Variation of plasma parameters in the magnetic surface, a neoclassical effect

F. Spineanu and M. Vlad
Bucharest

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Abstract

Work Sessions of Plasma Theory. Fourth meeting: Variation of equilibrium plasma parameters in the magnetic surface in tokamak. The previous discussions have prepared the solution of the neoclassical drift-kinetic equation in various regimes. It remained to pay more attention to the collision operators and go further to viscosity and its relation with the fluxes. It has been formulated a request to stop for a while on the variation of the parameters in the magnetic surface, a purely neoclassical problem, which however is essential to understand spontaneous spin-up and the equilibrium flows.Then some of the previous subjects will be repeated in this fourth meeting: diamagnetic flow, Pfirsch Schluter flow, geometric inertial factor, spontaneous spin-up, magnetic pumping. We will focus on the Stringer solution of the drift-kinetic equation for the perturbation to the equilibrium function, which reflects the neoclassical poloidal variation. We will also mention other approaches.

This text is a part of Lecture 4 from the **Work Session of Plasma Theory**. It is a ground for discussions, not to be taken as final.

1 Introduction to the subject of the equilibrium poloidal flows

See also the qualitative explanation in *Stringer*.

Any poloidal flow.

The momentum equation for species j

$$\begin{aligned} nm_j\left(\frac{\partial \mathbf{u}_j}{\partial t}+\left(\mathbf{u}_j\cdot\boldsymbol{\nabla}\right)\mathbf{u}_j\right) &= -\boldsymbol{\nabla}p_j-\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_j \\ &\quad +e_jn\mathbf{E}+e_jn\mathbf{u}_j\times\mathbf{B} \\ &\quad +\mathbf{R}_j \end{aligned}$$

leads to

$$n\mathbf{u}_j=\frac{1}{m_j\Omega_j}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}p_j$$

[It can be the diamagnetic current, with the significance which is given by the gyration $\times$ gradient of density.]

For reasons related to *toroidal geometry* this flow cannot have zero divergence.

Since this flow is actually a current, perpendicular to the magnetic field $B\widehat{\mathbf{n}}$, there is a non-zero divergence of this purely perpendicular electric current.

Then there is another current, *parallel* to $B\widehat{\mathbf{n}}$, whose divergence compensates for this one.

This is the Pfirsch Schluter current.

2 The Pfirsch Schluter current

The drift motions of electrons and ions $v_{drift,j}$ in *toroidal* field lead to *charge separation*.

In order to suppress this charge separation a *current flows along magnetic field lines*.

When there is resistivity (collisions) the neutralization of the charge separation by the parallel current is *incomplete*.

Then there is a residual electric field which still remains. This is $E_{\|}$ and is connected with $j_{\|}$ by $\eta \neq 0$.

This electric field induce an *enhancement* of the diffusion.

The enhancement comes from the *radial velocity* v_r that exists due to the coupling of the parallel electric field with the poloidal magnetic field, $v_r \sim \frac{E_{\|}}{B_\theta}$, in the Ohm's law

$$-\nabla_{\|}\phi + v_r B_\theta = \eta j_{\|}$$

The *radial velocity* v_r produces a radial flux $\Gamma_r = v_r n$. This is the factor q^2 which multiplies the classical diffusion.

The parallel current arising from the non-zero divergence of the *diamagnetic current*

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{j} &= 0 \\ \boldsymbol{\nabla}_{\perp} \cdot \mathbf{j}_{\perp} + \boldsymbol{\nabla}_{\|} \cdot \mathbf{j}_{\|} &= 0 \end{aligned}$$

Now taking the perpendicular current as resulting from the *diamagnetic* flows of electrons and ions, the parallel gradient can be written as

$$\begin{aligned} \boldsymbol{\nabla}_{\|} \cdot \mathbf{j}_{\|} &= \frac{1}{qR}\frac{\partial}{\partial\theta} j_{\|} \\ &= -\boldsymbol{\nabla}_{\perp} \cdot \mathbf{j}_{\perp} = -\boldsymbol{\nabla}_{\perp} \cdot \left(e\frac{1}{m\Omega}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p \right) \\ &= -\boldsymbol{\nabla}_{\perp} \cdot \left(e\frac{1}{|e|\, B}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p \right) \end{aligned}$$

Let us look to the last term. It is the perpendicular divergence of the *diamagnetic* flow.

Note that the operator of parallel derivative is

$$\nabla_{\parallel} \sim \frac{1}{qR}\frac{\partial}{\partial\theta}$$

and that the perpendicular current $\mathbf{j}_{\perp}$ is the *diamagnetic current, of ions + electrons*. **End.**

This is a *neoclassical* effect.

It is the magnetic field that has a space variation in the perpendicular direction. First we have

$$\begin{aligned}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p &= \left|\frac{dp}{dr}\right| (\widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_r) \\ &= -\widehat{\mathbf{e}}_\theta \left|\frac{dp}{dr}\right|\end{aligned}$$

Then, restricting the gradient to the part that contains B, we use the expression of the gradient operator expressed in the geometry of the toroidal region.

This part is repeated later in this text.

We have the *magnitude* of the magnetic field

$$B = \frac{B_0}{1 + \varepsilon \cos\theta}$$

and we must calculate the perpendicular divergence of the perpendicular current, which means

$$\boldsymbol{\nabla}\cdot \left(-\widehat{\mathbf{e}}_\theta \frac{1}{B}\left|\frac{dp}{dr}\right|\right)$$

and this is approximated by (B_0 is constant)

$$\boldsymbol{\nabla}\cdot \left(\widehat{\mathbf{e}}_\theta \frac{B_0}{B}\right) = \boldsymbol{\nabla}\cdot \left[\widehat{\mathbf{e}}_\theta \left(1 + \varepsilon \cos\theta\right)\right]$$

Here is the essential part of the calculation: there is a divergence of the diamagnetic "flow" that is exclusively due to the *geometry*. This has consequences in the balance of flows.

Here it is explained how this divergence is calculated.

In the orthogonal coordinates (r, θ, φ) we have the element of distance:

$$dl^2 = (dr)^2 + r^2 (d\theta)^2 + (R_0 + r\cos\theta)^2 d\varphi^2$$

which gives the coefficients

$$\begin{aligned} h_1 &= 1 \\ h_2 &= r \\ h_3 &= R_0 + r\cos\theta \end{aligned}$$

Then the divergence of a vector $\mathbf{a}$ is written

$$\boldsymbol{\nabla}\cdot\mathbf{a} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial r}\left(h_2 h_3 a_1\right) + \frac{\partial}{\partial\theta}\left(h_1 h_3 a_2\right) + \frac{\partial}{\partial\varphi}\left(h_1 h_2 a_3\right)\right)$$

which gives

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\left[\widehat{\mathbf{e}}_\theta\left(1+\varepsilon\cos\theta\right)\right] &= \frac{1}{r\left(R_0+r\cos\theta\right)}\frac{\partial}{\partial\theta}\left(\left(R_0+r\cos\theta\right)\left(1+\varepsilon\cos\theta\right)\right) \\
&= \frac{1}{r\left(R_0+r\cos\theta\right)}R_0\frac{\partial}{\partial\theta}\left[\left(1+\varepsilon\cos\theta\right)^2\right] \\
&= \varepsilon\frac{\left(-2\sin\theta\right)}{r}
\end{aligned}$$

From this result we get

$$\begin{aligned}
-\boldsymbol{\nabla}_\perp\cdot\mathbf{j}_\perp &= -\boldsymbol{\nabla}_\perp\left(dia\right) = \\
&= -\boldsymbol{\nabla}_\perp\cdot\left(e\frac{1}{m\Omega}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}p\right) \\
&= -\boldsymbol{\nabla}_\perp\left[\left(e\frac{1}{m\Omega}\right)\left(-\widehat{\mathbf{e}}_\theta\right)\left|\frac{dp}{dr}\right|\right] \\
&= \boldsymbol{\nabla}_\perp\cdot\left(\widehat{\mathbf{e}}_\theta\frac{B_0}{B}\right)\frac{1}{B_0}\left|\frac{dp}{dr}\right| \\
&= \varepsilon\frac{\left(-2\sin\theta\right)}{r}\frac{1}{B_0}\left|\frac{dp}{dr}\right| \\
&= \frac{r}{RB_0}\left|\frac{dp}{dr}\right|\frac{\partial}{r\partial\theta}\left(2\cos\theta\right)
\end{aligned}$$

Equating the two sides of the *current conservation* equation

$$\frac{1}{qR}\frac{\partial}{\partial\theta}j_\parallel = -\frac{r}{RB}e\left(\frac{dp}{dr}\right)\frac{\partial}{r\partial\theta}\left(2\cos\theta\right)$$

Integrating on the poloidal angle θ:

$$J_\parallel \equiv J_\parallel^{PS} = -\varepsilon\frac{2}{B_\theta}\frac{dp}{dr}\cos\theta$$

This is the Pfirsch Schluter current.

We note

$$\varepsilon\frac{1}{B_\theta} = \frac{r}{RB_\theta}\frac{B}{B} = q\frac{1}{B}$$

and the combination

$$\frac{1}{B}\frac{dp}{dr}$$

is clearly coming from the diamagnetic flow

$$n\mathbf{u}^{dia} = \frac{1}{m\Omega}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p$$

and remark that the Pfirsch Schluter current is

- parallel with B
- *proportional with q*
- harmonic on θ
- proportional with the diamagnetic current.

The first physical quantity that varies on the magnetic surface is the Pfirsch Schluter current.

There is a poloidal electric field related to this current

$$E_\theta = \frac{1}{\sigma_\parallel}\frac{B}{B_\theta}\left(-\varepsilon\frac{2}{B_\theta}\frac{dp}{dr}\cos\theta\right)$$

It is the projection on θ (poloidal) of the relationship $E_\parallel = J_\parallel/\sigma_\parallel$, with the factor of projection

$$E_\parallel\left(B/B_\theta\right) = E_\theta$$

As mentioned above there is this *electric field* that still exists after the parallel current $j_\parallel$ has tried to neutralize the charge separation produced by the non-zero divergence of the current of diamagnetic origin. This electric field is due to either

- finite resistivity $\eta = \sigma^{-1}$, or
- Landau damping

We must check the *toroidal* current.

$$\begin{aligned}
j_\varphi &= \left(\mathbf{j}^{dia,\perp}\right)_\varphi + \left(\mathbf{j}^{PS,\parallel}\right)_\varphi \\
&= \mathbf{j}_\perp^{dia}\cdot\widehat{\mathbf{e}}_\varphi + \mathbf{j}_\parallel^{PS}\cdot\widehat{\mathbf{e}}_\varphi \\
&= \left(\frac{1}{B}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p\right)\cdot\widehat{\mathbf{e}}_\varphi + \left(-\varepsilon\frac{2}{B_\theta}\frac{dp}{dr}\cos\theta\,\widehat{\mathbf{n}}\right)\cdot\widehat{\mathbf{e}}_\varphi
\end{aligned}$$

$$j_\varphi = \frac{1}{B}\frac{dp}{dr}\left[\left(\widehat{\mathbf{n}}\times\widehat{\mathbf{e}}_r\right)\cdot\widehat{\mathbf{e}}_\varphi\right] + \left(-\varepsilon\frac{2}{B_\theta}\frac{dp}{dr}\cos\theta\right)\left(\widehat{\mathbf{n}}\cdot\widehat{\mathbf{e}}_\varphi\right)$$

$$\widehat{\mathbf{n}}\cdot\widehat{\mathbf{e}}_\varphi = \cos\left(\widehat{\mathbf{n}}\ ,\ \widehat{\mathbf{e}}_\varphi\right) = \frac{B_T}{B}$$

$$
\begin{aligned}
\left(\widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_r\right) \cdot \widehat{\mathbf{e}}_\varphi &= \widehat{\mathbf{e}}_\perp \cdot \widehat{\mathbf{e}}_\varphi = \cos\left(\widehat{\mathbf{e}}_\perp \ , \ \widehat{\mathbf{e}}_\varphi\right) = -\sin\left(\widehat{\mathbf{e}}_\perp \ , \ \widehat{\mathbf{e}}_\theta\right) \\
&= -\frac{B_\theta}{B}
\end{aligned}
$$

$$
j_\varphi = \frac{1}{B}\frac{dp}{dr}\left(-\frac{B_\theta}{B}\right) + \left(-\varepsilon\frac{2}{B_\theta}\frac{dp}{dr}\cos\theta\right)\frac{B_T}{B}
$$

2.1 The derivation by Hirshman neoclassical current 1978

This is also in *models, bootstrap and diamagnetic*.

And in *plasma theory, bootstrap*.

The current

$$
\mathbf{j} = \mathbf{j}_\perp + j_\| \widehat{\mathbf{n}}
$$

$$
\mathbf{j} = \frac{1}{B}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p + j_\| \widehat{\mathbf{n}}
$$

where

$$
\mathbf{j}_\perp = \frac{1}{B}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p
$$

is the diamagnetic current.

The parallel current $j_\| \widehat{\mathbf{n}}$ is called "force-free". This is because the vector product with $\mathbf{B}$ is zero, they are parallel.

The magnetic field

$$
\mathbf{B} = \mathbf{B}_T + \mathbf{B}_p
$$

$$
\begin{aligned}
\mathbf{B}_T &= F(\psi)\, \boldsymbol{\nabla}\varphi \quad \text{toroidal} \\
\mathbf{B}_p &= \boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi \quad \text{poloidal}
\end{aligned}
$$

$$
2\pi\psi \equiv \text{poloidal flux}
$$

NOTE

The sense of introducing $F(\psi)$: the real expression for the toroidal magnetic field is

$$
\mathbf{B}_T = \frac{B_0}{h}\widehat{\mathbf{e}}_\varphi = B_0\frac{R_0}{R_0 h}\widehat{\mathbf{e}}_\varphi = B_0 R_0 \boldsymbol{\nabla}\varphi \sim F \boldsymbol{\nabla}\varphi
$$

END

The expression of the current is multiplied by the poloidal magnetic field $\mathbf{B}_p$,

$$\begin{aligned}
\mathbf{j}\cdot\mathbf{B}_p &= \left(\frac{1}{B}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}p\right)\cdot\mathbf{B}_p + j_{\|}\widehat{\mathbf{n}}\cdot\mathbf{B}_p \\
&= \frac{1}{B}\left(\mathbf{B}_p\times\widehat{\mathbf{n}}\right)\cdot\boldsymbol{\nabla}p + j_{\|}\left(\frac{\mathbf{B}}{B}\right)\cdot\mathbf{B}_p \\
&= \frac{1}{B}\left[\left(\boldsymbol{\nabla}\varphi\times\boldsymbol{\nabla}\psi\right)\times\frac{\mathbf{B}}{B}\right]\cdot\boldsymbol{\nabla}p + j_{\|}\frac{\mathbf{B}_T+\mathbf{B}_p}{B}\cdot\mathbf{B}_p \\
&= \frac{1}{B}\left[\left(\boldsymbol{\nabla}\varphi\times\boldsymbol{\nabla}\psi\right)\times\frac{\mathbf{B}_T+\mathbf{B}_p}{B}\right]\cdot\boldsymbol{\nabla}p + j_{\|}\frac{B_p^2}{B} \\
&= \frac{1}{B}\left[\left(\boldsymbol{\nabla}\varphi\times\boldsymbol{\nabla}\psi\right)\times\frac{F\left(\psi\right)\boldsymbol{\nabla}\varphi}{B}\right]\cdot\boldsymbol{\nabla}p + j_{\|}\frac{B_p^2}{B}
\end{aligned}$$

Since the pressure only has radial (ψ) variation, we have

$$\begin{aligned}
\left(\boldsymbol{\nabla}\varphi\times\boldsymbol{\nabla}\psi\right)\times\boldsymbol{\nabla}\varphi &= -\boldsymbol{\nabla}\varphi\left(\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\psi\right)+\boldsymbol{\nabla}\psi\left(\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\varphi\right) \\
&\rightarrow \left|\boldsymbol{\nabla}\varphi\right|^2\boldsymbol{\nabla}\psi
\end{aligned}$$

and

$$\boldsymbol{\nabla}p = \frac{dp}{d\psi}\boldsymbol{\nabla}\psi$$

The first term is

$$\begin{aligned}
& \frac{1}{B}\left[\left(\boldsymbol{\nabla}\varphi\times\boldsymbol{\nabla}\psi\right)\times\frac{F\left(\psi\right)\boldsymbol{\nabla}\varphi}{B}\right]\cdot\boldsymbol{\nabla}p \\
=& \frac{1}{B}\left|\boldsymbol{\nabla}\varphi\right|^2\left|\boldsymbol{\nabla}\psi\right|^2\frac{dp}{d\psi}F
\end{aligned}$$

where

$$\begin{aligned}
\left|\boldsymbol{\nabla}\varphi\right| &= \frac{1}{R} \\
\left|\boldsymbol{\nabla}\psi\right| &= RB_p
\end{aligned}$$

Then

$$\frac{1}{B}\frac{1}{R^2}R^2B_p^2\,\frac{dp}{d\psi}F = \frac{B_p^2}{B^2}\frac{dp}{d\psi}F$$

We have

$$\mathbf{j}\cdot\mathbf{B}_p = \frac{B_p^2}{B^2}\frac{dp}{d\psi}F + j_{\|}\frac{B_p^2}{B}$$

Introduce for the left hand side the notation

$$\begin{aligned}
K &\equiv \frac{\mathbf{j}\cdot\mathbf{B}_p}{B_p^2} \\
&= \frac{j_p}{B_p}\text{ proportional with the poloidal velocity}
\end{aligned}$$

and then divide by B_p^2 the equation. We can rewrite

$$j_{\|} = -\frac{Fp'}{B} + KB$$

with $p' = dp/d\psi$.

We **note** that $K = \mathbf{j}\cdot\widehat{\mathbf{e}}_\theta / B_\theta$ contains the projection of $\mathbf{j}$ on the poloidal direction. We can use this poloidal component to find what is its effect on the parallel direction. For this, we must multiply the poloidal component (part of K) with the angle B/B_θ,

$$j_{\|}^K = (\mathbf{j}\cdot\widehat{\mathbf{e}}_\theta) \times \frac{B}{B_\theta} = \frac{\mathbf{j}\cdot\widehat{\mathbf{e}}_\theta}{B_\theta} B = KB$$

This part must be added to the other component of parallel current, coming from $-Fp'/B$.

To prove that K is a function of only the surface function ψ.

The poloidal component of the Ampere law

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{B} &= \mu_0 \mathbf{j} \\ (\boldsymbol{\nabla} \times \mathbf{B}) \cdot \mathbf{B}_p &= \mu_0 \mathbf{j} \cdot \mathbf{B}_p \\ &= \mu_0 K B_p^2 \end{aligned}$$

The rotational of $\mathbf{B}$ is

$$\begin{aligned} \boldsymbol{\nabla}\times(\mathbf{B}_T + \mathbf{B}_p) &= \boldsymbol{\nabla}\times[F(\psi)\,\boldsymbol{\nabla}\varphi] \text{ (poloidal)} \\ &\quad +\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi) \text{ (toroidal)} \end{aligned}$$

Since this will be scalar-multiplied by $\mathbf{B}_p$, we have only the first term because the second term is a sum of vectors oriented along $\boldsymbol{\nabla}\varphi$ and respectively $\boldsymbol{\nabla}\psi$. Then

$$\begin{aligned} (\boldsymbol{\nabla} \times \mathbf{B}) \cdot \mathbf{B}_p &= \{\boldsymbol{\nabla}\times[F(\psi)\,\boldsymbol{\nabla}\varphi]\} \cdot \mathbf{B}_p \\ &= \left[\frac{dF}{d\psi}\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\varphi + F\boldsymbol{\nabla} \times \boldsymbol{\nabla}\varphi\right] \cdot \mathbf{B}_p \\ &= -F'\left(RB_p \frac{1}{R}\widehat{\mathbf{e}}_p\right) \cdot B_p \widehat{\mathbf{e}}_p \\ &= -F' B_p^2 \end{aligned}$$

and the Ampere's law is

$$-F' B_p^2 = \mu_0 K B_p^2$$

or

$$K = -\frac{F'}{\mu_0}$$

function of ψ

NOTE
that we have two expressions for K,

$$K=-\frac{1}{\mu_0}\frac{dF}{d\psi}=\frac{j_p}{B_p}$$

from the second we see that K is proportional with the poloidal velocity

$$K\sim v_p$$

END

Return to the equation for the parallel current

$$j_{\|}=-\frac{Fp'}{B}+KB$$

We multiply by B

$$j_{\|}B=-Fp'+KB^2$$

and note that F, p', K are functions of only the surface. Then we take surface averaging

$$\left\langle j_{\|}B\right\rangle=-Fp'+K\left\langle B^2\right\rangle$$

from where

$$K=\frac{Fp'}{\langle B^2\rangle}+\frac{\langle\mathbf{j}\cdot\mathbf{B}\rangle}{\langle B^2\rangle}$$

and this is replaced in the equation for $j_{\|}$,

$$\begin{aligned}
j_{\|} &= -\frac{Fp'}{B}+\left(\frac{Fp'}{\langle B^2\rangle}+\frac{\langle\mathbf{j}\cdot\mathbf{B}\rangle}{\langle B^2\rangle}\right)B \\
&= -\frac{Fp'}{B}\left(1-\frac{B^2}{\langle B^2\rangle}\right) \\
&\quad +\frac{\langle\mathbf{j}\cdot\mathbf{B}\rangle}{\langle B^2\rangle}B
\end{aligned}$$

or

$$j_{\|}=j_{PS}+j_{neo}$$

The first term is the Pfirsch Schluter current

$$j_{PS}=-\frac{Fp'}{B}\left(1-\frac{B^2}{\langle B^2\rangle}\right)$$

This current exists in all collisional regimes.

$$j_{PS}=-I\frac{1}{B}\frac{dp}{d\psi}\left(1-\frac{B^2}{\langle B^2\rangle}\right)$$

The other term has zero divergence.

3 The Pfirsch Schluter radial fluxes

The reference **Hazeltine Hinton review**.

The momentum equation

$$nm\frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \nabla \cdot \boldsymbol{\pi} + en\mathbf{E} + en\mathbf{u} \times \mathbf{B} + \mathbf{F}$$

is multiplied with

$$\begin{aligned} & -\frac{1}{e}\sqrt{g}\nabla\psi \times \nabla\theta \\ = & -\frac{1}{e}\sqrt{g}\frac{1}{q}B_T \ \widehat{\mathbf{e}}_\varphi \end{aligned}$$

where

$$g(\psi, \theta, \varphi) = \frac{1}{\left|\nabla\psi \cdot (\nabla\theta \times \nabla\varphi)\right|^2}$$

For circular surfaces

$$\begin{aligned} \nabla\psi &= RB_\theta \widehat{\mathbf{e}}_\psi \\ \nabla\theta &= \frac{1}{r}\widehat{\mathbf{e}}_\theta \end{aligned}$$

and

$$RB_\theta \frac{1}{r}\frac{B_T}{B_T} = \frac{1}{q}B_T$$

Then

$$\begin{aligned} g &= \frac{1}{\left|RB_\theta \ \frac{1}{r}\frac{1}{R}\right|^2} = \frac{q^2R^2}{B_T^2} \\ &= \frac{r^2}{B_\theta^2} \end{aligned}$$

and

$$\sqrt{g} = \frac{r}{B_\theta}$$

The term of scalar pressure

$$\begin{aligned} (-\nabla p) \cdot \left[-\frac{1}{e}\sqrt{g}\nabla\psi \times \nabla\theta\right] &= \left[\frac{1}{e}\sqrt{g}\frac{1}{q}B_T \ \widehat{\mathbf{e}}_\varphi\right] \cdot \nabla p \\ &= \frac{1}{e}\sqrt{\frac{q^2R^2}{B_T^2}}\frac{B_T}{q}\frac{\partial p}{R\partial\varphi} \\ &= \frac{1}{e}\frac{\partial p}{R\partial\varphi} = 0 \end{aligned}$$

The pressure has NO variation in the toroidal direction.

In this work the surface average of the parallel projection of the pressure tensor

$$\left\langle R^2 \boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a \right\rangle$$

is taken zero.

in **Honda** the lowest order is zero but there is a higher order

$$\left\langle R^2 \boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a^{(2)} \right\rangle \neq 0$$

where the viscous stress exists and transfers momentum radially from the parallel direction.

The term

$$\begin{aligned}
& (en\mathbf{u} \times \mathbf{B}) \cdot \left[-\frac{1}{e}\sqrt{g}\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta \right] \\
= & -n\left(\mathbf{u} \times \mathbf{B}\right) \cdot \sqrt{g}\frac{1}{q}B_T \ \widehat{\mathbf{e}}_\varphi \\
= & -n\mathbf{u}{\cdot}B_\theta \widehat{\mathbf{e}}_r \sqrt{\frac{q^2R^2}{B_T^2}}\frac{B_T}{q} \\
= & -n \ \mathbf{u}{\cdot}\left(RB_\theta \widehat{\mathbf{e}}_r\right) \\
= & -n \ \mathbf{u} \cdot \boldsymbol{\nabla}\psi
\end{aligned}$$

This is the origin of the radial current.

The current density $\mathbf{j}$ perpendicular on the magnetic surface, $\mathbf{j} = en\mathbf{u}_r$.

See **Honda** commented in *polarization notes* for the calculation of the radial current density: polarization $\mathbf{j}^{pol}$ and total $\mathbf{j}^{tot}$.

Returning to the scalar product of the equation of momentum, we consider lowest order where the time derivative is zero

$$\begin{aligned}
0 \quad = \quad & (en\mathbf{E} + \mathbf{F}) \cdot \left(-\frac{1}{e}\sqrt{g}\frac{1}{q}B_T \ \widehat{\mathbf{e}}_\varphi \right) \\
& -n \ \mathbf{u} \cdot \boldsymbol{\nabla}\psi
\end{aligned}$$

and now it is averaged over the surface

$$\langle n \ \mathbf{u} \cdot \boldsymbol{\nabla}\psi \rangle = -\frac{1}{e}\frac{1}{q}\left\langle \sqrt{g}\mathbf{B}_T \cdot (en\mathbf{E} + \mathbf{F}) \right\rangle$$

This expression contains the flux of particles both classical and neoclassical. It is the radial current.

The *friction* force $\mathbf{F}$ is separated in **Honda** in *Coulombian* and *non-Coulombian* $\mathbf{R}^C$ and $\mathbf{R}^{non-C}$.

The classical part is sustained by $\mathbf{F}_\perp$.

