \xi} \\ & -C\left(\widetilde{f}\right) \\ = \quad & \frac{v^2\left(1+\xi^2\right)}{2}\frac{1}{R}\sin\theta\frac{m}{T}\left[V_E + V_n + \left(\frac{v^2}{2T/m} - \frac{3}{2}\right)V_T\right] f_M \end{aligned}$$

The collision operator

$$\begin{aligned} C(f) &= \nu_c(x)\frac{\partial}{\partial\xi}\left(1-\xi^2\right)\frac{\partial f}{\partial\xi} \quad \text{(pitch angle scattering)} \\ &\quad +\xi\widehat{S}^{(1)}(f) \qquad \text{(slowing down)} \end{aligned}$$

where

$$x^2 = \frac{v^2}{2T/m}$$

$$\nu_c(x) = \frac{3\sqrt{2\pi}}{4}\nu_{ii}\frac{\left(1-\frac{1}{2x^2}\right)\operatorname{erf}(x) + \frac{\exp\left(-x^2\right)}{\sqrt{\pi}\ x}}{x^3}$$

$$\nu_{ii} = \frac{4\pi e^4}{m^2}\ln\Lambda\frac{n}{v_{th}^3}$$

The part of *slowing down* is

$$\begin{aligned} &\widehat{S}^{(1)}(f) \\ =\ & 3\left[\nu_c(x)-\nu_S(x)\right]\int_{-1}^{1} d\xi\ \xi\ f \\ & +3x\nu_S(x)\,f_M\frac{\int dx\ \nu_S(x)\ x^3\ \left(\int_{-1}^{1} d\xi\ \xi\ f\right)}{\int dx\ \nu_S(x)\ x^4\ f_M} \end{aligned}$$

$$\nu_S(x) = \nu_{ii}\ 2\frac{\operatorname{erf}(x) - \frac{2x\ \exp\left(-x^2\right)}{\sqrt{\pi}}}{x^3}$$

Normalizations

$$\widetilde{f} = \widehat{f}\ f_M$$

$$\widehat{t} = \frac{t}{qR/v_{th}}$$

$$\widehat{\nu}_c = \frac{\nu_c(x)}{qR/v_{th}}$$

$$\begin{aligned} \widehat{V}_E &= qV_E \\ \widehat{V}_n &= V_n\frac{q}{v_{th}} \\ \widehat{V}_T &= V_T\frac{q}{v_{th}} \end{aligned}$$

The expressions of the two parts of the collision operator remai the same but now

$$\nu_{ii} \rightarrow \widehat{\nu}_{ii} = \frac{\nu_{ii}}{qR/v_{th}}$$

Above it was derived the equation.
Now we examine the conservation laws.

The neutrality

$$\begin{aligned} \langle j_r \rangle &= \int R d\theta \int d^3v \ V_r \ f \\ &= 0 \end{aligned}$$

where

$$d^3v = 2\pi v^2 dv d\xi$$

and the integrals have the limits

$$\begin{aligned} x &\in [0, \infty] \\ \xi &\in [-1, 1] \end{aligned}$$

The neutrality is linearized

$$\frac{\partial \widehat{V}_E}{\partial \widehat{t}} + \frac{q^2}{2\pi^{3/2}} \int dx \ d\xi \ d\theta \ \left(1 + \xi^2\right) \ x^4 \exp\left(-x^2\right) \ \widehat{f} \ \sin\theta = 0$$

From now on remove the hat $\widehat{}$.

Define the *poloidally averaged macroscopic parallel velocity*

$$\overline{U}_{\parallel} \equiv \frac{\int d\theta \ \int d^3v \ v_{\parallel} f}{\int d\theta \ \int d^3v \ f}$$

The normalization

$$\overline{U}_{\parallel} \rightarrow \overline{U}_{\parallel} \frac{\varepsilon}{v_{th}}$$

$$\overline{U}_{\parallel} = \frac{\varepsilon}{\pi^{3/2}} \int d\theta \ d\xi \ dx \ x^3 \ \xi \ \exp\left(-x^2\right) \ f$$

The expansion in Legendre polynomials (which also means separation of variables $\left(\frac{v}{v_{th}} = x, \theta\right)$ and (ξ).

