Abstract

The exchange of momentum and energy in plasma is mediated by collisions. The balance of forces, which establishes a cuasi-equilibrium of flows and loss fluxes is governed by collisions. In the kinetic approach the details of collisional exchanges of momentum are complex and take into account the distinctive types of neoclassical orbits (trapped or circulating). We will examine the collisional processes first on general grounds then in a series of applications. This should help us understand not only the role of collisions but the way we must choose to place emphasis on certain particularities, i.e. to adapt the operators to various regimes. This should prepare us for the review of the connections between friction forces and transport fluxes. [These are Notes intended to help discussions of the Work Sessions of Plasma Theory; provisional, raw, text will be found on the web page http://florin.spineanu.free.fr]

1 Introduction

General comments

1.1 Basic elements Rosenbluth potentials

This is **Hazeltine Hinton RevModPhys.**

The collision operator

$$\begin{aligned} C_a &= -\gamma_a \left[\frac{\partial}{\partial v_\alpha} \left(f_a \frac{\partial h_a}{\partial v_\alpha} \right) \right. \\ & \qquad \left. - \frac{1}{2} \frac{\partial^2}{\partial v_\alpha \partial v_\beta} \left(f_a \frac{\partial^2 g_a}{\partial v_\alpha \partial v_\beta} \right) \right] \end{aligned}$$

where

$$\gamma_a \equiv 4\pi \frac{e^4}{m_a^2} \ln \Lambda$$

$$h_a = \sum_b \left(1 + \frac{m_a}{m_b} \right) \int d^3 v' \; \frac{f_b \left(\mathbf{v}' \right)}{\left| \mathbf{v}' - \mathbf{v} \right|}$$

$$g_a = \sum_b \int d^3 v' \; f_b \left(\mathbf{v}' \right) \left| \mathbf{v} - \mathbf{v}' \right|$$

The equations verified by the Rosenbluth potentials are

$$\frac{\partial^2}{\partial v_\alpha \partial v_\alpha} h_a = -4\pi \sum_b \left(1 + \frac{m_a}{m_b} \right) f_b$$

$$\frac{\partial^2}{\partial v_\alpha \partial v_\beta}\frac{\partial^2}{\partial v_\alpha \partial v_\beta} g_a = -8\pi \sum_b f_b$$

Approximations in the case

$$\begin{aligned} a &\equiv e \quad \text{(electrons)} \\ b &\equiv i \quad \text{(ions)} \\ \frac{m_e}{m_i} &\ll 1 \end{aligned}$$

then

$$\begin{aligned} g_e &= \int d^3v' \; f_i(\mathbf{v}') \left|\mathbf{v}-\mathbf{v}'\right| \\ &\approx \int d^3v' \; f_i(\mathbf{v}') \;\; v \left(1-\frac{\mathbf{v}\cdot\mathbf{v}'}{v^2}\right) \\ &= n_i v \;\left(1-\frac{\mathbf{u}_i\cdot\mathbf{v}}{v^2}\right) \end{aligned}$$

with

$$n\mathbf{u}_i = \int d^3v \; f_i(v) \;\; \mathbf{v}$$

It was assumed that

$$\frac{|\mathbf{u}_i|}{v} \ll 1$$

the motion of the ions is slow compared with the speed of the electrons.

It results

$$C_{ei} \approx 2\pi \frac{e^4}{m_e^2} \ln\Lambda \;\, n_i \;\, \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{U}(\mathbf{v}-\mathbf{u}_i)\cdot \frac{\partial}{\partial \mathbf{v}} f_e$$

with

$$U_{\alpha\beta}(\mathbf{x}) = \frac{1}{x^3}\left(x^2\delta_{\alpha\beta} - x_\alpha x_\beta\right)$$

The linearization of the collision operator

$$C_{ei}^{lin} \approx \gamma_e \;\, n_i \;\; \left\{ \frac{1}{2}\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{U}(\mathbf{v}) \cdot \frac{\partial}{\partial \mathbf{v}} f_e + \frac{2\mathbf{u}_i\cdot\mathbf{v}}{v^3}\frac{1}{v_{th,e}^2} f_{Me} \right\}$$

1.2 A Fokker Planck collision operator Hirshman Sigmar 1976 FP

The collisions impose to consider

- the test particles (species a) , and
- the background particles (species b, can be identical to a)

Due to collisions the distribution functions of a and b are changed. The change is considered small, therefore *linearization* is possible

$$f_a = f_{a0} + f_{a1}$$

where f_{a0} is in general Maxwellian.

The Fokker Planck collision operator for collisions of species a with the species b is

$$C_{ab}(f_a, f_b)$$

Since the various effects of dynamics (including collisions) are small relative to the Maxwellian, the linearization is possible

$$\begin{aligned} & C_{ab}(f_{a0} + f_{a1} \ , \ f_{b0} + f_{b1}) \\ = \ & C_{a,b}(f_{a0}, f_{b0}) \\ & + C_{ab}(f_{a1}, f_{b0}) + C_{ab}(f_{a0}, f_{b1}) \end{aligned}$$

When there is *thermal equilibration* this due to

$$C_{ab}(f_{a0}, f_{b0}) \sim -(T_{a0} - T_{b0})$$

If we place our interest on species a, the component $C_{ab}(f_{a1}, f_{b0})$ is the effect of collisions of the test particles a with the background b.

The other component $C_{ab}(f_{a0}, f_{b1})$ will produce an effect of change of the background gradients (and plasma parameters) due to the effect of the collisions with the test particles. It is rather usual to take this effect to be negligeable.

$$C_{ab}(f_{a0}, \ f_{b1}) \approx 0$$

This means to *assume* that only the distribution function of species a is modified by collisions. The background f_{b0} remains unperturbed.

For this *assumption* the operator FP (**Trubnikov**)

$$\begin{aligned} C_{ab}(f_{a1}, f_{b0}) \ = \ & \nu_{ab}^{defl} \ \mathcal{L} \ [f_{a1}] \\ & + \frac{1}{v^2} \frac{\partial}{\partial v^2} \left[v^3 \left(\frac{\mu_{ab}}{m_b} \ \nu_{ab}^{slowing} \ f_{a1} + \frac{1}{2} \nu_{ab}^{\parallel} \ v \ \frac{\partial f_{a1}}{\partial v} \right) \right] \end{aligned}$$

which contains

- deflection
- slowing down
- parallel diffusion

Here

$$\mu_{ab} \equiv \frac{m_a m_b}{m_a + m_b} \quad \text{reduced mass}$$

$$\nu_{ab}^{slowing}\left(v\right) \equiv \nu_{ab}\left(\frac{2T_{a0}}{T_{b0}}\right)\left(1+\frac{m_b}{m_a}\right)\frac{G\left(\frac{v}{v_{th,b}}\right)}{\frac{v}{v_{th,a}}}$$

$$\nu_{ab}^{defl}\left(v\right) = \nu_{ab}\frac{\Phi\left(\frac{v}{v_{th,b}}\right) - G\left(\frac{v}{v_{th,b}}\right)}{\left(\frac{v}{v_{th,a}}\right)^3}$$

$$\nu_{ab}^{\parallel}\left(v\right) = \nu_{ab}\ 2\frac{G\left(\frac{v}{v_{th,b}}\right)}{\left(\frac{v}{v_{th,a}}\right)^3}$$

with the frequency

$$\nu_{ab} = \frac{4\pi}{2^{3/2}}\frac{\left(e_a e_b\right)^2}{\sqrt{m_a}}\ln\Lambda\frac{n_{b0}}{T_a^{3/2}}$$

the Chandrasekhar function

$$G\left(x\right) = \frac{\Phi\left(x\right) - x\frac{d\Phi(x)}{dx}}{2x^2}$$

The energy exchange rate between the species (a) and (b) is

$$\nu_{ab}^{E}\left(v\right) = 2\nu_{ab}^{slowing}\left(v\right) - \left[2\nu_{ab}^{defl}\left(v\right) + \nu_{ab}^{\parallel}\left(v\right)\right]$$

The operator of pitch angle is

$$\mathcal{L} \equiv \frac{1}{2}\left(\mathbf{v}\times\boldsymbol{\nabla}\right)\cdot\left(\mathbf{v}\times\boldsymbol{\nabla}\right)$$

The *spherical harmonics*

$$Y_{lm}\left(\Omega\right)$$

$$\Omega \equiv \text{solid angle in velocity space}$$

are eigenfunctions

$$\mathcal{L}\ \{Y_{lm}(\Omega)\} = -\frac{1}{2} l(l+1)\ Y_{lm}(\Omega)$$

The use of only first component, $C_{ab}(f_{a1}, f_{b0})$ is justified if the effect of collisions is negligible on the background gradients. It is so for the first evolution after creation of hot ions from NBI.

But in general, and especially for *transport calculations* one has to take into account the change of the background distribution function, f_{1b} , which means $C_{ab}(f_{a0}, f_{b1})$.

The field particles are b.

$$\begin{aligned}
& C(f_{a0}, f_{b1}) \\
= \ & 4\pi\ \nu_{ab}\ v_{th,a}^3\ \frac{f_{a0}}{n_{b0}} \\
& \times \left[\frac{m_a}{m_b} f_{b1}(\mathbf{v}) + \frac{2v}{v_{th,a}^2}\left(1 - \frac{m_a}{m_b}\right) \left.\frac{d\phi_{b1}}{dv}\right|_{\xi} - \frac{2}{v_{th,a}^2}\phi_{b1}(\mathbf{v}) - \left(\frac{2v}{v_{th,a}^2}\right)^2 \left.\frac{d^2\psi_{b1}}{dv^2}\right|_{\xi} \right]
\end{aligned}$$

$$\phi_{b1}\ ,\ \psi_{b1} \equiv \text{perturbed Rosenbluth potentials}$$

The symmetry in velocity space suggests to expand the perturbed Rosenbluth potentials in series of *spherical harmonics*, $Y_{lm}(\Omega)$.

Actually the expansion is for the b-species perturbed distribution function

$$f_{b1} \sim Y_{lm}(\Omega)$$

and it is found

$$C_{ab}(f_{a0}, f_{b1} \sim Y_{lm}) \sim Y_{lm}$$

Taking an angle component (a harmonic function) Y_{lm} in the perturbed f_{b1} results in obtaining the same angle component Y_{lm} of the collision operator.

In a collision operator there are two parts:

- a test particle part
- a "field particle" portion

It is done a comparison between the *test particle* collision operator $C_{ab}(f_{a1}, f_{b0})$ and the *field particle* operator $C_{ab}(f_{a0}, f_{b1})$.

After expansion in harmonics Y_{lm} it is found that the *field particle* operator is affected by a factor

$$\sim \frac{1}{l^2}$$

compared with the *test particle* operator.

This suggests the magnitude of its effect.

An illustration of this order of magnitude.

Consider the case

$$\begin{aligned} v_{th,a} &\gg v_{th,b} \\ & \quad \text{thermal velocity of TEST particles} \\ &\gg \\ & \quad \text{thermal velocity of FIELD particles} \end{aligned}$$

This is a Lorentz case. Like in NBI.

$$\begin{array}{l} \text{TEST particles are fast } \ , \ v_{>} \\ \text{FIELD particles are slow } \ , \ v_{<} \end{array}$$

and there is a parameter

$$\frac{v_{<}}{v_{>}} \ll 1$$

The Rosenbluth potentials

$$\phi_{b1} \quad \text{and} \quad \psi_{b1}$$

are expanded in this small parameter $\frac{v_{<}}{v_{>}}$. It is found that the l-th spherical harmonic in $C_{ab}(f_{a0}, f_{b1})$ has coefficient

$$\left(\frac{v_{<}}{v_{>}}\right)^l$$

Since $\frac{v_{<}}{v_{>}} \ll 1$ this means that the higher order harmonics can be neglected.

the

$$l \geq 3$$

harmonics of the FIELD terms may be ignored in the linearized C_{ab} operator.

The conclusion of **HS1976** is that the following moments of the perturbed distribution function f_{b1} of the *field particles*

$$l = 0, 1, 2$$

which means

$$
\begin{array}{rcl}
 & & v^0 \ \text{(density)} \\
v^1 & \rightarrow & \mathbf{v} \ \text{(momentum)} \\
v^2 & \rightarrow & \mathbf{v}\ \mathbf{v} \ \text{(tensor)}
\end{array}
$$

can couple through the term $C_{ab}\left(f_{a0}, f_{b1}\right)$ and produce an effect on the shape of the distribution function f_{a1} of the TEST particles.

NOTE

We must observe the distinction

- expansion of the kinetic distribution function f in a series of terms with separation of variables, in v and in ξ.
- expansion of the collision operator in modes $Y_{lm}(\Omega)$ which is suggested by the symmetry of the distribution function in the space of velocity. See **Hirshman Sigmar 1976** for the evaluation of the number of model l that must be retained, based on $C \sim l^{-2}$.

END

The expansion in angular harmonics in velocity space (of the perturbation f_{b1} of the *field* particles)

$$
f_{b1} = \sum_{l=0}^{\infty} \sum_{|m| \leq l} f_b^{lm}(v)\ \ Y_{lm}(\Omega)
$$

where the coefficients are calculated, if the distribution function f_{b1} is known, as

$$
f_b^{lm}(v) = \int d\Omega'\ Y_{lm}(\Omega')\ \ f_{b1}(v, \Omega')
$$

This is introduced in the expression of the collision operator $C_{ab}\left(f_{a0}, f_{b1}\right)$ which then is

$$
C_{ab}\left(f_{a0}, f_{b1}\right) = \sum_{l=0}^{\infty} \sum_{|m| \leq l} Y_{lm}(\Omega)\ \ \mathcal{C}_{ab}\left[f_b^{(l)}(v)\right]
$$

This expression extracts the angular harmonics Y_{lm} from the collision operator (of the field f_{b1} particles) and introduces a linear operator $\mathcal{C}$ which only depends on a function $f_b^{(l)}$ that depends on v. The projection m does not occur here.

This will become now the object of studies.

The function-argument

$$f_b^{(l)}(v)$$

is expanded in a series of Sonine polynomials (generalized Laguerre polynomials) od $\left(j, l+\frac{1}{2}\right)$.

$$\begin{aligned} & L_j^{(l+1/2)}(x) \\ \text{of } x \quad \equiv \quad & \frac{v}{v_{th,b}} \end{aligned}$$

$$\begin{aligned} f_b^{(l)}(v) \quad = \quad & 2\frac{v^l}{\left(v_{th,b}^2\right)^l} F_{bj}^{(l)} \\ & \times L_j^{(l+1/2)}\left(x_b^2\right) \\ & \times f_{b0} \end{aligned}$$

with the series-coefficients which are formally expressed as inverse transformation

$$F_{bj}^{(l)} = \frac{\pi^{3/2}\ j!}{\left(j+l+\frac{1}{2}\right)!} \int_0^\infty v^2 dv\ \ (v)^l L_j^{(l+1/2)}\left(x_b^2\right)\ \frac{f_b^{(l)}(v)}{n_{b0}}$$

Since now we have an expression for $f_b^{(l)}(v)$ in terms of the coefficients (to the Laguerre polynomials) $F_{bj}^{(l)}$ we can return to the expression of $C_{ab}(f_{a0}, f_{b1})$ or, specifically, to the function $\mathcal{C}\left(f_b^{(l)}(v)\right)$ which are the coefficients of the harmonics Y_{lm} (in $C_{ab}(f_{a0}, f_{b1})$)

$$\begin{aligned} \mathcal{C}\left[f_b^{(l)}(v)\right] \quad = \quad & \sum_j 2F_{bj}^{(l)} \frac{1}{\left(v_{th,b}\right)^l} \\ & \times \nu_{ab}^{lj}(v) \\ & \text{with sum over the index } j \text{ of the Laguerre polynomials} \end{aligned}$$

here we have the "frequency"

$$\nu_{ab}^{lj}(v) \equiv \mathcal{C}\left[(x_b)^l\ \ L_j^{(l+1/2)}\left(x_b^2\right)\ f_{b0}\right]$$

Note the possibility to make an expansion in Sonine series of directly $\mathcal{C}_{ab}\left[f_b^{(l)}\right]$.

$$\mathcal{C}_{ab}\left[f_b^{(l)}\right] = 2\frac{\psi_{ab}^{lj}}{\left(v_{th,a}^2\right)^l} v^l\ L_j^{(l+1/2)}\left(x_a^2\right)\ \frac{f_{a0}}{n_{a0}}$$

a relation that can be reversed

$$\psi_{ab}^{lj} = \frac{\pi^{3/2}\ j!}{\left(j+l+\frac{1}{2}\right)!} \int_0^\infty v^2 dv\ \ (v)^l\ \ L_j^{(l+1/2)}\left(x_a^2\right)\ \mathcal{C}_{ab}\left[f_b^{(l)}\right]$$

The functions ψ_{ab}^{lj} are new versions of *restoring coefficients.*

They are now moments directly of the collisional operators $\mathcal{C}_{ab}$ and NOT of the distribution function f_{b1}.

End

1.3 Parameter of collisionality

The effective neoclassical collision frequency

$$\nu_{*i} = \varepsilon^{-3/2}\nu_i$$

and this is compared with the transit frequency

$$w_t = \frac{v_{th,i}}{qR}$$

Expansion in Legendre polynomials of the variable

$$\frac{v_\parallel}{v}$$

This is suggested by the form of the integrands in the *Rosenbluth potentials.* See below **Fokker Planck equation**.

1.4 Notes on collisions

In a paper of **Poli Peeters** on NTM for large banana widths, the collision operator is shown to be the analytic expression of a stochastic process with a Langevin equation.

Consider a test particle in a plasma of trapped and passing particles.

A trapped particle has higher $v_\perp$ and a collectivity of trapped particles will induce by collision a transfer of momentum which is of the same nature as the dominant momentum for trapped particles: perpendicular.

In a uniform plasma the transfer of perpendicular momentum to the test particle is equal in one and another direction. But if there is a *gradient* of pressure, a transfer of momentum in the perpendicular direction will be dominated by one direction.

In the presence of a gradient of density the transfer of *parallel* momentum from trapped particles to a test particle will have only *one* direction (similar with the unidirectionality of the diamagnetic flow, for the case of gyration).

A passing particle has higher $v_{\|}$ and a collectivity of passing particles will transfer momentum to a test particle mainly in the parallel direction.

In a uniform plasma the transfer of parallel momentum is equal in both direction since the passing particles move equally in both directions. In the presence of a gradient *there is no breaking of uniformity* of parallel momentum transfer.

If there is however a current in plasma (which means a directed motion of the passing particles) then there is a dominant transfer of energy.

1.5 The one dimensional model of Shanny Dowson Green PF 10 (1967) 1281.

Electrons in $1D$ with velocities $(U_x, U_y, U_z \rightarrow u_x, v_\perp \equiv U, v)$. Ions are infinitely heavy immobile and spread uniformly on x.

They say: the transverse motion is an energy reservoir for the logitudinal motion, and the immobile ions as a momentum reservoir.

No exchange of energy between electrons and ions.

The scattering of electrons by ions is just a pitch angle scattering, which connects the longitudinal and the transversal motions.

The motion of electron sheets is advanced in time according to equations without collisions. These equations for a discrete set of sheets describe the oscillation with plasma frequency around the equilibrium positions and motion in an electric field self-generated by the variation of the density of electrons (ions are uniformly distributed).

The collisions are introduced as follows.

A step of collisionles motion in time is Δt.

It is assumed that during Δt there are several small angle collisions with the total effect of NOT-changing the magnitude of the velocity but changing its orientation in velocity space.

Two angles describe the effect of a collision

- the angle of deflection in real space ϕ;
- the angle this deflection makes with the plane (U, v), denoted ψ, in the space of velocities.

There are two assumptions

the angle ψ made by the deflection with the (U, v) plane has uniform random distribution over $(0, 2\pi)$.

the angle ϕ has a distribution that is constructed such as to obtain the Spitzer formula for small angle scattering.

The differential probability for a scattering in the interval

$$(\phi, \phi + d\phi)$$

is

$$P(\phi)\, d\phi = \frac{\phi\ d\phi}{\langle \phi^2 \rangle \Delta t} \exp\left[-\frac{\phi^2}{2 \langle \phi^2 \rangle \Delta t} \right]$$

NOTE that dispersion is

$$\begin{aligned} \langle \phi^2 \rangle - \langle \phi \rangle^2 &\rightarrow \langle \phi^2 \rangle \\ \text{since } \langle \phi \rangle &= 0 \end{aligned}$$

END

The connection with the physics

$$\begin{aligned} \langle \phi^2 \rangle &= \frac{3}{2} \frac{\omega_p^2}{\Lambda} \ln \Lambda \\ \Lambda &= 6\pi \left(\frac{n}{Z} \right) \left(\frac{U^2 + v^2}{\omega_p^2} \right)^{3/2} \end{aligned}$$

At this moment this work attempts to construct a practical mechansim to produce the random functions ϕ and ψ that we need to describe the scattering.

They intend to use a random variable which has uniform distribution over the interval $(0, 1)$.

They introduce a function α which is dependent of ϕ in the following way: when ϕ covers all its interval, $\alpha(\phi)$ covers $(0, 1)$.

Assume the probability of finding α in the interval $d\alpha$ is 1.

Then

$$\begin{aligned} P(\alpha)\, d\alpha &= P(\phi)\, d\phi \\ d\alpha &= P(\phi)\, d\phi \\ \frac{d\alpha}{d\phi} d\phi &= P(\phi)\, d\phi \end{aligned}$$

Taking the condition

$$\phi = 0 \rightarrow \alpha = 0$$

one integrates the last equality

$$\alpha(\phi) = \int^{\phi} P(\phi') \, d\phi'$$

where we must use

$$P(\phi') \, d\phi' = \frac{\phi' \; d\phi'}{\langle \phi^2 \rangle \Delta t} \exp\left[-\frac{\phi'^2}{2 \langle \phi^2 \rangle \Delta t} \right]$$

This is the connection between the variable ϕ and the function α which we intend to use as a starting element to construct ϕ.

Assume we take a value r chosen randomly between 0 and 1, with uniform probability. Then we can find ϕ using the connection written above

$$\phi = \alpha^{-1}(r)$$

Then, from this connecyion it results that ϕ will have the expected distribution and all what we use is an input from a uniformly distributed random variable.

$$\alpha(\phi) = 1 - \exp\left(-\frac{\phi^2}{2 \langle \phi^2 \rangle \Delta t} \right)$$

and for a chosen value r we have

$$\phi = \left[-2 \langle \phi^2 \rangle \Delta t \ln(1 - r) \right]^{1/2}$$

Again the result is:

starting from a random variable r which is uniformly distributed over $(0,1)$ one gets a variable ϕ (the angle of deflection) whose distribution function is $P(\phi) \, d\phi$.

Later in the treatment adopted by **Boozer** we will see that the distribution function for the angle of deflection ϕ (or a related variable) is given by a Gaussian spreading such that the average is $\langle \phi \rangle = 0$ and the dispersion $\left[\langle \phi^2 \rangle - \langle \phi \rangle^2 \right] \Delta t$ grows in time.

It is proved further that this mechanism of constructing the random deflection angle ϕ from the uniformly distributed random r, produces the effect of a Lorentz collision operator

$$\frac{\partial}{\partial t} P(V, \theta, t) = \nu(V) \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \sin\theta \frac{\partial}{\partial \theta} P(V, \theta, t)$$

where

$$\begin{aligned} U &= V \cos\theta \\ v &= V \sin\theta \end{aligned}$$

and we recognize the angle θ as

$$\sin\theta = \frac{v}{V} = \frac{v_\perp}{V}$$

The collision frequency is velocity dependent

$$\nu = \frac{1}{2}\left\langle \phi^2 \right\rangle$$

Now it is shown that the mechanism of constructing the scattering based on a random uniformly distributed variable α with values $r \in (0,1)$ is equivalent to a Fokker Planck operator.

Consider a vector of velocity in the space (U_x, U_y, U_z).

The deflection is a small angle ϕ around this fixed velocity vector, *i.e.* a small cone with angle 2ϕ.

It is taken conventionally along the z axis which makes possible to write the components after collisions

$$\begin{aligned} V_x &= V\sin\phi\cos\psi \\ V_y &= V\sin\phi\sin\psi \\ V_z &= V\cos\phi \end{aligned}$$

The authors consider the probability that after a time Δt the angle of deflection is in the interval

$$[0,\phi]$$

and the angle ψ is in the interval

$$[0,\psi]$$

and write this probability as a product of terms separated, each dependent on a single angle variable

$$P_1(\phi,\psi) = \widetilde{P}_1(\phi)\,\widetilde{\widetilde{P}}_1(\psi)$$

where the first function, of ϕ, is the cumulated effect of the Gaussian with specified disperssion

$$\widetilde{P}_1(\phi) = \int^{\phi} \frac{\phi' d\phi'}{\left\langle \phi^2 \right\rangle \Delta t} \exp\left[-\frac{\phi'^2}{2\left\langle \phi^2 \right\rangle \Delta t}\right]$$

and for the second we use the uniform distribution of ψ, with normalization over the interval of values $(0, 2\pi)$.

$$\widetilde{\widetilde{P}}_2(\phi) = \frac{1}{2\pi} \int^{\psi} d\psi'$$

From this basis we want to find the probability that at time t the velocity is another vector $\mathbf{V}$.

The calculation of this probability consists of taking the same probability but for a vector which is close, $\mathbf{V} - \Delta\mathbf{V}$ and multiply with the probability for the small change of the vector $\mathbf{V}$, as it was calculated above. Then, *integrate* this product of the two probabilities over the intermediate angles $d\phi d\psi$,

$$P(\mathbf{V}, t) = \int\int P(\mathbf{V} - \Delta\mathbf{V}, t - \Delta t) \times P_1(\phi, \psi)\, d\phi d\psi$$

We now expand the probability in the integrand

$$\begin{aligned} P(\mathbf{V}, t) \quad = \quad & \int\int \Big[P(\mathbf{V}, t - \Delta t) \\ & - \Delta\mathbf{V} \cdot \frac{\partial P}{\partial \mathbf{V}} \\ & + (\Delta\mathbf{V}\ \Delta\mathbf{V}) : \frac{\partial^2 P}{\partial \mathbf{V}\ \partial \mathbf{V}} + \ldots \Big] \\ & \times P_1(\phi, \psi)\ \ d\phi d\psi \end{aligned}$$

At the limit $\Delta t \to 0$,

$$\begin{aligned} \frac{\partial P(\mathbf{V}, t)}{\partial t} \quad = \quad & \lim_{\Delta t \to 0} \sum_{i=1}^{3} \left(-\frac{\langle \Delta v_i \rangle}{\Delta t} \frac{\partial P}{\partial v_i} \right. \\ & \left. + \sum_{k=1}^{3} \frac{\langle \Delta v_i\ \Delta v_k \rangle}{\Delta t} \frac{\partial^2 P}{\partial v_i \partial v_k} + \ldots \right) \end{aligned}$$

The averages are defined as

$$\frac{\langle \Delta v_i \rangle}{\Delta t} = \frac{1}{\Delta t} \int\int d\phi d\psi\ \Delta v_i\ \widetilde{P}_1(\phi, \psi)\, \widetilde{\widetilde{P}}_1(\phi, \psi)$$

$$\frac{\langle \Delta v_i\ \Delta v_k \rangle}{\Delta t} = \frac{1}{\Delta t} \int\int d\phi d\psi\ \Delta v_i\ \Delta v_k\ \widetilde{P}_1(\phi, \psi)\, \widetilde{\widetilde{P}}_1(\phi, \psi)$$

the two probabilities, $\widetilde{P}_1$ and $\widetilde{\widetilde{P}}_1$ are known explicitely.

Then the integrations can be made.
The results are

$$\frac{\langle \Delta v_{x,y,z} \rangle}{\Delta t} = 0$$

$$\begin{aligned}
\frac{\langle \Delta v_i \Delta v_k \rangle_{i \neq k}}{\Delta t} &= 0 \\
\frac{\langle \Delta v_x^2 \rangle}{\Delta t} &= \frac{1}{2} \langle \phi^2 \rangle V^2 \\
\frac{\langle \Delta v_y^2 \rangle}{\Delta t} &= \frac{1}{2} \langle \phi^2 \rangle V^2 \\
\frac{\langle \Delta v_z^2 \rangle}{\Delta t} &= - \langle \phi^2 \rangle V^2
\end{aligned}$$

The components of the velocities have been written above in terms of the two angles (ϕ, ψ) and the magnitude V.

Then the derivatives

$$\frac{\partial P}{\partial v_i} \quad \text{and} \quad \frac{\partial^2 P}{\partial v_i \, \partial v_k}$$

can be expressed as derivatives to (ϕ, ψ). But they can be converted into derivatives to the two components

$$\begin{aligned}
\text{longitudinal } U &= V \cos \theta \\
\text{transversal } v &= V \sin \theta
\end{aligned}$$

relative to the direction x where the motion of the electron sheets takes place.

The formula for $\partial P / \partial t$ is now explicit, after inserting the averages of the changes of the velocities

$$\frac{\partial P (V, \theta, t)}{\partial t} = \frac{1}{2} \langle \phi^2 \rangle \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial P}{\partial \theta}$$

where

$$\langle \phi^2 \rangle = \frac{Z}{n} \frac{1}{8\pi} \frac{\omega_p^4}{V^3} \ln \Lambda^2$$

1.6 The Monte Carlo collision operator of Boozer

The equation

$$\frac{\partial f}{\partial t} = \frac{1}{2} \nu_d \frac{\partial}{\partial \xi} \left(1 - \xi^2 \right) \frac{\partial f}{\partial \xi}$$

where

$$\xi \equiv \frac{v_{\|}}{v}$$

$$\nu_d = \frac{3}{2}\sqrt{\frac{\pi}{2}}\nu_B \frac{\Phi(x) - \Psi(x)}{x^3}$$

for

$$\begin{aligned} x &\equiv \frac{v}{v_{th}} \\ &= \frac{v}{\sqrt{2T/m}} \end{aligned}$$

and the Braginski collision frequency

$$\begin{aligned} \nu_B &= \frac{4\sqrt{\pi}}{3} e^4 \frac{1}{\sqrt{m}} \Lambda \frac{n}{T^{3/2}} \\ &= \frac{\Lambda/10}{3 \times 10^6} \left(\frac{2}{A}\right)^{1/2} \frac{n}{T^{3/2}} \\ (\text{where } [n] &= cm^{-3},\ [T] = eV < \quad \Lambda = 18.4 \end{aligned}$$

The two functions

$$\begin{aligned} \Phi(x) &= \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-t^2\right) dt \quad \text{the error function} \\ \Psi(x) &= \frac{\Phi - x\frac{d\Phi}{dx}}{2x^2} \end{aligned}$$

Connected with that, the *thermal deflection rate* is

$$\nu_d(x = 1) = 1.182 \times \nu_B$$

The operator of collision constructed via the *binomial distribution.* Consider

$$\langle \xi \rangle = \int_{-1}^{1} \xi f \, d\xi$$

and

$$\langle \xi^2 \rangle = \int_{-1}^{1} \xi^2 f d\xi$$

We look for the equations verified by these two functions.

The purpose is to determine the time dependence of the average and of the dispersion of ξ, after collisions.

$$\begin{aligned}
\frac{d\left\langle \xi \right\rangle}{dt} &= \int_{-1}^{1} d\xi \ \frac{df}{dt}\xi \\
&= \int_{-1}^{1} d\xi \ \xi \ \frac{1}{2}\nu_d \frac{\partial}{\partial \xi}\left(1-\xi^2\right)\frac{\partial f}{\partial \xi} \\
&= \frac{1}{2}\nu_d \int_{-1}^{1} d\xi \left\{ \frac{\partial}{\partial \xi}\left[\xi\left(1-\xi^2\right)\frac{\partial f}{\partial \xi}\right] - \left(1-\xi^2\right)\frac{\partial f}{\partial \xi} \right\} \\
&= -\frac{1}{2}\nu_d \int_{-1}^{1} d\xi \ \left(1-\xi^2\right)\frac{\partial f}{\partial \xi} \\
&= -\frac{1}{2}\nu_d \int_{-1}^{1} d\xi \left\{ \frac{\partial}{\partial \xi}\left(1-\xi^2\right) f - \left(-2\xi\right) f \right\} \\
&= -\nu_d \int_{-1}^{1} d\xi \ \xi f \\
&= -\nu_d \left\langle \xi \right\rangle
\end{aligned}$$

short

$$\frac{d\left\langle \xi \right\rangle}{dt} = -\nu_d \left\langle \xi \right\rangle$$

Further

$$\frac{d}{dt}\left\langle \xi^2 \right\rangle = \nu_d \left(1 - 3\left\langle \xi^2 \right\rangle\right)$$

From $\left\langle \xi \right\rangle$ and $\left\langle \xi^2 \right\rangle$ we create the equation for the square of the standard deviation

$$\sigma^2 = \left\langle \xi^2 \right\rangle - \left\langle \xi \right\rangle^2$$

as

$$\frac{d\sigma^2}{dt} = \nu_d \left(1 - 3\left\langle \xi^2 \right\rangle + 2\left\langle \xi \right\rangle^2\right)$$

To use these equations of evolution of the average, average of the square and the square standard deviation, we assume the initial condition for f,

$$f\left(\xi, t=0\right) = \delta\left(t\right)\delta\left(\xi_0\right)$$

then

$$\frac{d\left\langle \xi \right\rangle}{dt} = -\nu_d \xi_0$$

$$\frac{d\sigma^2}{dt} = \nu_d \left(1 - 3\xi_0^2\right)$$

In time, the average of ξ will evolve and we expect that after a time t the function f will be a *Gaussian* centered at the new value of $\langle\xi\rangle$, and with a dispersion that has evolved

$$\xi(t) = \xi_0 (1 - \nu_d t)$$

$$\sigma(t) = \sqrt{(1 - \xi_0^2)\,\nu_d t}$$

This is a state that we take as reference.

We want to reproduce this state, using a mechanism that involves a random variable which we can easily manage.

For this, we assume that the change of the center ξ and of the standard deviation σ in time t is the result of a series of small angle collisions, which are steps in the pitch angle of *equal size* but of random sign.

Consider n trials consisting of choosing one of the two signs $+$ or $-$, with equal probability.

The distribution function to obtain m times the sign $+$ of n trials is

$$P(m) = \frac{1}{2^n}\frac{n!}{m!\,(n-m)!}$$

Now define the difference between the numbers of the signs $+$ and the signs $-$. The sing $+$ are m and the signs $-$ are $n-m$, so the difference

$$j = m - (n - m) = 2m - n$$

The distribution is

$$P(j) = \frac{1}{\sqrt{2\pi n}}\exp\left(-\frac{j^2}{2n}\right)$$

The standard deviation of this variable j is $\sqrt{n}$.

This is also a Gaussian function, like the distribution of the function $f(\xi, t)$, with center and dispersion found above.

Then we would like to map one Gaussian to another, by changing the ξ steps such as to obtain the probability (Gaussian) of binomial trials.

$$\xi = \sqrt{(1 - \xi_0^2)\,\nu_d \tau}$$

where τ is the length of time between the steps.

After n steps, the time is

$$t = n\tau$$

and the standard deviation of the ξ distribution will be the step size $\times\ \sqrt{n}$,

$$\sigma = \sqrt{(1 - \xi_0^2)\,\nu_d t}$$

If the pitch angle is changed

$$\text{from } \xi_0 \text{ to } \xi_n$$

after a time step τ,

$$\xi_n = \xi_0 \left(1 - \nu_d \tau\right) \pm \sqrt{\left(1 - \xi_0^2\right) \nu_d \tau}$$

then this mechanism reproduces the effect of a Lorentz collision operator.

The condition is

$$\nu_d \tau \ll 1$$

The choice $+$ or $-$ in this formula is arbitrary, with equal probability.

In Monte Carlo, the time step of integration τ must be small enough such that the change of the pitch angle to be small.

2 Collision operators

$$C(f, f) = \sum_k C_{jk}(f, f)$$

2.1 Fokker-Planck operator in the Landau form. Karney review for NBI

The operator of collisions is of *Fokker-Planck* form

$$\begin{aligned} C_{ab}(f) &= -\int d^3 v_b' \int d\Omega \; \sigma_{ab}(\Omega) \\ &\quad \times \left| \mathbf{v}_a - v_b' \right| \\ &\quad \times \left[f_a(\mathbf{v}_a) f_b(\mathbf{v}_b) - f_a(\mathbf{v}_a'') f_b(\mathbf{v}_b'') \right] \end{aligned}$$

wher

$$\begin{aligned} \mathbf{v}_a, \mathbf{v}_b' &\equiv \text{ velocities before collisions} \\ \mathbf{v}_a'', \mathbf{v}_b'' &\equiv \text{ velocities after collisions} \end{aligned}$$

$$\sigma_{ab} \equiv \text{ cross section}$$

The **Fokker-Planck** collision operator with the **Landau** collision integral

$$\begin{aligned} C_{jk}(f, f) &= c_{jk} \frac{\partial}{\partial \mathbf{v}} \cdot \int d^3 v' \frac{\partial^2 u}{\partial \mathbf{v} \partial \mathbf{v}} \\ &\quad \times \left(\frac{\partial f_j(\mathbf{v})}{\partial \mathbf{v}} f_k(\mathbf{v}') - \frac{m_j}{m_k} f_j(\mathbf{v}) \frac{\partial f_k(\mathbf{v}')}{\partial \mathbf{v}'} \right) \end{aligned}$$

$$c_{jk} = 2\pi \frac{e_j^2 e_k^2}{m_j^2} \ln \Lambda$$

where the relative velocity is

$$\mathbf{u} \equiv \mathbf{v} - \mathbf{v}'$$

$$\frac{\partial^2 u}{\partial \mathbf{v} \partial \mathbf{v}} = \frac{1}{u^3} \left(u^2 \mathbf{I} - \mathbf{u}\mathbf{u} \right)$$

The paper **Fokker Planck and quaslinear codes Karney**
The Fokker Planck equation in the presence of electric field

$$\frac{\partial f_a}{\partial t} + \frac{q_a}{m_a} \mathbf{E} \cdot \frac{\partial f_a}{\partial \mathbf{v}} = \sum_b C \left(f_a, f_b \right)$$

NOTES
We observe the absence of the spatial convective term

$$\mathbf{v} \cdot \frac{\partial f_a}{\partial \mathbf{x}}$$

and this means that the problem is *confined to the space of velocity.* This will happen in every spatial point $\mathbf{x}$ and possible variations with $\mathbf{x}$ come only from parameters like T_a or n_a.