The neoclassical part by $\mathbf{F}_{\|}$

$$\mathbf{F}_{\|} = F_{\|} \frac{\mathbf{B}}{B}$$

and we have in the RHS

$$-\frac{1}{e}\frac{1}{q}\sqrt{g}F_{\|}\frac{\mathbf{B}_T \cdot \mathbf{B}}{B}$$

In this expression one can use

$$I(\psi, \theta, \varphi) = \frac{1}{q}\sqrt{g}\mathbf{B}_T \cdot \mathbf{B}$$

and write

$$\frac{1}{q}\sqrt{g}F_{\|}\frac{\mathbf{B}_T \cdot \mathbf{B}}{B} = I\frac{F_{\|}}{B}$$

NOTE
In **Hirshman Sigmar review**, and
In **Honda** the identity to be used is

$$\frac{\widehat{\mathbf{n}} \times \boldsymbol{\nabla}\psi}{B} = \frac{I}{B}\widehat{\mathbf{n}} - R^2 \boldsymbol{\nabla}\varphi$$

END

We replace

$$\begin{aligned}
\langle n \ \mathbf{u} \cdot \boldsymbol{\nabla}\psi \rangle &= -\frac{1}{e}\frac{1}{q}\left\langle \sqrt{g}\mathbf{B}_T \cdot (en\mathbf{E} + \mathbf{F}) \right\rangle \\
&= -\frac{1}{e}\left\langle I\frac{F_{\|}}{B} + \frac{1}{q}\sqrt{g}\mathbf{B}_T \cdot en\mathbf{E} \ \frac{\mathbf{B}_T \cdot \mathbf{B}}{\mathbf{B}_T \cdot \mathbf{B}} \right\rangle \\
&= -\frac{1}{e}\left\langle I\frac{F_{\|}}{B} + \frac{1}{q}\sqrt{g}\mathbf{B}_T \cdot en\mathbf{E} \ \frac{\mathbf{B}_T \cdot \mathbf{B}}{\mathbf{B}_T \cdot \mathbf{B}} \right\rangle \\
&= -\frac{1}{e}\left\langle I\frac{F_{\|}}{B} + Ien\frac{\mathbf{B}_T \cdot \mathbf{E}}{\mathbf{B}_T \cdot \mathbf{B}} \right\rangle
\end{aligned}$$

Now we take

$$F_{\|} = p_e A_1$$

Then

$$\langle n \ \mathbf{u} \cdot \boldsymbol{\nabla}\psi \rangle = \frac{1}{e}p_e \left\langle I\frac{A_1}{B} \right\rangle - nI\left\langle \frac{\mathbf{B}_T \cdot \mathbf{E}}{\mathbf{B}_T \cdot \mathbf{B}} \right\rangle$$

The heat transport.

We now involve in these considerations the transport of the heat. Why? Because the fluxes and the forces are connected both directly and in cross-relations. The general form of the connection is a matrix.

The heat flux parallel to the line has the general form.

$$q_{i\|} = -T_i K_i A_i$$

On the other hand the expression of the heat flux has been found to be of the form

$$\begin{aligned} q_{i\|} &= -\frac{5}{2}\frac{1}{ZeB} I p_i \frac{\partial T_i}{\partial \psi} \\ &+ \widehat{L}_i(\psi) B \end{aligned}$$

where $\widehat{L}_i(\psi)$ is a formal function only dependent on the surface label.

From these two we obtain

$$A_i = \frac{5}{2}\frac{1}{ZeB}\ I \frac{1}{K_i} n_i \frac{\partial T_i}{\partial \psi} + \frac{\widehat{L}_i(\psi) B}{K_i T_i}$$

We know that A_i is a *force* therefore it has the nature of a gradient of a plasma variable or a combination of gradients. Then the surface average operates as a annihilator to the product of A_i and B. The condition is

$$\langle A_i B \rangle = 0$$

and allows to obtain a form of $\widehat{L}_i(\psi)$ which gives

$$A_i = \frac{5}{2}\frac{1}{ZeB}\frac{1}{K_i}\ I n_i \frac{\partial T_i}{\partial \psi}\left[I - \langle I \rangle \frac{B^2}{\langle B^2 \rangle}\right]$$

Since the *neoclassical* part of the flux (after excluding the classical part) projected on the normal to the surface and is

$$\langle \mathbf{q}_{iNC} \cdot \boldsymbol{\nabla}\psi \rangle = -\frac{5}{2}\frac{1}{e} p_i T_i \left\langle I \frac{A_i}{B} \right\rangle$$

we replace A_i here

$$\begin{aligned} \langle \mathbf{q}_{iNC} \cdot \boldsymbol{\nabla}\psi \rangle &= -1.6 \left(\frac{1}{Ze}\right)^2 \frac{m_i}{\tau_i} p_i \frac{\partial T_i}{\partial \psi} \\ &\times \left[\left\langle \frac{I^2}{B^2} \right\rangle - \frac{\langle I^2 \rangle}{\langle B^2 \rangle}\right] \end{aligned}$$

4 Why the equilibrium flows and currents must have variations in the surface

Toroidality.

The geometry imposes to the flows to have variations that are accompanied by loss of zero-divergence. See Pfirsch Schluter and Stringer.

First observation: the variations are small.

They are small due to the fact that the *toroidicity effect* (a gemoetrical effect) is small if

$$\varepsilon = \frac{r}{R} \ll 1$$

The most important part for a physical quantity is the surface-averaged part.

4.0.1 The equation of continuity

The equations involve in this approach the *neoclassical drifts* of the particles

$$\begin{aligned} \mathbf{v} &= \widehat{\mathbf{n}} v_{\parallel} + \mathbf{v}_{Dj} \\ &\quad + \mathbf{V}_E^{(0)} + \mathbf{V}_E^{(1)} + ... \end{aligned}$$

where

$$\begin{aligned} \mathbf{v}_{Dj} &= \frac{1}{\Omega_j} \widehat{\mathbf{n}} \times \frac{v_{\perp}^2/2 + v_{\parallel}^2}{R} \left(-\widehat{\mathbf{e}}_R\right) \\ &\approx -\frac{1}{e_j B} \frac{2T_j}{R} \widehat{\mathbf{e}}_{vert} \end{aligned}$$

it was taken

$$\frac{2T_j}{m_j} = v^2$$

The other velocity is *electric*

$$\begin{aligned} \mathbf{V}_E^{(0)} &= \frac{-\boldsymbol{\nabla}\phi^{(0)} \times \widehat{\mathbf{n}}}{B} \qquad \text{(radial)} \\ \mathbf{V}_E^{(1)} &= \frac{-\boldsymbol{\nabla}\phi^{(1)} \times \widehat{\mathbf{n}}}{B} \qquad (\sim \theta, \quad \text{poloidal}) \end{aligned}$$

In the equation of continuity we have the divergence of the flow of the particles caused by their *neoclassical drift*.

$$\boldsymbol{\nabla} \cdot (n \mathbf{v}_{Dj}) = -\frac{1}{e_j B} \frac{2T_j}{R} \frac{dn_0}{dr} \sin\theta$$

The divergence of the flow of particles moving with the electric velocity is

$$\begin{aligned} \boldsymbol{\nabla} \cdot \left(\mathbf{V}_E^{(0)}\right) &= \boldsymbol{\nabla} \cdot \left[\frac{-\boldsymbol{\nabla}\phi^{(0)} \times \mathbf{B}}{B^2}\right] \\ &= -\boldsymbol{\nabla}\left(\frac{1}{B^2}\right) \left[-\boldsymbol{\nabla}\phi^{(0)} \times \mathbf{B}\right] \\ &\quad + \frac{1}{B^2} \left(\boldsymbol{\nabla} \times \mathbf{B}\right) \cdot \boldsymbol{\nabla}\phi^{(0)} \end{aligned}$$

The last term is zero since the gradient of the 0− potential $\phi^{(0)}$ is almost radial and $\perp$ on $\mathbf{B}$.

The first term is purely geometrical, comes from the variation of the magnitude of the magnetic field. It is

$$-\boldsymbol{\nabla}\left(\frac{1}{B^2}\right)\left[-\boldsymbol{\nabla}\phi^{(0)}\times\mathbf{B}\right] = -\frac{2}{R}v_{E\theta}^{(0)}\sin\theta \quad \text{(geometry)}$$

The equation of continuity is

$$\begin{aligned}
&\left(\mathbf{V}_E^{(0)}\cdot\boldsymbol{\nabla}\right)n_1 \qquad \left(\text{acts like } \frac{\partial n_1(r,\theta)}{\partial\theta}\right)\\
&+\left(\mathbf{V}_E^{(1)}\cdot\boldsymbol{\nabla}\right)n_0 \qquad \left(\text{acts like } \left(E_\theta^{(1)}\times B_T\right)\frac{\partial n_0(r)}{\partial r}\right)\\
&+n_0\frac{\partial v_{\parallel j}}{\partial s}\\
=\;& \frac{2}{R}\left(\frac{1}{e_jB}\frac{dp_j}{dr}+n_0V_{E\theta}^{(0)}\right)\sin\theta
\end{aligned}$$

We **NOTE** that *we must admit* the variation in the magnetic surface of two quantities

- one is the density; it occurs as a *perturbation* $n_1(r,\theta)$ to the zeroth-order, *i.e.* surface-averaged, density $n_0(r)$
- the other is again the parallel velocity, here $v_{\parallel j}(r,s)$ where s is a variable along the line, therefore explores regions of different θ;

Using this form of the *continuity equation* Stringer obtains the Pfirsch Schluter current, and this results precisely from the term where the *neoclassical drift* flows have been expressed in terms of the gradient of the pressure (as if they would come from diamagnetic: they do not come from diamagnetic).

We write the equation of continuity for electrons and for ions and substract the two equations. One obtains

$$J_\parallel^{PS} = -2q\frac{1}{B_0}\frac{dp}{dr}\cos\theta \quad \text{Pfirsch Schluter current}$$

(compare with **Hirshman 1977)**

$$j_{PS} = -I\frac{1}{B}\frac{dp}{d\psi}\left(1-\frac{B^2}{\langle B^2\rangle}\right)$$

).

The two terms $\boldsymbol{\nabla}\cdot\mathbf{V}_E^{(0)} = -\frac{2}{R}V_{E\theta}^{(0)}\sin\theta$ cancel since they are identical for electrons and ions. The part that contains the *neoclassical drifts* is the source of the gradient of pressure dp/dr.

4.0.2 The equation of momentum conservation

The next equation is the momentum conservation where the basic flow is the *poloidal* electric velocity $V_{E\theta}^{(0)}$. The equation is

$$nm_i \left(V_{E\theta}^{(0)} \frac{\partial}{r\partial\theta} \right) v_{\parallel,i} (r,\theta) = -(T_i + T_e) \frac{\varepsilon}{q} \frac{\partial n_1 (r,\theta)}{r\partial\theta}$$

This balance of momenta involves the nonlinear static advection of the parallel velocity in the poloidal direction (by the poloidal electric velocity V_E) and the gradient of the pressure along the magnetic field line.

It is of the kind $(\mathbf{v} \cdot \boldsymbol{\nabla}) \mathbf{v} = -\boldsymbol{\nabla} p$.

This equation means that the static advection of the parallel velocity $v_{\parallel,i}$ by the poloidal velocity $V_{E\theta}^{(0)}$ is balanced by the parallel gradient of the pressure projected on the poloidal direction.

4.0.3 The Ohm's law

The Ohm's law

$$\eta j_{\parallel} = \frac{\varepsilon}{q} \frac{\partial}{\partial\theta} \left[-\phi^{(1)} (\theta) + \frac{T_e}{|e|} \frac{n_1 (\theta)}{n_0} \right]$$

where

$$j_{\parallel} = J_{\parallel}^{PS}$$

and

$$\frac{\varepsilon}{q} = \frac{B_\theta}{B_T} \quad \text{projection factor from } \theta \text{ direction to parallel}$$

The Pfirsch Schluter current $J_{\parallel}^{PS}$ allows now to write the system of equations for n_1 and $\phi^{(1)}$, perturbations on surface.

$$\begin{aligned} n_1 (\theta) &= n_0 \frac{1}{D} 2\varepsilon \frac{1}{V_{E\theta}^{(0)}} \left[-\left(v_i^{dia} + V_{E\theta}^{(0)} \right) \cos\theta \right. \\ &\qquad \left. + \eta \left(\frac{1}{n_0} \frac{dn_0}{dr} \right) \left(\frac{1}{B} \frac{dp}{dr} \right) \left(\frac{q^2}{\varepsilon^2} \right) \frac{r \sin\theta}{B} \right] \end{aligned}$$

where

$$v_j^{dia} = \frac{T_j}{e_j B} \left(\frac{1}{n_0} \frac{dn_0}{dr} \right)$$

$$c_s^2 = \frac{T_i + T_e}{m_i}$$

and the denominator

$$\begin{aligned} D &= 1 + \frac{v_e^{dia}}{V_{E\theta}^{(0)}} \\ &\quad - \frac{c_s^2}{\left(V_{E\theta}^{(0)} \right)^2} \frac{\varepsilon^2}{q^2} \end{aligned}$$

We **Note** that the combination

$$1+\frac{v_i^{dia}}{V_{E\theta}^{(0)}}$$

and the combination

$$1+\frac{v_e^{dia}}{V_{E\theta}^{(0)}}=1-\frac{\left|v_e^{dia}\right|}{V_{E\theta}^{(0)}}$$

occur in the expression of n_1. But, in D, the posible resonance $1-\frac{\left|v_e^{dia}\right|}{V_{E\theta}^{(0)}}=0$ only involves the *electrons*. It is avoided by the quantity

$$\begin{aligned} & -\frac{c_s^2}{\left(V_{E\theta}^{(0)}\right)^2}\frac{\varepsilon^2}{q^2} \\ = & -\frac{c_{s\theta}^2}{\left(V_{E\theta}^{(0)}\right)^2} \end{aligned}$$

which compares the poloidally projected sound velocity to the electric poloidal velocity.

4.1 A practical outcome: the radial particle flux is non-uniform on the surface

The radial flux of particles is calculated as an average over the magnetic surface, $\theta \in (0,2\pi)$. The quantity that is averaged is the local radial flux obtained as product between the density (zero-order n_0 and correction for variation in the surface, $n_1(\theta)$) and the radial velocity. The radial velocity is the *neoclassical drifts* $\left.v_{drift,j}\right|_r$ plus the electric contribution produced by the variation of the potential in the surface

$$\begin{aligned} \Gamma_{rj} &= nv_{rj} \\ &= \int_0^{2\pi}\frac{d\theta}{2\pi}(n_0+n_1)\left(\frac{1}{B_0}\frac{\partial\phi^{(1)}}{r\partial\theta}+\frac{T_j}{e_jB_0}\varepsilon\frac{2\sin\theta}{r}\right)(1+\varepsilon\cos\theta)^2 \end{aligned}$$

The second term in the second paranthesis is

$$\frac{T_j}{e_jB_0}\varepsilon\frac{2\sin\theta}{r}\approx \left.v_{drift,j}\right|_{radial}$$

consistent with the approximation adopted by Stringer for the neoclassical drift expressing $v_\perp^2/2+v_\parallel^2$ in terms of Temperature.

The result

$$
\begin{aligned}
\Gamma_{rj} &= n v_{rj} \\
&= q^2 \eta \; 2 n_0 \frac{1}{D} \left(\frac{1}{B_0} \frac{dp}{dr} \right) \frac{1}{B_0} \left[\frac{c_s^2 \frac{\varepsilon^2}{q^2}}{\left(V_{E\theta}^{(0)} \right)^2} + \frac{v_{ion}^{dia} - v_j^{dia}}{V_{E\theta}^{(0)}} \right]
\end{aligned}
$$

4.2 The equations for the currents and flows

The line

$$
ds^2 = h_r^2 dr^2 + h_\theta^2 d\theta^2 + h_\varphi^2 d\varphi^2
$$

The values for circular geometry are

$$
\begin{aligned}
h_r^C &= 1 \\
h_\theta^C &= r \\
h_\varphi^C &= R_0 + r \cos\theta
\end{aligned}
$$

The magnetic field will be assumed slightly more general than in circular surfaces, and we will return to this simple geometry by taking $B(r) = B_0$ =const.

$$
\begin{aligned}
B_r &= 0 \\
B_\theta &= \frac{b(r)}{h} = \frac{\varepsilon}{q} \frac{B_0}{h} \\
B_\varphi &= \frac{B_0}{h}
\end{aligned}
$$

where

$$
h = 1 + \varepsilon \cos\theta = \frac{R}{R_0} \quad , \quad \varepsilon = \frac{r}{R}
$$

and q is the safety factor. The current is (HLR)

$$
J_r = 0
$$

$$
J_\theta = 0
$$

$$
\begin{aligned}
J_\varphi &= -\frac{1}{b} \frac{dp}{dr} (1 + \varepsilon \cos\theta) \\
&= -\frac{1}{b(r)/h} \frac{dp}{dr} \\
&= -\frac{1}{B_\varphi} \frac{B_\varphi}{B_\theta} \frac{dp}{dr}
\end{aligned}
$$

Since

$$
\frac{B_\varphi}{B_\theta} \equiv \Theta^{-1} = \left(\frac{\varepsilon}{q} \right)^{-1}
$$

we have

$$\frac{\varepsilon}{q} J_\varphi = -\frac{h}{B_0(r)} \frac{dp}{dr}$$

which is identical with HassamKulsrud. We will find later below that the HK result is obtained from the **Grad Shafranov** equation.

We will also use

$$\mathbf{B} = B_\theta \widehat{\mathbf{e}}_\theta + B_\varphi \widehat{\mathbf{e}}_\varphi$$

from where we have

$$|\mathbf{B}| = \sqrt{B_\theta^2 + B_\varphi^2} = \frac{B_0}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}}$$

The perpendicular current $j_\perp$ comes from

$$\mathbf{j} \times \mathbf{B} = -\boldsymbol{\nabla} p$$

where

$$\begin{aligned}
\mathbf{j} \times \mathbf{B} &= j_\perp |\mathbf{B}| (-\widehat{\mathbf{e}}_r) = j_\perp \frac{B_0}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}} (-\widehat{\mathbf{e}}_r) \\
&= -\frac{dp}{dr}\widehat{\mathbf{e}}_r
\end{aligned}$$

from where

$$j_\perp = \frac{h}{B_0} \frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}} \frac{dp}{dr}$$

We notice that it is usual to work with two sets of projections of the current, (j_θ, j_φ) and $\left(j_\|, j_\perp\right)$. The connection is ensured by the expressions

$$\widehat{\mathbf{e}}_\varphi \cdot \widehat{\mathbf{e}}_\perp = -\frac{B_\theta}{|\mathbf{B}|} = -\frac{\frac{\varepsilon}{q}\frac{B_0}{h}}{\frac{B_0}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}}} = -\frac{\varepsilon}{q}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}$$

and

$$\widehat{\mathbf{e}}_\perp \cdot \widehat{\mathbf{e}}_\theta = \frac{B_\varphi}{|\mathbf{B}|} = \frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}$$

We use the two expressions (j_θ, j_φ) to obtain geometrically $j_\|$ as

$$j_\| = j_\theta \sin\alpha + j_\varphi \cos\alpha$$

where

$$\cos\alpha = \frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}} \ , \ \sin\alpha = \frac{\varepsilon}{q}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}$$

and, as derived above

$$
\begin{aligned}
j_\theta &= -\frac{1}{h}\frac{dB_0}{dr} \\
j_\varphi &= \frac{q}{\varepsilon}\left[-\frac{h}{B_0}\frac{dp(r)}{dr}\right]
\end{aligned}
$$

Then

$$j_\| = \frac{q}{\varepsilon}\left[-\frac{h}{B_0}\frac{dp(r)}{dr}\right]\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}$$

or

$$j_\| = \left[-\frac{h}{B_0}\frac{dp(r)}{dr}\right]\frac{q}{\varepsilon}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}$$

$$j_\| = \left[-\frac{h}{B_0}\frac{dp(r)}{dr}\right]\frac{q}{\varepsilon}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}$$

$$j_\| = \left[-\frac{h}{B_0(r)}\frac{dp(r)}{dr}\right]\frac{q}{\varepsilon}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}$$

That we must use the **Grad Shafranov** equation,

$$\frac{h}{r}\frac{\varepsilon}{q}B_0\frac{d}{dr}\left[\frac{r}{h}\frac{\varepsilon}{q}B_0\right] = -h^2\frac{dp(r)}{dr}$$

we use $B_\theta = \frac{\varepsilon}{q}\frac{B_0}{h}$ and divide by B_0 and h

$$\frac{\varepsilon}{q}\frac{1}{r}\frac{d}{dr}\left[rB_\theta(r)\right] = -\frac{h}{B_0}\frac{dp(r)}{dr}$$

We remind that the equilibrium equation

$$0 = -\boldsymbol{\nabla}p + \mathbf{j}\times\mathbf{B}$$

becomes the GS equation after using the Ampere's law

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu\mathbf{j}$$

projected on the toroidal (φ) direction

$$\mu_0\mathbf{j} = \boldsymbol{\nabla}\times\mathbf{B}\big|_\varphi = \frac{h_\varphi}{h_r h_\theta h_\varphi}\widehat{\mathbf{e}}_\varphi\left[\frac{\partial}{\partial r}(h_\theta B_\theta) - \frac{\partial}{\partial\theta}(h_r B_r)\right] = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)$$

and taking units such that $\mu_0 = 1$, we have

$$j_\varphi = \frac{1}{r}\frac{d}{dr}\left[rB_\theta(r)\right]$$

Replacing in the equilibrium equation we find

$$-\frac{h}{B_0}\frac{dp(r)}{dr} = \frac{\varepsilon}{q}J_\varphi$$

and note that actually the **Grad Shafranov** equation provides us with the explicit form of the toroidal component of the current.

We also note that the components of the current obey the zero-divergence condition (charge continuity)

$$\boldsymbol{\nabla}\cdot\mathbf{j} = 0$$

$$\frac{1}{h_r h_\theta h_\varphi}\left[\frac{\partial}{\partial r}(h_\theta h_\varphi\ j_r) + \frac{\partial}{\partial\theta}(h_r h_\varphi\ j_\theta) + \frac{\partial}{\partial\varphi}(h_r h_\theta\ j_\varphi)\right] = 0$$

Here we must insert $j_r = 0$ and assume axisymmetry

$$\frac{\partial}{\partial\varphi}(h\ j_\varphi) = 0$$

It results

$$\frac{1}{rh}\frac{\partial}{\partial\theta}(h\ j_\theta) = 0$$

where we use $j_\theta = 0$.

5 Poloidal nonuniformity of the profiles

This is a neoclassical effect.

The next fragment is also on *studies, Stringer*.

5.1 The physical picture of the Stringer effect

The qualitative picture has this formulation.

Start by recalling that the diamagnetic flows of electrons and ions are the result of the gradient of pressure combined with the magnetic field. These flows are perpendicular on the magnetic field line, which means that their main component is poloidal with just a small projection on the toroidal direction. The two flows, of electron and ions, are added into a current, the diamagnetic current

$$j_\perp \equiv j^{dia}$$

The toroidal geometry makes that an element of plasma in the diamagnetic flow undergoes variations of volume along the poloidal direction: contraction when the element moves from the low-field side of the tokamak toward the high-field side (from outboard to inboard) and then dilation when it makes the other half of the circumference. The diamagnetic current cannot have a zero-divergence $\boldsymbol{\nabla}\cdot j^{dia} \neq 0$, due to this variations.

In order to preserve the zero-divergence (equation of continuity of charge/current) one needs to accept the existence of another current, parallel with the magnetic field lines, with a non-zero divergence that compensate exactly the non-zero divergence of the diamagnetic current. It is in this way that the divergence of the *total* current is made equal to zero.

The current that we find necessary to compensate the non-zero divergence of the diamagnetic current is the Pfirsch Schluter current. We note that this argument for the existence of the Pfirsch Schluter current is based ona conservation constraint, $\nabla \cdot \mathbf{j} = 0$ and at this moment the dynamical factors that produce the flow of charges along the magnetic line, *i.e.* the PS current have not been examined. Starting from the expression of the diamagnetic current, taking into account the toroidal geometry in the operators and imposing the condition that the new (PS) current compensate the non-zero divergence of the diamagnetic current, one derives the expression of the PS current density. It has harmonic $(\cos\theta)$ variation of the amplitude on the poloidal angle and is in the direction of the magnetic field line. On the low field side it flows in one direction and on the high field side it flows in the opposite direction, just as $\cos\theta$ says.

Now a quantity with variation on the magnetic surface (variation with the poloidal angle θ) has been introduced in the plasma equations. This current is no more a function of only the surface label, r or ψ. It is also a function of θ. The variation on the magnetic surface is now part of the theory.

Necessarily, all the other quantities involved in the equations of balance must contain a variation in the magnetic surface, *i.e.* with θ. This may be small and can be seen as a correction to the value of a variable which is only a function of the surface label r or ψ. We are then led to work with plasma variables that have a main part that is a function constant on surfaces and in addition a small component which has also variation with θ.

Now with this new dependence $\sim (r, \theta)$ of any plasma variable: density, electric potential (but not temperature because we have not yet involved the heat flow) we return to the equations of moments of plasma.

The equation of continuity will contain $n_0(r)$ and $n_1(r,\theta)$. Also the velocity, which is supposed to be $E \times B$, *i.e.* produced by an electric field will be $v_0(r) = -\frac{1}{B_0}\frac{\partial \phi_0}{\partial r}$ plus the correction for the variation over the surface $v_1(r,\theta) = -\frac{1}{B_0}\frac{\partial \phi_1}{\partial r}$ where $\phi_1 = \phi_1(r,\theta)$. Intuitively, the rotation of plama in poloidal direction is not uniform: there are intervals of time where the rotation is slow and other intervals where the motion is accelerated. The comparison later made by **Hassam Drake** is with a circular motion of a pendule.

The equation for momentum balance is an equation for the velocity $v_0 + v_1$. Here we can see more clearly that it really was necessary to return to the equation *with* the assumption that the plasma variables contain a part that depends on θ. We see that by the occurence of the term

$$e_j n \mathbf{v}_1 \times \mathbf{B}$$

which is $j_1 \times B$ and now j_1 contains the Pfirsch Schluter current density which has a dependence of θ as $\cos\theta$. All variables that are involved in the equation of momentum and equally in the equation of continuity must have, we conclude, a part that has a trigonometric dependence on θ.

Indeed the solution for $n_1(r,\theta)$ can be expressed as terms that are coefficients of $\cos\theta$ and $\sin\theta$ sometimes denoted n_{1c} and n_{1s}.

The correction $\phi_1(r,\theta)$ is obtained from neutrality.

The next step is to include the solutions for $n_1(r,\theta)$ and $\phi_1(r,\theta)$, beside the constant-on-surface functions $n_0(r)$, $\phi_0(r)$, to calculate the fluxes of particles across the magnetic surface. The flux is

$$\left.\mathbf{\Gamma}\right|_r = n\left.\mathbf{v}^{drift}\right|_r$$

and these factors can be formally calculated for each species, ions and electrons.

It is better to integrate over surface these fluxes (here simply by integrating on θ between 0 and 2π) since we have in this way the flux of ions and of electrons that flow out from the volume enclosed by a magnetic surface. This will bring the problem of neutrality if the two fluxes are not equal. In this process we *note* that there are products of trigonometric functions, like $\cos\theta\sin\theta$ and $(\sin\theta)^2$. From the integration over θ some of them will cancel by periodicity $(\cos\theta\sin\theta\to 0)$ and some of them will give finite effect $\left(\sin^2\theta\to\frac{1}{2}\right)$. This effect results from the combination of the trigonometric variation of

- the plasma variable on surface, like $n_1(r,\theta)\sim\cos\theta$ and $\sin\theta$; and
- the variation of the *drift velocity* projected on the radial direction (perpendicular on the magnetic surface) $v_r^{drift}\sim\sin\theta$

and we note that the second factor is a pure neoclassical effect.

Now these (averaged on θ) fluxes of electrons and ions are compared.

They are not equal.

The ambipolarity is however a constraint: one has to have equal fluxes of electrons and of ions leaving the volume bounded by a magnetic surface. If not, the plasma inside the volume bounded by the magnetic surface will not remain neutral.

To reach ambipolarity a certain parameter in the expression of fluxes must be found. The ambipolarity becomes an equation for this parameter. This is v_0 or the gradient of the potential Φ_0 constant on surfaces, or equivalently, the radial electric field $-\boldsymbol{\nabla}\Phi_0$. When this gradient has a certain value, the two fluxes are equal and the ambipolarity is verified.

And now we examine the conclusion: in order the fluxes to be ambipolar, one has to admit the existence of a radial electric field with a particular magnitude. This means that the plasma rotates $E\times B$.

There is poloidal rotation.

This is the main conclusion of Stringer analysis.

We also note that there is no external factor except for the toroidality. The plasma spontaneously rotates, by the simple existence of the neoclassical consequences of the toroidality.

It is an intrinsic effect.

