Notes on equilibrium neoclassical flows in tokamak and spontaneous poloidal spin-up

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April 30, 2023

Abstract

This is the basic level of organization of the plasma at equilibrium in tokamak. It corresponds to the neoclassical flows.

Possible application to helical entrainement of tungsten toward the center, carried by poloidal rotation of Stringer effect.

This text is a part of Lecture 1 from the **Work Session of Plasma Theory**. It is a ground for discussions, not to be taken as final.

1 Introduction to the equilibrium flows

Diamagnetic, Pfirsch Schluter, radial electric field and polarization

2 The diamagnetic flow Hazeltine Hinton RMP

In HH RMP the perpendicular flows

$$(n\mathbf{u}_{\perp})_1 = \frac{1}{eB}\left(\widehat{\mathbf{n}} \times \boldsymbol{\nabla}\overline{p} + e\overline{n}\ \boldsymbol{\nabla}\overline{\Phi}\right)$$

and

$$(\mathbf{q}_{\perp})_1 = \frac{5}{2}\frac{1}{eB}\ \overline{p}\ \widehat{\mathbf{n}} \times \boldsymbol{\nabla}\overline{T}$$

These flows must have zero-divergence.

Therefore *parallel return flows* are necessary to ensure

$$\begin{aligned} \boldsymbol{\nabla}\cdot(n\mathbf{u})_1 &= 0 \\ \boldsymbol{\nabla}\cdot(\mathbf{q})_1 &= 0 \end{aligned}$$

To identify parallel flows that are part of the *total* flows $(n\mathbf{u})_1$ and $(\mathbf{q})_1$, one adopts genereic expressions for these total flows

$$\begin{aligned} (n\mathbf{u})_1 &= \widehat{K}\ \mathbf{B} && \text{parallel} \\ &\quad +\widetilde{K}\ \boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\Theta && \text{toroidal} \end{aligned}$$

Asking zero-divergence

$$\frac{1}{\sqrt{g}}\frac{\partial \widetilde{K}}{\partial \varphi} = -\mathbf{B}\cdot\boldsymbol{\nabla}\widehat{K}$$

From the perpendicular flow

$$\widetilde{K} = -\frac{1}{e}\left(\frac{\partial \overline{p}}{\partial \psi} + e\overline{n}\;\frac{\partial \overline{\Phi}}{\partial \psi}\right)\sqrt{g}$$

The quantities involved here are average over surface and do not depend on φ. Then

$$\nabla_{\|}\widehat{K} = 0$$

and

$$\begin{aligned} & \widehat{\mathbf{n}}\cdot(n\mathbf{u})_1 \qquad \text{parallel flow} \\ = \quad & \left(nu_{\|}\right)_1 \\ = \quad & -\frac{1}{eB}\;I\;\left(\frac{\partial \overline{p}}{\partial \psi} + e\overline{n}\;\frac{\partial \overline{\Phi}}{\partial \psi}\right) \\ & \qquad + \widehat{K}\left(\psi\right)\;B \end{aligned}$$

where

$$I \equiv \sqrt{g}\;\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\Theta\cdot\mathbf{B}$$

The diamagnetic flux and the Pfirsch Schluter flux.

In **Hazeltine Ware impurity electrostatic trapping**.

$$\Gamma_a = \int\frac{d\theta}{2\pi}h\;\int d^3v\;v_{D,r}\;f_a$$

$$v_{D,r} = -\frac{T_a}{e_a B}\frac{1}{R}\left[\frac{v^2}{v_{th,a}^2}\left(1+\xi^2\right)\sin\theta + h\frac{\partial}{\partial\theta}\left(\frac{R}{r}\frac{e_a\widetilde{\Phi}}{T_i}\right)\right]$$

$$x = \frac{R}{r}\frac{e_a\widetilde{\Phi}}{T_a} = \frac{m_a}{Z_a}\frac{1}{T_i}\left[u_E^2 + \left(\overline{u}-u_E\right)^2\right]\cos\theta$$

where

$$u_E = -\frac{1}{B_{\theta 0}}\frac{d\overline{\Phi}}{dr}$$

$$V_{\|0} = \overline{u} + \widetilde{u}\left(\theta\right)$$

$$\begin{aligned} \widetilde{u}\left(\theta\right) \quad = \quad & u_E\frac{r}{R}\cos\theta \\ & + \left(u_E - \overline{u}\right)\left(\frac{\widetilde{n}}{\overline{n}} + \frac{r}{R}\cos\theta\right) \end{aligned}$$

and

$$\overline{V}_{\parallel i} \approx V_{\parallel Z} \equiv \overline{u}$$

The stress anisotropy

$$\mathbf{P}_{\parallel} - \mathbf{P}_{\perp} = \int d^3 v \, \frac{mv^2}{2} \left(3\xi^2 - 1\right) \, f$$

The Pfirsch Schluter flux

$$\Gamma_{PS} = -\frac{1}{eB_0} \frac{1}{r} \left\langle h^2 \left(\frac{\partial \mathbf{P}_{\parallel}}{\partial \theta} + e\overline{n} \frac{\partial \widetilde{\Phi}}{\partial \theta} \right) \right\rangle$$

$$\langle \rangle \equiv \int \frac{d\theta}{2\pi} (...)$$

3 Ware pinch of trapped particles

The paper by **Ware PRL 1970.**

Inductive toroidal electric field

$$E_{\varphi} = -\frac{\partial A_{\varphi}}{\partial t}$$

Conservation of the *generalized momentum*

$$\frac{d}{dt} \left[R \left(\frac{mv_{\parallel} B_{\varphi}}{B} + eA_{\varphi} \right) \right] = 0$$

Consider the two turning points of the banana, at *different* minor radius

$$r \text{ and } r + \delta r$$

and the times when they are reached

$$t \text{ and } t + \delta t$$

The conservation of the generalized momentum, reduced for

$$v_{\parallel} = 0$$

is

$$(\delta r) \frac{\partial}{\partial r} (RA_{\varphi}) + (\delta t) \, R \frac{\partial A_{\varphi}}{\partial t} = 0$$

since in the gradient

$$\boldsymbol{\nabla} (RA_{\varphi})$$

there is only the component along r.

$$\frac{1}{R} \frac{\partial}{\partial r} (RA_{\varphi}) = -B_{\theta}$$

These formulas lead to

$$(\delta r)\, B_\theta + (\delta t)\, E_\varphi = 0$$

or

$$\frac{\delta r}{\delta t} = -\frac{E_\varphi}{B_\theta}$$

The conclusion is that all particles that are trapped drift toward the magnetic axis with this velocity.

Explanation by **Ware**

the field E_φ shifts the center of the banana in that half-torus (above or below the equatorial plane) where the drift

$$\mathbf{B} \times \boldsymbol{\nabla} B$$

has a radial projection that points inward.

3.1 The Pfirsch Schluter current

A good explanation is in **Stringer PRL.**

See also *bootstrap and diamagnetic*, after Hirshman current.

The drift motions of electrons and ions $v_{drift,j}$ in *toroidal* field lead to *charge separation*.

In order to suppress this charge separation a *current flows along magnetic field lines*.

When there is resistivity (collisions) the neutralization of the charge separation by the parallel current is *incomplete*.

Then there is a residual electric field which still remains. This is $E_\|$ and is connected with $j_\|$ by $\eta \neq 0$.

This electric field induce an *enhancement* of the diffusion. The enhancement comes from the *radial velocity* v_r that exists due to the coupling of the parallel electric field with the poloidal magnetic field, in the Ohm's law $-\nabla_\| \phi + v_r B_\theta = \eta j_\|$. The *radial velocity* v_r produces a radial flux $\Gamma_r = v_r n$. This is the factor q^2 which multiplies the classical diffusion.

Now, in regimes where collisions are rare, instead of collisional resistivity, it is the *Landau damping* that is invoked.

Stringer PRL mentions the fact that the variation of ion's density on the magnetic surface is related to finite *ion inertia*. This means $(\mathbf{v} \cdot \boldsymbol{\nabla})\, \mathbf{v}$.

The parallel current arising from the non-zero divergence of the *diamagnetic current*

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{j} &= 0 \\ \boldsymbol{\nabla}_\perp \cdot \mathbf{j}_\perp + \boldsymbol{\nabla}_\| \cdot \mathbf{j}_\| &= 0 \end{aligned}$$

Now taking the perpendicular current as resulting from the *diamagnetic* flows of electrons and ions, the parallel gradient can be written as

$$\begin{aligned}
\boldsymbol{\nabla}_{\|} \cdot \mathbf{j}_{\|} &= \frac{1}{qR}\frac{\partial}{\partial\theta} j_{\|} \\
&= -\boldsymbol{\nabla}_{\perp} \cdot \mathbf{j}_{\perp} = -\boldsymbol{\nabla}_{\perp} \cdot \left(e\frac{1}{m\Omega}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p \right) \\
&= -\boldsymbol{\nabla}_{\perp} \cdot \left(e\frac{1}{|e|\, B}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p \right)
\end{aligned}$$

Let us look to the last term. It is the perpendicular divergence of the *diamagnetic* flow.

Note that the operator of parallel derivative is

$$\nabla_{\|} \sim \frac{1}{qR}\frac{\partial}{\partial\theta}$$

and that the perpendicular current $\mathbf{j}_{\perp}$ is the *diamagnetic current, of ions + electrons*. **End.**

This is a *neoclassical* effect.

It is the magnetic field that has a space variation in the perpendicular direction. First we have

$$\begin{aligned}
\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p &= \left| \frac{dp}{dr} \right| (\widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_r) \\
&= -\widehat{\mathbf{e}}_{\theta} \left| \frac{dp}{dr} \right|
\end{aligned}$$

Then, restricting the gradient to the part that contains B, we use the expression of the gradient operator expressed in the geometry of the toroidal region.

This part is repeated later in this text.

We have the *magnitude* of the magnetic field

$$B = \frac{B_0}{1+\varepsilon\cos\theta}$$

and we must calculate the perpendicular divergence of the perpendicular current, which means

$$\boldsymbol{\nabla} \cdot \left(-\widehat{\mathbf{e}}_{\theta}\frac{1}{B}\left| \frac{dp}{dr} \right| \right)$$

and this is approximated by (B_0 is constant)

$$\boldsymbol{\nabla} \cdot \left(\widehat{\mathbf{e}}_{\theta}\frac{B_0}{B} \right) = \boldsymbol{\nabla} \cdot \left[\widehat{\mathbf{e}}_{\theta}\left(1+\varepsilon\cos\theta\right) \right]$$

Here is the essential part of the calculation: there is a divergence of the diamagnetic "flow" that is exclusively due to the *geometry* of the magnetic field

$$\frac{B_0}{B} = 1+\varepsilon\cos\theta$$

It is the variation of the toroidal magnetic field as

$$B \sim \frac{1}{R}$$

We note that it is the gyrofrequency $\Omega = eB/m$ which introduces a variation of the diamagnetic current in the surface. It is the fact that the charges of the diamagnetic current advance transversally to the magnetic field (always sitting in the magnetic surface) that there is a non-zero divergence of the diamagnetic current. If the motion of the charges were strictly vertical such that $B =$const, then there is no divergence.

This has consequences in the balance of flows.

Here it is explained how this divergence is calculated.
In the orthogonal coordinates (r, θ, φ) we have the element of distance:

$$dl^2 = (dr)^2 + r^2 (d\theta)^2 + (R_0 + r\cos\theta)^2 d\varphi^2$$

which gives the coefficients

$$\begin{aligned} h_1 &= 1 \\ h_2 &= r \\ h_3 &= R_0 + r\cos\theta \end{aligned}$$

Then the divergence of a vector $\mathbf{a}$ is written

$$\boldsymbol{\nabla} \cdot \mathbf{a} = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial r} (h_2 h_3 a_1) + \frac{\partial}{\partial \theta} (h_1 h_3 a_2) + \frac{\partial}{\partial \varphi} (h_1 h_2 a_3) \right)$$

which gives

$$\begin{aligned} \boldsymbol{\nabla} \cdot [\widehat{\mathbf{e}}_\theta (1 + \varepsilon\cos\theta)] &= \frac{1}{r (R_0 + r\cos\theta)} \frac{\partial}{\partial \theta} ((R_0 + r\cos\theta)(1 + \varepsilon\cos\theta)) \\ &= \frac{1}{r (R_0 + r\cos\theta)} R_0 \frac{\partial}{\partial \theta} \left[(1 + \varepsilon\cos\theta)^2 \right] \\ &= \varepsilon \frac{(-2\sin\theta)}{r} \end{aligned}$$

From this result we get

$$
\begin{aligned}
-\boldsymbol{\nabla}_\perp \cdot \mathbf{j}_\perp &= -\boldsymbol{\nabla}_\perp (dia) = \\
&= -\boldsymbol{\nabla}_\perp \cdot \left(e\frac{1}{m\Omega}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p \right) \\
&= -\boldsymbol{\nabla}_\perp \left[\left(e\frac{1}{m\Omega} \right) (-\widehat{\mathbf{e}}_\theta) \left| \frac{dp}{dr} \right| \right] \\
&= \boldsymbol{\nabla}_\perp \cdot \left(\widehat{\mathbf{e}}_\theta \frac{B_0}{B} \right) \frac{1}{B_0} \left| \frac{dp}{dr} \right| \\
&= \varepsilon \frac{(-2\sin\theta)}{r} \frac{1}{B_0} \left| \frac{dp}{dr} \right| \\
&= \frac{r}{RB_0} \left| \frac{dp}{dr} \right| \frac{\partial}{r\partial\theta} (2\cos\theta)
\end{aligned}
$$

NOTE ON An alternative calculation

$$
-\boldsymbol{\nabla}_\perp \cdot \left(e\frac{1}{m\Omega}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} p \right) = -\boldsymbol{\nabla}_\perp \cdot \left[\left(e\frac{1}{m\Omega} \right) \left(-\widehat{\mathbf{e}}_\theta \left| \frac{dp}{dr} \right| \right) \right]
$$

Taking factor $|dp/dr|$ we have to calculate

$$
\begin{aligned}
\boldsymbol{\nabla}_\perp \cdot \left(e\frac{1}{m\Omega}\widehat{\mathbf{e}}_\theta \right) &= \boldsymbol{\nabla}_\perp \cdot \left(\frac{1}{B}\widehat{\mathbf{e}}_\theta \right) \\
&= \boldsymbol{\nabla}_\perp \left(\frac{1}{B} \right) \cdot \widehat{\mathbf{e}}_\theta + \frac{1}{B} \left(\boldsymbol{\nabla}_\perp \cdot \widehat{\mathbf{e}}_\theta \right)
\end{aligned}
$$

The first term is

$$
\begin{aligned}
\boldsymbol{\nabla}_\perp \left(\frac{1}{B} \right) &= -\frac{1}{B^2} \boldsymbol{\nabla}_\perp B \\
&= -\frac{1}{B^2} \boldsymbol{\nabla}_\perp \left(B_0 \frac{R_0}{R} \right) \\
&= -\frac{1}{B^2} B_0 R_0 \left(-\frac{1}{R^2} \boldsymbol{\nabla}_\perp R \right) \\
&= \frac{B_0}{B^2} \frac{R_0}{R^2} \widehat{\mathbf{e}}_R
\end{aligned}
$$

We take separately

$$
\begin{aligned}
\boldsymbol{\nabla}_\perp R &= \left(\widehat{\mathbf{e}}_r \frac{1}{h_r} \frac{\partial}{\partial r} + \widehat{\mathbf{e}}_\theta \frac{1}{h_\theta} \frac{\partial}{\partial \theta} + \widehat{\mathbf{e}}_\varphi \frac{1}{h_\varphi} \frac{\partial}{\partial \varphi} \right) (R_0 + r\cos\theta) \\
&= \left(\widehat{\mathbf{e}}_r \frac{\partial}{\partial r} + \widehat{\mathbf{e}}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \widehat{\mathbf{e}}_\varphi \frac{1}{R_0 + r\cos\theta} \frac{\partial}{\partial \varphi} \right) (R_0 + r\cos\theta) \\
&= \widehat{\mathbf{e}}_r \cos\theta + \widehat{\mathbf{e}}_\theta (-\sin\theta) \\
&= \widehat{\mathbf{e}}_R
\end{aligned}
$$

Here we should decide if the angle θ is measured from the equatorial plane or from the symmetry axis of the torus. Above it was considered that θ is measured from the equatorial plane towards the higher z direction.

$$\frac{1}{B^2}\frac{B_0R_0}{R^2} = B_0R_0\frac{(1+\varepsilon\cos\theta)^2}{B_0^2}\frac{1}{(R_0+r\cos\theta)^2} = \frac{1}{B_0R_0}$$

The first term is then

$$\boldsymbol{\nabla}_\perp\left(\frac{1}{B}\right)\cdot\widehat{\mathbf{e}}_\theta = \frac{1}{B_0R_0}\widehat{\mathbf{e}}_R\cdot\widehat{\mathbf{e}}_\theta = \frac{1}{B_0R_0}(-\sin\theta) = \frac{1}{B_0R_0}\frac{\partial}{\partial\theta}(\cos\theta)$$

The second term is $\frac{1}{B}\left(\boldsymbol{\nabla}_\perp\cdot\widehat{\mathbf{e}}_\theta\right)$ and contains the *divergence* of the versor

$$\begin{aligned}
&\boldsymbol{\nabla}_\perp\cdot\widehat{\mathbf{e}}_\theta \\
= \;& \frac{1}{h_rh_\theta h_\varphi}\left\{\frac{\partial}{\partial r}\left[h_\theta h_\varphi\left(\widehat{\mathbf{e}}_\theta\right)_r\right]+\frac{\partial}{\partial\theta}\left[h_rh_\varphi\left(\widehat{\mathbf{e}}_\theta\right)_\theta\right]+\frac{\partial}{\partial\varphi}\left[h_rh_\theta\left(\widehat{\mathbf{e}}_\theta\right)_\varphi\right]\right\} \\
= \;& \frac{1}{r\left(R_0+r\cos\theta\right)}\left\{\frac{\partial}{\partial\theta}\left[h_rh_\varphi\left(\widehat{\mathbf{e}}_\theta\right)_\theta\right]\right\} \\
= \;& \frac{1}{r\left(R_0+r\cos\theta\right)}\frac{\partial}{\partial\theta}\left[\left(R_0+r\cos\theta\right)\right] \\
= \;& \frac{1}{R_0+r\cos\theta}\frac{\partial}{\partial\theta}(\cos\theta)
\end{aligned}$$

The perpendicular divergence of the diamagnetic current is

$$\begin{aligned}
-\boldsymbol{\nabla}_\perp\cdot\mathbf{j}_\perp &= -\boldsymbol{\nabla}_\perp\cdot\left(e\frac{1}{m\Omega}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}p\right) \\
&= -\boldsymbol{\nabla}_\perp\left(e\frac{1}{m\Omega}\right)\cdot\left(-\widehat{\mathbf{e}}_\theta\left|\frac{dp}{dr}\right|\right) \\
&= \left|\frac{dp}{dr}\right|\left[\frac{1}{B_0R_0}\frac{\partial}{\partial\theta}(\cos\theta)+\right. \\
&\qquad\left.+\frac{1}{B}\frac{1}{R_0+r\cos\theta}\frac{\partial}{\partial\theta}(\cos\theta)\right] \\
&= \left|\frac{dp}{dr}\right|\frac{1}{B_0R_0}\frac{\partial}{\partial\theta}(2\cos\theta)
\end{aligned}$$

and obtain the same result.