If there is an external *wave* that injects energy in the population a then we must include another term in the equation. It is a *wave-induced quasilinear flux in velocity space*: some particles will be in resonance with the wave and will increase some of their velocity components (implicitely the energy) and the configuratio of the distribution function in the velocity space $\mathbf{v}$ will change). This is represented as the velocity-space-divergence of a flux in the velocity space

$$\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{S}^{wave}$$

The fact that the divergence of the flux $\mathbf{S}^{wave}$ is not zero is clearly due to the existence of a *source* in velocity space: this is the momentum and energy coming from exterior with the wave.

Another observation is about the term of electric acceleration which clearly has effect in the velocity space

$$\frac{q_a}{m_a} \mathbf{E} \cdot \frac{\partial f_a}{\partial \mathbf{v}}$$

It can be written as a velocity-space-divergence

$$\begin{aligned}
\frac{q_a}{m_a}\mathbf{E}\cdot\frac{\partial f_a}{\partial \mathbf{v}} &= \frac{\partial}{\partial \mathbf{v}}\cdot\left(\frac{q_a}{m_a}\mathbf{E}\ f_a\right) \\
&= \frac{\partial}{\partial \mathbf{v}}\cdot\mathbf{S}^{elec}
\end{aligned}$$

Finally the collision term

$$C\left(f_a, f_b\right) = -\boldsymbol{\nabla}\cdot\mathbf{S}_c^{a/b}$$

if the collisions are small-angle.

It results that the Fokker Planck equation is

$$\frac{\partial f_a}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{S} = 0$$

where the flux in the velocity space

$$\mathbf{S} = \mathbf{S}_c + \mathbf{S}^{wave} + \mathbf{S}^{elec}$$

and the collisional flux is produced by contributions

$$\mathbf{S}_c = \sum_b \mathbf{S}_c^{a/b}$$

The Landau form of the collision operator

$$\mathbf{S}_c^{a/b} \equiv \text{flux rate of collisions } (c) \text{ of test population } a \text{ with the background population } b$$

$$\mathbf{S}_c^{a/b} = \frac{1}{8\pi}\frac{q_a^2 q_b^2}{\varepsilon_0^2}\frac{1}{m_a}\ln\Lambda^{a/b}\int d^3v'\ \mathbf{U}\left(\mathbf{u}\right)\cdot\left(f_a\left(\mathbf{v}\right)\frac{1}{m_b}\frac{\partial f_b\left(\mathbf{v}'\right)}{\partial \mathbf{v}'} - f_b\left(\mathbf{v}'\right)\frac{1}{m_a}\frac{\partial f_a\left(\mathbf{v}\right)}{\partial \mathbf{v}}\right)$$

where

$$\begin{aligned}
\mathbf{U}\left(\mathbf{u}\right) &= \frac{1}{u}\mathbf{I} - \frac{\mathbf{u}\cdot\mathbf{u}}{u^3} \\
\mathbf{u} &= \mathbf{v} - \mathbf{v}'
\end{aligned}$$

The Rosenbluth potentials

$$\phi_b\left(\mathbf{v}\right) = -\frac{1}{4\pi}\int d^3v'\ \frac{f_b\left(\mathbf{v}'\right)}{\left|\mathbf{v}-\mathbf{v}'\right|}$$

$$\psi_b(\mathbf{v}) = -\frac{1}{8\pi}\int d^3v' \ \left|\mathbf{v}-\mathbf{v}'\right| f_b(\mathbf{v})$$

The name *potentials* is suggested by equations of Poisson type

$$\boldsymbol{\nabla}_{\mathbf{v}}^2 \phi_b(\mathbf{v}) = f_b(\mathbf{v})$$

$$\boldsymbol{\nabla}_{\mathbf{v}}^2 \psi_b(\mathbf{v}) = \phi_b(\mathbf{v})$$

the Landau collision operator can be reexpressed

$$\begin{aligned}\mathbf{S}_c^{a/b} &= -\mathbf{D}_c^{a/b}\cdot\boldsymbol{\nabla}_{\mathbf{v}} f_a(\mathbf{v}) \\ &\quad +\mathbf{F}_c^{a/b} f_b(\mathbf{v})\end{aligned}$$

where

$$\begin{aligned}\mathbf{D}_c^{a/b} &= -4\pi\Gamma^{a/b}\frac{1}{n_b}\boldsymbol{\nabla}_{\mathbf{v}} \ \boldsymbol{\nabla}_{\mathbf{v}} \ \psi_b(\mathbf{v}) \\ &\quad \text{a tensor of diffusion in } \mathbf{v} \text{ space}\end{aligned}$$

$$\begin{aligned}\mathbf{F}_c^{a/b} &= -4\pi\Gamma^{a/b}\frac{1}{n_b}\boldsymbol{\nabla}_{\mathbf{v}} \ \phi_b(\mathbf{v}) \\ &\quad \text{velocity of convection in } \mathbf{v} \text{ space}\end{aligned}$$

The notations

$$\Gamma^{a/b} = \frac{1}{4\pi}\frac{q_a^2 q_b^2}{\varepsilon_0^2}\frac{1}{m_a^2}\ln\Lambda^{a/b}$$

here we note (**from Karney**)

$$\nu_{ab} = \frac{\Gamma^{a/b}}{v_{th,a}^3}$$

The integrands in the two *Rosebluth potentials* contain

$$\left|\mathbf{v}-\mathbf{v}'\right| = \sqrt{v^2+v'^2-2v \ v' \ \cos(\mathbf{v},\mathbf{v}')}$$

To find the angle between the two velocityis it is assumed that one of them, $\mathbf{v}$ is taken as reference. If this velocity is directed along the magnetic field

$$\mathbf{v} \parallel \mathbf{B}$$

then the other velocity is

$$\mathbf{v}' = (v', \theta')$$

$$
\begin{aligned}
|\mathbf{v}-\mathbf{v}'| &= v^2\sqrt{1+\left(\frac{v'}{v}\right)^2-2\left(\frac{v'}{v}\right)\cos\theta'} \\
&= v^2\sqrt{1+x^2-2x\cos\theta'}
\end{aligned}
$$

The expression of the radical also occurs at denominator in the first Rosenbluth potential.

This integrands suggest to use the expansion in series of Legendre polynomials.

The occurence of the expression

$$
\frac{1}{\sqrt{x^2-2x\cos\theta+1}}
$$

in the Rosenbluth potential suggests the expansion in Legendre functions.

In **Morse Feshbach page 1574 ch11**

$$
\begin{aligned}
\frac{\exp(ikR)}{R} &= ik\sum_{n=0}^{\infty}(2n+1)\,P_n(\cos\theta)\,j_n(kr)\,h_n(kr_0) \\
\text{for } r_0 &> r>0
\end{aligned}
$$

where

$$
R=\sqrt{r^2+r_0^2-2rr_0\cos\theta}
$$

with the particular form

$$
\exp(ikr\cos\theta)=\sum_{n=0}^{\infty}(2n+1)\,i^nP_n(\cos\theta)\,j_n(kr)
$$

And, in **Morse Feshbach page 1734 ch12** we find

$$
\frac{1}{r_{12}}=\sum_{n=0}^{\infty}P_n(\cos\theta_{12})\times\begin{cases}\frac{r_1^n}{r_2^{n+1}} & \text{for } r_2>r_1 \\ \frac{r_2^n}{r_1^{n+1}} & \text{for } r_1>r_2\end{cases}
$$

NOTES on the spherical harmonic functions

$$
Y_{l,m}(\theta,\varphi)=\left[\frac{(l-m)!}{(l+m)!}\frac{2l+1}{4\pi}\right]^{1/2}\exp(im\varphi)\ \ P_l^m(\cos\theta)
$$

The equation for the Legendre functions $P_\nu(x)$

$$
\left(1-x^2\right)\frac{d^2w}{dx^2}-2x\frac{dw}{dx}+\nu(\nu+1)\,w=0
$$

and the equation for the associated Legendre functions $P_{\nu}^{\mu}(x)$

$$\left(1-x^2\right)\frac{d^2w}{dx^2}-2x\frac{dw}{dx}+\left[\nu\left(\nu+1\right)-\frac{\mu}{1-x^2}\right]w=0$$

END

Coordinates in the velocity space

$$\begin{array}{c} \left(v_{\perp},v_{\parallel},\varphi\right) \\ \text{cylindrical} \\ \varphi \text{ is azimuthal around } \quad \parallel \quad \text{direction} \end{array}$$

and

$$\begin{array}{c} (v,\theta,\varphi) \\ \text{spherical} \end{array}$$

where

$$\begin{aligned} \theta &\equiv \text{angle between } \mathbf{v} \text{ and the magnetic field (pitch angle)} \\ \varphi &\equiv \text{azimuthal angle, around the magnetic field direction} \end{aligned}$$

The connections

$$\begin{aligned} v^2 &= v_{\parallel}^2+v_{\perp}^2 \\ \cos\theta &= \frac{v_{\parallel}}{v} \end{aligned}$$

Using these coordinates the *divergence of the velocity-space-flux* is

$$\frac{\partial}{\partial\mathbf{v}}\cdot\mathbf{S}=\frac{1}{v_{\perp}}\frac{\partial}{\partial v_{\perp}}\left(v_{\perp}S_{\perp}\right)+\frac{\partial}{\partial v_{\parallel}}S_{\parallel}$$

with the components of the vector-flux

$$\begin{aligned} \mathbf{S}_c^{a/b} &= -\mathbf{D}_c^{a/b}\cdot\boldsymbol{\nabla}_{\mathbf{v}}f_a\left(\mathbf{v}\right) \\ & \quad +\mathbf{F}_c^{a/b}f_b\left(\mathbf{v}\right) \end{aligned}$$

expressed in parallel/perpendicular coordinates (we remove the indices)

$$\begin{aligned} S_{\perp} &= -D_{\perp\perp}\frac{\partial f}{\partial v_{\perp}}-D_{\perp\parallel}\frac{\partial f}{\partial v_{\parallel}} \\ & \quad +F_{\perp}\ f \end{aligned}$$

$$
\begin{aligned}
S_{\parallel} &= -D_{\parallel\perp}\frac{\partial f}{\partial v_{\perp}} - D_{\parallel\parallel}\frac{\partial f}{\partial v_{\parallel}} \\
&\quad +F_{\parallel}\ f
\end{aligned}
$$

In the other system

$$(v, \theta)$$

$$\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{S} = \frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2 S_v\right) + \frac{1}{\sin\theta}\frac{\partial}{v\partial\theta}\left(\sin\theta\ S_{\theta}\right)$$

$$
\begin{aligned}
S_{v} &= -D_{vv}\frac{\partial f}{\partial v} - D_{v\theta}\frac{\partial f}{v\partial\theta} \\
&\quad +F_{v}\ f
\end{aligned}
$$

$$
\begin{aligned}
S_{\theta} &= -D_{\theta v}\frac{\partial f}{\partial v} - D_{\theta\theta}\frac{\partial f}{v\partial\theta} \\
&\quad +F_{\theta}\ f
\end{aligned}
$$

Karney shows how to connect these components of the tensor of "velocity-space-diffusion"

$$\begin{pmatrix} D_{\perp\perp} \\ D_{\perp\parallel} \\ D_{\parallel\perp} \\ D_{\parallel\parallel} \end{pmatrix} = \mathbf{M}\cdot \begin{pmatrix} D_{vv} \\ D_{v\theta} \\ D_{\theta v} \\ D_{\theta\theta} \end{pmatrix}$$

and

$$\begin{pmatrix} F_{\perp} \\ F_{\parallel} \end{pmatrix} = \mathbf{N}\cdot \begin{pmatrix} F_{v} \\ F_{\theta} \end{pmatrix}$$

The two matrices are

$$
\begin{aligned}
\mathbf{M} &= \mathbf{M}^{-1} \\
&= \begin{pmatrix} s^2 & sc & sc & c^2 \\ sc & -s^2 & c^2 & -sc \\ sc & c^2 & -s^2 & -sc \\ c^2 & -sc & -sc & s^2 \end{pmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{N} &= \mathbf{N}^{-1} \\
&= \begin{pmatrix} s & c \\ c & -s \end{pmatrix}
\end{aligned}
$$

with the notations

$$
\begin{aligned}
s &\equiv \sin\theta \\
c &= \cos\theta
\end{aligned}
$$

The expansion of the function $f(v,\theta)$ is

$$
f(v,\theta) = \sum_{l=0}^{\infty} f^{(l)}(v)\, P_l(\cos\theta)
$$

The coefficient can be obtained by the orthogonality properties of the Legendre functions

$$
f^{(l)}(v) = \frac{2l+1}{2} \int_0^{\pi} d\theta \; f(v,\theta)\, P_l(\cos\theta) \sin\theta
$$

Using this and the expansion of the expression $|\mathbf{v}-\mathbf{v}'|$ is $P_l(\cos\theta)$, one obtains the l-th component of the first Rosenbluth potential

$$
\phi_b^{(l)}(v) = -\frac{1}{2l+1} \left[\int_0^{v} dv' \; \frac{v'^{l+2}}{v^{l+1}} f_b^{(l)}(v') + \int_v^{\infty} dv' \; \frac{v^l}{v'^{l-1}} f_b^{(l)}(v') \right]
$$

and the l-th component of the second Rosenbluth potential

$$
\psi_b^{(l)}(v) = \frac{1}{2(4l^2-1)} \Bigg[\int_0^{v} dv' \; \frac{v'^{l+2}}{v^{l-1}} \left(1 - \frac{l - \frac{v'^2}{2}}{l + \frac{3}{2}v^2} \right) f_b^{(l)}(v') + \int_v^{\infty} dv' \; \frac{v^l}{v'^{l-3}} \left(1 - \frac{l - \frac{v^2}{2}}{l + \frac{3}{2}v'^2} \right) f_b^{(l)}(v'
$$

Particular cases.

The isotropic background.

This means

$$
f_b(\mathbf{v}) = f_b(v)
$$

with no dependence of f_b on θ. This only refers to the *background*, which is sometimes called *field* particles.

The Rosenbluth potentials are also isotropic,

$$
\begin{aligned}
\phi &= \phi(v) \\
\psi &= \psi(v)
\end{aligned}
$$

The flux vector of the Fokker Planck equation is composed of the part coming from the *diagonal* diffusion tensor and the convection tensor

$$\begin{aligned} S^{a/b}_{c-v} &= -D^{a/b}_{c-vv}\,\frac{\partial f_a}{\partial v} \\ &\quad +F^{a/b}_{c-v} f_a \end{aligned}$$

and the part of diffusion which is *not diagonal*

$$S^{a/b}_{c-\theta} = -D^{a/b}_{c-\theta\theta}\frac{\partial f_a}{v\partial\theta}$$

The expressions of the diffusion tensor components is now easier

$$D^{a/b}_{c-vv} = \frac{4\pi}{3}\Gamma^{a/b}\frac{1}{n_b}\left[\int_0^v dv'\ \frac{v'^4}{v^3}\ f_b\left(v'\right) + \int_v^\infty dv'\ v' f_b\left(v'\right)\right]$$

$$D^{a/b}_{c-\theta\theta} = \frac{4\pi}{3}\Gamma^{a/b}\frac{1}{n_b}\left[\int_0^v dv'\ \frac{v'^2}{2v^3}\left(3v^2 - v'^2\right)\ f_b\left(v'\right) + \int_v^\infty dv'\ f_b\left(v'\right)\right]$$

$$F^{a/b}_{c-v} = -\frac{4\pi}{3}\Gamma^{a/b}\frac{1}{n_b}\frac{m_a}{m_b}\int_0^v dv'\ \frac{3v'^3}{v^2}\ f_b\left(v'\right)$$

there is still another, most interesting case, where the distribution of the background population is Maxwellian.

this case can be calculated explicitely (**Trubnikov**)

$$D^{a/b}_{c-vv} = \Gamma^{a/b}\frac{1}{2v}\left[\frac{\mathrm{erf}\left(u\right)}{u^2} - \frac{1}{u}\frac{d\,\mathrm{erf}\left(u\right)}{du}\right]$$

$$D^{a/b}_{c-\theta\theta} = \Gamma^{a/b}\frac{1}{4v}\left[\left(2 - \frac{1}{u^2}\right)\mathrm{erf}\left(u\right) + \frac{1}{u}\frac{d\,\mathrm{erf}\left(u\right)}{du}\right]$$

$$F^{a/b}_{c-v} = -\Gamma^{a/b}\frac{1}{v^2}\frac{m_a}{m_b}\left[\mathrm{erf}\left(u\right) - u\frac{d\,\mathrm{erf}\left(u\right)}{du}\right]$$

with the notations

$$\mathrm{erf}\left(u\right) = \frac{2}{\sqrt{\pi}}\int_0^u dx\ \exp\left(-x^2\right)$$

$$\frac{d\,\mathrm{erf}\left(u\right)}{du} = \frac{2}{\sqrt{\pi}}\exp\left(-u^2\right)$$

$$u \equiv \frac{v}{\sqrt{2}v_{th,b}}$$

2.2 Landau collision frequency for the electron-ions collisions

From **Galeev Sagdeev**

$$St_{ei}\left\{f_e\right\} = \frac{\partial}{\partial v_\alpha}\frac{2\pi e^4 Z^2 \ln\Lambda}{m_e^2} n\left(\frac{\delta_{\alpha\beta}}{v} - \frac{v_\alpha v_\beta}{v^3}\right)\left(\frac{\partial f_e}{\partial v_\beta} - \frac{v_\beta}{T} f_e\right)$$

$$\nu_{ei} = \frac{16\sqrt{\pi}e^4 Z^2 \ln\Lambda}{3m_e^2}\frac{n}{v_{th}^3}$$

2.3 Collision operator in Rutherford 1970

$$C\left[f_0\ ,\ \overline{f}_1\right] = \sum_k C_{jk}\left[f_0, \overline{f}_1\right]$$

$$\begin{aligned} C_{jk}\left[f_0\ ,\ \overline{f}_1\right] &= c_{jk}\ RB_T\frac{v_\parallel}{\Omega_i}\left(\frac{\partial}{\partial\epsilon}\frac{\mathbf{v}}{v_\parallel} + \frac{\partial}{B\partial\mu}\frac{\mathbf{v}_\perp}{v_\parallel}\right) \\ &\quad \times\int d^3v'\ \frac{\partial^2 u}{\partial\mathbf{v}\ \partial\mathbf{v}}\cdot\widehat{\mathbf{n}}\left(f_{0k}\frac{\partial f_{0j}}{\partial\psi} - \frac{e_j}{e_k}f_{0j}\frac{\partial f_{0k}}{\partial\psi}\right) \end{aligned}$$

The collision operator is expanded according to the representation of the function

$$\begin{aligned} f &= f_0 + f_1 + \ldots\ \text{(classic and neoclassic from orbits)} \\ &\quad + g_1 + \ldots\ \text{(collisions)} \end{aligned}$$

then

$$C = C\left[f_0, \overline{f}_1\right] + C\left[f_0, g_1\right]$$

This formula is adapted for collisions electron-ions and the function g_1,

$$C_{ei}\left[f_0\ ,\ g_1\right] = \nu_{ei}\left(v\right)\frac{v_\parallel}{B}\frac{\partial}{\partial\mu}\left(\mu v_\parallel\frac{\partial g_{1e}}{\partial\mu}\right)$$

where

$$\begin{aligned} \nu_{ei}\left(v\right) &= 2c_{ei}\ \frac{n}{v^3} \\ \text{with } c_{jk} &= 2\pi\frac{e_j^2 e_k^2}{m_j^2}\ln\Lambda \\ &\quad \text{Lorentz gas approximation} \end{aligned}$$

and

$$
\begin{aligned}
C_{jk}\left[f_0,\overline{f}_1\right] &= c_{jk}\frac{1}{\Omega_j}RB_T\ v_{\parallel} \\
&\times\left(\frac{\partial}{\partial\epsilon}\frac{\mathbf{v}}{v_{\parallel}}+\frac{\partial}{B\partial\mu}\frac{\mathbf{v}_{\perp}}{v_{\parallel}}\right)\int d^3v'\ \frac{\partial^2 u}{\partial\mathbf{v}\partial\mathbf{v}'}\left(f_{0k}\frac{\partial f_{0j}}{\partial\psi}-\frac{e_j}{e_k}f_{0j}\frac{\partial f_{0k}}{\partial\psi}\right)
\end{aligned}
$$

One finds

$$
C_{ei}\left[f_0\ ,\ g_1\right]=\nu_{ei}\left(v\right)\frac{v_{\parallel}}{B}\frac{\partial}{\partial\mu}\left(\mu v_{\parallel}\frac{\partial g_{1e}}{\partial\mu}\right)
$$

and

$$
C_{ei}\left[f_0,\overline{f}_1\right]=\nu_{ei}\ \frac{2v_{\parallel}}{eB/m_e}RB_T\ \frac{1}{n}\frac{\partial n}{\partial\psi}f_{Me}
$$

The structure seems similar to

$C_{ei}=$ (Lorentz pitch angle) + (Hirshman Sigmar Clarke-type)

2.4 Krook collision operator

The simplest model

$$
C\left(g\right)=-\nu_k g
$$

2.5 Coulomb collisions

For Coulomb collisions, scattering through the angle $\Delta v/v$ occurs in the time

$$
\tau\sim\frac{1}{\nu}\left(\frac{\Delta v}{v}\right)^2
$$

where ν is the collision frequency for the scattering through $\pi/2$ angle.

2.6 Lorentz collisions

From **Callen**.

It is assumed that the background particles are

- imobile and
- are infinitely heavy and

- are distributed randomly in space

The *test particle* is light compared with the background particles.
Then the result is NOT transfer of energy but *deflection* of the trajectory.
electrons and ions.

An electron travels toward a heavy ions.

$$\mathbf{x} = vt\widehat{\mathbf{e}}_z$$

Relative to a referential with origin at the ion's position the trajectory of the electron will reach the plane at a distance b and will make an angle with the "vertical" axis

$$\mathbf{x} = b\left(\widehat{\mathbf{e}}_x \cos\zeta + \widehat{\mathbf{e}}_y \sin\zeta\right) + vt\widehat{\mathbf{e}}_z$$

and the distance is

$$|\mathbf{x}| = \sqrt{b^2 + v^2t^2}$$

The electric force exerced by the ion on the electron is

$$\mathbf{F} = -\frac{Z_i e^2}{4\pi\varepsilon_0}\frac{\mathbf{x}}{|\mathbf{x}|^3}$$

Due to this force exerced on the electron during its motion in the region around the ion, there will be a change in the *momentum* of the electron

$$\begin{aligned}
m\Delta\mathbf{v} &= \int_{-\infty}^{\infty} dt\ \mathbf{F} \\
&= \int_{-\infty}^{\infty} dt\left(-\frac{Z_i e^2}{4\pi\varepsilon_0}\frac{\mathbf{x}}{|\mathbf{x}|^3}\right) \\
&= \frac{Z_i e^2}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} dt\frac{b\left(\widehat{\mathbf{e}}_x \cos\zeta + \widehat{\mathbf{e}}_y \sin\zeta\right) + vt\widehat{\mathbf{e}}_z}{\left(b^2 + v^2t^2\right)^{3/2}}
\end{aligned}$$

We note that the integration over the last term is zero due to the symmetry $t \in (-\infty, +\infty)$. And

$$\int_{-\infty}^{\infty} dt\frac{1}{\left(b^2 + v^2t^2\right)^{3/2}} = \frac{2}{vb^2}$$

then the two possible (vectorial) parts are

$$\begin{aligned}
m\Delta v_{\perp} &= \frac{2Z_i e^2}{4\pi\varepsilon_0}\frac{1}{vb}\left(\widehat{\mathbf{e}}_x \cos\zeta + \widehat{\mathbf{e}}_y \sin\zeta\right) \\
m\Delta v_{\parallel} &= 0
\end{aligned}$$

However in the next order the parallel momentum is affected in the collision, and this is because of the conservation of the energy

$$\frac{1}{2}m\left|\mathbf{v}\right|^2=\frac{1}{2}m\left|\mathbf{v}+\Delta\mathbf{v}\right|^2$$

$$\begin{aligned}
\Delta v_{\parallel} &= -\frac{1}{2v}\left(\Delta v_{\perp}\right)^2 \\
&= -\frac{1}{2v}\left(\frac{2Z_i e^2}{4\pi\varepsilon_0}\frac{1}{mvb}\right)^2 \\
&= -\frac{2Z_i^2 e^4}{\left(4\pi\varepsilon_0\right)^2 m^2 b^2}\frac{1}{v^3}
\end{aligned}$$

In a population of ions

$$\begin{aligned}
\left\langle F_{\parallel}\right\rangle &= \frac{1}{2}m\frac{d}{dt}\left\langle v_{\parallel}\right\rangle \\
&= n_i v\int d\zeta\int bdb\ m\left\langle \Delta v_{\parallel}\right\rangle \\
&= -4\pi n_i\frac{Z_i^2 e^4}{\left(4\pi\varepsilon_0\right)^2 m}\frac{1}{v^2}\int\frac{bdb}{b^2}
\end{aligned}$$

It appears the need to calculate

$$\int_{b_{\min}}^{b_{\max}}\frac{db}{b}=\left.\ln b\right|_{b_{\min}}^{b_{\max}}$$

To find the limits $b_{\min,\max}$. The minimum distance corresponds to the equality between the electrostatic potential energy and the kinetic energy

$$\frac{q_e q_i}{|x|}=\frac{mv^2}{2}\ \text{ at }\ |x|=b_{\min}$$

$$b_{\min}=\frac{Ze^2}{mv^2/2}\sim\frac{Ze^2}{T_e}=\frac{1}{n\lambda_D^2}$$

Taking into account quantum effects,

$$b_{\min}=\max\left[\frac{Ze^2}{T_e},\frac{\hbar}{mv}\right]$$

The maximum impact parameter is the Debye radius, because at larger distance the electrostatic field of the ion is screened and there is no interaction

$$b_{\max}\sim\lambda_D$$

then

$$\int_{b_{\min}}^{b_{\max}} \frac{db}{b} = \ln b|_{b_{\min}}^{b_{\max}} = \ln \frac{b_{\max}}{b_{\min}} \equiv \ln \Lambda \sim 17$$

Now we can return to the calculation of the friction force in the parallel direction

$$\begin{aligned} m\frac{dv_{\|}}{dt} &= m\frac{\langle \Delta v_{\|} \rangle}{\Delta t} = \langle F_{\|} \rangle \\ &= -\left[4\pi n_i \frac{Z_i^2 e^4}{(4\pi\varepsilon_0)^2 m^2} \frac{1}{v^3} \ln \Lambda \right] m v_{\|} \\ &\equiv -\nu_{ei} m v_{\|} \end{aligned}$$

where

$$\begin{aligned} \nu_{ei} &= 4\pi n_i \frac{Z_i^2 e^4}{(4\pi\varepsilon_0)^2 m^2} \frac{1}{v^3} \ln \Lambda \\ &= \frac{4\pi n_e Z_i e^4}{(4\pi\varepsilon_0)^2 m^2} \ln \Lambda \quad \text{(neutrality)} \\ &= \frac{\omega_{pe} \ln \Lambda}{n_e \lambda_D^2} \left(\frac{T_e}{m_e v^2/2} \right)^{3/2} \end{aligned}$$

Now consider the drifting electrons $(V_{\|})$ described in kinetic theory by a Maxwellian

$$\begin{aligned} f_e(v) &= \frac{n_e}{(2\pi T_e/m_e)^{3/2}} \exp\left[-\frac{(\mathbf{v} - V_{\|}\widehat{\mathbf{e}}_z)^2}{2T_e/m_e} \right] \\ &\approx n_e \left(\frac{m_e}{2\pi T_e} \right)^{3/2} \exp\left[-\frac{v^2}{2T_e/m_e} \right] \left(1 - \frac{m_e v_{\|} V_{\|}}{T_e} + ... \right) \end{aligned}$$

The total friction force

$$\int d^3 v \; f_e(v) \langle F_{\|} \rangle = -m_e n_e \nu_e V_{\|}$$

where

$$\begin{aligned} \nu_e &= \frac{4\sqrt{2\pi} n_i Z_i^2 e^4}{3\sqrt{m_e} (4\pi\varepsilon_0)^2 T_e^{3/2}} \ln \Lambda = \frac{4}{3} \frac{\sqrt{2\pi}}{\sqrt{m_e}} \frac{e^4 Z_i^2}{(4\pi\varepsilon_0)^2} \ln \Lambda \frac{n_i}{T_e^{3/2}} \\ &= \frac{1}{\tau_e} \end{aligned}$$

NOTE

See **Landau Lifshitz Mechanics** for the Rutherford cross section $\sin^{-4}\left(\frac{\theta}{2}\right)$.
END

A simple Lorentz model for the *friction force* is given in **Stacey 1992 Poloidal Rotation**

$$\mathbf{R} = -n_j m_j \sum_{k \neq j} \nu_{jk} \left(\mathbf{v}_j - \mathbf{v}_k\right)$$

valid when

$$m_j \gg m_k$$

which means that a heavy particles moves in an environment of light particles.

For comparison, in **Hirschman 1977**, the friction force on species a is defined as the mean rate of collisional momentum generation

$$\begin{aligned} R_a &= \int d^3v \; m_a v_{\|} \; C_a \\ \text{where } \; C_a &= \sum_b C_{ab} \end{aligned}$$

Similar for the heat "friction"

$$H_a = \int d^3v \; m_a v_{\|} \; \left[\left(\frac{v}{v_{th,a}}\right)^2 - \frac{5}{2}\right] \; C_a$$

2.7 Collision operator for NBI of Cordey Houghton 1973

This part should also be included in NBI.

NOTE
there is a difference between the NBI ions and the α particles.

The NBI ions are generated with a certain direction (of the beam) and they are NOT isotropic. NBI ion population is *anisotropic*.

The α particles are *isotropic*.

In the FP eq for NBI ions the convective derivative does not usually appear. The evolution is in the velocity space (slowing down, pitch angle scattering) for the transfer of energy to the background plasma.

For α particles, the FP includes the spatial derivative (the convection) from the beginning and the neoclassical effects are present.
END

The Fokker Plank eq.

- in uniform magnetic field (no toroidality, no trapped particles)
- axisymmetry in velocity space
- use of the Rosenbluth potentials g and h;
- the background ions and electrons are cold Maxwellians

The eq

$$
\begin{aligned}
& \frac{1}{\Gamma_{hot}} \frac{\partial f}{\partial t} \\
= & \frac{1}{v^2} \frac{\partial}{\partial v} \left(v^2 \ f \ \frac{\partial h}{\partial v} \right) \\
& + \frac{1}{2} \frac{1}{v^2} \frac{\partial}{\partial \left(v^2 \right)} \left(v^2 \ f \ \frac{\partial^2 g}{\partial v^2} \right) \\
& + \frac{1}{2} \frac{1}{v^2} \frac{\partial}{\partial \xi^2} \left[f \ \frac{1}{v} \left(1 - \xi^2 \right) \frac{\partial g}{\partial v} \right] \\
& + \frac{1}{2} \frac{1}{v^2} \frac{\partial}{\partial v} \left(-2f \ \frac{\partial g}{\partial v} \right) \\
& + \frac{1}{2} \frac{1}{v^2} \frac{\partial}{\partial \xi} \left(2f \ \xi \frac{1}{v} \frac{\partial g}{\partial v} \right)
\end{aligned}
$$

The Rosenbluth potentials

$$
\begin{aligned}
h &= \frac{4}{\sqrt{\pi}} \sum_{j=e,i} n_j \left(\frac{m_{hot} + m_j}{m_j v_j^3} \right) \\
& \times \left[\int_0^v dv' \ v'^2 \frac{1}{v} \exp \left(- \frac{v'^2}{v_{th,j}^2} \right) + \int_v^\infty dv' \ v' \ \exp \left(- \frac{v'^2}{v_{th,j}^2} \right) \right]
\end{aligned}
$$

$$
\begin{aligned}
g = & \frac{4}{\sqrt{\pi}} \sum_{j=i,e} \frac{n_j}{v_{th,j}^3} \left[v \int_0^v dv' \exp \left(- \frac{v'^2}{v_{th,j}^2} \right) v'^2 \left(1 + \frac{v'^2}{3v^2} \right) \right. \\
& \left. + \int_v^\infty dv' \ \exp \left(- \frac{v'^2}{v_{th,j}^2} \right) v'^3 \left(1 + \frac{v^2}{3v'^2} \right) \right]
\end{aligned}
$$

with the coefficient

$$
\Gamma_{hot} = 4\pi \frac{Z_{hot}^2 e^4}{m_{hot}^2} \log \Lambda
$$

Range of velocities

$$v_{th,i} \ll v_{hot} \ll v_{th,e}$$

This will simplify the expressions

$$\begin{aligned}
\tau_s \frac{\partial f}{\partial t} &= \frac{1}{v^2}\frac{\partial}{\partial v}\left[\left(v_c^3 + v^3\right)\ f\right] \\
&+ \frac{1}{2}\frac{m_i}{m_{hot}}\frac{v_c^3}{v^3}\frac{\partial}{\partial \xi}\left[\left(1-\xi^2\right)\frac{\partial f}{\partial \xi}\right] \\
&+ \widetilde{S}\left(v - v_0\right)\ \times \delta\left(\xi - \xi_0\right)\ \tau_s
\end{aligned}$$

where

$$\begin{aligned}
v_c &= \left(\frac{3\sqrt{\pi}}{4}\frac{m_e}{m_i}\right)^{1/3}\ v_{th,e} \\
&= \text{the value of the hot ion velocity} \\
&\quad \text{at which the rate of transfer of energy} \\
&\quad \text{from the hot ions to the background electrons} \\
&\quad \text{equals the rate of transfer of energy} \\
&\quad \text{from the hot ions to the background ions}
\end{aligned}$$

$$v_0 = \text{peak of the injection velocity}$$

$$\tau_s = \frac{m_i}{m_{hot}} v_c^3\ n\ \Gamma_{hot}$$

The source is constant and all particles are injected at the angle with the magnetic field which is

$$\arccos \xi_0 = \arccos\left(\frac{v_{\parallel 0}}{v_0}\right)$$

NOTE that in **Fowler code** the term of slowing down (the first, not the pitch angle scattering) is explicitely separated into two contributions, from electrons and from ions.

The equation in **Fowler** is (see **NBI**)

$$\begin{aligned}
\tau_s \frac{\partial f}{\partial t} &= -\frac{\tau_s}{\tau_{cx}} f \qquad\qquad \text{charge exchange} \\
&+\frac{1}{x^2}\frac{\partial}{\partial x}\left[\left(x^3 - 2Bx + x_c^3 + \frac{C}{x^2}\right) f\right] \qquad \text{drag} \\
&+\frac{1}{x^2}\frac{\partial^2}{\partial x^2}\left[\left(Bx^2 + \frac{C}{x}\right) f\right] \qquad \text{diffusion in velocity} \\
&+\frac{D}{x^3}\left(1 - \frac{D_1}{x^2} + D_2 x\right)\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) \text{ angular scattering} \\
&+\tau_s \sum_l \dot{n}_{fl} S_l (x,\theta) \quad \text{source of injected ions}
\end{aligned}$$

without electric field E^* and without compression of the plasma column. Here

$$\begin{aligned}
x &= \frac{v}{v_0} \\
v_0 &= \text{the speed of the NBI ions}
\end{aligned}$$

$$x_e = \frac{v_e}{v_0} \ , \ x_i = \frac{v_i}{v_0}$$

$$x_c = \text{critical velocity} = \frac{v_c}{v_0}$$

The terms that we discuss are the "drag" and the "diffusion in velocity". The coefficients in these terms are

$$\begin{aligned}
B &= \frac{1}{2}\frac{m_e}{m_{fast}} x_e^2 \\
C &= \frac{1}{2}\frac{m_i}{m_{fast}} x_i^2 x_c^3
\end{aligned}$$

and this shows the separation between electrons and ions.

END of the NOTE

The solution of the equation for f.

It is adopted the series expansion AND separation of variables v and ξ ,

$$\begin{aligned}
f &= \sum_{n=0}^{\infty} A_n (v) \ P_n (\xi) \\
P_n (\xi) &\equiv \text{Legendre polynomials}
\end{aligned}$$

Approximation adopted by **Cordey Houghton** for the range of velocities

$$v > v_c$$

Here is the domain where the NBI ions are slowed down collisionally by the *electrons*. Then the term of *pitch angle scattering* can be neglected.