There is another explanation of the Pfirsch Schluter effect, which describes it as a dynamical effect. This is in contrast with the one presented above which just makes use of the conservation $\nabla \cdot \mathbf{j} = 0$ of the current. The Pfirsch Schluter current appears as a necessity to obtain the conservation of charge. The explanation given by **Stringer PRL** is more physical.

Later Hassam and Drake and Kleva have introduced the external factor, $\Gamma(\theta)$, possibly due to the turbulence. The rotation exists in this case.

5.2 The variation of density and potential on the surface Stringer 1991

This is **pfb3 1991**.

It is a detailed form of the PRL of 1969.

Stringer calculates the correction to the distribution function that is associated with the variation of n and Φ on magnetic surfaces. This variation is a result of toroidality. The input is therefore the drift of the particles.

The treatment is *drift-kinetic*.

$$\begin{aligned} f_j &= f_j^{(0)}\left(r, v_\parallel, v_\perp^2\right) \\ &\quad + f_j^{(1)}\left(r, \theta, v_\parallel, v_\perp^2\right) + \ldots \end{aligned}$$

$$\Phi(r,\theta) = \Phi^{(0)}(r) + \Phi^{(1)}(r,\theta) + \ldots$$

The guiding center velocity

$$\begin{aligned} \mathbf{V}_j &= v_\parallel \widehat{\mathbf{n}} \\ &\quad + \mathbf{V}_D \\ &\quad + \mathbf{V}^{(0)} + \mathbf{V}^{(1)} + \ldots \end{aligned}$$

where

$$\mathbf{V}_D = -\frac{1}{\Omega_j}\frac{v_\perp^2/2 + v_\parallel^2}{R}\left(\frac{B_\varphi}{B}\widehat{\mathbf{e}}_z - \frac{B_z}{B}\widehat{\mathbf{e}}_\varphi\right)$$

with

$$\Omega_j = \frac{e_j B}{m_j}$$

The drift is mainly vertical,

$$\widehat{\mathbf{e}}_R \times \widehat{\mathbf{n}}$$

In any case the vertical magnetic field is very small

$$\frac{B_z}{B} \ll 1$$

and the velocities are

$$\begin{aligned}
\mathbf{V}^{(0)} &= \frac{1}{B}\frac{d\Phi^{(0)}}{dr} \\
\mathbf{V}^{(1)} &= \frac{-\boldsymbol{\nabla}\Phi^{(1)} \times \widehat{\mathbf{n}}}{B}
\end{aligned}$$

$$\begin{aligned}
\Theta &\equiv \frac{B_\theta}{B_\varphi} \\
&= \frac{\varepsilon}{q} = O(\varepsilon) \\
&\ll 1
\end{aligned}$$

The diamagnetic velocities

$$\begin{aligned}
v_{*j} &= \frac{T_{0j}}{e_j B}\frac{d\ln n_0}{dr} \\
v_{*j}^T &= \frac{1}{e_j B}\frac{dT_{0j}}{dr}
\end{aligned}$$

Take the parallel velocity

$$v_\parallel = \sqrt{\frac{2}{m_j}\left(\epsilon - \mu B - e_j \Phi\right)}$$

then

$$\begin{aligned}
\frac{dv_\parallel}{dt} &= -\frac{1}{m_j v_\parallel}\left(\mathbf{V}\cdot\boldsymbol{\nabla}\right)\left(\mu B + e_j\Phi\right) \\
&= -\frac{1}{m_j v_\parallel}\left(V^{(0)}\frac{\partial}{r\partial\theta} + v_\parallel \frac{\partial}{\partial l_\parallel} + V_r\frac{\partial}{\partial r}\right)\left(e_j\Phi + \mu B\right) \\
&= -\frac{e_j}{m_j}\frac{B_\theta}{B_\varphi}\frac{\partial \Phi^{(1)}}{r\partial\theta} - \varepsilon\left(\frac{B_\theta}{B_\varphi}\frac{v_\perp^2}{2} - V^{(0)}v_\parallel\right)\frac{\sin\theta}{r}
\end{aligned}$$

The first term $-\frac{e_j}{m_j}\frac{B_\theta}{B_\varphi}\frac{\partial\Phi^{(1)}}{r\partial\theta}$ comes from $v_\parallel \frac{\partial}{\partial l_\parallel} = v_\parallel \frac{1}{qR}\frac{\partial}{\partial\theta}$ which is applied on $\Phi^{(1)}(r,\theta)$.

This comes from

$$\begin{aligned}
\frac{dv_\parallel}{dt} &= \left(\frac{\partial}{\partial t} + \left(\mathbf{V}\cdot\boldsymbol{\nabla}\right)\right)v_\parallel = \left(\mathbf{V}\cdot\boldsymbol{\nabla}\right)v_\parallel \\
&= \left(\mathbf{V}\cdot\boldsymbol{\nabla}\right)\sqrt{\frac{2}{m_j}\left(\epsilon - \mu B - e_j\Phi\right)} \\
&= \left(\mathbf{V}\cdot\boldsymbol{\nabla}\right)\frac{1}{2}\frac{1}{\sqrt{\frac{2}{m_j}\left(\epsilon - \mu B - e_j\Phi\right)}}\frac{2}{m_j}\left(-\mu B - e_j\Phi\right) \\
&= -\frac{1}{m_j}\frac{1}{v_\parallel}\left(\mathbf{V}\cdot\boldsymbol{\nabla}\right)\left(\mu B + e_j\Phi\right)
\end{aligned}$$

The derivation along the magnetic field line is

$$\begin{aligned}\frac{\partial}{\partial l_{\|}} &= \nabla_{\|} = \frac{B_\theta}{B_\varphi}\frac{\partial}{r\partial\theta} \\ &= \frac{1}{qR}\frac{\partial}{\partial\theta}\end{aligned}$$

The radial drift velocity takes into account the existence of the perturbation of the electric potential

$$\begin{aligned}V_r &= -\frac{1}{B}\frac{\partial\Phi^{(1)}}{r\partial\theta} \\ &\quad -\frac{1}{\Omega_j}\frac{v_\perp^2/2+v_\|^2}{R}\sin\theta\end{aligned}$$

The drift-kinetic equation

$$\frac{\partial f_j}{\partial t} + \left(\mathbf{V}_j\cdot\boldsymbol{\nabla}\right)f_j + \frac{\partial f_j}{\partial v_\|}\frac{dv_\|}{dt} + \frac{\partial f_j}{\partial v_\perp^2}\frac{dv_\perp^2}{dt} = 0$$

The drift kinetic equation is linearized to order ε.
the result is

$$\begin{aligned}f_j^{(1)} &= \frac{1}{V^{(0)}+\Theta v_\|}\left\{\left[\frac{\Phi^{(1)}}{B} - \frac{1}{\Omega_j}\varepsilon\left(\frac{v_\perp^2}{2}+v_\|^2\right)\cos\theta\right]\frac{\partial f_j^{(0)}}{\partial r}\right. \\ &\quad -e_j\frac{v_\perp^2}{v_{th,j}^2}\left(V^{(0)}+\Theta v_\|\right)f_j^{(0)}\cos\theta \\ &\quad \left. +\left[\frac{e_j}{m_j}\Theta\Phi^{(1)} - \varepsilon\left(\Theta\frac{v_\perp^2}{2} - V^{(0)}v_\|\right)\right]\frac{\partial f_j^{(0)}}{\partial v_\|}\right\}\end{aligned}$$

We note the poloidal velocity, composed of the electric velocity $V_E^{(0)}$ and of the poloidal projection of the parallel velocity $\Theta v_\|$.

This combination, representing the poloidal velocity, should be almost zero

$$V_E^{(0)} + \frac{\varepsilon}{q}v_\| \approx 0$$

(see **Galeev Sagdeev Liu Novakovskii** for a precise formula $\sim\sqrt{\kappa^2-\sin^2\frac{\theta}{2}}$).

This correction to the distribution function contains

- the effect of the drift of the particles v_D.
- the effect of the presence of a potential constant on the magnetic surfaces $\Phi^{(0)}$.

- the effect of a variation of the electric potential in the surface, $\Phi^{(1)}$.

A term

$$\frac{\Phi^{(1)}}{B}\frac{\partial f_j^{(0)}}{\partial r}$$

is the radial advection due to the potential $\Phi^{(1)}$, of the equilibrium distribution function.

A term

$$\left[\Theta \frac{e_j \Phi^{(1)}}{m_j}\right] \frac{\partial f_j^{(0)}}{\partial v_{\|}}$$

is the acceleration in parallel velocity produced by the electric field correction $e_j \Phi^{(1)}$ after being projected along the parallel direction by $\Theta \equiv B_\theta / B_\varphi$.

Therefore these two terms contain the effect of $\Phi^{(1)}$ on the distribution function (necessarily of zero-order $f_j^{(0)}$ since $\Phi^{(1)}$ is itself small).

The variation of the electric potential in the surface $\Phi^{(1)}$ is determined from the condition of neutrality

$$n_e = n_i$$

We have to obatin the densities by integrating over the velocity space

$$\int dv_{\|} \int dv_{\perp}^2$$

The integration over $v_{\perp}^2$ can be done.

The integration over $v_{\|}$ is complicated by the singularities of the denominator

$$\frac{1}{V^{(0)} + \Theta v_{\|}}$$

and this integration must be treated like Landau singularity.

The distribution function that is to be integrated is the Maxwell function. Then one introduces the *definition*

$$\frac{1}{n_0} \int_{-\infty}^{\infty} \frac{F_j^{(0)}}{v_{\|} - W} \left(v_{\|}^s\right) dv_{\|} \equiv K_s \left(\frac{W}{v_{th,j}}\right)$$

These functions are expressed through the *plasma dispersion function*.

The relations are

$$\begin{aligned} K_s \left(\frac{W}{v_{th,j}}\right) &= W \; K_{s-1} \left(\frac{W}{v_{th,j}}\right) \\ &+ J_s \end{aligned}$$

where

$$J_s = \begin{cases} (s-2)(s-4)\ldots 1 \; \left(\frac{v_{th,j}}{2}\right)^{\frac{s-1}{2}} & \text{for} \quad s \text{ odd} \\ 0 & \text{for} \quad s \text{ even} \end{cases}$$

The connection with the Plasma Dispersion Function is

$$\begin{aligned} K_0\left(\frac{W}{v_{th,j}}\right) &= \frac{1}{W}\left[I\left(\frac{W}{v_{th,j}}\right)-1\right] \\ K_1\left(\frac{W}{v_{th,j}}\right) &= I\left(\frac{W}{v_{th,j}}\right) \end{aligned}$$

where

$$\begin{aligned} I(z) &= 1-2z\exp\left(-z^2\right)\int_0^z dt\exp\left(-t^2\right) \\ &\quad +i\sqrt{\pi}\ z\exp\left(-z^2\right) \end{aligned}$$

NOTE that we have here the Principal value and the singularity $i\pi\delta$ which, after integration, gives the Landau term

The expression of the distribution function $f_j^{(1)}$ will be integrated over the velocity space to obtain the densities.

Then neutrality will be invoked, obtaining an equation for the potential $\Phi^{(1)}$.

Definitions

$$\begin{aligned} V_{*n,j} &= \frac{T_{0j}}{e_j B}\frac{1}{n_0}\frac{dn_0}{dr} \\ V_{*T,j} &= \frac{T_{0j}}{e_j B}\frac{1}{T_{0j}}\frac{dT_{0j}}{dr} \end{aligned}$$

Stringer finds that the supplementary velocity $V_j^{(1)}$ induced by the variation of the potential $\Phi^{(1)}$ in the surface is a fraction of the diamagnetic velocity

$$V_j^{(1)} \sim \varepsilon V_{*n,j}$$

The density is

$$\begin{aligned} \frac{n_j^{(1)}}{n_0} &= \frac{e_j\Phi^{(1)}}{T_j} \\ &\quad \times\left[\frac{1}{V^{(0)}}\left(V_{*n,j}-\frac{V_{*T,j}}{2}\right)(1-I_j)-I_j-\frac{V^{(0)}V_{*T,j}}{v_{th,j}^2\Theta^2}I_j\right] \\ &\quad +\varepsilon\exp(i\theta)\left[\left(1+\frac{V_{*n,j}}{V^{(0)}}+\frac{V_{*T,j}}{2V^{(0)}}\right)(I_j-1)\right. \\ &\qquad +2z_j^2 I_j\left(1+\frac{V_{*n,j}}{V^{(0)}}\right) \\ &\qquad \left.+z_j^2\frac{V_{*T,j}}{V^{(0)}}\left(1+2z_j^2 I_j\right)\right] \end{aligned}$$

The new notations are

$$z_j \equiv -\frac{V^{(0)}}{v_{th,j}\Theta}$$

$$I_j \equiv I(z_j)$$

Note that

$$z_j = -\frac{V_E^{(0)}\frac{\varepsilon}{q}}{v_{th,j}} = -\frac{V^{pol}\ \text{(projected on parallel direction)}}{\text{(thermal velocity)}}$$

After calculation of $n_e^{(1)}$ and $n_i^{(1)}$ it is invoked the neutrality.

The equation for neutrality becomes an equation for the perturbation to the uniform electric potential on the surface: $\Phi^{(1)}(r,\theta)$.

6 Electrostatic trapping of impurities (Hazeltine Ware)

Hazeltine Ware Phys Fluids 19, 8, 1976, 1163.

6.1 General

The collisional parameter

$$\begin{aligned}\widehat{\nu} &= r\frac{\nu_{e,i}}{\frac{B_\theta}{B}v_{th,e,i}} \\ &= \frac{\nu_{e,i}\,[1/s]}{\frac{1}{qR}\ [1/m]\,v_{th,i}\left[\frac{m}{s}\right]} = \text{nondimensional}\end{aligned}$$

Note that in the denominator it is a poloidal projection of the thermal velocity, which can be assumed to be along the parallel direction. This poloidal velocity is then combined with the length on the poloidal direction, $\sim r$ and this indicates a typical "poloidal" time scale, or frequency. It is then compared with the collision frequency.

In **Novakovskii** (*polarization*) the parameter is

$$\begin{aligned}\widehat{\nu} &= \frac{r\ \nu_{ii}}{\Theta v_{th,i}} \\ &= \frac{\nu(x)}{v_{th}/(qR)}\end{aligned}$$

where $x = v/v_{th}$.

Plateau regime

$$\varepsilon^{3/2} < \widehat{\nu}_{e,i} < 1$$

with $\nu_{e,i} \equiv$Coulombian collisions.

Order of magnitude of the variation of the electrostatic potential on the surface

$$\frac{|e|\,\widetilde{\Phi}}{T} \sim \varepsilon = \frac{r}{R}$$

other variables

$$R = R_0 h$$
$$h = 1 + \frac{r}{R_0}\cos\theta \approx 1 + \frac{r}{R}\cos\theta$$

One defines

$$\begin{aligned}
R_0\left(r\right) &= \text{the major radius measured} \\
&\quad\ \text{from the center of the surface} \\
&\quad\ \text{with radius } r
\end{aligned}$$

$$\frac{dR_0\left(r\right)}{dr} \sim \frac{r}{R_0} \ll 1$$
$$\mathbf{B} = \mathbf{B}_T + \mathbf{B}_\theta$$
$$B_T = \frac{B_0}{h} \approx B_0\left(1 - \frac{r}{R}\cos\theta\right)$$
$$B_p \equiv B_\theta = B_{p0}\left(1 + \Lambda\frac{r}{R}\cos\theta\right)$$
$$1 + \Lambda = -\frac{R_0}{r}\frac{dR_0}{dr}$$

The Shafranov shift

$$-\Lambda = 1 + \frac{1}{\varepsilon}\frac{dR_0\left(r\right)}{dr}$$

The surfaces with centers located at $R_0\left(r\right)$ and with small radius r are NOT concentric. Their centers are shifted.

$$\begin{aligned}
\Theta &= \frac{B_p}{B} \\
&\approx \Theta_0\left[1 + \left(1 + \Lambda\right)\frac{r}{R}\cos\theta\right] \\
\Theta_0 &= \frac{B_{p0}}{B_0} \text{ independent of } \theta
\end{aligned}$$

Flux surface average

$$\begin{aligned}
\left\langle F\right\rangle &= \oint\frac{d\theta}{2\pi}\left(1 - \frac{r}{R}\Lambda\cos\theta\right)F \\
&\quad\ \text{flux surface average}
\end{aligned}$$

where Λ is Shafranov shift. Properties

$$\langle \mathbf{B}\cdot\boldsymbol{\nabla}F\rangle = 0$$

or

$$\begin{aligned}
& B_0\left\langle \frac{1}{h}\ \nabla_{\|}F\right\rangle \\
= \quad & B_0\oint\frac{d\theta}{2\pi}\frac{\left(1-\frac{r}{R}\Lambda\cos\theta\right)}{1+\frac{r}{R}\cos\theta}\frac{d}{dl_{\|}}F
\end{aligned}$$

with

$$\frac{d}{dl_{\|}} = \frac{1}{qR}\frac{\partial}{\partial\theta}$$

If we take the Shafranov shift, $\Lambda = -1$ then

$$\begin{aligned}
\langle \mathbf{B}\cdot\boldsymbol{\nabla}F\rangle &= B_0\left\langle \frac{1}{h}\ \nabla_{\|}F\right\rangle \\
&= B_0\oint\frac{d\theta}{2\pi}\frac{1+\frac{r}{R}\cos\theta}{1+\frac{r}{R}\cos\theta}\frac{1}{qR}\frac{\partial}{\partial\theta}F \\
&= \frac{B_0}{qR_0}\oint\frac{d\theta}{2\pi}\frac{1}{qR}\frac{\partial F}{\partial\theta} = 0
\end{aligned}$$

For a gradient,

$$\langle\boldsymbol{\nabla}\cdot\mathbf{F}\rangle = \frac{1}{rR_0}\frac{\partial}{\partial r}\left[\frac{r}{B_{p0}}\langle RB_p\ F_r\rangle\right]$$

or

$$\begin{aligned}
\langle\boldsymbol{\nabla}\cdot\mathbf{F}\rangle &= \frac{1}{rR_0}\frac{\partial}{\partial r}\left[\frac{r}{B_{p0}}\left\langle F_r\ R_0h\ \frac{B_{p0}}{h}\right\rangle\right] \\
&= \frac{1}{rR_0}\frac{\partial}{\partial r}\left[rR_0\langle F_r\ \rangle\right]\quad\left(\text{recall }\frac{dR_0(r)}{dr} = -\frac{r}{R_0}(1+\Lambda)\ll 1\right) \\
&\approx \frac{1}{r}\frac{\partial}{\partial r}\left[r\langle F_r\rangle\right]
\end{aligned}$$

Notation for *poloidal* average

$$\overline{F} \equiv \oint\frac{d\theta}{2\pi}F$$

poloidal average

note the difference between *flux surface average* $\langle F\rangle$ and poloidal average $\overline{F}$. There is a h factor in the flux surface average.

$$\widetilde{F}(r,\theta) = F(r,\theta)-\overline{F}(r)$$

$$\widetilde{F}(r,\theta) = F_c\cos\theta + F_s\sin\theta$$

The flux surface average is re-written

$$\langle F\rangle = \overline{F}-\frac{\Lambda}{2}\frac{r}{R}F_c$$

6.2 The equilibrium with impurity species

Everything below is for impurities, and should carry the index Z.

The pressure tensor of the impurities

$$\begin{aligned}\mathbf{P} &= \int d^3v\ m\mathbf{v}\mathbf{v}\ f \\ P_{\alpha\beta} &= \int d^3v\ m v_\alpha v_\beta\ f\end{aligned}$$

The collisional friction force

$$\mathbf{F} = \int d^3v\ m\mathbf{v}\ C(f)$$

The momentum equation, at equilibrium

$$0 = -\boldsymbol{\nabla}\cdot\mathbf{P} + Zen(-\boldsymbol{\nabla}\phi) + Zen\mathbf{V}\times\mathbf{B} + \mathbf{F}$$

(see also **Helander 3999**).

The gradients of pressure (forces) of impurities are small

$$\boldsymbol{\nabla}\ln p \leq \frac{Ze}{T}\boldsymbol{\nabla}\Phi$$

There are two flows on the perpendicular direction to the magnetic line

- the gradient of the pressure (diamagnetic flow)
- the gradient of the electric potential on the surface

These perpendicular flows impose the existence of parallel flows

6.2.1 A. Poloidal electric flow

Perpendicular on the magnetic line.

Now it is assumed that there is an electrostatic potential $\Phi^{(0)}$ whose average over the poloidal angle is $\overline{\Phi}(r)$ and this combines with the main magnetic field $B\widehat{\mathbf{n}}$ to induce a perpendicular flow of the impurities

$$\mathbf{V}_{\perp 0} = \frac{-\boldsymbol{\nabla}\overline{\Phi}\times\widehat{\mathbf{n}}}{B_0}$$

The existence of the perpendicular flow requires parallel flows to compensate the non-zero divergence.

6.2.2 B. Zero-divergence of the flow on surface imposes the existence of the parallel flow

This is the moment when the poloidal flow is connected with the toroidal flow, in a relationship similar to *diamagnetic* flow and *Pfirsch-Schluter* flow.

There is a *zero-order* velocity on the magnetic surface

$$\mathbf{V}_0 = \mathbf{V}_{\|0} + \mathbf{V}_{\perp 0}$$

The total divergence of the flow is zero

$$\boldsymbol{\nabla}\cdot \left(n\mathbf{V}_{\perp 0} + n\mathbf{V}_{\|0}\right) = 0$$

The parallel flow is separated into a part that does not depend on poloidal angle θ and a θ-dependent part

$$\begin{aligned} V_{\|0} &\equiv u \\ &= \overline{u}(r) + \widetilde{u}(\theta) \end{aligned}$$

It is also defined

$$u_E = \frac{1}{B_{\theta 0}} \left(-\frac{d\overline{\Phi}}{dr}\right)$$

This is a *toroidal* velocity. Since $\overline{\Phi}(r)$ is given and is not dynamic, the toroidal velocity u_E is fixed. It does NOT depend on θ.

u_E is the toroidal projection of the perpendicular *electric* velocity $\mathbf{V}_{\perp 0} = \frac{-\boldsymbol{\nabla}\overline{\Phi}\times\widehat{\mathbf{n}}}{B_0} = \frac{1}{B_0}\left(-\frac{d\overline{\Phi}}{dr}\right)$ by multiplication with B_0/B_{p0}.

The order of magnitude, suggested by the continuity equation

$$\frac{\widetilde{n}}{\overline{n}} \sim \frac{\widetilde{u}}{\overline{u}}$$

From the equation of continuity

$$\boldsymbol{\nabla}\cdot (nV) = 0$$

we obtain the part of the *parallel* velocity, $\widetilde{u}$, which varies on the magnetic surface.

$$\begin{aligned} \widetilde{u}(\theta) &= u_E \frac{r}{R}\cos\theta \\ &\quad + (u_E - \overline{u})\left(\frac{\widetilde{n}}{\overline{n}} + \frac{r}{R}\cos\theta\right) \end{aligned}$$

NOTE

Later in the paper, it is mentioned the case of the ST device, where

$$\overline{u} \approx 0$$

Then this formula becomes

$$\begin{aligned} \widetilde{u}\left(\theta\right) &= u_E \; 2\frac{r}{R}\cos\theta \\ & \quad +u_E\frac{\widetilde{n}}{\overline{n}} \end{aligned}$$

which means Pfirsch-Schluter-like but the perpendicular flow is given by the radial gradient of electric potential and NOT of the pressure.

The other term is from the zero-divergence of the variations

$$\widetilde{n}u_E + \overline{n}\widetilde{u} \approx 0$$

END

Let us see how is composed. The first term and the last term in the second paranthesis

$$u_E \times 2\frac{r}{R}\cos\theta$$

is of the same nature as the Pfirsch Schluter current

$$\begin{aligned} J_{\parallel}^{PS} &= -\varepsilon\frac{2}{B_\theta}\frac{dp}{dr}\cos\theta \\ &= -\frac{dp}{dr}\frac{1}{B_{p0}} \times 2\frac{r}{R}\cos\theta \end{aligned}$$

where instead of the diamagnetic velocity (used in Pfirsch Schluter zero-divergence condition) it is used the *toroidal* velocity u_E induced by the gradient of the given electrostatic potential $\overline{\Phi}\left(r\right)$.

$$\frac{1}{B_{p0}}\left(-\frac{dp}{dr}\right) \rightarrow u_E = \frac{1}{B_{p0}}\left(-\frac{d\overline{\Phi}}{dr}\right)$$

However the velocity involved in PS is diamagnetic (i.e. perpendicular $\sim$ poloidal) and is

$$n\mathbf{v}_{dia} = \frac{1}{m\Omega}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p \sim \frac{1}{B_T} \times \left(-\frac{dp}{dr}\right)\widehat{\mathbf{e}}_\theta$$

and the parallel electric velocity u_E is of the same nature as diamagnetic if projected on poloidal direction

$$\begin{aligned} u_E &\sim \widehat{\mathbf{e}}_\parallel \\ \text{projected on } \theta \text{ is } u_E \times \frac{B_\theta}{B} &\rightarrow v_{dia} \\ u_E &\rightarrow \frac{B}{B_\theta}v_{dia} \end{aligned}$$

The additional terms in the expression of $\widetilde{u}$ (resulted from continuity)

$$\left(u_E - \overline{u}\right)\frac{\widetilde{n}}{\overline{n}} + \left(-\overline{u}\right)\frac{r}{R}\cos\theta$$

The last term

$$(-\overline{u})\,\frac{r}{R}\cos\theta \rightarrow -\overline{u}\;\overline{n}\;\frac{r}{R}\cos\theta$$

comes from a purely geometric part of the equation of continuity, where the divergence has to introduce the metric coefficients, h. None of $\overline{u}\;\overline{n}$ are affected by derivation to θ.

The other

$$(u_E - \overline{u})\,\frac{\widetilde{n}}{\overline{n}}$$

can be reduced, in the absence of the electric potential, to

$$\overline{u}\widetilde{n}$$

This difference will sometimes be considered small $u_E - \overline{u} \approx 0$.

6.2.3 C. The parallel velocity induced by radial gradient of pressure

For the ions of hydrogen.

The lowest order *parallel* velocity is

$$\frac{1}{v_{th,i}}\left(\overline{V}_{\|i} - u_E\right) = -\frac{T_i}{eB_p}\frac{1}{v_{th,i}}\left[\frac{d}{dr}\ln p_i + \kappa\frac{d}{dr}\ln T_i\right]$$

where κ is function only of r (this is Hazeltine)

The first term in the RHS is the diamagnetic velocity projected onto the parallel direction, *i.e.* $\times\frac{B}{B_\theta}$.

The second is Hazeltine's poloidal velocity, projected on the parallel direction.

Note Later it will be adopted $\overline{V}_{\|i} \approx \overline{u}$ (after assuming that the collisional coupling between the impurities and hydrogen ions makes their velocities almost equal) so that the difference in the LHS is

$$(\overline{u} - u_E)$$

that will occur several times below. (Later it will be shown that this difference is very small)

The fact that

$$\overline{V}_{\|i} \;\text{ (for the hydrogen ions)} \approx \overline{u} \quad \text{(for the impurities } Z\text{)}$$

is an important assumption: it means that the collisions produce an exchange of momentum between background hydrogen ions and the impurity ions which makes them to flow in the same way.

END

Note

The coefficient κ is obtained by Hazeltine in three regimes of collisionality, see **Novakovskii**

End.

Comment on the parallel flow induced solely by the gradient of pressure and magnetic field (no trapped/circulating particles) We must first be sure that the toroidal velocity induced by the radial gradient of pressure can be identified with $\overline{V}_{\parallel i} - u_E$.