END OF NOTE on the alternative calculation

Equating the two sides of the *current conservation* equation

$$\frac{1}{qR}\frac{\partial}{\partial\theta}j_\parallel = -\frac{r}{RB}e\left(\frac{dp}{dr}\right)\frac{\partial}{r\partial\theta}(2\cos\theta)$$

Integrating on the poloidal angle θ:

$$J_\parallel = -\varepsilon\frac{2}{B_\theta}\frac{dp}{dr}\cos\theta$$

This is the Pfirsch Schluter current.

We note

$$\varepsilon\frac{1}{B_\theta}=\frac{r}{RB_\theta}\frac{B}{B}=q\frac{1}{B}$$

and the combination

$$\frac{1}{B}\frac{dp}{dr}$$

is clearly coming from the diamagnetic flow

$$n\mathbf{v}^{dia}=\frac{1}{m\Omega}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}p$$

and remark that the Pfirsch Schluter current is

- parallel with B
- *proportional with* q
- harmonic on θ
- proportional with the diamagnetic current.

There is a poloidal electric field related to this current

$$E_\theta=\frac{1}{\sigma_\parallel}\frac{B}{B_\theta}\left(-\varepsilon\frac{2}{B_\theta}\frac{dp}{dr}\cos\theta\right)$$

It is the projection on θ (poloidal) of the relationship $E_\parallel=J_\parallel/\sigma_\parallel$, with the factor of projection

$$E_\parallel\left(B/B_\theta\right)=E_\theta$$

As mentioned above there is this *electric field* that still exists after the parallel current $j_\parallel$ has tried to neutralize the charge separation produced by the non-zero divergence of the current of diamagnetic origin. This electric field is due to either

- finite resistivity $\eta=\sigma^{-1}$, or
- Landau damping

3.2 Note on the similar effect with the current of direct ion loss

Part of the new ions, of high energy, produced by NBI, are trapped on banana orbits. They produce two electric currents.

First there is a direct loss, due to the large banana, larger than the distance from the creation and the plasma edge. In DIII-D it was estimated that at counter-injection, about half of the new ions are lost in one bounce period. This is a current, J^{loss}, directed from the plasma core toward the edge and, due to the symmetry, almost along the equatorial plane.

The second contribution comes from the newly created, hot trapped ions, that remain inside but perform a displacement from the point of creation (charge exchange, ionization) to the "center" of the banana orbit. This initial, transitory part of their motion is unique and is an electric current. Multiplied by the number of new trapped NBI ions, it gives a significant current J^{ini}, also laying in the equatorial plane.

Before considering the *return* current and the establishment of a stationary electric field, we note a geometrical effect which is analogue to the relation "diamagnetic current" - "Pfirsch Schluter current".

The problem may be confined to the equatorial plane. The geometry is $ds^2 = dr^2 + r^2 d\varphi^2$ with $h_r = 1$, $h_\varphi = r$. The charges of the total current $J_r = J^{loss} + J^{ini}$ starts from a surface element $dA = rdrd\varphi$ and move radially to an expanded element of area on the equatorial plane, $dA' = r'drd\varphi$, where $dA' > dA$ since $r' > r$. This DOES NOT yet means a non-zero divergence. If a number of N charges of $|e|$ moves from the center of a disc toward the edge of the disc, then

$$J_r = \frac{N\,|e|}{\Delta t \times \Delta l_\perp}$$

is the electric charge that traverses in time Δt a segment $\Delta l_\perp$ perpendicular on r, $\Delta l_\perp = rd\theta$. If the velocity c of the charges is constant, then $\Delta t = \frac{dr}{c}$ where dr is any UNIFORM subdivision of the radius.

$$J_r = \frac{N\,|e|}{(dr/c) \times rd\theta} = \frac{N\,|e|\,c}{drd\theta}\frac{1}{r} = \frac{\text{const}}{r}$$

This only expresses the fact that a constant source of charges from $r = 0$ is spreaded while it advences radially. The divergence in cylindrical coordinates of this radial current is zero

$$\begin{aligned} \nabla \cdot \mathbf{J}_r &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\text{const}}{r}\right) \\ &= 0 \end{aligned}$$

There is no need for any other current, since there is nothing to correct: the current is conserved and the divergence is zero.

We see the difference between this example and the Pfirsch Schluter example.

In the PS case, the amplitude of the current is modified but not because the charges are agglomerated in a smaller area during their poloidal advancement on the surface. The amplitude of the current is modified since there is a dependence on the magnetic field and this has changed when the package of charges moves in the surface from low field to high field. This is the reason for a non-zero divergence.

If we want to have a similar situation for the equatorial current due to banana widening and direct loss, then we must take into account the local dependence on the magnetic field.

Consider the widening of the orbit of a new ion until it reaches the final, periodic, motion on banana. It lies in the equatorial plane and has a radial direction

$$J_r^{eqt} \approx -\frac{1}{2} |e| \dot{n}_0^{new-ions} \left(\frac{\partial S}{\partial r} \right) \rho_i^2 q^2 \varepsilon^{-1/2}$$

This current depends on r through the source

$$\dot{\Gamma}(r) \equiv \dot{n}_0^{new-ions} \left(\frac{\partial S}{\partial r} \right)$$

and also through

$$\rho_i^2 q^2 \varepsilon^{-1/2} \approx \frac{2T}{|e|^2} m_i \frac{1}{B_\theta^2} \varepsilon^{3/2}$$

Since

$$\mathbf{\nabla} \cdot \mathbf{J}_r^{eqt}(r) = \frac{1}{r} \frac{\partial}{\partial r} \left[r J_r^{eqt}(r) \right] \neq 0$$

there is a current flowing along the parallel direction which must compensate and keep $\mathbf{\nabla} \cdot \mathbf{J}^{tot} = 0$. Although we know and we will introduce later a limitation on the poloidal angle - due to limited extension of sources - the operator of divergence cannot include $\partial / r\partial\theta$ since the current is only radial. We write

$$\nabla_\| j_\|^c = -\frac{1}{r} \frac{\partial}{\partial r} \left[r J_r^{eqt}(r) \right]$$

We note that

$$B_\theta = \frac{b(r)}{h}$$

we have

$$\frac{1}{B_\theta^2} \varepsilon^{3/2} = \frac{1}{b^2(r)} h^2 \varepsilon^{3/2} = \frac{1}{b^2(r)} \varepsilon^{3/2} (1 + 2\varepsilon \cos\theta)$$

Then

$$\begin{aligned}
\frac{1}{qR} \frac{\partial}{\partial \theta} j_\|^c &= \frac{1}{r} \frac{\partial}{\partial r} \left\{ r \left[\frac{1}{2} |e| \dot{\Gamma}(r) \frac{2T}{|e|^2} m_i \frac{1}{B_\theta^2} \varepsilon^{3/2} \right] \right\} \\
&= \frac{1}{r} \frac{\partial}{\partial r} \left\{ r \frac{1}{2} \dot{\Gamma} \frac{2T}{|e|} m_i \frac{1}{b^2(r)} \varepsilon^{3/2} (1 + 2\varepsilon \cos\theta) \right\} \\
&= F(r) + G(r) \cos\theta
\end{aligned}$$

where

$$F(r) \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\dot{\Gamma}\frac{T}{|e|}m_i\frac{1}{b^2(r)}\varepsilon^{3/2}\right)$$
$$G(r) \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{1}{2}\dot{\Gamma}\frac{2T}{|e|}m_i\frac{1}{b^2(r)}\varepsilon^{5/2}\right)$$

This gives a compensating parallel current

$$j_{\|}^{c} = qRF(r)\int_{-\Delta\theta/2}^{\Delta\theta/2} d\theta' + qRG(r)\sin\theta$$

This is the analog of the Pfirsch Schluter current. We have placed a limit $\Delta\theta$ to the poloidal extension since the source is however limited poloidally. This current, in contrast with the PS current changes sign above and below the equatorial plane. Then on one side of it it will reduce the PS current and on the other side will increase it. This moves the maximum of the PS current from $\theta = 0$ to finite θ. The comparison of the magnitudes is

$$\left|\frac{qR}{r}\frac{\partial}{\partial r}\left(r\;\dot{\Gamma}\frac{2T}{|e|}m_i\frac{1}{b^2(r)}\varepsilon^{5/2}\right)\right| \text{ versus}$$
$$\left|\varepsilon\frac{2}{B_\theta}\frac{dp}{dr}\right|$$

$$\frac{2qR}{\Delta r\,|e|\,b^2(r)}\varepsilon^{5/2}\frac{\left|\dot{n}_0^{new-ions}\left(\frac{\partial S}{\partial r}\right)\right|}{n}(nT)\,m_i \times \frac{\Delta r}{2\varepsilon\frac{h}{b(r)}(nT)}$$
$$\sim \quad qR\frac{1}{|e|\,b(r)\,h}\varepsilon^{3/2}\frac{\left|\dot{n}_0^{new-ions}\left(\frac{\partial S}{\partial r}\right)\right|}{n}$$

$$\frac{rB_T}{RB_\theta}\frac{1}{|e|\,(b(r)/h)\,h^2}\varepsilon^{3/2}m_i = qR\frac{h^2}{|e|\,B_\theta}m_i\varepsilon^{3/2}$$
$$\approx qR\frac{1}{\Omega_{i\theta}}\varepsilon^{3/2}$$

and the other factor

$$\frac{\left|\dot{n}_0^{new-ions}\left(\frac{\partial S}{\partial r}\right)\right|}{n} \approx \frac{1}{\Delta t}\frac{\delta n^{new-ion}}{n}\frac{1}{\Delta r}$$

and their product

$$qR\frac{1}{\Omega_{i\theta}}\varepsilon^{3/2} \times \frac{1}{\Delta t}\frac{\delta n^{new-ion}}{n}\frac{1}{\Delta r}$$
$$= q\varepsilon^{3/2}\frac{R}{\Delta r}\frac{1}{\Omega_{i\theta}\Delta t}\frac{\delta n^{new-ion}}{n}$$

we have

$$
\begin{aligned}
\Delta t &> \Omega_{\theta i}^{-1} \\
\frac{1}{\Omega_{\theta i}\Delta t} &< 1
\end{aligned}
$$

and

$$
\frac{\left|j_{\parallel}^{c}\right|}{\left|j^{PS}\right|} = \frac{|\text{compensating current density}|}{|\text{PS current density}|} \sim 1
$$

or comparable.

3.3 Resistive plasma equilibrium of flows and currents (Stringer PRL)

The equations involve in this approach the *neoclassical drifts* of the particles

$$
\begin{aligned}
\mathbf{v} &= \widehat{\mathbf{n}} v_{\parallel} + \mathbf{v}_{Dj} \\
&\quad + \mathbf{V}_E^{(0)} + \mathbf{V}_E^{(1)} + ...
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{v}_{Dj} &= \frac{1}{\Omega_j}\widehat{\mathbf{n}} \times \frac{v_{\perp}^2/2 + v_{\parallel}^2}{R}\left(-\widehat{\mathbf{e}}_R\right) \\
&\approx -\frac{1}{e_j B}\frac{2T_j}{R}\widehat{\mathbf{e}}_{vert}
\end{aligned}
$$

it was taken

$$
\frac{2T_j}{m_j} = v^2
$$

The other velocity is *electric*

$$
\begin{aligned}
\mathbf{V}_E^{(0)} &= \frac{-\nabla\phi^{(0)} \times \widehat{\mathbf{n}}}{B} \\
\mathbf{V}_E^{(1)} &= \frac{-\nabla\phi^{(1)} \times \widehat{\mathbf{n}}}{B}
\end{aligned}
$$

In the equation of continuity we have the divergence of the flow of the particles caused by their *neoclassical drift*.

$$
\nabla\cdot\left(n\mathbf{v}_{Dj}\right) = -\frac{1}{e_j B}\frac{2T_j}{R}\frac{dn_0}{dr}\sin\theta
$$

The divergence of the flow of particles moving with the electric velocity is

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\left(\mathbf{V}_E^{(0)}\right) &= \boldsymbol{\nabla}\cdot\left[\frac{-\boldsymbol{\nabla}\phi^{(0)}\times\mathbf{B}}{B^2}\right] \\
&= -\boldsymbol{\nabla}\left(\frac{1}{B^2}\right)\left[-\boldsymbol{\nabla}\phi^{(0)}\times\mathbf{B}\right] \\
&\quad +\frac{1}{B^2}\left(\boldsymbol{\nabla}\times\mathbf{B}\right)\cdot\boldsymbol{\nabla}\phi^{(0)}
\end{aligned}$$

The last term is zero since the gradient of the potential is almost radial and $\perp$ on $\mathbf{B}$.

The first term is purely geometrical, comes from the variation of the magnitude of the magnetic field. It is

$$-\boldsymbol{\nabla}\left(\frac{1}{B^2}\right)\left[-\boldsymbol{\nabla}\phi^{(0)}\times\mathbf{B}\right] = -\frac{2}{R}v_{E\theta}^{(0)}\sin\theta$$

The equation of continuity is

$$\begin{aligned}
&\left(\mathbf{V}_E^{(0)}\cdot\boldsymbol{\nabla}\right)n_1+\left(\mathbf{V}_E^{(1)}\cdot\boldsymbol{\nabla}\right)n_0 \\
&+n_0\frac{\partial v_{\parallel j}}{\partial s} \\
=\ &\frac{2}{R}\left(\frac{1}{e_j B}\frac{dp_j}{dr}+n_0V_{E\theta}^{(0)}\right)\sin\theta
\end{aligned}$$

This equation is familiar. In **Hassam Drake** we find

$$\begin{aligned}
&\frac{\partial n_1}{\partial t}+V_E\frac{\partial n_1}{r\partial\theta}+n_0V_E\left(-2\varepsilon\frac{\sin\theta}{r}\right) \\
&+n_0\nabla_{\parallel}u_{\parallel} \\
=\ &S-\frac{1}{r}\frac{\partial}{\partial r}\left(r\Gamma_r\right)
\end{aligned}$$

and the differences are clear:

Stringer does not expect a time variation of the perturbation of the density profile $n_1(r,\theta)$ in *time*. Or, **Hassam Drake** are looking for spin-up and the time variation must be kept. In the induced spin-up the variation of the density in the surafec $n_1(r,\theta,t)$ *has time variation*.

Stringer does not include *sources*. Indeed for spontaneous spin-up the sources are not stricly needed. **Hassam Drake** invoke the variation of the sources on θ for spin-up.