The remaining equation is solved for the source term

$$\begin{aligned}
\widetilde{S}(v - v_0) &= S \frac{\exp\left[-\frac{(v-v_0)^2}{\delta^2}\right]}{v^2 \delta \sqrt{\pi}} \\
S &\equiv \text{number of particles injected per second} \\
& \quad \text{per unit volume}
\end{aligned}$$

We note that the source is *smeared out* with extension in velocity space around the central NBI-ions velocity v_0 of width δ.

Define

$$v^*(v,t) = \left[\left(v^3 + v_c^3\right)\exp\left(\frac{3t}{\tau_s}\right) - v_c^3\right]^{1/3}$$

The solution is

$$\begin{aligned}
f &= \tau_s \frac{1}{v^3 + v_c^3} \delta(\xi - \xi_0) \int_v^{v^*} dv' \ v'^2 \ \widetilde{S}(v' - v_0) \\
&= \frac{1}{2} \tau_s \frac{1}{v^3 + v_c^3} S \ \delta(\xi - \xi_0) \\
& \qquad \times \left[\operatorname{erf} c\left(\frac{v - v_0}{\delta}\right) - \operatorname{erf} c\left(\frac{v^* - v_0}{\delta}\right)\right]
\end{aligned}$$

where

$$\begin{aligned}
\operatorname{erf} c(x) &= 1 - \operatorname{erf}(x) \\
&= \text{complimentary error function}
\end{aligned}$$

The last expression is written

$$f \equiv r(v) \ \delta(\xi - \xi_0)$$

For large time

$$\begin{aligned}
t &\rightarrow \infty \\
v^* &\rightarrow \infty
\end{aligned}$$

the second erf function is zero and for

$$\begin{aligned} \frac{v - v_0}{\delta} &\gg 1 \\ (\delta &\equiv \text{half width of the source}) \end{aligned}$$

Then

$$f = \tau_s \frac{1}{v^3 + v_c^3} S \ \delta\left(\xi - \xi_0\right)$$

This is a solution obtained after the approximations have reduced the problem to *one dimension*.

Exact solution by series: little angular spreading for

$$v > v_c$$

This means that the scattering and slowing down of the hot ions *on electrons* does not produce spreading in

$$\xi = \frac{v_\parallel}{v}$$

This is because the ions are heavy.

The spreading begins when the hot ions slow down and scatter on background ions, equally heavy.

The paper also discusses the formation of an electric field due to the separation of charges after ionization.

2.8 Fokker Planck equation for NBI ions Cordey Core

Also in *MBI.tex*.

The next level.

The expression contains

$$\frac{\partial^2 f}{\partial u^2} \ , \ \frac{\partial f}{\partial u} \ , \ f \ , \ \text{source}$$

They correspond to

- diffusion in velocity space $\frac{\partial^2 f}{\partial u^2}$
- drag (slowing down) $\frac{\partial f}{\partial u}$

The solution is obtained with the expansion and separation of variables.
The equation
only velocity space and charge-exchange

$$f \equiv \text{ hot ion distribution}$$

$$\begin{aligned} & \frac{\partial f}{\partial t}+\frac{Z_{hot}eE^*}{m_{hot}}\left(\frac{\left(1-\xi^2\right)}{v}\frac{\partial f}{\partial \xi}+\xi\frac{\partial f}{\partial v}\right) \\ = \quad & C\left\{\frac{1}{2v^2}\frac{\partial}{\partial v^2}\left(v^2\frac{\partial^2 g}{\partial v^2}f\right)-\frac{1}{v^2}\frac{\partial}{\partial v}\left[f\left(v^2\frac{\partial h}{\partial v}+\frac{\partial g}{\partial v}\right)\right]\right. \\ & \qquad \left.-\frac{1}{2v^3}\frac{\partial g}{\partial v}\frac{\partial}{\partial \xi}\left(1-\xi^2\right)\frac{\partial f}{\partial \xi}\right\} \\ & -\frac{1}{\tau_{cx}}f \qquad \text{charge exchange with background} \\ & +S\delta\left(v-v_0\right)K\left(\xi\right) \quad \text{source and spreading in } \xi \end{aligned}$$

where

$$g,h \equiv \text{Rosenbluth potentials}$$

$$C=\frac{4\pi e^4 Z_{hot}^2}{m_{hot}^2}\log\Lambda$$

The modified electric field

$$E^*=E\left(1-\frac{1}{Z}\right)$$

$$\xi=\frac{v_{\parallel}}{v}$$

$$Z=\sum\frac{n_i Z_i^2}{n_e} \quad \text{Z effectiv}$$

NOTE

again the absence of the convective derivative, which is justified by the anisotropy of the beam, in contrast with the case of α particles.

Therefore no *drift* hence no neoclassical term.

END

Explanation regarding the term with the electric field,

$$\frac{Z_{hot}eE^*}{m_{hot}}\left(\frac{\left(1-\xi^2\right)}{v}\frac{\partial f}{\partial \xi}+\xi\frac{\partial f}{\partial v}\right)$$

There are two parts.

One is the *acceleration* of the new hot ions by the electric field

$$\frac{Z_{hot}eE^*}{m_{hot}}\frac{\left(1-\xi^2\right)}{v}\frac{\partial f}{\partial \xi}$$

where ZeE $\sim$force, $\frac{ZeE}{m_{hot}}$ $\sim$acceleration, $\times\frac{\partial f}{\partial v}$ $\sim$change of f by E. Factor $\sim v_\perp^2/v$.

The other is the *drag exerced against the new ions* by the electrons in their motion in the electric field

$$\frac{Z_{hot}eE^*}{m_{hot}}\xi\frac{\partial f}{\partial v}$$

Electrons move in opposite direction than the (fast) ions and exert a friction.

To obtain the acceleration of the hot ions and the drag by the (oppositely moving) electrons the Rosenbluth potentials are expressed as

$$\begin{aligned} h &= h_0 + h_1(v)\,\xi \ \times E \\ g &= g_0 + g_1(v)\,\xi \ \times E \end{aligned}$$

The two functions h_1 and g_1 are determined through the method for Spitzer Harm.

The *spread* of the source in the pitch angle $\xi \equiv v_\parallel/v$ is represented by $K(\xi)$.

The Rosenbluth potentials.

Are simplified by assuming

$$v_i \ll v_{hot} \ll v_e$$

The theory is extended in the *energy of the hot ions* subjected to slowing down, up to

$$v_{hot}|^{\text{limit}} \sim 1.5 \times v_{th,i}$$

(probably in **Fowler** to $2 \times v_{th,i}$).

$$v^2\frac{\partial h}{\partial v} + \frac{\partial g}{\partial v} = -\frac{4}{3\sqrt{\pi}}\frac{m_{hot}}{m_e}v^3\frac{n_e}{v_{th,e}^3} - m_{hot}\sum_{j(ions)}\frac{n_j Z_j^2}{m_j}$$

$$\frac{\partial g}{\partial v} = \sum n_j Z_j^2 = n_e Z_{eff}$$

$$v^2\frac{\partial^2 g}{\partial v^2} = \frac{4}{3\sqrt{\pi}}v^2\frac{n_e}{v_{th,e}} + \frac{1}{v}\sum_{j(ions)} n_j Z_j^2 v_{th,j}^2$$

These approximations allow to simplify the expressions inside the collision operator, RHS

$$\frac{1}{2v^2}\frac{\partial}{\partial v^2}\left(v^2\frac{\partial^2 g}{\partial v^2}f\right) \rightarrow \frac{1}{2v^2}\frac{\partial}{\partial v^2}\left[\left(A_1 v^2 + \frac{A_2}{v}\right)f\right]$$

and

$$\begin{aligned} & -\frac{1}{v^2}\frac{\partial}{\partial v}\left[f\left(v^2\frac{\partial h}{\partial v} + \frac{\partial g}{\partial v}\right)\right] \\ \rightarrow & \\ & -\frac{1}{v^2}\frac{\partial}{\partial v}\left[f\ \left(-A_3\ v^3 - A_4\right)\right] \end{aligned}$$

and

$$\begin{aligned} & -\frac{1}{2v^3}\frac{\partial g}{\partial v}\frac{\partial}{\partial \xi}\left(1-\xi^2\right)\frac{\partial f}{\partial \xi} \\ \rightarrow & \\ & -\frac{1}{2v^3}\ A_5\ \frac{\partial}{\partial \xi}\left(1-\xi^2\right)\frac{\partial f}{\partial \xi} \\ = & \ -\frac{1}{2v^3}\ A_5\ \mathcal{L}\left[f\right] \end{aligned}$$

where we have denoted constants A_i for

$$\begin{aligned} A_1 &= \frac{4}{3\sqrt{\pi}}\frac{n_e}{v_{th,e}} \\ A_2 &= \frac{1}{v}\sum_{j(ions)} n_j Z_j^2 v_{th,j}^2 \end{aligned}$$

$$\begin{aligned} A_3 &= \frac{4}{3\sqrt{\pi}}\frac{m_{hot}}{m_e}\frac{n_e}{v_{th,e}^3} \sim \frac{n}{T^{3/2}} \\ A_4 &= m_{hot}\sum_{j(ions)}\frac{n_j Z_j^2}{m_j} \end{aligned}$$

$$A_5 = n_e Z_{eff}$$

An approximation based on neutrality

$$\sum_{j(ions)} \frac{n_j Z_j^2}{m_j} \approx \frac{n_e}{m_i}$$

$$\sum_{j(ions)} n_j Z_j^2 v_{th,j}^2 = n_e v_{th,i}^2$$

and $Z_1 = 1$ (basic ions are hydrogen). Notation

$$\begin{aligned} u &\equiv \frac{v}{v_0} \\ v_0 &\equiv \text{velocity at birth} \end{aligned}$$

Then

$$\begin{aligned} & \delta a \frac{\partial^2 f}{\partial u^2} \qquad \text{(diffusion)} \\ & +b \frac{\partial f}{\partial u} \qquad \text{(drag)} \\ & +d\ f \\ & +\delta a r\ \mathcal{L}(f) - \frac{\eta}{u}\left(1-\xi^2\right)\frac{\partial f}{\partial \xi} \\ = & -\tau_s S\ \delta(u-1) K(\xi) \end{aligned}$$

where

$$\begin{aligned} a(u) &= 1 + \frac{2\beta}{u^3} \\ b(u) &= \frac{u_c^3}{u^3} + u - \eta\xi - 4\delta\frac{\beta}{u^3} + 4\delta\frac{1}{u} \end{aligned}$$

$$\begin{aligned} d(u) &= 3 - \frac{\tau_s}{\tau_{cx}} + \frac{4\delta\beta}{u^5} + \frac{2\delta}{u^2} \\ r(u) &= \frac{m_i Z_{eff}}{2m_{hot}} \frac{u_c^3}{u^3} \frac{1}{\delta\ a} \end{aligned}$$

$$u_c^3 = \frac{v_c^3}{v_0^3}$$

This u_c is the critical velocity where the transfer of energy from the hot ions to the background electrons changes to transfer of energy from hot ions to background ions.

$$v_c = \left(\frac{3\sqrt{\pi}}{4}\frac{m_e}{m_i}\right)^{1/3} v_{th,e}$$

The parameter

$$\eta = \frac{Z_{hot} e E^*}{m_{hot} v_0} \tau_s$$

This ratio is (acceleration of hot ions due to E^*) / $\left(\frac{v_0}{\tau_s}\right)$. Or, change of the velocity due to acceleration by E^* in a collision time compared (divided to) the initial velocity. This parameter η is small, the effect = change of velocity of the hot ion, of the acceleration by electric E^* field in a collision time τ_s is much smaller than the initial velocity of the hot ion. $\eta \ll 1$.

In the WKBJ solution of the simplified equation (after these approximations) the terms with η will be neglected.

Further, the Legendre operator

$$\mathcal{L} \equiv \frac{\partial}{\partial \xi} \left(1 - \xi^2\right) \frac{\partial}{\partial \xi} \quad \text{pitch angle operator}$$

In the approximation of the equation such as to apply WKBJ this term will be replaced by a constant, α^2.

The parameter

$$\delta = \frac{1}{2} \frac{m_e v_{th,e}^2}{m_{hot} v_0^2} = \frac{1}{2} \frac{T_e}{\epsilon_0}$$

This is a definition for $\epsilon_0 \sim m_{hot} v_0^2 / 2$, the initial energy of a hot ion. δ is the ratio of the electron thermal energy to the hot ion initial energy. *This will be considered a small parameter.* $\delta \ll 1$. Note that it multiplies the diffusion term $\delta a \frac{\partial^2 f}{\partial u^2}$, and the pitch angle operator.

Other constants

$$\beta = 0.66 \frac{v_{th,e} \; v_{th,i}^2}{v_0^3}$$

$$\begin{aligned} \tau_s &= \frac{3}{16\sqrt{\pi}} \frac{m_e m_{hot}}{e^4 Z_h^2 \ln \Lambda} \frac{v_{th,e}^3}{n_e} \\ &\sim \frac{T_e^{3/2}}{n_e} \quad \text{(usual factor)} \end{aligned}$$

The expression contains

$$\frac{\partial^2 f}{\partial u^2} \; , \; \frac{\partial f}{\partial u} \; , \; f \; , \; \text{source}$$

They correspond to

- diffusion in velocity space $\frac{\partial^2 f}{\partial u^2}$
- drag (slowing down) $\frac{\partial f}{\partial u}$

The solution is obtained with the expansion and separation of variables.

The term

$$b\frac{\partial f}{\partial u}$$

is eliminated by the transformation

$$f = y\left(u, \xi\right)\ \exp\left(-\int_1^u\ du\ \frac{b}{2\delta a}\right)$$

obtaining

$$\begin{aligned}
& \frac{\partial^2 y}{\partial u^2} \\
& -\left[\frac{b^2}{4\delta^2 a^2} - \frac{d}{\delta a} + \frac{1}{2\delta}\frac{\partial}{\partial u}\left(\frac{b}{a}\right)\right] y \\
& +\frac{\exp\left(\int_1^u\ du\ \frac{b}{2\delta a}\right)}{\delta a} \\
& \times\left(\mathcal{L}\ \left\{y\ \exp\left(-\int_1^u\ du\ \frac{b}{2\delta a}\right)\right\} \text{ pitch angle operator, of } f\right. \\
& \qquad \left. -\eta\left(1-\xi^2\right)\frac{\partial}{\partial \xi}\left[y\ \exp\left(-\int_1^u\ du\ \frac{b}{2\delta a}\right)\right]\right) \\
= \quad & -\tau_s\ \frac{S\ \exp\left(\int_1^u du\ \frac{b}{2\delta a}\right)\ \delta\left(u-1\right)\ K\left(\xi\right)}{\delta a}
\end{aligned}$$

The parameter δ is small

$$\delta \ll 1$$

An approximation

$$\begin{aligned}
& \text{replace } \mathcal{L} \text{ by} \\
\alpha^2 \quad = \quad & \text{constant}
\end{aligned}$$

and

terms of order η are neglected

Then

$$\frac{\partial^2 y}{\partial u^2} + Qy + P\alpha^2 = \text{source}$$

where Q and P are notations for the coefficients, and the solutions are

$$y_{\pm} = \frac{1}{\sqrt{D}} \exp\left(\pm \int_1^u du \; D\right)$$

$$\begin{aligned} D^2 &= \frac{b^2}{4\delta^2 a^2} - \frac{d}{\delta a} + \frac{1}{2\delta}\frac{\partial}{\partial u}\left(\frac{b}{a}\right) \\ &\quad +\alpha^2 \frac{1}{\delta a} \end{aligned}$$

From this solution one can return to f,

$$f_{\pm} = \frac{A_{\pm}}{\sqrt{D}} \exp\left[\int_1^u du \; \left(-\frac{b}{2\delta a} \pm D\right)\right]$$

A. the function f_- must be retained for the velocities above the injection velocity v_0. The distribution function exists in this region because there was *diffusion* in the velocity space of the hot ions. In other treatments this diffusion which creates velocities higher than that of the beam v_0 is not considered.

$$f_- \sim \exp\left[-\frac{1}{\delta}\int_1^u du \; \frac{b}{a}\right]$$

Boundary condition

$$v^3 f \rightarrow 0 \;\; \text{at} \;\; u \rightarrow \infty$$

B. the function f_+ must be retained for $v < v_0$.

$$f_+ \sim \exp\left[\int du \; \frac{\alpha^2 - d}{b}\right]$$

Boundary condition : f finite at $u = 0$.

Solution in this region is obtained by expansion in the small parameter η,

$$y = y^0 + \eta \; y^1 + \eta^2 \; y^2 + ...$$

The equation is

$$\begin{aligned} & \frac{\partial^2 y^0}{\partial u^2} - h y^0 + r \; \mathcal{L}\left[y^0\right] \\ = \; & -\tau_s \; S \; \exp\left(\int_1^u du \; \frac{b}{2\delta a}\right) \delta\left(u-1\right) \; K\left(\xi\right) \frac{1}{\delta a} \end{aligned}$$

where

$$h(u) = \frac{b^2}{4\delta^2 a^2} - \frac{d}{\delta a} + \frac{1}{2\delta}\frac{\partial}{\partial u}\left(\frac{b}{a}\right)$$

Solution is obtained by expansion in Legendre polynomials

$$y^0 = \sum_{n=0}^{\infty} c_n^{(0)} \; P_n(\xi)$$

2.9 The Fokker-Planck equation for NBI+ICRH Cox Start

Also in *NBI.*

The generation of current by ICRH. The increase of the energy of the *minority* (heated) ions leads to reduction of the collisionality of these ions with the electrons. The background ions preserve the collisionality, so there will be a difference - a flux - resulting for the two ion types.

Corrected scenario: start with NBI but with different ions, like He^3 or heavy ions. Then ICRH.

The Fokker Planck

$$\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial t}\right)_{wave} + \left(\frac{\partial f}{\partial t}\right)_C + S_f(v,\xi)$$

The collisions between the hot ions and a *Maxwellian* plasma.
The magnetic field is uniform.
Coulomb collision operator for

$$v_{th,i} \ll v_{hot} \ll v_{th,e}$$

$$\left(\frac{\partial f}{\partial t}\right)_C = \frac{1}{\tau_s}\left[b(u)\frac{\partial f}{\partial u} + df + r(u)\; Lf\right]$$

where

$$\begin{aligned} u &\equiv \frac{v}{v_0} \\ v_0 &\equiv \text{injection velocity} \end{aligned}$$

$$b(u) = \frac{u_c^3}{u^2} + u$$

$$u_c^3 = \frac{v_c^3}{v_0^3}$$

$$= \frac{3\sqrt{\pi}}{4}\ \frac{m_e}{m_i}\overline{Z}\ \frac{v_{th,e}^3}{v_0^3}$$

$$d = 3 - \frac{\tau_s}{\tau_{cx}}$$

$$\tau_s = \frac{1}{4}\frac{3}{4\sqrt{\pi}}\ m_e m_h \frac{1}{Z_h^2 e^2}\frac{1}{\ln\Lambda}\ \frac{v_{th,e}^3}{n_e}$$

$$r\left(u\right) = \beta\frac{u_c^3}{u^3}$$

$$\beta = \frac{1}{2}\frac{m_i}{m_h}\frac{Z_{eff}}{\overline{Z}}$$

$$Z_{eff} = \sum_i \frac{n_i Z_i^2}{n_e}$$

$$\overline{Z} = \sum_i \frac{m_h}{m_i}\frac{n_i Z_i^2}{n_e}$$

$$L \equiv \frac{\partial}{\partial\xi}\left(1-\xi^2\right)\frac{\partial}{\partial\xi}$$

$$\xi \equiv \frac{v_\parallel}{v}$$

The effect of the wave

$$\left(\frac{\partial f}{\partial t}\right)_W = D_c\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\left[v_\perp\frac{\partial f}{\partial v_\perp}\right]\delta\left(\omega-\Omega_c\right)$$

After change of variables

$$\left(v_\parallel, v_\perp\right) \rightarrow (u,\xi)$$
$$\left(u = \frac{v}{v_0}\right)$$

$$\left(\frac{\partial f}{\partial t}\right)_w = \frac{\gamma}{\tau_s}\frac{1}{u} \times\left[\left(1-\xi^2\right)u\frac{\partial^2 f}{\partial u^2} + \left(1+\xi^2\right)\frac{\partial f}{\partial u} - 2\xi\left(1-\xi^2\right)\frac{\partial^2 f}{\partial u\partial\xi} + \frac{\xi}{u}\ \frac{\partial}{\partial\xi}\left(1-\xi^2\right)\xi\frac{\partial f}{\partial\xi}\right]$$

$$\gamma = \frac{D_c}{v_0^2} \tau_s$$

Without the wave

$$\begin{aligned}
& b(u) \frac{\partial F}{\partial u} + dF \\
& + \beta \frac{u_c^3}{u^3} \frac{\partial}{\partial \xi} \left(1 - \xi^2\right) \frac{\partial F}{\partial \xi} \\
& + \tau_s S \frac{K(\xi)}{2\pi K_0 v_0^3} \ \delta(u-1) \\
= \ & 0
\end{aligned}$$

The solution

$$F(u,\xi) = \sum_{n=0}^{\infty} a_n^0(u) \ \ P_n(\xi)$$

2.10 Collision operators with impurities (Dobrowolny Nocentini)

The paper **Dobrowolny Nocentini NF 1974, page 831.**

Collisions within the same species

$$\begin{aligned}
C_{ss}(f_0|f_1) \ = \ & \nu_{ss} \left\{ \frac{v_\parallel}{B} \frac{\partial}{\partial \mu} \left[\mu v_\parallel \frac{\partial f_1^s}{\partial \mu} \right] \right. \\
& \left. + \frac{v_\parallel f_0^s}{\int d^3 v \ \nu_{ss} v_\parallel^2 f_0^s} \int d^3 v' v_\parallel' \ \nu_{ss}(v') f_1^s(v') \right\}
\end{aligned}$$

where

$$\nu_{ss} = \nu_{ss}^{(0)} \left(\frac{v_{th,s}}{v}\right)^3 h\left(\frac{v}{v_{th,s}}\right)$$

where

$$\nu_{ss}^{(0)} = \frac{4\pi e_s^2 e_{s'}^2}{m_s^2} \ln \Lambda \frac{N_{s'}}{v_{th,s}^3}$$

and

$$h(z) = \frac{1}{\sqrt{\pi} z} \exp\left(-z^2\right) + \left(1 - \frac{1}{2z^2}\right) \frac{2}{\sqrt{\pi}} \int_0^z dt \exp\left(-t^2\right)$$

The collisions of particles s with particles s' with

$$m_s \ll m_{s'}$$
$$\text{(electrons) with (ions)}$$

$$C_{ss'}(f_0|f_1) = \nu_{ss'}\left\{\frac{v_\parallel}{B}\frac{\partial}{\partial\mu}\left[\mu v_\parallel \frac{\partial f_1^s}{\partial \mu}\right] + v_\parallel f_0^s \frac{m_s}{T_s}\frac{1}{N_{s'}}\int d^3v' f_1^{s'}(v')\right\}$$

(a $v_\parallel$ should be inserted as factor in the integral in the RHS).

NOTE

In **Connor 1973** the operator of collision is ADOPTED in the form

$$C\left(\widehat{f_j}\right) = \sum_k C_{jk}\left(\widehat{f_j}, \widehat{f_k}\right)$$

$$C_{jk} = \nu_{jk}(x_j)\left[v_\parallel \frac{\partial}{\partial\mu} v_\parallel \frac{\mu}{B}\frac{\partial}{\partial\mu}\widehat{f_j} + v_\parallel F_{M,j}\frac{\int d^3v\ v_\parallel \nu_{kj}\ \widehat{f_k}}{\int d^3v\ v_\parallel^2\ \nu_{kj}\ F_{M,k}}\right]$$

where

$$\nu_{jk}(x_j) \equiv \text{collision freq. for diffusion in pitch angle (Trubnikov)}$$

$$f_j = F_{M,j} + \widehat{f_j}$$

$$\widehat{f_j} \equiv \text{first Larmor radius correction to the Maxwellian}$$

It looks that they should compare

$$\text{(Nocentini)}\ \frac{m_s}{T_s}\frac{1}{N_{s'}} \rightarrow \frac{1}{\int d^3v\ v_\parallel^2\ \nu_{kj}\ F_{M,k}}\ \text{(Connor1973)}$$

The part that, inside $\nu_{jk}(x_j)$ which depends on velocity (and is integrated over in the numerator and denominator) is, in **Connor1973**

$$x_j^{-3/2} h(x_k)$$

END

2.11 Collision operator used by Taguchi for poloidal rotation damping in the *plateau* regime

The banana regime. The equation

$$v_{\parallel}\frac{B_{\theta}}{B}\frac{\partial f_i^{(1)}}{r\partial\theta}+\mathbf{v}_{D,i}\cdot\boldsymbol{\nabla}\psi\ \left(x^2-\frac{3}{2}\right)\frac{1}{T_i}\frac{dT_i}{d\psi}\ f_i^{(0)}+C_{ii}\left(f_i^{(1)}\right)$$

with notations

$$\begin{aligned}\mu &= \frac{v_{\perp}^2}{2B}\\ \sigma &\equiv \frac{v_{\parallel}}{\left|v_{\parallel}\right|}\end{aligned}$$

$$x\equiv\frac{v}{v_{th,i}}$$

$$v_{th,i}=\left(\frac{2T_i}{m_i}\right)^{1/2}$$

The radial projection of the neoclassical drift velocity

$$\mathbf{v}_{D,i}\cdot\boldsymbol{\nabla}\psi=\frac{I}{e_i/m_i}v_{\parallel}\ \nabla_{\parallel}\left(\frac{v_{\parallel}}{B}\right)$$

NOTE in **Wong Hinton** the formula includes rotation

$$\mathbf{v}_D\cdot\boldsymbol{\nabla}\psi=\frac{m}{e}v_{\parallel}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\left(I\frac{v_{\parallel}}{B}+\omega R^2\right)$$

where

$$v_{\parallel}=\left\{2\left[\epsilon-\mu B-\frac{e}{m}\widetilde{\Phi}_0+\frac{\omega^2R^2}{2}\right]\right\}^{1/2}$$

END

In order to take into account the collisions one expands the *ion* distribution function of *first order*, $f_i^{(1)}$ is a series

$$f_i^{(1)}=f_{i,0}^{(1)}+\nu_{ii}\ f_{i,1}^{(1)}+...$$

the collision frequency being

$$\nu_{ii}=4\pi\frac{e^4}{m_i^2}\ln\Lambda\ \frac{n_i}{v_{th,i}^3}$$

The lowest order

$$f_{i,0}^{(1)} = -\frac{I}{e_i/m_i} v_{th,i} \left[\xi \; x \left(x^2 - \frac{3}{2} \right) f_i^{(0)} + g\left(x, \lambda, \sigma\right) \right] \frac{1}{T_i} \frac{dT_i}{d\psi}$$

where

$$\begin{aligned} \xi &\equiv \frac{v_\parallel}{v} \\ \lambda &= \frac{1-\xi^2}{B} = \frac{v_\perp^2}{v^2} \frac{1}{B} \end{aligned}$$

The correction g exists only for circulating particles

$$\begin{aligned} g &\neq 0 \;\; \text{for } \lambda < \lambda_c \text{ circulating particles} \\ g &= 0 \;\; \text{for } \lambda_c < \lambda \\ \text{where } \lambda_c &= \frac{1}{B_{\max}} \end{aligned}$$

Imposing the *periodicity* one obtains

$$\int_0^{2\pi} d\theta \; \frac{1}{v_\parallel} \frac{B}{B_\theta} \; C_{ii} \left[\frac{1}{B} \xi \; x \left(x^2 - \frac{3}{2} \right) f_i^{(0)} + g \right] = 0$$

The paper of **Taguchi** arrives at an equation of the form

$$\left\langle \frac{B}{v_\parallel} \left[C_a \left(f_{a1}^{(0)} \right) - \left(\frac{\partial f_{a1}^{(0)}}{\partial t} + e_a \frac{\partial \Phi}{\partial t} \frac{\partial f_{a1}^{(0)}}{\partial \epsilon} \right)_{v=\mathrm{ct}} \right] \right\rangle = 0$$

(See also **Hirshman Sigmar Clarke 1976** further below).

We **note** here the presence of the time variation of the electrostatic potential

$$e_a \frac{\partial \Phi}{\partial t} \frac{\partial f_{a1}^{(0)}}{\partial \epsilon}$$

This is justified since we are concerned here with the decay of the poloidal rotation. Then, there is a radial electric field $E_r(t)$ and this electric field has time variation $\partial E_r / \partial t$, as in a damping.

See **Novakovskii** in *rotation.tex*.

The distribution function is

$$\begin{aligned} f_{a1}^{(0)} &= -I \frac{v_\parallel}{\Omega_{ca}} \frac{\partial f_{a0}}{\partial \psi} \\ &\quad + g_a\left(\mu, \sigma, v, \psi, t\right) \end{aligned}$$

(**note** that the first term is

$$\begin{aligned} I\frac{v_{\|}}{\Omega_{ci}}\frac{\partial f_0}{\partial \psi} &= RB_{\varphi}\frac{v_{\|}}{\frac{eB}{m_i}}\frac{1}{RB_{\theta}}\frac{\partial f_0}{\partial r} \\ &\approx \frac{v_{\|}}{\Omega_{c\theta}}\frac{\partial f_0}{\partial r} = \rho_{\theta}\frac{\partial f_0}{\partial r} \end{aligned}$$

the usual neoclassical correction. In **Hirshman 1976** it is called *diamagnetic* part)

The equation is actually the solubility condition for the equation written for the function of higher order

$$f_{a1}^{(1)} \equiv \text{ order } \quad (\delta_a)^1 \quad \text{and} \quad (\nu_a^*)^1$$

usual neoclassical expansion.

$$C_a\left(f_{a1}^{(0)}\right)$$

is very important in the equation for $f_{a1}^{(0)}$ which becomes actually an equation for g_a.

The variable to write the *pitch angle* scattering is

$$\xi \equiv \frac{v_{\|}}{v}$$

The property

$$C_a\left[\varphi\left(v\right)P_l\left(\xi\right)\right] = P_l\left(\xi\right)C_a^l\left[\varphi\left(v\right)\right]$$

The Legendre polynomial $P_l\left(\xi \equiv \frac{v_{\|}}{v}\right)$ is an eigenfunction of the *linearized collision operator.*

See **HirshmanSigmar1976** for $C\left[f\right]$, with $f \sim Y_{lm}\left(\Omega\right) \rightarrow C_a \sim Y_{lm}$.

The *separability* of the collision operator into a product of a function of ξ and a function of v is mentioned by **Coredy NBI**.

Approximations.

For high order of the Legendre polynomials

$$l^2 \gg 1$$

the pitch angle scattering terms are a good approximation for the collision operator C_a^l.

$$C_a^l \approx -\frac{l\left(l+1\right)}{2}\nu_a^D\left(v\right)$$

$$\nu_a^D(v) = \sum_b \nu_{ab}^D(v)$$

$$\nu_{ab}^D(v) = \nu_{ab} \frac{1}{\left(\frac{v}{v_{a,th}}\right)^3} \left[\mathrm{erf}\left(\frac{v}{v_{a,th}}\right) - G_c\left(\frac{v}{v_{b,th}}\right) \right]$$

and for

$$x \equiv \frac{v}{v_{th}}$$

$$G_c(x) = \frac{1}{2x^2} \left[\mathrm{erf}(x) - \frac{2x}{\sqrt{\pi}} \exp\left(-x^2\right) \right]$$

Then the approximation becomes possible

$$\begin{aligned} C_a\left(f_{a1}^{(0)}\right) &\approx \nu_a^D(v) L\left(f_{a1}^{(0)}\right) \\ &+ \sum_{l=0}^{l=2} P_l(\xi) \left[C_a^l(f_l) + \frac{l(l+1)}{2} \nu_a^D(v) f_l \right] \end{aligned}$$

The functions f_l are projections of the unknown perturbation of the distribution function $f_{a1}^{(0)}$ on the basis of the Legendre polynomials

$$\begin{aligned} f_l &= \frac{2l+1}{2} \int_{-1}^{+1} d\xi P_l(\xi) f_{a1}^{(0)} \\ \text{limits are } \xi &= \pm 1 \text{ strong circulating } v_\parallel = \pm v \end{aligned}$$

The operator L is the *pitch-angle scattering operator*,

$$L = \frac{v_\parallel}{B} \frac{\partial}{\partial \mu} \mu v_\parallel \frac{\partial}{\partial \mu}$$

In the equation for the solvability condition we have to insert the *collision* operator in the approximate form and the expression of the function $f_{a1}^{(0)}$ in terms of g_a. It is obtained

$$\begin{aligned} & \nu_p \left\langle \frac{B}{\sqrt{1-\lambda B}} \right\rangle g_a \\ & + 2\nu_a^D(v) \frac{\partial}{\partial \lambda} \lambda \left\langle \sqrt{1-\lambda B} \right\rangle \frac{\partial g_a}{\partial \lambda} \\ = \; & -\sigma \left(C_a^1 + \nu_a^D(v)\right) K_a \\ & + \sigma I \frac{B}{\Omega_{ca}} \left[C_a^1\left(v \frac{\partial f_{a0}}{\partial \psi}\right) - \frac{e_a}{T_a} v f_{a0} \frac{\partial^2 \Phi}{\partial t \partial \psi} \right] \end{aligned}$$

where

$$\begin{aligned}
\lambda &\equiv \frac{\sqrt{1-\xi^2}}{B} = \frac{\sqrt{1-\left(v_\parallel/v\right)^2}}{B} \\
&= \frac{v_\perp}{v}\frac{1}{B\left(\mathbf{x}\right)} \\
&= \text{pitch-angle variable } \sim \frac{v_\perp}{v} h
\end{aligned}$$

Note. The *variable* λ controlls the extension in the velocity space of the *CIRCULATING* particles

$$\begin{aligned}
\lambda_c &= \frac{1}{B_{\max}} \\
&\quad \text{along a magnetic field line}
\end{aligned}$$

The critical value λ_c of the variable $\lambda = \frac{v_\perp}{v} h \times \frac{1}{B_0} = \frac{v_\perp}{v}\frac{1}{B(\mathbf{x})}$ occurs when all velocity is perpendicular $v_\perp = v$, no parallel velocity is left, and this means *turning points* $v_\parallel = 0$ of the banana. This occurs along the orbit where the magnetic field is maximum, *i.e.* closest to the inner limit of plasma.

And

$$\begin{aligned}
-\lambda_c &< \lambda < \lambda_c \\
&\quad \text{circulating}
\end{aligned}$$

Small λ around the value 0 means small perpendicular velocity, *i.e.* sufficiently high *parallel* velocity, *i.e.* circulating particles.

This is actually a small region of the velocity space. **End.**

NOTE in other papers

$$\begin{aligned}
\lambda &= \frac{v_\perp^2}{v^2} h \\
&= \frac{\mu}{w}
\end{aligned}$$

END

To study the time dependent damping (variation in time of the distribution function under the action of the magnetic pumping), one has to assume a decay of the part of the distribution function that exists only for the *untrapped* (circulating) particles g_a, which is assumed to have a time variation as

$$g_a \sim \exp\left(-\nu_p t\right)$$

The fact that only the term

$$l = 1$$

survives in the sum over indices of the Legendre polynomials is due to the fact that the terms with $l \neq 1$ involve integrations of odd functions in σ.

The remaining component, for $l = 1$ is written in terms of K_a as

$$K_a(v, \psi, t) = \frac{3}{2} \langle B^2 \rangle \int_0^{\lambda_c} d\lambda \; g_a(\lambda, \sigma = 1, v, \psi, t)$$

and the limit of the *circulating* region are given by

$$\lambda_c = \frac{1}{B_{\max}}$$

$$\begin{aligned} -\lambda_c &< \lambda < \lambda_c \\ & \text{(circulating)} \end{aligned}$$

The solution of the differential equation for g_a is obtained in several steps.

First it is assumed that the right hand side is a *source*.

Then it is taken the left hand side and is treated as an *eigenfunction* and *eigenvalue* problem

$$\frac{d}{d\lambda} \lambda \sqrt{1 - \lambda B} \frac{dG}{d\lambda} + \frac{1}{2} \left\langle \frac{B}{\sqrt{1 - \lambda B}} \right\rangle \kappa G = 0$$

with

$$\begin{aligned} G &\equiv \text{eigenfunction} \\ \kappa &\equiv \text{eigenvalue} \end{aligned}$$

and

$$\begin{aligned} \int_0^{\lambda_c} d\lambda \left\langle \frac{B}{\sqrt{1 - \lambda B}} \right\rangle G_n G_m &= 0 \\ \text{for } \kappa_n &\neq \kappa_m \end{aligned}$$

From **Taguchi ion thermal conductivity banana regime**

The coordinates

$$(\psi, \varphi, \chi)$$

where

$$\chi \equiv \text{poloidal angle-like variable}$$

one can take $\chi = \theta$.