NOTE

The equation for $\overline{V}_{\parallel i} - u_E$ (above) is derived in Eq.(61) of **hazeltine rotation plasma collisional**

$$nU_1 = -\frac{I}{eB} p \left[\frac{n'}{n} + \frac{e\Phi'}{T} + \frac{T_i'}{T_i} \left(1 + \frac{B^2}{\langle B^2 \rangle} (1.8 + 0.27b) \right) \right]$$

where

$$B_\varphi = \frac{I}{R}$$

$$n' = \frac{dn}{d\psi}$$

$$b \equiv \left\langle \frac{\left(\nabla_\parallel B\right)^2}{B^2} \right\rangle \frac{B^2}{\left\langle \left(\nabla_\parallel B\right)^2 \right\rangle}$$

There are three regimes and three expressions for the *term with the gradient of the ion temperature.*

Neglecting the part with the gradient of the temperature (which makes the difference between the three regimes) we retain only the first term, which is the same in all the three regimes. Also, we will ignore the term with Φ' since in our treatment this is substracted $\overline{V}_{\parallel i} - u_E$. Approx

$$\begin{aligned} nU_1 &= -\frac{1}{eB_\theta} p \frac{d \ln n}{dr} \\ &= -\frac{T}{eB_\theta} n \left(\frac{1}{n} \frac{dn}{dr} \right) \end{aligned}$$

or

$$U_1 = -\frac{T}{eB_\theta} \left(\frac{1}{n} \frac{dn}{dr} \right)$$

END

We also note that the first term (ignoring the denominator $v_{th,i}$) of $\left(\overline{V}_{\parallel i} - u_E\right)$ is

$$\frac{T_i}{eB_p} \frac{p_i'}{p_i} \sim \frac{T_i}{eB_p} T_i \frac{\partial n}{\partial r} \frac{1}{nT_i} = \frac{T}{eB_\theta} \left(\frac{1}{n} \frac{dn}{dr} \right)$$

This is identical with the main part of the previous expression for the parallel flow, of **Hazeltine**. It results that

$$\overline{V}_{\parallel i} - u_E = U_1$$

This flow is simply induced by the relaxation of the pressure gradient. The pressure gradient combines with the poloidal magnetic field and produce together a flow in the parallel ($\sim$ toroidal) direction.

Both (1) the gradient of pressure and (2) the poloidal field (or the toroidal current j_φ) have definite directions and their combination gives an effectively ORIENTED flow. Even if the ions move equally in parallel and anti-parallel directions, the resulting flow is unequivocally oriented. There is no cancelling due to the lack of preffered direction of motion of ions parallel and anti-parallel with $\mathbf{B}$.

We NOTE that this flow does NOT include any reference to trapped/circulating particles.

In some sense it is a situation analogous to the poloidal flow and its ambiguous relationship with the diamagnetic flow due to gyration. There, the radial gradient of pressure produces, collisionally, a force acting on a test particle from small radius (high p) to larger radius (low p). This force $\mathbf{F}$ is directed radially and combined with $\mathbf{B}$ to produce the diamagnetic flow

$$nu_{dia}\widehat{\mathbf{e}}_\theta = \frac{1}{eB}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p$$

In this perspective (simply derived from the equation of conservation of momentum) there is no reference to the gyration of particles and to the unbalanced flux of momentum through a plane perpendicular on small radius.

Similarly, in the case of the toroidal flow as derived by **Hazeltine** the gradient of pressure produces a force $\mathbf{F}$ by more collisional momentum transfer from the side with higher pressure (on a test particle) and this is combined with the poloidal magnetic field to produce the toroidal flow. No need to introduce here trapped/circulating particles.

Obviously, in this perspective it just is necessary to solve the kinetic equation in a neoclassical expansion, then multiply f by $v_\parallel$ and integrate on d^3v to obtain the flow (similarly, - the bootstrap).

We note the separation of the electric velocity u_E and the right hand side is the diamagnetic part. This is projected using

$$\frac{B}{B_p}$$

on the parallel direction.

6.2.4 D. Collisional friction

Collisional friction between the two species leads to

$$\overline{V}_{\parallel i} \approx V_{\parallel z} \equiv \overline{u}$$

This is an essential assumption, which makes the impurity ions something like passive particles.

Compared with the *thermal velocity of the impurities* the flow velocities

$$u_E \quad \text{and} \quad \overline{u}$$

are of the same magnitude

$$\frac{u_E}{v_{th,z}} \sim \frac{\overline{u}}{v_{th,z}} \sim 1$$

6.2.5 E. The conservation of the momentum

The parameter of an expansion is the inverse of the impurity electric charge.

$$Z \sim 8$$

$$\frac{1}{Z} \ll 1$$

In zeroth order Z^0, the impurity pressure tensor

$$\mathbf{P} = \mathbf{I}p_0 + mn_0 \ \mathbf{V}_0 \ \mathbf{V}_0$$

With this expression the equation of momentum conservation at equilibrium for the impurities becomes is projected along the parallel direction

$$\widehat{\mathbf{n}} \cdot \{-\boldsymbol{\nabla} \cdot \mathbf{P} + Zen\,(-\boldsymbol{\nabla}\phi) + Zen\mathbf{V} \times \mathbf{B} + \mathbf{F}\} = 0$$

neglecting the friction $\mathbf{F}$. Here the pressure, the convective flow and the electric force along the magnetic field line are balanced.

$$0 = -mn_0\widehat{\mathbf{n}} \cdot (\mathbf{V}_0 \cdot \boldsymbol{\nabla})\,\mathbf{V}_0 - \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}p_0 - Zen_0 \ \nabla_{\|}\widetilde{\Phi}$$

(**note** in **Helander** there is also the friction $\mathbf{R}_{\|Z}$ with impurities).

The second term is

$$\widehat{\mathbf{n}} \cdot (\mathbf{V}_0 \cdot \boldsymbol{\nabla})\,\mathbf{V}_0 = \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\frac{V_0^2}{2} - \widehat{\mathbf{n}} \cdot [\mathbf{V}_0 \times (\boldsymbol{\nabla} \times \mathbf{V}_0)]$$

where

$$\begin{aligned}
\mathbf{V}_0 &= \mathbf{V}_{\|0} + \mathbf{V}_{\perp 0} \\
&= \overline{u} + \widetilde{u} + u_E \\
&\quad + \frac{-\boldsymbol{\nabla}\overline{\Phi} \times \widehat{\mathbf{n}}}{B_0}
\end{aligned}$$

NOTE in **Shaing Sanuki**

$$\mathbf{B}\cdot\left[\left(\mathbf{V}\cdot\boldsymbol{\nabla}\right)\mathbf{V}\right]=\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{1}{2}V^{2}\right)+\left(\mathbf{V}\times\mathbf{B}\right)\cdot\left(\boldsymbol{\nabla}\times\mathbf{V}\right)$$

END

Then the parallel momentum conservation is converted into the variation on the poloidal direction

$$\frac{m\left(\overline{u}-u_{E}\right)}{T}\frac{\partial\widetilde{u}}{\partial\theta}+\frac{m}{T}u_{E}\overline{u}\,\frac{r}{R}\sin\theta=-\frac{1}{p}\frac{\partial\widetilde{p}}{\partial\theta}-\frac{Ze}{T}\frac{\partial\widetilde{\Phi}}{\partial\theta}$$

recall $\overline{u}$ is an average parallel velocity, along $\mathbf{B}$. And u_E is the *toroidal* velocity due to the radial gradient of the electric potential Φ_0, combined with B_θ.

We note here that the derivation in the parallel direction is expressed through the actual variation, which is poloidal, θ. The variation of parameters in the magnetic surface takes place along this θ. This is axisymmetry.

We also **Note** that the term

$$\frac{m}{T}u_{E}\overline{u}\,\frac{r}{R}\sin\theta$$

comes from *geometric* factor h under operator of divergence, from

$$\left(u_{\parallel}\nabla_{\parallel}\right)u_{\parallel}$$

And the term

$$\frac{m\left(\overline{u}-u_{E}\right)}{T}\frac{\partial\widetilde{u}}{\partial\theta}$$

has the same origin, the parallel convection derivation.

Comment on this equation.

It is a balance of momenta along the parallel direction, where the momenta with only existence on the poloidal direction (due to variations $\widetilde{\Phi}\left(\theta\right)$ and $\widetilde{n}\left(\theta\right)$ on the surface) are converted using factors of projection.

As equation along the magnetic field line they look similar to the ones used in *drift wave* theory, where in addition the collisional friction is included for balance. A similar momentum balance along the magnetic line is considered by **Helander** with the inclusion of $R_{\parallel Z}$, friction between the impurities and the background ions. However there the stationary $\left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{v}$ is further neglected and only the electric field $-\nabla_{\parallel}\widetilde{\Phi}$, the pressure $\nabla_{\parallel}p$ and this friction $R_{\parallel Z}$ are retained (similar to the drift wave treatment).

Here it is considered that the relative collisional exchange of momentum (friction) $R_{\parallel}$ may not be included but the stationary flow $\left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{v}$ must be.

The expression is integrated over the angle θ,

$$\frac{Ze\widetilde{\Phi}}{T}=-\frac{\overline{p}}{p}+\frac{m}{T}\left[u_{E}\overline{u}\,\frac{r}{R}\cos\theta+\left(u_{E}-\overline{u}\right)\widetilde{u}\right]$$

NOTE

Here we will insert, later, the variation $\widetilde{u}(\theta)$ of the *parallel* velocity as derived above,

$$\begin{aligned}\widetilde{u} &= u_E \frac{r}{R}\cos\theta \\ &\quad + (u_E - \overline{u})\left(\frac{\widetilde{n}}{\overline{n}} + \frac{r}{R}\cos\theta\right)\end{aligned}$$

This expression for the variation of the electrostatic potential $\widetilde{\Phi}$ on the surface is dependent on the averaged parallel velocity $\overline{u}$, on the electric velocity $u_E \sim \frac{d\Phi^{(0)}}{dr}$ along the toroidal direction, and the variation of the density with θ, which is $\widetilde{n}/\overline{n}$. Then, simplifying, we have a connection between $\widetilde{\Phi}$ and $\widetilde{n}/\overline{n}$. This is useful since we replace one with the other.

Something similar occurs at **Helander** where the Boltzmannian density allows to replace the potential with the density. The condition of neutrality will next be used to connect the impurity density with the electron and ion densities.

END

6.2.6 F. The conservation of energy for the impurities.

The energy is dominated by *mass flow*, the terms with velocity $\mathbf{V}_0$.

$$\boldsymbol{\nabla}\cdot\left(\frac{3}{2}p_0\mathbf{V}_0\right) + p_0\left(\boldsymbol{\nabla}\cdot\mathbf{V}_0\right) = 0$$

The first term is the divergence of the flux flow with velovity $\mathbf{V}_0$ of density of energy (p_0). The second term is the change in the energy per volume due to the compressibility of the velocity of impurities. The origin is the variation of the velocity on the surface. This is non-zero-divergence of the velocity. What our equations have asked is the equation of continuity which involves the flux nV and not only V. Both n and V have variation on the surface.

$$\frac{\widetilde{p}_Z}{\overline{p}_Z} = \frac{5}{3}\frac{\widetilde{n}_Z}{\overline{n}_Z}$$

the new parameter

$$\begin{aligned}x &\equiv \frac{R}{r}\frac{e\widetilde{\Phi}}{T_i} \\ &= \frac{\frac{e\widetilde{\Phi}}{T_i}}{\varepsilon} \quad \text{to be compared with 1.}\end{aligned}$$

The following expression is obtained from the variation of the electrostatic potential on surface

$$\frac{Ze\widetilde{\Phi}}{T_i}$$

obtained before, in which we insert the expression of the variation of the toroidal velocity, $\widetilde{u}$ and replace the ratio of the pressures of impurities with the ratio of the densities of the impurities.

It results

$$\begin{aligned} x &= \frac{m_Z}{ZT_i}\left\{\left[u_E^2+(\overline{u}-u_E)^2\right]\cos\theta \right. \\ &\qquad \left. +\left[\left((\overline{u}-u_E)^2-\frac{5}{3}\right)\frac{T_i}{m_Z}\right]\frac{R}{r}\frac{\widetilde{n}_Z}{\overline{n}_Z}\right\} \end{aligned}$$

Later it will be used the part

$$x_c$$

which is the $\cos\theta$ term of x

$$x_c = \frac{m_Z}{ZT_i}\left[u_E^2+(\overline{u}-u_E)^2\right]$$

NOTE

A typical case is

$$\overline{u}\approx 0$$

from where we find

$$\begin{aligned} x &\approx 2\frac{m_Z}{ZT_i}u_E^2\ \cos\theta = \frac{4}{Z}\left(\frac{u_E}{v_{th,Z}}\right)^2 \\ &\approx 0.5 \end{aligned}$$

or

$$\frac{\frac{e\widetilde{\Phi}}{T_i}}{\varepsilon}\sim 0.5$$

END

6.2.7 G. Neutrality and assumption that the densities are Boltzmann

(similar in **Helander 3999**)

The average and the perturbation of impurity densities are eliminated by

1. neutrality

$$Z\widetilde{n}_Z = \widetilde{n}_e - \widetilde{n}_i$$

2. and the assumption that the densities are Boltzmannian.

For the perturbed density, we extract explicitely the Boltzmann distribution and leave a correction $n_{1e,i}$ which must account of the neoclassical drift. This part is neglected being smaller compared with the effect of the variation of the electrostatic potential $\widetilde{\Phi}$.

$$\begin{aligned}\widetilde{n}_{a=i,e} &= -\overline{n}_{a=i,e}\frac{e_a\widetilde{\Phi}}{T_a}+n_{1a} \\ a &= i,e\end{aligned}$$

Retaining the $\widetilde{\Phi}$ term

$$\begin{aligned}& x\left\{1-\frac{\overline{n}_i+\kappa\overline{n}_e}{Z^2 n_{\overline{n}_z}}\left[m_Z\frac{(\overline{u}-u_E)^2}{T_i}-\frac{5}{3}\right]\right\} \\ =& \frac{m_Z}{ZT_i}\left[u_E^2+(\overline{u}-u_E)^2\right]\cos\theta\end{aligned}$$

where

$$\kappa=\frac{T_i}{T_e}$$

The term $1/Z^2$ is neglected

$$x=\frac{m_z}{ZT_i}\left[u_E^2+(\overline{u}-u_E)^2\right]\cos\theta$$

This is the first relationship between x and $\overline{u}$.

Estimation of the order of magnitude of the variation of the density of impurity on a magnetic surface

$$\frac{\widetilde{n}_Z}{\overline{n}_Z}=x\left(\frac{r}{R}\right)\frac{\overline{n}_i+\frac{T_i}{T_e}\overline{n}_e}{Z\overline{n}_Z}$$

We note that, if x is of order unity, then $x\left(\frac{r}{R}\right)\sim\left(\frac{R}{r}\right)\frac{e\widetilde{\Phi}}{T_i}\times\left(\frac{r}{R}\right)=\frac{e\widetilde{\Phi}}{T_i}$.

We also see that

$$x\sim\frac{1}{Z}\frac{1}{T_Z/m_Z}u_E^2\cos\theta\sim\frac{1}{Z}\frac{u_E^2}{v_{th,Z}^2}\cos\theta$$

and the variation of the electrostatic potential (energy of an electron, relative to the temperature)

$$\frac{e\widetilde{\Phi}}{T_i}\sim\varepsilon\frac{1}{Z}\frac{u_E^2}{v_{th,Z}^2}\cos\theta$$

If we compare the parallel velocity induced by the static part of potential $\overline{\Phi}(r)$, which is $u_E=-\frac{1}{B_\theta}\frac{d\overline{\Phi}}{dr}$ with the thermal velocity of the impurity ions, $v_{th,Z}$ we can assume that

$$\frac{u_E}{v_{th,Z}}\sim\frac{\overline{u}}{v_{th,Z}}\sim 1$$

(eq.28 in **Hazeltine**). This also means that

$$\overline{u} - u_E \approx 0$$

which eliminates the second term in the expression of x.

Then

$$\frac{e\widetilde{\Phi}}{T_i} \sim \varepsilon \cos\theta$$

(later it is defined

$$x \sim \frac{\frac{e\widetilde{\Phi}}{T_i}}{\varepsilon} \sim 1$$

Remarks

At this moment we collect few aspects of the derivation

- the poloidal *ELECTRIC* $d\overline{\Phi}/dr$ flow imposes a toroidal flow (to compensate for non-zero divergence) which has *poloidal variation* just like Pfirsch Schluter.
- the velocity of the impurity ions is approx the same as the velocity of the background ions, due to collisional coupling, $\frac{\overline{u}}{v_{th,Z}} \sim 1$
- the uniformized parallel velocity $\overline{u} = \overline{V}_{\parallel,i}$ is approx the same as the *ELECTRIC* parallel velocity u_E.
- the uniformized parallel velocity $\overline{u}$ and the thermal velocity of impurities $v_{th,Z}$ are approximately equal.
- the final approximative result: the variation of the electrostatic potential (normalized to T_i) is of order ε and has $\cos\theta$ variation.

6.3 Variation on surface, electrostatic trapping Hazeltine Ware

Two distribution functions

$$f_e \text{ and } f_i$$

in the *plateau* regime

$$\begin{aligned} \left(\frac{r}{R}\right)^{3/2} &< \widehat{\nu}_{e,i} < 1 \\ \widehat{\nu}_{e,i} &\equiv r\frac{1}{(B_p/B)\, v_{th,e,i}}\nu_{e,i}^{Coulomb} \end{aligned}$$

The second line: the thermal velocity of the electrons, ions, is mainly parallel with the field. It is projected on the poloidal diraction by multiplying with

B_p/B. Then this *poloidal velocity* is made relative to the dimension of the small radius, r . This gives a *time* of travel over the distance r, which is

$$\frac{r}{(B_p/B)\, v_{th,e,i}} \sim \text{time}$$

Finally this time is compared with a frequency of collision: the Coulomb collision frequency of electrons (ions),

$$\nu_{e,i}^{Coulomb}$$

The adimensional quantity that arises is the ratio between the time to propagate *thermally* on the poloidal circumference at radius r, $\frac{r}{(B_p/B)v_{th,e,i}}$ and the time between two Coulombian collisions $1/\nu_{e,i}^{Coulomb}$. This adimensional measure is smaller than 1 if the time of poloidal "thermal" travel is shorter than the time between two collisions. The collisions are *rare* in this case.

The comparison and the reasoning look similar to the considerations on time-scales done by **Helander 3999** where it is concluded that the analysis can be restrained at the *magnetic surface*. This is because the *radial diffusion*, which is due to collisions is slower than the flows in the surface.

The electrostatic trapping is treated here as comparable to the magnetic trapping, *banana*.

The pressure gradients of the impurity are considered small

$$|\boldsymbol{\nabla} p_Z| \ll Ze\Phi$$

so it will be ignored in the first calculations.

$$\begin{aligned} \mathbf{V}_{\perp}^{(0)} &= \frac{1}{(B_0/h)} \widehat{\mathbf{n}} \times \boldsymbol{\nabla}\overline{\Phi}(r) \\ &\quad \text{poloidal velocity} \end{aligned}$$

Due to the geometry, the *density conservation* will impose the presence of parallel compensating flows

$$\boldsymbol{\nabla}\cdot \left(n\mathbf{V}_{\perp}^{(0)} + n\mathbf{V}_{\parallel}^{(0)} \right) = 0$$

It is clear that $\mathbf{V}_{\parallel}^{(0)}$ will have an average part (constant on surface) and a part that depends on θ.

$$\begin{aligned} V_{\parallel}^{(0)} &\equiv u \\ &= \overline{u} + \widetilde{u}(\theta) \end{aligned}$$

An electrostatic potential that is constant on surfaces and has radial variation $\Phi^{(0)}(r)$ will produce a poloidal flow.

This poloidal flow has a parallel projection

$$u_E = -\frac{1}{B_\theta}\frac{d\Phi^{(0)}(r)}{dr}$$

The part of the parallel velocity that depends on θ is

$$\widetilde{u}(\theta) = u_E \frac{r}{R}\cos\theta + (u_E - \overline{u})\left(\frac{\widetilde{n}}{n} + \frac{r}{R}\cos\theta\right)$$

Velocity space variables

$$\begin{aligned} v &= |\mathbf{v}| \\ \xi &= \frac{v_\parallel}{v} \end{aligned}$$

The drift kinetic equation

$$\begin{aligned} &\left(\xi + \frac{e}{m\Omega_p v}\frac{\partial\Phi}{\partial r}\right)\left(\frac{\partial f}{\partial\theta} - \frac{e}{m}\frac{\partial\Phi}{\partial\theta}\frac{\partial f}{\partial v}\right) \\ &-\frac{v}{\Omega_p}\left[\frac{1+\xi^2}{2}\frac{r}{R}\sin\theta + \frac{e}{mv^2}\frac{\partial\Phi}{\partial\theta}\right]\left(\frac{\partial f}{\partial r} - \frac{e}{mv}\frac{\partial\Phi}{\partial r}\frac{\partial f}{\partial v}\right) \\ &-\frac{1-\xi^2}{2}\left(\frac{r}{R}\sin\theta + \frac{2e}{mv^2}\frac{\partial\Phi}{\partial\theta}\right)\frac{\partial f}{\partial\xi} \\ &+\frac{e}{m}\frac{r\xi}{\Theta v}\frac{E_0}{h}\frac{\partial f}{\partial v} \\ = \;&\frac{r}{\Theta v}C(f) \end{aligned}$$

The first paranthesis

$$\frac{v_\parallel}{v} + \frac{1}{v}\frac{e}{m\left(\frac{eB_p}{m}\right)}\frac{\partial\Phi}{\partial r}$$

can be written

$$\begin{aligned} &= \frac{1}{v}\left(v_\parallel + \frac{B}{B_p}\frac{1}{B}\frac{\partial\Phi}{\partial r}\right) \\ &= \frac{1}{v}\left(v_\parallel - \frac{1}{\Theta}v_E\right) \end{aligned}$$

We **NOTE** the occurence of the resonant factor, $v_\parallel - \frac{1}{\Theta}v_E$. Since it is divided to v which is almost parallel, this fraction is actually a projection as *poloidal* velocity. It is a convective part $\mathbf{v}\cdot\boldsymbol{\nabla} \sim v_\theta \frac{\partial}{\partial\theta}$.

Taking from the RHS the factors, we re-write this paranthesis as

$$\frac{\Theta v}{r} \times \frac{1}{v}\left(v_\parallel - \frac{1}{\Theta}v_E\right) = \frac{1}{r}\frac{B_p}{B}\left(v_\parallel - \frac{B}{B_p}v_E\right) = \frac{1}{r}V_{poloidal}$$

Now let us look to what is applied this *convection operator*, of the nature $(\mathbf{v}\cdot\boldsymbol{\nabla}) f$.

We find, as normal, the spatial poloidal gradient $\frac{\partial f}{\partial\theta}$ which is to be convected by $V_{poloidal}$. Including the factors that can be taken from the RHS,

$$\frac{\Theta v}{r}\times\frac{1}{v}\left(v_{\parallel}-\frac{1}{\Theta}v_E\right)\times\frac{\partial f}{\partial\theta}=V_{poloidal}\times\frac{\partial f}{r\partial\theta}$$

The second term in the same paranthesis, $-\frac{e}{m}\frac{\partial\Phi}{\partial\theta}\frac{\partial f}{\partial v}$, is however energetic, it comes from the change in the velocity space produced by the electric force that acts on particles. The term becomes more explicit after taking into account the factors that in the original equations multiply the collision operator, in the RHS.

It will result later, from the comparison between Eqs.47 and 40 in **Hazeltine** that there is a factor missing at the denominator, v

$$\begin{aligned}
& \frac{\Theta v}{r}\times\xi\times\left(-\frac{e}{mv}\frac{\partial\Phi}{\partial\theta}\frac{\partial f}{\partial v}\right)\\
= & \frac{B_\theta}{B}\times v_{\parallel}\times\left(-\frac{e}{mv}\frac{\partial\Phi}{r\partial\theta}\right)\frac{\partial f}{\partial v}\\
= & \frac{v_\theta}{v}\times\frac{eE_\theta}{m}\frac{\partial f}{\partial v}
\end{aligned}$$

The *force* is composed of the electric field $-\frac{e}{m}\frac{\partial\Phi}{r\partial\theta}=\frac{eE_\theta}{m}$, like $\frac{e\mathbf{E}}{m}\frac{\partial f}{\partial v}$.

It is question of the *acceleration* of particles in the potential that is *variable on the surface.*

From the first paranthesis, we take now the second term

$$\frac{e}{m\Omega_p v}\frac{\partial\Phi}{\partial r}$$

and transfer the factors from the RHS, and apply it on the energetic term of the second paranthesis, including the velocity that was missing

$$\begin{aligned}
& \frac{\Theta v}{r}\times\frac{e}{m\Omega_p v}\frac{\partial\Phi}{\partial r}\times\left(-\frac{e}{mv}\frac{\partial\Phi}{\partial\theta}\frac{\partial f}{\partial v}\right)\\
= & \frac{B_p}{B}\frac{e}{m\Omega_p}\frac{\partial\Phi}{\partial r}\times\left(-\frac{e}{mv}\frac{\partial\Phi}{r\partial\theta}\frac{\partial f}{\partial v}\right)=\frac{e}{m\Omega}\frac{\partial\Phi}{\partial r}\times\frac{1}{v}\left(\frac{eE_\theta}{m}\frac{\partial f}{\partial v}\right)\\
= & \frac{1}{B}\frac{\partial\Phi}{\partial r}\times\frac{1}{v}\left(\frac{eE_\theta}{m}\frac{\partial f}{\partial v}\right)\\
= & -\frac{v_\theta}{v}\times\left(\frac{eE_\theta}{m}\frac{\partial f}{\partial v}\right)
\end{aligned}$$

It has precisely the same structure as the first term, arising from ξ.

We repeat the conclusions

Together, these two terms that are in the first paranthesis

$$\begin{aligned} & \frac{\Theta v}{r} \times \left(\xi + \frac{e}{m\Omega_p v} \frac{\partial \Phi}{\partial r} \right) \\ = & V_{poloidal} \times \frac{1}{r} \end{aligned}$$

compose a poloidal velocity $V_{poloidal}$ and when they act on

$$\frac{\partial f}{\partial \theta}$$

represent the convective displacement, like

$$\mathbf{v} \cdot \boldsymbol{\nabla} \sim V_{poloidal} \frac{\partial}{r \partial \theta}$$

Further, they compose the poloidal velocity and also act on the energetic term

$$\frac{\Theta v}{r} \times \left(\xi + \frac{e}{m\Omega_p v} \frac{\partial \Phi}{\partial r} \right) \left(-\frac{e}{mv} \frac{\partial \Phi}{\partial \theta} \frac{\partial f}{\partial v} \right)$$

and generate terms of the kind

$$-V_{poloidal} \times \left(\frac{eE_\theta}{m} \frac{\partial f}{\partial v} \right)$$

This means: the motion with velocity $V_{poloidal}$ against the poloidal electric field E_θ produces energetic changes in velocity space for the distribution function.

THE THIRD PARANTHESIS is a radial convective operator, transporting radially the gradient of the distribution function, $\sim v_r \frac{\partial}{\partial r}$. The radial convection is done by the (radial projection of the) neoclassical drift velocity $\mathbf{v}_D$ and by the electric velocity due to variation of $\Phi(r, \theta)$ on surface (*i.e.* $E_\theta \times B_T$).