The introduction of the set of Legendre polynomials

$$\widehat{f}(\theta,\ x,\xi)=\sum F_n(x,\theta)\ \ P_n(\xi)$$

(**note** the similarity with the expansions in problems where the *pitch angle* scattering is important, like **Cordey**).

The collision operator

$$\widehat{S}\left[\sum B_n(x)\,P_n(\xi)\right]=-\sum\left(\widehat{\nu}_nB_n\right)\ \ P_n(\xi)$$

where

$$\begin{aligned}\widehat{\nu}_n(x)\,B_n &= n(n+1)\left[\widehat{\nu}_n-\delta_{1,n}\left(\widehat{\nu}_c-\widehat{\nu}_S\right)\right]B_n\\ &\quad -x\widehat{\nu}_S\ \delta_{1,n}\frac{\int dx\ x^3\widehat{\nu}_SB_n}{\int dx\ x^4\ \widehat{\nu}_S\exp\left(-x^2\right)}\end{aligned}$$

The procedure:

- use the expansion in series of Legendre polynomials P_n in the equation
- multiply the equation by P_n
- integrate over ξ on the interval of orthogonality of the Legendre polynomials

  $$[-1,1]$$

- it results a system of coupled equation for the functions F_n

In the regime dominated by banana.
In equilibrium, the distribution of banana is obtained

$$\begin{aligned}&\xi\ x\ \frac{\partial\widehat{f}}{\partial\theta}\\ &-\varepsilon\frac{x\left(1-\xi^2\right)}{2}\sin\theta\frac{\partial\widehat{f}}{\partial\xi}\\ =\ &x^2\left(1+\xi^2\right)\sin\theta\ \left[\widehat{V}_E+\widehat{V}_n+\left(x^2-\frac{3}{2}\right)\widehat{V}_T\right]\exp\left(-x^2\right)\end{aligned}$$

The time derivative has been removed (at equilibrium) and the collisions neglected (banana).

The evolution is slow

$$\frac{\partial}{\partial t}\ll\omega_{bounce}$$

and

$$\widehat{\nu} \ll \omega_{bounce}$$

The *banana quasi-steady state* solution

$$\begin{aligned}\widehat{f} &= -\xi \ x \ \left(1+\varepsilon\cos\theta\right)\frac{2}{\varepsilon}\left[\widehat{V}_E+\widehat{V}_n+\left(x^2-\frac{3}{2}\right)\widehat{V}_T\right]\exp\left(-x^2\right) \\ &\quad +C\end{aligned}$$

where c is constant on surfaces.

Define

$$F\left(\xi,\theta\right)=\int dx \ x^2\widehat{f}$$

then

$$F\sim\xi\left(1+\varepsilon\cos\theta\right)\left(V_E+V_n+0.5V_T\right)$$

At point θ the trapping condition is

$$|\xi|\leq\sqrt{\varepsilon\left(1+\cos\theta\right)}$$

14 The viscosity in cuasi-axisymmetric toroidal system Sugama Nishimura

The first correction to the distribution function

$$\begin{aligned}f_a &= f_{Ma}\left[1+\frac{e_a}{T_a}\int^{l_{parallel}}\frac{dl_{\|}}{B}\left(BE_{\|}-\frac{B^2}{\langle B^2\rangle}\langle BE_{\|}\rangle\right)\right] \\ &\quad +f_a^{(1)}\end{aligned}$$

The first part contains the Boltzmann distrbution correction,

$$\frac{e\phi}{T}$$

in the potential

$$\phi\sim\int dl_{\|} \ E_{\|}$$

but calculated above a level of averege.

$$\begin{aligned}f_a^{(1)} &\equiv \text{deviation from Maxwellian,} \\ &\quad\ \text{that produces neoclassical transport}\end{aligned}$$