The field

$$\mathbf{B} = I\boldsymbol{\nabla}\varphi + \boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi$$

The drift-kinetic equation

$$v_{\|}\frac{B_{\chi}}{B}\frac{\partial f_{i1}}{\partial \chi} - C_{ii}\left(f_{i1}\right) = -\mathbf{v}_{Di}\cdot\boldsymbol{\nabla}\psi\left[\left(x^2-\frac{3}{2}\right)\frac{\partial}{\partial\psi}\ln T_i\right]f_{i0}$$

NOTE

The paranthesis in the RHS

$$\left(x^2-\frac{3}{2}\right)\frac{\partial}{\partial\psi}\ln T_i$$

is just a part from the radial (ψ) derivative of the Maxwellian distribution function. The density should be there too. The expression

$$\frac{\partial}{\partial\psi}n$$

is combined with the given expression

$$\begin{aligned}\frac{\partial}{\partial\psi}\ln p_i &= \frac{1}{nT_i}\frac{\partial}{\partial\psi}\left(nT_i\right) = \frac{1}{nT_i}T_i\frac{\partial n}{\partial\psi} + \frac{1}{nT_i}n\frac{\partial T_i}{\partial\psi}\\ &= \frac{\partial}{\partial\psi}\ln n + \frac{\partial}{\partial\psi}\ln T_i\end{aligned}$$

Combined with the rest

$$\begin{aligned}&-\frac{\partial}{\partial\psi}\ln p_i + \left(x^2-\frac{3}{2}\right)\frac{\partial}{\partial\psi}\ln T_i\\ =\;& -\frac{\partial}{\partial\psi}\ln n - \frac{\partial}{\partial\psi}\ln T_i + \left(x^2-\frac{3}{2}\right)\frac{\partial}{\partial\psi}\ln T_i\\ =\;& -\frac{\partial}{\partial\psi}\ln n + \left(x^2-\frac{5}{2}\right)\frac{\partial}{\partial\psi}\ln T_i\\ =\;& -\frac{\partial}{\partial\psi}\ln n - L_1^{(3/2)}\left(x_a^2\right)\frac{\partial}{\partial\psi}\ln T_i\end{aligned}$$

from where

$$\begin{aligned}-\frac{\partial}{\partial\psi}\ln p_i + \left(x^2-\frac{3}{2}\right)\frac{\partial}{\partial\psi}\ln T_i &= -\frac{\partial}{\partial\psi}\ln n - L_1^{(3/2)}\left(x_a^2\right)\frac{\partial}{\partial\psi}\ln T_i\\ \left(x^2-\frac{3}{2}\right)\frac{\partial}{\partial\psi}\ln T_i + \frac{\partial}{\partial\psi}\ln n &= \frac{\partial}{\partial\psi}\ln p_i - L_1^{(3/2)}\left(x_a^2\right)\frac{\partial}{\partial\psi}\ln T_i\end{aligned}$$

and we note that the LHS is

$$\begin{aligned}\left(x^2-\frac{3}{2}\right)\frac{\partial}{\partial\psi}\ln T_i+\frac{\partial}{\partial\psi}\ln n &= \frac{\partial f_M}{\partial\psi} \\ &= \frac{\partial}{\partial\psi}\frac{n(\psi)}{\left[2\pi\frac{T_i(\psi)}{m_i}\right]^{3/2}}\exp\left[-\frac{m_i v^2}{2T_i(\psi)}\right]\end{aligned}$$

which means

$$\frac{\partial f_M}{\partial\psi}=\frac{\partial}{\partial\psi}\ln p_i - L_1^{(3/2)}\left(x_a^2\right)\frac{\partial}{\partial\psi}\ln T_i$$

Then instead of

$$\left[\left(x^2-\frac{3}{2}\right)\frac{\partial}{\partial\psi}\ln T_i\right]f_M \rightarrow \frac{\partial f_M}{\partial\psi}$$

we have

$$\left[\frac{\partial}{\partial\psi}\ln p_i - L_1^{(3/2)}\left(x_a^2\right)\frac{\partial}{\partial\psi}\ln T_i\right]f_M=\frac{\partial f_M}{\partial\psi}$$

and in **Hirshman Sigman Clarke** this is

$$\left[A_{1a}-L_1^{(3/2)}\left(x_a^2\right)A_{2a}\right]f_M$$

with the notations

$$\begin{aligned}A_{1a} &= \nabla_\parallel \ln p_a - \frac{e_a}{T_a}E_\parallel \\ A_{2a} &= \nabla_\parallel T_a\end{aligned}$$

This is because the coefficient here is

$$-\mathbf{v}_{Di}\cdot\boldsymbol{\nabla}\psi\left[\left(x^2-\frac{3}{2}\right)\frac{\partial}{\partial\psi}\ln T_i\right]f_{i0}$$

and in **HSC** is

$$v_\parallel\left[A_{1a}-L_1^{(3/2)}\left(x_a^2\right)A_{2a}\right]f_M$$

END

The variables are

$$\epsilon,\mu,\sigma$$

and

$$\mathbf{v}_{Di}\cdot\boldsymbol{\nabla}\psi=\frac{m_i I}{e_i}\mathbf{v}\cdot\boldsymbol{\nabla}\left(\frac{v_\parallel}{B}\right)$$

or

$$\mathbf{v}_D\cdot\boldsymbol{\nabla}\psi = I v_\parallel \widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta\frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{\Omega}\right)$$

where we notice the replacement

$$\mathbf{v}\cdot\boldsymbol{\nabla} \quad \rightarrow \quad v_{\|}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta\frac{\partial}{\partial\theta}$$

and we know that

$$\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta=\frac{1}{qR}$$

Then we conclude that the operator $\mathbf{v}\cdot\boldsymbol{\nabla}$ when it is applied on a quantity that has mostly poloidal variation θ, is replaced by

$$\mathbf{v}\cdot\boldsymbol{\nabla} \quad \rightarrow \; v_{\|}\frac{1}{qR}\frac{\partial}{\partial\theta}$$

and this is the case with

$$\frac{v_{\|}}{\Omega_{ci}}$$

We also have

$$\frac{1}{qR}\frac{\partial}{\partial\theta}=\nabla_{\|}$$

NOTE. We find later in **Rosenbluth Hinton NBI** the expression

$$\mathbf{v}_D\cdot\boldsymbol{\nabla}\psi=Iv_{\|}\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta\right)\frac{\partial}{\partial\theta}\left(\frac{v_{\|}}{\Omega_c}\right)$$

where

$$\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta=\frac{1}{qR}$$

$$v_{Dr}\left(RB_\theta\right)=I\frac{v_{\|}}{qR}\frac{\partial}{\partial\theta}\left(\frac{v_{\|}}{\Omega}\right)$$

Now we must make the connection between the two.

the above formula is

$$\begin{aligned} v_{Dr}\left(RB_\theta\right) &= RB_\varphi\frac{v_{\|}}{\frac{rB_\varphi}{RB_\theta}R}\frac{\partial}{\partial\theta}\left(\frac{v_{\|}}{\Omega}\right) \\ v_{Dr} &= v_{\|}\frac{\partial}{r\partial\theta}\left(\frac{v_{\|}}{\Omega}\right) \end{aligned}$$

We know that the velocity is an independent coordinate of the phase space $(\mathbf{x},\mathbf{v})$. Then the spatial derivative applied on $v_{\|}/B$ will affect B and will affect $v_{\|}$ since

$$v_{\|}=\mathbf{v}\cdot\widehat{\mathbf{n}}$$

and the direction of the magnetic line $\widehat{\mathbf{n}}$ depends on the position. In particular it depends on θ,

$$\widehat{\mathbf{n}} = \widehat{\mathbf{n}}(\theta)$$

The magnitude of the magnetic field is

$$\begin{aligned} |\mathbf{B}| &= B \approx \frac{B_0}{1+\varepsilon\cos\theta} \\ &= \frac{B_0}{R/R_0} \text{ where } B_0 \text{ and } R_0 \text{ are constants} \end{aligned}$$

and also depends on θ.

We can take

$$\begin{aligned} |\boldsymbol{\nabla}\psi| &= RB_\theta \equiv \text{function of only } \psi \\ I &= RB_\varphi \equiv \text{function of only } \psi \end{aligned}$$

From these two formulas it does not result that

$$\begin{aligned} |\boldsymbol{\nabla}\psi| &= RB_\theta \frac{RB_\varphi}{RB_\varphi} = I\frac{B_\theta}{B_\varphi}\frac{R}{r}\frac{r}{R} = I\varepsilon\frac{1}{q} \\ &= I\frac{r}{qR} \end{aligned}$$

and then the left hand side is

$$\mathbf{v}_{Di}\cdot\boldsymbol{\nabla}\psi = v_{Dr}RB_\theta\frac{\partial}{\partial r}$$

and we must use

$$\mathbf{v}\cdot\boldsymbol{\nabla}\,(\text{applied on a function that depends on } \theta) = v_\parallel\frac{1}{qR}\frac{\partial}{\partial\theta}$$

$$\begin{aligned} \frac{m_i I}{e_i}\mathbf{v}\cdot\boldsymbol{\nabla}\left(\frac{v_\parallel}{B}\right) &= RB_\varphi\frac{1}{\frac{e_i}{m_i}}\times \\ &\quad \times v_\parallel\frac{1}{qR}\frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{B}\right) \\ &= RB_\varphi v_\parallel\frac{1}{qR}\frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{\Omega_{ci}}\right) \end{aligned}$$

and taking the two sides

$$\begin{aligned} v_{Dr}RB_\theta &= RB_\varphi v_\parallel\frac{1}{qR}\frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{\Omega_{ci}}\right) \\ v_{Dr} &= \frac{B_\varphi}{B_\theta}v_\parallel\frac{1}{qR}\frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{\Omega_{ci}}\right) \end{aligned}$$

From the first equation of $\mathbf{v}_{Di} \cdot \boldsymbol{\nabla}\psi$ we have obtained

$$v_{Dr} = v_\parallel \frac{\partial}{r\partial\theta}\left(\frac{v_\parallel}{\Omega_{ci}}\right)$$

which must be compared with the second expression

$$\begin{aligned}
& I\frac{v_\parallel}{qR}\frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{\Omega}\right) \\
= \; & RB_\varphi \frac{1}{\frac{rB_T}{RB_\theta}R} v_\parallel \frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{\Omega_{ci}}\right) \\
=
\end{aligned}$$

A formula that is known

$$\mathbf{v}_D \cdot \boldsymbol{\nabla}\psi = I v_\parallel \frac{1}{qR}\frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{\Omega}\right)$$

and

$$v_{D,r}\left|\boldsymbol{\nabla}\psi\right| = I v_\parallel \frac{1}{qR}\frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{\Omega_{ci}}\right)$$

and it is known that

$$\left|\boldsymbol{\nabla}\psi\right| = I\frac{r}{qR}$$

we have

$$v_{Dr} = v_\parallel \frac{\partial}{r\partial\theta}\left(\frac{v_\parallel}{\Omega_{ci}}\right)$$

2.12 Collision operator used by Cordey Connor for the NBI heating

The Fokker-Planck equation is

$$\begin{aligned}
& \frac{\partial f_h}{\partial t} - v_\parallel \frac{B_\theta}{B_T}\frac{\partial f_h}{r\partial\theta} \\
= \; & \frac{1}{\tau_s}\left\{ \frac{2}{w^{1/2}}\frac{\partial}{\partial w}\left[\left(w_c^{3/2} + w^{3/2}\right) f_h\right] \right. \\
& + 2\mu \frac{w_c^{3/2} + w^{3/2}}{w^{3/2}}\frac{\partial f_h}{\partial\mu} \\
& \left. + \frac{m_i}{m_h}\left(\frac{w_c}{w}\right)^{3/2}\frac{v_\parallel}{B}\frac{\partial}{\partial\mu}\left(\mu v_\parallel \frac{\partial f}{\partial\mu}\right)\right\} \\
& + S
\end{aligned}$$

(see also **GAFFEY**) where f_h is the *hot* ion (NBI) distribution function,

$$\begin{aligned}
w &= \frac{v^2}{2} = \epsilon - \frac{e\phi}{m} \quad \text{(energy per unit mass of the particle)} \\
\mu &= \frac{v_\perp^2}{2B} \\
v_\parallel &= \sqrt{2\,(w - \mu B)}
\end{aligned}$$

the slowing-down time

$$\tau_s = \frac{3}{4\sqrt{2\pi}} \frac{m_h}{\sqrt{m_e}} \frac{1}{e^4 Z_h^2} \frac{1}{\ln\Lambda} \frac{T_e^{3/2}}{n}$$

The critical energy is

$$w_c = \left(\frac{3\sqrt{\pi}}{4} Z\right)^{2/3} \left(\frac{m_i}{m_e}\right)^{1/3} \frac{m_h}{m_i} T_e$$

2.13 The pitch-angle and energy collision operators Ware Hazeltine

The paper is on anomalous pinch due to the *variation of the electric potential on the magnetic surface.*

The determination of

$$\widetilde{n}\,(\theta)$$

goes through the calculation of the distribution function

$$\begin{aligned}
& v_\parallel \frac{\partial f_1}{\partial x_\parallel} \\
& -\frac{3\sqrt{\pi}}{8}\nu_e \left(\frac{v_{th,e}^3}{v^3}\right) \frac{\partial}{\partial \xi}\left(1-\xi^2\right)\frac{\partial f_1}{\partial \xi} \qquad \text{(pitch angle)} \\
& -\frac{1}{\sqrt{2}}\nu_{ee} v_{th,e}^2\, f_0\, \frac{\partial^2}{\partial v^2}\left(\frac{f_1}{f_0}\right) \qquad \text{(energy)} \\
= \; & \frac{1}{B}\frac{\partial \widetilde{\phi}\,(\theta)}{r\partial\theta} f_0 \left[\frac{n'}{n} - \frac{3}{2}\frac{T'}{T} + \frac{e\left(E_r - v_\parallel B_\theta\right)}{T_e}\right] \quad \left(\begin{array}{c}\text{radial advection of the} \\ \text{equilibrium function } f_0\end{array}\right) \\
\equiv \; & A \sin\theta \quad \text{(notation)}
\end{aligned}$$

where

$$\nu_{ee} = \frac{1}{\sqrt{2}\tau_e}$$
$$\nu_e = \nu_{ei} + \nu_{eZ} = \frac{Z_{eff}}{\tau_e}$$

Note. For comparison

$$\lambda = \frac{1}{B}\frac{v_\perp}{v}$$

2.14 Collision operator used by Helander Fulop

The paper is **current driven by asymmetric poloidal fuelling Helander Fulop**

$$\begin{aligned} C_e(f_1) &= C_{ee}(f_1) \\ &+\nu_{Defl,ei}\frac{m_e v_\parallel V_{\parallel,i}}{T_e} f_0 \text{ (collisional coupling with the flow)} \\ &+\nu_{Defl,ei}\frac{2}{B}\frac{1}{v^2}v_\parallel \frac{\partial}{\partial\lambda} v_\parallel v \frac{\partial f_1}{\partial\lambda} \quad \text{(pitch angle)} \end{aligned}$$

where *ion-electron deflection frequency* is

$$\nu_{Defl,ei} = \frac{3\sqrt{\pi}}{4}\frac{1}{\tau_{ei}}\left(\frac{v}{v_{th,e}}\right)^3$$

Note that we recognize here

$$\begin{aligned} \frac{m_e v_\parallel V_{\parallel,i}}{T_e} f_0 &= \frac{2 v_\parallel V_{\parallel,i}}{v_{th,e}^2} f_0 \\ &= \text{correction due to displaced Maxwellian} \end{aligned}$$

Formulas are in *impurities.tex* since it is about the poloidal asymmetry of the density of impurities.

2.15 Collisional operator used for Pfirsch-Schluter friction (Hazeltine Ware)

The paper of **Hazeltine Ware** uses the collision operator for the ion-impurity ion

$$
\begin{aligned}
C_{iZ} &= \frac{3\sqrt{2\pi}}{8}\frac{1}{\tau_i}\left(\frac{v_{th,i}}{v}\right)^3\frac{Z^2 n_Z}{n_i} \\
&\times\left[\frac{\partial}{\partial\xi}\left(1-\xi^2\right)\frac{\partial f_i}{\partial\xi}+\frac{4\xi v u_Z}{v_{th,i}^2}f_{0i}\right]
\end{aligned}
$$

[pitch angle plus collisions with a relative flow (shifted Maxwellian)]

$$
\xi \equiv \frac{v_\parallel}{v}
$$

where

$$
\begin{aligned}
v &\equiv \text{ion velocity} \\
u_Z &\equiv \text{velocity of impurity ion}
\end{aligned}
$$

and

$$
f_{0i} = f_{Mi} = \frac{n_{0i}}{\left(\pi v_{th,i}^2\right)^{3/2}}\exp\left(-\frac{v^2}{v_{th,i}^2}\right)
$$

$$
f_i = f_{0i}\left(1+\frac{2\xi v u_Z}{v_{th,i}^2}\right)+g
$$

Note that this is identical with the operator used by **Helander 1999** for the non-uniform distribution of the impurities on the surface. The first part is

$$
f_{0i}+\frac{2v_\parallel u_Z}{v_{th,i}^2}f_{0i}
$$

and is identical with the form from **Hisrchman 1976.**

A shifted Maxwellian, see **Hirshman Sigmar**.

NOTE

About the term

$$
\frac{2v_\parallel u_Z}{v_{th,i}^2}f_{0i}
$$

is explained in **neutral valanju clavin hazeltine solano** as "the first order correction of a displaced Maxwellian". The distribution function there

$$
f = f_M + f_{displ} + g
$$

$$f_{displ} = \frac{2V_{i\|} v\xi}{v_{th,i}^2} f_M$$

where

$$v\xi = v_{\|}$$

Therefore this term is there to take into account a flow of the ions, $V_{i\|} \equiv u_Z$.

Other variables in that treatment of *charge exchange*:

$$\begin{aligned} \omega_{trans} &= \frac{v\xi}{qR} = \frac{v_{\|}}{qR} \\ & \text{the transit frequency} \end{aligned}$$

$$\begin{aligned} Mf &\equiv \text{mirror force} \\ &= -\frac{1}{2}\frac{r}{R}\omega_{trans}\frac{1}{\xi}\sin\theta\left(1-\xi^2\right)\frac{\partial f}{\partial \xi} \end{aligned}$$

END

2.16 The operator of collision used by Frieman 1970

This is

$$\begin{aligned} C_{ab} &= -2\pi \ln \Lambda \frac{e^4}{m_a}\frac{\partial}{\partial \mathbf{v}} \cdot \\ & \cdot \int d\mathbf{v}' \cdot \mathbf{U} \cdot \left[f_0^{(a)}(\mathbf{v}) \left(\frac{1}{m_b} \frac{\partial f_1^{(b)}(\mathbf{v}')}{\partial \mathbf{v}'} + \mathbf{v} \frac{f_1^{(b)}(\mathbf{v}')}{T} \right) \right. \\ & \qquad \left. - f_0^{(b)}(\mathbf{v}') \left(\frac{1}{m_a} \frac{\partial f_1^{(a)}(\mathbf{v})}{\partial \mathbf{v}} + \mathbf{v}' \frac{f_1^{(a)}(\mathbf{v})}{T} \right) \right] \end{aligned}$$

where

$$\mathbf{U} = \frac{1}{u}\delta_{ab} - \frac{u_a u_b}{u^3}$$

$$\mathbf{u} = \mathbf{v} - \mathbf{v}'$$

Now it is expanded in the mass ratio

$$\frac{m_e}{m_i} \ll 1$$

$$
\begin{aligned}
C_{ei} &= 2\pi \ln \Lambda \frac{e^4}{m_e^2} N_0^i \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{U}\left(\mathbf{v}'=0\right) \cdot \frac{\partial f_1^{(e)}\left(\mathbf{v}\right)}{\partial \mathbf{v}} \\
& -2\pi \ln \Lambda \frac{e^4}{m_e} N_0^i \left(\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{U}\left(\mathbf{v}'=0\right) \; f_0^{(e)}\left(\mathbf{v}\right) \right) \cdot \mathbf{V}_1^{(i)} \\
& +...O\left(m_e/M_i\right)^{1/2}
\end{aligned}
$$

where

$$
\mathbf{V}_1^{(i)} \equiv \text{first order ion velocity}
$$

The comparison

$$
\begin{aligned}
C_{ee} &\sim C_{ei} \\
C_{ii} &\sim \sqrt{\frac{M_i}{m_e}}\, C_{ie}
\end{aligned}
$$

2.17 Collision operator used by Santarius Hinton

It is Lorentz, for **pitch-angle scattering**

$$
C\left(f\right) = \nu_{ei}\left(v\right) \frac{1}{2} \frac{\partial}{\partial \xi} \left(1-\xi^2\right) \frac{\partial f}{\partial \xi}
$$

where

$$
\xi \equiv \frac{v_\parallel}{v}
$$

$$
\nu_{ei}\left(v\right) = \frac{3\sqrt{\pi}}{4} \frac{1}{\tau_e} \frac{v_{th,e}^3}{v^3}
$$

(**note** it is identical to **Helander 1999**)

$$
\tau_e = \frac{3}{16\sqrt{\pi}} \frac{m_e^2}{\left|e\right|^4} \frac{1}{\ln \Lambda} \frac{v_{th,e}^3}{n_i}
$$

we recognize in the last factor

$$
\nu \sim \frac{n}{T^{3/2}}
$$

It is interesting that **Santarius Hinton**, although are studying drift waves, they want to keep the neoclassical motion of the particles.

[Just as **Frieman Rutherford**, etc., it was considered that a correct derivation of the drift wave theory must start from a trajectory of the particle given by the neoclassical theory.]

The motion of the particle in a wave-perturbation analysis represents the characteristic of the operator Fokker-Planck. The trajectory is important since the operator acting on the perturbed distribution function is the derivation along the trajectory. Usually it is taken very simple

$$x = vt$$

leading to the *propagator* which is the inverse of the derivative along the trajectory

$$G \sim \frac{1}{i\omega - ik_{\|}v_{\|}}$$

After that one considers something that makes the particle NOT to be in phase with the wave: collisions

$$\frac{1}{i\omega - ik_{\|}v_{\|} + \nu}$$

But now, one considers a more complicated trajectory, taking into account also the *drift motion.*

It is the retain of the drift motion in the orbit of the generic particle (*i.e.* in the propagator),

$$\mathbf{v}_D \cdot \boldsymbol{\nabla} f_e^{(1)}$$

that produces the term

$$\cos\theta + \widehat{s}\theta \sin\theta$$

where

$$\widehat{s} = \frac{rq'}{q}$$

The *boundary between trapped and untrapped particles*

$$\left(\frac{v_{\|}}{v}\right)^2 = \xi^2 = \frac{\epsilon \cos^2(\theta/2)}{1 + \varepsilon \cos\theta}$$

2.18 Collision operator detailed in Bolton Ware

The objective is the calculation of the heat flux of the ions.

$$v_{\|}\frac{B_\theta}{B}\frac{\partial f_1}{r\partial\theta} - C(f_1) = -v_{Dr}\frac{\partial f_0}{\partial r}$$

where

$$v_{Dr} = -\frac{1}{\Omega_c}\frac{\mu B + v_{\|}^2}{R}\sin\theta$$

(**note** here $v_\| \nabla_\| f_1$ is expressed as usual by the projection on the poloidal direction).

Now a change of variables is made

$$\left.\begin{array}{c}\mathbf{x}\\ \varepsilon\\ \mu\end{array}\right\}\rightarrow\left\{\begin{array}{c}\mathbf{x}\\ v\\ \xi\equiv\frac{v_\|}{v}\end{array}\right.$$

Since ξ contains a direction $\|$ the derivative

$$\frac{\partial f}{\partial\theta}\rightarrow\frac{\partial f}{\partial\theta}+\frac{\partial f}{\partial\xi}\frac{\partial\xi}{\partial\theta}$$

One has to use (**Novakovski**)

$$\mathbf{B}\equiv\left(0,\frac{\varepsilon}{q}\frac{B_0}{h},\frac{B_0}{h}\right)$$

and

$$\begin{aligned}v_\| &= \mathbf{v}\cdot\widehat{\mathbf{n}}=v_\varphi\widehat{n}_\varphi+v_\theta\widehat{n}_\theta\\ \xi &= \frac{v_\|}{v}=\frac{v_\varphi\widehat{n}_\varphi+v_\theta\widehat{n}_\theta}{v}\end{aligned}$$

$$\begin{aligned}\widehat{n}_\varphi &= 1\times\frac{B_\varphi}{B}\\ \widehat{n}_\theta &= 1\times\frac{B_\theta}{B}\end{aligned}$$

We use

$$\begin{aligned}B_r &= 0\\ B_\theta &= \frac{b(r)}{1+(r/R)\cos\theta}=\frac{\varepsilon}{q}\frac{B_0}{h}\\ B_\varphi &= \frac{B_0}{1+(r/R)\cos\theta}=\frac{B_0}{h}\end{aligned}$$

and

$$B\approx\frac{B_0}{h}$$

Then approximately

$$\begin{aligned}\widehat{n}_\theta &= \frac{B_\theta}{B}\\ &= \frac{\frac{\varepsilon}{q}\frac{B_0}{h}}{\frac{B_0}{h}}=\frac{\varepsilon}{q}\end{aligned}$$

where only ε depends on θ, through

$$\varepsilon = \frac{r}{R} = \frac{r}{R_0 + r\cos\theta}$$

$$\widehat{n}_\theta = \frac{r}{qR_0}\left(1 - \frac{r}{R_0}\cos\theta\right)$$

$$\begin{aligned}
\frac{\partial}{\partial\theta}\widehat{n}_\theta &= \frac{r^2}{qR_0^2}\sin\theta \\
&\approx \frac{\varepsilon}{q}r\frac{\sin\theta}{R_0} \quad \text{(here one should expand } R/R_0\text{)} \\
&= \widehat{n}_\theta\frac{r}{R_0}\sin\theta \\
&= \widehat{n}_\theta\varepsilon\sin\theta \quad \text{(again } R/R_0\text{)}
\end{aligned}$$

we can write this

$$\frac{\partial}{\partial\theta}\ln\widehat{n}_\theta \approx \varepsilon\sin\theta$$

Further

$$\widehat{n}_\varphi = \frac{B_\varphi}{B} \approx \frac{\frac{B_0}{h}}{\frac{B_0}{h}}$$

Now we return to

$$\xi = \frac{v_\parallel}{v} = \frac{v_\varphi\widehat{n}_\varphi + v_\theta\widehat{n}_\theta}{v}$$

$$\begin{aligned}
\frac{\partial\xi}{\partial\theta} &= \frac{1}{v}\left[v_\varphi\frac{\partial}{\partial\theta}\widehat{n}_\varphi + v_\theta\frac{\partial}{\partial\theta}\widehat{n}_\theta\right] \\
&= \frac{1}{v}\left[v_\theta\widehat{n}_\theta\varepsilon\sin\theta\right] \\
&= \frac{v_\theta}{v}\varepsilon\sin\theta \times \frac{\varepsilon}{q}
\end{aligned}$$

This is

$$\frac{\partial\xi}{\partial\theta} = \frac{v_\theta}{v}\varepsilon\sin\theta \times \frac{B_\theta}{B}$$

NOTE

We have to calculate

$$\frac{\partial\xi}{\partial\theta} = \frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{v}\right) = \frac{1}{v}\frac{\partial v_\parallel}{\partial\theta}$$

We should use from **Hazeltine.**

Starting with

$$
\begin{aligned}
\epsilon &= \frac{v^2}{2} + e\phi \\
v_\parallel &= \sqrt{2\left(\epsilon - \mu B - e\phi\right)} \\
\boldsymbol{\nabla} v_\parallel &= -\frac{\mu \boldsymbol{\nabla} B + \frac{e}{m}\boldsymbol{\nabla}\phi}{v_\parallel}
\end{aligned}
$$

assuming $\phi = 0$, (which cannot be done if we have rotation or damping of the rotation)

$$
\begin{aligned}
\frac{\partial v_\parallel}{\partial \theta} &= -\mu \frac{1}{v_\parallel}\frac{\partial B}{\partial \theta} \\
&= -\frac{v_\perp^2}{2B}\frac{1}{v_\parallel}\frac{\partial}{\partial\theta}\frac{B_0}{h} \\
&\approx -\frac{v_\perp^2/2}{v_\parallel}\frac{1}{B_0/h}B_0\frac{\partial}{\partial\theta}\left(1-\varepsilon\cos\theta\right) \\
&= -\frac{v_\perp^2/2}{v_\parallel}\left(1+\varepsilon\cos\theta\right)\left(\varepsilon\sin\theta\right) \\
&= -\frac{v_\perp^2/2}{v_\parallel}\varepsilon\sin\theta - \frac{v_\perp^2/2}{v_\parallel}\varepsilon^2\cos\theta\sin\theta \\
&= -\frac{v_\perp^2/2}{v_\parallel}\varepsilon\sin\theta - \frac{v_\perp^2/2}{v_\parallel}\varepsilon^2\frac{1}{2}\sin\left(2\theta\right)
\end{aligned}
$$

The result is

$$
\begin{aligned}
\frac{\partial \xi}{\partial \theta} &= \frac{1}{v}\frac{\partial v_\parallel}{\partial\theta} \\
&= -\frac{1}{v}\frac{v_\perp^2/2}{v_\parallel}\varepsilon\sin\theta - \frac{1}{v}\frac{v_\perp^2/2}{v_\parallel}\varepsilon^2\frac{1}{2}\sin\left(2\theta\right)
\end{aligned}
$$

We factor out $\xi^{-1} = \left(v_\parallel/v\right)^{-1}$ which means

$$
\frac{1}{\xi}\frac{\partial\xi}{\partial\theta} = \frac{1}{\xi}\left[-\frac{v_\perp^2/2}{v^2}\varepsilon\sin\theta - \frac{v_\perp^2/2}{v^2}\varepsilon^2\frac{1}{2}\sin\left(2\theta\right)\right]
$$

and it only remains a problem of sign for the second term.

END

NOTE

Up to this point we have

$$\begin{aligned} v_{\|}\frac{B_\theta}{B}\frac{\partial f_1}{r\partial\theta} - C\left(f_1\right) &= -v_{Dr}\frac{\partial f_0}{\partial r} \\ v_{\|}\frac{B_\theta}{B}\frac{1}{r}\left[\frac{\partial f_1}{\partial\theta} + \frac{\partial f_1}{\partial\xi}\frac{\partial\xi}{\partial\theta}\right] - C\left(f_1\right) &= -v_{Dr}\frac{\partial f_0}{\partial r} \end{aligned}$$

and one multiplies with

$$\frac{1}{v}\times\frac{B}{B_\theta}r$$

such as to form the variable ξ in front of the first term

$$\begin{aligned} &\xi\frac{\partial f_1}{\partial\theta} + v_{\|}\frac{B_\theta}{B}\frac{1}{r}\left(\frac{1}{v}\times\frac{B}{B_\theta}r\right)\frac{\partial f_1}{\partial\xi}\left[-\frac{v_\perp^2/2}{v^2}\varepsilon\sin\theta - \frac{v_\perp^2/2}{v^2}\varepsilon^2\frac{1}{2}\sin\left(2\theta\right)\right] \\ &-\left(\frac{1}{v}\times\frac{B}{B_\theta}r\right)C\left(f_1\right) \\ = &\left(\frac{1}{v}\times\frac{B}{B_\theta}r\right)\left(-\frac{1}{\Omega_c}\frac{\mu B + v_{\|}^2}{R}\sin\theta\right)\frac{\partial f_0}{\partial r} \end{aligned}$$

$$\begin{aligned} &\xi\frac{\partial f_1}{\partial\theta} \\ &-\xi\frac{\varepsilon}{2}\frac{v_\perp^2}{v^2}\left[\sin\theta + \frac{\varepsilon}{2}\sin\left(2\theta\right)\right]\frac{\partial f_1}{\partial\xi} \\ &-\frac{1}{v}\frac{B}{B_\theta}rC\left(f_1\right) \\ = &-\varepsilon\frac{1}{\Omega_{c\theta}}\frac{v_\perp^2/2 + v_{\|}^2}{v}\sin\theta\frac{\partial f_0}{\partial r} \end{aligned}$$

The units are OK here: adimensional.

In **Bolton Ware** it is

$$\begin{aligned} &\xi\frac{\partial f_1}{\partial\theta} - \frac{\varepsilon}{2}\left(1-\xi^2\right)\left(\sin\theta - \frac{\varepsilon}{2}\sin 2\theta\right)\frac{\partial f_1}{\partial\xi} - \frac{B}{B_\theta}r\ C\left(f_1\right) \\ = &\frac{1}{\Omega_{c\theta}}\varepsilon\frac{v_\perp^2/2 + v_{\|}^2}{v}\sin\theta\frac{\partial f_0}{\partial r} \end{aligned}$$

It looks that the term with $C\left(f_1\right)$ needs a factor $\frac{1}{v}$.

We also note that the second tem is

$$\begin{aligned} &-\frac{1}{2}\varepsilon\frac{v^2 - v_{\|}^2}{v^2}\sin\theta + \frac{\varepsilon^2}{4}\frac{v^2 - v_{\|}^2}{v^2}\sin\left(2\theta\right) \\ = &-\varepsilon\frac{v_\perp^2/2}{v^2}\sin\theta + \varepsilon\frac{v_\perp^2/2}{v^2}\frac{1}{2}\sin\left(2\theta\right) \end{aligned}$$

The collision operator is the *linearized* form of the Fokker-Planck collision operator.

Collisions with electrons are neglected since the terms that they contribute are of order $(m_e/m_i)^{1/2}$.

$$C(f) = C(f_1, f_0) + C(f_0, f_1)$$

where

$$\begin{aligned} C(f_1, f_0) &= \frac{3\sqrt{2\pi}}{4} \left\{ [\Phi(u) - G(u)] \frac{1}{2} \frac{1}{u^3} \frac{\partial}{\partial \xi} \left(1-\xi^2\right) \frac{\partial f}{\partial \xi} \right. \\ & \left. + \frac{1}{u^2} \frac{\partial}{\partial u} u^2 \left(\frac{2G(u)}{u} \right) \left(\frac{1}{2} \frac{\partial f}{\partial u} + uf \right) \right\} \end{aligned}$$

$$u = \frac{v}{v_{th,i}}$$

$$v_{th,i} = \sqrt{\frac{2T_i}{m_i}}$$

$$\tau = \frac{3}{4\sqrt{\pi}} \frac{m_i^{1/2}}{e^4 \ln \Lambda} \frac{T_i^{3/2}}{n_i}$$

$$\Phi(u) = \text{error function}$$

$$G(u) = \frac{\Phi(u) - u \frac{d\Phi}{du}}{2u^2}$$

Chandrasekhar function

Note for a detailed derivation see **Galeev Sagdeev, Rosenbluth**.

The second part,

$$C(f_0, f_1)$$

requires a special treatment.

Expand the distribution function f in a series of Legendre polynomials of the argument

$$\xi \equiv \frac{v_\parallel}{v}$$

$$f = \sum_{l=0}^{\infty} g_l(v, \theta) P_l(\xi)$$

Consider the collision operator $S(f)$ and take into account that it is linearized

$$\begin{aligned} S(f) &= S\left(\sum_{l=0}^{\infty} g_l(v,\theta)\, P_l(\xi)\right) \\ &= \sum_{l=0}^{\infty} S_l(g_l(v,\theta))\, P_l(\xi) \end{aligned}$$

We note the appearence of a set of functions S_l . These functions, coefficients of the Legendre polynomials in the series, must be calculated.

To determine the functions $S_l(g_l(v,\theta))$ one starts by defining a set of new functions

$$\Delta_j^l(v)$$

and a set of coefficients

$$\psi_j^l$$

and the functions $S_l(g_l(v,\theta))$ are expressed as a series

$$\begin{aligned} & S_l(g_l(v,\theta)) \\ = & \sum_{j=0}^{N} \psi_j^l [g_l(v,\theta)]\ \Delta_j^l\left[\delta_j^l(v)\right] \\ & +\Delta_{N+1}^l(g_l) \end{aligned}$$

For the moment the othernew set of functions

$$\delta_j^l(v)$$

is not specified.

The orthogonality

$$\begin{aligned} \Delta_{j+1}^l [g_l(v,\theta)] &= -\psi_j^l [g_l(v,\theta)]\ \Delta_j^l\left[\delta_j^l(v)\right] \\ &\quad +\Delta_j^l [g_l(v,\theta)] \end{aligned}$$

$$\Delta_0^l [g_l(v,\theta)] = S_l [g_l(v,\theta)]$$

$$\psi_j^l = \frac{\int dv\ w_j^l\ \Delta_j^l [g_l(v,\theta)]}{\int dv\ w_j^l\ \Delta_j^l\left[\delta_j^l(v)\right]}$$

2.19 Collision operator used by Wiley Hinton

It is about run-away and acceleration of electrons in the plasma

The collision operator in the equation

$$\frac{\partial f_e}{\partial \tau} + E\frac{\partial f_e}{\partial u_\parallel} = D\left(f_e\right) + I\left(f_e\right)$$

with

$$\begin{aligned} D\left(f_e\right) &= A\left(u\right)\frac{1}{2u^3}\frac{\partial}{\partial \xi}\left(1-\xi^2\right)\frac{\partial}{\partial \xi}f_e \\ &\quad + \frac{1}{u^3}\frac{\partial}{\partial u}u^2\frac{B\left(u\right)}{u^3}\left(\frac{1}{2}\frac{\partial f_e}{\partial u} + uf_e\right) \end{aligned}$$

and

$$E\frac{\partial f_e}{\partial u_\parallel} = E\left(\xi\frac{\partial f_e}{\partial u} + \frac{1-\xi^2}{u}\frac{\partial f_e}{\partial \xi}\right)$$

where

$$\begin{aligned} u &= \frac{v}{v_e} \\ E &= \frac{\parallel -\text{electric field}}{E_{Dreicer}} \\ \xi &= \frac{v_\parallel}{v} \\ v_e &= \left(\frac{2T_e}{m_e}\right)^{1/2} \end{aligned}$$

The collision operator is linearized for an electric field much smaller than Dreicer field.