The terms

$$\left(\mathbf{V}_E^{(0)}\cdot\boldsymbol{\nabla}\right)n_1 = V_E\frac{\partial n_1}{r\partial\theta}$$

are the same.

the parallel derivatives are the same

$$n_0 \frac{\partial v_{\| j}}{\partial s} = n_0 \nabla_{\|} u_{\|}$$

Now, the terms (from Stringer)

$$\frac{2}{R} \left(\frac{1}{e_j B} \frac{dp_j}{dr} + n_0 V_{E\theta}^{(0)} \right) \sin\theta$$

and (after transfering the term in the same side for comparison, the sign changes) respectively (from HD)

$$n_0 V_E \left(2\varepsilon \frac{\sin\theta}{r} \right)$$

We can recognize the last term

$$\begin{aligned} & & \frac{2}{R} n_0 V_{E\theta}^{(0)} \sin\theta \quad , \text{ since} \\ n_0 V_E \left(2\varepsilon \frac{\sin\theta}{r} \right) & = & \frac{2}{R} n_0 V_E \sin\theta \end{aligned}$$

But the first term

$$\frac{2}{R} \frac{1}{e_j B} \frac{dp_j}{dr} \sin\theta$$

is *not present at Hassam Drake*. Its presence at Stringer is the result of :

- using the neoclassical drift of the particles, $v_{drift,j}$. These drifts are considered responsible for a flow
- approximation of the square-velocity part in the $v_{drift,j}$ which is $v_{\perp}^2/2 + v_{\|}^2 \sim \frac{2T_j}{m_j}$.

By this approximation, the expression of the velocity appears similar with that of the *diamagnetic* flow.

But the point of departure is NOT the diamagnetic flow.

Most probable, Stringer did not want to use the *diamagnetic* flow since this has the serious obstacle that it cannot have a *divergence*. There is no meaning to apply the operator of divergence to the diamagnetic "flow" since the guiding centers are fixed.

On the contrary, the *particle drifts* $v_{drift,j}$ exist and the flows (electron ions) created by these *neoclassical drifts* produce the Pfirsch Schluter parallel current in the end.

$[-\phi]$

Using this form of the *continuity equation* Stringer obtains the Pfirsch Schluter current, and this results precisely from the term where the *neoclassical drift* flows have been expressed in terms of the gradient of the pressure (as if they would come from diamagnetic: they do not come from diamagnetic).

We write the equation of continuity for electrons and for ions and substract the two equations. One obtains

$$
\begin{aligned}
j_{\|} &= -2q\frac{1}{B_0}\frac{dp}{dr}\cos\theta \\
&\quad \text{Pfirsch Schluter current}
\end{aligned}
$$

The two terms $\boldsymbol{\nabla}\cdot\mathbf{V}_E^{(0)} = -\frac{2}{R}V_{E\theta}^{(0)}\sin\theta$ cancel since they are identical for electrons and ions. The part that contains the *neoclassical drifts* is the source of the gradient of pressure dp/dr.

The next equation is the momentum conservation where the basic flow is the electric velocity $V_{E\theta}^{(0)}$. The equation is

$$
nm_i\left(V_{E\theta}^{(0)}\frac{\partial}{r\partial\theta}\right)v_{\|,i} = -\left(T_i+T_e\right)\frac{\varepsilon}{q}\frac{\partial n_1}{r\partial\theta}
$$

This balance of momenta involves the inertial static advection of the parallel velocity and the gradient of the pressure along the magnetic field line. In the right hand side there is the gradient of the pressure along the magnetic line. Since the latter can only come from *poloidal variation*, the poloidal derivative is projected on the parallel direction.

The Ohm's law

$$
\eta j_{\|} = \frac{\varepsilon}{q}\frac{\partial}{\partial\theta}\left[-\phi^{(1)}+\frac{T_e}{|e|}\frac{n_1}{n_0}\right]
$$

The first term is the electric field resulting from the variation of the electrostatic potential in the surface, $\phi^{(1)}\left(r,\theta\right)$. The second term is of the nature $\mathbf{v}\times\mathbf{B}$ (as expected in the Ohm's law) and is the parallel projection of the poloidal velocity sustained by the variation of the pressure, through density $n_1\left(r,\theta\right)$ in the magnetic surface, $\nabla_{\|}p_1\left(r,\theta\right)\sim T_e\nabla_{\|}n_1\left(r,\theta\right)$.

The Pfirsch Schluter current $j_{\|}$ allows now to write the system of equations for n_1 and $\phi^{(1)}$, perturbations on surface.

$$
\begin{aligned}
n_1 &= n_0\frac{1}{D}2\varepsilon\frac{1}{V_{E\theta}^{(0)}}\left[-\left(v_i^{dia}+V_{E\theta}^{(0)}\right)\cos\theta\right. \\
&\qquad\left.+\eta\left(\frac{1}{n_0}\frac{dn_0}{dr}\right)\left(\frac{1}{B}\frac{dp}{dr}\right)\left(\frac{q^2}{\varepsilon^2}\right)\frac{r\sin\theta}{B}\right]
\end{aligned}
$$

where

$$
v_j^{dia} = \frac{T_j}{e_jB}\left(\frac{1}{n_0}\frac{dn_0}{dr}\right)
$$

$$
c_s^2 = \frac{T_i+T_e}{m_i}
$$

and the denominator

$$\begin{aligned} D &= 1+\frac{v_e^{dia}}{V_{E\theta}^{(0)}} \\ &\quad -\frac{c_s^2}{\left(V_{E\theta}^{(0)}\right)^2}\frac{\varepsilon^2}{q^2} \end{aligned}$$

We **Note** that the combination

$$1+\frac{v_i^{dia}}{V_{E\theta}^{(0)}}$$

and the combination

$$1+\frac{v_e^{dia}}{V_{E\theta}^{(0)}}=1-\frac{\left|v_e^{dia}\right|}{V_{E\theta}^{(0)}}$$

occur in the expression of n_1. But, in D, the posible resonance $1-\frac{\left|v_e^{dia}\right|}{V_{E\theta}^{(0)}}=0$ only involves the *electrons*. It is avoided by the quantity

$$\begin{aligned} &-\frac{c_s^2}{\left(V_{E\theta}^{(0)}\right)^2}\frac{\varepsilon^2}{q^2} \\ = &-\frac{c_{s\theta}^2}{\left(V_{E\theta}^{(0)}\right)^2} \end{aligned}$$

which compares the poloidally projected sound velocity to the electric poloidal velocity.

We also **NOTE** the presence of the combination of terms

$$1-\frac{\left|v_e^{dia}\right|}{V_{E\theta}^{(0)}}\sim 1-\frac{\left|v_e^{dia}\right|}{u}$$

that is also present in the expression of a drift vortex moving in plasma. This is the factor that defines the *effective Larmor radius* (see our works on the role of this effective ρ_s)

$$\frac{1}{\left(\rho_s^{eff}\right)^2}=\frac{1}{\rho_s^2}\left(1-\frac{v^{dia}}{u}\right)$$

It is then useful to reflect to the explanation given by **Stringer** to this term.

The radial flux of particles is calculated as an average over the magnetic surface, $\theta\in(0,2\pi)$. The quantity that is averaged is the local radial flux obtained as product between the density (zero-order n_0 plus the correction for variation in the surface, $n_1(\theta)$) and the radial velocity. The radial velocity is the *neoclassical drifts* $v_{drift,j}\big|_r$ plus the electric contribution produced by the variation of the potential in the surface

$$\begin{aligned}\Gamma_{rj} &= n v_{rj} \\ &= \int_0^{2\pi} \frac{d\theta}{2\pi}\left(n_0+n_1\right)\left(\frac{1}{B_0}\frac{\partial\phi^{(1)}}{r\partial\theta}+\frac{T_j}{e_j B_0}\varepsilon\frac{2\sin\theta}{r}\right)\left(1+\varepsilon\cos\theta\right)^2\end{aligned}$$

The second term in the second paranthesis is

$$\frac{T_j}{e_j B_0}\varepsilon\frac{2\sin\theta}{r}\approx v_{drift,j}\big|_{radial}$$

consistent with the approximation adopted by Stringer for the neoclassical drift expressing $v_\perp^2/2+v_\parallel^2$ in terms of Temperature.

The result

$$\begin{aligned}\Gamma_{rj} &= n v_{rj} \\ &= q^2\eta\ 2n_0\frac{1}{D}\left(\frac{1}{B_0}\frac{dp}{dr}\right)\frac{1}{B_0}\left[\frac{c_s^2\frac{\varepsilon^2}{q^2}}{\left(V_{E\theta}^{(0)}\right)^2}+\frac{v_{ion}^{dia}-v_j^{dia}}{V_{E\theta}^{(0)}}\right]\end{aligned}$$

3.4 The equations for the currents and flows

The papers by **Rosenbluth Lee Hazeltine PRL** and PF71

We keep the variation of the magnitude of the magnetic field $B_0(r)$ with the radius, $\frac{dB_0(r)}{dr}\neq 0$, but later we will return to tokamak case.

The line

$$ds^2 = h_r^2 dr^2 + h_\theta^2 d\theta^2 + h_\varphi^2 d\varphi^2$$

The values for circular geometry are

$$\begin{aligned}h_r^C &= 1 \\ h_\theta^C &= r \\ h_\varphi^C &= R_0 + r\cos\theta\end{aligned}$$

The magnetic field will be assumed slightly more general than in circular surfaces, and we will return to this simple geometry by taking $B(r)=B_0=$const.

$$\begin{aligned}B_r &= 0 \\ B_\theta &= \frac{b(r)}{h}=\frac{\varepsilon}{q}\frac{B_0(r)}{h} \\ B_\varphi &= \frac{B_0(r)}{h}\end{aligned}$$

where

$$h = 1 + \varepsilon \cos\theta = \frac{R}{R_0} \quad , \quad \varepsilon = \frac{r}{R}$$

and q is the safety factor. The current is (HLR)

$$J_r = 0$$

The poloidal current, derived exclusively from the Ampere's law $\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j}$ assuming that the magnetic field only depends on r, neglects the basic equilibrium $0 = -\boldsymbol{\nabla} p + \mathbf{j} \times \mathbf{B}$.

$$J_\theta = -\frac{1}{1+\varepsilon\cos\theta}\frac{dB_0(r)}{dr} = -\frac{1}{h}\frac{dB_0(r)}{dr}$$

$$\begin{aligned}
J_\varphi &= -\frac{B_0}{b(r)}\frac{dB_0}{dr}\frac{1}{1+\varepsilon\cos\theta} - \frac{1}{b}\frac{dp}{dr}(1+\varepsilon\cos\theta) \\
&= -\frac{1}{h}\frac{B_0(r)/h}{b(r)/h}\frac{dB_0(r)}{dr} - \frac{1}{b(r)/h}\frac{dp}{dr} \\
&= -\frac{1}{h}\frac{B_\varphi}{B_\theta}\frac{dB_0(r)}{dr} - \frac{1}{B_\varphi}\frac{B_\varphi}{B_\theta}\frac{dp}{dr}
\end{aligned}$$

Since

$$\frac{B_\varphi}{B_\theta} \equiv \Theta^{-1} = \left(\frac{\varepsilon}{q}\right)^{-1}$$

we have

$$\frac{\varepsilon}{q}J_\varphi = -\frac{1}{h}\frac{dB_0(r)}{dr} - \frac{h}{B_0(r)}\frac{dp}{dr}$$

which is identical with HassamKulsrud. We will find later below that the HK result is obtained from the **Grad Shafranov** equation.

We will also use $\mathbf{B} = B_\theta \widehat{\mathbf{e}}_\theta + B_\varphi \widehat{\mathbf{e}}_\varphi$ from where we have

$$|\mathbf{B}| = \sqrt{B_\theta^2 + B_\varphi^2} = \frac{B_0(r)}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}}$$

The perpendicular current $j_\perp$ comes from

$$\mathbf{j} \times \mathbf{B} = -\boldsymbol{\nabla} p$$

where

$$\begin{aligned}
\mathbf{j} \times \mathbf{B} &= j_\perp |\mathbf{B}| (-\widehat{\mathbf{e}}_r) = j_\perp \frac{B_0(r)}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}}(-\widehat{\mathbf{e}}_r) \\
&= -\frac{dp}{dr}\widehat{\mathbf{e}}_r
\end{aligned}$$

from where

$$j_{\perp}=\frac{h}{B_{0}\left(r\right)}\frac{1}{\sqrt{1+\frac{\varepsilon^{2}}{q^{2}}}}\frac{dp}{dr}$$

We notice that it is usual to work with two sets of projections of the current, (j_{θ},j_{φ}) and $\left(j_{\parallel},j_{\perp}\right)$. The connection is ensured by the expressions

$$\widehat{\mathbf{e}}_{\varphi}\cdot\widehat{\mathbf{e}}_{\perp}=-\frac{B_{\theta}}{|\mathbf{B}|}=-\frac{\frac{\varepsilon}{q}\frac{B_{0}(r)}{h}}{\frac{B_{0}(r)}{h}\sqrt{1+\frac{\varepsilon^{2}}{q^{2}}}}=-\frac{\varepsilon}{q}\frac{1}{\sqrt{1+\frac{\varepsilon^{2}}{q^{2}}}}$$

and

$$\widehat{\mathbf{e}}_{\perp}\cdot\widehat{\mathbf{e}}_{\theta}=\frac{B_{\varphi}}{|\mathbf{B}|}=\frac{1}{\sqrt{1+\frac{\varepsilon^{2}}{q^{2}}}}$$

We use the two expressions (j_{θ},j_{φ}) to obtain geometrically $j_{\parallel}$ as

$$j_{\parallel}=j_{\theta}\sin\alpha+j_{\varphi}\cos\alpha$$

where

$$\cos\alpha=\frac{1}{\sqrt{1+\frac{\varepsilon^{2}}{q^{2}}}}\ ,\ \sin\alpha=\frac{\varepsilon}{q}\frac{1}{\sqrt{1+\frac{\varepsilon^{2}}{q^{2}}}}$$

and, as derived above

$$\begin{aligned}
j_{\theta} &= -\frac{1}{h}\frac{dB_{0}\left(r\right)}{dr}\\
j_{\varphi} &= \frac{q}{\varepsilon}\left[-\frac{1}{h}\frac{dB_{0}\left(r\right)}{dr}-\frac{h}{B_{0}\left(r\right)}\frac{dp\left(r\right)}{dr}\right]
\end{aligned}$$

Then

$$j_{\parallel}=\left[-\frac{1}{h}\frac{dB_{0}\left(r\right)}{dr}\right]\frac{\varepsilon}{q}\frac{1}{\sqrt{1+\frac{\varepsilon^{2}}{q^{2}}}}+\frac{q}{\varepsilon}\left[-\frac{1}{h}\frac{dB_{0}\left(r\right)}{dr}-\frac{h}{B_{0}\left(r\right)}\frac{dp\left(r\right)}{dr}\right]\frac{1}{\sqrt{1+\frac{\varepsilon^{2}}{q^{2}}}}$$

or

$$\begin{aligned}
j_{\parallel} &= \left[-\frac{1}{h}\frac{dB_{0}\left(r\right)}{dr}\right]\left(\frac{\varepsilon}{q}\frac{1}{\sqrt{1+\frac{\varepsilon^{2}}{q^{2}}}}+\frac{q}{\varepsilon}\frac{1}{\sqrt{1+\frac{\varepsilon^{2}}{q^{2}}}}\right)\\
&\quad+\left[-\frac{h}{B_{0}\left(r\right)}\frac{dp\left(r\right)}{dr}\right]\frac{q}{\varepsilon}\frac{1}{\sqrt{1+\frac{\varepsilon^{2}}{q^{2}}}}
\end{aligned}$$

$$\begin{aligned}
j_{\parallel} &= \frac{q}{\varepsilon}\left[-\frac{1}{h}\frac{dB_{0}\left(r\right)}{dr}\right]\frac{1}{\sqrt{1+\frac{\varepsilon^{2}}{q^{2}}}}\left(\frac{\varepsilon^{2}}{q^{2}}+1\right)\\
&\quad+\left[-\frac{h}{B_{0}\left(r\right)}\frac{dp\left(r\right)}{dr}\right]\frac{q}{\varepsilon}\frac{1}{\sqrt{1+\frac{\varepsilon^{2}}{q^{2}}}}
\end{aligned}$$

$$j_{\|} = \frac{q}{\varepsilon}\left[-\frac{1}{h}\frac{dB_0(r)}{dr}\right]\sqrt{1+\frac{\varepsilon^2}{q^2}} + \left[-\frac{h}{B_0(r)}\frac{dp(r)}{dr}\right]\frac{q}{\varepsilon}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}$$

This is precisely the result of **Hassam Kulsrud**.

Hazeltine Lee Rosenbluth PF71

That we must use the **Grad Shafranov** equation, which in **Hassam Kulsrud** is

$$\frac{h}{r}\frac{\varepsilon}{q}B_0(r)\frac{d}{dr}\left[\frac{r}{h}\frac{\varepsilon}{q}B_0(r)\right] = -B_0(r)\frac{dB_0(r)}{dr} - h^2\frac{dp(r)}{dr}$$

we use $B_\theta = \frac{\varepsilon}{q}\frac{B_0(r)}{h}$ and divide by B_0 and h

$$\frac{\varepsilon}{q}\frac{1}{r}\frac{d}{dr}\left[rB_\theta(r)\right] = -\frac{1}{h}\frac{dB_0(r)}{dr} - \frac{h}{B_0}\frac{dp(r)}{dr}$$

We remind that the equilibrium equation

$$0 = -\boldsymbol{\nabla}p + \mathbf{j}\times\mathbf{B}$$

becomes the GS equation after using the Ampere's law

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu\mathbf{j}$$

projected on the toroidal (φ) direction

$$\mu_0\mathbf{j} = \boldsymbol{\nabla}\times\mathbf{B}\big|_\varphi = \frac{h_\varphi}{h_r h_\theta h_\varphi}\widehat{\mathbf{e}}_\varphi\left[\frac{\partial}{\partial r}(h_\theta B_\theta) - \frac{\partial}{\partial\theta}(h_r B_r)\right] = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)$$

and taking units such that $\mu_0 = 1$, we have

$$j_\varphi = \frac{1}{r}\frac{d}{dr}\left[rB_\theta(r)\right]$$

Replacing in the equilibrium equation we find

$$-\frac{1}{h}\frac{dB_0(r)}{dr} - \frac{h}{B_0}\frac{dp(r)}{dr} = \frac{\varepsilon}{q}J_\varphi$$

and note that actually the **Grad Shafranov** equation provides us with the explicit form of the toroidal component of the current.