The equation

$$\widehat{V}_{\parallel}\left[f_a^{(1)}\right] + \mathbf{v}_{Da}\cdot\boldsymbol{\nabla} f_{Ma} - \frac{e_a}{T_a}v_{\parallel}\; B\frac{\left\langle BE_{\parallel}\right\rangle}{\left\langle B^2\right\rangle} f_{Ma} = C_a^{lin}\left[f_a^{(1)}\right]$$

$$C_a^{lin}\left[f_a^{(1)}\right] = \sum_b \left\{ C_{ab}\left[f_a^{(1)}, f_{Mb}\right] + C_{ab}\left[f_{Ma}, f_b^{(1)}\right]\right\}$$

$$C_{ab} \equiv \text{Landau collision operator}$$

The term

$$-\frac{e_a}{T_a}v_{\parallel}\; B\frac{\left\langle BE_{\parallel}\right\rangle}{\left\langle B^2\right\rangle} f_{Ma}$$

is energetic. It is the work that must be done by the particle against an electric field $E_{\parallel}$. The energetic effect of particle - field $E_{\parallel}$ interaction leads to a change of the equilibrium distribution function (Maxwellian) by *acceleration* of particles and displacement in the velocity space.

The operator

$$\widehat{V}_{\parallel}\left[\right] \equiv v\xi\nabla_{\parallel} - \frac{1}{2}v\left(1-\xi^2\right)\nabla_{\parallel}\ln B\frac{\partial}{\partial\xi}$$

The first term is

$$v_{\parallel}\nabla_{\parallel}$$

and the second is the energetic change induced by the variation of $v_{\parallel}$ along the particle orbit

$$\frac{1}{2}v\left(1-\xi^2\right)\nabla_{\parallel}\ln B\frac{\partial}{\partial\xi} = \frac{dv_{\parallel}}{dt}\frac{\partial f}{\partial v_{\parallel}}$$

NOTE

In the drift kinetic equation we find

$v_{\parallel}\nabla_{\parallel} \rightarrow$ parallel convection of the θ-dependent part of f

$\mathbf{v}_{Da}\cdot\boldsymbol{\nabla} f_{Ma} \rightarrow$ advection of the Maxwellian equilibrium function by the neoclassical particle drift

$\frac{dv_{\parallel}}{dt}\frac{\partial f}{\partial v_{\parallel}} \rightarrow$ convection in velocity space determined by variation of $v_{\parallel}$ along the line

This is a separation of effects:

- the *magnetic drifts* and the *electric* $E \times B$ drift are in $\mathbf{v}_D$; this term induces the correction to the Maxwellian with the order ρ_θ, distance on which the equilibrium distribution is advected by $\mathbf{v}_D$.
- the *magnetic mirror* effect, which makes transfers between *parallel* and *perpendicular* energies are in the energetic term $\partial f / \partial v_\parallel$.

END

The variables are $(\mathbf{x}, \epsilon, \mu)$ where ϵ contains the potential ϕ. Then

$$(\mathbf{x}, v, \xi)$$

The expansion in *Legendre polynomials*

$$\begin{aligned} P_0(\xi) &= 1 \\ P_1(\xi) &= \xi \\ P_2(\xi) &= \frac{3}{2}\xi^2 - \frac{1}{2} \end{aligned}$$

$$F(\mathbf{x}, v, \xi) = \sum_{l=0}^{\infty} F^{(l)}(\mathbf{x}, v, \xi)$$

where

$$F^{(l)}(\mathbf{x}, v, \xi) = P_l(\xi) \frac{2l+1}{2} \int_{-1}^{1} d\xi' P_l(\xi') F(\mathbf{x}, v, \xi')$$

Further there will be necessary an expansion in *Laguerre polynomials*.

The $l = 1$ Legendre component of the first order correction to the distribution function $f_a^{(1)(l=1)}$ will be expanded in Laguerre $L_j^{(3/2)}(x_a^2)$, with

$$\begin{aligned} L_0^{3/2}\left(x_a^2\right) &= 1 \\ L_1^{3/2}\left(x_a^2\right) &= \frac{5}{2} - x_a^2 \end{aligned}$$

NOTE

The expansion of $f_a^{(1)}$ (neoclassical correction to the Maxwellian) in Legendre polynomials is in *velocity space* and is suggested by the form of the *pitch angle* part of the collision operator. This expansion introduces a *separation* of the variables v and $\xi = \frac{v_\parallel}{v}$.