With

$$A\left(u\right) = A_e\left(u\right) + A_I\left(u\right)$$

$$A_e\left(u\right) = \Phi\left(x\right) - G\left(x\right)\Big|_{x=\frac{v}{v_e}}$$

$$A_I\left(u\right) = Z\left(\Phi\left(x\right) - G\left(x\right)\right)\Big|_{x=\frac{v}{v_I}}$$

and

$$\frac{B\left(u\right)}{u^3} = \frac{B_e\left(u\right)}{u^3} + \frac{B_I\left(u\right)}{u^3}$$

$$\frac{B_e\left(u\right)}{u^3} = \frac{2}{x}G\left(x\right)\Big|_{x=\frac{v}{v_e}}$$

$$\frac{B_I(u)}{u^3} = Z\left(\frac{v_e}{v_I}\right)\frac{2}{x}G(x)\Big|_{x=\frac{v}{v_I}}$$

where

$$\Phi(x) \equiv \text{error function}$$

$$\begin{aligned} G(x) &\equiv \text{Chandrasekhar function} \\ &= \frac{\Phi(x) - c\frac{d\Phi}{dx}}{2x^2} \end{aligned}$$

It is the paper **steady state electron distributionWiley Hinton**.

The new parameter refers to the *runaway electrons,* and introduces the Dreicer field

$$\begin{aligned} E_D &= \frac{1}{e}m_e v_{th,e}\frac{1}{\tau_e} \\ &= \frac{\text{force}}{\text{charge}} \end{aligned}$$

and

$$\eta \equiv \frac{E}{E_D}$$

The drift kinetic equation is

$$\frac{\partial f_e}{\partial \tau} + \eta\frac{\partial f_e}{\partial u_\parallel} = D(f_e) + I(f_e)$$

where

$$u \equiv \frac{v}{v_{th,e}}$$

$$\tau \equiv \frac{t}{\tau_s}$$

and collisions

$$\begin{aligned} D(f_e) &= A(u)\frac{1}{2u^3}\frac{\partial}{\partial\xi}\left(1-\xi^2\right)\frac{\partial}{\partial\xi}f_e \\ &\quad + \frac{1}{u^2}\frac{\partial}{\partial u}u^2 B(u)\frac{1}{u^3}\left(\frac{1}{2}\frac{\partial f_e}{\partial u} + u f_e\right) \end{aligned}$$

The second term in the Right hand Side is

$$\eta\frac{\partial f_e}{\partial u_\parallel} = \eta\left(\frac{\partial f_e}{\partial u} + \frac{1-\xi^2}{u}\frac{\partial f_e}{\partial \xi}\right)$$

where

$$\xi = \frac{v_\parallel}{v}$$

The distribution function f_e is normalized to

$$\frac{n}{v_{th,e}^3}$$

which is the *physical* factor in the expression of the frequency of collisions.

2.20 Collision operator for impurity-ion friction (Helander1999)

This is from **Helander 1999**.

The parallel friction between the hydrogen ions and the impurities.

The operator of collision

$$\begin{aligned} C_{iz} &= \frac{1}{2}\nu_{iz}(v)\frac{\partial}{\partial\xi}\left(1-\xi^2\right)\frac{\partial}{\partial\xi} \\ &\quad +\nu_{iz}\frac{m_i v_\parallel V_{\parallel z}}{T_i} f_{Mi} \end{aligned}$$

where the frequency is

$$\nu_{iz}(v) = \frac{3}{4}\sqrt{\pi}\frac{1}{\tau_{iz}}\left(\frac{v_{Ti}}{v}\right)^3$$

NOTE

The second part is

$$\begin{aligned} & \nu_{iz} \times \frac{m_i v_\parallel V_{\parallel z}}{T_i} f_{Mi} \\ = & \nu_{iz} \times \frac{2 v_\parallel V_{\parallel z}}{2T_i/m_i} f_{Mi} \\ = & \nu_{iz} \times \frac{2 v_\parallel V_{\parallel z}}{v_{th,i}^2} f_{Mi} \end{aligned}$$

and we recognize the term in the distribution function which comes from the *displaced Maxwellian* due to the ion flow.

END

NOTE

In **Connor 1973** the collision operator is

$$C\left(\widehat{f_j}\right) = \sum_k C_{jk}\left(\widehat{f_j}, \widehat{f_k}\right)$$

$$C_{jk} = \nu_{jk}(x_j)\left[v_\parallel \frac{\partial}{\partial \mu} v_\parallel \frac{\mu}{B}\frac{\partial}{\partial \mu}\widehat{f}_j \;+\; v_\parallel F_{M,j}\frac{\int d^3v\ v_\parallel \nu_{kj}\ \widehat{f}_k}{\int d^3v\ v_\parallel^2\ \nu_{kj}\ F_{M,k}}\right]$$

END

NOTE that the frequency $\nu_{iz}(v)$ is NOT the inverse of the collision time τ_{iz} but includes factors that depend on the velocity v. **END.**

The result of calculation of the friction (the definition is from **Hirschman 1976**) between *background ions* and *impurity ions*,

$$\begin{aligned} R_{\parallel z} &= -\int m_i v_\parallel C_{iz}\left(f_i^{(1)}\right) d^3v \\ &= -p_i \frac{I}{\Omega_i}\frac{1}{\tau_{iz}}\left(\frac{d\ln p_i}{d\psi} - \frac{3}{2}\frac{d\ln T_i}{d\psi}\right) + \frac{m_i n_i}{\tau_{iz}}\left(u - \frac{K_z}{n_z}\right)B \end{aligned}$$

where

$$u = \frac{\tau_{iz}}{n_i B}\int v_\parallel\ \nu_{iz}\ h_i\ d^3v$$

(note a similar definition is adopted in **Connor1973** and in **Helander3999**). The volume in velocity space

$$d^3v = \frac{B}{v_\parallel} d\epsilon_0 d\mu$$

2.21 Collision operator used by Rosenbluth Hinton for transversal NBI

See **rotation.tex** for NBI transversal and for alpha-sustained rotation.

The bounce-averaged collision operator is

$$\begin{aligned} \overline{C}_{fast} f &= \nu_s \frac{2m_i}{m_{fast}}\frac{v_c^3}{v^3}\frac{1}{I_2(\lambda)}\frac{\partial}{\partial \lambda}\lambda I_1(\lambda)\frac{\partial f}{\partial \lambda} + \\ &\quad + \nu_s \frac{1}{v^2}\frac{\partial}{\partial v}\left(v^3 f\right) \end{aligned}$$

where

$$I_1(\lambda) = \int_{-\pi}^{\pi}\frac{d\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}\xi$$

$$I_2(\lambda) = \int_{-\pi}^{\pi}\frac{d\theta}{\widehat{\mathbf{b}}\cdot\boldsymbol{\nabla}\theta}\frac{1}{\xi}$$

These surface-averages of $v_\parallel$ and $\frac{1}{v_\parallel}$ are calculated in **Connor1973** and it is invoked the result from **Oberman Hinton**.

And in **Rosenbluth Hazeltine Hinton** in appendix.

2.22 Ion-impurity collisions (Hirshman Sigmar Clarke 1976)

The equation for the species a is

$$\begin{aligned}&\frac{\partial f_a}{\partial t}+\left(v_\parallel \widehat{\mathbf{n}}+\mathbf{v}_{d,a}\right)\cdot \boldsymbol{\nabla} f_a+\left[\mu\frac{\partial B}{\partial t}+\frac{e_a}{m_a}\left(v_\parallel E_\parallel^{(A)}+\frac{\partial \phi}{\partial t}\right)\right]\frac{\partial f_a}{\partial \epsilon}\\ =\ &\sum_b C_{ab}\left(f_a,f_b\right)\end{aligned}$$

where

$$\epsilon=\frac{1}{2}v^2+\frac{e_a}{m_a}\phi$$

with

$$e_a=Z_a\left|e\right|$$

$$\begin{aligned}\mathbf{E}^{(A)} &= E_{\varphi 0}\frac{R_0}{R}\widehat{\mathbf{e}}_\varphi\\ &\sim E_{\varphi 0}\frac{1}{h\left(\theta\right)}\widehat{\mathbf{e}}_\varphi\end{aligned}$$

the electric field produced by induction by the tokamak transformer.

We **NOTE** the presence of a term $\partial B/\partial t$ which comes from the consideration of the possibility of *plasma compression*. An equation for $\partial B/\partial t$ is in **Hinton Waltz** paper on the heating due to gyrokinetic turbulence.

The drift velocity

$$\begin{aligned}\mathbf{v}_{d,a} &= -v_\parallel \widehat{\mathbf{n}}\times\boldsymbol{\nabla}\left(\frac{v_\parallel}{\Omega_{ca}}\right)\\ &\quad +\frac{v_\parallel^2}{\Omega_{ca}}\frac{1}{B}\left(\boldsymbol{\nabla}\times\mathbf{B}\right)_\perp\end{aligned}$$

and, since the drift velocity will multiply the spatial gradient of the zero-order distribution function,

$$\mathbf{v}_{d,a}\cdot\boldsymbol{\nabla} f_{a0}$$

and since $f_{a0} = f_{Ma}$ has only variation in a direction perpendicular on the magnetic surface

$$\boldsymbol{\nabla} f_{Ma} = \boldsymbol{\nabla}\psi \frac{\partial f_{Ma}}{\partial \psi}$$

we will have to use

$$\begin{aligned}
\mathbf{v}_{d,a} \cdot \boldsymbol{\nabla}\psi &= \left[-v_\parallel \widehat{\mathbf{n}} \times \boldsymbol{\nabla} \left(\frac{v_\parallel}{\Omega_{ca}} \right) + \frac{v_\parallel^2}{\Omega_{ca}} \frac{1}{B} \left(\boldsymbol{\nabla} \times \mathbf{B} \right)_\perp \right] \cdot \boldsymbol{\nabla}\psi \\
&= \left[-v_\parallel \widehat{\mathbf{n}} \times \boldsymbol{\nabla} \left(\frac{v_\parallel}{\Omega_{ca}} \right) \right] \cdot \boldsymbol{\nabla}\psi \\
&= v_\parallel \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \left(I \frac{v_\parallel}{\Omega_{ca}} \right) \\
&= v_\parallel \nabla_\parallel \left(I \frac{v_\parallel}{\Omega_{ca}} \right)
\end{aligned}$$

(in this form the term $\mathbf{v}_{d,a} \cdot \boldsymbol{\nabla} f$ which actually is $\mathbf{v}_{d,a} \cdot \boldsymbol{\nabla}\psi \frac{\partial f_{Ma}}{\partial \psi}$ can be coupled to $v_\parallel \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} f_{a1}$).

The second term in the drift velocity is

$$\begin{aligned}
\frac{v_\parallel^2}{\Omega_{ca}} \frac{1}{B} \left(\boldsymbol{\nabla} \times \mathbf{B} \right)_\perp &= \frac{v_\parallel^2}{\Omega_{ca}} \frac{1}{B} \mathbf{j}_\perp \\
&\sim v_\parallel^2 \frac{1}{B^2} v_\perp \quad \rightarrow \text{ polarization ?}
\end{aligned}$$

The expansion

$$f_a = f_{a0} + f_{a1} + ...$$

$$f_{a0} = f_{Ma} = n_a(\psi) \frac{1}{\left[\pi \frac{2T_a(\psi)}{m_a} \right]^{3/2}} \exp\left[-\frac{\epsilon}{\frac{T_a(\psi)}{m_a}} \right]$$

$$n_a(\psi) = n_{a0} \exp\left[\frac{e_a \phi}{T_{a0}(\psi)} \right]$$

The equation for the first order

$$\begin{aligned}
& v_\parallel \nabla_\parallel f_{a1} + v_\parallel \nabla_\parallel \left(I \frac{v_\parallel}{\Omega_{ca}} \frac{\partial f_{a0}}{\partial \psi} \right) \\
& -v_\parallel \frac{e_a E^{(A)}}{T_a} f_{a0} \qquad \text{(work against the parallel } E\text{)} \\
= & \sum_b C_{ab}(f_{a1}, f_{b1})
\end{aligned}$$

Note that here we have taken $\partial f_{a0}/\partial\psi$ inside the paranthesis that comes from the drift velocity. Then it is convenient to separate from f_{a1} the part resulting from the drift motion $\mathbf{v}_D$ advection of the equilibrium $f_{a0} = f_{Ma}$ distribution function.

$$\begin{aligned} f_{a1} &= -I\frac{v_\parallel}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial\psi} \quad \left(\text{neoclassical } \frac{\rho_\theta}{L_n}\right) \\ &\quad + g_a\left(\epsilon,\mu,\psi\right) \end{aligned}$$

where (**Hirshman Sigmar Clarke**)

$$\begin{aligned} -I\frac{v_\parallel}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial\psi} &\equiv \text{diamagnetic response of the species } a \\ g_a &\equiv \text{collisional response of the species } a \\ g_a &= f_{a1} + I\frac{v_\parallel}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial\psi} \end{aligned}$$

(see **Rutherford).**

The "diamagnetic" term is

$$\rho_\theta\frac{\partial f_{Ma}}{\partial r}$$

the first part of the neoclassical correction to the equilibrium distribution function. For g_a the collisions are essential.

Note that the modulation along the magnetic field line : the *energy* term $v_\parallel\left(-e\nabla_\parallel\widetilde{\Phi}^{(1)}\right)\frac{\partial f^{(0)}}{\partial\epsilon}$ (which gives a term $1/T$) . This term, resulting from the motion of the particle in an electrostatic potential with variation on the magnetic surface, appears in **Helander3999** and is important for *impurity* ions. Is not explicit here.

Instead, there is a similar *energy* term with the parallel electric field which is induced $E^{(A)}$.

Consider the surface average

$$\langle A\left(\mathbf{x}\right)\rangle = \frac{2\pi}{\frac{\partial V}{\partial\psi}}\oint d\chi\frac{A}{\boldsymbol{\nabla}\chi\cdot\mathbf{B}}$$

$$\frac{\partial V}{\partial\psi} = 2\pi\oint d\chi\frac{1}{\boldsymbol{\nabla}\chi\cdot\mathbf{B}}$$

Returning to the drift-kinetic equation, we have

$$v_\parallel\nabla_\parallel\left(f_{a1} + I\frac{v_\parallel}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial\psi}\right) = v_\parallel\frac{e_aE^{(A)}}{T_a}f_{a0} + \sum_b C_{ab}\left(f_{a1},f_{b1}\right)$$

We divide by $v_\parallel$ and note that the surface average is applied to

$$\begin{aligned} & \oint d\chi \frac{1}{\mathbf{B}\cdot\boldsymbol{\nabla}\chi} \nabla_\parallel () \\ = & \oint d\theta \frac{1}{B_\theta \frac{1}{r}} \frac{d}{dl_\parallel} () \end{aligned}$$

but

$$dl_\parallel = \frac{B}{B_\theta} dl_\theta = r d\theta \frac{B}{B_\theta}$$

Then

$$\frac{d}{dl_\parallel} = \frac{B_\theta}{B} \frac{d}{r d\theta}$$

and

$$\begin{aligned} \oint d\theta \frac{1}{B_\theta \frac{1}{r}} \frac{d}{dl_\parallel} () &= \oint d\theta \frac{1}{B_\theta \frac{1}{r}} \frac{B_\theta}{B} \frac{d}{r d\theta} () \\ &= \oint d\theta \frac{1}{B} \frac{d}{d\theta} () \end{aligned}$$

The magnitude of B is

$$B \approx \frac{B_0}{h} = B_0 \frac{1}{1+\varepsilon\cos\theta}$$

and B is a function of θ. But it is of order ε and multiples a quantity which is also of order ε. Then $\frac{1}{B}$ can be inserted in the paranthesis

$$\begin{aligned} \nabla_\parallel \left(f_{a1} + I \frac{v_\parallel}{\Omega_{ca}} \frac{\partial f_{a0}}{\partial \psi} \right) &= \frac{1}{v_\parallel} \left[v_\parallel \frac{e_a E^{(A)}}{T_a} f_{a0} + \sum_b C_{ab} \left(f_{a1}, f_{b1} \right) \right] \\ \frac{B_\theta}{B} \frac{d}{r d\theta} \left(f_{a1} + I \frac{v_\parallel}{\Omega_{ca}} \frac{\partial f_{a0}}{\partial \psi} \right) &= \frac{1}{v_\parallel} \left[v_\parallel \frac{e_a E^{(A)}}{T_a} f_{a0} + \sum_b C_{ab} \left(f_{a1}, f_{b1} \right) \right] \\ B_\theta \frac{d}{r d\theta} \left(f_{a1} + I \frac{v_\parallel}{\Omega_{ca}} \frac{\partial f_{a0}}{\partial \psi} \right) &= \frac{B}{v_\parallel} \left[v_\parallel \frac{e_a E^{(A)}}{T_a} f_{a0} + \sum_b C_{ab} \left(f_{a1}, f_{b1} \right) \right] \end{aligned}$$

Now we take the surface average

$$\left\langle B_\theta \frac{d}{r d\theta} \left(f_{a1} + I \frac{v_\parallel}{\Omega_{ca}} \frac{\partial f_{a0}}{\partial \psi} \right) \right\rangle = \left\langle \frac{B}{v_\parallel} \left[v_\parallel \frac{e_a E^{(A)}}{T_a} f_{a0} + \sum_b C_{ab} \left(f_{a1}, f_{b1} \right) \right] \right\rangle$$

In the left hand side we replace

$$
\begin{aligned}
\langle(...)\rangle &= \frac{2\pi}{V'}\oint d\theta \frac{1}{B_\theta \frac{1}{r}}(...) \\
\left\langle B_\theta \frac{d}{rd\theta}\left(f_{a1} + I\frac{v_\parallel}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial \psi}\right)\right\rangle &= \frac{2\pi}{V'}\oint d\theta \frac{1}{B_\theta \frac{1}{r}} B_\theta \frac{d}{rd\theta}\left(f_{a1} + I\frac{v_\parallel}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial \psi}\right) \\
&= 0
\end{aligned}
$$

due to periodicity.

This is what results

$$
\left\langle \frac{B}{v_\parallel}\left[v_\parallel \frac{e_a E^{(A)}}{T_a} f_{a0} + \sum_b C_{ab}\left(f_{a1}, f_{b1}\right)\right]\right\rangle = 0
$$

(see also **Cordey, Taguchi**)

The expression of the collision operator.

HSC write

$$
\begin{aligned}
C_{ab}\left(f_{a1}, f_{b1}\right) &= \nu_{ab}^{defl}\ \frac{v_\parallel}{B}\frac{\partial}{\partial \mu}\mu v_\parallel \frac{\partial}{\partial \mu} f_{a1} \\
&+ \left[\nu_{ab}^{defl} - \nu_{ab}^{slowing}\right] \frac{v_\parallel u_{a1}(v)}{v^2}\ f_{a0} \\
&+ \nu_{ab}^{slowing}\ \frac{2v_\parallel}{v_{th,a}^2} r_{ba}\ \ f_{a0}
\end{aligned}
$$

The definitions

The <u>*deflection*</u> frequency

$$
\nu_{ab}^{defl} = \nu_{ab}\frac{\Phi\left(\frac{v}{v_{th,b}}\right) - G\left(\frac{v}{v_{th,b}}\right)}{\left(\frac{v}{v_{th,a}}\right)^3}
$$

The <u>*slowing down*</u> frequency

$$
\nu_{ab}^{slowing} = 2\frac{T_{a0}}{T_{b0}}\left(1 + \frac{m_b}{m_a}\right)\ \nu_{ab}\ \frac{G\left(\frac{v}{v_{th,b}}\right)}{\left(\frac{v}{v_{th,a}}\right)}
$$

The frequency of Coulombian collisions of a with b,

$$
\nu_{ab} = \frac{4\pi}{2^{3/2}}\frac{e_a^2 e_b^2}{\sqrt{m_a}}\ln\Lambda \frac{n_{b0}}{T_a^{3/2}}
$$

The function

$$G(x) = \frac{\Phi(x) - x\frac{d\Phi(x)}{dx}}{2x^2}$$
the Chandrasekhar function

where

$$\Phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x dt \exp\left(-t^2\right)$$
error function

In the formulas above it has been introduced

$$u_{a1}(v) = \frac{1}{f_{a0}}\frac{3}{4\pi}\int v_{\parallel} f_{a1} d\Omega^V$$
$$d\Omega^V \equiv \text{solid angle in velocity space}$$

$$d\Omega^V = \pi \sum_{\sigma=\pm 1} \frac{B d\mu}{\epsilon}\frac{1}{\left|v_{\parallel}\right|/v}$$

This is an average flow velocity in the parallel direction.

Let us note

$$d^3v = \frac{B}{v_{\parallel}} d\epsilon d\mu$$

the *momentum restoring* coefficient

$$r_{ab} \equiv \frac{\int d^3v \; m_b \nu_{ba}^{slowing} \; f_{b1}}{m_a n_{a0}\left\{\nu_{ab}^{slowing}\right\}}$$

The following operator is introduced

$$\{F_{ab}(v)\} \equiv 2\int d^3v \; \left(\frac{v_{\parallel}}{v_{th,a}}\right)^2 \; \frac{f_{a0}}{n_{a0}} \; F_{ab}(v)$$

To solve the drift-kinetic equation

$$\lambda \equiv \mu \frac{\left\langle B^2\right\rangle^{1/2}}{\epsilon - \frac{e_a \phi(\psi)}{m_a}}$$

Here $\langle B^2\rangle^{1/2}$ is used instead of B_0 It will leave a factor h after dividing by B from μ.

And choose

$$f_{a1} = \widehat{f}_{a1} f_{a0}$$

results

$$\begin{aligned}
\widehat{f}_{a1} &= -Iv_\parallel \frac{1}{\frac{e_a\langle B^2\rangle^{1/2}}{m_a}} \frac{\langle B^2\rangle^{1/2}}{B} \frac{\partial}{\partial\psi} \ln f_{a0} \\
&\quad + I \frac{1}{\frac{e_a\langle B^2\rangle^{1/2}}{m_a}} \Theta^{circ} \left[V_\parallel \frac{\partial f_{a0}}{\partial\psi} \right] \text{(is zero for trapped)} \\
&\quad + \Theta^{circ} \left[\frac{2V_\parallel}{v_{th,a}^2} \left(\frac{e_a \overline{E}_\parallel^{(A)}}{m_a} \frac{1}{\nu_a^{defl}} + \left(1 - \frac{\nu_a^{slow}}{\nu_a^{defl}} \right) \frac{1}{2x_a^2} \overline{u}_{a1}(v) + \sum_b \overline{r}_{ab} \frac{\nu_{ab}^{slow}}{\nu_{ab}^{defl}} \right) \right]
\end{aligned}$$

The terms that contain the Heaviside function

$$\Theta^{circ}$$

only exist for circulating particles.

The notations are

$$\frac{\langle B^2\rangle^{1/2}}{B} \equiv h$$

$$x_a = \frac{v}{v_{th,a}}$$

$$V_\parallel(\lambda) = \int_\lambda^{\lambda_c} d\lambda \, \frac{\frac{v^2}{2}}{\langle v_\parallel \rangle}$$

for

$$\begin{aligned}
\lambda &= \frac{\mu}{\epsilon} B_0 \\
&= \frac{v_\perp^2}{2B(x)} \frac{1}{v^2/2} \langle B^2 \rangle^{1/2} \\
&= \frac{v_\perp^2}{v^2} h
\end{aligned}$$

Small λ means small perpendicular velocity and higher parallel velocity, *i.e.* <u>circulating</u> particles.

$$\begin{aligned}
\lambda_c &= \frac{\langle B^2\rangle^{1/2}}{B_c} \\
&< 1
\end{aligned}$$

$$\begin{aligned} B_c(\psi) &= \text{maximum of } B \text{ in a surface } \psi \\ &\equiv B_{\max}(\psi) \end{aligned}$$

and

$$\nu_a^{defl} = \sum_b \nu_{ab}^{defl}$$

$$\nu_a^{slow} = \sum_b \nu_{ab}^{slow}$$

a new avareging

$$\overline{A} \equiv \left\langle \frac{A}{h} \right\rangle$$

To obtain an explicit expression for $\widehat{f}_{a1}$ we must eliminate u_{a1}.

The present form of $\widehat{f}_{a1}$, which contains $\overline{u}_{a1}$ is introduced in the formula for u_{a1}, $u_{a1}(v) = \frac{1}{f_{a0}} \frac{3}{4\pi} \int v_\parallel f_{a1} d\Omega^V$, and the integration over the angular space $d\Omega$ is performed. The result is an equation for u_{a1} whose result is

$$\begin{aligned} \frac{\overline{u}_{a1}(v)}{x_a^2} &= -f_T\, I \frac{v_{th,a}^2}{\widehat{\Omega}_a} \frac{\partial}{\partial \psi} \ln f_{a0} \times \frac{\nu_a^{defl}}{\nu_a} \\ &\quad + 2 f_c \left(\frac{e_a \overline{E}_\parallel^{(A)}}{m_a} \frac{1}{\nu_a} + \sum_b \overline{r}_{ab} \frac{\nu_{ab}^{slow}}{\nu_a} \right) \end{aligned}$$

where

$$\frac{3}{4\pi} \int d\Omega^V \; v_\parallel \; \mathbf{\Theta}^{circ} \left[V_\parallel \right] = f_c \; v^2 \frac{1}{h}$$

Here the

$$\text{the Heaviside function } \mathbf{\Theta}^{circ} []$$

will select the region of phase space where only circulating particles are. For this region, only $V_\parallel$ defined above is selected.

The proportions of circulating and trapped particles

$$\begin{aligned} f_c &= \frac{3}{4} \int_0^{\lambda_c} \frac{\lambda d\lambda}{\left\langle \sqrt{1 - \frac{\lambda}{h}} \right\rangle} \\ f_T &= 1 - f_c \\ &\approx 1.46 \times \sqrt{\varepsilon} \end{aligned}$$

$$\nu_a = f_c \nu_a^{slow} + f_T \nu_a^{defl}$$

Let us **NOTE** that

$$\begin{aligned}
1-\frac{\lambda}{h} &= 1-\frac{\mu}{\epsilon}B_0\frac{1}{h}=1-\frac{v_\perp^2}{v^2} \\
&= \frac{v_\parallel^2}{v^2} \\
\text{and } \sqrt{1-\frac{\lambda}{h}} &= \frac{v_\parallel}{v}=\xi
\end{aligned}$$

The function $\overline{u}_{a1}(v)$ is replaced in the expression of $\widehat{f}_{a1}$.

$$\begin{aligned}
\widehat{f}_{a1} &= -I\frac{1}{\widehat{\Omega}_a}hv_\parallel\frac{\partial}{\partial\psi}\ln f_{a0} \\
&\quad +I\frac{1}{\widehat{\Omega}_a}\mathbf{\Theta}^{circ}\left[V_\parallel\right]\frac{\partial}{\partial\psi}\ln f_{a0} \\
&\quad +\mathbf{\Theta}^{circ}\left[\frac{2V_\parallel}{v_{th,a}^2}\left(\frac{e_a\overline{E}_\parallel^{(A)}}{m_a}\frac{1}{\nu_a}+\sum_b\frac{\nu_{ab}^{slow}}{\nu_a}\overline{r}_{ba}\right)\right]
\end{aligned}$$

The *restoring coefficients* must still be determined.
Here $\widehat{\Omega}_a=\frac{e_a\left\langle B^2\right\rangle^{1/2}}{m_a}$.

$$\begin{aligned}
V_\parallel(\lambda) &\equiv \int_\lambda^{\lambda_c}d\lambda\;\frac{v^2/2}{\left\langle v_\parallel\right\rangle} \\
&\quad \text{integration over part } [\lambda,\lambda_c] \text{ of circulating particles}
\end{aligned}$$

definition that involves the number of trapped particles

$$\begin{aligned}
&\frac{3}{4\pi}\int d\Omega\;v_\parallel\;\Theta\left[V_\parallel\right] \\
&= v^2\frac{1}{h}\;f_{circ}
\end{aligned}$$

$$\begin{aligned}
f_{circ} &= \frac{3}{4}\int_0^{\lambda_c}\frac{\lambda\;d\lambda}{\left\langle\sqrt{1-\frac{\lambda}{h}}\right\rangle} \\
&\leq 1 \;\;\text{fraction of circulating particles}
\end{aligned}$$

We note

$$\left\langle\sqrt{1-\frac{\lambda}{h}}\right\rangle=\left\langle\sqrt{1-\frac{v_\perp^2}{v^2}}\right\rangle=\left\langle\frac{v_\parallel}{v}\right\rangle\sim\frac{1}{v}\left\langle v_\parallel\right\rangle$$

and

$$\begin{aligned} f_{trapped} &= 1 - f_{circ} \\ &\approx 1.46 \times \sqrt{\frac{r}{R}} \end{aligned}$$

2.23 The radial transport fluxes

The expressions

$$\Gamma_a = \left\langle \int d^3v \ \left(\mathbf{v}_{Da} \cdot \boldsymbol{\nabla}\psi \right) \ f_a \right\rangle$$

$$Q_a = \left\langle \int d^3v \ \left(\mathbf{v}_{Da} \cdot \boldsymbol{\nabla}\psi \right) \ \left(m_a \epsilon - e_a \Phi \right) \ f_a \right\rangle$$

and

$$\begin{aligned} \Gamma_a &= -I(\psi) \left\langle \frac{\sum_b R_{ab}}{m_a \Omega_a} \right\rangle \\ &\quad -I(\psi) \left\langle \frac{n_{0a} \ E_\parallel^{(A)}}{B} \right\rangle \end{aligned}$$

where

$$R_{ab} \equiv \int d^3v \ \left(m_a v_\parallel \right) \ C_{ab}\left(f_a^{(1)} \ , \ f_b^{(1)} \right)$$

parallel collision FRICTION force
between species a and b

and one finds

$$\begin{aligned} -I(\psi) \left\langle \frac{\sum_b R_{ab}}{m_a \Omega_a} \right\rangle &\sim R \times F_{ab} \sim \frac{R m v_\parallel}{t} \\ &\sim \text{toroidal angular momentum generation} \end{aligned}$$

Calculate

- the distribution function $f_a^{(1)}$

- the linearized collision operator $C_{ab}\left(f_a^{(1)}\ ,\ f_b^{(1)}\right)$
- then the Friction forces R_{ab},
- finally the radial particle flux Γ_a,

2.23.1 The distribution function

For the first order distribution function $f_a^{(1)}$,

$$\begin{aligned}
\widehat{f}_a^{(1)} \quad = \quad & -I\frac{v_\parallel}{\Omega_a}h\ \frac{\partial}{\partial\psi}\ln f_{0a} \\
& +\frac{\nu_s^a}{\nu_a}\ I\frac{\Theta\left(V_\parallel\right)}{\Omega_a}\ \frac{\partial}{\partial\psi}\ln f_{0a} \\
& +\frac{2}{\alpha_a^2}\ \ \Theta\left(V_\parallel\right)\ \frac{e_a}{m_a}\overline{E}_\parallel^{(A)} \\
& +\frac{2}{\alpha_a^2}\ \ \Theta\left(V_\parallel\right)\ \sum_b\frac{\nu_s^{ab}}{\nu_a}\ \overline{r}_{ab}
\end{aligned}$$

with ν_s^{ab} ≡slowing down freq. due to b, $\nu_s^a \equiv \sum_b \nu_s^{ab}$ slowing down due to ALL species b; and $\nu_a = f_{circ}\ \nu_s^a + f_{trapped}\nu_a^{defl}$, freq. due to slowing down on circulating and pitch angle from trapped.

Note that $\nu_{slowing}^{ab}$ does NOT depend on the distribution function. So is ν_s^a and ν_a.

2.23.2 The collision operator

The formula

$$\begin{aligned}
C_{ab}\left(f_{a1}, f_{b1}\right) \quad = \quad & \nu_{ab}^{defl}\ \frac{v_\parallel}{B}\frac{\partial}{\partial\mu}\mu v_\parallel\frac{\partial}{\partial\mu}f_{a1} \\
& +\left[\nu_{ab}^{defl}-\nu_{ab}^{slowing}\right]\ \frac{v_\parallel u_{a1}\left(v\right)}{v^2}\ \ f_{a0} \\
& +\nu_{ab}^{slowing}\ \frac{2v_\parallel}{v_{th,a}^2}r_{ba}\ \ f_{a0}
\end{aligned}$$

uses

- $f_a^{(1)}$ in the pitch angle operator,

- the flow velocity u_{a1} is expressed in terms of $f_a^{(1)}$;
- the momentum restoring coefficient r_{ab} on species a from the species b is defined with the help of $f_b^{(1)}$;

the other quantities in this formula, ν_{ab}^{defl}, $\nu_{ab}^{slowing}$ do not depend on $f_a^{(1)}$.

2.23.3 Friction forces

Using the colision operator $C\left[f_a^{(1)}, f_b^{(1)}\right]$ where we have inserted $f_a^{(1)}$ we will now caclulate formally R_{ab}.

$$R_{ab} \equiv \int d^3v \ \left(m_a v_\parallel\right) \ C_{ab}\left[f_a^{(1)} \ , \ f_b^{(1)}\right]$$

It is the friction related to the relative parallel flows of particles a and b.

2.23.4 The radial particle flux

The part

$$\Gamma_a^{friction} = -I\left(\psi\right) \ \left\langle \frac{\sum_b R_{ab}}{m_a \Omega_a} \right\rangle$$

is due to the friction between flowing particle along parallel direction.

2.24 Comments on the fluxes

Due to

$$\frac{\partial}{\partial \psi} \ln f_{0a}$$

it will appear in the fluxes the coefficients

$$A_{1a}$$
$$A_{2a}$$
$$A_{3a}$$

The approximation

$$\overline{r}_{ab} = -I \frac{d\Phi}{d\psi} \frac{1}{\left\langle B^2 \right\rangle^{1/2}}$$

this is

$$\frac{E_r}{B_\theta}$$

and the flows are toroidal.

Introduced in $f_a^{(1)}$ one can calculate the flows.

2.25 Collisions multiple-ion species (Hirschman 1977)

2.25.1 Classical transport since the regime is Pfirsch Schluter

This starts with the parallel balance of forces

$$\begin{aligned} & v_\parallel \left[\left(\nabla_\parallel \ln p_a - \frac{e_a}{T_a} E_\parallel \right) \right. \\ & \qquad \left. - L_1^{(3/2)} \left(x_a^2 \right) \left(\nabla_\parallel \ln T_a \right) \right] f_{Ma} \\ = & \sum_b \left[C_{ab} \left(f_{a1}, f_{Mb} \right) + C \left(f_{Ma}, f_{b1} \right) \right] \end{aligned}$$

The two *generalized forces* are

$$\begin{aligned} A_{1a} & \equiv \nabla_\parallel \ln p_a - \frac{e_a}{T_a} E_\parallel \\ A_{2a} & \equiv \nabla_\parallel \ln T_a \end{aligned}$$

It is clear that the equation is not trivial for only *variation of the plasma variables in the magnetic surface* (see **Stringer**).

[**Note** the use of the Laguerre polynomials is made explicit also in **Shaing Hirshman Zarnstorff** viscosity, where the viscous stress and friction forces are expressed in terms of flows u and q/p and a system of equation results. This involves the coefficients $\mathcal{M}$ and $\mathcal{N}$. **End**.]

Then the equation shows balance at equilibium of forces, only permitted by collisions.

This is a typical situation of Pfirsch Schluter regime since the latter is exclusively dnetermined by variation of the basic plasma variables in the magnetic surface.

Also, the Pfirsch Schluter regime is strongly collisional and the bananas are not visible. Then the perturbation to the distribution function that results from the necessary variation of plasma variables in the magnetic surface f_{a1} is only governed by the collisional friction in response to the parallel gradients of pressure, temperature, transported (advected) by the parallel velocity. We have no reason to introduce in the balance the term of advection by the drift velocity of the radially-varying Maxwellian $v_D \partial f_{Ma}/\partial r$. The drift is not visible.

$$\begin{aligned}
x_a^2 &= \left(\frac{v}{v_{th,a}}\right)^2 \\
v_{th,a} &= \left(\frac{2T_a}{m_a}\right)^{1/2} \\
L_k^{(3/2)}\left(x_a^2\right) &\equiv \text{Sonin polynomials of order } 3/2 \text{ and index } k
\end{aligned}$$

We write the expressions of the Laguerre polynomials

$$L_0^{(3/2)}\left(x_a^2\right) = 1$$

$$L_1^{(3/2)}\left(x_a^2\right) = \frac{5}{2} - x_a^2$$

$$L_2^{(3/2)}\left(x_a^2\right) = \frac{35}{8} - \frac{7}{2}x_a^2 + \frac{1}{2}x_a^4$$

NOTE
Let us compare **Hirschman Sigmar Clarke** (see above)

$$v_\parallel \nabla_\parallel \left(f_{a1} + I\frac{v_\parallel}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial \psi}\right) = v_\parallel \frac{e_a E^{(A)}}{T_a} f_{a0} + \sum_b C_{ab}\left(f_{a1}, f_{b1}\right)$$

with **Hirschman 1977** (multiple-ion species)

$$\begin{aligned}
& v_\parallel \left[\left(\nabla_\parallel \ln p_a - \frac{e_a}{T_a}E_\parallel\right)\right. \\
& \qquad \left. - L_1^{(3/2)}\left(x_a^2\right)\left(\nabla_\parallel \ln T_a\right)\right] f_{Ma} \\
&= \sum_b \left[C_{ab}\left(f_{a1}, f_{Mb}\right) + C\left(f_{Ma}, f_{b1}\right)\right]
\end{aligned}$$

We must note that the second equation the Left Hand Side is

$$\sim v_\parallel \nabla_\parallel f_{Ma}$$

which is possible since the plasma parameters inside f_{Ma}, the density and the temperature, have spatial variations along the magnetic field lines, *i.e.* have variation on magnetic surface.