Consider the velocity, after including the factors from the RHS

$$\begin{aligned} & \frac{\Theta v}{r} \times \frac{v}{\Omega_p} \left[\frac{1+\xi^2}{2} \frac{r}{R} \sin\theta + \frac{e}{mv^2} \frac{\partial \Phi}{\partial \theta} \right] \\ = & \frac{B_p}{B} \frac{1}{r} v \times v \frac{m}{eB_p} \frac{1}{v^2} r \left[\left(v_\parallel^2 + \frac{v_\perp^2}{2} \right) \frac{1}{R} \sin\theta + \frac{e}{m} \frac{\partial \Phi}{r \partial \theta} \right] \end{aligned}$$

the first part is

$$\begin{aligned} & \frac{m}{eB} \times \left(v_\parallel^2 + \frac{v_\perp^2}{2} \right) \frac{1}{R} \sin\theta \\ = & \frac{1}{\Omega} \left(v_\parallel^2 + \frac{v_\perp^2}{2} \right) \frac{1}{R} \sin\theta \\ = & \left[\mathbf{v}_{drift} \right]_{radial} \end{aligned}$$

The second

$$\begin{aligned}
& \frac{\Theta v}{r} \times v \frac{m}{eB_p} \frac{1}{v^2} r \times \frac{e}{m} \frac{\partial \Phi}{r \partial \theta} \\
= & \frac{1}{B} \frac{\partial \Phi}{r \partial \theta} \quad = -\frac{E_\theta}{B_T} \\
= & \left[\frac{1}{B} \frac{\partial \Phi}{r \partial \theta} \right]_{radial}
\end{aligned}$$

The two terms have the same nature,

$$\begin{aligned}
& \frac{\Theta v}{r} \times \frac{v}{\Omega_p} \left[\frac{1+\xi^2}{2} \frac{r}{R} \sin\theta + \frac{e}{mv^2} \frac{\partial \Phi}{\partial \theta} \right] \\
= & V_{radial}
\end{aligned}$$

coming from two sources: the radial projection of the neoclassical drift and respectively the radial velocity induced by the poloidal variation of the potential Φ.

Acting on the first term we get

$$V_{radial} \frac{\partial}{\partial r} f$$

is a radial convection.

The same terms that are in this paranthesis (and are equivalent with V_{radial}) act on the term that contains $\partial f / \partial v$, which is an energetic effect.

$$\begin{aligned}
& \frac{\Theta v}{r} \times \frac{v}{\Omega_p} \left[\frac{1+\xi^2}{2} \frac{r}{R} \sin\theta + \frac{e}{mv^2} \frac{\partial \Phi}{\partial \theta} \right] \left(-\frac{e}{mv} \frac{\partial \Phi}{\partial r} \frac{\partial f}{\partial v} \right) \\
= & V_{radial} \times \left(\frac{1}{v} \frac{eE_r}{m} \right) \frac{\partial f}{\partial v}
\end{aligned}$$

Two factors compose a derivative to the energy, since

$$\begin{aligned}
& \frac{eE_r}{m} \frac{\partial f}{\partial v} \\
= & \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} = \mathbf{v} \cdot \frac{\mathbf{F}}{m} \frac{\partial f}{\partial \epsilon}
\end{aligned}$$

where ϵ is the energy per unit mass. But

$$\frac{\partial}{\partial \epsilon} = \frac{\partial}{\partial v} \frac{\partial v}{\partial \epsilon}$$

with

$$\epsilon \sim \frac{v^2}{2}, \quad v = \sqrt{2\epsilon}, \; \frac{\partial v}{\partial \epsilon} = \sqrt{2} \frac{1}{2\sqrt{\epsilon}} = \sqrt{2} \frac{1}{2} \sqrt{\frac{2}{v^2}} = \frac{1}{2} 2 \frac{1}{v} = \frac{1}{v}$$

$$\frac{\partial f}{\partial \epsilon} = \frac{\partial v}{\partial \epsilon}\frac{\partial f}{\partial v} = \frac{1}{v}\frac{\partial f}{\partial v}$$

then

$$\begin{aligned}\mathbf{v}\cdot\frac{\mathbf{F}}{m}\frac{\partial f}{\partial \epsilon} &= v\frac{eE_r}{m}\left(\frac{1}{v}\right)\frac{\partial f}{\partial v} \\ &= \frac{eE_r}{m}\frac{\partial f}{\partial v}\end{aligned}$$

and the verification is OK. Indeed the full expression

$$V_{radial} \times \left(\frac{1}{v}\right)\frac{eE_r}{m}\frac{\partial f}{\partial v}$$

is

$$\frac{V_{radial}}{v} \times \left(\mathbf{a}\cdot\frac{\partial f}{\partial \mathbf{v}}\right)$$

The nature of this term is *energetic*: the motion in the radial direction, with V_{radial}, is doen against the radial electric field E_r and this means exchange of energy and modifications in the velocity space of f.

Who is producing the radial flow?

One part is from $\mathbf{v}_{D,radial}$ and the other is the combination $E_\theta \times B_T$. The small variation of the electrostatic potential on surface, $\widetilde{\Phi}(\theta)$, produces poloidal electric fields $\widetilde{E}(\theta)$ which combined with toroidal magnetic field B_T produce radial electric drifts. Togenther with the neoclassical drifts $\mathbf{v}_D$, the radial displacement is made against the radial electric field $\frac{d\Phi^{(0)}}{dr}$ and it modifies the energy.

The next term is

$$\frac{\Theta v}{r} \times \left[-\frac{\left(1-\xi^2\right)}{2}\left(\frac{r}{R}\sin\theta + \frac{2e}{mv^2}\frac{\partial \Phi}{\partial \theta}\right)\right]\frac{\partial f}{\partial \xi}$$

and it is energetic. We have

$$\begin{aligned}\xi &= \frac{v_\parallel}{v} = \frac{v_\parallel}{\sqrt{v_\parallel^2 + v_\perp^2}} = \frac{v_\parallel}{\sqrt{v_\parallel^2 + 2\mu B}} \\ v_\parallel^2\xi^2 + 2\mu B\xi^2 &= v_\parallel^2, \quad v_\parallel^2\left(1-\xi^2\right) = 2\mu B\xi^2\end{aligned}$$

$$v_\parallel = \sqrt{2\mu B}\frac{\xi}{\sqrt{1-\xi^2}}$$

$$\begin{aligned}\left.\frac{\partial v_\parallel}{\partial \xi}\right|_{\mathbf{x}=\mathbf{ct},\mu=ct} &= \sqrt{2\mu B}\frac{\sqrt{1-\xi^2} - \xi\frac{\frac{1}{2}(-2\xi)}{\sqrt{1-\xi^2}}}{1-\xi^2} \\ &= \sqrt{2\mu B}\frac{\sqrt{1-\xi^2} + \frac{\xi^2}{\sqrt{1-\xi^2}}}{1-\xi^2} = \sqrt{2\mu B}\frac{1-\xi^2+\xi^2}{\left(1-\xi^2\right)^{3/2}}\end{aligned}$$

$$\left.\frac{\partial v_{\|}}{\partial \xi}\right|_{\mathbf{x}=\mathbf{ct},\mu=ct} = v_{\perp}\frac{1}{\left(\frac{v_{\perp}^2}{v^2}\right)^{3/2}} = v\left(\frac{v^2}{v_{\perp}^2}\right)$$

We must return to ξ,

$$\left.\frac{\partial v_{\|}}{\partial \xi}\right|_{\mathbf{x}=\mathbf{ct},\mu=ct} = v\frac{v^2}{v^2-v_{\|}^2} = \frac{v}{1-\xi^2}$$

and

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial v_{\|}}\frac{\partial v_{\|}}{\partial \xi} = \frac{v}{1-\xi^2}\frac{\partial}{\partial v_{\|}}$$

Returning

$$\begin{aligned}
&\frac{\Theta v}{r} \times \left[-\frac{\left(1-\xi^2\right)}{2}\left(\frac{r}{R}\sin\theta + \frac{2e}{mv^2}\frac{\partial \Phi}{\partial \theta}\right)\right]\frac{\partial f}{\partial \xi} \\
=\ & \frac{B_p}{B}v\left[-\frac{1-\xi^2}{2}\left(\frac{1}{R}\sin\theta + \frac{2e}{m}\frac{1}{v^2}\frac{\partial \Phi}{r\partial \theta}\right)\right]\frac{v}{1-\xi^2}\frac{\partial f}{\partial v_{\|}} \\
=\ & \left(-\frac{B_p}{B}v^2\frac{\sin\theta}{2R} + \frac{B_p}{B}\frac{eE_\theta}{m}\right)\frac{\partial f}{\partial v_{\|}}
\end{aligned}$$

If we take the relationship between velocity components as

$$\frac{B_p}{B}v_\theta = v_{\|}$$

then

$$\frac{\partial}{\partial v_{\|}} = \frac{\partial}{\partial v_\theta}\frac{\partial v_\theta}{\partial v_{\|}} = \frac{B}{B_p}\frac{\partial}{\partial v_\theta}$$

or

$$\frac{B_p}{B}\frac{\partial f}{\partial v_{\|}} = \frac{\partial f}{\partial v_\theta}$$

The term becomes

$$\left(-v^2\frac{\sin\theta}{2R} + \frac{eE_\theta}{m}\right)\frac{\partial f}{\partial v_\theta}$$

it is, approximately, the "accelerations" that modify the energy on poloidal direction:

- the neoclassical drift, projected on the poloidal direction
- the electric field that results from the poloidal variation of the electrostatic potential $\widetilde{\Phi}(\theta)$.

The electric field is taken as

$$\begin{aligned}
E_\varphi &= \frac{E_0}{h} \\
\text{where } \frac{\partial E_0}{\partial \theta} &= 0
\end{aligned}$$

The third line in the kinetic equation is an energetic term, it contains $\frac{\partial f}{\partial \xi}$.

To solve the equation for the distribution function

$$f = f_0 + f_1 + ...$$

The lowest order is zero

$$f_0 = f_M = \frac{n_0(\theta)}{\left(\sqrt{\pi} v_{th}\right)^3} \exp\left(-\frac{v^2}{v_{th}^2}\right)$$

In contrast with basic cases, here the distribution function is Maxwellian in velocity space but has spatial variation due to the variation of the density, $n^{(0)}(\theta)$. This variation is assumed a Boltzmannian response of the density to the electrostatic potential on the surface $\widetilde{\Phi}(\theta)$.

The density

$$\frac{\partial f_0}{\partial \theta} = \frac{e}{mv} \frac{\partial \Phi}{\partial \theta} \frac{\partial f_0}{\partial v}$$

$$\frac{\widetilde{n}_0(\theta)}{\overline{n}_0(r)} = -\frac{e\widetilde{\Phi}}{T}$$

(Boltzmann distribution)

6.4 The collision operators Hazeltine Ware electrostatic trapping

See also *plasma, general, impurities* **DIGRESSION**, compared with **Connor 1973**.

The collisions of electrons with the ions and with the impurities The operator is

$$C_{ei} + C_{eZ} = \frac{1}{\tau_e} \frac{3\sqrt{\pi}}{8} \frac{n_e}{\overline{n}_e} Z_{eff} \frac{v_{th,e}^3}{v^3} \left[\frac{\partial}{\partial \xi}\left(1-\xi^2\right)\frac{\partial f_e}{\partial \xi} + \frac{4u_d \xi v}{v_{th,e}^2} f_{Me}\right]$$

where

$$\tau_e \equiv \text{electron collision time for a pure plasma}$$

$$\tau_e = \frac{3}{16\sqrt{\pi}} \frac{m_e^2}{e^4} \frac{1}{\ln\Lambda} \frac{v_{th,e}^3}{\overline{n}_e}$$

We **NOTE** the classical structure,

$$\frac{1}{\tau_e} \sim \frac{n}{T^{3/2}}$$

and

$$Z_{eff} = \frac{n_i + Z^2 n_Z}{n_e}$$

$$
\begin{aligned}
u_d &\equiv \text{center of charge velocity} \\
&= \frac{n_i V_{\|i} + Z^2 n_Z V_{\|Z}}{n_i + Z^2 n_Z}
\end{aligned}
$$

We note that u_d is a *parallel* velocity.

There is a simplifying approximation

$$u_d = V_{\|Z} = u$$

Remark in **Hazeltine Ware**:

"the term from the operator C_e which involves u acts as a driving inhomogeneous term in the equation for the distribution function f". This part (containing u) is separated

$$C_e = C_0\left(f_e\right) + C_*$$

where

$$C_* = \frac{1}{\tau_e} \frac{3\sqrt{\pi}}{2} \frac{n_e Z_{eff}}{\overline{n}_e} \frac{v_{th,e} u \xi}{v^2} f_{Me}$$

where

$$
\begin{aligned}
C_0 &\equiv \text{the homogeneous collision operator} \\
\text{after taking } u_d &= 0
\end{aligned}
$$

which means without the displacement of the Maxwellian.

The ion collision operator The collisions between ions and impurities

$$C_{iZ} = \frac{1}{\tau_i} \frac{3\sqrt{2\pi}}{8} \frac{Z^2 n_Z}{\overline{n}_i} \frac{v_{th,i}^3}{v^3} \left[\frac{\partial}{\partial \xi}\left(1-\xi^2\right)\frac{\partial f_i}{\partial \xi} + \frac{4u\xi v}{v_{th,i}^2} f_{Mi}\right]$$

where

$$\tau_i = \tau_e \times \sqrt{2}\sqrt{\frac{m_i}{m_e}} \left(\frac{T_i}{T_e}\right)^{3/2} \frac{\overline{n}_e}{\overline{n}_i}$$

The collisions between the ions and the electrons

$$C_{ie} = -\frac{\overline{F}_{ei}}{p_i} f_{Mi} v \xi$$

with

$$
\begin{aligned}
\overline{F}_{ei} &\equiv \int d^3v \; m_e v \xi \; C_{ei} \\
&\quad \text{electron-ion friction force} \\
&\quad \text{in the parallel direction}
\end{aligned}
$$

We **NOTE** from above that C_{ie} is proportional with the *parallel* velocity. And $\overline{F}_{ei}$ is the force that result from the *parallel* exchange of momentum. **END.**

The momentum balance for the electrons

$$\begin{array}{rcl}\overline{F}_e & = & \left(1+\frac{Z^2\overline{n}_Z}{\overline{n}_i}\right)\overline{F}_{ei} \approx e\overline{n}_e E_0 \\ \text{(collision force)} & \sim & \text{(electric force)}\end{array}$$

Then the term C_{ie} can be combined with the term that contains E_0, leading to the replacement

$$E_* = E_0\left(1-\frac{1}{Z_{eff}}\right) \quad \text{instead of} \quad E_0$$

NOTE
A similar definition

$$\nu_{kj} \leftrightarrow$$

6.5 Kinetic equation, electrons

Return to kinetic. We take the *first order*.

$$\begin{aligned} & \xi\left(\frac{\partial f_1}{\partial\theta}+\frac{e}{mv}\frac{\partial\Phi}{\partial\theta}\frac{\partial f_1}{\partial v}\right) \\ & -\frac{1-\xi^2}{2}\left(\frac{r}{R}\sin\theta+\frac{2e}{mv^2}\frac{\partial\Phi}{\partial\theta}\right)\frac{\partial f_1}{\partial\xi}-\frac{r}{\Theta v}C_0\left(f_1\right) \\ = \; & \frac{r}{\Theta v}C_* \\ & +\frac{e}{T}\frac{r\xi}{\Theta}\frac{E_0}{h}f_0 \\ & +\frac{v}{\Omega_p}\left[\frac{1+\xi^2}{2}\frac{r}{R}\sin\theta+\frac{e}{mv^2}\frac{\partial\Phi}{\partial\theta}\right]\left(\frac{\partial f_0}{\partial r}-\frac{e}{mv}\frac{\partial\Phi}{\partial r}\frac{\partial f_0}{\partial v}\right)\end{aligned}$$

We note that the first paranthesis is the *convection* in the poloidal direction of the perturbation f_1.

$$\begin{aligned} & \frac{\Theta v}{r}\times\left(\frac{\partial f_1}{\partial\theta}+\frac{e}{mv}\frac{\partial\Phi}{\partial\theta}\frac{\partial f_1}{\partial v}\right) \\ = \; & \frac{B_p}{B}v_\parallel\left(\frac{\partial f_1}{r\partial\theta}+\frac{e}{mv}\frac{\partial\Phi}{r\partial\theta}\frac{\partial f_1}{\partial v}\right)\end{aligned}$$

Consider that the parallel velocity is the largest.

$$v_\theta = v_\parallel\cos\alpha = v_\parallel\sin\beta = v_\parallel\frac{B_p}{B}$$

then

$$v_\theta \left(\frac{\partial f_1}{r\partial\theta} + \frac{e}{mv} \frac{\partial \Phi}{r\partial\theta} \frac{\partial f_1}{\partial v} \right)$$

$$v_\theta \frac{\partial f_1}{r\partial\theta} + \frac{v_\theta}{v} \times \frac{eE_\theta}{m} \frac{\partial f_1}{\partial v}$$

The first term is a convection.

The second term is energetic and expresses the change in the velocity space generated by the accelerations produced by the poloidal electric field.

The other term that includes action of the perturbed f_1 is

$$\frac{\Theta v}{r} \times \left[-\frac{1-\xi^2}{2} \left(\frac{r}{R} \sin\theta + \frac{2e}{mv^2} \frac{\partial \Phi}{\partial\theta} \right) \frac{\partial f_1}{\partial \xi} \right]$$

and is *energetic*.

It consists of the accelerations produced by the *neoclassical drifts* and by the *poloidal electric field* on the distribution function on $v_\parallel$ *i.e.* $f_1(\xi)$.

The rest is *source* and involves the Maxwellian (zero-order) distribution function.

6.5.1 The parallel flow in the Maxwellian

The term

$$\frac{r}{\Theta v} C_*$$

is treated by introducing a *displacement* to the Maxwellian, which means a *flow* $v_\parallel$,

$$f_{displ} = \frac{2uv\xi}{v_{th}} f_0$$

with

$$C_0(f_{displ}) = -C_*$$

6.5.2 The induction electric field part of the distribution function (Spitzer Harm)

For the second term

$$\frac{e}{T} \frac{r\xi}{\Theta} \frac{E_0}{h} f_0$$

one needs to introduce the Spitzer function f_s, by the definition

$$\overline{C}_0(f_s) = -\frac{ev\xi}{T} E_0 f_M$$

with $\overline{C}_0$ being the θ averaged operator

$$f_s = -\frac{ev_{th,e}}{T_e} E_0 \tau_e \; f_{Me} \xi S_e \left(\frac{v}{v_{th,e}} \right)$$

The function S_e has been calculated and is in Tables.

The Spitzer-Harm function can be represented in terms of polynomials

$$S_e = A\left(Z_0\right) \left(\frac{v}{v_{th,e}} \right)^2 \left[1 + a\left(Z_0\right) \left(\frac{v}{v_{th,e}} \right)^2 \right]$$

where the functions A and a are tabulated.

6.5.3 The rest of the distribution function

The part of the collision operator that depends on θ is

$$\widetilde{C}_0\left(f_s\right) = \frac{3\sqrt{\pi}}{2} \frac{v_{th}^2}{v^3} \frac{\widetilde{n}_e Z_{eff}}{\overline{n}_e} \frac{eE_0}{m_e} f_{0e} \; \xi S_e$$

Return to the equation

$$\begin{aligned} & C_0\left(f_1\right) + ev\xi E_0 \frac{1}{T_e} f_{0e} + C_* \\ = \; & C_0\left(f_1 - f_d\right) - \overline{C}_0\left(f_s\right) \\ = \; & \overline{C}_0\left(g\right) + \widetilde{C}_0\left(f_s\right) + ...O\left(r^2/R^2\right) \end{aligned}$$

with the definition

$$\begin{aligned} g &= f_1 - f_s - f_d \\ &\sim O\left(\frac{r}{R}\right) \end{aligned}$$

The collision operator

$$\begin{aligned} & \widetilde{C}_0\left(f_s\right) \\ = \; & \text{driving term for } g \end{aligned}$$

Then

$$g_e = \int^{\theta} d\theta' \; \frac{r}{\Theta v \xi} \widetilde{C}_0\left(f_s\right) + g_p$$

According to **HW** the function g_p is the perturbation that results from all driving terms in the kinetic equation, which are different of $\widetilde{C}\left(f_s\right)$.

In velocity space g_p has a localized form, in a region where the slow electrons are resonant

$$\xi \sim \widehat{\nu}^{1/3}$$

To solve the kinetic equation: the collisions suppress the singularity $v_{\|} = 0$,

Kinetic function for electrons To lowest order in r/R,

$$\begin{aligned} f_{1e} &= f_{de} \quad \text{(displacement with Maxwellian)} \\ &\quad +f_{se} \quad \text{(Spitzer)} \\ &\quad +g_e \quad \text{(what remains)} \end{aligned}$$

or

$$\begin{aligned} f_{1e} &= \text{displacement term} \\ &\quad + \text{Spitzer function} \\ &\quad + \text{solution of drift-kinetic eq.} \end{aligned}$$

$$\frac{g_e}{f_{Me}} = \pi\delta\left(\xi\right) G_e + P\left(\frac{1}{\xi}\right) \int^{\theta} d\theta' G_e\left(\theta'\right)$$

The function G_e is

$$\begin{aligned} & G_e\left(\theta\right) \\ = \; & \frac{3\sqrt{\pi}}{4}\frac{r}{\Theta}\frac{\widetilde{n}_e Z_{eff}}{\overline{n}_e}\frac{eE_0}{T_e}\left(\frac{v_{th,e}}{v}\right)^4 S_e\xi \\ & +\frac{r}{R}\frac{1}{v}\left\{\left(1+\xi^2\right)\left(\frac{v^2}{v_{th,e}^2}\sin\theta - \kappa\frac{\partial x}{\partial\theta}\right)\left(\overline{u} + \frac{T_e}{m_e}\frac{f_{se}}{\xi v\, f_{Me}}\right)\right. \\ & -\left[\frac{v^2}{v_{th,e}^2}\left(1+\xi^2\right)\sin\theta - \kappa\frac{\partial x}{\partial\theta}\right]\left[u_{pe} + u_E + \frac{1}{eB_p}\frac{dT_e}{dr}\left(\frac{v^2}{v_{th,e}^2} - 3\right)\right] \\ & \left. -2\xi^2\left(\frac{R}{r}\frac{\partial\overline{u}}{\partial\theta}\frac{v^2}{v_{th,e}^2} + \frac{\kappa}{2}\overline{u}\frac{\partial x}{\partial\theta}\right)\right\} \end{aligned}$$

The velocities

$$\begin{aligned} u &= \overline{u} + \widetilde{u} \\ &= V_{\parallel Z} \end{aligned}$$

and

$$\kappa \equiv \frac{T_i}{T_e}$$

$$x \equiv \frac{R}{r}\frac{e\widetilde{\Phi}}{T_i}$$

$$u_E = -\frac{1}{B_p}\frac{d\overline{\Phi}}{dr} \quad \text{toroidal velocity due to radial } \Phi \text{ gradient}$$

$$\begin{aligned} u_{pe} \;\equiv\; & \frac{T_e}{eB_p}\left(\frac{d}{dr}\ln p_e + \frac{1}{2}\frac{d}{dr}\ln T_e\right) \\ & \text{parallel velocity whose poloidal projection} \\ & \text{is the diamagnetic velocity} \end{aligned}$$

Kinetic function for H ions Similar

$$\begin{aligned} f_{1i} &= f_{di} \quad \text{(displacement in the Maxwellian)} \\ &\quad + f_{si} \quad \text{(Spitzer standard)} \\ &\quad + g_i \end{aligned}$$

where

$$\begin{aligned} f_{di} &= \frac{2uv\xi}{v_{th,i}} f_{Mi} \\ &\quad \text{displacement} \quad \left(\sim uv_{\parallel}\right) \end{aligned}$$

$$\begin{aligned} f_{si} &= \frac{1}{\sqrt{2}} \tau_i \; v_{th,i} \frac{eE_*}{T_i} f_{Mi} S_i \xi \\ &\quad \text{Spitzer} \end{aligned}$$

Now there are two functions $S_{e,i}$ related to the Spitzer problem.
Here it is question of the *ion Spitzer* problem.
They can be expressed through a single function S,

$$S\left(\frac{v}{v_{th}}, Z\right) \quad \text{tabulated}$$

then

$$\begin{aligned} S_e\left(\frac{v}{v_{th,e}}\right) &= S\left(\frac{v}{v_{th,e}}, Z_{eff}\right) \\ S_i\left(\frac{v}{v_{th,i}}\right) &= S\left(\frac{v}{v_{th,i}}, \frac{Z^2 n_Z}{n_i}\right) \end{aligned}$$

A similar velocity

$$\begin{aligned} u_{pi} &= -\frac{T_i}{eB_p}\left(\frac{d}{dr}\ln p_i + \frac{1}{2}\frac{d}{dr}\ln T_i\right) \\ &\quad \text{parallel velocity whose poloidal projection} \\ &\quad \text{is diamagnetic velocity} \end{aligned}$$

The perturbation is

$$\frac{g_i}{f_{Mi}} = \pi\delta\left(\xi\right) G_i + P\left(\frac{1}{\xi}\right) \int^{\theta} d\theta' \; G_i\left(\theta'\right)$$

The expression of the function $G_i(\theta)$ is

$$\begin{aligned}
& G_i(\theta) \\
= \; & -\frac{3\sqrt{\pi}}{4}\frac{r}{\Theta}\frac{Z^2\widetilde{n}_Z}{\overline{n}_i}\frac{eE_*}{T_i}\frac{\xi}{(v/v_{th,i})^4}S_i \\
& +\frac{r}{R}\frac{1}{v}\left\{\left(1-\xi^2\right)\left(\frac{v^2}{v_{th,i}^2}\sin\theta+\frac{\partial x}{\partial\theta}\right)\left(\overline{u}+\frac{T_i}{m_i}\frac{f_{si}}{f_{Mi}}\right)\right. \\
& -\left[\left(1+\xi^2\right)\frac{v^2}{v_{th,i}^2}\sin\theta+\frac{\partial x}{\partial\theta}\right]\left[u_{pi}+u_E-\frac{1}{eB_p}\frac{dT_i}{dr}\left(\frac{v^2}{v_{th,i}^2}-3\right)\right] \\
& \left.-2\xi^2\left[\frac{R}{r}\frac{v^2}{v_{th,i}^2}\frac{\partial\widetilde{u}}{\partial\theta}-\frac{1}{2}\overline{u}\frac{\partial x}{\partial\theta}\right]\right\}
\end{aligned}$$

The ion's velocity

$$n_iV_{\|i}=\int d^3v\ v\xi\ f_{1i}$$

To this velocity f_{si} contributes a term of order $(m_e/m_i)^{1/2}$; and g_i contributes only a term of order (r/R).

$$\overline{V}_{\|i}\approx\overline{u}=\overline{V}_{\|Z}$$

6.6 The fluxes

The <u>*radial*</u> *guiding center drift velocity* is

$$v_d^{radial}=-\frac{T_a}{e_aB_p}\frac{1}{R}\left[\frac{v^2}{v_{th,e}^2}\left(1+\xi^2\right)\sin\theta+h\frac{\partial x}{\partial\theta}\right]$$

Comment We recall the definition of x

$$x\equiv\frac{R}{r}\frac{e\widetilde{\Phi}}{T_i}=\frac{\frac{e\widetilde{\Phi}}{T_i}}{\varepsilon}$$

and the result for x (from the *parallel* momentum conservation, converted to θ, $e\widetilde{\Phi}/T_i$, after integration over θ and inserting $\widetilde{u}$, using neutrality and Boltzmann distribution for densities)

$$\begin{aligned}
x \;=\; & \frac{m_Z}{ZT_i}\left\{\left[u_E^2+(\overline{u}-u_E)^2\right]\cos\theta\right. \\
& \left.+\left[\left((\overline{u}-u_E)^2-\frac{5}{3}\right)\frac{T_i}{m_Z}\right]\frac{R}{r}\frac{\widetilde{n}_Z}{\overline{n}_Z}\right\}
\end{aligned}$$

the term

$$
\begin{aligned}
& -\frac{T_a}{e_a B_p}\frac{1}{R}h\frac{\partial x}{\partial\theta} \\
= & -\frac{T_a}{e_a B_p}\frac{1}{R}h\frac{R}{r}\frac{e}{T_i}\frac{\partial\widetilde{\Phi}}{\partial\theta} \\
= & -\frac{B}{B_\theta}h\left[\frac{1}{B}\frac{\partial\widetilde{\Phi}}{r\partial\theta}\right] = -h\frac{B}{B_\theta}\widetilde{E}_\theta
\end{aligned}
$$

Remark that

$$
\begin{aligned}
\frac{T_a}{e_a B_p}\frac{1}{R}\frac{v^2}{v_{th,e}^2}\left(1+\xi^2\right)\sin\theta &= \frac{T_a/m_a}{\Omega_{a\theta}}\frac{1}{v_{th,e}^2}\frac{1}{R}v^2 2\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{v^2}\sin\theta \\
&= \frac{B}{B_\theta}\frac{1}{\Omega_a}\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{R}\sin\theta \quad \left(\text{since } \frac{2T_a}{m_a} = v_{th,a}^2\right) \\
&= \frac{B}{B_\theta}v_{drift}\sin\theta \\
&= \frac{B}{B_\theta}\left[\mathbf{v}^{drift}\right]_{radial}
\end{aligned}
$$

The stress anisotropy is

$$
\begin{aligned}
\mathbf{P}_\parallel - \mathbf{P}_\perp &= \int d^3v \; f \;\; \frac{mv^2}{2}\left(3\xi^2-1\right) \\
&= \int d^3v \; f \;\; \frac{mv^2}{2}\frac{2v_\parallel^2 - v_\perp^2}{v^2}
\end{aligned}
$$

The flux is

$$
\Gamma_a = \int \frac{d\theta}{2\pi}h\int d^3v \; v_{D,r} \; f_a
$$

the flux is the surface average of the flow generated by the neoclassical drift of particles.

$$
\Gamma = \Gamma_{PS} + \Gamma_{neo} + \Gamma_E
$$

where

$$
\Gamma_{PS} = -\frac{1}{eB_0}\overline{h^2\left(\frac{\partial\mathbf{P}_\parallel}{r\partial\theta} + e\overline{n}\frac{\partial\Phi}{r\partial\theta}\right)}
$$

$$
\Gamma_{neo} = \frac{1}{eB_0}\frac{1}{R}\overline{\sin\theta\left(\mathbf{P}_\parallel - \mathbf{P}_\perp\right)}
$$

which is valid in plateau or banana regimes.

$$
\Gamma_e = -\frac{1}{B_0}\overline{h^2\widetilde{n}\frac{\partial\widetilde{\Phi}}{r\partial\theta}}
$$

(**note** bar is the integral over θ).