We also note that the components of the current obey the zero-divergence condition (charge continuity)

$$\boldsymbol{\nabla}\cdot\mathbf{j} = 0$$

$$\frac{1}{h_r h_\theta h_\varphi}\left[\frac{\partial}{\partial r}(h_\theta h_\varphi\, j_r) + \frac{\partial}{\partial\theta}(h_r h_\varphi\, j_\theta) + \frac{\partial}{\partial\varphi}(h_r h_\theta\, j_\varphi)\right] = 0$$

Here we must insert $j_r = 0$ and assume axisymmetry

$$\frac{\partial}{\partial \varphi} (h \; j_\varphi) = 0$$

It results

$$\frac{1}{rh} \frac{\partial}{\partial \theta} (h \; j_\theta) = 0$$

where we use $j_\theta = -\frac{1}{1+\varepsilon \cos\theta} \frac{dB_0(r)}{dr}$. We obtain an identity

$$\frac{\partial}{\partial \theta} \frac{dB_0(r)}{dr} = 0$$

The order of magntitude is

$$\frac{dB_0}{dr} \sim \frac{dp}{dr} \sim b^2 \sim \frac{a^2}{R^2}$$

3.4.1 The velocities

In HLR PRL there is a radial velocity

$$v_r$$

In the treatment of the Pfirsch Schluter current the radial velocity v_r exists ONLY if there is a resistivity $\eta \neq 0$ in the Ohm's law that, according to **Stringer PRL** prevents the exact neutralization of the charge along the magnetic field line.

In another paper the radial flux

$$\Gamma_r = n v_{Dr}^{transp}$$

is calculated on the base of the distribution function determined by solving the drift-kinetic equation and of the particle neoclassical drift $v_{De,i}$, integrated over the velocity space.

4 Externally sustained flows and the neoclassical constraints

Also in **impurity accumulation.**

The diamagnetic rotation of electrons and ions is a consequence of the gradient of pressure. The combined rotation of electrons and ions is a poloidal current, the diamagnetic current. Due to the geometry, the poloidal current must be accompanied by a toroidal current such as to maintain the zero-divergence $\nabla \cdot \mathbf{j} = 0$. An important aspect is the absence of a significant radial component of the electric current $j_r \approx 0$. Then the conservation zero-divergence only involves the parallel and the perpendicular components, or, the toroidal and the poloidal currents. The toroidal current required by $\nabla \cdot \mathbf{j} = 0$ is Pfirsch Schluter. There is a clear relationship between the diamagnetic current (or, its source, the gradient of the pressure) and the toroidal PS current. The expression of the poloidal (diamagnetic) current in terms of dp/dr and the *geometry* determine the expression of the PS current.

Now we can generalize this connection:

if there is a factor that creates poloidal rotation, the poloidal flow MUST still be accompanied by a toroidal flow, in order to preserve the zero-divergence of the fluxes in the magnetic surface. If we assume that $j_r \approx 0$ in general, then the connection will only involve the poloidal current (given) and the toroidal current, induced. This happens, for example, when there is an Internal transport Barrier (ITB) consisting of a poloidal sheared rotation of plasma on a radially limited region; it arises a toroidal current.

The connection poloidal-toroidal is governed by geometry and therefore it is universal.

If the toroidal flow is larger than the PS flow then a poloidal flow which is larger than the diamagnetic flow will occur. However the poloidal flow is strongly damped by the magnetic pumping and there should be a balance: the drive on the poloidal flow, exerted by an externally sustained toroidal flow (like NBI), is constrained by the magnetic pumping. Then the toroidal flow itself cannot increase freely.

However this still has an alternative: $j_r \approx 0$ can be locally broken and only the integrated (on the surface) form remains valid, as in **Hassam Drake spontaneous spin up.**

We must admit that the magnetic pumping acts like a limiting factor against the external mechanism that tries to drive the toroidal rotation.

NOTE that this scheme seems to cover the problem rised before: the geometrical connection between poloidal flow and toroidal flow.

This problem has two stages:

The first stage

1. first we start from classical diamagnetic $\rightarrow$ Pfirsch Schluter connection. We note that it can be extended to any poloidal flow $\rightarrow$ toroidal flow.

2. then we note that $A \equiv B$ means $B \equiv A$ or, a toroidal flow will equally induce a poloidal flow, from geometrical constraints

3. we are led to evaluate the magnetic damping, which will act directly on the poloidal flow and indirectly on the toroidal flow, when this is at the origin of everything.

The second stage

1. a poloidal flow, sustained by some "external" cause, can produce an *inverse Stringer effect*, producing a nonuniformity of the radial flux of particles on θ. For this to be seen as the consequence of an equivalence, we must review the Stringer effect and eliminate the idea that the poloidal nonuniformity of the flux *induces* the poloidal rotation; instead we must adopt the idea that the poloidal nonuniformity $\sim \theta$ of the radial flux $\Gamma(r,\theta)$ *is accompanied* by a poloidal rotation. Then the possibility to look in reversed terms apears normal.

2. a toroidal flow first induces a poloidal flow (from geometry) and this poloidal flow will induce a nonuniformity on θ of the flux Γ.

5 Poloidal nonuniformity of the profiles

This is a neoclassical effect.

Stringer.

There is a text: **plasma, theory, variation in surface**.

5.1 Helander high-Z impurities

This is also in *impurities.tex*.

The paper **Helander 1999 high Z impurities**.

The lowest order drift-kinetic equation

$$v_{\parallel}\nabla_{\parallel}f_a = \sum_b C_{ab}\left(f_a, f_b\right)$$

inplies uniform distribution on the magnetic surface, $f_a(\psi)$ =const.

Define

$$\frac{1}{\tau_{ab}} = \frac{e_a e_b}{\left(4\pi\varepsilon_0\right)^2}\ln\Lambda\frac{1}{m_a^2}\times n_b\frac{1}{v_T^>}\frac{1}{v_{Ta}^2}$$

where

$$v_T^> \equiv \text{ the largest of the thermal velocities } v_{Ta} \text{ and } v_{Tb}$$

An assumption: the density of high-Z impurities is large such that

$$\begin{array}{c} \text{frequency of high-}Z\text{ - ion collisions} \\ \sim \\ \text{frequency of ion-ion collisions} \end{array}$$

Another assumption

$$\begin{aligned} Z_{eff}-1 &= \frac{n_z z^2}{n_i} \\ &\sim O(1) \end{aligned}$$

The time scale for establishing equilibrium in the parallel direction, for high-Z impurities

$$\tau_{\parallel} \sim \frac{L_{\parallel}^2}{v_{Tz}^2 \tau_{zz}}$$

We note : $v_{Tz}\tau_{zz}$ =distance travelled by a high-Z ion between two collisions. $L_{\parallel}/\left(v_{Tz}\tau_{zz}\right)$ how many collisions occur on $L_{\parallel}$. And $L_{\parallel}/v_{Tz}$ =time for an ion to travel on this distance. The ratio

$$\text{(number of collisions)} \times \text{(time to travel)} \sim \left(L_{\parallel}/\left(v_{Tz}\tau_{zz}\right)\right) \times \left(L_{\parallel}/v_{Tz}\right) = \tau_{\parallel} \text{ (to equilibrate)}$$

Time for cross-field particle transport

$$\begin{aligned} \tau_{\perp} &\sim \frac{L_{\perp}^2}{\left(\rho_z^2/\tau_{zi}\right)} \\ &= \frac{L_{\perp}^2}{D} \end{aligned}$$

The ratio

$$\frac{\tau_{\parallel}}{\tau_{\perp}} = \frac{\left(Z_{eff}-1\right)}{z^{3/2}} \Delta^2 << 1$$

and this allows to analyze the parallel equilibration on each magnetic surface separately.

It is assumed that the electrostatic potential is constant on magnetic surfaces

$$\Phi\left(\psi\right) \approx \text{const}$$

NOTE

This means that there is a poloidal rotation. In other papers this is

$$V_0 \widehat{\mathbf{e}}_{\theta} \approx \frac{-\boldsymbol{\nabla}\Phi_0 \times \widehat{\mathbf{n}}}{B}$$

and is involved in denominators like

$$\begin{aligned} &V_0 + \Theta v_{\parallel} \\ &\text{for PARTICLES} \end{aligned}$$

in expressions for the perturbation of the distribution function, f_{1j}. These denominators produce singularities when there is integration over the parallel velocity to generate the density for applying the neutrality. The singularities are treated by the Landau damping procedure, with two terms, one consists of the Plasma Dispersion Function and the other is the residuum. It is of the same nature like $1/x = \mathbf{P}\left(1/x\right) + i\pi\delta\left(x\right)$, **Rozhanski Tendler**, Galeev, etc.

END

NOTE

In **Galeev Sagdeev Liu Novakovskii**

$$v_{\parallel} = \frac{B}{B_{\theta}} V_E + \sim \sqrt{\kappa^2 - \sin^2 \frac{\theta}{2}}$$

END.

However the toroidicity effects and the distribution of high-Z impurities produce a variation of the potential on the magnetic surface,

$$\Phi_1 (\theta)$$

The drift kinetic equation

$$v_{\parallel} \nabla_{\parallel} \left(f_i^{(1)} + \frac{I}{\Omega_{ci}} v_{\parallel} \frac{\partial f_{Mi}}{\partial \psi} \right) + v_{\parallel} \nabla_{\parallel} \left(\frac{|e| \, \Phi^{(1)}}{T_i} \right) f_{Mi} = C_i \left(f_i^{(1)} \right)$$

Note. let us comment the form of the drift-kinetic equation. The *parallel* advection (through $v_{\parallel}$) of the perturbation to the distribution function $f_j^{(1)}$ is connected with the *radial* advection of the equilibrium distribution function f_{Mj} done by the neoclassical drift (radial). In addition there is an energetic effect, created by something new: the electric potential is NOT uniform on the magnetic surface and its variation along the line, $\parallel$, along the parallel direction, together with the high parallel velocity $v_{\parallel}$ gives a energetic/time effect $E_{\parallel}^{(1)} v_{\parallel}$ which acts on the equilibrium distribution function f_{Mj} through $\partial f_{Mj} / \partial \epsilon$. This is why we get $1/T_j$.

We would be tempted to use the name *adiabatic part* for $\frac{I}{\Omega_{ci}} v_{\parallel} \frac{\partial f_{Mi}}{\partial \psi}$.
However we do not have wave or instability evolution.
It is simply static effect.
End.

To first order the LHS can be integrated

$$\begin{aligned} f_i^{(1)} &= -\frac{I}{\Omega_{ci}} v_{\parallel} \frac{\partial f_{Mi}}{\partial \psi} - \frac{|e| \, \Phi^{(1)}}{T_i} f_{Mi} \\ &\quad + h_i (\epsilon_0, \mu, \psi, \sigma) \end{aligned}$$

where

$$h_i \equiv 0 \text{ for TRAPPED}$$

The velocity is mainly due to the *zero*-order electric potential uniform on the surface.

$$\begin{aligned} \mathbf{V}_{\perp z} &= \frac{-\boldsymbol{\nabla} \Phi_0 \times \widehat{\mathbf{n}}}{B} \\ &= \frac{d\Phi_0}{d\psi} \left(\frac{I}{B} \widehat{\mathbf{n}} - R^2 \boldsymbol{\nabla} \varphi \right) \end{aligned}$$

The paranthesis results from the vector of magnetic field.

the equation of continuity for the high-Z ions

$$\boldsymbol{\nabla}\cdot\left(n_z \mathbf{V}_z\right) = 0$$

The parallel velocity results from this equation of continuity

$$\begin{aligned} V_{\parallel z} &= -\frac{I}{B}\frac{d\Phi^{(0)}}{d\psi} \\ &\quad +K_z\left(\psi\right)\frac{B}{n_z} \end{aligned}$$

The second term introduces a function $K_z\left(\psi\right)$ proportional with the flux function. It is a constant of integration of the operator of gradient in the parallel direction.

NOTE
this latter term is important.
This is because the first term

$$-\frac{I}{B}\frac{d\Phi^{(0)}}{d\psi} = -\frac{RB_\varphi}{B}\frac{d\Phi^{(0)}}{d\psi}$$

is the projection of the poloidal rotation along the parallel direction. For this we remind that

$$\frac{\partial}{\partial\psi} = \frac{1}{RB_\theta}\frac{\partial}{\partial r}$$

Then

$$\begin{aligned} -\frac{I}{B}\frac{d\Phi^{(0)}}{d\psi} &= -\frac{RB_\varphi}{B}\frac{d\Phi^{(0)}}{d\psi} = -\frac{RB_\varphi}{B}\frac{1}{RB_\theta}\frac{\partial\Phi^{(0)}}{\partial r} = -\frac{B_\varphi}{B_\theta}\left[\frac{1}{B}\frac{\partial\Phi^{(0)}}{\partial r}\right] \\ &= \frac{B_\varphi}{B_\theta}V_\theta^{(0)} \end{aligned}$$

This is indeed the *projection of the poloidal velocity along the direction parallel to the magnetic line.*

Or, if there is another reason for parallel motion of the fluid, one will find it only as an addition to this projected poloidal rotation. This is the reason of $K\left(\psi\right)$.
END

NOTE
We remark that the *kinetic* treatment and the *fluid* eqquations are ere MIXED. We calculate the correction to the distribution function $f_j^{(1)}$ but we use the equation of continuity.
END

The parallel friction between the hydrogen ions and the impurities.

The operator of collision

$$\begin{aligned} C_{iz} &= \frac{1}{2}\nu_{iz}(v)\frac{\partial}{\partial\xi}\left(1-\xi^2\right)\frac{\partial}{\partial\xi} \quad \text{(pitch angle)} \\ &\quad +\nu_{iz}\frac{m_i v_\| V_{\|z}}{T_i}f_{Mi} \quad \text{(collisional coupling with a parallel flow)} \end{aligned}$$

where the frequency is

$$\nu_{iz}(v) = \frac{3}{4}\sqrt{\pi}\frac{1}{\tau_{iz}}\left(\frac{v_{Ti}}{v}\right)^3$$

The result of calculation of the friction

$$\begin{aligned} R_{\|z} &= -\int m_i v_\| C_{iz}\left(f_i^{(1)}\right)d^3v \quad \text{(parallel friction force)} \\ &= -p_i\frac{I}{\Omega_i}\frac{1}{\tau_{iz}}\left(\frac{d\ln p_i}{d\psi}-\frac{3}{2}\frac{d\ln T_i}{d\psi}\right) \\ &\quad +\frac{m_i n_i}{\tau_{iz}}\left(u-\frac{K_z}{n_z}\right)B \end{aligned}$$

where

$$u = \frac{\tau_{iz}}{n_i B}\int v_\| \ \nu_{iz} \ h_i \ d^3v$$

is a velocity that results from the integration of the second term in C_{iz} using the expression for $V_{\|z}$ which contains K_z.

NOTE

This result is interesting.

We have an explicit form of the integral over the collisional operator, with the objective to calculate the *friction* **force**.