This is NOT the first equation, where the neoclassical aspect $\rho_\theta \partial f_{Ma}/\partial r$ is present.

Then we say that in this approach the variation *of plasma variables n, T_a , p_a in the magnetic surface, along the magnetic lines leads to parallel advection $v_{\parallel}\nabla_{\parallel}$ which is only compensated (balanced) by collisions.*

The present approach is related to Pfirsch Schluter regime, where there is high collisionality and NO trace of banana. Therefore the perturbation to the distribution function f_{a1} does not contain a neoclassical part $\rho_\theta \partial f_{Ma}/\partial r$.

The theory developed by **Hirschman Sigmar Clarke** retains both terms

$$\begin{array}{c} v_{\parallel}\nabla_{\parallel}f^{(1)} \quad \text{and} \\ \mathbf{v}_{drift}\cdot\boldsymbol{\nabla}f^{(0)} \end{array}$$

the second brings the basic neoclassical perturbation of drift-velocity-advection of the Maxwellian distribution which has radial variation. This perturbation has the factor ρ_θ .

$$\begin{aligned} & \mathbf{v}_{Drift}\cdot\boldsymbol{\nabla}f_{Ma} \\ = & \ \mathbf{v}_{drift}\cdot\boldsymbol{\nabla}\psi \ \times\frac{\partial f_{Ma}}{\partial\psi} \\ = & \ v_{\parallel}\nabla_{\parallel}\left(I\frac{v_{\parallel}}{\Omega_{ca}}\right)\times\frac{\partial f_{Ma}}{\partial\psi} \end{aligned}$$

In **Hirschman 1977** the Maxwellian is considered as parametric function of the *(1) density* and *(2) temperature*. Then there is a parallel variation of the Maxwellian distribution, through these paramaters. Then the operator

$$v_{\parallel}\nabla_{\parallel}f_M$$

provides the parallel advection (by the very high parallel velocity) of the small perturbation of the parameters *in the surface.*

$$\begin{aligned} n(r,\theta) &= n_0(r)+n_1(r,\theta) \\ T(r,\theta) &= T_0(r)+T_1(r,\theta) \end{aligned}$$

This advection is halted (balanced) by collisions.

This is in the deep collisional regime Pfirsch Schluter.

END

2.25.2 Method of moments

The solution of the equation for the perturbation to the distribution function f_{a1} is obtained by the method of *moments.*

The equation is multiplied by

$$m_a v_\parallel \; L_k^{(3/2)} \left(x_a^2\right)$$

and integrated over the velocity space.

$$\begin{aligned} & p_a A_{1a} \delta_{0k} \\ & -\frac{5}{2} p_a A_{2a} \delta_{1k} \\ = & \int d^3 v \; m_a v_\parallel L_k^{(3/2)} \; C_a \end{aligned}$$

for

$$C_a = \sum_b C_{ab}$$

The first moment, with $L_0^{(3/2)} \left(x_a^2\right)$,

$$\nabla_\parallel p_a - e_a n_a E_\parallel = R_a$$

The second moment, with $L_1^{(3/2)} \left(x_a^2\right)$,

$$\frac{5}{2} n_a \nabla_\parallel T_a = H_a$$

The superior moments $k \geq 2$ produce *constraining equations*

$$\begin{aligned} \int d^3 v \; m_a v_\parallel \; L_k^{(3/2)} \left(x_a^2\right) \; C_a &= 0 \\ k &\geq 2 \end{aligned}$$

because these terms $L_{k \geq 2}^{(3/2)} \left(x_a^2\right)$ are missing in the left hand side of the balance of parallel forces.

Two friction forces R_a and H_a can be defined directly

$$R_a = \int d^3 v \; m_a v_\parallel \; C_a$$

$$H_a = \int d^3 v \; m_a v_\parallel \; \left[\left(\frac{v}{v_{th,a}} \right)^2 - \frac{5}{2} \right] \; C_a$$

NOTE the first moment, selects

$$k = 0$$

in the term

$$p_a A_{1a} \delta_{0k}$$

and this leaves

$$p_a \nabla_\| \ln p_a = \nabla_\| p_a$$

This is the variation of the pressure in the surface. There will be also the term with the electric field $E_\|$. But what is important is the balance

$$\nabla_\| p_a \sim R_a$$

which expresses exactly what we have defined as present regime: balance of variation of parameters in the surface by collisions.

END

The perturbation to the distribution function is expanded in a series of *flows* u_{ak} with coefficients Sonin polynomials

$$f_{a1} = \frac{2v_\|}{v_{th,a}^2} f_{Ma} \sum_{k=0}^{\infty} u_{ak} \ L_k^{(3/2)} \left(x_a^2\right)$$

We **NOTE** the following. In the equation expressing the balance of forces along the parallel direction, the perturbation to the distribution function f_{a1} is only contained in the collisional operator, in the right hand side. The left hand side consists of parallel forces: the parallel gradient of the pressure; they come from assuming that the parameters of a maxwellian distribution function (local) have small variation along the parallel direction and the operator

$$v_\| \nabla_\|$$

acting on the full distribution function

$$f_{Ma}$$

will extract these forces that must further be balanced by collisional friction (right hand side).

Now, the collision operator will be linearized, $f_{Ma} + f_{a1}$.

It is natural to expect that the perturbation f_{a1} will have as coefficients

$$v_\|$$
$$\text{and} \ \ f_{Ma}$$

then

$$f_{a1} \sim v_\| f_{Ma} \times ...$$

END

Only the first two terms in the Sonine-polynomials expansion have a name: flow velocity $u_{a\|}$ and heat flow $\frac{q_{a\|}}{p_a}$.

The expansion in *flows* u_{ak} with Sonin polynomials as coefficients is

$$\begin{aligned}
f_{a1} &= \frac{2v_\|}{v_{th,a}^2} f_{Ma} \left[L_0^{(3/2)}\left(x_a^2\right)\ u_{a\|} \qquad \left(\text{flow } u_{a\|}\right) \right. \\
&\qquad -\frac{2}{5} L_1^{(3/2)}\left(x_a^2\right) \frac{q_{a\|}}{p_a} \qquad \left(\text{flow } \frac{q_{a\|}}{p_a}\right) \\
&\qquad \left. + \sum_{k=2}^{\infty} L_k^{(3/2)}\left(x_a^2\right)\ u_{ak} \right] \qquad \left(\text{higher order flows } u_{ak}\right)
\end{aligned}$$

where the *flows* corresponding to higher $k \geq 2$ Sonine polynomials in the expansion are defined as

$$u_{ak} = \frac{1}{n_a} \frac{\int d^3v\ L_k^{(3/2)}\left(x_a^2\right)\ v_\| f_{a1}}{\left\{ \left[L_k^{(3/2)}\left(x_a^2\right) \right]^2 \right\}}$$

with the notation

$$\left\{ A\left(v\right) \right\} = \frac{8}{3\sqrt{\pi}} \int d^3v\ x^4 \exp\left(-x^2\right)\ \times A\left(x v_{th,a}\right)$$

for a function of velocity v. This symbol occurs from the condition of normalization of the Sonine polynomials, where there is a *weight function* of the set of functions.

We NOTE that here we have already assumed that the equilibrium distribution functions are MAXWELLIAN.

This is what makes interesting the Sonine polynomials: the wight function for orthonormality is an exponential like a Maxwell distribution.

The coefficients of the $k = 0$ Sonine polynomial is

$$u_{\|a} = \frac{1}{n_a} \int d^3v\ v_\| f_{a1}$$

parallel flow velocity

The coefficient of the $k = 1$ Sonine polinomial is

$$q_{\|a} = \int d^3v\ \left(\frac{mv^2}{2} - \frac{5}{2} T_a \right) v_\| f_{a1}$$

parallel random heat flux

Here "random" means "statistically determined".

We now have an expression for the perturbation to the distribution function, as a series of *flows* with Sonine polynomials as coefficients.

This is NOT yet a solution.

It is not a solution because the terms of the expansion consist of integrals over the velocity space of product of Laguerre (Sonine) polynomials and the perturbation function f_{a1}, exactly what we want to find.

Then it is a symbolical solution.

We can replace this "solution" in the collision operator, after linearization $f_{Ma} + f_{a1}$.

After doing this, we can *purely formally* integrate in the expressions defining the friction flow and the friction of heat.

This will make the two flows $u_{a\|}$ and $q_{a\|}/p_a$ to be exhibited in the expressions of R_a and H_a. The coefficients are numerical, a matrix denoted l_{ij}^{ab}.

$$R_a = \sum_b \left[l_{11}^{ab}\; u_{b\|} + \frac{2}{5} l_{12}^{ab} \; \left(\frac{q_{b\|}}{p_b} \right) \right]$$

$$H_a = \sum_b \left[l_{21}^{ab}\; u_{b\|} + \frac{2}{5} l_{22}^{ab} \; \left(\frac{q_{b\|}}{p_b} \right) \right]$$

The new coefficients

$$l_{1j}^{ab} \equiv \text{classical friction}$$

And

$$\begin{aligned} \left(l_{ij}^{ab} \right) &\equiv (\text{classical friction coeff.}) \\ &= (\text{classical trapsort coeff})^{-1} \end{aligned}$$

The series expansion in $L_k^{(3/2)}\left(x_a^2\right)$ is *infinite.*

It must be cut at a finite number of terms.

The $N = 0$ (no higher terms, just $u_{a\|}$ and $q_{a\|}/p_a$ flows) is incorect.

Take $N = 1$, which means to include just one (higher) flow, u_{a2} in addition

$$\begin{aligned} u_{a0} &= u_{a\|} \\ u_{a1} &= \frac{q_{a\|}}{p_a} \\ u_{a2} &= \text{the only higher order flow retained} \end{aligned}$$

Then

$$\begin{aligned} f_{a1}^{(N=1)} &= \frac{2v_\parallel}{v_{th,a}^2} f_{Ma} \left[L_0^{(3/2)} \left(x_a^2\right) \ u_{a\parallel} \right. \\ & \qquad -\frac{2}{5} L_1^{(3/2)} \left(x_a^2\right) \ \frac{q_{a\parallel}}{p_a} \\ & \qquad \left. + L_2^{(3/2)} \left(x_a^2\right) \ u_{a2} \right] \end{aligned}$$

The flow u_{a2} must be calculated from the constraints, in terms of the first two flows.

$$\int d^3v \ v_\parallel L_2^{(3/2)} \left(x_a^2\right) \ C_a \left[f_{a1}^{(N=1)} \ , \ f_{b1}^{(N=1)} \right] = 0$$

After separation of the unknown functions (flows) u_{a2},

$$\begin{aligned} u_{a2} &= -\sum_b \alpha_{ab} \ u_{b\parallel} \\ & \quad +\frac{2}{5} \sum_b \beta_{ab} \ \left(\frac{q_{b\parallel}}{p_b} \right) \\ & \quad +\sum_b \Delta_{ab} \ u_{b2} \end{aligned}$$

where

$$\alpha_{ab} = \frac{1}{M_a^{22} + N_{aa}^{22}} \left[M_a^{20} \ \delta_{ab} + \left(\frac{v_{th,a}}{v_{th,b}} \right) \ N_{ab}^{20} \right]$$

$$\beta_{ab} = \frac{1}{M_a^{22} + N_{aa}^{22}} \left[M_a^{21} \ \delta_{ab} + \left(\frac{v_{th,a}}{v_{th,b}} \right) \ N_{ab}^{21} \right]$$

$$\Delta_{ab} = \frac{1}{M_a^{22} + N_{aa}^{22}} \left[-\left(\frac{v_{th,a}}{v_{th,b}} \right) \ N_{ab}^{22} \ \left(1 - \delta_{ab}\right) \right]$$

The matrices are obtained by integration over the velocity space

$$M_{ab}^{jk} = \int d^3v \ \left(\frac{v_\parallel}{v_{th,a}} \right) L_j^{(3/2)} \left(x_a^2\right) \ C_{ab} \left[\frac{2v_\parallel}{v_{th,a}} f_{Ma} L_k^{(3/2)} \left(x_a^2\right) \quad , \quad f_{Mb} \right]$$

$$N_{ab}^{jk} = \int d^3v \ \left(\frac{v_\parallel}{v_{th,a}} \right) L_j^{(3/2)} \left(x_a^2\right) \ C_{ab} \left[\ f_{Ma} \quad , \quad \frac{2v_\parallel}{v_{th,b}} f_{Mb} L_k^{(3/2)} \left(x_b^2\right) \right]$$

The symbol

$$M_a^{jk} = \sum_b M_{ab}^{jk}$$

For the elements of the matrices we have

$$\begin{aligned} M_{ab}^{jk} &= M_{ab}^{kj} \\ T_a N_{ab}^{jk} &= T_b N_{ba}^{kj} \end{aligned}$$

These elements can be calculated explicitely in the case of Maxwellian f_{Ma}, if the collision operator is Coulombian linearized.

Then we have an algebraic equation for the flows

$$u_{a2}$$

which can be solved by inverting a matrix.

Two coefficients appear in this algebraic inversion

$$\widehat{\alpha}_{ab} = \alpha_{ab} + \sum_c \Delta_{ac}\alpha_{cb} + ...$$

$$\widehat{\beta}_{ab} = \beta_{ab} + \sum_c \Delta_{ac}\beta_{cb} + ...$$

Using these coefficients we find the first higher flow u_{a2} in terms of the lower flows $u_{b\|}$ and $\frac{q_{b\|}}{p_b}$ of *all the other species.*

$$u_{a2} = -\sum_b \widehat{\alpha}_{ab}\ u_{b\|} + \frac{2}{5}\sum_b \widehat{\beta}_{ab}\left(\frac{q_{b\|}}{p_b}\right)$$

Returning to the relations between the friction forces and the flows

$$l_{11}^{ab} = m_a\left[M_a^{00}\ \delta_{ab} - M_a^{02}\ \widehat{\alpha}_{ab} + \left(\frac{v_{th,a}}{v_{th,b}}\right)\ N_{ab}^{00} - \sum_s \left(\frac{v_{th,a}}{v_{th,s}}\right)\ N_{as}^{02}\ \widehat{\alpha}_{sb}\right]$$

$$l_{12}^{ab} = m_a\left[-M_a^{01}\ \delta_{ab} + M_a^{02}\ \widehat{\beta}_{ab} - \left(\frac{v_{th,a}}{v_{th,b}}\right)\ N_{ab}^{01} + \sum_s \left(\frac{v_{th,a}}{v_{th,s}}\right)\ N_{as}^{02}\ \widehat{\beta}_{sb}\right]$$

$$l_{22}^{ab} = m_a\left[M_a^{11}\ \delta_{ab} - M_a^{12}\ \widehat{\beta}_{ab} + \left(\frac{v_{th,a}}{v_{th,b}}\right)\ N_{ab}^{11} - \sum_s \left(\frac{v_{th,a}}{v_{th,s}}\right)\ N_{as}^{12}\ \widehat{\beta}_{sb}\right]$$

The balance of the friction forces

$$R_{ab} + R_{ba} = 0$$

leads to

$$m_a v_{th,a} M_{ab}^{0j} + m_b v_{th,b} N_{ba}^{0j} = 0$$

2.25.3 Calculation of the matrix elements M_{ab}^{ik} and N_{ab}^{ij}

The method is to use the generating function of the Sonine polynomials $L_k^{(3/2)}\left(x_a^2\right)$.

$$\sum_{j,k=0} \xi^j \eta^k \; M_{ab}^{jk} = -\frac{n_a}{\tau_{ab}} \overline{m}_{ab}\left(\xi, \eta\right)$$

$$\sum_{j,k=0} \xi^j \eta^k \; N_{ab}^{jk} = \frac{T_a}{T_b} \frac{v_{th,a}}{v_{th,b}} \frac{n_a}{\tau_{ab}} \overline{n}_{ab}\left(\xi, \eta\right)$$

2.25.4 The transport in the Pfirsch Schluter regime

$$\Gamma_a^{\psi} = \left\langle \int d^3 v \;\mathbf{v} \cdot \boldsymbol{\nabla}\psi \; f_a \right\rangle$$

$$\frac{q_a^{\psi}}{T_a} = \left\langle \int d^3 v \;\mathbf{v} \cdot \boldsymbol{\nabla}\psi \; \left[x_a^2 - \frac{5}{2}\right] \; f_a \right\rangle$$

3 Collisions in the Neutral Beam Injection (Gaffey)

This is **Gaffey J. Plasma Phys. 1976, 116, 149–1169.**

3.1 The Fokker Planck equation

This is

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} f + \frac{e}{m}\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \cdot \frac{\partial f}{\partial \mathbf{v}} = C\left(f, f\right)$$

The **Landau** form of the **Fokker Planck** collision operator. The test species i collides with background species j.

$$\begin{aligned} C\left(f_i, f_j\right) &= 2\pi e_i^2 e_j^2 \ln\Lambda \\ &\quad \times \frac{1}{m_i}\frac{\partial}{\partial \mathbf{v}_i} \cdot \int d^3 v_j \;\boldsymbol{\omega} \cdot \left(\frac{1}{m_i}\frac{\partial}{\partial \mathbf{v}_i} - \frac{1}{m_j}\frac{\partial}{\partial \mathbf{v}_j}\right) f_i f_j \end{aligned}$$

[**note** the symmetry of the integrand in this expression]

where

$$\begin{aligned} \boldsymbol{\omega} &= \frac{\partial^2 g}{\partial \mathbf{v}_i \partial \mathbf{v}_i} \\ &= \frac{1}{g}\,\mathbf{I} - \frac{\mathbf{g}\,\mathbf{g}}{g^3} \end{aligned}$$

g is the absolute value of the relative velocity

$$\begin{aligned} g &= |\mathbf{g}| \\ &= |\mathbf{v}_i - \mathbf{v}_j| \end{aligned}$$

$$\mathbf{g}_{ij} = \mathbf{v}_i - \mathbf{v}_j$$

of course, not a tensor.

A property of the tensor $\boldsymbol{\omega}$ is

$$\frac{\partial}{\partial \mathbf{v}_i} \cdot \boldsymbol{\omega} = -\frac{\partial}{\partial \mathbf{v}_j} \cdot \boldsymbol{\omega}$$

The second term in paranthesis can factor out $\boldsymbol{\omega} f_i$,

$$\boldsymbol{\omega} f_i \cdot \left(-\frac{1}{m_j} \frac{\partial f_j}{\partial \mathbf{v}_j} \right)$$

and is possible an integration by parts, leading to

$$\begin{aligned} & C\left(f_i, f_j\right) \\ = \; & 2\pi e_i^2 e_j^2 \frac{1}{m_i^2} \ln \Lambda \\ & \times \frac{\partial}{\partial \mathbf{v}_i} \cdot \left[\frac{\partial f_i}{\partial \mathbf{v}_i} \cdot \int d^3 v_j \; \boldsymbol{\omega} \; f_j \right. \\ & \qquad \left. - \frac{m_i}{m_j} f_i \frac{\partial}{\partial \mathbf{v}_i} \cdot \int d^3 v \; \boldsymbol{\omega} \; f_j \right] \end{aligned}$$

It is defined

$$F\left(x_{ij}\right) = \frac{1}{n_0} \frac{1}{v_{th,j}} \int d^3 v_j \; f_j \; \mathbf{g}_{ij}$$

Note that

$$\begin{aligned} \mathbf{g}_{ij} &= \mathbf{v}_i - \mathbf{v}_j \\ & \quad \text{not a tensor, just a vector} \end{aligned}$$

Another definition

$$x_{ij} \equiv \frac{v_i}{v_{th,j}}$$

Then

$$\begin{aligned} & C\left(f_i, f_j\right) \\ = \; & 2\pi e_i^2 e_j^2 \frac{1}{m_i^2} \ln \Lambda \; \times n_{0j} v_{th,j} \\ & \times \frac{\partial}{\partial \mathbf{v}_i} \cdot \left[\frac{\partial f_i}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 F\left(x_{ij}\right)}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} \right. \\ & \qquad \left. - \frac{m_i}{m_j} f_i \frac{\partial}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 F\left(x_{ij}\right)}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} \right] \end{aligned}$$

Now we will use the variable x_{ij}.
This will transform the derivatives of F.

$$\frac{\partial^2 F(x_{ij})}{\partial \mathbf{v}_i\, \partial \mathbf{v}_i} = \frac{\partial F(x_{ij})}{\partial x_{ij}} \frac{\partial^2 x_{ij}}{\partial \mathbf{v}_i\, \partial \mathbf{v}_i} + \frac{\partial^2 F(x_{ij})}{\partial x_{ij}^2} \frac{\partial x_{ij}}{\partial \mathbf{v}_i} \frac{\partial x_{ij}}{\partial \mathbf{v}_i}$$

The other term is more complicated since we have to take the "divergence" in the velocity space for the species i, i.e.

$$\frac{\partial}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 F(x_{ij})}{\partial \mathbf{v}_i\, \partial \mathbf{v}_i}$$

Here we replace $\frac{\partial^2 F(x_{ij})}{\partial \mathbf{v}_i\, \partial \mathbf{v}_i}$ by the expression in the previous formula.

$$\frac{\partial}{\partial \mathbf{v}_i} \cdot \left[\frac{\partial F(x_{ij})}{\partial x_{ij}} \frac{\partial^2 x_{ij}}{\partial \mathbf{v}_i\, \partial \mathbf{v}_i} + \frac{\partial^2 F(x_{ij})}{\partial x_{ij}^2} \frac{\partial x_{ij}}{\partial \mathbf{v}_i} \frac{\partial x_{ij}}{\partial \mathbf{v}_i} \right]$$

For the first term

$$\begin{aligned} & \frac{\partial}{\partial \mathbf{v}_i} \cdot \left[\frac{\partial F(x_{ij})}{\partial x_{ij}} \frac{\partial^2 x_{ij}}{\partial \mathbf{v}_i\, \partial \mathbf{v}_i} \right] \\ = \quad & \frac{\partial^2 F(x_{ij})}{\partial x_{ij}^2} \frac{\partial x_{ij}}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 x_{ij}}{\partial \mathbf{v}_i\, \partial \mathbf{v}_i} + \frac{\partial F(x_{ij})}{\partial x_{ij}} \frac{\partial}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 x_{ij}}{\partial \mathbf{v}_i\, \partial \mathbf{v}_i} \end{aligned}$$

The second term in the paranthesis will lead to

$$\begin{aligned} & \frac{\partial}{\partial \mathbf{v}_i} \cdot \left[\frac{\partial^2 F(x_{ij})}{\partial x_{ij}^2} \frac{\partial x_{ij}}{\partial \mathbf{v}_i} \frac{\partial x_{ij}}{\partial \mathbf{v}_i} \right] \\ = \quad & \frac{\partial^3 F(x_{ij})}{\partial x_{ij}^3} \frac{\partial x_{ij}}{\partial \mathbf{v}_i} \cdot \frac{\partial x_{ij}}{\partial \mathbf{v}_i} \frac{\partial x_{ij}}{\partial \mathbf{v}_i} \\ & + \frac{\partial^2 F(x_{ij})}{\partial x_{ij}^2} \frac{\partial}{\partial \mathbf{v}_i} \cdot \frac{\partial x_{ij}}{\partial \mathbf{v}_i} \frac{\partial x_{ij}}{\partial \mathbf{v}_i} \end{aligned}$$

The derivatives on the function $F(x_{ij})$ can be made more explicit

$$
\begin{aligned}
& C(f_i, f_j) \\
= \;& 2\pi e_i^2 e_j^2 \frac{1}{m_i^2} \ln\Lambda \;\; \times n_{0j} v_{th,j} \\
& \times \frac{\partial}{\partial \mathbf{v}_i} \cdot \left[\frac{\partial f_i}{\partial \mathbf{v}_i} \cdot \left(\frac{\partial F(x_{ij})}{\partial x_{ij}} \frac{\partial^2 x_{ij}}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} + \frac{\partial^2 F(x_{ij})}{\partial x_{ij}^2} \frac{\partial x_{ij}}{\partial \mathbf{v}_i} \frac{\partial x_{ij}}{\partial \mathbf{v}_i} \right) \right. \\
& - \frac{m_i}{m_j} f_i \left(\frac{\partial F(x_{ij})}{\partial x_{ij}} \frac{\partial}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 x_{ij}}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} \right. \\
& \qquad + \frac{\partial^2 F(x_{ij})}{\partial x_{ij}^2} \frac{\partial}{\partial \mathbf{v}_i} \cdot \frac{\partial x_{ij}}{\partial \mathbf{v}_i} \frac{\partial x_{ij}}{\partial \mathbf{v}_i} + \frac{\partial^2 F(x_{ij})}{\partial x_{ij}^2} \frac{\partial x_{ij}}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 x_{ij}}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} \\
& \qquad \left. \left. + \frac{\partial^3 F(x_{ij})}{\partial x_{ij}^3} \frac{\partial x_{ij}}{\partial \mathbf{v}_i} \cdot \frac{\partial x_{ij}}{\partial \mathbf{v}_i} \frac{\partial x_{ij}}{\partial \mathbf{v}_i} \right) \right]
\end{aligned}
$$

Gaffey calculates the derivatives of x_{ij},

$$
\begin{aligned}
\frac{\partial x_{ij}}{\partial \mathbf{v}_i} &= \frac{1}{v_{th,j}} \frac{\mathbf{v}_i}{v_i} \\
\frac{\partial}{\partial \mathbf{v}_i} \cdot \frac{\partial x_{ij}}{\partial \mathbf{v}_i} &= \frac{1}{v_{th,j}} \frac{2}{v_i} \\
\frac{\partial^2 x_{ij}}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} &= \frac{1}{v_{th,j}} \frac{1}{v_i^3} \left(v_i^2 \, \mathbf{I} - \mathbf{v}_i \, \mathbf{v}_i \right) \\
\frac{\partial}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 x_{ij}}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} &= -\frac{1}{v_{th,j}} \frac{2\mathbf{v}_i}{v_i^3}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial x_{ij}}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 x_{ij}}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} &= 0 \\
\frac{1}{v_{th,j}} \frac{\mathbf{v}_i}{v_i} \cdot \frac{1}{v_{th,j}} \frac{1}{v_i^3} \left(v_i^2 \, \mathbf{I} - \mathbf{v}_i \, \mathbf{v}_i \right) &= 0
\end{aligned}
$$

The final form (up to this moment of calculations)

$$
\begin{aligned}
& C(f_i, f_j) \\
= \;& \Gamma_{ij} \frac{\partial}{\partial \mathbf{v}_i} \cdot \left[\frac{\partial f_i}{\partial \mathbf{v}_i} \cdot \left(F' \frac{\partial^2 v_i}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} + F'' \frac{\mathbf{v}_i \, \mathbf{v}_i}{v_i^3} x_{ij} \right) \right. \\
& \qquad \left. + 2 \frac{m_i}{m_j} f_i \, \frac{\mathbf{v}_i}{v_i^3} \, \left(F' - x_{ij} \, F'' - F''' \, \frac{x_{ij}^2}{2} \right) \right]
\end{aligned}
$$

with the coefficient

$$\Gamma_{ij} = 2\pi e_i^2 e_j^2 \frac{1}{m_i^2} \ln \Lambda \; n_{0j}$$

3.2 Simplifying assumptions

The assumptions that are specific to NBI, adopted by **Gaffey**

- the density of NBI ions is much smaller than that of the background ions: there are no collisions between the NBI ions
$$n_{beam} \ll n_{background}$$
- the geometry of the beam: axisymmetric relative to **B**.
- Maxwellian distributions for f_i and f_j.
- the velocity of the NBI ions is between the thermal-ion and thermal-electron
$$v_{th,i} \ll v_b \ll v_{th,e}$$
which means that the energy of the beam ions is greater than the ion temperature but not a factor of
$$\frac{m_i}{m_e}$$
greater than the electron temperature

The calculations can be made explicit. Starting from the definition

$$F\left(x_{ij}\right) = \frac{1}{n_0} \frac{1}{v_{th,j}} \int d^3 v_j \; f_j \; \mathbf{g}_{ij}$$

we have

$$\begin{aligned} F\left(x\right) &= \left(x + \frac{1}{2x}\right) \Phi\left(x\right) + \frac{1}{2} \frac{d\Phi\left(x\right)}{dx} \\ \Phi\left(x\right) &\equiv \text{error function} \end{aligned}$$

Approximative limitting values for the function F are

$$\begin{aligned} F &\approx \frac{2}{\sqrt{\pi}} \left(1 + \frac{1}{3} x^2 - \frac{1}{30} x^4 + ...\right) \\ \text{for } x &\ll 1 \\ F &\approx x + \frac{1}{2x} - \frac{\exp\left(-x^2\right)}{2\sqrt{\pi} x^4} \left(1 - \frac{3}{x^2} + ...\right) \\ \text{for } x &\gg 1 \end{aligned}$$

3.3 Beam colliding with electrons and with ions

Explicit case: NBI beam ions colliding with background *electrons*, case $x \ll 1$

$$\begin{aligned} & C\left(f_b, f_e\right) \\ = & \frac{4}{3\sqrt{\pi}} \Gamma_{be} \frac{x_{be}}{v} \frac{\partial}{\partial \mathbf{v}} \cdot \left[\frac{\partial f_b}{\partial \mathbf{v}} + \frac{m_b}{m_e} \frac{2x_{be}^2}{v} \frac{\mathbf{v}}{v} f_b\right] \end{aligned}$$

and after neglecting the first term in paranthesis (the second has a coefficient $m_b/m_e \gg 1$)

$$C\left(f_b, f_e\right) = \frac{8}{3\sqrt{\pi}} \Gamma_{be} \frac{x_{be}^3}{v^3} \frac{m_b}{m_e} \left(v \frac{\partial f_b}{\partial v} + 3 f_b\right)$$

Explicit case: beam ions colliding with background *ions*, case $x \gg 1$

$$C\left(f_b, f_i\right) = \Gamma_{bi} \frac{\partial}{\partial \mathbf{v}} \cdot \left[\frac{\partial f_b}{\partial \mathbf{v}} \cdot \left(\frac{\partial^2 v}{\partial \mathbf{v}\, \partial \mathbf{v}} \left(1 - \frac{1}{2x_{bi}^2}\right) + \frac{\mathbf{v}\ \mathbf{v}}{v^2} \frac{1}{x_{bi}^2}\right) + \frac{m_b}{m_i} \frac{2\mathbf{v}}{v^3} f_b\right]$$

We have adopted the ordering

$$x_{bi} = \frac{v_b}{v_{th,i}} \gg 1$$

Then

$$\begin{aligned} & C\left(f_b, f_i\right) \\ = & \Gamma_{bi} \frac{\partial}{\partial \mathbf{v}} \cdot \left[\frac{\partial^2 v}{\partial \mathbf{v}\, \partial \mathbf{v}} \cdot \frac{\partial f_b}{\partial \mathbf{v}} + \frac{m_b}{m_i} \frac{2\mathbf{v}}{v^3} f_b\right] \end{aligned}$$

Another assumption is that f_b is axisymmetric in velocity space relative to the direction of $\mathbf{B}$. This introduces $\xi = v_{\parallel}/v$.

$$\begin{aligned} & C\left(f_b, f_i\right) \\ = & \Gamma_{bi} \frac{1}{v^3} \left[\frac{\partial}{\partial \xi} \left(1 - \xi^2\right) \frac{\partial f_b}{\partial \xi} + \frac{m_b}{m_i} 2v \frac{\partial f_b}{\partial v}\right] \end{aligned}$$

The two contributions to the collisional term for f_b are included

$$\begin{aligned} \left(\frac{\partial f_b}{\partial t}\right)_{\mathrm{coll}} = & \frac{1}{v^3} \left[\frac{8}{3\sqrt{\pi}} \Gamma_{be} \left(v \frac{\partial f_b}{\partial v} + 3 f_b\right)\right. \\ & \left. + \sum_i \Gamma_{bi} \left(\frac{m_b}{m_i} 2v \frac{\partial f_b}{\partial v} + \frac{\partial}{\partial \xi} \left(1 - \xi^2\right) \frac{\partial f_b}{\partial \xi}\right)\right] \end{aligned}$$

A new parameter is introduced: the velocity of the beam neutrals v_c for which the drag exerted by collisions beam-electrons equals the drag exerted by collisions beam-background ions

$$\begin{aligned} v_c &= \left(\frac{3\sqrt{\pi}}{4} \frac{m_e}{m_b} Z_1 \right)^{1/3} \\ &= 0.09 \left(\frac{m_H}{m_b} Z_1 \right)^{1/3} v_{th,e} \end{aligned}$$

where

$$Z_1 = \sum_i \frac{n_i}{n_e} \frac{Z_i^2 m_b}{m_i}$$

The expression in terms of v_c is

$$\left(\frac{\partial f_b}{\partial t} \right)_{\mathrm{col}\,l} = \frac{1}{\tau_s} \frac{1}{v^3} \left[v \frac{\partial}{\partial v} \left(v^3 + v_c^3 \right) f_b + Z_2 \frac{v_c^3}{2} \frac{\partial}{\partial \xi} \left(1 - \xi^2 \right) \frac{\partial f_b}{\partial \xi} \right]$$

with

$$Z_2 = \sum_i \frac{n_i}{n_e} \frac{Z_i^2}{Z_1}$$

$$\tau_s = \frac{v_c^3}{2\Gamma_{be} Z_1} = \frac{M_b^2 v_c^3}{4\pi e^2 e_b^2} \ln \Lambda \times \frac{1}{Z_1} \frac{1}{n_e}$$

3.4 Time asymptotic solution for the distribution function

Gaffey solves the equation.

Time asymptotic solution means time stationarity and the derivation to time is zero.

3.4.1 No source

Adopting the assumption that the distribution f_b is axisymmetric relative to $\mathbf{B}$ the following term is zero

$$\begin{aligned} & \frac{e_b}{m_b} \left(\mathbf{v} \times \mathbf{B}_0 \right) \cdot \frac{\partial f_b}{\partial \mathbf{v}} \\ = & -\Omega_b \frac{\partial f_b}{\partial \zeta} = 0 \end{aligned}$$

where ζ is the gyro-phase. Then the **Fokker Planck** equation is simply the zero-effect of the collisions

$$\begin{aligned}\left(\frac{\partial f_b}{\partial t}\right)_{\mathrm{col}\,l} &= \frac{1}{\tau_s}\frac{1}{v^3}\left[v\frac{\partial}{\partial v}\left(v^3+v_c^3\right)f_b + Z_2\frac{v_c^3}{2}\frac{\partial}{\partial \xi}\left(1-\xi^2\right)\frac{\partial f_b}{\partial \xi}\right] \\ &= 0\end{aligned}$$

It is remarked that the second term is the Legendre operator. This suggests that the expansion in Legendre polynomials is useful.

Then one expands $f_b(\xi)$ in series of Legendre polynomials

$$f_b(v,\xi) = \sum_{l=0}^{\infty} f_l(v)\,P_l(\xi)$$

The reason of this expansion comes from the form of the Rosenbluth potentials where the following expression appears

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{\sqrt{x^2-2x\cos\theta+1}}$$

which expands in Legendre polynomials. In different words, the pitch angle scattering has the form of the operator from the equation that defines the Legendre polynomials.

This expansion separates in every term the factors with different dependencies: coefficients $f_l(v)$ depend on the modulus v and the dependence on the pitch angle variable ξ is taken by the polynomials. Inversly

$$f_l(v) = \frac{2l+1}{2}\int_{-1}^{1} f_b(v,\xi)\,P_l(\xi)\,d\xi$$

The **Fokker Planc**k equation becomes a set of un-coupled first-order ordinary differential equations for the function-coefficients $f_l(v)$

$$v\frac{\partial f_l(v)}{\partial v} + \frac{3v^2-\frac{1}{2}Z_2 l(l+1)v_c^3}{v^3+v_c^3}f_l(v) = 0$$

The solution

$$f_l(v) = \frac{A_l}{v^3+v_c^3}\left(\frac{v^3}{v^3+v_c^3}\right)^{\frac{1}{6}l(l+1)Z_2}$$

where A_l are constants determined by boundary conditions.

Note that this equation is without sources.

3.4.2 With the source

Now we consider the introduction of a source.

We still are at stationarity, $\partial/\partial t = 0$.