The expansion in Laguerre polynomials is suggested by the derivatives of the Maxwellian function which produce combinations like $\frac{5}{2} - x_a^2$, functions of x_a^2. These occur when we calculate the *forces* that drive the *fluxes*. Forces are gradients of parameters, or, electric field.

END

Consider the Laguerre polynomials for the first $l = 1$ Legendre component of $f_a^{(1)(l=1)}$.

First, the Legendre transformation that identifies the series $l = 0, ...$ and retains $l = 1$ component

$$f_a^{(1)(l=1)} = \frac{v_\parallel}{v} \frac{3}{2} \int_{-1}^{1} d\left(\frac{v_\parallel}{v}\right) \frac{v_\parallel}{v} f_a^{(1)}$$

It is a "projection" on the function

$$P_1(\xi) = \xi = \frac{v_\parallel}{v}$$

Next, we express the same $l = 1$ component of $f_a^{(1)}$, *i.e.* $f_a^{(1)(l=1)}$, as results from the integration over $v_\parallel / v$, and identify two terms that are *flows*.

$$\begin{aligned} f_a^{(1)(l=1)} &= \frac{2}{v_{th,a}} \xi x_a \left[u_{\parallel a} + \left(x_a - \frac{5}{2} \right) \frac{2}{5} \frac{q_{\parallel a}}{p_a} \right] f_{Ma} \\ &\quad + f_a^{(1)(l=1)(j \geq 2)} \end{aligned}$$

where

$$u_{\parallel a} = \frac{1}{n_a} \int d^3v \; v_\parallel f_a^{(1)}$$

$$q_{\parallel a} = T_a \int d^3v \; v_\parallel \left(x_a^2 - \frac{5}{2} \right) f_a^{(1)}$$

NOTE

The first order neoclassical correction to the Maxwellian, $f_a^{(1)}$, is the solution of the drift-kinetic equation written above and needs explicit form of the collision operator, of $\mathbf{v}_D$, etc.

However here the first Legendre component is shown to be expressed through the mass flow and the heat. The $l = 1$ first Legendre component of $f_a^{(1)}$, *i.e.* $f_a^{(1)(l=1)}$ is defined through an integral over the first Legendre polynomial

$$P_1(\xi) = \xi = \frac{v_\parallel}{v}$$

Then to calculate the component one has to integrate the given function multiplied by the particular element of the set of functions, here the Legendre polynomial ξ. The result contains two flows since the integrations coincide with the definitions of these two flows.

$$\begin{array}{c} u_{\|a} \\ \dfrac{q_{\|a}}{p_a} \end{array}$$

This is how we obtain the first distribution function $f_a^{(1)(l=1)}$.

Since we have its expression in terms of the two flows (matter, heat), we consider the *fluid moments* of the drift kientic equations.

The drift kinetic equation is multiplied by

$$\begin{array}{c} 1 \text{ and} \\ \dfrac{1}{2} m_a v^2 \end{array}$$

and integrated over velocity space

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{u}_a &= 0 \\ \boldsymbol{\nabla} \cdot \mathbf{q}_a &= 0 \end{aligned}$$

The flows are composed of parallel and of perpendicular components

$$\mathbf{u}_a = u_{\|a} \widehat{\mathbf{n}} + \mathbf{u}_{\perp a}$$

$$\mathbf{q}_a = q_{\|a} \widehat{\mathbf{n}} + \mathbf{q}_{\perp a}$$

The perpendicular components are diamagnetic

$$\begin{aligned} \mathbf{u}_{\perp a} &= \frac{1}{e_a B} X_{a1} \boldsymbol{\nabla} s \times \widehat{\mathbf{n}} \\ &= \frac{1}{e_a B} \left(-\frac{1}{n_a} \frac{\partial p_a}{\partial s} - e_a \frac{\partial \phi}{\partial s} \right) \end{aligned}$$

We see that

$$\frac{1}{e_a B} \boldsymbol{\nabla} s \times \widehat{\mathbf{n}} \left(-\frac{1}{n_a} \frac{\partial p_a}{\partial s} \right) = \frac{1}{e_a B} \widehat{\mathbf{n}} \times \boldsymbol{\nabla} p_a \ \text{ dia}$$