END

The volume

$$d^3v = \frac{B}{v_\|}d\epsilon_0 d\mu$$

the parallel momentum

$$\begin{aligned} & \widehat{\mathbf{n}}\cdot m_z n_z\left(\mathbf{V}_z\cdot\boldsymbol{\nabla}\right)\mathbf{V}_z \quad \text{(static parallel convective term)} \\ = \ & -\nabla_\| p_z - \widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_z \ \text{(parallel gradient of pressures)} \\ & -n_z\left|e\right| z \ \nabla_\|\Phi^{(1)} \quad \text{(electric field in } \| \text{ direction)} \\ & +R_{\|z} \quad \text{(collisional friction)} \end{aligned}$$

The parallel heat balance

$$
\begin{array}{rcl}
 & & \frac{3}{2} n_z \left(\mathbf{V}_z \cdot \boldsymbol{\nabla}\right) T_z \quad \text{(static parallel convective term)} \\
 & = & -p_z \left(\boldsymbol{\nabla} \cdot \mathbf{V}_z\right) \quad \text{(compressibility, source of heat)} \\
 & & -\boldsymbol{\nabla} \cdot \mathbf{q}_z \quad \text{(divergence of heat flow)} \\
-\boldsymbol{\pi}_z & : & \boldsymbol{\nabla}\ \mathbf{V}_z \quad \text{(pressure term)} \\
 & & +Q_{zi} \quad \text{(source)}
\end{array}
$$

The compressional heating

$$
p_z \left(\boldsymbol{\nabla} \cdot \mathbf{V}_z\right) \sim \delta p_z \frac{v_{Ti}}{L_{\|}}
$$

The divergence of the *diamagnetic* HEAT flow

$$
\begin{array}{rl}
 & \boldsymbol{\nabla} \cdot \mathbf{q}_z \\
= & \boldsymbol{\nabla} \cdot \left(\frac{5}{2} p_z \frac{1}{|e| B} \widehat{\mathbf{n}} \times \boldsymbol{\nabla} T_z\right) \\
\sim & \delta p_z \frac{v_{Ti}}{z L_{\|}}
\end{array}
$$

Most important, the ion-impurity energy equilibration: we calculate the ratio between the heat resulting from *compressibility* of the plasma, $\boldsymbol{\nabla} \cdot \mathbf{v} \neq 0$, and the heat exchanges between ions and impurities

$$
\begin{array}{rcl}
p_z \frac{\boldsymbol{\nabla} \cdot \mathbf{V}_z}{Q_{zi}} & \sim & \frac{z \boldsymbol{\nabla} \cdot \mathbf{q}_z}{Q_{zi}} \\
 & \sim & \frac{\delta p_z \left(\frac{v_{Ti}}{L_{\|}}\right)}{n_z \frac{T_i - T_z}{\tau_{zi}}} \sim \frac{\delta}{z \widehat{\nu}_{ii}} \frac{T_i}{T_i - T_z}
\end{array}
$$

It leads to

$$
\frac{T_i - T_z}{T_i} \sim \frac{\delta}{z \widehat{\nu}_{ii}} \ll 1
$$

The equation

$$
n_z\ z \left|e\right| \nabla_{\|} \Phi^{(1)} + T_i \nabla_{\|} n_z = R_{\| z}
$$

this is the fundamental equation, expressing the balance along the magnetic field line. **NOTE** that it is also the basic equation for the dynamics of the electrons in the case of the drift instabilities, **Diamond Lee**. **END.**

There is an identity resulting from the averaging along the parallel direction of the above equation

The solubility condition is obtained by multiplying by B and averaging over the magnetic surface

$$\left\langle BR_{\|z}\right\rangle = 0$$

where we use the average operator as

$$\langle ... \rangle = \frac{\oint \frac{d\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta} (...)}{\oint \frac{d\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}}$$

It is determined the parallel flow K_z.

It is inserted in the parallel balance, in the friction force

$$\begin{aligned}
& \left(T_i + \frac{n_z z^2}{2n_0} T_0\right) \nabla_{\|} n_z \\
= \quad & -p_i \frac{I}{\Omega_{ci}} \frac{1}{\tau_{iz}} \left(\frac{d}{d\psi}\ln p_i - \frac{3}{2}\frac{d}{d\psi}\ln T_i\right)\left(1 - \frac{\langle n_z\rangle}{n_z}\frac{B^2}{\langle B^2\rangle}\right) \quad \text{(friction)} \\
& + \frac{m_i n_i}{\tau_{iz}} \frac{1}{n_z} u \left(n_z - \frac{\left\langle n_z B^2\right\rangle}{B^2}\right) B \quad \text{(friction)}
\end{aligned}$$

There is a connection between the potential and the density of impurities

$$n_z = \frac{n_z - n}{z} = \frac{2n_0}{z}\frac{|e|\,\Phi^{(1)}}{T_0}$$

where

$$\frac{2n_0}{T_0} \equiv \frac{n_{e0}}{T_e} + \frac{n_{i0}}{T_i}$$

NOTE remember the same equation adopted by **Wiley Hinton** for the response of plasma to an applied electric field, a problem that geneeralizes the Spitzer one. **END.**

New definitions

$$\begin{aligned}
n &\equiv \frac{n_z}{\langle n_z\rangle} \\
b &= \frac{B}{\left\langle B^2\right\rangle^{1/2}} \\
\alpha &\equiv \frac{\langle n_z\rangle z^2 T_0}{2n_0 T_i}
\end{aligned}$$

$$\gamma \equiv \frac{-\frac{|e|u}{IT_i}\left\langle B^2\right\rangle}{\frac{d}{d\psi}\ln p_i - \frac{3}{2}\frac{d}{d\psi}\ln T_i}$$

a new coordinate

$$d\vartheta = \frac{\langle \mathbf{B} \cdot \boldsymbol{\nabla}\theta \rangle}{\mathbf{B} \cdot \boldsymbol{\nabla}\theta} d\theta$$

The parallel momentum

$$(1+\alpha n) \frac{\partial n}{\partial \vartheta} = g \left[n - b^2 + \gamma \left(n - \left\langle n b^2 \right\rangle \right) b^2 \right]$$

where

$$g = -z^2 \frac{IB}{\Omega_{ci}} \frac{1}{\tau_{ii}} \frac{1}{\langle \mathbf{B} \cdot \boldsymbol{\nabla}\theta \rangle} \left(\frac{d}{d\psi} \ln p_i - \frac{3}{2} \frac{d}{d\psi} \ln T_i \right)$$

Two extreme situations, according to the magnitude of g.

First case the pressure and the temperature gradients are small

$$g \ll 1$$

Then, assuming

$$\left(n - \left\langle n b^2 \right\rangle \right) b^2 \sim O(g) \ll 1$$

and the density is almost constant on the magnetic surface

$$n \approx 1$$

it results

$$(1+\alpha) \frac{\partial n}{\partial \vartheta} = g \left(1 - b^2 \right)$$

Here we remind that

$$b \equiv \frac{B}{\left\langle B^2 \right\rangle^{1/2}}$$

and this means that the term b^2 has the poloidal variation of B^2.

It results that there is an asymmetry which is *up-down*.

The second case

$$g \gg 1$$

then

$$n = n_0 + n_1$$

$$n_0 = \frac{\gamma}{1 - \left\langle \frac{1}{1+\gamma b^2} \right\rangle} \frac{b^2}{1+\gamma b^2}$$

which is in-out asymmetry.

the correction n_1 is *up-down* asymmetry.

5.1.1 The variation of the transport rate of the Hydrogen ions on the poloidal direction

The flux

$$\langle \mathbf{\Gamma}_i \cdot \boldsymbol{\nabla}\psi \rangle = \left\langle \frac{I}{|e|\, B} R_{\| z} \right\rangle$$

5.2 The variation of the electric potential and density on the magnetic surface (Stringer PRL)

The text is also in *Stringer.tex*, in *Studies, Plasma.* And *Neoclass2.tex.*

The *particle* drift velocity in **Stringer PRL**

$$\begin{aligned} \mathbf{v}_{drift,i} &= -\frac{T_i}{|e|\, B}\frac{1}{R}\widehat{\mathbf{e}}_z = -\frac{1}{\Omega_i}\frac{T_i/m_i}{R}\widehat{\mathbf{e}}_z \\ & \quad \text{(vertical)} \end{aligned}$$

This is an approximation of the neoclassical drift of a particle.

The divergence of the flux associated with the particles' drifts

$$\boldsymbol{\nabla}\cdot (n\mathbf{v}_{drift,i}) = -\frac{1}{\Omega_i}\frac{T_i/m_i}{R}\frac{dn_0}{dr}\sin\theta$$

The electric velocity

$$\mathbf{v}_E = \frac{-\boldsymbol{\nabla}\phi \times \widehat{\mathbf{n}}}{B}$$

The divergence

$$\boldsymbol{\nabla}\cdot (\mathbf{v}_E) = -\frac{2}{R} v_{E\theta}\sin\theta$$

where it was assumed that $\boldsymbol{\nabla} \times \mathbf{B} \cdot \boldsymbol{\nabla}\phi = 0$.

The multiple scales

$$\begin{aligned} \mathbf{v}_j &= v_{\| j}\widehat{\mathbf{n}} + \mathbf{v}_{drift,j} \\ & \quad + \mathbf{v}_E \\ & \quad + \mathbf{v}_1 + ... \end{aligned}$$

where

$$\mathbf{v}_1 = \frac{-\boldsymbol{\nabla}\phi_1 \times \widehat{\mathbf{n}}}{B}$$

The **equation of continuity**

$$\boldsymbol{\nabla}\cdot (n\mathbf{v}_i) = 0$$

$$
\begin{aligned}
& \left(\mathbf{v}_E \cdot \boldsymbol{\nabla}\right) n_1 + \left(\mathbf{v}_1 \cdot \boldsymbol{\nabla}\right) n_0 + n_0 \left(\frac{\partial v_{\|}}{\partial l_{\|}}\right) \\
= \; & \frac{1}{\Omega_i} \frac{T_i/m_i}{R} \frac{dn_0}{dr} 2\sin\theta \\
& + n_0 v_{E\theta} \frac{1}{R} 2\sin\theta
\end{aligned}
$$

We recognize in the first term in the RHS

$$
\begin{aligned}
\frac{1}{\Omega_i} \frac{T_i/m_i}{R} \frac{dn_0}{dr} 2\sin\theta &= \frac{2}{R} \frac{p_i'}{|e|\, B} \sin\theta \\
&= -\boldsymbol{\nabla} \cdot \left(n v_{drift,i}\right) \\
&= -\text{the divergence of the guiding center drift fluid flow}
\end{aligned}
$$

and the second term is the divergence of the flow produced by the electric field.

The second condition is the **momentum balance along the parallel direction, with inclusion of the ion inertia**

$$
n m_i v_{E\theta} \frac{\partial v_{\|i}}{r\partial\theta} = -\left(T_i + T_e\right) \frac{\varepsilon}{q} \frac{\partial n_1}{r\partial\theta}
$$

We note the occurence of the *projection factor* $\Theta \equiv \varepsilon/q$.

The third condition is the **Ohm's law projected onto the parallel direction.**

Equivalently, this is the electron momentum balance,

$$
0 = -\boldsymbol{\nabla} p_e - |e|\, n\mathbf{E} - |e|\, n\mathbf{v}_e \times \mathbf{B} + \eta \mathbf{v}_e
$$

$$
\eta \mathbf{j} = \mathbf{E} + \mathbf{v} \times \mathbf{B}
$$

$$
\begin{aligned}
\eta j_{\|} &= -\frac{\varepsilon}{q} \frac{\partial \phi_1}{\partial\theta} \\
&\quad + \frac{\varepsilon}{q} \frac{\partial}{\partial\theta} \left(\frac{T_e}{n_0\, |e|} n_1\right)
\end{aligned}
$$

For the last term, we note that n_0 does not depend on θ, nor T_e. It is like a Boltzmann correction to the electrostatic potential:

besides the correction to the electric potential $\phi_1(\theta)$ *there is a correction to the density* n_1 which is not described by the potential ϕ_1.

The equations of continuity are written for electrons and for ions and they are substracted

$$
\begin{aligned}
j_{\|} &= -q \frac{2p'}{B_0} \cos\theta \\
& \quad \text{the Pfirsch Schluter current}
\end{aligned}
$$

Since we have $j_{\parallel}$ we can use the Ohm's law and obtain equations for ϕ_1 and n_1

$$\begin{aligned} n_1 &= \frac{1}{D}\varepsilon n_0 \frac{1}{v_{E\theta}} \\ &\times \left[-\left(v_{*i}+v_{E\theta}\right)\cos\theta \right. \\ &\left. +\eta\frac{1}{B}\frac{dp}{dr}\frac{1}{B}\left(\frac{1}{n_0}\frac{dn_0}{dr}\right)\left(\frac{\varepsilon}{q}\right)^2 \sin\theta \right] \end{aligned}$$

where

$$\begin{aligned} D &\equiv 1+\frac{v_{*e}}{v_{E\theta}} \\ &-\frac{c_s^2}{v_{E\theta}^2}\left(\frac{\varepsilon}{q}\right)^2 \end{aligned}$$

The diamagnetic velocities contain their sign

$$v_{*j} = \frac{T_j/m_j}{\Omega_j}\frac{1}{n_0}\frac{dn_0}{dr}$$

and

$$c_s^2 = \frac{T_e+T_i}{m_i}$$

5.3 The variation of density and potential on the surface Stringer 1991

This is **pfb3 1991**.

It is a detailed form of the PRL of 1969.

Stringer calculates the correction to the distribution function that is associated with the variation of n and Φ in magnetic surfaces. This variation is a result of toroidality. The input is therefore the drift of the particles.

The treatment is *drift-kinetic*.

$$\begin{aligned} f_j &= f_j^{(0)}\left(r, v_{\parallel}, v_{\perp}^2\right) \\ &\quad +f_j^{(1)}\left(r,\theta, v_{\parallel}, v_{\perp}^2\right)+... \end{aligned}$$

$$\Phi\left(r,\theta\right) = \Phi^{(0)}\left(r\right)+\Phi^{(1)}\left(r,\theta\right)+...$$

The guiding center velocity

$$\begin{aligned} \mathbf{V}_j &= v_{\parallel}\widehat{\mathbf{n}} \\ &+\mathbf{V}_D \\ &+\mathbf{V}^{(0)}+\mathbf{V}^{(1)}+... \end{aligned}$$

where

$$\mathbf{V}_D = -\frac{1}{\Omega_j}\frac{v_\perp^2/2 + v_\parallel^2}{R}\left(\frac{B_\varphi}{B}\widehat{\mathbf{e}}_z - \frac{B_z}{B}\widehat{\mathbf{e}}_\varphi\right)$$

with

$$\Omega_j = \frac{e_j B}{m_j}$$

The drift is mainly vertical,

$$\begin{pmatrix} \widehat{\mathbf{e}}_R & & \\ & B_\varphi & B_z \\ & \widehat{\mathbf{e}}_\varphi & \widehat{\mathbf{e}}_z \end{pmatrix} = \mathbf{B}_{(\varphi,z)} \times \widehat{\mathbf{n}}\big|_R$$

?

In any case the vertical magnetic field is very small

$$\frac{B_z}{B} \ll 1$$

and the velocities are

$$\begin{aligned} \mathbf{V}^{(0)} &= \frac{1}{B}\frac{d\Phi^{(0)}}{dr} \\ \mathbf{V}^{(1)} &= \frac{-\boldsymbol{\nabla}\Phi^{(1)} \times \widehat{\mathbf{n}}}{B} \end{aligned}$$

One uses the notation that is typical for russian articles

$$\begin{aligned} \Theta &\equiv \frac{B_\theta}{B_\varphi} \\ &= \frac{\varepsilon}{q} = O(\varepsilon) \\ &\ll 1 \end{aligned}$$

The diamagnetic velocities

$$\begin{aligned} v_{*j} &= \frac{T_{0j}}{e_j B}\frac{d\ln n_0}{dr} \\ v_{*j}^T &= \frac{1}{e_j B}\frac{dT_{0j}}{dr} \end{aligned}$$

Take the parallel velocity

$$v_\parallel = \sqrt{\frac{2}{m_j}\left(\epsilon - \mu B - e_j\Phi\right)}$$

then

$$\begin{aligned}
\frac{dv_{\parallel}}{dt} &= -\frac{1}{m_j v_{\parallel}} (\mathbf{V}\cdot\boldsymbol{\nabla}) (e_j\Phi + \mu B) \\
&= -\frac{1}{m_j v_{\parallel}} \left(V^{(0)} \frac{\partial}{r\partial\theta} + v_{\parallel}\frac{\partial}{\partial l_{\parallel}} + V_r \frac{\partial}{\partial r} \right) (e_j\Phi + \mu B) \\
&= -\frac{e_j}{m_j}\frac{B_\theta}{B_\varphi}\frac{\partial\Phi^{(1)}}{r\partial\theta} - \varepsilon \left(\frac{B_\theta}{B_\varphi}\frac{v_\perp^2}{2} - V^{(0)} v_{\parallel} \right) \frac{\sin\theta}{r}
\end{aligned}$$

This comes from

$$\begin{aligned}
\frac{dv_{\parallel}}{dt} &= \left(\frac{\partial}{\partial t} + (\mathbf{V}\cdot\boldsymbol{\nabla}) \right) v_{\parallel} = (\mathbf{V}\cdot\boldsymbol{\nabla}) v_{\parallel} \\
&= (\mathbf{V}\cdot\boldsymbol{\nabla}) \sqrt{\frac{2}{m_j}(\epsilon - \mu B - e_j\Phi)} \\
&= (\mathbf{V}\cdot\boldsymbol{\nabla}) \frac{1}{2} \frac{1}{\sqrt{\frac{2}{m_j}(\epsilon - \mu B - e_j\Phi)}} \frac{2}{m_j} (-\mu B - e_j\Phi) \\
&= -\frac{1}{m_j}\frac{1}{v_{\parallel}} (\mathbf{V}\cdot\boldsymbol{\nabla}) (\mu B + e_j\Phi)
\end{aligned}$$

The derivation along the magnetic field line is

$$\begin{aligned}
\frac{\partial}{\partial l_{\parallel}} &= \nabla_{\parallel} = \frac{B_\theta}{B_\varphi}\frac{\partial}{r\partial\theta} \\
&= \frac{1}{qR}\frac{\partial}{\partial\theta}
\end{aligned}$$

The radial drift velocity takes into account the existence of the perturbation of the electric potential

$$\begin{aligned}
V_r = &-\frac{1}{B}\frac{\partial\Phi^{(1)}}{r\partial\theta} \\
&-\frac{1}{\Omega_j}\frac{v_\perp^2/2 + v_{\parallel}^2}{R}\sin\theta
\end{aligned}$$

and the particle drifts.