The source term contributes to the equilibrium balance

$$\begin{aligned}\left(\frac{\partial f_b}{\partial t}\right)_{\mathrm{col}\,l} &= \frac{1}{\tau_s}\frac{1}{v^3}\left[v\frac{\partial}{\partial v}\left(v^3+v_c^3\right)f_b + Z_2\frac{v_c^3}{2}\frac{\partial}{\partial\xi}\left(1-\xi^2\right)\frac{\partial f_b}{\partial\xi}\right] + S\left(v,\xi\right)\\ &= 0\end{aligned}$$

Now both $f_b\left(v,\xi\right)$ and $S\left(v,\xi\right)$ are expanded in series of Legendre polynomials

$$v\frac{\partial f_l\left(v\right)}{\partial v} + \frac{3v^2-\frac{1}{2}Z_2 l\left(l+1\right)v_c^3}{v^3+v_c^3}f_l\left(v\right) = -\frac{v^3}{v^3+v_c^3}\tau_s S_l\left(v\right)$$

The coefficients of the series of the source

$$S_l\left(v\right) = \frac{2l+1}{2}\int_{-1}^{1}d\xi\ S\left(v,\xi\right)P_l\left(\xi\right)$$

The solution to the equations of the set, uses the solution of the *homogeneous* equation, obtained above

$$f_l\left(v\right) = -\tau_s\frac{1}{v^3+v_s^3}\left(\frac{v^3}{v^3+v_c^3}\right)^{\frac{1}{6}l(l+1)Z_2}\int^{v}v^2dv\ S_l\left(v\right)\left(\frac{v^3+v_c^3}{v^3}\right)^{\frac{1}{6}l(l+1)Z_2}$$

The simplified representation of the beam as consisting of only one energy and one pitch angle

$$S\left(v,\xi\right) = S^0\frac{1}{v^2}\delta\left(v-v_b\right)\delta\left(\xi-\xi_b\right)$$

the integration can be done explicitely.

Then the solution is

$$\begin{aligned}f_l\left(v\right) &= -\frac{2l+1}{2}\tau_s S^0\frac{1}{v^3+v_c^3}P_l\left(\xi_b\right)P_l\left(\xi\right)\\ &\quad\times\left[\left(\frac{v^3}{v_c^3}\right)\left(\frac{v_b^3+v_c^3}{v^3+v_c^3}\right)\right]^{\frac{1}{6}l(l+1)Z_2}\\ &\quad\times\Theta\left(v-v_b\right)\end{aligned}$$

where Θ is the Heaviside function.

Introducing the coefficients in the series

$$
\begin{aligned}
f_b(v,\xi) &= S^0\tau_s \frac{1}{v^3+v_c^3}\sum_{l=0}^{\infty}\frac{2l+1}{2}P_l(\xi_b)P_l(\xi) \\
&\times\left[\left(\frac{v^3}{v_c^3}\right)\left(\frac{v_b^3+v_c^3}{v^3+v_c^3}\right)\right]^{\frac{1}{6}l(l+1)Z_z} \\
&\times\Theta(v_b-v)
\end{aligned}
$$

Notes of **Gaffey** on this solution

- the function $f_b(v,\xi)$ has NO ions travelling faster than the injection velocity v_b: there is NO parallel diffusion. Instead we have perpendicular diffusion. Test of consistency, the total number density

$$
\begin{aligned}
N_b &= \int_0^\infty v^2 dv \int_{-1}^{1} d\xi\ f_b(b,\xi) \\
&= S^0\tau_s \int_0^{v_b} dv \frac{1}{v^3+v_c^3} \\
&= S^0\frac{\tau_s}{3}\ln\left(\frac{v_b^3+v_c^3}{v_c^3}\right) \\
&= S^0\tau_0(v_b)
\end{aligned}
$$

3.5 Time evolution of the beam distribution

Now we consider the *dynamical* state where the source and the collisions evolve together to establish a balance (which we have considered before)

This means to solve

$$
\begin{aligned}
\tau_s\frac{\partial f_b(v,\xi,t)}{\partial t} &= \frac{1}{v^2}\frac{\partial}{\partial v}\left[\left(v^3+v_c^3\right)f_b(v,\xi,t)\right] \\
&+\frac{Z_2 v_c^3}{2v^3}\frac{\partial}{\partial\xi}\left(1-\xi^2\right)\frac{\partial f_b(v,\xi,t)}{\partial\xi} \\
&+\tau_s S(v,\xi,t)
\end{aligned}
$$

for the distribution function $f_b(v,\xi,t)$ of the ions of the beam.

After expansion

$$
\begin{aligned}
\tau_s\frac{\partial f_l(v,t)}{\partial t} &= \frac{1}{v^2}\frac{\partial}{\partial v}\left(v^3+v_c^3\right)f_l(v,t) \\
&-\frac{l(l+1)}{2}\frac{v_c^3}{v^3}Z_2 f_l(v,t) \\
&+\tau_s S_l(v,t)
\end{aligned}
$$

To solve the time evolution one has to use a *Laplace transformation* of the function.

The complex variable associated to the time in this Laplace transform is p.

It is assumed that

$$f_l\left(v,t=0\right)=0$$

then

$$\begin{aligned}
& \frac{1}{v^2}\frac{\partial}{\partial v}\left(v^3+v_c^3\right)\widetilde{f}_l\left(v,p\right) \\
& -\frac{l\left(l+1\right)}{2}\frac{v_c^3}{v^3}Z_2\widetilde{f}_l\left(v,p\right) \\
& -\tau_s p\widetilde{f}_l\left(v,p\right) \\
= & -\tau_s\widetilde{S}_l\left(v,p\right)
\end{aligned}$$

The equation in v is integrated, obtaining as solution the Laplace-component p of the transformed distribution

$$\begin{aligned}
\widetilde{f}_l\left(v,p\right) &= \tau_s\frac{1}{v^3+v_c^3}\left(\frac{v^3}{v^3+v_c^3}\right)^{\frac{1}{6}l(l+1)Z_2}\left(v^3+v_c^3\right)^{\frac{1}{3}p\tau_s} \\
& \times\int_0^\infty v'^2dv'\ \frac{1}{\left(v'^3+v_c^3\right)^{\frac{1}{3}p\tau_s}}\widetilde{S}_l\left(v',p\right)\left(\frac{v'^3+v_c^3}{v'^3}\right)^{\frac{1}{6}l(l+1)Z_2}
\end{aligned}$$

This response must further be inverted Laplace

$$\begin{aligned}
f_l\left(v,t\right) &= \tau_s\frac{1}{v^3+v_c^3}\left(\frac{v^3}{v^3+v_c^3}\right)^{\frac{1}{6}l(l+1)Z_2} \\
& \times\int_0^\infty v'^2dv'\ \left(\frac{v'^3+v_c^3}{v'^3}\right)^{\frac{1}{6}l(l+1)Z_2} \\
& \qquad\times\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dp\ \widetilde{S}_l\left(v',p\right)\exp\left\{p\left[t-\frac{\tau_s}{3}\ln\left(\frac{v'^3+v_c^3}{v^3+v_c^3}\right)\right]\right\}
\end{aligned}$$

A technical property

$$\begin{aligned}
& \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dp\ \widetilde{S}_l\left(v',p\right)\exp\left\{p\left[t-\frac{\tau_s}{3}\ln\left(\frac{v'^3+v_c^3}{v^3+v_c^3}\right)\right]\right\} \\
= & \ S_l\left[v',t-\frac{\tau_s}{3}\ln\left(\frac{v'^3+v_c^3}{v^3+v_c^3}\right)\right]
\end{aligned}$$

since the LHS is just the inverse Laplace transform but instead of the exponent pt one has

$$\begin{aligned} pt &\rightarrow \\ & p\left[t - \frac{\tau_s}{3}\ln\left(\frac{v'^3 + v_c^3}{v^3 + v_c^3}\right)\right] \end{aligned}$$

Then the expression of the coefficients f_l is

$$\begin{aligned} f_l(v,t) &= \tau_s \frac{1}{v^3 + v_c^3}\left(\frac{v^3}{v^3 + v_c^3}\right)^{\frac{1}{6}l(l+1)Z_2} \\ & \times \int_v^\infty v'^2 dv' \left(\frac{v'^3 + v_c^3}{v'^3}\right)^{\frac{1}{6}l(l+1)Z_2} \\ & \qquad \times S_l\left[v', t - \frac{\tau_s}{3}\ln\left(\frac{v'^3 + v_c^3}{v^3 + v_c^3}\right)\right] \end{aligned}$$

This formula permits to find the time interval for slowing down between

- the initial velocity v', injected at a time $\tau_0(v')$, and
- a velocity v at time t,

The connection is

$$\frac{1}{3}\tau_s \ln\left(\frac{v'^3 + v_c^3}{v^3 + v_c^3}\right) = \tau_0(v') - \tau_0(v)$$

where the following time variable is defined

$$\tau_0(v) = \frac{1}{3}\tau_s \ln\left(\frac{v^3 + v_c^3}{v_c^3}\right)$$

This is the time for a particle injected with the velocity v to slow down to the final velocity $= 0$.

3.6 A simplified assumption for the beam distribution injected

Assume the beam is switched on at $t = 0$ and keeps constant source thereafter

$$\begin{aligned} S(v,\xi,t) &= S^0 \frac{1}{v^2}\delta(v - v_b)\delta(\xi - \xi_b) \\ & \times\Theta(t) \end{aligned}$$

Then

$$\begin{aligned}
f_b(v,\xi,t) &= S^0\tau_s \frac{1}{v^3+v_c^3}\Theta\left[t-\frac{\tau_s}{3}\ln\left(\frac{v'^3+v_c^3}{v^3+v_c^3}\right)\right] \\
&\quad \times\sum_{l=0}^{\infty}\frac{2l+1}{2}P_l(\xi)\,P_l(\xi_b) \\
&\qquad\qquad \times\left[\left(\frac{v^3}{v_c^3}\right)\left(\frac{v_b^3+v_c^3}{v^3+v_c^3}\right)\right]^{\frac{1}{6}l(l+1)Z_2} \\
&\qquad\qquad \times\Theta(v_b-v)
\end{aligned}$$

There is a travelling front in velocity space with no particles with velocity

$$\begin{aligned}
v &< v_{\min}(t) \\
v_{\min}(t) &= \left[\left(v_b^3+v_c^3\right)\exp\left(-\frac{3t}{\tau_s}\right)-v_c^3\right]^{1/3} \\
&\text{for} \\
t &< \tau_0(v_b)=\frac{1}{3}\tau_s\ln\left(\frac{v_b^3+v_c^3}{v_c^3}\right)
\end{aligned}$$

No particles with velocity higher than v_b (since there is no parallel diffusion).

3.7 Charge exchange

The equation will contain a new term, with an expression which depends on a typical time for charge exchange

$$\begin{aligned}
\tau_s\frac{\partial f_b(v,\xi,t)}{\partial t} &= \frac{1}{v^2}\frac{\partial}{\partial v}\left[\left(v^3+v_c^3\right)f_b(v,\xi,t)\right] \\
&\quad +\frac{Z_2v_c^3}{2v^3}\frac{\partial}{\partial\xi}\left(1-\xi^2\right)\frac{\partial f_b(v,\xi,t)}{\partial\xi} \\
&\quad -\frac{\tau_s}{\tau_{cx}(v)}f_b \quad \text{(this is charge exchange term)} \\
&\quad +\tau_s S(v,\xi,t)
\end{aligned}$$

The charge exchange is present through the term

$$\begin{aligned}
&\tau_{cx}(v) \\
=\ &\text{mean charge exchange life-time}
\end{aligned}$$

and has expression

$$\frac{1}{\tau_{cx}}=n_n\sigma_{cx}(v)\ \ v$$

where $n_n \equiv$ neutral particle density. An expression for the cross section

$$\sigma_{cx}(v) = 21 \times 10^{-16} \frac{1}{0.2 \times \left[\frac{1}{2}mv^2\right] + 1} \quad \left(cm^2\right)$$

the quantity

$$\left[\frac{1}{2}mv^2\right] \quad \text{is measured in } keV$$

An indication is

$$\begin{aligned} E_b &\equiv \text{energy of the beam} \\ &= \frac{1}{2}m_b v_b^2 \\ &\gg 5 \;\; (keV) \end{aligned}$$

The new term modifies the solution obtained in the case without CX by multiplication with a factor

$$\exp\left\{-\tau_s \int_v^{v_b} dv \; \frac{1}{\tau_{cx}(v)} \frac{v^2}{v^3 + v_c^3}\right\}$$

Gaffey mentions orders of magnitude for ORMAK

$$\begin{aligned} \tau_s &\approx 10...80 \;\; (m\,\text{sec}) \\ \tau_{cx} &\approx 10 \;\; (m\,\text{sec}) \\ n_0 &\approx 10^{15} \;\; \left(m^{-3}\right) \text{ density of neutrals} \end{aligned}$$

Density of the beam particles (ions)

$$N_b(t) = \int_v^\infty v^2 dv \int_{-1}^1 d\xi \; f_b(v, \xi, t)$$

For the case where we neglect the charge exchange

$$\begin{aligned} N_b(t) &= S^0 \int_0^{\tau_0(v_b)} \Theta\left[t + \tau_0(v) - \tau_0(v_b)\right] \\ &= \begin{cases} S^0 \tau_0(v_b) & \text{for } \;\; t \geq \tau_0(v_b) \;\; \text{density is saturated} \\ S^0 t & \text{for } \;\; t \leq \tau_0(v_b) \end{cases} \end{aligned}$$

The second line shows the *linear increase* of the density of ions originated in the NBI, up to the limit of time $\tau_0(v_b)$. The increase is obviously given by the source, which is switched on at $t = 0$ and held constant thereafter.

Gaffey explains why there is saturation while the physical process is: slowing down up to $v = 0$. The particles with $v = 0$ are accumulated but the analytic treatment does not include them.

The density of beam ions with *charge exchange* included is

$$N_b(t) = S^0 \tau_s \times \left\{ \begin{array}{ll} 1 - \exp\left(-\frac{\tau_0(v_b)}{\tau_{cx}}\right) & \text{for} \quad t \geq \tau_0(v_b) \\ 1 - \exp\left(-\frac{t}{\tau_{cx}}\right) & \text{for} \quad t \leq \tau_0(v_b) \end{array} \right.$$

We assume that

- the charge exchange time τ_{cx} is taken independent of the velocity v
- the spurce is δ form as before

the solution

$$\begin{aligned} f_b(v, \xi, t) &= S^0 \tau_s \frac{1}{v^3 + v_c^3} \left(\frac{v^3 + v_c^3}{v_b^3 + v_c^3} \right)^{\frac{\tau_s}{3\tau_{cx}}} \\ & \times \Theta \left[t - \frac{\tau_s}{3} \ln \left(\frac{v_b^3 + v_c^3}{v^3 + v_c^3} \right) \right] \\ & \times \sum_{l=0}^{\infty} \frac{2l+1}{2} P_l(\xi) P_l(\xi_b) \left[\frac{v^3}{v_b^3} \left(\frac{v_b^3 + v_c^3}{v^3 + v_c^3} \right) \right]^{\frac{1}{6} l(l+1) Z_2} \Theta(v_b - v) \end{aligned}$$

The charge exchange is important when

$$\tau_{cx} \leq \frac{\tau_s}{3}$$

NOTE see the comments of **Horton Kishimoto Kim Tajima JT60** about the relative magnitudes of the terms in the sum for f_b, distribution of beam ions. **END.**

3.8 Gaffey: electric field

The effect of the electric field on the NBI particles.

To take into account the electric field we have the term

$$\frac{e}{m_b} \mathbf{E} \cdot \frac{\partial}{\partial \mathbf{v}} f_b$$

and we express the gradient in the space of the velocity

$$
\begin{aligned}
& \widehat{\mathbf{e}}_{\|} \cdot \frac{\partial}{\partial \mathbf{v}} f_b \\
= \; & \xi \frac{\partial f_b}{\partial v} + \frac{1}{v}\left(1-\xi^2\right) \frac{\partial f_b}{\partial \xi}
\end{aligned}
$$

$$
\begin{aligned}
& \tau_s \left\{ \frac{\partial f_b}{\partial t} + \frac{e}{m_b} E_{\|}^{eff} \left[\xi \frac{\partial f_b}{\partial v} + \frac{1}{v}\left(1-\xi^2\right) \frac{\partial f_b}{\partial \xi} \right] \right\} \\
= \; & \frac{1}{v^2} \frac{\partial}{\partial v}\left(v^3+v_c^3\right) f_b + \frac{1}{2}\frac{v_c^3}{v^3} Z_2 \frac{\partial}{\partial \xi}\left(1-\xi^2\right) \frac{\partial f_b}{\partial \xi} \\
& -\frac{\tau_s}{\tau_{cx}} f_b \\
& +\tau_s S
\end{aligned}
$$

The effective electric field is

$$
E_{\|}^{eff} = E_{\|}\left(1 - \frac{Z_b}{Z_1 Z_2}\right)
$$

It includes the neoclassical correction due to electron motion opposite to the ion in the electric field.

The acceleration due to the electric field is in the term

$$
\frac{e}{m_b} E_{\|}^{eff} \left[\xi \frac{\partial f_b}{\partial v} + \frac{1}{v}\left(1-\xi^2\right) \frac{\partial f_b}{\partial \xi} \right]
$$

It will be necessary to use properties of the Legendre polynomials

$$
\begin{aligned}
(2l+1)\,\xi P_l(\xi) &= (l+1)\,P_{l+1}(\xi) + l P_{l-1}(\xi) \\
(2l+1)\,P_l(\xi) &= \frac{dP_{l+1}(\xi)}{d\xi} - \frac{dP_{l-1}(\xi)}{d\xi} \\
\frac{d}{d\xi}\left(1-\xi^2\right)\frac{dP_l(\xi)}{d\xi} + l(l+1)\,P_l(\xi) &= 0 \;\; \text{the equation of Legendre}
\end{aligned}
$$

The term with the electric field *couples* the l-th component with the two sides: $l-1$ and $l+1$.

$$
\begin{aligned}
& \frac{e}{m_b} E_{\|}^{eff} \tau_s \frac{1}{v} \left[\left(\frac{l+1}{2l+3} v \frac{df_{l+1}}{dv} + \frac{(l+1)(l+2)}{2l+3} f_{l+1} \right) \right. \\
& \qquad \left. + \left(\frac{l}{2l-1} v \frac{df_{l-1}}{dv} - \frac{l(l-1)}{2l-1} f_{l-1} \right) \right] \\
= \; & \frac{1}{v^2} \frac{d}{dv}\left[\left(v^3+v_c^3\right) f_l\right] - \frac{l(l+1)}{2} \frac{v_c^3}{v^3} f_l \\
& +\tau_s S_l \frac{1}{v^2} \delta\left(v_b - v\right)
\end{aligned}
$$

A definition

$$S_l \equiv S^0 \frac{2l+1}{2} P_l(\xi_b)$$

the effect of the electric field is contained in

$$\begin{aligned} & \frac{e}{m_b} E_{\parallel}^{eff} \tau_s \frac{1}{v_b} \\ = \; & E_{\parallel}^{eff} \frac{m_b v_c^3}{4\pi n_e (eZ_b)^2 (eZ_1)} \frac{1}{v_b} \ln\Lambda \\ = \; & \frac{14.8}{Z_b^2 Z_1^{1/3}} \left(\frac{m_b}{m_H}\right)^{1/3} \frac{E_{\parallel}^{eff}}{E_{Dreicer}} \frac{v_c}{v_b} \end{aligned}$$

where the electric field is much smaller than the Dreicer field

$$E_{Dreicer} = \frac{4\pi e^3}{m_e} \ln\Lambda \times \frac{n_e}{v_{th,e}^2}$$

Then *the effect of the electric field can be considered a perturbation.*

$$f_l(v) = f_l^{(0)}(v) + f_l^{(1)}(v)$$

The zeroth order for a δ source results

$$\begin{aligned} f_l^{(0)}(v) &= \tau_s S_l \frac{1}{v^3+v_c^3} \left[\left(\frac{v^3}{v_b^3}\right) \left(\frac{v_b^3+v_c^3}{v^3+v_c^3}\right) \right]^{\frac{1}{6}l(l+1)Z_2} \\ & \times \Theta(v_b - v) \end{aligned}$$

The first order has the advantage that the l components are no more coupled

$$\begin{aligned} & \frac{1}{v^2}\frac{d}{dv}\left(v^3+v_c^3\right) f_l^{(1)} - \frac{l(l+1)}{2}\frac{v_c^3}{v^3} Z_2 f_l^{(1)} \\ = \; & \frac{e}{m_b} E_{\parallel}^{eff} \tau_s \frac{1}{v} \left[\frac{l+1}{2l+3} \left(v \frac{df_{l+1}^{(0)}}{dv} + (l+2) f_{l+1}^{(0)} \right) \right. \\ & \left. + \frac{l}{2l-1} \left(v \frac{df_{l-1}^{(0)}}{dv} - (l-1) f_{l-1}^{(0)} \right) \right] \end{aligned}$$

the method of solution of this equation, **Gaffey**.

The form of the equation suggests that it is necessary to use an *integrating factor.*

The previous situations that this method has been applied is when we added the *source* term

$$v\frac{df_l}{dv}+\frac{3v^2-\frac{l(l+1)}{2}Z_2\ v_c^3}{v^3+v_c^3}f_l=-\frac{v^3}{v^3+v_c^3}\tau_s S_l(v)$$

with solution

$$f_l(v)=-\tau_s\frac{1}{v^3+v_c^3}\left(\frac{v^3}{v^3+v_c^3}\right)^{\frac{1}{6}l(l+1)Z_2}\int^v v^2dv\ S_l(v)\left(\frac{v^3+v_c^3}{v^3}\right)^{\frac{1}{6}l(l+1)Z_2}$$

Then, the integrating factor is, as above

$$\text{integrating factor }=\left(\frac{v^3+v_c^3}{v^3}\right)^{\frac{1}{6}l(l+1)Z_2}$$

and the term in the RHS is a "source"

$$\begin{aligned}
f_l^{(1)}(v) &= \frac{e}{m_b}E_\parallel^{eff}\tau_s\\
&\times\frac{1}{v^3+v_c^3}\left(\frac{v^3}{v^3+v_c^3}\right)^{\frac{1}{6}l(l+1)Z_2}\\
&\times\int^v vdv\ \left(\frac{v^3+v_c^3}{v^3}\right)^{\frac{1}{6}l(l+1)Z_2}\left[\frac{l+1}{2l+3}\left(v\frac{df_{l+1}^{(0)}}{dv}+(l+2)f_{l+1}^{(0)}\right)\right.\\
&\qquad\left.+\frac{l}{2l-1}\left(v\frac{df_{l-1}^{(0)}}{dv}-(l-1)f_{l-1}^{(0)}\right)\right]
\end{aligned}$$

We have the solution $f_l^{(0)}(v)$ in the zeroth order but we note the presence of the *derivatives* and we want to eliminate them from this expression, using the differential equation, exact, before expansion

$$\begin{aligned}
&f_l^{(1)}(v)\\
=\ &\frac{e}{m_b}E_\parallel^{eff}\tau_s\times\frac{1}{v^3+v_c^3}\left(\frac{v_b^3}{v_b^3+v_c^3}\right)\\
&\times\left[\frac{l+1}{2l+3}\tau_s S_{l+1}+\frac{l}{2l-1}\tau_s S_{l-1}\right]\left[\left(\frac{v^3}{v_b^3}\right)\left(\frac{v_b^3+v_c^3}{v^3+v_c^3}\right)\right]^{\frac{1}{6}l(l+1)Z_2}\\
&\times\Theta(v_b-v)\\
&-\frac{e}{m_b}E_\parallel^{eff}\tau_s\ f_l^{(0)}(v)\int_v^{v_b}vdv\ \frac{1}{v^3+v_c^3}\frac{1}{f_l^{(0)}(v)}\\
&\qquad\times\left[\frac{l+1}{2l+3}\left(\frac{(l+1)(l+2)v_c^3Z_2}{2(v^3+v_c^3)}-\frac{3v^3}{v^3+v_c^3}+l+2\right)f_{l+1}^{(0)}(v)\right.\\
&\qquad\left.+\frac{l}{2l-1}\left(\frac{l(l-1)v_c^3Z_2}{2(v^3+v_c^3)}-\frac{3v^3}{v^3+v_c^3}-(l-1)\right)f_{l-1}^{(0)}(v)\right]
\end{aligned}$$

3.9 Details of calculations Gaffey

the integral

$$F\left(x_{ij}\right)=\frac{1}{n_0}\frac{1}{v_{th,j}}\int d^3v_j\ f\left(v_j\right)g_{ij}$$

where

$$\begin{aligned} g_{ij} &= \left|\mathbf{v}_i-\mathbf{v}_j\right| \\ x_{ij} &= \frac{v_i}{v_{th,j}} \end{aligned}$$

For a Maxwellian plasma

$$f\left(v_j\right)=n_{0j}\frac{1}{\pi^{3/2}v_{th,j}^3}\exp\left(-\frac{v_j^2}{v_{th,j}^2}\right)$$

The velocity space coordinates

$$\begin{aligned} F\left(x_{ij}\right) &= \frac{2}{\sqrt{\pi}}\frac{1}{v_{th,j}^4}\int_0^\infty v_j^2dv_j\ \exp\left(-\frac{v_j^2}{v_{th,j}^2}\right) \\ &\quad\times\int_0^\pi d\theta\ \sin\theta\ \sqrt{v_i^2-2v_iv_j\cos\theta+v_j^2} \end{aligned}$$

The z axis was chose to coincide with the velocity $\mathbf{v}_i$.

$$F\left(x_{ij}\right)=\frac{2}{3\sqrt{\pi}}\frac{1}{v_{th,j}^4}\int_0^\infty v_jdv_j\ \frac{1}{v_i}\exp\left(-\frac{v_j^2}{v_{th,j}^2}\right)\left[\left(v_i+v_j\right)^3-\left|v_i-v_j\right|^3\right]$$

The domain of integration of v_j is divided relative to v_i,

$$\begin{aligned} F\left(x_{ij}\right) &= \frac{2}{3\sqrt{\pi}}\frac{1}{v_{th,j}^4}\left[\frac{1}{v_i}\int_0^{v_i}v_j^2dv_j\ \left(3v_i^2+v_j^2\right)\exp\left(-\frac{v_j^2}{v_{th,j}^2}\right)\right. \\ &\quad\left.+\int_{v_i}^\infty v_jdv_j\ \left(v_i^2+3v_j^2\right)\exp\left(-\frac{v_j^2}{v_{th,j}^2}\right)\right] \end{aligned}$$

Now introduce the new variables

$$\begin{aligned} t &\equiv \frac{v_j}{v_{th,j}} \\ x_{ij} &= \frac{v_i}{v_{th,j}} \end{aligned}$$

then

$$F(x) = \frac{4}{3\sqrt{\pi}} \left(\frac{1}{x} \int_0^x dt\ t^2 \left(t^2 + 3x^2\right) e^{-t^2} + \int_x^\infty dt\ t \left(3t^2 + x^2\right) e^{-t^2} \right)$$

$$F(x) = \left(x + \frac{1}{2x} \right) \Phi(x) + \frac{1}{2} \frac{d\Phi(x)}{dx}$$

where

$$\begin{aligned} \Phi(x) &= \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-t^2\right) dt \\ \frac{d\Phi(x)}{dx} &= \frac{2}{\sqrt{\pi}} \exp\left(-x^2\right) \end{aligned}$$

The derivatives of the relative velocity
This is

$$g = |\mathbf{g}| = |\mathbf{v}_i - \mathbf{v}_j|$$

$$\frac{\partial g}{\partial \mathbf{v}_i} = \frac{\mathbf{g}}{g}$$

$$\begin{aligned} \frac{\partial^2 g}{\partial \mathbf{v}_i\ \partial \mathbf{v}_i} &= \frac{1}{g^3} \left(g^2\ \mathbf{I} - \mathbf{g}\ \mathbf{g}\right) \\ &\equiv \boldsymbol{\omega} \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial \mathbf{v}_i} \cdot \boldsymbol{\omega} &= \frac{\partial}{\partial \mathbf{v}_i} \left[\frac{1}{g^3} \left(g^2\ \mathbf{I} - \mathbf{g}\ \mathbf{g}\right) \right] \\ &= -2 \frac{\mathbf{g}}{g^3} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2}{\partial \mathbf{v}_i \partial \mathbf{v}_i} : \boldsymbol{\omega} &= \frac{\partial}{\partial \mathbf{v}_i} \cdot \left(-2 \frac{\mathbf{g}}{g^3} \right) \\ &= 0 \end{aligned}$$

Define

$$x_{ij} \equiv \frac{v_i}{v_{th,j}}$$

$$\frac{\partial x_{ij}}{\partial \mathbf{v}_i} = \frac{1}{v_{th,j}} \frac{\partial v_i}{\partial \mathbf{v}_i} = \frac{1}{v_{th,j}} \frac{\mathbf{v}_i}{v_i}$$

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{v}_i} \cdot \frac{\partial x_{ij}}{\partial \mathbf{v}_i} &= \frac{1}{v_{th,j}} \frac{\partial}{\partial \mathbf{v}} \cdot \frac{\partial}{\partial \mathbf{v}_i} v_i \\
&= \frac{1}{v_{th,j}} \frac{2}{v_i}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 x_{ij}}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} &= \frac{1}{v_{th,j}} \frac{\partial^2 v_i}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} \\
&= \frac{1}{v_{th,j}} \left(\frac{1}{v_i} \mathbf{I} - \frac{\mathbf{v}_i \, \mathbf{v}_i}{v_i^3} \right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 x_{ij}}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} &= \frac{1}{v_{th,j}} \frac{\partial}{\partial \mathbf{v}_i} \cdot \left(\frac{1}{v_i} \mathbf{I} - \frac{\mathbf{v}_i \, \mathbf{v}_i}{v_i^3} \right) \\
&= -\frac{2}{v_{th,j}} \frac{\mathbf{v}_i}{v_i^3}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial x_{ij}}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 x_{ij}}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} &= \frac{1}{v_{th,j}} \frac{\mathbf{v}_i}{v_i} \cdot \left(\frac{1}{v_i} \mathbf{I} - \frac{\mathbf{v}_i \, \mathbf{v}_i}{v_i^3} \right) \\
&= 0
\end{aligned}$$

An important expression

$$\frac{\partial}{\partial \mathbf{v}_i} \cdot \left[\frac{\partial f_b}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 v_i}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} \right]$$

is calculated introducing the variables in the velocity space

$$(\xi, \theta, \varphi)$$

$$\begin{aligned}
\xi &= \cos\theta \\
&= \frac{\mathbf{v}_i}{v_i} \cdot \widehat{\mathbf{e}}_B \\
&= \frac{v_\parallel}{v} \\
&\quad \text{we do not write "species"} - i
\end{aligned}$$

then

$$\begin{aligned}
\frac{\partial \xi}{\partial \mathbf{v}_i} &= \frac{\partial}{\partial \mathbf{v}_i} \frac{\mathbf{v}_i}{v_i} \cdot \widehat{\mathbf{e}}_B \\
&= \frac{\partial^2 v}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} \cdot \widehat{\mathbf{e}}_B
\end{aligned}$$

It is assumed that the distribution function f_b does not depend on the angle φ.

$$\frac{\partial f_b}{\partial \varphi} = 0$$

$$\frac{\partial f_b}{\partial \mathbf{v}_i} = \frac{\partial f_b}{\partial v}\frac{\partial v}{\partial \mathbf{v}_i} + \frac{\partial f_b}{\partial \xi}\frac{\partial \xi}{\partial \mathbf{v}_i}$$

We multiply this formal equality by

$$\frac{\partial^2 v}{\partial \mathbf{v}_i\, \partial \mathbf{v}_i}$$

It results

$$\begin{aligned}\frac{\partial f_b}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 v}{\partial \mathbf{v}_i\, \partial \mathbf{v}_i} &= \frac{\partial f_b}{\partial v}\frac{\partial v}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 v}{\partial \mathbf{v}_i\, \partial \mathbf{v}_i} \\ &\quad + \frac{\partial f_b}{\partial \xi}\frac{\partial \xi}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 v}{\partial \mathbf{v}_i\, \partial \mathbf{v}_i}\end{aligned}$$

we take into account, for the second term

$$\frac{\partial \xi}{\partial \mathbf{v}_i} = \frac{\partial^2 v}{\partial \mathbf{v}_i\, \partial \mathbf{v}_i} \cdot \widehat{\mathbf{e}}_B$$

and for the first term

$$\begin{aligned}\frac{\partial x_{ij}}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 x_{ij}}{\partial \mathbf{v}_i\, \partial \mathbf{v}_i} &= \frac{1}{v_{th,j}}\frac{\mathbf{v}_i}{v_i} \cdot \left(\frac{1}{v_i}\mathbf{I} - \frac{\mathbf{v}_i\, \mathbf{v}_i}{v_i^3}\right) \\ &= 0\end{aligned}$$

where we can take the particular case, for clarity

$$\begin{aligned}\mathbf{v}_j &= 0 \\ x_{ij} &= |\mathbf{v}_i| = v_i\end{aligned}$$

Then the first term in the RHS of the equation is zero and we only have the second one

$$\begin{aligned}\frac{\partial f_b}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 v_i}{\partial \mathbf{v}_i\, \partial \mathbf{v}_i} &= \frac{\partial f_b}{\partial \xi}\frac{\partial \xi}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 v_i}{\partial \mathbf{v}_i\, \partial \mathbf{v}_i} \\ &= \frac{\partial f_b}{\partial \xi}\left(\frac{\partial^2 v_i}{\partial \mathbf{v}_i\, \partial \mathbf{v}_i} \cdot \widehat{\mathbf{e}}_B\right) \cdot \frac{\partial^2 v_i}{\partial \mathbf{v}_i\, \partial \mathbf{v}_i} \\ &= \frac{\partial f_b}{\partial \xi}\widehat{\mathbf{e}}_B \cdot \frac{1}{v_i}\frac{\partial^2 v_i}{\partial \mathbf{v}_i\, \partial \mathbf{v}_i}\end{aligned}$$

$$
\begin{aligned}
& \frac{\partial}{\partial \mathbf{v}} \cdot \left[\frac{\partial f_b}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 v}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} \right] \\
= & \left[\frac{\partial^2 f_b}{\partial \xi^2} \widehat{\mathbf{e}}_B \cdot \frac{1}{v_i} \frac{\partial^2 v_i}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} + \frac{1}{v} \frac{\partial f_b}{\partial \xi} \frac{\partial}{\partial \mathbf{v}} \right] \cdot \frac{\partial^2 v_i}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} \cdot \widehat{\mathbf{e}}_B \\
= & \left(1 - \xi^2\right) \frac{1}{v^3} \frac{\partial^2 f_b}{\partial \xi^2} - \frac{2\xi}{v^3} \frac{\partial f_b}{\partial \xi} \\
= & \frac{1}{v^3} \frac{\partial}{\partial \xi} \left[\left(1 - \xi^2\right) \frac{\partial f_b}{\partial \xi} \right]
\end{aligned}
$$

Very important formula.

Another one used by Gaffey.

$$
\frac{\partial}{\partial \mathbf{v}} \cdot \left(\frac{\mathbf{v}}{v^3} f_b \right) = \frac{1}{v^2} \frac{\partial f_b}{\partial v}
$$

4 The *multispecies* Fokker Planck equation for NBI Mirin, McCoy

The form of the equation is in the article.

If there is an additional source, like RF, **Mirin McCoy** add the term

$$
\begin{aligned}
R_a = & D_a \left\{ \frac{1}{v^2} \frac{\partial}{\partial v} \left(v^2 \sin\theta \frac{\partial f_a}{\partial v} + v \sin\theta \cos\theta \frac{\partial f_a}{\partial \theta} \right) \right. \\
& \left. + \frac{1}{v^2 \sin\theta} \frac{\partial}{\partial \theta} \left(v \cos\theta \sin^2\theta \frac{\partial f_a}{\partial v} + \sin\theta \cos^2\theta \frac{\partial f_a}{\partial \theta} \right) \right\}
\end{aligned}
$$

with the coefficient

$$
D_a = D_{0a} \exp\left(-\frac{v^2}{c^2} \right)
$$

5 Impurity inward transport Burrell 1980

See also *impurities.tex*.

The paper is **NF20, 1021 (1980) Burrell**.
It is about the action that should be taken to *reverse the transport of impurity from inward to outward.*

Ohkawa proposal: gas puff localized at bottom, to create a poloidally asymmetric particle source.

Explanation by **Burrell**.

The vertical drifts of the ions combines with the radial gradients of density and pressure to create *variation of plasma variables in the magnetic surface.*

This is now well known and described and includes the Pfirsch Schluter current.

An impurity ion is acted upon by two frictional forces:

- a slippage friction force, which results from the different velocities of the backgroud ions and of the impurity ions along the magnetic lines

$$\begin{aligned} & \text{slippage friction} \\ \sim \; & v_{\parallel i} - v_{\parallel Z} \end{aligned}$$

  To fix our image, we consider that the ion neoclassical drift $\mathbf{v}_D$ is vertical. This means that the main magnetic field points into the page and the current also points in the same direction. This will create The helicity of the magnetic field toward right-down. The flow of the impurities results from the motion of the ions along this direction (as the current) and from the fact that this collisional pressure dominates a collisional resistance from Z ions. Then the impurity ions will have an induced flow that in projection is seen as downward.

- a friction force resulting from the dependence of the collisional force on the temperature along the magnetic field line. This second frictional force involves, therefore, the variation of plasma variables in the magnetic surface. The variables of plasma that have significant neoclassical (intrinsic) variation in the magnetic surface are density $n_1(r,\theta)$, and the electrostatic potential $\phi_1(r,\theta)$. But the temperature is higher at the low-field side and this means that the collisional force exerted from the low field side toward regions at smaller $R = R_0 + r\cos\theta$, is smaller than the collisional force exerted from smaller R toward low field side $R = R_0 + r$. This is therefore a force towards the low field side, opposed to the slippage force.