And

$$\begin{aligned} \frac{\mathbf{q}_{\perp a}}{p_a} &= \frac{5}{2} \frac{1}{e_a B} X_{a2} \boldsymbol{\nabla} s \times \widehat{\mathbf{n}} \\ &= \frac{5}{2} \frac{1}{e_a B} \left(-\frac{\partial T_a}{\partial s} \right) \boldsymbol{\nabla} s \times \widehat{\mathbf{n}} \end{aligned}$$

Here again

$$\frac{5}{2}\frac{1}{e_a B}\left(-\frac{\partial T_a}{\partial s}\right)\boldsymbol{\nabla} s \times \widehat{\mathbf{n}} = \frac{5}{2}\frac{1}{e_a B}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} T_a$$

The flow incompressibility can be integrated since the perpendicular component is known

$$u_{\|a} = \frac{\left\langle u_{\|a} B\right\rangle}{\left\langle B^2\right\rangle} B + \frac{1}{e_a} X_{a1} \widetilde{U}$$

$$\frac{2}{5}\frac{q_{\|a}}{p_a} = \frac{2}{5}\frac{1}{p_a}\frac{\left\langle q_{\|a} B\right\rangle}{\left\langle B^2\right\rangle} B + \frac{1}{e_a} X_{a2} \widetilde{U}$$

where $\widetilde{U}$, related to Pfitsch Schluter, is solution of

$$\begin{aligned}
\mathbf{B} \cdot \boldsymbol{\nabla}\left(\frac{\widetilde{U}}{B}\right) &= (\mathbf{B} \times \boldsymbol{\nabla} s) \cdot \boldsymbol{\nabla}\left(\frac{1}{B^2}\right) \\
\left\langle B \widetilde{U}\right\rangle &= 0
\end{aligned}$$

Note the fact that they are related with Pfirsch Schluter: they represent the lowest order of perpendicular flows, so necessarily PS. **End.**

The equation is composed of the following terms, examind separately. The LHS may be derived from

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \left(\widetilde{U}\, \widehat{\mathbf{n}}\right) &= \boldsymbol{\nabla} \cdot \left(\widetilde{U}\, \frac{\mathbf{B}}{B}\right) \\
&= \boldsymbol{\nabla} \cdot \mathbf{B}\, \left(\frac{\widetilde{U}}{B}\right) + \mathbf{B} \cdot \boldsymbol{\nabla}\left(\frac{\widetilde{U}}{B}\right) \\
&= \mathbf{B} \cdot \boldsymbol{\nabla}\left(\frac{\widetilde{U}}{B}\right)
\end{aligned}$$

For the RHS $(\mathbf{B} \times \boldsymbol{\nabla} s) \cdot \boldsymbol{\nabla}\left(\frac{1}{B^2}\right)$ we start from

$$\boldsymbol{\nabla} \cdot \left(\frac{\mathbf{B} \times \boldsymbol{\nabla} s}{B^2}\right) = \frac{1}{B^2} \boldsymbol{\nabla} \cdot (\mathbf{B} \times \boldsymbol{\nabla} s) + (\mathbf{B} \times \boldsymbol{\nabla} s) \cdot \boldsymbol{\nabla}\left(\frac{1}{B^2}\right)$$

For the first term we apply

$$\boldsymbol{\nabla} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\boldsymbol{\nabla} \times \mathbf{a}) - \mathbf{a} \cdot (\boldsymbol{\nabla} \times \mathbf{b})$$

and obtain

$$\begin{aligned}\frac{1}{B^2}\boldsymbol{\nabla}\cdot(\mathbf{B}\times\boldsymbol{\nabla}s) &= \frac{1}{B^2}\boldsymbol{\nabla}s\cdot(\boldsymbol{\nabla}\times\mathbf{B})-\mathbf{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{\nabla}s)\\ &= \frac{1}{B^2}\boldsymbol{\nabla}s\cdot(\boldsymbol{\nabla}\times\mathbf{B})\end{aligned}$$