The drift-kinetic equation

$$\frac{\partial f_j}{\partial t} + (\mathbf{V}_j\cdot\boldsymbol{\nabla}) f_j + \frac{\partial f_j}{\partial v_{\parallel}}\frac{dv_{\parallel}}{dt} + \frac{\partial f_j}{\partial v_\perp^2}\frac{dv_\perp^2}{dt} = 0$$

Note that we expect time variation since there will be flows.
It is question of spontaneous spin-up of Stringer.
Then there will be variations of $v_{\parallel}$ and of $v_\perp^2$ when the plasma will traverse the regions with variable magnetic field.

The drift kinetic equation is linearized to order ε.
the result is

$$
\begin{aligned}
f_j^{(1)} &= \frac{1}{V^{(0)} + \Theta v_\parallel} \left\{ \left[\frac{\Phi^{(1)}}{B} - \frac{1}{\Omega_j} \varepsilon \left(\frac{v_\perp^2}{2} + v_\parallel^2 \right) \cos\theta \right] \frac{\partial f_j^{(0)}}{\partial r} \right. \\
& -e_j \frac{v_\perp^2}{v_{th,j}^2} \left(V^{(0)} + \Theta v_\parallel \right) f_j^{(0)} \cos\theta \\
& \left. + \left[\frac{e_j}{m_j} \Theta \Phi^{(1)} - \varepsilon \left(\Theta \frac{v_\perp^2}{2} - V^{(0)} v_\parallel \right) \right] \frac{\partial f_j^{(0)}}{\partial v_\parallel} \right\}
\end{aligned}
$$

The first term

$$
\frac{\Phi^{(1)}}{B} - \frac{1}{\Omega_j} \varepsilon \left(\frac{v_\perp^2}{2} + v_\parallel^2 \right) \cos\theta
$$

is v_r and consists of two parts:

- first $E_\theta \times B_\varphi$ where $E_\theta \sim \nabla_\theta \Phi^{(1)}$ is a poloidal electric field coming from the variation of the electrostatic potential in the surface.
- The second term is the radial projection of the neoclassical drifts of the particles

The second term

$$
-e_j \frac{v_\perp^2}{v_{th,j}^2} \left(V^{(0)} + \Theta v_\parallel \right) f_j^{(0)} \cos\theta
$$

is purely geometric, as in the derivation of the Pfirsch Schluter current drom the diamagnetic flow.

The third term has been derived above

$$
\frac{dv_\parallel}{dt} = -\frac{e_j}{m_j} \frac{B_\theta}{B_\varphi} \frac{\partial \Phi^{(1)}}{r \partial \theta} - \varepsilon \left(\frac{B_\theta}{B_\varphi} \frac{v_\perp^2}{2} - V^{(0)} v_\parallel \right) \frac{\sin\theta}{r}
$$

We note the poloidal velocity, composed of the electric velocity $V_E^{(0)}$ and of the poloidal projection of the parallel velocity $\Theta v_\parallel$.

This combination, representing the poloidal velocity, should be almost zero

$$
V_E^{(0)} + \frac{\varepsilon}{q} v_\parallel \approx 0
$$

This correction to the distribution function contains

- the effect of the drift of the particles v_D.

- the effect of the presence of a potential constant on the magnetic surfaces $\Phi^{(0)}$.
- the effect of a variation of the electric potential in the surface, $\Phi^{(1)}$.

A term

$$\frac{\Phi^{(1)}}{B}\frac{\partial f_j^{(0)}}{\partial r}$$

is the radial advection due to the potential $\Phi^{(1)}$, of the equilibrium distribution function.

A term

$$\left[\Theta\frac{e_j\Phi^{(1)}}{m_j}\right]\frac{\partial f_j^{(0)}}{\partial v_{\parallel}}$$

is the acceleration of the parallel velocity produced by the electric field correction $e_j\Phi^{(1)}$ projected along the parallel direction by $\Theta \equiv B_\theta/B_\varphi$.

Therefore these two terms contain the effect of $\Phi^{(1)}$ on the distribution function (necessarily of zero-order $f_j^{(0)}$ since $\Phi^{(1)}$ is itself small).

The variation of the electric potential in the surface $\Phi^{(1)}$ is determined from the condition of neutrality

$$n_e = n_i$$

We have to obatin the densities by integrating over the velocity space

$$\int dv_{\parallel} \int dv_{\perp}^2$$

The integration over $v_{\perp}^2$ can be done.

The integration over $v_{\parallel}$ is complicated by the singularities of the denominator

$$\frac{1}{V^{(0)} + \Theta v_{\parallel}}$$

and this integration must be treated like Landau singularity.

The distribution function that is to be integrated is the Maxwell function. Then one introduces the *definition*

$$\frac{1}{n_0}\int_{-\infty}^{\infty}\frac{F_j^{(0)}}{v_{\parallel}-W}\ \left(v_{\parallel}^s\right)\ dv_{\parallel} \equiv K_s\left(\frac{W}{v_{th,j}}\right)$$

where s is an integral exponent.

These functions are expressed through the *plasma dispersion function.*

The relations are

$$\begin{aligned} K_s\left(\frac{W}{v_{th,j}}\right) &= W\ K_{s-1}\left(\frac{W}{v_{th,j}}\right) \\ &\quad +J_s \end{aligned}$$

where

$$J_s = \begin{cases} (s-2)(s-4)\ldots 1 \ \left(\frac{v_{th,j}}{2}\right)^{\frac{s-1}{2}} & \text{for} \quad s \text{ odd} \\ 0 & \text{for} \quad s \text{ even} \end{cases}$$

The connection with the Plasma Dispersion Function is

$$\begin{aligned} K_0\left(\frac{W}{v_{th,j}}\right) &= \frac{1}{W}\left[I\left(\frac{W}{v_{th,j}}\right) - 1\right] \\ K_1\left(\frac{W}{v_{th,j}}\right) &= I\left(\frac{W}{v_{th,j}}\right) \end{aligned}$$

where

$$\begin{aligned} I(z) &= 1 - 2z\exp\left(-z^2\right)\int_0^z dt \exp\left(-t^2\right) \\ &\quad + i\sqrt{\pi}\ z\exp\left(-z^2\right) \end{aligned}$$

NOTE that we have here the Principal value and the singularity $i\pi\delta$ which, after integration, gives the Landau term

The expression of the distribution function $f_j^{(1)}$ will be integrated over the velocity space to obtain the densities.

Then neutrality will be invoked, obtaining an equation for the potential $\Phi^{(1)}$.

Definitions

$$\begin{aligned} V_{*n,j} &= \frac{T_{0j}}{e_j B}\frac{1}{n_0}\frac{dn_0}{dr} \\ V_{*T,j} &= \frac{T_{0j}}{e_j B}\frac{1}{T_{0j}}\frac{dT_{0j}}{dr} \end{aligned}$$

Stringer finds that the supplementary velocity $V_j^{(1)}$ induced by the variation of the potential $\Phi^{(1)}$ in the surface is a fraction of the diamagnetic velocity

$$V_j^{(1)} \sim \varepsilon V_{*n,j}$$

The density is

$$\begin{aligned} \frac{n_j^{(1)}}{n_0} &= \frac{e_j\Phi^{(1)}}{T_j} \\ &\quad \times\left[\frac{1}{V^{(0)}}\left(V_{*n,j} - \frac{V_{*T,j}}{2}\right)(1 - I_j) - I_j - \frac{V^{(0)}V_{*T,j}}{v_{th,j}^2\Theta^2}I_j\right] \\ &\quad + \varepsilon\exp(i\theta)\left[\left(1 + \frac{V_{*n,j}}{V^{(0)}} + \frac{V_{*T,j}}{2V^{(0)}}\right)(I_j - 1)\right. \\ &\qquad + 2z_j^2 I_j\left(1 + \frac{V_{*n,j}}{V^{(0)}}\right) \\ &\qquad \left. + z_j^2\frac{V_{*T,j}}{V^{(0)}}\left(1 + 2z_j^2 I_j\right)\right] \end{aligned}$$

The new notations are

$$z_j \equiv -\frac{V^{(0)}}{v_{th,j}\Theta}$$

$$I_j \equiv I(z_j)$$

Note that

$$z_j = -\frac{V_E^{(0)}\frac{\varepsilon}{q}}{v_{th,j}} = -\frac{V^{pol} \ \text{(projected on parallel direction)}}{\text{(thermal velocity)}}$$

After calculation of $n_e^{(1)}$ and $n_i^{(1)}$ it is invoked the neutrality.

The equation for neutrality becomes an equation for the perturbation to the uniform electric potential on the surface: $\Phi^{(1)}(r,\theta)$.

6 Radial electric field Rozhansky Tendler

This is **PF-B 4(7), 1877 (1992).**

The momentum equation

$$nm_i\frac{d\mathbf{u}_i}{dt} = -\nabla(p_e+p_i) - \nabla\cdot\boldsymbol{\pi}_i + \mathbf{j}\times\mathbf{B} + \mathbf{F}$$

where

$$\begin{aligned} \nabla\cdot\boldsymbol{\pi} &\equiv \text{viscosity} \\ \mathbf{F} &\equiv \text{external force, NBI} \end{aligned}$$

The field

$$\mathbf{B} = \frac{\varepsilon}{q}\frac{B_0}{h}\widehat{\mathbf{e}}_\theta + \frac{B_0}{h}\widehat{\mathbf{e}}_\varphi$$

after averaging with

$$\langle()\rangle = \int_0^{2\pi}\frac{d\theta}{2\pi}(1+\varepsilon\cos\theta)\ ()$$

The *parallel* projection of the momentum is

$$\left\langle nm_i\mathbf{B}\cdot\frac{d\mathbf{u}_i}{dt}\right\rangle = -\langle\mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_i\rangle^{neo}$$

The toroidal component of the momentum equation is averaged and gives the *radial current*

$$\langle j_r\rangle = \frac{q}{\varepsilon}\left\langle nm_i\frac{1}{B^2}\mathbf{B}_\varphi\cdot\frac{d\mathbf{u}_i}{dt}\right\rangle + \frac{q}{\varepsilon}\left\langle\frac{1}{B^2}\mathbf{B}_\varphi\cdot\nabla\cdot\boldsymbol{\pi}_i\right\rangle^{neo}$$

The average is important.

At stationarity the equation $\left\langle nm_i \mathbf{B}\cdot \frac{d\mathbf{u}_i}{dt}\right\rangle = -\left\langle \mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_i\right\rangle^{neo}$ leads to

$$\left\langle \mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_i\right\rangle^{neo} = 0$$

which only allows an average poloidal velocity composed of

$$v_\theta = V_E^{(0)} + u_{pi}^{(dia)} + \frac{\varepsilon}{q}\overline{u}_\varphi$$

where

$$\overline{u}_\varphi = \left\langle \frac{B_0}{B_\varphi} u_\varphi \right\rangle$$

In standard neoclassical theory the *radial current* averaged is zero.

$$\boldsymbol{\pi}_i^{(neo)} = \left(p_{\parallel i} - p_{\perp i}\right)\left(\widehat{\mathbf{n}}\,\widehat{\mathbf{n}} - \frac{1}{3}\mathbf{I}\right)$$

7 Multiple equilibria poloidal rotation (Ware Wiley)

It is **PF 24(5) 936 (1981)**.

The equation in first order

$$\frac{\rho_{\theta i}}{L}$$

is

$$\begin{aligned}
\rho\frac{\partial \overline{V}_\theta}{\partial t} &= -\overline{\frac{\left(\widetilde{P}_\parallel + \widetilde{P}_\perp\right)\sin\theta}{R}} \\
&= 0
\end{aligned}$$

where $\widetilde{\ }$ is the part of a function that varies on θ.

We note that in the composition of the pressure it has been included the neoclassical drift, here projected on the poloidal plane.

$$\begin{aligned}
\rho\frac{\partial \overline{V}_\theta}{\partial t} &= -\frac{\sqrt{\pi}}{2} n m_i v_{th,i}\frac{r}{R^2}\left[\overline{V}_\parallel - \frac{E_r}{B_\theta} + \frac{T_i}{eB_\theta}\left(\frac{1}{n}\frac{dn}{dr} - \frac{3}{2}\frac{1}{T}\frac{dT_i}{dr}\right)\right] \\
&= -\frac{\sqrt{\pi}}{2} n m_i v_{th,i}\frac{q}{R}\left(\overline{V}_\theta + \frac{1}{2e}\frac{dT_i}{dt}\right) \\
&= 0
\end{aligned}$$

(we notice in the second line the factor ε/q that was needed to project on poloidal direction) since

$$\overline{V}_\theta = \frac{B_\theta}{B}\overline{V}_\parallel - \frac{E_r}{B} + \frac{1}{enB}\frac{dp_i}{dr}$$

The equailibrium condition is

$$\begin{aligned} & \overline{V}_{\parallel} - \frac{E_r}{B_\theta} + \frac{T_i}{eB_\theta}\left(\frac{1}{n}\frac{dn}{dr} - \frac{3}{2}\frac{1}{T}\frac{dT_i}{dr}\right) \\ = \quad & 0 \end{aligned}$$

or

$$\overline{V}_\theta = -\frac{1}{2}\frac{1}{e}\frac{dT_i}{dt}$$

the coefficient 1/2 is different for different regimes see **Hazeltine**.

$$\mathbf{P} = \sum_j \left(\mathbf{P}_j + \boldsymbol{\pi}_j + n_j m_j \mathbf{V}_j \ \mathbf{V}_j\right)$$

where

$$\mathbf{P}_j = \int d^3v \ m_j \left(\mathbf{v} - \mathbf{V}_j\right) \ \left(\mathbf{v} - \mathbf{V}_j\right) \overline{f}_j^{\zeta}$$

the distribution function is averaged over the gyrophase ζ.

$$\boldsymbol{\pi}_j = \int d^3v \ m_j \left(\mathbf{v} - \mathbf{V}_j\right) \ \left(\mathbf{v} - \mathbf{V}_j\right) \widetilde{f}_j^{\zeta}$$

The distribution function is here the part that depends on the gyrophase ζ.

8 Poloidal rotation Hassam Kulsrud

8.1 Basic equations

The equations

$$\begin{aligned} nm_i\left(\frac{\partial \mathbf{v}}{\partial t} + \left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{v}\right) \quad &= \quad -\boldsymbol{\nabla} p - \boldsymbol{\nabla}\cdot\boldsymbol{\Pi} \\ & \quad\quad +\mathbf{j}\times\mathbf{B} \end{aligned}$$

where

$$\boldsymbol{\Pi} = -3\eta_0\left(\widehat{\mathbf{n}}\ \widehat{\mathbf{n}} - \frac{1}{3}\right)\left(\widehat{\mathbf{n}}\ \widehat{\mathbf{n}} - \frac{1}{3}\right) : \boldsymbol{\nabla}\ \mathbf{v}$$

$$p = n\left(T_e + T_i\right)$$

The heat equation is expressed in terms of the *entropy*

$$nT\left(\frac{\partial s}{\partial t} + \left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)s\right) = -\boldsymbol{\nabla}\cdot\mathbf{q} - \boldsymbol{\Pi}\colon\boldsymbol{\nabla}\ \mathbf{v}$$

with the flux of heat

$$\mathbf{q} = -\chi\frac{1}{B}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}T$$

note that this is *parallel*

$$q_{\|}=\chi_{\|}\frac{1}{B}\nabla_{\|}T$$

The Ohm's law

$$\mathbf{E}+\mathbf{v}\times\mathbf{B}=\eta\mathbf{j}$$

and

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{B} &= \mu_0\mathbf{j}\\ \boldsymbol{\nabla}\times\mathbf{E} &= 0\\ \boldsymbol{\nabla}\cdot\mathbf{B} &= 0\end{aligned}$$

The magnetic field is (more general than circular)

$$\mathbf{B}=\left(0,\frac{b\left(r\right)}{h},\frac{B_0\left(r\right)}{h}\right)$$

The following averaging operator is introduced

$$\left\langle f\right\rangle=\frac{\int\frac{dS}{|\boldsymbol{\nabla}p|}f}{\int\frac{dS}{|\boldsymbol{\nabla}p|}}$$

The equation of continuity

$$\frac{\partial}{\partial t}\left\langle n\right\rangle+\frac{1}{\int\frac{dS}{|\boldsymbol{\nabla}p|}}\frac{\partial}{\partial p}\int\mathbf{dS}\cdot n\mathbf{v}=0$$

The equation for the *circulation*. This is the equation for the surface average of the parallel velocity,but actually it is on *mixed helicity*.

$$\begin{aligned}&\frac{\partial}{\partial t}\left\langle\mathbf{v}\cdot\mathbf{B}\right\rangle+\frac{1}{\int\frac{dS}{|\boldsymbol{\nabla}p|}}\frac{\partial}{\partial p}\int\mathbf{dS}\cdot\mathbf{v}\ \mathbf{v}\cdot\mathbf{B}\\ =\ &-\mathbf{v}\cdot\left\langle\boldsymbol{\nabla}\times\left(\eta\mathbf{j}-\widehat{\mathbf{e}}_z\frac{\mathcal{E}}{h}\right)\right\rangle\\ &-\left\langle\frac{3\eta_0}{nm_i}\mathbf{B}\cdot\boldsymbol{\nabla}\ln B\ \mathbf{v}\cdot\boldsymbol{\nabla}\ln B\right\rangle\\ &+\frac{1}{m_i}\sum_{e,i}\left\langle T\mathbf{B}\cdot\boldsymbol{\nabla}s\right\rangle\end{aligned}$$

The equation for toroidal momentum

$$\frac{\partial}{\partial t}\left\langle nm_iRh\ \mathbf{v}\cdot\widehat{\mathbf{e}}_{\varphi}\right\rangle+\frac{1}{\int\frac{dS}{|\boldsymbol{\nabla}p|}}\frac{\partial}{\partial p}\int\mathbf{dS}\cdot\mathbf{v}\ \left(nm_iRh\ \mathbf{v}\cdot\widehat{\mathbf{e}}_{\varphi}\right)=0$$

The equation for the *entropy* assuming that the species is adiabatic

$$\begin{aligned} & \langle n \rangle \frac{\partial s}{\partial t} + \frac{1}{\int \frac{dS}{|\boldsymbol{\nabla} p|}} \int \mathbf{dS} \cdot n\mathbf{v} \left(\frac{\partial s}{\partial p} \right) \\ = & \left\langle \chi \left(\frac{\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} T}{T} \right)^2 \right\rangle \\ & + \left\langle \frac{3\eta_0}{T} \left(\mathbf{v} \cdot \boldsymbol{\nabla} \ln B \right)^2 \right\rangle \end{aligned}$$

For the averaging operator we have

$$\frac{1}{\int \frac{dS}{|\boldsymbol{\nabla} p|}} \int \mathbf{dS} \cdot \mathbf{v} \ f = \frac{dp}{dr} \langle v_r \ f \rangle$$

For different functions f the quantity $\langle v_r \ f \rangle$ is derived by averaging the toroidal component of the Ohm's law.