The second is explained as follows:
the collisional cross section depends on the velocity as

$$\frac{n_Z}{v_{th,Z}^3}$$

and this makes a difference between regions where the temperature (*i.e.* the thermal velocity $v_{th,Z}$) is different. In the expression of Burrell, the hot regions *hit less* than the cold regions. This creates a directed frictional force, from the cold toward the hot regions, along the magnetic line.

This *thermal frictional force* opposes the *slippage frictional force.*

Since usually the slippage force is larger than the thermal frictional force, there is *dominant flow of impurities downward.*

There is a slight accumulation of impurities at the bottom of the tokamak.

This is a poloidal nonuniformity of the density of impurities.

Combined with the equilibrium gradients leads to *inward transport of Z ions.*

See **Connor 1973** for an explanation based on ambipolarity.

Ohkawa proposal:

inject hydrogen ions at the bottom, since in their motion they will reverse the sense of the slippage force.

In addition, since there is local cooling, the thermal force will be enhanced, since it points from cold to hot along the line. This time we enhance the collisional friction by raising the force due to decrease of the temperature at the bottom.

As result, there will be a *flow of impurities upward.* (reversed)

6 Notes on collisions

6.1 Collision frequency

6.1.1 Neoclassical collision frequency

The (neoclassical) critical collision frequency. The factor ν_*/ν is essentially the **number of banana orbits a trapped particle can make before being scattered out the region of trapped particles and becoming passing.**

At the limit of the *plateau* to *banana* regime

$$\nu_* = \varepsilon^{3/2} \frac{v}{qR}$$

In **Rosenbluth Hazeltine Hinton** the collision frequency *needed* for scattering a trapped particle into a circulating orbit is estimated as

$$\begin{aligned}\nu_{eff} &\approx \frac{1}{\varepsilon}\nu_c \\ &(\nu_c \equiv \text{classical collision frequency})\end{aligned}$$

It results that it is necessary to have more frequent collision (compared with the classical case) if we want to convert a trapped particle into a circulating one.

In **Novakovskii** it is reminded that the trapped particles are less collisional than the circulating ones. This is the cause for which at a change of the gradient of temperature the trapped particles react easily and rapidly while the circulating particles are slower in reacting. This difference at the boundary in the velocity space between trapped and circulating induces a discontinuity and *DAMPING* of poloidal rotation. The cause is therefore the *friction* between circulating and trapped.

6.1.2 Landau collision frequency for the electron-ions collisions

$$St_{ei}\{f_e\} = \frac{\partial}{\partial v_\alpha}\frac{2\pi e^4 n Z^2 \ln\Lambda}{m_e^2}\left(\frac{1}{v}\delta_{\alpha\beta} - \frac{v_\alpha v_\beta}{v^3}\right)\left(\frac{\partial f_e}{\partial v_\beta} - \frac{v_\beta}{T}f_e\right)$$

$$\nu_{ei} = \frac{16\sqrt{\pi}e^4 Z^2 \ln\Lambda}{3m_e^2}\frac{n}{v_{th}^3}$$

The expression for Landau collision integral in **Hinton Rosenbluth 1973**.

$$C_{ab}(f_a\ ,\ f_b) = -c_{ab}\frac{1}{m_a}\frac{\partial}{\partial \mathbf{v}}\cdot\int d^3v'\ \mathfrak{U}\cdot\left[\frac{1}{m_b}\frac{\partial f_b}{\partial \mathbf{v}'}f_a(\mathbf{v}) - \frac{1}{m_a}\frac{\partial f_a}{\partial \mathbf{v}}f_b(\mathbf{v}')\right]$$

where the tensor is

$$\begin{aligned}\mathfrak{U} &\equiv \frac{1}{u}\mathbf{I} - \frac{\mathbf{u}\,\mathbf{u}}{u^3} \\ \text{for }\mathbf{u} &= \mathbf{v} - \mathbf{v}'\end{aligned}$$

and

$$c_{ab} = 2\pi e_a^2 e_b^2 \ln\Lambda$$

6.1.3 Coulomb collisions

For Coulomb collisions, scattering through the angle $\Delta v/v$ occurs in the time

$$\tau \sim \frac{1}{\nu}\left(\frac{\Delta v}{v}\right)^2$$

where ν is the collision frequency for the scattering through $\pi/2$ angle.

6.1.4 Collision operator for trapped - circulating particles

See below.

Consider only the **pitch-angle scattering** operator

$$C(f) = \frac{\nu_D}{2}\frac{\partial}{\partial \xi}\left(1-\xi^2\right)\frac{\partial f}{\partial \xi}$$

where ν_D is the deflection collision frequency and

$$\xi = \frac{v_{\parallel}}{v}$$

is the **pitch angle**.

Change of variables

$$(\psi, \xi) \rightarrow (\psi_*, \omega)$$

Keeping the highest order derivatives, one obtains

$$C(f) \simeq \frac{\nu_D}{2}\left[(v\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta)^2\frac{\partial^2 f}{\partial \omega^2} - 2\frac{Iv^2\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta}{\Omega S}\frac{\partial^2 f}{\partial\omega\partial\psi_*} + \left(\frac{Iv}{\Omega S}\right)^2\frac{\partial^2 f}{\partial \psi_*^2}\right]$$

Here it has been approximated $1-\xi^2 \approx 1$. This is valid since **it is the barely circulating $\xi \ll 1$ and the barely trapped particles which contribute to the ion orbit loss.**

A new variable is introduced, $\widehat{\omega}$ which makes possible to connect these expressions which will finally give f with the *neoclassical distribution functionfor ions*:

$$\omega = \sigma\widehat{\omega}\;\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta\;\left(1-k\sin^2\frac{\theta}{2}\right)^{1/2}$$

Then

- $k < 1$ corresponds to *poloidally circulating particles,* and
- $1 < k < \infty$ corresponds to the *poloidally trapped particles.*

Here the **direction** of the variable ω is $\sigma = \pm 1$.

Another change of variables is suggested by the form of the **drift flux function**. We take

$$\psi_* = \psi_0 - \frac{I}{\Omega S}\widetilde{\omega}_0$$

assuming

$$\varepsilon \ll 1 \text{ but } |S|\,\varepsilon < 1$$

then

$$\begin{aligned}\widehat{\omega}^2 \;=\;& \overline{\omega}_0^2 \frac{B_0^2}{B_x^2} + \\ & 2SE + \\ & 2\,|S|\,\varepsilon\left[\mu B_0 + \left(\overline{\omega}_0 \frac{B_0}{B_x} - I\frac{1}{B_0}\frac{\partial\Phi}{\partial\psi}\right)^2\right]\end{aligned}$$

where

$$E = E - \mu B_0 - \frac{e\Phi}{m_i} - \frac{1}{2}\left(\overline{\omega}_0 \frac{B_0}{B_x} - I\frac{1}{B_0}\frac{\partial\Phi}{\partial\psi}\right)^2$$

The notation $\widetilde{\omega}_0$ represents $\overline{\omega}_0 B/B_x$ where

$$\overline{\omega}_0 = v_{\parallel} + I\frac{1}{B_0}\frac{\partial\Phi}{\partial\psi}$$

evaluated at $\psi = \psi_0$ and $\theta = 0$ or π (depending on where the particles are trapped: $\theta = 0$ for inside of a tokamak, $\theta = \pi$ if the particle is trapped outside of the tokamak). B_x is the value of B evaluated at $\theta = 0$ or $\theta = \pi$.

6.2 Collisional relaxation (Fundamenski Garcia)

6.2.1 Test particle s colliding with field particles s'

The expressions of the rates in **Fundamenski Garcia.**

Definition: collision rate between a test particle and the field particles is the time needed to change the direction of the test particles by a right angle from their initial flow direction.

This is 90 degrees - deflection.

This is therefore *momentum* exchange. I.E. deflection NOT slowing down.

The slowing down rate for a *test particle*

$$\frac{\partial v_{\parallel s}}{\partial t} = -\nu_{ss'}^{t}\, v_{\parallel s}$$

$$\begin{aligned}\nu_{ss'}^{t} &= -\frac{\partial}{\partial t}\ln v_{\| s} \\ &= 4\gamma_{ss'}\frac{1}{m_s^2}\left(1+\frac{m_s}{m_{s'}}\right)\frac{\Psi\left(v\right)}{v}\times\frac{n_{s'}}{v_{th,s'}^3}\end{aligned}$$

The rate of transversal deflection of a test particle

$$\begin{aligned}\nu_{ss'}^{t,\perp} &= \frac{1}{v_s^2}\frac{\partial}{\partial t}v_{s\perp}^2 \\ &= \frac{2D_{ss'}^{\perp}}{v_s^2} \\ &= 4\gamma_{ss'}\frac{1}{m_s^2}\frac{\Phi\left(v\right)-\Psi\left(v\right)}{v^3}\times\frac{n_{s'}}{v_{th,s'}^3}\end{aligned}$$

The rate of dispersion or parallel diffusion of a test particle

$$\begin{aligned}\nu_{ss'}^{t,\|} &= \frac{1}{v_s^2}\frac{\partial}{\partial t}v_{s\|}^2 \\ &= \frac{D_{ss'}^{\|}}{v_s^2} \\ &= 4\gamma_{ss'}\frac{1}{m_s^2}\frac{\Psi\left(v\right)}{v^3}\times\frac{n_s}{v_{th,s'}^3}\end{aligned}$$

The dispersion is caused by collisions that change randomly the velocity $v_{\| s}^2$, around an average value.

The rate of energy loss of a test particle

$$\begin{aligned}\nu_{ss'}^{t,\epsilon} &= \frac{1}{K_s^2}\frac{\partial}{\partial t}K_s^2 \\ &= 16\gamma_{ss'}\frac{1}{m_s^2}\frac{\Psi\left(v\right)}{v^3}\times\frac{n_{s'}}{v_{th,s'}^3}\end{aligned}$$

We have separated systematically the physical factor

$$\frac{n_{s'}}{v_{th,s'}^3}\sim\frac{n_{s'}}{T_{s'}^{3/2}}$$

as it occurs in numerical calculations.

The condition of conservation of energy imposes

$$\nu_{ss'}^{t,\epsilon}\left(v\right)+\nu_{ss'}^{t,\perp}\left(v\right)+\nu_{ss'}^{t,\|}\left(v\right)=2\nu_{ss'}^{t}\left(v\right)$$

The kinetic energy of the test particle (s)

$$K_s = \frac{m_s v_s^2}{2}$$

The quantities that specifies the *field particles*

$$n_{s'} \ , \ \mathbf{V}_{s'} \ , \ T_{s'}$$

$$\begin{aligned}
n_{s'} &= \int d^3v \ f_{s'} \\
n_{s'}\mathbf{V}_{s'} &= \int d^3v \ \mathbf{v} f_{s'} \\
\frac{3}{2} n_{s'} T_{s'} &= \int d^3v \ \frac{1}{2} m_{s'} \left|\mathbf{v} - \mathbf{V}_s\right|^2 \ f_{s'}
\end{aligned}$$

The velocity v_s of the test particle is normalized to the thermal velocity of the field particles

$$\upsilon = \frac{v_s}{v_{th,s'}}$$

$$v_{th,s'} = \sqrt{\frac{2T_{s'}}{m_{s'}}}$$

The error function

$$\Phi(\upsilon) = \frac{2}{\sqrt{\pi}} \int_0^{\upsilon} d\xi \ \exp\left(-\xi^2\right)$$

The Chandrasekhar function

$$\Psi(\upsilon) = \frac{1}{2\upsilon^2} \left[\Phi - \upsilon \frac{d\Phi}{d\upsilon}\right]$$

Fundameski Garcia explain the limiting values taken by these functions.

For a velocity of the *test particle* much less than the thermal velocity of the field particles

$$\begin{aligned}
v_s &\ll v_{th,s'} \\
\upsilon &= \frac{v_s}{v_{th,s'}} \ll 1
\end{aligned}$$

the error function and the Chandraskhar function are close to zero

$$\lim_{\upsilon \to 0} \Phi(\upsilon) \to \frac{2}{\sqrt{\pi}} \upsilon \to 0$$

$$\lim_{\upsilon\to 0}\Psi(\upsilon) \to \frac{2}{3\sqrt{\pi}}\upsilon \to 0$$

Then, the rate of *slowing down* tends to a constant

$$\begin{aligned}
\nu_{ss'}^{t} &\to 4\gamma_{ss'}\frac{1}{m_s^2}\left(1+\frac{m_s}{m_{s'}}\right)\frac{\frac{2}{3\sqrt{\pi}}\upsilon}{\upsilon}\times\frac{n_{s'}}{v_{th,s'}^3} \\
&\to 4\gamma_{ss'}\frac{1}{m_s^2}\left(1+\frac{m_s}{m_{s'}}\right)\frac{2}{3\sqrt{\pi}}\times\frac{n_{s'}}{v_{th,s'}^3} \\
&= \text{const}
\end{aligned}$$

We have the series expansion

$$\begin{aligned}
\text{erf}(z) &= \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,(2n+1)}z^{2n+1} \\
&= \frac{2}{\sqrt{\pi}}\left[z-\frac{1}{3}z^3+\frac{1}{10}z^5-...\right]
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dz}\text{erf}(z) &\approx \frac{2}{\sqrt{\pi}}\left[1-z^2+\frac{1}{2}z^4-...\right] \\
&= \frac{2}{\sqrt{\pi}}\exp\left(-z^2\right)
\end{aligned}$$

Let is try

$$\begin{aligned}
&\text{erf}(z) - z\frac{d}{dz}\text{erf}(z) \\
=& \frac{2}{\sqrt{\pi}}\left[\left(z-\frac{1}{3}z^3+\frac{1}{10}z^5-...\right)-z\left(1-z^2+\frac{1}{2}z^4-...\right)\right] \\
=& \frac{2}{\sqrt{\pi}}\left[z-\frac{1}{3}z^3+\frac{1}{10}z^5-...-z+z^3-\frac{1}{2}z^5\right] \\
=& \frac{2}{\sqrt{\pi}}\left[\frac{2}{3}z^3-\frac{2}{5}z^5+...\right]
\end{aligned}$$

and

$$\begin{aligned}
\Psi(z) &= \frac{1}{2z^2}\left[\text{erf}(z)-z\frac{d}{dz}\text{erf}(z)\right] \\
&= \frac{1}{2z^2}\frac{2}{\sqrt{\pi}}\left[\frac{2}{3}z^3-\frac{2}{5}z^5+...\right] \\
&= \frac{2}{\sqrt{\pi}}\left[\frac{1}{3}z-\frac{z^3}{5}+...\right]
\end{aligned}$$

And the limit close to zero is

$$\lim_{\upsilon\to 0}\Psi(\upsilon)=\frac{2}{3\sqrt{\pi}}\upsilon\to 0$$

Further

$$\begin{aligned}\frac{\Phi(z)-\Psi(z)}{z^3} &= \frac{\frac{2}{\sqrt{\pi}}\left[z-\frac{1}{3}z^3+\frac{1}{10}z^5-...\right]-\frac{2}{\sqrt{\pi}}\left[\frac{1}{3}z-\frac{z^3}{5}+...\right]}{z^3}\\ &= \frac{2}{\sqrt{\pi}}\frac{1}{z^3}\left[\frac{2}{3}z-\frac{2}{15}z^3-...\right]\end{aligned}$$

We obtain the possibility to express the limits of the formulas for the rates

$$\lim_{\upsilon\to 0}\frac{\Psi(\upsilon)}{\upsilon}\to\frac{2}{3\sqrt{\pi}}\frac{2}{3}\frac{1}{\upsilon^2}\to\infty$$

$$\begin{aligned}\frac{\Phi(\upsilon)-\Psi(\upsilon)}{\upsilon^3} &\to \infty\text{ for }\ \upsilon\to 0\\ & \text{divergent}\end{aligned}$$

Note this should be connected with the statement that the magnetic helicity $\mathbf{A}\cdot\mathbf{B}$ is conserved when the resistivity goes to zero, but the mechanical helicity $\mathbf{v}\cdot\boldsymbol{\omega}$ is not conserved since the zero of collisionality means divergent rate of *transversal dispersion*. **END.**

NOTE
Another calculation

$$\begin{aligned}\Psi(z) &= \frac{1}{2z^2}\left[\text{erf}(z)-z\frac{d}{dz}\text{erf}(z)\right]\\ &= \frac{1}{2z^2}\frac{2}{\sqrt{\pi}}\left[z-\frac{1}{3}z^3+\frac{1}{10}z^5-...\right]\\ &\quad -\frac{1}{2z}\frac{2}{\sqrt{\pi}}\left[1-z^2+\frac{1}{2}z^4-...\right]\\ &= \frac{2}{\sqrt{\pi}}\left[\frac{1}{2z}-\frac{1}{6}z+\frac{1}{20}z^3\right.\\ &\quad\left.-\frac{1}{2z}+\frac{1}{2}z-\frac{1}{4}z^3+...\right]\end{aligned}$$

$$\Psi(z)=\frac{2}{\sqrt{\pi}}\left[\frac{1}{3}z-\frac{z^3}{5}+...\right]$$

END

Other limiting values (remember the velocity is normalized to the thermal velocity, $v \sim \frac{v}{v_{th,a}}$)

$$\lim_{v \to \infty} \Phi(v) = \text{erf}(\infty) = 1$$

$$\lim_{v \to \infty} \Psi(v) = 0$$

and two particular valus

$$\begin{aligned} \Phi(1) &\approx 0.8 \\ \Psi(1) &\approx 0.2 \end{aligned}$$

The constant in SI

$$\gamma_{ss'} = \frac{1}{8\pi\varepsilon_0^2} e_s^2 e_{s'}^2 \ln \Lambda_{ss'}$$

The logarithm

$$\ln \Lambda_{ss'} = \ln \left(\frac{r_{\max}}{r_{\min}} \right)$$

The *minimum* distance in the collisions is the *larger* between

- the de Broglie length

$$\frac{\hbar}{2\mu_{ss'}} \frac{1}{\langle u \rangle}$$

- the classical distance of closest approach

$$\frac{1}{4\pi\varepsilon_0^2} \frac{e_s e_{s'}}{\mu_{ss'}} \frac{1}{\langle u \rangle^2}$$

where

$$\mathbf{u} = \mathbf{v}_s - \mathbf{v}_{s'}$$

is the relative velocity

$$\langle \rangle \equiv \text{average over } f_s \text{ and } f_{s'}$$

$$\begin{aligned} \mu_{ss'} &= \frac{1}{\frac{1}{m_s} + \frac{1}{m_{s'}}} \\ &= \text{reduced mass} \end{aligned}$$

The maximum distance is the Debye length

$$\begin{aligned} r_{\max} &= \lambda_D^{eff} \\ &= \left(\frac{\varepsilon_0 T_s}{\sum_a n_a e_a^2} \right)^{1/2} \end{aligned}$$

6.2.2 Fluid exchange of momentum: a test-particle species s colliding with a field particle species s'

This time the test particle s is not an isolated (*i.e.* individualized) particle, - it is an ensemble of species s particles with the distribution function

$$f_s$$

and the field particles have the distribution function

$$f_{s'}$$

The force of friction on the species s by collisions with the species s' is

$$\mathbf{F}_{ss'} = \int d^3v \; m_s \mathbf{v} \; C_{ss'} \left(f_s, f_{s'} \right)$$

which is expressed as a linear dependence on the relative *average* flow velocities

$$\mathbf{F}_{ss'} = -m_s n_s \nu_{ss'} \left(\mathbf{V}_s - \mathbf{V}_{s'} \right)$$

This is the definition of $\nu_{ss'}$.

There is a flow velocity $\mathbf{V}_s$ for the fluid of species s.

The momentum balance for the species s will include this friction with s' ,

$$\begin{aligned} m_s n_s \left(\frac{\partial}{\partial t} + \mathbf{V}_s \cdot \boldsymbol{\nabla} \right) \mathbf{V}_s &= -\boldsymbol{\nabla} p_s - \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_s \\ &\quad + e_s n_s \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right) \\ &\quad + \sum_{s'} \mathbf{F}_{ss'} \end{aligned}$$

The fluid quantities are defined as

$$n_s = \int d^3v \; f_s$$

$$n_s \mathbf{V}_s = \int d^3v \; \mathbf{v} f_s$$

$$\frac{3}{2} n_s T_s = \int d^3v \; \frac{1}{2} m_s \left| \mathbf{v} - \mathbf{V}_s \right|^2 \; f_s$$

Taking separately only the friction force effect, the change of the momentum is

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_s \cdot \boldsymbol{\nabla} \right) \mathbf{V}_s = -\sum \nu_{ss'} \left(\mathbf{V}_s - \mathbf{V}_{s'} \right)$$

the force is proportional with the difference in the flow velocities of the two fluids.

Explicit form of the collision operator. The form of the collision operator

$$C_{ss} \left(f_s, f_{s'} \right)$$

will permit to calculate the coefficient $\nu_{ss'}$.

For large $\Lambda_{ss'}$, the Landau-Boltzmann operator

$$C_{ss'} = \gamma_{ss'} \frac{1}{m_s} \frac{\partial}{\partial \mathbf{v}} \cdot \int d^3v' \; f_s \left(\mathbf{v} \right) f_{s'} \left(\mathbf{v}' \right) \; \left(\frac{1}{u} \mathbf{I} - \frac{\mathbf{u} \, \mathbf{u}}{u^3} \right) \; \chi_{ss'} \left(\mathbf{v}, \mathbf{v}' \right)$$

$$\chi_{ss'} \left(\mathbf{v}, \mathbf{v}' \right) = \frac{1}{m_s} \frac{\partial}{\partial \mathbf{v}} \ln f_s - \frac{1}{m_{s'}} \frac{\partial}{\partial \mathbf{v}'} \ln f_{s'}$$

This form for Coulombian collisions can be written in the form Fokker-Planck

$$\begin{aligned} C_{ss'} &= -\frac{\partial}{\partial \mathbf{v}} \cdot \left[\mathbf{A}_{ss'} f_s - \frac{1}{2} \frac{\partial}{\partial \mathbf{v}} \cdot \left(\mathbf{D}_{ss'} f_s \right) \right] \\ &= -\frac{\partial}{\partial \mathbf{v}} \cdot \left(\mathbf{A}_{ss'} f_s \right) + \frac{1}{2} \frac{\partial^2}{\partial \mathbf{v} \partial \mathbf{v}} : \mathbf{D}_{ss'} \; f_s \end{aligned}$$

where

$$\begin{aligned} \mathbf{A}_{ss'} &\sim \frac{\left\langle \Delta \mathbf{v} \right\rangle_{ss'}}{\Delta t} \quad \text{dynamical friction} \\ \mathbf{D}_{ss'} &\sim \frac{\left\langle \Delta \mathbf{v} \Delta \mathbf{v} \right\rangle_{ss'}}{\Delta t} \quad \text{velocity space diffusion} \end{aligned}$$

6.2.3 The Rosenbluth potentials

The expression for the collision operator of a population s of *test* particles with a population s' of field particles

$$\begin{aligned} C_{ss'} &= -\gamma_{ss'}\frac{1}{m_s}\frac{\partial}{\partial \mathbf{v}}\cdot\int d^3v'\ f_s(\mathbf{v})\, f_{s'}(\mathbf{v}')\left(\frac{1}{u}\mathbf{I}-\frac{\mathbf{u}\,\mathbf{u}}{u^3}\right)\times \\ & \quad \left[\frac{1}{m_s}\frac{\partial}{\partial \mathbf{v}}\ln f_s(\mathbf{v})-\frac{1}{m_{s'}}\frac{\partial}{\partial \mathbf{v}'}\ln f_{s'}\right] \\ &= -\frac{\partial}{\partial \mathbf{v}}\cdot\left(\mathbf{A}_{ss'}f_s\right)+\frac{1}{2}\frac{\partial^2}{\partial \mathbf{v}\partial \mathbf{v}}:\mathbf{D}_{ss'}\ f_s \end{aligned}$$

can be rexpressed using the Rosenbluth potentials.

$$g_{s'}(\mathbf{v}) = \int d^3v'\ u\ f_{s'}(\mathbf{v}')$$

$$h_{s'}(\mathbf{v}) = \int d^3v'\ \frac{1}{u}\ f_{s'}(\mathbf{v}')$$

where

$$u = |\mathbf{u}| = |\mathbf{v}-\mathbf{v}'|$$

The *dynamical friction* vector

$$\mathbf{A}_{ss'} = 2\gamma_{ss'}\frac{1}{m_s^2}\left(1+\frac{m_s}{m_{s'}}\right)\frac{\partial}{\partial \mathbf{v}}h_{s'}$$

The *velocity-space diffusion* tensor

$$\mathbf{D}_{ss'} = 2\gamma_{ss'}\frac{1}{m_s^2}\frac{\partial^2}{\partial \mathbf{v}\ \partial \mathbf{v}}g_{s'}$$

Now one takes the simplest case where the *field particles* have

- Maxwellian distribution,
- no flow $V_{s'} = 0$,
- temperature $T_{s'}$

then

$$g_{s'M}(\mathbf{v}) = n_{s'}\frac{v_{th,s'}}{2v}\left[v\frac{d\Phi}{dv}+\left(1+2v^2\right)\Phi(v)\right]$$

and

$$h_{s'M}(\mathbf{v}) = n_{s'} \frac{1}{v_{th,s'}} \frac{\Phi(v)}{v}$$

and with these formukas the *dynamical friction* vector and the *velocity-space diffusion* tensor can be calculated

$$\mathbf{A}_{ss'M} = n_{s'} \frac{2}{m_{s'}} \gamma_{ss'} \frac{\mathbf{v}}{v^3} \left[\Phi(v) - v \frac{d\Phi}{dv} \right]$$

$$\mathbf{D}_{ss'M} = n_{s'} \frac{1}{m_{s'}} \frac{1}{m_s^2} \gamma_{ss'} \frac{T_{s'}}{v^3} \left[\mathbf{I}\ F_1(v) + 3 \frac{\mathbf{v}\ \mathbf{v}}{v^2} F_2(v) \right]$$

where

$$F_1(v) = v \frac{d\Phi}{dv} + \left(2v^2 - 1\right) \Phi(v)$$

$$F_2(v) = \left(1 - \frac{2}{3} v^2\right) \Phi(v) - v \frac{d\Phi}{dv}$$

The collision operator with field species Maxwellian The replacement of the two expressions (vector and tensor) in the Collision Operator

$$C_{ss'M} = -n_{s'} \gamma_{ss'} \frac{2}{m_s m_{s'}} \frac{1}{v_{th,s'}} \frac{1}{v_{th,s}^2} \left(1 - \frac{T_{s'}}{T_s}\right) \left[\frac{\Phi}{v} - \left(\frac{T_s/m_s}{T_{s'}/m_{s'}} \right) \frac{d\Phi}{dv} \right] f_{sM}$$

6.3 Estimate of the ion orbit loss rate in tokamak (Shaing [?])

See also **Yushmanov Horton**.

To explain the H-mode in tokamaks. The loss of ions is localized in the high energy part of the distribution function, since here the ions are less collisional. Being less collisional, they have rather clear banana trajetories and in their motion they can hit the limiter.

But the expulsion of a "hot" ion from the plasma is simultaneously compensated (electrically) by the entrance of an ion from the exterior of the plasma toward the interior. This influx is driven by the ion viscosity which is essentially determined by a non-zero rotation of the plasma. It concerns the lower temperature ions, *i.e.* the ions of the Pfirsch-Schluter-plateau, more collisional. This again rises the problem of Stix : it is necessary to have the poloidal rotation + collisions in order to get a radial electric current. This current will participate in the radial component of the $\nabla \times \mathbf{B}$ equation, to compensate (together with the polarization electric current) the outflux of directly lost hot ions. Also in Rosenbluth it is the current of compensation of the lost very-hot-ions at NBI.

This problem of balance of ion radial currents can be re-stated: the loss of ions is the primary process; but the plasma establishes an electric radial field andthe corresponding rotation in order to obtain, through the ion-viscosity, a radial electric current of ions which balances the outgoing ion flux.

- the *outgoing* ion orbit loss flux (in **banana regime**) , is balanced by
- incoming *viscosity driven* ion flux (plateau-Pfirsch-Schluter)

to mentain ambipolarity at the steady state.

Nota. It must be understood that the *disappearence of an ion (* whose banana orbit hits the wall or the limitator) from the plasma **should not be seen as a current directed to the wall**. One expects that the ion which will replace the lost ion, will come from the plasma border toward the centre. So, there is a current of response, the so-called *incoming current.*

The effect of viscosity is separated in two in order to emphasize the two different effects:

1. "viscosity-driven flux" is the flux driven by the viscosity contributed by the particles in the regime **plateau-Pfirsch-Schluter**. Actually is the flux of ions of replacement.

2. "ion orbit loss flux" is the flux driven by the viscosity of the particles in the banana regime. (This is because the loss of ions is due to the difference in the radial drifts of the electrons and the ions, when the drifts are not too much perturbed by the collisions , *i.e.* in the banana regime). But it is not clear how the viscosity is involved in the **direct loss of banana ions to the limiter**.

When the two fluxes (outcoming and incoming) are integrated over the velocity space, they *approximately cancel* each other. The **net** ion flux cancels to order $\sqrt{m_e/m_i}$ *when E_r is determined proprely from the momentum balance equation.*

NOTE. In the paper Shaing insists on the difference between the *ion loss current* and *the plasma current density.* The later is the current formed by ions which replaces the ions lost by the intersection of their banana with the wall.

The torque associated to the ion orbit loss flux is counterbalanced by the torque associated with the viscosity. At steady state there is no net radial current across flux surfaces and there is no net torque applied on the plasma.

Various currents which constitutes the radial plasma current:

- the **ion orbit loss current** $e\Gamma_{orbit}$
- the **viscosity-driven** current;
- the **polarization** current;

At *steady state*:

- the radial current density j_r which is proportional with $\partial E_r/\partial t$ **and** the polarization current vanish
- the ion orbit loss current $e\Gamma_{orbit}$ is balanced by the viscosity-driven flux

The equation used by Shaing

$$\left(v_{\|}\widehat{\mathbf{n}}+\mathbf{V}_E\right)\cdot\boldsymbol{\nabla}\theta\frac{\partial f}{\partial\theta}+\mathbf{v}_d\cdot\boldsymbol{\nabla}\psi\frac{\partial f}{\partial\psi}=C\left(f\right)$$

where $\mathbf{V}_E$ is the electric velocity, which means a *rotation*. The equation is simply $\left(v_{\|}\widehat{\mathbf{n}}+\mathbf{v}_d+\mathbf{V}_E\right)\cdot\boldsymbol{\nabla}f=C\left(f\right)$: *without explicit time derivative* (since the process here is not dynamic, as for example when we study the decay of plasma rotation by the torque generated from the radial electric current of ions (compared to electrons) in the presence of **rotation** and **collisions**, see Stix; also, without *energetic term* in the drift kinetic equation; this should be accounted for in other cases, as for example the **transit time magnetic pumping**, where the viscosity is noncollisional. This appears in [**?**] where the equation is $(u\widehat{\mathbf{n}}+\mathbf{v}_d+\mathbf{V})\cdot\boldsymbol{\nabla}f+\dot{w}\partial f/\partial w=0$; and the equation is so because there is a strong plasma rotation composed of : *diamagnetic*, *electric* and *parallel*.

The $\mathbf{v}\cdot\boldsymbol{\nabla}f$ part of the equation (left hand side) The first part in the equation is of this form because the drift kinetic distribution function f is function of only the **poloidal** θ coordinate and **radial** r coordinate. The variables are (the ion charge is e, *i.e.* $e=|e|$):

$$\begin{aligned}
& \psi,\theta \\
E \;=\; & \frac{v^2}{2}-\frac{|e|}{m_i}\Phi \\
& \mu
\end{aligned}$$

The effects of orbit squeezing can be taken into account by employing a new coordinate ψ_* istead of ψ.

$$\psi_* = \psi - \frac{I}{S\Omega}\left(v_\parallel + \frac{I}{\Omega}\frac{B^2}{B_0^2}\frac{e}{m_i}\frac{\partial \Phi}{\partial \psi}\right)$$

This defines the *drift surface.*

The **squeezing factor** is

$$S = 1 + \left(\frac{I}{\Omega_0}\right)^2 \frac{e}{m_i}\frac{\partial^2 \Phi}{\partial \psi^2}$$

The quantity Ω_0 is Ω calculated at the magnetic axis. The shear of the electric field Φ'' (or $\frac{\partial^2 \Phi}{\partial \psi^2}$) is considered constant over the width of the banana orbit.

It can be shown that the projection of the parallel + electric velocities on the poloidal direction

$$\omega = (v_\parallel \widehat{\mathbf{n}} + \mathbf{V}_E) \cdot \boldsymbol{\nabla}\theta \simeq -\frac{I}{\Omega}\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta \frac{\partial \psi_*/\partial \psi}{\partial \psi_*/\partial E}$$

and the radial projection of the drift velocity (using ψ as surface label) is expressible from derivatives of the *drift surface variable* ψ^*.

$$\mathbf{v}_d \cdot \boldsymbol{\nabla}\psi = \frac{I}{\Omega}\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta \frac{\partial \psi_*/\partial \theta}{\partial \psi_*/\partial E}$$

The meaning of the notation is

$$I = R^2 \mathbf{B} \cdot \boldsymbol{\nabla}\varphi = R B_\varphi$$

The drift associated with the poloidal field variation $\partial B/\partial \theta$ have been neglected. Taking this relation into the drift-kinetic equation we get

$$(v_\parallel \widehat{\mathbf{n}} + \mathbf{V}_E) \cdot \boldsymbol{\nabla}\theta \frac{\partial f}{\partial \theta} + \mathbf{v}_d \cdot \boldsymbol{\nabla}\psi \frac{\partial f}{\partial \psi} \simeq \omega \left. \frac{\partial f}{\partial \theta}\right|_{\psi_*, E, \mu}$$

We note that ω is a frequency which has the role of a coordinate which combines the parallel velocity and electric $\mathbf{E} \times \mathbf{B}$ particle velocities **projected on the poloidal direction**, divided at the local small radius r. **Note** We must check the flux variable ψ_* with the "drift surface" variable, as introduced in the review of Hazeltine and Hinton.

Since this is defined in terms of the particle motion, it is difficult to see how is connected with the Kulikovskii surfaces. The latter are related with surfaces where $dl_\parallel/B$ =const.

The collision operator Consider only the **pitch-angle scattering** operator

$$C(f) = \frac{\nu_D}{2}\frac{\partial}{\partial \xi}\left(1-\xi^2\right)\frac{\partial f}{\partial \xi}$$

where ν_D is the deflection collision frequency and

$$\xi = \frac{v_\parallel}{v}$$

is the **pitch angle**.

Change of variables

$$(\psi, \xi) \rightarrow (\psi_*, \omega)$$

Keeping the highest order derivatives, one obtains

$$C(f) \simeq \frac{\nu_D}{2}\left[(v\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta)^2\frac{\partial^2 f}{\partial \omega^2} - 2\frac{Iv^2\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta}{\Omega S}\frac{\partial^2 f}{\partial\omega\partial\psi_*} + \left(\frac{Iv}{\Omega S}\right)^2\frac{\partial^2 f}{\partial\psi_*^2}\right]$$

Here it has been approximated $1-\xi^2 \approx 1$. This is valid since **it is the barely circulating and the barely trapped particles which contribute to the ion orbit loss.**

A new variable is introduced, $\widehat{\omega}$ which makes possible to connect these expressions which will finally give f with the *neoclassical distribution function for ions*:

$$\omega = \sigma\widehat{\omega}\ \widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta\ \left(1 - k\sin^2\frac{\theta}{2}\right)^{1/2}$$

Then

- $k < 1$ corresponds to *poloidally circulating particles,* and
- $1 < k < \infty$ corresponds to the *poloidally trapped particles.*

Here the **direction** of the variable ω is $\sigma = \pm 1$.

Another change of variables is suggested by the form of the **drift flux function**. We take

$$\psi_* = \psi_0 - \frac{I}{\Omega S}\widetilde{\omega}_0$$

assuming

$$\varepsilon \ll 1 \ \text{ but } \ |S|\,\varepsilon < 1$$

then

$$\begin{aligned}\widehat{\omega}^2 &= \overline{\omega}_0^2 \frac{B_0^2}{B_x^2} \\ &\quad +2SE \\ &\quad +2\left|S\right|\varepsilon\left[\mu B_0+\left(\overline{\omega}_0\frac{B_0}{B_x}-I\frac{1}{B_0}\frac{\partial\Phi}{\partial\psi}\right)^2\right]\end{aligned}$$

where

$$E = E - \mu B_0 - \frac{e\Phi}{m_i} - \frac{1}{2}\left(\overline{\omega}_0\frac{B_0}{B_x}-I\frac{1}{B_0}\frac{\partial\Phi}{\partial\psi}\right)^2$$

The notation $\widetilde{\omega}_0$ represents $\overline{\omega}_0 B/B_x$ where

$$\overline{\omega}_0 = v_\parallel + I\frac{1}{B_0}\frac{\partial\Phi}{\partial\psi}$$

evaluated at $\psi = \psi_0$ and $\theta = 0$ or π (depending on where the particles are trapped: $\theta = 0$ for inside of a tokamak, $\theta = \pi$ if the particle is trapped outside of the tokamak). B_x is the value of B evaluated at $\theta = 0$ or $\theta = \pi$.

For ω see *viscosity* where the approach of **Shaing** is detailed.