This is the electrostatic flux. It results from the combination of the *radial velocity* produced by the poloidal electric field $\frac{\partial \widetilde{\Phi}}{r\partial\theta}$ associated to the nonuniformity on θ together with the main magnetic field, $\widetilde{E} \times B$ and the *variation* of the density, $\widetilde{n}(\theta)$. If there is an unfavorable phase between the radial velocity (which is static, as is $\widetilde{E}$) and $\widetilde{n}$ then the flux does not exist or is small.

The situation is similar to the Stringer effect.

But in this case $\widetilde{\Phi}$ is produced statically by the impurities and the ions, in contrast to Stringer where it is the *poloidal nonuniformity* of the radial diffusion flux which is combined with the neoclassical $\cos\theta$ factor to produce the *torque*.

The Pfirsch Schluter flux arises from the poloidal variation of the parallel stress, which is calculated in the order-1 expansion

$$\begin{aligned}
\mathbf{P}_{\parallel 1} &= \int d^3v \; m \; v_{\parallel} \; v_{\parallel} \;\; f_1 \\
&= \int d^3v \; mv^2\xi^2 f_1
\end{aligned}$$

and the poloidal variation is

$$\frac{\partial \mathbf{P}_{\parallel e}}{\partial\theta} = -\frac{3\sqrt{\pi}}{4}\frac{r}{\Theta}\frac{\widetilde{n}_e}{\overline{n}_e} Z_{eff}\frac{m_e e}{T_e} E_0 \int d^3v \left(\frac{v_{th,e}}{v}\right)^4 v^2\xi^2 f_{Me} \; S_e$$

Estimation

$$\Gamma_{PSe} = \frac{E_0}{B_{p0}}\overline{n}_e \frac{r}{R}\frac{n_{gc}}{\overline{n}_Z}$$

where

$$n_{gc} \equiv \cos\theta \text{ component of } \widetilde{n}_Z$$

and for ions

$$\Gamma^{(i)}_{PS} = \frac{E_*}{B_{p0}}\overline{n}_i \frac{r}{R}\frac{n_{gc}}{\overline{n}_Z}$$

Questions to be discussed.

Why the Pfirsch-Schluter *flux* of particle across the surfaces must include the toroidal external electric field E_0 ?

6.6.1 Poloidal variation of the ion density $\widetilde{n}_i$

Using the equation for the function G_e one can calculate

$$\begin{aligned}
\widetilde{n}_i &= \widetilde{n}_0 \\
&\quad + \sqrt{\pi}\frac{1}{v_{th,i}}\overline{n}_i \left\{ \Delta u_i \left(\sin\theta + \frac{\partial x}{\partial\theta}\right) - \frac{1}{eB_p}\frac{dT_i}{dr}\left(\sin\theta + 2\frac{\partial x}{\partial\theta}\right) \right. \\
&\qquad \left. + \frac{eE_*\tau_i\overline{n}_i}{\sqrt{2}m_i Z^2\overline{n}_Z}\left[\left(\alpha_{2i} + \frac{5}{2}\right)\sin\theta + \frac{\partial x}{\partial\theta}\right]\right\} \\
&\quad -3\alpha_{2i}\frac{eE_*}{T_i}\frac{r}{R}\overline{n}_i \frac{\partial}{\partial\theta}\left(\frac{\widetilde{n}_Z}{\overline{n}_Z}\right)
\end{aligned}$$

and

$$\Delta u_i = \overline{u} - u_{pi} - u_E$$

and α_{2i} is in α_2 from the Tables, calculated for

$$Z_* = \frac{Z^2 \overline{n}_Z}{\overline{n}_i}$$

The velocity Δu_i is the *parallel ion velocity* after elimination from it of the toridal velocity sustained by the radial gradients and B_θ and of the static electric potential $\overline{\Phi}(r)$ and B_θ. This remaining parallel velocity is indeed of the nature of *Pfirsch Schluter* and moreover results from the variation of the parameters on the surface.

6.6.2 The electrostatic flux of the ions

The *electrostatic flux* of the IONS is

$$\Gamma_E = \frac{1}{B_p} \overline{h^2 \widetilde{n} \frac{\partial \widetilde{\Phi}}{r \partial \theta}}$$

$$\begin{aligned}
\Gamma_{E,i} \quad = \quad & \frac{\sqrt{\pi}}{4} \rho_i \frac{r}{R^2} \overline{n}_i \left\{ \Delta u_i \left(x_c - 2\overline{x^2} \right) - \frac{1}{eB_p} \frac{dT_i}{dr} \left(x_c - 4\overline{x^2} \right) \right. \\
& \left. + \tau_i \frac{e}{\sqrt{2}} \frac{\overline{n}_i}{m_i Z^2 \overline{n}_Z} E_* \left[\left(\alpha_{2i} + \frac{5}{2} \right) x_c - 2\overline{x^2} \right] \right\} \\
& + 3 \frac{1}{B_p} \frac{r}{R} E_* \alpha_{2i} \frac{\overline{n_Z x}}{\overline{n}_Z} \overline{n}_i
\end{aligned}$$

6.6.3 The electrostatic flux of the electrons

This is calculated in a similar way

$$\begin{aligned}
\Gamma_{E,e} \quad = \quad & \frac{\sqrt{\pi}}{4} \rho_e \frac{r}{R^2} \overline{n}_e \kappa \left\{ \Delta u_e \left(x_c + 2\kappa \overline{x^2} \right) + \frac{1}{eB_p} \frac{dT_e}{dr} \left(x_c + 4\kappa \overline{x^2} \right) \right. \\
& \left. - \tau_e \frac{e}{m_e Z_{eff}} E_0 \left[x_c \left(\alpha_{2e} + \frac{5}{2} \right) + 2\kappa \overline{x^2} \right] \right\} \\
& - 3\alpha_{2e} \frac{E_0}{B_p} \frac{r}{R} \kappa \frac{\overline{n_Z x}}{\overline{n}_Z} \overline{n}_e
\end{aligned}$$

where

$$\kappa = \frac{T_i}{T_e}$$

and

$$\Delta u_e = \overline{u} - \overline{u}_{pe} - u_E$$

and α_{2e} is α_2 from the Table calculated for

$$Z_* = Z_{eff}$$

6.7 The Neoclassical fluxes

These fluxes are determined from the anisotropy of the pressure tensor.

The expression to be used is

$$\mathbf{P}_{\parallel} - \mathbf{P}_{\perp} = \int d^3v \ f \ \frac{mv^2}{2} \left(3\xi^2 - 1\right)$$

and

$$\Gamma_{neo} = \frac{1}{eB_0} \frac{1}{R} \overline{\sin\theta \left(\mathbf{P}_{\parallel} - \mathbf{P}_{\perp}\right)}$$

6.7.1 The neoclassical flux of the ions

This is

$$\begin{aligned} \Gamma_{neo,i} \quad &= \quad \frac{\sqrt{\pi}}{2} \rho_i \frac{r}{R^2} \overline{n}_i \left\{ \Delta u_i \left(1 - \frac{x_c}{2}\right) + \frac{1}{eB_p} \frac{dT_i}{dr} \frac{x_c}{2} \right. \\ & \left. +\tau_i \frac{1}{\sqrt{2}} \frac{e}{m_i} \frac{n_i}{Z^2 \overline{n}_Z} E_* \left[\alpha_{1i} - \frac{x_c}{2} \left(\alpha_{2i} + \frac{5}{2} \right) \right] \right\} \end{aligned}$$

6.7.2 The neoclassical flux of electrons

This is

$$\begin{aligned} \Gamma_{neo,e} \quad &= \quad \frac{\sqrt{\pi}}{2} \rho_e \frac{r}{R^2} \overline{n}_e \left\{ \Delta u_e \left(1 + \kappa \frac{x_c}{2}\right) + \frac{1}{eB_p} \frac{dT_e}{dr} \frac{x_c}{2} \right. \\ & \left. -\tau_e \frac{e}{m_e Z_{eff}} E_0 \left[\alpha_{1e} + \kappa \frac{x_c}{2} \left(\alpha_{2e} + \frac{5}{2} \right) \right] \right\} \end{aligned}$$

6.8 The current

The distribution function is defined in the framework of the Z-ion rest frame

$$\widehat{f}_a \quad , \ a = e, i$$

The current

$$(nu)_a = n_a u + \int d^3v \ v\xi \ \widehat{f}_{1a}$$

Now we return to the situation of equilibrium of a current flowing in a collisional plasma with the distribution function that verifies the Spitzer-Harm equilibrium

$$\overline{C}_0 \left(f_s\right) = -ev\xi \ E_0 \frac{f_M}{T}$$

with a formal expression of the Spitzer Harm function

$$\begin{aligned} f_s &= -v_{th,e}\ \tau_e\ \frac{eE_0}{T_e} f_{Me}\ \xi \\ & \times S_e\left(\frac{v}{v_{th,e}}\right) \end{aligned}$$

where S_e is in Tables.

This definition of a collisional equilibrium is taken as basis for extracting the expression for

$$v\xi = v_{\parallel}$$

or

$$v_{\parallel} = v\xi = -\frac{T}{eE_0}\overline{C}_0\left(f_s\right)\frac{1}{f_M}$$

It results

$$\left(nu\right)_a = n_a u - \frac{T_a}{e_a E_\varphi}\int d^3v\ \overline{C}_{0a}\left(f_{sa}\right)\ \frac{\widehat{f}_{1a}}{f_{0a}}$$

where $f_{0a} \equiv f_{Ma}$.

this can be transformed using the self-adjointness of $\overline{C}_{0a}$, as

$$\left(nu\right)_a = n_a u - \frac{T_a}{e_a E_\varphi}\int d^3v\ \frac{f_{sa}}{f_{0a}}\ \overline{C}_{0a}\left(\widehat{f}_{1a}\right)$$

The distribution function in the Z-ion frame is approximated as

$$\widehat{f}_{1a} = f_{1a} - \frac{2uv\xi}{v_{th,a}^2} f_{Ma}$$

They differ through the displacement part.

The collision operators

$$C\left(f_1\right) = \overline{C}_0\left(\widehat{f}\right) + \widetilde{C}\left(\widehat{f}\right)$$

and take

$$\overline{C}_0\left(\widehat{f}\right) = C\left(f_1\right) - \widetilde{C}\left(\widehat{f}\right)$$

This is inserted in the expression of the flux

$$\left(nu\right)_a = n_a u + \left(nu\right)_{1a} + \left(nu\right)_{2a}$$

The corresponding parts are

$$\left(nu\right)_{a1} = -\frac{T_a}{e_a E_\varphi}\int d^3v\ \frac{f_{sa}}{f_{0a}}\ C_{0a}\left(f_{1a}\right)$$

$$\left(nu\right)_{a2} = \frac{T_a}{e_a E_\varphi}\int d^3v\ \frac{f_{sa}}{f_{0a}}\ \widetilde{C}_{0a}\left(\widehat{f}_{1a}\right)$$

The parallel current is

$$J_{\parallel} = J_1 + J_2$$

where

$$J_{m=1,2} = e\left[(nu)_{i,m} - (nu)_{e,m}\right]$$

Now we must use the kinetic equation for the first order correction to the distribution function. From this equation we must extract the collision operator. From

$$\begin{aligned}
& \xi\left(\frac{\partial f_1}{\partial \theta} + \frac{e}{mv}\frac{\partial \Phi}{\partial \theta}\frac{\partial f_1}{\partial v}\right) \\
& -\frac{1-\xi^2}{2}\left(\frac{r}{R}\sin\theta + \frac{2e}{mv^2}\frac{\partial \Phi}{\partial \theta}\right)\frac{\partial f_1}{\partial \xi} - \frac{r}{\Theta v}C_0\left(f_1\right) \\
= \; & \frac{r}{\Theta v}C_* \\
& +\frac{e}{T}\frac{r\xi}{\Theta}\frac{E_0}{h}f_0 \\
& +\frac{v}{\Omega_p}\left[\frac{1+\xi^2}{2}\frac{r}{R}\sin\theta + \frac{e}{mv^2}\frac{\partial \Phi}{\partial \theta}\right]\left(\frac{\partial f_0}{\partial r} - \frac{e}{mv}\frac{\partial \Phi}{\partial r}\frac{\partial f_0}{\partial v}\right)
\end{aligned}$$

we extract

$$\begin{aligned}
& C_0\left(f_1\right) \\
= \; & \frac{\Theta v}{r} \times \left[\xi\left(\frac{\partial f_1}{\partial \theta} + \frac{e}{mv}\frac{\partial \Phi}{\partial \theta}\frac{\partial f_1}{\partial v}\right)\right. \\
& \left. -\frac{1-\xi^2}{2}\left(\frac{r}{R}\sin\theta + \frac{2e}{mv^2}\frac{\partial \Phi}{\partial \theta}\right)\frac{\partial f_1}{\partial \xi}\right] \\
& -\frac{e}{T}v\xi E_{\varphi} f_0
\end{aligned}$$

and the first component of flow is

$$\begin{aligned}
(nu)_1 \; = \; & -\frac{T_a}{e_a E_{\varphi}}\int d^3v \; \frac{f_{sa}}{f_{0a}} \\
& \times \frac{\Theta v}{r} \times \left[\xi\left(\frac{\partial f_1}{\partial \theta} + \frac{e}{mv}\frac{\partial \Phi}{\partial \theta}\frac{\partial f_1}{\partial v}\right)\right. \\
& \left. -\frac{1-\xi^2}{2}\left(\frac{r}{R}\sin\theta + \frac{2e}{mv^2}\frac{\partial \Phi}{\partial \theta}\right)\frac{\partial f_1}{\partial \xi}\right] \\
& -\frac{e}{T}v\xi E_{\varphi} f_0
\end{aligned}$$

Hazeltine comments that the terms multiplied by ξ are from Pfirsch Schluter current.

The last term gives the Spitzer current $J_{\parallel s}$.

7 The Pfirsch Schluter friction (Hazeltine Ware)

The paper follows the preceding one.

The collisional friction is

$$\mathbf{F}_a = \int d^3v \; m_a \mathbf{v} \; C_a(f)$$

it is the force resulting from momentum transfer via collisions, in every interval of time.

$$F_{\parallel} = \widehat{\mathbf{n}} \cdot \mathbf{F}$$

produces the neoclassical diffusion.

The operator of collision

$$\begin{aligned} C_{iZ} &= \frac{1}{\tau_i} \frac{3\sqrt{2\pi}}{8} \left(\frac{v_{th,i}}{v} \right)^3 \frac{Z^2 n_Z}{n_i} \\ &\quad \times \left[\frac{\partial}{\partial \xi} \left(1 - \xi^2\right) \frac{\partial f_i}{\partial \xi} + \frac{4\xi v u_Z}{v^2} f_{Mi} \right] \end{aligned}$$

where

$$\tau_i = \frac{3}{16} \sqrt{\frac{2}{\pi}} \frac{m_i^2}{e^4} \frac{1}{\ln \Lambda} \times \frac{v_{th,i}^3}{\overline{n}_i}$$

(with the known structure

$$\frac{1}{\tau_i} \sim \frac{n}{T^{3/2}}$$

).

And

$$u_Z \equiv \text{parallel velocity of impurity ions}$$

The distribution function

$$f_i = f_{Mi} \left(1 + \frac{2 u_Z \xi v}{v_{th,i}^2} \right) + g$$

The force of *friction* between ions and impurities is

$$\widetilde{F}_{i\parallel} = -\frac{1}{\tau_i} \frac{3\sqrt{2\pi}}{4} Z^2 n_Z m_i v_{th,i} \int_{-1}^{1} d\xi \; \xi \; \int_0^\infty ds \left[\frac{d}{ds} \operatorname{erf}(s) \right] g(s, \xi) \frac{1}{f_{Mi}}$$

with

$$s \equiv \frac{v}{v_{th,i}}$$

The equation for g is

$$\xi \frac{\partial g}{\partial \theta} - \frac{r}{\Theta v_{th,i}} \frac{1}{s} C_0(g) = G(\xi, s) f_{Mi}$$

From the preceding work

$$\begin{aligned}\left(\frac{R}{r}\right)G_1 &= s\times\frac{1}{v_{th,i}}\left\{\frac{1}{eB_\theta}\frac{dT_i}{dr}\left[\frac{\partial x}{\partial\theta}+\left(1+\xi^2\right)\left(s^2-3\right)\sin\theta\right]\right.\\ &\quad -\left(u_p+u_E\right)\left(1+\xi^2\right)\sin\theta\\ &\quad \left.+\overline{u}_Z\left(1-\xi^2\right)\sin\theta-2\xi^2\frac{\partial\overline{u}_Z}{\partial\theta}\frac{R}{r}\right\}\end{aligned}$$

$$\left(\frac{R}{r}\right)G_2=\frac{1}{s}\times\frac{1}{v_{th,i}}\left(\Delta u-\frac{3}{eB_\theta}\frac{dT_i}{dr}\frac{\partial x}{\partial\theta}\right)$$

where

$$u_\theta=-\frac{T}{eB_\theta}\left(\frac{d}{dr}\ln p+\frac{1}{2}\frac{d}{dr}\ln T\right)$$

$$u_E=\frac{1}{B_p}\left(-\frac{d\Phi}{dr}\right)$$

$$\Delta u=\overline{u}_Z-u_p-u_E$$

$$p=n_iT_i$$

The separation is made because

$$\begin{aligned}G_1 &\sim s \ \text{ as } \ s\to 0\\ G_2 &\sim \frac{1}{s} \ \text{ as } \ s\to 0\end{aligned}$$

The solution of the equation for g is

$$\begin{aligned}\frac{g_j}{f_{Mi}} &= \mathrm{P}\left(\frac{1}{\xi}\right)\int^\theta G_j d\theta'+\pi\delta\left(\xi\right)G_j\\ \text{for } j &= 1,2\end{aligned}$$

The singularity can be removed if there is a substantial toroidal velocity

$$\eta=\frac{u_E}{v_{th,i}}$$

which leads to a different form of the equation for the singular g_2,

$$\left(\xi+\frac{\eta}{s}\right)\frac{\partial g_2}{\partial\theta}-\frac{r}{\Theta v_{th,i}}\frac{1}{s}C_0\left(g_2\right)=G_2f_{Mi}$$

After solving for $g_{1,2}$ one can express the poloidally-varying friction force on ions

$$\begin{aligned}\widetilde{F}_{i\|} &= \nu_{iZ}\times n_im_i\frac{r}{R}\cos\theta\\ &\quad\times\left[\frac{1}{eB_\theta}\frac{dT_i}{dr}\left(4+3x^{\cos}\left(\frac{1}{2}-\gamma\right)\right)\right.\\ &\quad +2\left(u_p+u_E\right)-\overline{u}_Z\\ &\quad \left.-\frac{R}{r}\widetilde{u}_Z^{\cos}+\gamma x^{\cos}\left(\overline{u}_Z-u_E-u_p\right)\right]\end{aligned}$$

where γ is defined from the solution of equation for g_2.

8 Poloidal electric field in tokamak Chang

There are thw trappings

- magnetic, defined by
$$\lambda = \frac{\mu B_0}{\epsilon} = \frac{v_\perp^2}{v^2} h$$
- electrostatic, due to the variation of the potential in the magnetic surface
$$\phi = \langle \phi \rangle + \widetilde{\phi}(\theta)$$

The invariants

$$\begin{aligned} \epsilon &= \frac{v^2}{2} + \frac{q}{m}\phi \\ \mu &= \frac{v_\perp^2}{2B} \end{aligned}$$

for

$$\begin{aligned} q &= e > 0 \quad \text{for ions} \\ &= -e < 0 \quad \text{for electrons} \end{aligned}$$

When the trapping is *magnetic*, for reference it is adopted

$$\widetilde{\phi}(\theta = 0) = 0$$

The parameter λ-critic for trapping is modified

$$\lambda \geq \frac{B_0}{B(\pi)} \left[1 - \frac{q}{m} \frac{\phi(\theta = \pi)}{\epsilon} \right]$$

This is λ-crtitic

$$\begin{aligned} \lambda_{crit,B} \quad \equiv \quad & \frac{B_0}{B(\pi)} \left[1 - \frac{q}{m} \frac{\phi(\theta = \pi)}{\epsilon} \right] \\ & \text{energy-dependent} \end{aligned}$$

The boundary between the trapped and circulating particles is now dependent on the energy of the particles.

If the energy ϵ of the particle is very low

$$\frac{q}{m}\phi \leq \epsilon < \frac{q}{m}\widetilde{\phi}$$

the critical λ is zero and the particle is trapped (electrostatically) for any pitch angle.

The *electrostatic trapping* takes place at the low field side, *i.e.* inside the toroidal cross section when

$$\begin{aligned} e\widetilde{\phi}\,(\theta = \pi) &< e\phi\,(\theta = 0) \\ &\text{this locates the ION trapping} \\ &\text{in the region inside the torus} \end{aligned}$$

Then instead of the adoption of the previous reference $\widetilde{\phi}\,(\theta = 0) = 0$ in the case of *magnetic* trapping, now, for *electrostatic* trapping the adopted reference angle is

$$\phi\,(\pi) = 0$$

This means that at the outer region (low field side) of the poloidal cross section there is high electric potential, $\phi\,(\theta = 0)$ =high. The ions are repulsed from this region. [The electric field $E_\theta = -\frac{d\phi(\theta)}{dl_\theta}$ is directed from $\theta = 0$ toward $\theta = \pi$. Then the ions are pushed from the region $\theta \sim 0$ toward $\theta \sim \pi$. This is the trapping].

The conditions of trapping when

$$q\widetilde{\phi}\,(0) > q\widetilde{\phi}\,(\pi)$$

is

$$\begin{aligned} x_\parallel^2 &< \frac{r}{R_0}\left(X_0 - x_\perp^2\right)(1 - \cos\theta) \\ &\text{for } E \text{ trapping} \end{aligned}$$

and

$$\begin{aligned} x_\parallel^2 &< \frac{r}{R_0}\left(x_\perp^2 - X_0\right)(1 + \cos\theta) \\ &\text{for } B \text{ trapping} \end{aligned}$$

where

$$x \equiv \frac{v}{v_{th}}$$

and

$$\frac{q\widetilde{\phi}}{T} = \frac{r}{R}\,X_0 \cos\theta$$

When

$$q\widetilde{\phi}\,(0) < q\widetilde{\phi}\,(\pi)$$

there is onlt B trapping on the outside of the torus. The region of trapping is defined

$$x_\parallel^2 < \frac{r}{R_0}\left(x_\perp^2 + X_0\right)(1 + \cos\theta)$$

The kinetic equation

$$v_{\parallel}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}f+\mathbf{v}_D\cdot\boldsymbol{\nabla}f+qv_{\parallel}E_{\parallel}\frac{\partial f}{\partial\epsilon}=C\left[f\right]$$

It is assumed

$$\frac{e\widetilde{\phi}}{T} \quad \sim \quad \underset{\text{this is big}}{\frac{r}{R}}$$

The zeroth order

$$v_{\parallel}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}f_0=C\left[f_0\right]$$

with solution Maxwell Boltzmann

$$f_0=\frac{N}{\left(2\pi\frac{T}{m}\right)^{3/2}}\exp\left(-\frac{\epsilon}{T/m}\right)$$

the density is

$$N=\overline{n}\ \exp\left(\frac{q\left\langle\phi\right\rangle}{T}\right)$$

and

$$\begin{aligned}\epsilon &= \frac{v^2}{2}\\ &\quad +\frac{q}{m}\left\langle\phi\right\rangle+\frac{q}{m}\widetilde{\phi}\left(\theta\right)\end{aligned}$$

This expression of energy produces a *factor*

$$f_0\sim\exp\left[-\frac{\frac{q}{m}\widetilde{\phi}}{\frac{T}{m}}\right]$$

For the electrons

$$\begin{aligned}& v_{\parallel}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}f_{e1}+\mathbf{v}_D\cdot\boldsymbol{\nabla}f_{e0}+v_{\parallel}eE_{\parallel}\ \frac{1}{T_e}f_{e0}\\ =&\ \left(C_{ee}^{lin}+C_{ei}^{lin}\right)\ f_{e1}\end{aligned}$$

with

$$\mathbf{v}_D\cdot\boldsymbol{\nabla}\psi=Iv_{\parallel}\widehat{\mathbf{n}}\cdot\nabla\left(\frac{v_{\parallel}}{\Omega_e}\right)$$

Here occurs the distinction
the derivative of the distribution function is taken at
constant ϵ, and NOT at
constant $w=v^2/2$ *kinetic energy*

Substitution

$$\begin{aligned} f_{e1} &= \frac{2v_\parallel u_{i\parallel}}{v_{th,e}^2} f_{e0} \\ &- \frac{e}{T_e} f_{e0} \int_0^{l_{pol}} \frac{dl_{pol}}{B_{pol}} \left(BE_\parallel - \frac{B^2}{\langle B^2 \rangle} \langle BE_\parallel \rangle \right) \\ &+ v_\parallel \ f_{se} \left[\frac{B}{\langle B^2 \rangle} \langle E_\parallel B \rangle \right] \\ &+ H_e \end{aligned}$$

After the substitution, the new function H_e is represented in terms of *dynamic forces*, A_{ne}, as

$$H_e = \sum_{n=1}^{4} g_{ne} \ A_{ne}$$

Here $f_{se} \equiv$ the Spitzer function.
The substitution lead to

$$v_\parallel \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} g_{ne} - C_e^{lin} [g_{ne}] = -\alpha_n f_{e0}$$

The notations

$$\alpha_n \equiv v_\parallel \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \left(v_\parallel \ \gamma_n \right)$$

$$\gamma_1 \equiv -\frac{h}{|\Omega_{e,pol}|}$$

$$\gamma_2 \equiv \left(\frac{\epsilon^*}{T} - \frac{5}{2} \right) \gamma_1$$

$$\gamma_3 \equiv \frac{\widehat{f_{se}}}{h}$$

$$\begin{aligned} \gamma_4 &= -\frac{1}{h} |\Omega_{e,pol}| \\ & \text{the inverse of } \gamma_1 \end{aligned}$$

and

$$\epsilon^* = \epsilon - e \langle \phi \rangle = \frac{1}{2} v^2 + e \widetilde{\phi} (\theta)$$

The spatial derivation $\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}$ is done at *constant energy* ϵ^*.