Start from

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j}$$

with

$$\mathbf{E} = -\boldsymbol{\nabla} \phi + \frac{\mathcal{E}}{h} \widehat{\mathbf{e}}_\varphi$$

where an external, inductive, electric field is considered, $\mathcal{E}$, toroidal.

First we multiply by $\mathbf{B}$ the Ohm's law

$$\mathbf{E} \cdot \mathbf{B} = \eta j_\parallel |\mathbf{B}|$$

and replace

$$\begin{aligned} \left(-\boldsymbol{\nabla} \phi + \frac{\mathcal{E}}{h} \widehat{\mathbf{e}}_\varphi \right) \cdot \mathbf{B} &= \eta j_\parallel |\mathbf{B}| \\ \mathcal{E} \frac{B_\varphi}{h} &= \eta j_\parallel |\mathbf{B}| \end{aligned}$$

The magnitude is

$$|\mathbf{B}| = \frac{B_0(r)}{h} \sqrt{1 + \frac{\varepsilon^2}{q^2}}$$

The equation becomes

$$\mathcal{E} \frac{B_0(r)}{h^2} = \eta j_\parallel \frac{B_0(r)}{h} \sqrt{1 + \frac{\varepsilon^2}{q^2}}$$

after averaging

$$\mathcal{E} = \frac{\left\langle \eta j_\parallel \frac{B_0(r)}{h} \sqrt{1 + \frac{\varepsilon^2}{q^2}} \right\rangle}{\left\langle \frac{B_0(r)}{h^2} \right\rangle}$$

the previously derived expression of the parallel current

$$\begin{aligned} j_{\|} &= \frac{q}{\varepsilon}\left[-\frac{1}{h}\frac{dB_0(r)}{dr}\right]\sqrt{1+\frac{\varepsilon^2}{q^2}} \\ &+\left[-\frac{h}{B_0(r)}\frac{dp(r)}{dr}\right]\frac{q}{\varepsilon}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}} \end{aligned}$$

$$j_{\|}=\frac{q}{\varepsilon}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}h\left[-\frac{dB_0}{dr}\frac{1}{h^2}\left(1+\frac{\varepsilon^2}{q^2}\right)-\frac{1}{B_0}\frac{dp}{dr}\right]$$

(we **note** that this is a more accurate expression for the Pfirsch Schluter current, which now contains the possible non-zero derivative $dB_0/dr \neq 0$).

The numerator of the expression of $\mathcal{E}$ is the average of

$$\begin{aligned} & \eta j_{\|}\frac{B_0(r)}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}} \\ =\ & \eta\frac{B_0(r)}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}}\frac{q}{\varepsilon}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}h\left[-\frac{dB_0}{dr}\frac{1}{h^2}\left(1+\frac{\varepsilon^2}{q^2}\right)-\frac{1}{B_0}\frac{dp}{dr}\right] \\ =\ & \eta B_0(r)\frac{q}{\varepsilon}\left[-\frac{dB_0}{dr}\frac{1}{h^2}\left(1+\frac{\varepsilon^2}{q^2}\right)-\frac{1}{B_0}\frac{dp}{dr}\right] \end{aligned}$$

Then

$$\begin{aligned} \mathcal{E} &= \frac{\left\langle \eta j_{\|}\frac{B_0(r)}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}}\right\rangle}{\left\langle\frac{B_0(r)}{h^2}\right\rangle} \\ &= \frac{1}{B_0}\frac{1}{\left\langle\frac{1}{h^2}\right\rangle}\eta B_0\frac{q}{\varepsilon}\left\langle-\frac{dB_0}{dr}\frac{1}{h^2}\left(1+\frac{\varepsilon^2}{q^2}\right)-\frac{1}{B_0}\frac{dp}{dr}\right\rangle \end{aligned}$$

Consider an arbitrary function f of plasma variables.

We take the φ (toroidal) component of the Ohm's law

$$\begin{aligned} \frac{\mathcal{E}}{h}+(\mathbf{v}\times\mathbf{B_0})_{\varphi} &= \eta j_{\varphi} \\ \frac{\mathcal{E}}{h}+v_r B_{\theta} &= \eta j_{\varphi} \end{aligned}$$

and multiply by

$$\frac{f}{B_{\theta}}$$

$$\eta j_{\varphi}\frac{f}{B_{\theta}}-\frac{\mathcal{E}}{h}\frac{f}{B_{\theta}}=v_r f$$

and average over surface

$$\left\langle \left(\eta j_\varphi - \frac{\mathcal{E}}{h} \right) \frac{f}{B_\theta} \right\rangle = \langle v_r f \rangle$$

Now we use

$$\frac{\varepsilon}{q} j_\varphi = -\frac{1}{h} \frac{dB_0(r)}{dr} - \frac{h}{B_0} \frac{dp(r)}{dr}$$

We return to

$$\begin{aligned} \langle v_r f \rangle &= \left\langle \left(\eta j_\varphi - \frac{\mathcal{E}}{h} \right) \frac{f}{B_\theta} \right\rangle \\ &= \eta \left\langle j_\varphi \frac{f}{B_\theta} \right\rangle - \left\langle \frac{\mathcal{E}}{h} \frac{f}{B_\theta} \right\rangle \end{aligned}$$

and take into account that $\mathcal{E}$ is already averaged.

$$\begin{aligned} \langle v_r f \rangle &= \eta \left\langle f \frac{1}{B_\theta} \frac{q}{\varepsilon} \left[-\frac{1}{h} \frac{dB_0(r)}{dr} - \frac{h}{B_0} \frac{dp(r)}{dr} \right] \right\rangle \\ & \quad -\mathcal{E} \left\langle f \frac{1}{hB_\theta} \right\rangle \end{aligned}$$

The second term can be expressed as

$$-\mathcal{E} \left\langle f \frac{1}{hB_\theta} \right\rangle$$

where

$$\frac{1}{hB_\theta} = \frac{1}{h \frac{b(r)}{h}} = \frac{1}{b(r)}$$

and is factored out from the averaging.

$$\begin{aligned} & -\mathcal{E} \frac{1}{b(r)} \langle f \rangle \\ = & -\frac{1}{B_0} \frac{1}{\left\langle \frac{1}{h^2} \right\rangle} \eta B_0 \frac{q}{\varepsilon} \left\langle -\frac{dB_0}{dr} \frac{1}{h^2} \left(1 + \frac{\varepsilon^2}{q^2} \right) - \frac{1}{B_0} \frac{dp}{dr} \right\rangle \frac{1}{b(r)} \langle f \rangle \end{aligned}$$

and note that

$$\frac{B_0}{b} = \frac{B_0/h}{b/h} = \frac{B_\varphi}{B_\theta} = \Theta^{-1} = \frac{q}{\varepsilon}$$

We have

$$\begin{aligned} & -\mathcal{E} \frac{1}{b(r)} \langle f \rangle \\ = & \ \eta \frac{1}{B_0} \left(\frac{q}{\varepsilon} \right)^2 \frac{\left\langle \frac{dB_0}{dr} \frac{1}{h^2} \left(1 + \frac{\varepsilon^2}{q^2} \right) + \frac{1}{B_0} \frac{dp}{dr} \right\rangle}{\left\langle \frac{1}{h^2} \right\rangle} \end{aligned}$$

NOTE that this almost the same as the second term in the Appendix C of **Hassam Kulsrud**, with the difference that they have $B_0 \frac{dp}{dr}$ instead of $\frac{1}{B_0}\frac{dp}{dr}$. There is a problem of units in HK. To be checked more carefully. **END.**

Working the first term we remind that

$$\frac{\varepsilon}{q} = \Theta = \frac{B_\theta}{B_\varphi}$$

$$\frac{1}{B_\theta} = \frac{q}{\varepsilon}\frac{1}{B_\varphi} = \frac{q}{\varepsilon}\frac{h}{B_0(r)}$$

then

$$\begin{aligned}
& f\frac{1}{B_\theta}\frac{q}{\varepsilon}\left[-\frac{1}{h}\frac{dB_0(r)}{dr} - \frac{h}{B_0}\frac{dp(r)}{dr}\right] \\
= \quad & f\frac{q}{\varepsilon}\frac{h}{B_0(r)}\frac{q}{\varepsilon}\left[-\frac{1}{h}\frac{dB_0(r)}{dr}\right] + f\frac{q}{\varepsilon}\frac{h}{B_0(r)}\frac{q}{\varepsilon}\left[-\frac{h}{B_0}\frac{dp(r)}{dr}\right] \\
= \quad & -f\left(\frac{q}{\varepsilon}\right)^2\frac{1}{B_0}\frac{dB_0}{dr} \\
& -f\left(\frac{q}{\varepsilon}\right)^2\frac{1}{B_0^2}\frac{dp}{dr}h^2
\end{aligned}$$

One can factorize from the averaging operator all factors that only depend on ψ (*i.e.* on the radius r).

$$\begin{aligned}
& \eta\left\langle f\frac{1}{B_\theta}\frac{q}{\varepsilon}\left[-\frac{1}{h}\frac{dB_0(r)}{dr} - \frac{h}{B_0}\frac{dp(r)}{dr}\right]\right\rangle \\
= \quad & -\left(\frac{q}{\varepsilon}\right)^2\frac{1}{B_0}\frac{dB_0}{dr}\langle f\rangle - \left(\frac{q}{\varepsilon}\right)^2\frac{1}{B_0^2}\frac{dp}{dr}\left\langle fh^2\right\rangle \\
= \quad & -\eta\left(\frac{q}{\varepsilon}\right)^2\frac{1}{B_0}\left[\frac{dB_0}{dr}\langle f\rangle + \frac{1}{B_0}\frac{dp}{dr}\left\langle fh^2\right\rangle\right]
\end{aligned}$$

This is indeed the first term in **HK** appendix C.

Finally, the expression of the average is

$$\begin{aligned}
& \langle fv_r\rangle \\
= \quad & -\eta\left(\frac{q}{\varepsilon}\right)^2\frac{1}{B_0}\left[\frac{dB_0}{dr}\langle f\rangle + \frac{1}{B_0}\frac{dp}{dr}\left\langle fh^2\right\rangle\right] \\
& +\eta\frac{1}{B_0}\left(\frac{q}{\varepsilon}\right)^2\frac{\left\langle\frac{dB_0}{dr}\frac{1}{h^2}\left(1+\frac{\varepsilon^2}{q^2}\right)+\frac{1}{B_0}\frac{dp}{dr}\right\rangle}{\left\langle\frac{1}{h^2}\right\rangle}
\end{aligned}$$

or

$$\begin{aligned}
& \langle fv_r\rangle \\
= \quad & \eta\left(\frac{q}{\varepsilon}\right)^2\frac{1}{B_0}\left\{-\left[\frac{dB_0}{dr}\langle f\rangle + \frac{1}{B_0}\frac{dp}{dr}\left\langle fh^2\right\rangle\right] + \frac{\left\langle\frac{dB_0}{dr}\frac{1}{h^2}\left(1+\frac{\varepsilon^2}{q^2}\right)+\frac{1}{B_0}\frac{dp}{dr}\right\rangle}{\left\langle\frac{1}{h^2}\right\rangle}\right\}
\end{aligned}$$

The order of magntitude is

$$\frac{dB}{dr} \sim \frac{dp}{dr} \sim b^2 \sim \frac{a^2}{R^2}$$

The components of the magnetic field $\mathbf{B}$ and of the current $\mathbf{J}$ are related by the equation

$$0 = -\boldsymbol{\nabla} p + \mathbf{J} \times \mathbf{B}$$

The other equation is

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}$$

It results the **Grad Shafranov** equation

$$\frac{1}{h_r h_\theta} \frac{\partial}{\partial r} \left[\frac{h_\theta}{h_r h_\varphi} b(r) \right] = -\frac{B}{b(r)} \frac{dB}{dr} \frac{1}{h_\varphi} - \frac{h_\varphi}{b(r)} \frac{dp}{dr}$$

The RHS is J_φ.
The LHS is $\boldsymbol{\nabla} \times \mathbf{B}$, component along φ.

8.2 Averaged equations for poloidal and toroidal rotation

The equations

$$\begin{aligned} nm_i \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla}) \mathbf{v} \right) &= -\boldsymbol{\nabla} p - \boldsymbol{\nabla} \cdot \boldsymbol{\Pi} \\ &\quad + \mathbf{j} \times \mathbf{B} \end{aligned}$$

where

$$\boldsymbol{\Pi} = -3\eta_0 \left(\widehat{\mathbf{n}}\, \widehat{\mathbf{n}} - \frac{1}{3} \right) \left(\widehat{\mathbf{n}}\, \widehat{\mathbf{n}} - \frac{1}{3} \right) : \boldsymbol{\nabla}\, \mathbf{v}$$

$$p = n (T_e + T_i)$$

The heat equation is expressed in terms of the *entropy*

$$nT \left(\frac{\partial s}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla}) s \right) = -\boldsymbol{\nabla} \cdot \mathbf{q} - \boldsymbol{\Pi} \colon \boldsymbol{\nabla}\, \mathbf{v}$$

with the flux of heat

$$\mathbf{q} = -\chi \frac{1}{B} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} T$$

The Ohm's law

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j}$$

and

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{B} &= \mu_0 \mathbf{j} \\ \boldsymbol{\nabla} \times \mathbf{E} &= 0 \\ \boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \end{aligned}$$

The magnetic field is (more general than circular)

$$\mathbf{B} = \left(0, \frac{b\,(r)}{h}, \frac{B_0\,(r)}{h}\right)$$

NOTE this is more than the tokamak: in tokamak

$$\frac{dB_0}{dr} = 0$$

One can also include the Shafranov shift.
END

The following averaging operator is introduced

$$\langle f \rangle = \frac{\int \frac{dS}{|\boldsymbol{\nabla} p|} f}{\int \frac{dS}{|\boldsymbol{\nabla} p|}}$$

The equation of continuity involves the averaging of the divergence of a flux of particles, $n\mathbf{v}$.

$$\frac{\partial}{\partial t} \langle n \rangle + \frac{1}{\int \frac{dS}{|\boldsymbol{\nabla} p|}} \frac{\partial}{\partial p} \int \mathbf{dS} \cdot n\mathbf{v} = 0$$

Let us note here that the integrand of the operator at numerator has changed and a derivation to p has been introduced instead. This form of the average will be treated in detail below.