Returning to the term

$$\begin{aligned}\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{B}\times\boldsymbol{\nabla}s}{B^2}\right) &= (\mathbf{B}\times\boldsymbol{\nabla}s)\cdot\boldsymbol{\nabla}\left(\frac{1}{B^2}\right)\\ &\quad+\frac{1}{B^2}\boldsymbol{\nabla}s\cdot(\boldsymbol{\nabla}\times\mathbf{B})\end{aligned}$$

or

$$(\mathbf{B}\times\boldsymbol{\nabla}s)\cdot\boldsymbol{\nabla}\left(\frac{1}{B^2}\right)=\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{B}\times\boldsymbol{\nabla}s}{B^2}\right)-\frac{1}{B^2}\boldsymbol{\nabla}s\cdot(\boldsymbol{\nabla}\times\mathbf{B})$$

Now the equality is

$$\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{\widetilde{U}}{B}\right)=(\mathbf{B}\times\boldsymbol{\nabla}s)\cdot\boldsymbol{\nabla}\left(\frac{1}{B^2}\right)$$

becomes

$$\boldsymbol{\nabla}\cdot\left(\widetilde{U}\;\widehat{\mathbf{n}}\right)=\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{B}\times\boldsymbol{\nabla}s}{B^2}\right)-\frac{1}{B^2}\boldsymbol{\nabla}s\cdot(\boldsymbol{\nabla}\times\mathbf{B})$$

Remark that in tokamak, where $\mathbf{B}$ is along s, $\mathbf{B}\parallel\boldsymbol{\nabla}s$ the first term is cancelled. The second is

$$\frac{1}{B^2}\widehat{\mathbf{e}}_{\parallel}\;\mu_0\mathbf{j}$$

The divergence of the parallel flow $\widetilde{U}$ is a parallel current.

NOTE

The $l=1$ component (projection) $f_a^{(1)(l=1)}$ of the first order neoclassical distribution function is expressed in terms of flows $u_{\|a}$ and $q_{\|a}/p_a$.

However this is a *formal* result.

The two flows are simple definitions, in that context.

The flows are integrations over the first order neoclassical distribution $f_a^{(1)}$. And this is NOT known yet. One has to solve the drift kinetic equation.

END

NOTE

The flows cannot be yet given in terms of $f_a^{(1)}$.

The flows are expressed by very general properties, they are composed of parallel and perpendicular parts.

The perpendicular part is *diamagnetic*.

The parallel part includes an effect of Pfirsch Schluter flow.

END

Now we go forward to solve the drift kinetic equation.

The component of the Legendre expansion of $f_a^{(1)}$ that will produce the effective perturbation is for $l \geq 2$.

The notation is g_a,

$$g_a = f_a^{(1)} - f_a^{(1)(l=1)}$$

The equation for the distribution function is

$$V_\parallel \left[f_a^{(1)} \right] + \mathbf{v}_D \cdot \boldsymbol{\nabla} f_{Ma} + \frac{e_a}{T_a} v_\parallel B \frac{\left\langle B E_\parallel \right\rangle}{\left\langle B^2 \right\rangle} f_{Ma} = C_a^{lin} \left[f_a^{(1)} \right]$$

and this becomes now an equation for g_a,

$$V_\parallel \left[g_a \right] - C_a^{lin} \left[g_a \right] = H_a^{(l=1)} + H_a^{(l=2)}$$

with

$$H_a^{(l=1)} = \frac{e_a}{T_a} v_\parallel B \frac{\left\langle B E_\parallel \right\rangle}{\left\langle B^2 \right\rangle} f_{Ma} + C_a^{lin} \left[f_a^{(1)(l=1)} \right]$$

$$\begin{aligned} H_a^{(l=2)} &= f_{Ma} \frac{1}{T_a} \left[\sigma_{Ua} \left\{ \frac{\left\langle u_{\parallel a} B \right\rangle}{\left\langle B^2 \right\rangle} + \frac{2}{5} \frac{1}{p_a} \frac{\left\langle q_{\parallel a} B \right\rangle}{\left\langle B^2 \right\rangle} \left(x_a^2 - \frac{5}{2} \right) \right\} \right. \\ & \left. + \sigma_{Xa} \left\{ X_{a1} + X_{a2} \left(x_a^2 - \frac{5}{2} \right) \right\} \right] \end{aligned}$$