9 Poloidal rotation Hassam Kulsrud

Also in *Stringer*.

9.1 Basic equations

The equations

$$\begin{aligned} nm_i\left(\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}\right) &= -\boldsymbol{\nabla}p-\boldsymbol{\nabla}\cdot\boldsymbol{\Pi} \\ & +\mathbf{j}\times\mathbf{B} \end{aligned}$$

where

$$\boldsymbol{\Pi}=-3\eta_0\left(\widehat{\mathbf{n}}\,\widehat{\mathbf{n}}-\frac{1}{3}\right)\left(\widehat{\mathbf{n}}\,\widehat{\mathbf{n}}-\frac{1}{3}\right):\boldsymbol{\nabla}\ \mathbf{v}$$

$$p=n\left(T_e+T_i\right)$$

The heat equation is expressed in terms of the *entropy*

$$nT\left(\frac{\partial s}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})\,s\right)=-\boldsymbol{\nabla}\cdot\mathbf{q}-\boldsymbol{\Pi}\colon\boldsymbol{\nabla}\ \mathbf{v}$$

with the flux of heat

$$\mathbf{q}=-\chi\frac{1}{B}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}T$$

note that this is *parallel*

$$q_\parallel=\chi_\parallel\frac{1}{B}\nabla_\parallel T$$

The Ohm's law

$$\mathbf{E}+\mathbf{v}\times\mathbf{B}=\eta\mathbf{j}$$

and

$$\begin{aligned} \boldsymbol{\nabla}\times\mathbf{B} &= \mu_0\mathbf{j} \\ \boldsymbol{\nabla}\times\mathbf{E} &= 0 \\ \boldsymbol{\nabla}\cdot\mathbf{B} &= 0 \end{aligned}$$

The magnetic field is (more general than circular)

$$\mathbf{B}=\left(0,\frac{b\,(r)}{h},\frac{B_0}{h}\right)$$

The following averaging operator is introduced

$$\langle f\rangle=\frac{\int\frac{dS}{|\boldsymbol{\nabla}p|}f}{\int\frac{dS}{|\boldsymbol{\nabla}p|}}$$

The equation of continuity

$$\frac{\partial}{\partial t}\langle n\rangle+\frac{1}{\int\frac{dS}{|\boldsymbol{\nabla}p|}}\frac{\partial}{\partial p}\int\mathbf{dS}\cdot n\mathbf{v}=0$$

The equation for the *circulation*

$$\begin{aligned}
& \frac{\partial}{\partial t}\left\langle \mathbf{v}\cdot\mathbf{B}\right\rangle + \frac{1}{\int \frac{dS}{|\boldsymbol{\nabla} p|}}\frac{\partial}{\partial p}\int \mathbf{dS}\cdot\mathbf{v}\ \mathbf{v}\cdot\mathbf{B} \\
= \quad & -\mathbf{v}\cdot\left\langle \boldsymbol{\nabla}\times\left(\eta\mathbf{j}-\widehat{\mathbf{e}}_z\frac{\mathcal{E}}{h}\right)\right\rangle \\
& -\left\langle \frac{3\eta_0}{nm_i}\mathbf{B}\cdot\boldsymbol{\nabla}\ln B\ \mathbf{v}\cdot\boldsymbol{\nabla}\ln B\right\rangle \\
& +\frac{1}{m_i}\sum_{e,i}\left\langle T\mathbf{B}\cdot\boldsymbol{\nabla} s\right\rangle
\end{aligned}$$

The equation for toroidal momentum

$$\frac{\partial}{\partial t}\left\langle nm_iRh\ \mathbf{v}\cdot\widehat{\mathbf{e}}_\varphi\right\rangle + \frac{1}{\int \frac{dS}{|\boldsymbol{\nabla} p|}}\frac{\partial}{\partial p}\int \mathbf{dS}\cdot\mathbf{v}\ \ \left(nm_iRh\ \mathbf{v}\cdot\widehat{\mathbf{e}}_\varphi\right) = 0$$

The equation for the *entropy* assuming that the species is adiabatic

$$\begin{aligned}
& \left\langle n\right\rangle\frac{\partial s}{\partial t} + \frac{1}{\int \frac{dS}{|\boldsymbol{\nabla} p|}}\int \mathbf{dS}\cdot\ n\mathbf{v}\left(\frac{\partial s}{\partial p}\right) \\
= \quad & \left\langle \chi\left(\frac{\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla} T}{T}\right)^2\right\rangle \\
& +\left\langle \frac{3\eta_0}{T}\left(\mathbf{v}\cdot\boldsymbol{\nabla}\ln B\right)^2\right\rangle
\end{aligned}$$

For the averaging operator we have

$$\frac{1}{\int \frac{dS}{|\boldsymbol{\nabla} p|}}\int \mathbf{dS}\cdot\mathbf{v}\ f = \frac{dp}{dr}\left\langle v_r\ f\right\rangle$$

For different functions f the quantity $\left\langle v_r\ f\right\rangle$ is derived by averaging the toroidal component of the Ohm's law.

Start from

$$\mathbf{E}+\mathbf{v}\times\mathbf{B}=\eta\mathbf{j}$$

with

$$\mathbf{E}=-\boldsymbol{\nabla}\phi+\frac{\mathcal{E}}{h}\widehat{\mathbf{e}}_\varphi$$

where an external, inductive, electric field is considered, $\mathcal{E}$, toroidal.

First we multiply by $\mathbf{B}$ the Ohm's law

$$\mathbf{E}\cdot\mathbf{B}=\eta j_\parallel\left|\mathbf{B}\right|$$

and replace

$$\begin{aligned}
\left(-\boldsymbol{\nabla}\phi+\frac{\mathcal{E}}{h}\widehat{\mathbf{e}}_\varphi\right)\cdot\mathbf{B} &= \eta j_\parallel\left|\mathbf{B}\right| \\
\mathcal{E}\frac{B_\varphi}{h} &= \eta j_\parallel\left|\mathbf{B}\right|
\end{aligned}$$

The magnitude is

$$|\mathbf{B}| = \frac{B_0}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}}$$

The equation becomes

$$\mathcal{E}\frac{B_0}{h^2} = \eta j_{\parallel}\frac{B_0}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}}$$

after averaging

$$\mathcal{E} = \frac{\left\langle \eta j_{\parallel}\frac{B_0}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}}\right\rangle}{\left\langle \frac{B_0}{h^2}\right\rangle}$$

the previously derived expression of the parallel current

$$j_{\parallel} = \left[-\frac{h}{B_0(r)}\frac{dp(r)}{dr}\right]\frac{q}{\varepsilon}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}$$

The numerator of the expression of $\mathcal{E}$ is the average of

$$\begin{aligned}
& \eta j_{\parallel}\frac{B_0}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}} \\
= \quad & \eta\frac{B_0}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}}\frac{q}{\varepsilon}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}h\left[-\frac{1}{B_0}\frac{dp}{dr}\right] \\
= \quad & \eta B_0\frac{q}{\varepsilon}\left[-\frac{1}{B_0}\frac{dp}{dr}\right]
\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{E} &= \frac{\left\langle \eta j_{\parallel}\frac{B_0}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}}\right\rangle}{\left\langle \frac{B_0}{h^2}\right\rangle} \\
&= \frac{1}{B_0}\frac{1}{\left\langle\frac{1}{h^2}\right\rangle}\eta B_0\frac{q}{\varepsilon}\left\langle -\frac{1}{B_0}\frac{dp}{dr}\right\rangle
\end{aligned}$$

Consider an arbitrary function f of plasma variables.

We take the φ (toroidal) component of the Ohm's law

$$\begin{aligned}
\frac{\mathcal{E}}{h} + (\mathbf{v}\times\mathbf{B_0})_{\varphi} &= \eta j_{\varphi} \\
\frac{\mathcal{E}}{h} + v_r B_{\theta} &= \eta j_{\varphi}
\end{aligned}$$

and multiply by

$$\frac{f}{B_{\theta}}$$

$$\eta j_\varphi \frac{f}{B_\theta} - \frac{\mathcal{E}}{h}\frac{f}{B_\theta} = v_r f$$

and average over surface

$$\left\langle \left(\eta j_\varphi - \frac{\mathcal{E}}{h} \right) \frac{f}{B_\theta} \right\rangle = \langle v_r f \rangle$$

Now we use

$$\frac{\varepsilon}{q} j_\varphi = -\frac{h}{B_0}\frac{dp(r)}{dr}$$

We return to

$$\begin{aligned} \langle v_r f \rangle &= \left\langle \left(\eta j_\varphi - \frac{\mathcal{E}}{h} \right) \frac{f}{B_\theta} \right\rangle \\ &= \eta \left\langle j_\varphi \frac{f}{B_\theta} \right\rangle - \left\langle \frac{\mathcal{E}}{h}\frac{f}{B_\theta} \right\rangle \end{aligned}$$

and take into account that $\mathcal{E}$ is already averaged.

$$\begin{aligned} \langle v_r f \rangle &= \eta \left\langle f \frac{1}{B_\theta} \frac{q}{\varepsilon} \left[-\frac{h}{B_0}\frac{dp(r)}{dr} \right] \right\rangle \\ & \quad -\mathcal{E} \left\langle f \frac{1}{hB_\theta} \right\rangle \end{aligned}$$

The second term is

$$-\mathcal{E} \left\langle f \frac{1}{hB_\theta} \right\rangle$$

where

$$\frac{1}{hB_\theta} = \frac{1}{h\frac{b(r)}{h}} = \frac{1}{b(r)}$$

and is factored out from the averaging. After this we replace the expression of $\mathcal{E}$,

$$-\mathcal{E}\frac{1}{b(r)} \langle f \rangle = -\frac{1}{B_0}\frac{1}{\left\langle \frac{1}{h^2} \right\rangle} \eta B_0 \frac{q}{\varepsilon} \left\langle -\frac{1}{B_0}\frac{dp}{dr} \right\rangle \frac{1}{b(r)} \langle f \rangle$$

and note that

$$\frac{B_0}{b} = \frac{B_0/h}{b/h} = \frac{B_\varphi}{B_\theta} = \Theta^{-1} = \frac{q}{\varepsilon}$$

We have

$$\begin{aligned} & -\mathcal{E}\frac{1}{b(r)} \langle f \rangle \\ &= \eta \frac{1}{B_0} \left(\frac{q}{\varepsilon} \right)^2 \frac{\left\langle \frac{1}{B_0}\frac{dp}{dr} \right\rangle}{\left\langle \frac{1}{h^2} \right\rangle} \langle f \rangle \end{aligned}$$

Working the first term we remind that

$$\frac{\varepsilon}{q} = \Theta = \frac{B_\theta}{B_\varphi}$$

$$\frac{1}{B_\theta} = \frac{q}{\varepsilon}\frac{1}{B_\varphi} = \frac{q}{\varepsilon}\frac{h}{B_0(r)}$$

then

$$\begin{aligned}
& f\frac{1}{B_\theta}\frac{q}{\varepsilon}\left[-\frac{h}{B_0}\frac{dp(r)}{dr}\right] \\
= \quad & f\frac{q}{\varepsilon}\frac{h}{B_0(r)}\frac{q}{\varepsilon}\left[-\frac{h}{B_0}\frac{dp(r)}{dr}\right] \\
= \quad & -f\left(\frac{q}{\varepsilon}\right)^2\frac{1}{B_0^2}\frac{dp}{dr}h^2
\end{aligned}$$

One can factorize from the averaging operator all factors that only depend on ψ (*i.e.* on the radius r).

$$\begin{aligned}
& \eta\left\langle f\frac{1}{B_\theta}\frac{q}{\varepsilon}\left[-\frac{h}{B_0}\frac{dp(r)}{dr}\right]\right\rangle \\
= \quad & -\left(\frac{q}{\varepsilon}\right)^2\frac{1}{B_0^2}\frac{dp}{dr}\left\langle fh^2\right\rangle \\
= \quad & -\eta\left(\frac{q}{\varepsilon}\right)^2\frac{1}{B_0}\left[\frac{1}{B_0}\frac{dp}{dr}\left\langle fh^2\right\rangle\right]
\end{aligned}$$

Finally, the expression of the average is

$$\begin{aligned}
& \left\langle fv_r\right\rangle \\
= \quad & -\eta\left(\frac{q}{\varepsilon}\right)^2\frac{1}{B_0}\left[\frac{1}{B_0}\frac{dp}{dr}\left\langle fh^2\right\rangle\right] \\
& +\eta\frac{1}{B_0}\left(\frac{q}{\varepsilon}\right)^2\frac{\left\langle\frac{1}{B_0}\frac{dp}{dr}\right\rangle}{\left\langle\frac{1}{h^2}\right\rangle}\left\langle f\right\rangle
\end{aligned}$$

or

$$\begin{aligned}
& \left\langle fv_r\right\rangle \\
= \quad & \eta\left(\frac{q}{\varepsilon}\right)^2\frac{1}{B_0}\left\{-\frac{1}{B_0}\frac{dp}{dr}\left\langle fh^2\right\rangle + \frac{\left\langle\frac{1}{B_0}\frac{dp}{dr}\right\rangle}{\left\langle\frac{1}{h^2}\right\rangle}\left\langle f\right\rangle\right\}
\end{aligned}$$

The order of magntitude is

$$\frac{dB}{dr} \sim \frac{dp}{dr} \sim b^2 \sim \frac{a^2}{R^2}$$

This must be taken as a basis for the averages that will involve a function f.

$$\begin{aligned} & \langle f v_r \rangle \\ = \quad & \eta \left(\frac{q}{\varepsilon} \right)^2 \frac{1}{B_0} \left[\frac{1}{B_0} \frac{dp}{dr} \left(- \langle f h^2 \rangle + \frac{\langle f \rangle}{\left\langle \frac{1}{h^2} \right\rangle} \right) \right] \end{aligned}$$

The reason to factor out the gradient of the pressure comes from the definition of the average

$$\frac{\int \mathbf{dS} \cdot \mathbf{v} \; f}{\int \frac{\mathbf{dS}}{|\boldsymbol{\nabla} p|}} = \frac{dp}{dr} \langle f v_r \rangle$$

We introduce the notations

$$v_D \equiv -\eta \frac{1}{B_0} \frac{dp}{dr}$$

It has similar parametric dependence as the diamagnetic velocity but contains the resistivity η. It is the *resistive classical flow.*Then

$$\langle f v_r \rangle = v_D \frac{1}{B_0} \left[\left(\frac{q}{\varepsilon} \right)^2 \left(\langle f h^2 \rangle - \frac{\langle f \rangle}{\left\langle \frac{1}{h^2} \right\rangle} \right) \right]$$

NOTE the presence of the *resistivity* as FACTOR to the entire expression, *i.e.* again we see that v_r owes its existence to the resistivity that, in the Ohm's law, introduces the imperfect neutralization via parallel currents of the charge separation induced by the different drifts of electrons and ions. (Stringer PRL). **END.**

9.2 Application to adapted new form of the averaged equations

The equation of continuity

$$\frac{\partial}{\partial t} \langle n \rangle + \frac{1}{\int \frac{dS}{|\boldsymbol{\nabla} p|}} \frac{\partial}{\partial p} \int \mathbf{dS} \cdot n \mathbf{v} = 0$$

means

$$\frac{1}{\int \frac{dS}{|\boldsymbol{\nabla} p|}} \int \mathbf{dS} \cdot n \mathbf{v} = \frac{dp}{dr} \langle n v_r \rangle$$

$$\frac{\partial n}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left(r n \langle v_r \rangle \right) = 0$$

To calculate this averaged over the surface we use the prefiously derived equation for

$$f = 1$$

and obtain

$$\langle v_r \rangle = v_D q^2 \frac{1}{\varepsilon^2} \left(\langle h^2 \rangle - \frac{1}{\left\langle \frac{1}{h^2} \right\rangle} \right)$$

and introduce the notation

$$\alpha_1 \equiv \frac{1}{2\varepsilon^2} \left(\langle h^2 \rangle - \frac{1}{\left\langle \frac{1}{h^2} \right\rangle} \right)$$

then

$$\langle v_r \rangle = v_D 2 q^2 \alpha_1$$

The equation of continuity becomes

$$\frac{\partial n}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left[rn \left(v_D 2 q^2 \alpha_1 \right) \right] = 0$$

We note that α_1 is close to 1.

In a similar way, it is obtained the time evolution of the toroidal component of the flow. Define

$$v_t \equiv \langle h v_\varphi \rangle$$

We must repeat the calculation made for v_r . The velocity v_φ is obtained from the φ projection of the Ohm's law, *i.e.* after multiplying it with $\widehat{\mathbf{e}}_\varphi$ we take the average. We will need the component j_φ of the current, already derived. Finally

$$\begin{aligned} v_t &\equiv \langle h v_\varphi \rangle \\ &= \frac{q}{\varepsilon} \left(v_p - v_E \langle h^2 \rangle \right) \end{aligned}$$

where

$$v_E = \frac{1}{B_0} \frac{\partial \phi}{r \partial \theta}$$

and the poloidal rotation velocity v_θ is expressed through the function v_p that only depends on the magnetic surface (ψ)

$$v_\theta = \frac{v_p(r)}{h}$$

The projection of the rotation velocity perpendicular on the magnetic line is

$$\begin{aligned} v_\perp &= v_E \frac{h}{\sqrt{1 + \frac{\varepsilon^2}{q^2}}} \\ &= \frac{1}{B_0/h} \frac{\partial \phi}{r \partial \theta} \frac{1}{\sqrt{1 + \frac{\varepsilon^2}{q^2}}} = \frac{\partial \phi}{r \partial \theta} \frac{1}{B} \end{aligned}$$

As before, together with (v_θ, v_φ) it is possible to work with $(v_\parallel, v_\perp)$.

The equation for a combination of v_p and v_t has been derived from the average of the equation for the *circulation* $\mathbf{v}\cdot\mathbf{B}$ by Hassam and Drake.

$$\begin{aligned}
& \frac{\partial}{\partial t}\left[v_t + \left(\alpha_3 + 2q^2\alpha_1\right)\frac{\varepsilon}{q}v_p\right] \\
& +\frac{1}{r}\frac{\partial}{\partial r}\left\{r\left[\left(v_D 2q^2\alpha_1\right)\left(v_t + \left(\alpha_3 + 2q^2\alpha_1\right)\frac{\varepsilon}{q}v_p\right)\right] - 2q^2\alpha_1\frac{v_p}{\varepsilon/q}\right\} \\
= & -\frac{3}{2}\alpha_4\frac{\eta_0}{nm_iR^2}\frac{\varepsilon}{q}v_p \\
& +\Xi
\end{aligned}$$

By Ξ we note the terms related to thermal diffusion of the adiabatic species of particles (electrons). The notations are

$$\begin{aligned}
\alpha_3 &= \left\langle \frac{1}{h^2}\right\rangle \\
\alpha_4 &= \left(2R^2\right)\left\langle \frac{1}{h^2}\left(\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}\frac{\partial}{r\partial\theta}\ln h\right)^2\right\rangle
\end{aligned}$$

An equation for v_t results from the momentum conservation projected along the toroidal direction then averaged

$$\frac{\partial}{\partial t}\langle nv_t\rangle + \frac{1}{r}\frac{\partial}{\partial r}\left(rn\left[\left(v_D 2q^2\alpha_1\right)v_t + v_D 2q^2\alpha_2\left(v_t - v_p\frac{q}{\varepsilon}\right)\right]\right) = 0$$

where

$$\alpha_2 = \frac{1}{2\varepsilon^2}\left[\langle h^4\rangle\frac{1}{\langle h^2\rangle} - \langle h^2\rangle\right]$$

9.3 Equation for poloidal rotation

The previous calculations allow to write down the equation for the evolution of the poloidaal velocity

$$\begin{aligned}
\left(1+\frac{1}{2q^2}\right)\frac{\partial \ln v_p}{\partial t} &= -\frac{q^2}{\varepsilon^2}v_D\frac{1}{n}\frac{dn}{dr} \\
& \quad -\frac{3}{4}\frac{\eta_0}{nm_iq^2R^2} + \Xi'
\end{aligned}$$

The symbol Ξ' is introduced to represent the effect of the thermal conductivity of the electrons

$$\Xi' \sim \chi_e$$

Since v_t is connected with v_p we have the equation for it

$$\frac{\partial v_t}{\partial t} = \frac{1}{2n}\frac{1}{r}\frac{\partial}{\partial r}\left(rnq^2v_D\frac{q}{\varepsilon}v_p\right)$$

10 Spontaneous poloidal rotation (instability)

10.1 Detailed treatment

The equation of continuity

$$\begin{aligned}\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) &= 0 \\ \frac{\partial n}{\partial t} + \mathbf{B} \cdot \nabla \left(\frac{nv_{\parallel}}{B}\right) + \nabla \cdot (n\mathbf{v}_{\perp}) &= 0\end{aligned}$$

As it is written it shows that we will calculate the parallel gradient of the parallel velocity.

The second equation

$$\begin{aligned}T\mathbf{B} \cdot \nabla n &= -nm_i \mathbf{B} \cdot \mathbf{v} : \nabla \mathbf{v} \\ -\mathbf{B}\nabla : \mathbf{\Pi} & \\ & -nm_i \frac{\partial (\mathbf{B} \cdot \mathbf{v})}{\partial t}\end{aligned}$$

We recognize here the momentum equation

$$nm_i \frac{\partial \mathbf{v}}{\partial t} + nm_i (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla (nT) - \nabla \cdot \mathbf{\Pi}$$

considering T constant, multiply by $\mathbf{B}$. This corresponds to the equation for the *circulation* $\mathbf{v} \cdot \mathbf{B}$.

The equation of Ohm, in the absence of resistivity

$$\eta \mathbf{j} = -\nabla \phi + \mathbf{v} \times \mathbf{B}$$

$\eta = 0$, multiplied by $\mathbf{B}$ is

$$\mathbf{B} \cdot \nabla \phi = 0$$

the potential is constant on magnetic surfaces.

$$\mathbf{B} \cdot \nabla \left(\frac{j_{\parallel}}{B}\right) = -\nabla_{\perp} \cdot \mathbf{j}_{\perp}$$

The perpendicular current is extracted from the equation of momentum. For this, in contrast to previous multiplication by $\mathbf{B}$ we multiply vectorially by $\mathbf{B}$

$$\mathbf{j}_{\perp} = \frac{1}{B^2} \mathbf{B} \times \left(T \nabla n + \nabla \cdot \mathbf{\Pi} + nm_i \frac{d\mathbf{v}}{dt}\right)$$

This is essentially the *diamagnetic current*.

the equilibrium is defined by the functions that are *flux-functions*

$$\begin{aligned} & n(r) \\ V_p(r) &= \langle v_\theta h \rangle \\ V_t &= \langle v_\varphi h \rangle \end{aligned}$$

The average over the flux surface is

$$\langle f \rangle = \int \frac{d\theta}{2\pi} \; h \; f$$

the equilibrium state means

$$\begin{aligned} \frac{\partial}{\partial t} &\equiv 0 \\ \mathbf{\Pi} &= 0 \quad \text{pressure is isotropic} \\ \mathbf{R}_\perp &= 0 \quad \text{no friction} \end{aligned}$$

the equations under this equilibrium assumption lead to

$$v_\theta = \frac{V_p(r)}{h}$$

$$\begin{aligned} v_\varphi \approx\; & V_t - 2qV_p \cos\theta \\ & + \varepsilon \left[V_t \cos\theta + 2qV_p \left(1 + \frac{1}{4} \cos 2\theta \right) \right] \end{aligned}$$

The first equation says that the poloidal rotation is the rotation uniform on surface V_p modulated by

$$h = 1 + \varepsilon \cos\theta$$

Then

$$\langle nRv_\varphi \rangle = nR_0 V_t$$

$$\left\langle v_\parallel \frac{B}{B_0} \right\rangle = V_t + \frac{\varepsilon}{q} \left(1 + 2q^2 \right) V_p$$

The equations

$$\frac{\partial n}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left(rn\overline{v}_r \right) = 0$$

$$\frac{\partial}{\partial t} \left[nV_t \right] + \frac{1}{r} \frac{\partial}{\partial r} \left(r \; n \left[V_t \overline{v}_r - qV_p \widetilde{v}_r \right] \right) = 0$$

$$\begin{aligned} & \frac{\partial}{\partial t} \left[V_t + \frac{\varepsilon}{q} \left(1 + 1q^2 \right) V_p \right] \\ & + \overline{v}_r \frac{\partial V_t}{\partial r} - \widetilde{v}_r \frac{\partial}{\partial r} \left[qV_p \right] \\ & + \text{ (magnetic pumping)} \\ = \; & 0 \end{aligned}$$

The velocity is associated to a flux of transport of particles, across the surfaces (in radial direction). The flux is generated by the collisional friction that acts perpendicular to the magnetic field line

$$nv_r = \frac{R_\perp}{|e| B}$$

Then the average and the variable part

$$\overline{v}_r = \langle v_r \rangle$$

and

$$\widetilde{v}_r = \langle 2\cos\theta \ v_r \rangle$$

The origin of the poloidal spin-up : *the existence of the variation of the flux of transport with poloidal angle*

Equivalently,

$$\widetilde{v}_r \neq 0$$

The equation

$$\begin{aligned} & \frac{\varepsilon}{q}\left(1+2q^2\right)\left(\frac{\partial V_p}{\partial t}+\gamma_{MP}V_p\right) \\ & +qV_p\frac{\partial}{\partial r}\left(r \ n\widetilde{v}_r\right) \\ = \ & 0 \end{aligned}$$

the logic of the instability that consists of poloidal spin-up

Assume there is a poloidal velocity.

Due to the toroidality and

$$\boldsymbol{\nabla}\cdot\mathbf{v}=0$$

the poloidal rotation (with compression - distension of volume alternatively in low-field and high-field sides) necessarly is accompanied by toroidal flows that ensure the preservation of the incompressibility.

the toroidal flows have a spatial distribution which is harmonic in the poloidal section. It is Pfirsch Schluter flow and current.

The friction $R_\perp$ is modulated in the surface by these flows.

The friction generates transport fluxes Γ_r which are themselves modulated in the surface but for reasons that are independent of the Pfirsch-Schluter harmonic flows. The radial velocity they induce is also modulated, it is

$$\begin{aligned} \Gamma_r &= nv_r \\ v_r &= -D\left(1+\delta\cos\theta\right) \\ & \quad -v_0\frac{r}{a} \end{aligned}$$

From the combination between the two independent poloidal modulations

$$\begin{array}{rcl} \Gamma_r & \sim & f(\theta) \\ \text{Pfrirsch-Schluter flow} & \sim & g(\theta) \end{array}$$

it is induced a variation of the radial velocity

$$\widetilde{v}_r = \langle 2\cos\theta \ v_r \rangle$$

This combination acts like a drive (a torque) in the equation for the poloidal velocity V_p (function of surface ψ).

The higher the angular matching between the poloidal variation of transport rate $\Gamma_r(\theta)$ with the harmonic Pfirsch Schluter flow $\cos\theta$, the higher the drive of poloidal rotation.

If the poloidal rotation is enhanced by this drive $qV_p \frac{\partial}{\partial r}(r \ n\widetilde{v}_r)$ then the amplitude of the harmonic compensatory Pfirsch Schluter flows increases

then the poloidal drive is still higher.

11 Spontaneous spin-up (Hassam Drake)

Also in **Stringer**.

The equations.

The continuity

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla}\cdot(n\mathbf{u}_\perp) + \mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{nu_\parallel}{B}\right) = S - \frac{1}{r}\frac{\partial}{\partial r}(r\Gamma_r)$$

This equation is important for the derivation of the expression for the Pfirsch Schluter current.

This is because it introduces the *divergence of the flux of particles*, of the flow. This is where the *geometrical* poloidal compression and dilation will enter the dynamics. In the term $\boldsymbol{\nabla}\cdot[\widehat{\mathbf{e}}_\theta(1+\varepsilon\cos\theta)]$.

The momentum for all plasma (the mass is taken m_i), isothermal

$$nm_i\left[\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}\right] = -T\boldsymbol{\nabla}n + \mathbf{j}\times\mathbf{B} - m_iS\mathbf{u}$$

The current conservation is essential in connecting the perpendicular current (diamagnetic) with the parallel current (Pfirsch Schluter)

$$\boldsymbol{\nabla}\cdot\mathbf{j} = 0$$

The Ohm's law. Here, without the resistivity. This means that the radial velocity v_r will be attributed to another reason for which the charge neutrality cannot be fully suppressed by the parallel currents. The reason may be the *Landau damping* which acts when the collisionality is low. This appears in **Stringer** where a kinetic treatment allows to calculate the variation of the density and of the potential on the magnetic surface, by integrating over the velocity space the distribution functions of electron and ions and imposing neutrality. During the integration, one has to traverse the singularity $v_{\parallel} - \frac{\varepsilon}{q} v_{\theta}^{E} = 0$.

$$-\boldsymbol{\nabla}\phi + \mathbf{u} \times \mathbf{B} = 0$$

The magnetic field is

$$\mathbf{B} = \boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\varphi + I(\psi)\, \boldsymbol{\nabla}\varphi$$

Relative to the work **Hassam Kulsrud** here it is assumed that the electrons and ions are *isothermal*.

$$S(r,\theta) \equiv \text{particle source}$$

It is interesting to note how the *source* extracts from the momentum a part which is proportional with $m_i \mathbf{u}$ through S.

The radial flux

$$\begin{aligned}
\Gamma_r &= \langle\langle \widetilde{n} \widetilde{v}_r \rangle\rangle \\
&= -D(r,\theta) \frac{\partial n}{\partial r}
\end{aligned}$$

An object of study is the *circulation*.

This is obtained taking the projection in the *parallel* direction of the equation of momentum conservation. It is interesting that the variation of the density in the parallel direction (for isothermal plasma) gives the pressure that opposes to the geometrical advection of the flow, $\mathbf{B}\cdot(\mathbf{u}\cdot\boldsymbol{\nabla})\,\mathbf{u}$, which can be static. The imbalance gives $\frac{\partial}{\partial t}(\mathbf{u}\cdot\mathbf{B})$.

$$\begin{aligned}
\frac{\partial}{\partial t}(\mathbf{u}\cdot\mathbf{B}) + \mathbf{B}\cdot(\mathbf{u}\cdot\boldsymbol{\nabla})\,\mathbf{u} &= -c_s^2 \mathbf{B}\cdot\boldsymbol{\nabla}\ln n \quad \text{(parallel pressure)} \\
&\quad -\frac{S}{n}\mathbf{u}\cdot\mathbf{B} \quad \text{(external source of momentum)}
\end{aligned}$$

The poloidal component of the equation for the plasma momentum

$$\mathbf{B}_{pol}\cdot\left(n m_i \frac{d\mathbf{u}}{dt} + T\boldsymbol{\nabla}n\right) = \frac{B_{\varphi}}{R}\mathbf{j}\cdot\boldsymbol{\nabla}\psi \quad \text{(radial current)}$$

We note that

$$\mathbf{B}_{pol} = \boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\varphi$$

and the product $\mathbf{j}\cdot\boldsymbol{\nabla}\psi$ extracts the radial current.

Now comes the constraint that will provide the third equation: the total radial current traversing a magnetic surface must be zero.

Integrated over a magnetic surface the zero-divergence of the current density leads to

$$\int_{flux\_surf} \mathbf{ds} \cdot \mathbf{j} = \int \frac{ds}{|\boldsymbol{\nabla}\psi|} (\mathbf{j} \cdot \boldsymbol{\nabla}\psi) = 0$$

where

$$\begin{aligned} \mathbf{ds} &= ds \ \widehat{\mathbf{e}}_r \\ &= 2\pi R r d\theta \ \widehat{\mathbf{e}}_r \end{aligned}$$

then

$$\int_{flux\_surf} \frac{ds}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol} \cdot \left(n m_i \frac{d\mathbf{u}}{dt} + T \boldsymbol{\nabla} n \right) = 0$$

This equation, derived from current conservation

$$\boldsymbol{\nabla} \cdot \mathbf{j} = 0$$

will be used to derive the *time variation of the poloidal velocity.*

It is assumed first that the plasma velocity is smaller than the sound velocity.
The equilibrium, in zero order

$$\frac{\partial n_0}{\partial t} = 0$$

the equilibrium density is constant in time

$$\mathbf{u}_{\perp 0} \cdot \boldsymbol{\nabla} n_0 = 0$$

The perpendicular advection of the equilibrium density is zero: the density does not vary along *perpendicular* direction.

$$0 = T \mathbf{B}_0 \cdot \boldsymbol{\nabla} n_0$$

The equilibrium density does not vary parallel with the magnetic field line

$$\int d\theta \ \frac{1}{r} \frac{\partial n_0}{\partial \theta} = 0$$

If there is a poloidal variation of the density along the poloidal direction, the periodicity must be taken into account.

$$0 = -\boldsymbol{\nabla}\phi_0 + \mathbf{u}_0 \times \mathbf{B}_0$$

The Ohm's law without *resistivity.*

This means that the lowest order density is constant on the surfaces

$$n_0(r)$$

and the velocity which is perpendicular on the magnetic field is contained in the magnetic surface. It is the electric velocity

$$\begin{aligned}\mathbf{u}_{\perp 0} &= V_E \widehat{\mathbf{e}}_\theta \\ &= \frac{1}{B}\frac{d\phi_0}{dr}\widehat{\mathbf{e}}_\theta\end{aligned}$$

This velocity V_E is poloidal.

the first order in ε will reveal the presence of a perturbation of the density on the magnetic surface, n_1.

Also we will have to work with the parallel velocity $u_\parallel$.

$$\begin{aligned}&\frac{\partial n_1}{\partial t} + V_E \frac{\partial n_1}{r\partial\theta} + n_0 V_E \left(-2\varepsilon \frac{\sin\theta}{r}\right) \\ &+ n_0 \nabla_\parallel u_\parallel \\ = \;& S - \frac{1}{r}\frac{\partial}{\partial r}\left(r\Gamma_r\right)\end{aligned}$$

We see here that the poloidal rotation velocity V_E is very active: it carries the perturbation of the density on the surface

$$V_E \frac{\partial n_1}{r\partial\theta}$$

However we know that the poloidal velocity V_E is actually the diamagnetic velocity, $nv_{Dia} = 1/\left(m\Omega\right) dp/dr$. And this is equal with $-\boldsymbol{\nabla}\phi$, and from here we can introduce the symbol V_E. To understand the third term we should remember

$$\begin{aligned}\boldsymbol{\nabla}\cdot\left[\widehat{\mathbf{e}}_\theta\left(1+\varepsilon\cos\theta\right)\right] &= \frac{1}{r\left(R_0 + r\cos\theta\right)}\frac{\partial}{\partial\theta}\left(\left(R_0 + r\cos\theta\right)\left(1+\varepsilon\cos\theta\right)\right) \\ &= \varepsilon\frac{\left(-2\sin\theta\right)}{r}\end{aligned}$$

The factor $h = 1 + \varepsilon\cos\theta$ comes from the magnitude of the magnetic field $B = B_0/h$. The divergence is calculated for the poloidal flow resulting from the electric velocity V_E that carries the density $n_0 + n_1$. Both quantities do not have variation in this order but the *geometry* is essential.

Termenul $\nabla_\parallel u_\parallel$

The parallel momentum

$$n_0 m_i \left(\frac{\partial u_\parallel}{\partial t} + V_E \frac{\partial u_\parallel}{r\partial\theta}\right) = -T\nabla_\parallel n_1$$

Note that it is here that the *parallel viscosity* $\mathbf{\Pi}$ should appear to introduce the *magnetic damping*. Shaing, etc. **End.**

The parallel gradient is

$$\nabla_{\parallel} = \frac{1}{qR}\frac{\partial}{\partial\theta}$$

Finally the condition

$$\int d\theta \; B_{\theta 0}\frac{\partial n_1}{r\partial\theta} = 0$$

Since $B_{\theta 0}$ is actually constant on the surface and is taken out the integral the condition is trivially satisfied in this order.

The condition satisfied trivially at the first order must be recalculated in higher order, *i.e.* two, ε^2.

The equation to be used is

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{j} &= 0 \\ \text{or, the integral form } \int_{flux\_surf} \mathbf{ds}\cdot\mathbf{j} &= \mathbf{0} \end{aligned}$$

$$\int_{flux\_surf} \frac{ds}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol}\cdot\left(nm_i\frac{d\mathbf{u}}{dt} + T\boldsymbol{\nabla}n\right) = 0$$

derived from the condition of zero-divergence of the current.

The part

$$\int_{flux\_surf} \frac{ds}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol}\cdot\left(T\boldsymbol{\nabla}n\right)$$

will be calculated as

$$\begin{aligned} R^2 &\approx 1 \\ \frac{ds}{|\boldsymbol{\nabla}\psi|} &\sim \text{order } 1 \\ \mathbf{B}_{pol}\cdot\boldsymbol{\nabla}n &\sim \text{order } 1 \end{aligned}$$

An approximation

$$\mathbf{B}_{pol}\cdot\boldsymbol{\nabla}n \approx B_\theta\frac{\partial n_1}{r\partial\theta}$$

and

$$|\boldsymbol{\nabla}\psi| = 2\pi R B_\theta$$

In the product

$$\mathbf{B}_{pol}\cdot\left(nm_i\frac{d\mathbf{u}}{dt}\right)$$

we only retain

$$\mathbf{B}_{pol}\cdot\left(nm_i\frac{\partial\mathbf{u}}{\partial t}\right)$$

since $\mathbf{B}_{pol}\cdot(\mathbf{u}\cdot\boldsymbol{\nabla})\,\mathbf{u}$ is of higher order. This term will provide the time variation of the *poloidal velocity* $V_E\,(r,t)$.

We also have

$$ds = 2\pi R r d\theta$$

The integration of the first part is

$$\begin{aligned}
& \int_{flux_surf} \frac{ds}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol} \cdot \left(n m_i \frac{d\mathbf{u}}{dt} \right) \\
\approx & \int_{flux_surf} \frac{2\pi R r d\theta}{2\pi R B_\theta} B_\theta n m_i \frac{\partial V_E}{\partial t} \\
= & \left(2\pi\right) r n m_i \frac{\partial V_E}{\partial t}
\end{aligned}$$

the integration of the second term

$$\begin{aligned}
& \int_{flux_surf} \frac{ds}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol} \cdot \left(T \boldsymbol{\nabla} n \right) \\
= & T \int_{flux_surf} \frac{ds}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol} \cdot \boldsymbol{\nabla} \left[n_0 \left(r\right) + n_1 \left(r, \theta\right) \right] \\
= & T \int_{flux_surf} \frac{ds}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol} \cdot \boldsymbol{\nabla} n_1 \left(r, \theta\right)
\end{aligned}$$

we make an integration by parts and take into account the periodicity

$$\begin{aligned}
& []_0^{2\pi} - T \int_{flux_surf} n_1 \boldsymbol{\nabla} \cdot \left(\frac{ds}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol} \right) \\
= & -T \int_{flux_surf} n_1 \boldsymbol{\nabla} \cdot \left(\frac{2\pi R r d\theta}{2\pi R B_\theta} R^2 B_\theta \widehat{\mathbf{e}}_\theta \right)
\end{aligned}$$

we must find

$$\begin{aligned}
& \int_{flux_surf} \frac{ds}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol} \cdot \left(T \boldsymbol{\nabla} n \right) \\
= & -T \int_{flux_surf} n_1 \boldsymbol{\nabla} \cdot \left(\frac{ds}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol} \right) \qquad \text{(integration by parts)} \\
= & -T r \int n_1 d\theta \boldsymbol{\nabla} \cdot \left(h \widehat{\mathbf{e}}_\theta \right) \\
= & -T r \int d\theta \left(\varepsilon \frac{-2 \sin\theta}{r} \right) n_1
\end{aligned}$$

Then

$$\int_{flux\_surf} \frac{ds}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol} \cdot \left(n m_i \frac{d\mathbf{u}}{dt} + T \boldsymbol{\nabla} n \right) = 0$$

$$
\begin{aligned}
(2\pi)\, rnm_i \frac{\partial V_E}{\partial t} + \left\{ -Tr \int d\theta \left(\varepsilon \frac{-2\sin\theta}{r} \right) n_1 \right\} &= 0 \\
(2\pi)\, rnm_i \frac{\partial V_E}{\partial t} &= -2T\varepsilon \int d\theta \sin\theta\ n_1 \\
\frac{\partial V_E}{\partial t} &= -\frac{1}{r}\varepsilon \frac{c_s^2}{n_0} \int \frac{d\theta}{2\pi} \sin\theta\ n_1
\end{aligned}
$$

New notation

$$
N \equiv \frac{n_1}{n_0}
$$

the equations

$$
\begin{aligned}
& \frac{\partial N}{\partial t} + V_E \frac{\partial N}{r\partial\theta} + V_E \left(-2\varepsilon \frac{\sin\theta}{r} \right) \\
& +\nabla_{\|} u_{\|} \\
= \quad & \frac{S}{n_0} - \frac{1}{n_0} \frac{1}{r} \frac{\partial}{\partial r} \left(r\Gamma_r \right)
\end{aligned}
$$

$$
\frac{\partial u_{\|}}{\partial t} + V_E \frac{\partial u_{\|}}{r\partial\theta} = -c_s^2 \nabla_{\|} N
$$

$$
\frac{\partial V_E}{\partial t} = c_s^2 \int \frac{d\theta}{2\pi} N \left(-2\varepsilon \frac{\sin\theta}{r} \right)
$$

NOTE

Let us stop to make a comparison between this (**Hassam Drake**) system prepared for the spontaneous spin-up and the **Stringer PRL** system.

We note that the *time variation* in the equation for

- the density, $\partial n_1/\partial t$, and
- the velocity

  $$
  nm_i V_{E\theta}^{(0)} \frac{\partial v_{i\|}}{r\partial\theta} = -\left(T_e + T_i \right) \frac{\varepsilon}{q} \frac{\partial n_1}{r\partial\theta}
  $$

  this equation shows the balance of momentum carried by the "static advected" velocity (*i.e.* space variation of the velocity, $(\mathbf{v} \cdot \boldsymbol{\nabla})\, \mathbf{v}$) with the *pressure*. The projection is made on poloidal direction.

is *absent* at **Stringer**.

Since Hassam Drake work for spin-up driven by external source (poloidally asymmetric) the explicit time variation must be retained.

The main term however in the formulation Hassam Drake is still $V_E \frac{\partial u_{\|}}{r\partial\theta}$ (later $\widehat{u}_E \frac{\partial \widetilde{u}_{\|}}{r\partial\theta}$) which is the same as in Stringer. This term will be the main part of the expansion around the equilibrium static state.

The equilibrium static state at Hassam Drake is

$$\begin{aligned}\nabla_{\|}\widetilde{u}_{\|} &= \widetilde{F} \\ \nabla_{\|}\widetilde{N} &= 0\end{aligned}$$

and the expansion introduces new, small, quantities

$$\widehat{u}_E \ , \ \widehat{u}_{\|} \ , \ \widehat{N}$$

with the system

$$\begin{aligned}-2\varepsilon\widehat{u}_E\frac{\sin\theta}{r}+\nabla_{\|}\widehat{u}_{\|} &= 0 \\ \frac{\partial\widehat{u}_{\|}}{\partial t}+\widehat{u}_E\frac{\partial\widetilde{u}_{\|}}{r\partial\theta} &= -c_s^2\nabla_{\|}\widehat{N} \\ \frac{\partial\widehat{u}_E}{\partial t} &= -\frac{2\varepsilon c_s^2}{r}\int\frac{d\theta}{2\pi}\widehat{N}\sin\theta\end{aligned}$$

See the explanations below.
END

The functions that must be determined

$$N\left(r,\theta,t\right) \quad , \quad u_{\|}\left(r,\theta,t\right) \quad , \quad V_E\left(r,t\right)$$

The global balance is obtained by integrating over the surface $\int\frac{d\theta}{2\pi}\left(...\right)$.

$$\begin{aligned}\frac{\partial\overline{N}}{\partial t} &= \frac{\overline{S}}{n_0}-\frac{1}{n_0}\frac{\partial}{r\partial\theta}\left(r\overline{\Gamma}_r\right) \\ \frac{\partial\overline{u}_{\|}}{\partial t} &= 0\end{aligned}$$

After introducing the average over surfaces, the new variables are the differences that have variations in the surfaces

$$\widetilde{f}=f-\overline{f}$$

The source in the surface is

$$F\equiv\frac{S-\frac{1}{r}\frac{\partial}{\partial r}\left(r\Gamma_r\right)}{n_0}$$

The state that is taken as reference is the absence of the poloidal rotation

$$V_E^{ref}=0$$

and this reduces the equations to

$$\begin{aligned} N^{ref} &= \overline{N} \\ \text{i.e.} \quad \widetilde{N}^{ref} &= 0 \quad \text{(no variation of the density in the surface)} \end{aligned}$$

$$\nabla_{\|} \widetilde{u}_{\|}^{ref} = \widetilde{F}$$

$$\nabla_{\|} \widetilde{N}^{ref} = 0 \ \text{ from where } \widetilde{N} = 0$$

The variation of the parallel velocity along the magnetic line (equivalently, in the magnetic surface) is obtained in terms of the source

$$\begin{aligned} \nabla_{\|} \widetilde{u}_{\|}^{ref} &= \widetilde{F} \\ \frac{1}{qR} \frac{\partial}{\partial \theta} \widetilde{u}_{\|}^{ref} &= \widetilde{F} \\ \widetilde{u}_{\|}^{ref} &= qR \int d\theta' \ \widetilde{F} \end{aligned}$$

Consider a perturbation of this reference state

$$\begin{aligned} V_E &= V_E^{ref} + \widehat{V}_E \\ u_{\|} &= \widetilde{u}_{\|}^{ref} + \widehat{u}_{\|} \\ N &= \overline{N} + \widetilde{N}^{ref} + \widehat{N} \end{aligned}$$

This will induce a time variation of the poloidal (electric) velocity and of the density N and of the parallel velocity.

However the time variation is assumed to be slower than the sound speed

$$\frac{\partial}{\partial t} \ll \frac{c_s}{qR}$$

The time variation for N is neglected and the equation for density becomes a balance

$$\widehat{V}_E \left(-2\varepsilon \frac{\sin \theta}{r} \right) + \nabla_{\|} \widehat{u}_{\|} = 0$$

$$\frac{\partial \widehat{u}_{\|}}{\partial t} + \widehat{V}_E \frac{\partial \widetilde{u}_{\|}^{ref}}{r \partial \theta} = -c_s^2 \nabla_{\|} \widehat{N}$$

$$\frac{\partial \widehat{V}_E}{\partial t} = c_s^2 \int \frac{d\theta}{2\pi} \widehat{N} \left(-2\varepsilon \frac{\sin \theta}{r} \right)$$

Note the preservation of the poloidal derivative of the *reference* parallel velocity in the second equation. This reference value of the paralel velocity is fixed by the radial flux and the source of particles. It exists only because these sources and fluxes are *NOT constant on the poloidal circumference.*

This set of equations can be integrated.

The operator that must be made explicit is

$$\nabla_{\|} = \frac{1}{qR}\frac{\partial}{\partial\theta}$$

Then, since $\widehat{V}_E$ is constant on magnetic surfaces, the first equation is

$$\begin{aligned}
\widehat{V}_E\left(-2\varepsilon\frac{\sin\theta}{r}\right) + \nabla_{\|}\widehat{u}_{\|} &= 0 \quad \text{or} \\
\frac{1}{qR}\frac{\partial}{\partial\theta}\widehat{u}_{\|} &= \widehat{V}_E\left(2\varepsilon\frac{\sin\theta}{r}\right) \\
\widehat{u}_{\|} &= -2qR\varepsilon\frac{\cos\theta}{r}\widehat{V}_E
\end{aligned}$$

This is introduced in the second equation

$$\begin{aligned}
\frac{\partial}{\partial t}\left[-2qR\varepsilon\frac{\cos\theta}{r}\widehat{V}_E\right] + \widehat{V}_E\frac{\partial\widetilde{u}_{\|}^{ref}}{r\partial\theta} &= -c_s^2\nabla_{\|}\widehat{N} \\
&= -c_s^2\frac{1}{qR}\frac{\partial\widehat{N}}{\partial\theta}
\end{aligned}$$

and is integrated

$$\begin{aligned}
-c_s^2\widehat{N} &= -2\left(qR\right)^2\varepsilon\frac{1}{r}\frac{\partial\widehat{V}_E}{\partial t}\int^{\theta} d\theta'\cos\theta' \\
&\quad +\widehat{V}_E\frac{qR}{r}\widetilde{u}_{\|}^{ref}
\end{aligned}$$

we ignore for the moment the constant of integration which should be a function of surface.

This is introduced in the equation for the time variation of $\widehat{V}_E$, the third equation

$$\begin{aligned}
\frac{\partial\widehat{V}_E}{\partial t} &= c_s^2\int\frac{d\theta}{2\pi}\widehat{N}\left(-2\varepsilon\frac{\sin\theta}{r}\right) \\
&= \int\frac{d\theta}{2\pi}\left(2\varepsilon\frac{\sin\theta}{r}\right)\left\{-2\left(qR\right)^2\varepsilon\frac{1}{r}\frac{\partial\widehat{V}_E}{\partial t}\int^{\theta} d\theta'\cos\theta' + \widehat{V}_E\frac{qR}{r}\widetilde{u}_{\|}^{ref}\right\} \\
&= -4\left(qR\right)^2\varepsilon^2\frac{\partial\widehat{V}_E}{\partial t}\frac{1}{r^2}\int\frac{d\theta}{2\pi}\sin\theta\sin\theta \\
&\quad +2\varepsilon\frac{qR}{r^2}\widehat{V}_E\int\frac{d\theta}{2\pi}\sin\theta\widetilde{u}_{\|}^{ref}
\end{aligned}$$

The first term

$$-4\left(qR\right)^2\varepsilon^2\left(\frac{\partial\widehat{V}_E}{\partial t}\right)\frac{1}{r^2}\int\frac{d\theta}{2\pi}\sin\theta\sin\theta = -4q^2R^2\frac{r^2}{R^2}\left(\frac{\partial\widehat{V}_E}{\partial t}\right)\frac{1}{r^2}\frac{1}{2} = -2q^2\left(\frac{\partial\widehat{V}_E}{\partial t}\right)$$

and the second

$$
\begin{aligned}
2\varepsilon\frac{qR}{r^2}\widehat{V}_E &= 2\frac{r}{R}q\frac{R}{r}\frac{1}{r}\widehat{V}_E \\
&= \frac{2q}{r}\widehat{V}_E
\end{aligned}
$$

$$
\left(1+2q^2\right)\frac{\partial\widehat{V}_E}{\partial t} = \frac{2q}{r}\widehat{V}_E\int\frac{d\theta}{2\pi}\sin\theta\widetilde{u}_\parallel^{ref}
$$

In this equation we replace the reference state for the parallel velocity, which is fixed by the source and the flux, both these contributions being retained with their variation along the poloidal direction

$$
\left(1+2q^2\right)\frac{\partial\widehat{V}_E}{\partial t} = \widehat{V}_E \quad \times\frac{1}{\varepsilon^2}2q^2\left[\frac{1}{n_0}\int\frac{d\theta}{2\pi}S\cos\theta - \frac{1}{n_0}\frac{1}{r}\frac{\partial}{\partial r}\left(r\int\frac{d\theta}{2\pi}\Gamma_r\cos\theta\right)\right]
$$

we can easily recognize that an integration by parts have been made in the right hand side.

NOTE
How is generated this *inertia factor* $1+2q^2$.
We have seen that the first integration

$$
\widehat{u}_\parallel = -2qR\varepsilon\frac{\cos\theta}{r}\widehat{V}_E
$$

actually obtains the Pfirsch Schluter parallel current, with a poloidal flow given by $\widehat{V}_E$. This PS flow has a coefficient q, as usual.

The second equation has the RHS term $-c_s^2\frac{1}{qR}\frac{\partial\widehat{N}}{\partial\theta}$ and this introduces the second q factor. They now multiply the term $\partial\widehat{u}_\parallel/\partial t$.

The enhancement of the *radial diffusion* with a factor q^2, the known characteristics of the Pfirsch Schluter "regime" has the same origin. We note however that it is not yet clear what means PS regime.
END

12 Numerical study of the Stringer rotation

We have developed a numerical framework that incorporates the effect of poloidal variation of the rate of particle and energy losses. The equation for poloidal rotation under the Stringer effect is solved and the rate of rotation is compared with the *transit time magnetic pumping* damping.

13 Notes

The paper of **Stringer 1969** calculates the radial flux of particles taking into account the nonuniformity of the particle density $n_1(\theta)$ and of electric potential $\phi_1(\theta)$ on the magnetic surface. These result from the neoclassical drifts and the equations of continuity, the poloidal projected equation of momentum, in the presence of an equilibrium radial electric field represented by a potential $\phi_0(r)$. In the regime of low collisions instead of collisional resistivity that permits to use the Ohm law for the parallel current projected on poloidal direction, it is invoked the *kinetic* process of Landau damping.

The result is

$$nv_{Di} = -\frac{\sqrt{\pi}}{8}\varepsilon\frac{\rho_i^2 v_{th,i}}{r}\frac{1}{q}\left(1+\frac{v_0}{U_{in}}\right)\exp\left(-z_i^2\right)\left[1+\frac{S(S+\tau)}{F^2+L^2}\right]\frac{dn_0}{dr}$$

where

$$S \equiv 1+\tau+2z_i^2\left(1+\frac{U_{en}}{v_0}\right)$$

$$z_j \equiv -\frac{v_0}{v_{th,j}}\frac{q}{\varepsilon}$$

This is

$$\frac{1}{v_{th,j}}\frac{q}{\varepsilon} = \frac{1}{v_{th,j}}\frac{B_T}{B_\theta}$$

$$v_{th,j}\frac{B_\theta}{B_T} = v_{th,j}^{\theta}$$

the projection of the thermal (assumed parallel) velocity on the poloidal direction. And

$$v_0\widehat{\mathbf{e}}_\theta = \frac{-\boldsymbol{\nabla}\phi_0\times\widehat{\mathbf{n}}}{B}$$

$$z_j = -\frac{v_0}{v_{th,j}^{\theta}}$$

Later it is found that

$$\frac{v_0}{U_{ni}} = -1+\frac{1+\tau}{1+2z_i^2+2z_i^4}\left(\tau\frac{m_e}{m_i}\right)^{1/2}\exp\left(-z_i^2\right)$$

Then the diamagnetic and the electric rotations are almost equal in magnitude and opposite.

When

$$1+\frac{v_{dia}}{v_E}\approx 0$$

the neoclassical diffusion vanishes.