The equation for the *circulation*

$$\begin{aligned}
& \frac{\partial}{\partial t} \langle \mathbf{v} \cdot \mathbf{B} \rangle + \frac{1}{\int \frac{dS}{|\boldsymbol{\nabla} p|}} \frac{\partial}{\partial p} \int \mathbf{dS} \cdot \mathbf{v}\ \mathbf{v} \cdot \mathbf{B} \\
= \quad & -\mathbf{v} \cdot \left\langle \boldsymbol{\nabla} \times \left(\eta \mathbf{j} - \widehat{\mathbf{e}}_z \frac{\mathcal{E}}{h} \right) \right\rangle \\
& - \left\langle \frac{3\eta_0}{n m_i} \mathbf{B} \cdot \boldsymbol{\nabla} \ln B\ \mathbf{v} \cdot \boldsymbol{\nabla} \ln B \right\rangle \\
& + \frac{1}{m_i} \sum_{e,i} \langle T \mathbf{B} \cdot \boldsymbol{\nabla} s \rangle
\end{aligned}$$

The equation for toroidal momentum

$$\frac{\partial}{\partial t} \langle n m_i R h\ \mathbf{v} \cdot \widehat{\mathbf{e}}_\varphi \rangle + \frac{1}{\int \frac{dS}{|\boldsymbol{\nabla} p|}} \frac{\partial}{\partial p} \int \mathbf{dS} \cdot \mathbf{v}\ \ (n m_i R h\ \mathbf{v} \cdot \widehat{\mathbf{e}}_\varphi) = 0$$

The equation for the *entropy* assuming that the species is adiabatic

$$\begin{aligned}
& \langle n \rangle \frac{\partial s}{\partial t} + \frac{1}{\int \frac{dS}{|\boldsymbol{\nabla} p|}} \int \mathbf{dS} \cdot n\mathbf{v} \left(\frac{\partial s}{\partial p} \right) \\
= \quad & \left\langle \chi \left(\frac{\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} T}{T} \right)^2 \right\rangle \\
& + \left\langle \frac{3\eta_0}{T} \left(\mathbf{v} \cdot \boldsymbol{\nabla} \ln B \right)^2 \right\rangle
\end{aligned}$$

For the averaging operator we have

$$\frac{1}{\int \frac{dS}{|\boldsymbol{\nabla} p|}} \int \mathbf{dS} \cdot \mathbf{v}\ f = \frac{dp}{dr} \langle v_r\ f \rangle$$

For different functions f the quantity $\langle v_r\ f \rangle$ is derived by averaging the toroidal component of the Ohm's law.

Start from

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j}$$

with

$$\mathbf{E} = -\boldsymbol{\nabla} \phi + \frac{\mathcal{E}}{h} \widehat{\mathbf{e}}_{\varphi}$$

The electrostatic potential ϕ is a function of r and θ. The field $\mathcal{E}/h$ is *toroidal* electric field due to the transformer.

First we multiply by $\mathbf{B}$ the Ohm's law

$$\mathbf{E} \cdot \mathbf{B} = \eta j_{\|} |\mathbf{B}|$$

and replace

$$\begin{aligned} \left(-\boldsymbol{\nabla} \phi + \frac{\mathcal{E}}{h} \widehat{\mathbf{e}}_{\varphi} \right) \cdot \mathbf{B} &= \eta j_{\|} |\mathbf{B}| \\ \mathcal{E} \frac{B_{\varphi}}{h} &= \eta j_{\|} |\mathbf{B}| \end{aligned}$$

The magnitude is

$$|\mathbf{B}| = \frac{B_0(r)}{h} \sqrt{1 + \frac{\varepsilon^2}{q^2}}$$

and

$$B_{\varphi} = \frac{B_0(r)}{h}$$

The equation becomes

$$\mathcal{E} \frac{B_0(r)}{h^2} = \eta j_{\|} \frac{B_0(r)}{h} \sqrt{1 + \frac{\varepsilon^2}{q^2}}$$

after averaging

$$\mathcal{E} = \frac{\left\langle \eta j_{\|} \frac{B_0(r)}{h} \sqrt{1 + \frac{\varepsilon^2}{q^2}} \right\rangle}{\left\langle \frac{B_0(r)}{h^2} \right\rangle}$$

the previously derived expression of the parallel current

$$\begin{aligned} j_{\|} &= \frac{q}{\varepsilon} \left[-\frac{1}{h} \frac{dB_0(r)}{dr} \right] \sqrt{1 + \frac{\varepsilon^2}{q^2}} \\ &\quad + \left[-\frac{h}{B_0(r)} \frac{dp(r)}{dr} \right] \frac{q}{\varepsilon} \frac{1}{\sqrt{1 + \frac{\varepsilon^2}{q^2}}} \end{aligned}$$

$$j_{\|}=\frac{q}{\varepsilon}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}h\left[-\frac{dB_0}{dr}\frac{1}{h^2}\left(1+\frac{\varepsilon^2}{q^2}\right)-\frac{1}{B_0}\frac{dp}{dr}\right]$$

The numerator of the expression of $\mathcal{E}$ is the average of

$$\begin{aligned}
& \eta j_{\|}\frac{B_0(r)}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}} \\
= \; & \eta\frac{B_0(r)}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}}\frac{q}{\varepsilon}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}h\left[-\frac{dB_0}{dr}\frac{1}{h^2}\left(1+\frac{\varepsilon^2}{q^2}\right)-\frac{1}{B_0}\frac{dp}{dr}\right] \\
= \; & \eta B_0(r)\frac{q}{\varepsilon}\left[-\frac{dB_0}{dr}\frac{1}{h^2}\left(1+\frac{\varepsilon^2}{q^2}\right)-\frac{1}{B_0}\frac{dp}{dr}\right]
\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{E} &= \frac{\left\langle \eta j_{\|}\frac{B_0(r)}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}}\right\rangle}{\left\langle\frac{B_0(r)}{h^2}\right\rangle} \\
&= \frac{1}{B_0}\frac{1}{\left\langle\frac{1}{h^2}\right\rangle}\eta B_0\frac{q}{\varepsilon}\left\langle-\frac{dB_0}{dr}\frac{1}{h^2}\left(1+\frac{\varepsilon^2}{q^2}\right)-\frac{1}{B_0}\frac{dp}{dr}\right\rangle
\end{aligned}$$

Now we want to calculate the flux across the magnetic surface ψ of some quantity f.

The flux is locally

$$v_r f$$

and we want to find the surface average of it

$$\langle v_r f\rangle$$

Return to a previous calculation (this part starts by copy/paste)

Next we take the φ (toroidal) component

$$\begin{aligned}
\frac{\mathcal{E}}{h}+(\mathbf{v}\times\mathbf{B_0})_{\varphi} &= \eta j_{\varphi} \\
\frac{\mathcal{E}}{h}+v_r B_\theta &= \eta j_{\varphi}
\end{aligned}$$

This is the Ohm's law projected on toroidal direction. In the RHS there is toroidal electric field which - if combined with the poloidal magnetic field B_θ will produce a *radial* velocity. This is what we need.

This approach looks rather strange. We would expect to start from the drift velocity of the particles, projected on radial direction.

It is multiplied by

$$\frac{f}{B_\theta}$$

$$\eta j_{\varphi}\frac{f}{B_{\theta}}-\frac{\mathcal{E}}{h}\frac{f}{B_{\theta}}=v_{r}f$$

This is

$$\begin{aligned}
v_{r} &= \eta j_{\varphi}\frac{1}{B_{\theta}}-\frac{\mathcal{E}}{h}\frac{1}{B_{\theta}}=\frac{\eta j_{\varphi}-\frac{\mathcal{E}}{h}}{B_{\theta}}=\frac{\text{toroidal electric field}}{\text{poloidal magnetic field}}\\
&= \frac{E^{tor}}{B^{pol}}
\end{aligned}$$

The radial flux (v_r) is attributed to the toroidal electric field combined with the poloidal magnetic field.

This does NOT exclude the neoclassical origin of v_r (which is however a *fluid* quantity).

The neoclassical aspect enters through the contribution to the toroidal electric field that is due to the charge accumulation along a magnetic field line because of the different neoclassical drifts of the charges. The difference of the drifts give a current (along the line) and this one multiplied with the resistivity gives an electric field (**Stringer**) . This electric field projected on toroidal direction and combined with B_θ gives a radial drift, v_r which will produce radial fluxes of any plasma parameter.

averaged

$$\left\langle\left(\eta j_{\varphi}-\frac{\mathcal{E}}{h}\right)\frac{f}{B_{\theta}}\right\rangle=\langle v_{r}f\rangle$$

Now we use

$$\frac{\varepsilon}{q}j_{\varphi}=-\frac{1}{h}\frac{dB_{0}\left(r\right)}{dr}-\frac{h}{B_{0}}\frac{dp\left(r\right)}{dr}$$

Except for the first term which is usually taken 0 in tokamak, this formula is the *diamagnetic* flow projected on toroidal direction. Since the diamagnetic flow

$$\sim\frac{1}{m\Omega}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}p$$

is *perpendicular* on the magnetic field line the projection is made by

$$\frac{\varepsilon}{q}=\frac{B_{\theta}}{B_{T}}$$

multiplying the toroidal current j_φ.

We return to

$$\begin{aligned}
\langle v_{r}f\rangle &= \left\langle\left(\eta j_{\varphi}-\frac{\mathcal{E}}{h}\right)\frac{f}{B_{\theta}}\right\rangle\\
&= \eta\left\langle j_{\varphi}\frac{f}{B_{\theta}}\right\rangle-\left\langle\frac{\mathcal{E}}{h}\frac{f}{B_{\theta}}\right\rangle
\end{aligned}$$

and take into account that $\mathcal{E}$ is already averaged.

$$\begin{aligned}\langle v_r f\rangle &= \eta\left\langle f\frac{1}{B_\theta}\frac{q}{\varepsilon}\left[-\frac{1}{h}\frac{dB_0(r)}{dr}-\frac{h}{B_0}\frac{dp(r)}{dr}\right]\right\rangle \\ &\quad -\mathcal{E}\left\langle f\frac{1}{hB_\theta}\right\rangle\end{aligned}$$

The second term can be expressed as

$$-\mathcal{E}\left\langle f\frac{1}{hB_\theta}\right\rangle$$

where

$$\frac{1}{hB_\theta}=\frac{1}{h\frac{b(r)}{h}}=\frac{1}{b(r)}$$

and is factored out from the averaging

$$-\mathcal{E}\left\langle f\frac{1}{hB_\theta}\right\rangle=-\mathcal{E}\frac{1}{b(r)}\langle f\rangle$$

For the other term

$$\eta\left\langle f\frac{1}{B_\theta}\frac{q}{\varepsilon}\left[-\frac{1}{h}\frac{dB_0(r)}{dr}-\frac{h}{B_0}\frac{dp(r)}{dr}\right]\right\rangle$$

we take out of the factors that only depend on surface

$$\begin{aligned}&\eta\frac{q}{\varepsilon}\left\langle f\frac{1}{\frac{b(r)}{h}}h\left[-\frac{1}{h^2}\frac{dB_0(r)}{dr}-\frac{1}{B_0}\frac{dp(r)}{dr}\right]\right\rangle \\ =\;&\eta\frac{q}{\varepsilon}\frac{1}{b(r)}\left\langle f\;h^2\left[-\frac{1}{h^2}\frac{dB_0(r)}{dr}-\frac{1}{B_0}\frac{dp(r)}{dr}\right]\right\rangle \\ =\;&\eta\frac{q}{\varepsilon}\frac{1}{b(r)}\left\langle f\;\frac{1}{\frac{1}{h^2}}\left[-\frac{1}{h^2}\frac{dB_0(r)}{dr}-\frac{1}{B_0}\frac{dp(r)}{dr}\right]\right\rangle\end{aligned}$$

We multiply and divide by B_0 the factor outside the average

$$\eta\frac{q}{\varepsilon}\frac{1}{b(r)}\frac{B_0}{B_0}$$

and note that

$$\frac{B_0}{b}=\frac{B_0/h}{b/h}=\frac{B_\varphi}{B_\theta}=\Theta^{-1}=\frac{q}{\varepsilon}$$

which gives for the factor

$$\eta\frac{q}{\varepsilon}\frac{1}{b(r)}\frac{B_0}{B_0}=\eta\frac{1}{B_0}\left(\frac{q}{\varepsilon}\right)^2$$

Assumption about first order $\langle v_r f\rangle = 0$.

Here follows an assumption about the function (until now arbitrary) f which is supposed to have a zero flux across the magnetic surface

$$\langle v_r f \rangle \sim 0$$

in the lowest order.

NOTE that this is the basic condition $\langle j_r \rangle = 0$ used to determine the rate of damping $\partial V_E / \partial t$ of the poloidal rotation **Novakovskii**. **END.**

This transforms the equation of $\langle v_r f \rangle$ in a balance between the two RHS terms.

$$\begin{aligned}
0 &= -\mathcal{E} \frac{1}{b(r)} f + \eta f \frac{1}{B_\theta} \frac{q}{\varepsilon} \left[-\frac{1}{h} \frac{dB_0(r)}{dr} - \frac{h}{B_0} \frac{dp(r)}{dr} \right] \\
0 &= -\mathcal{E} \frac{1}{b(r)} f + \eta \frac{1}{B_0} \left(\frac{q}{\varepsilon} \right)^2 f \, \frac{1}{\frac{1}{h^2}} \left[-\frac{1}{h^2} \frac{dB_0(r)}{dr} - \frac{1}{B_0} \frac{dp(r)}{dr} \right] \\
0 &= -\mathcal{E} \frac{1}{b(r)} f \left(\frac{1}{h^2} \right) + \eta \frac{1}{B_0} \left(\frac{q}{\varepsilon} \right)^2 f \, \left[-\frac{1}{h^2} \frac{dB_0(r)}{dr} - \frac{1}{B_0} \frac{dp(r)}{dr} \right]
\end{aligned}$$

and in this order the surface average is splitted

$$-\mathcal{E} \frac{1}{b(r)} \langle f \rangle \left\langle \frac{1}{h^2} \right\rangle = -\eta \frac{1}{B_0} \left(\frac{q}{\varepsilon} \right)^2 \langle f \rangle \left\langle -\frac{1}{h^2} \frac{dB_0(r)}{dr} - \frac{1}{B_0} \frac{dp(r)}{dr} \right\rangle$$

$$\begin{aligned}
& -\mathcal{E} \frac{1}{b(r)} \langle f \rangle \\
= & -\eta \frac{1}{B_0} \left(\frac{q}{\varepsilon} \right)^2 \langle f \rangle \frac{1}{\left\langle \frac{1}{h^2} \right\rangle} \left\langle -\frac{dB_0}{dr} \frac{1}{h^2} \left(1 + \frac{\varepsilon^2}{q^2} \right) - \frac{1}{B_0} \frac{dp}{dr} \right\rangle
\end{aligned}$$

We have

$$\begin{aligned}
& -\mathcal{E} \frac{1}{b(r)} \langle f \rangle \\
= & \ \eta \frac{1}{B_0} \left(\frac{q}{\varepsilon} \right)^2 \frac{\left\langle \frac{dB_0}{dr} \frac{1}{h^2} \left(1 + \frac{\varepsilon^2}{q^2} \right) + \frac{1}{B_0} \frac{dp}{dr} \right\rangle \langle f \rangle}{\left\langle \frac{1}{h^2} \right\rangle}
\end{aligned}$$

These operations allow us to eliminate the electric field $\mathcal{E}$.

In higher order there is a nonzero radial (v_r) flux of f. We will replace now what we have obtained for $-\mathcal{E} \frac{1}{b(r)} \langle f \rangle$ in the lowest order.

Working the first term we remind that

$$\frac{\varepsilon}{q} = \Theta = \frac{B_\theta}{B_\varphi}$$

$$\frac{1}{B_\theta} = \frac{q}{\varepsilon} \frac{1}{B_\varphi} = \frac{q}{\varepsilon} \frac{h}{B_0(r)}$$

then

$$\begin{aligned}
& f\frac{1}{B_\theta}\frac{q}{\varepsilon}\left[-\frac{1}{h}\frac{dB_0(r)}{dr}-\frac{h}{B_0}\frac{dp(r)}{dr}\right] \\
= \; & f\frac{q}{\varepsilon}\frac{h}{B_0(r)}\frac{q}{\varepsilon}\left[-\frac{1}{h}\frac{dB_0(r)}{dr}\right]+f\frac{q}{\varepsilon}\frac{h}{B_0(r)}\frac{q}{\varepsilon}\left[-\frac{h}{B_0}\frac{dp(r)}{dr}\right] \\
= \; & -f\left(\frac{q}{\varepsilon}\right)^2\frac{1}{B_0}\frac{dB_0}{dr} \\
& -f\left(\frac{q}{\varepsilon}\right)^2\frac{1}{B_0^2}\frac{dp}{dr}h^2
\end{aligned}$$

One can factorize from the averaging operator all factors that only depend on ψ (*i.e.* on the radius r).

$$\begin{aligned}
& \eta\left\langle f\frac{1}{B_\theta}\frac{q}{\varepsilon}\left[-\frac{1}{h}\frac{dB_0(r)}{dr}-\frac{h}{B_0}\frac{dp(r)}{dr}\right]\right\rangle \\
= \; & -\left(\frac{q}{\varepsilon}\right)^2\frac{1}{B_0}\frac{dB_0}{dr}\langle f\rangle-\left(\frac{q}{\varepsilon}\right)^2\frac{1}{B_0^2}\frac{dp}{dr}\left\langle fh^2\right\rangle \\
= \; & -\eta\left(\frac{q}{\varepsilon}\right)^2\frac{1}{B_0}\left[\frac{dB_0}{dr}\langle f\rangle+\frac{1}{B_0}\frac{dp}{dr}\left\langle fh^2\right\rangle\right]
\end{aligned}$$

This is indeed the first term in **HK** appendix C.

Finally, the expression of the average is

$$\begin{aligned}
& \langle fv_r\rangle \\
= \; & -\eta\left(\frac{q}{\varepsilon}\right)^2\frac{1}{B_0}\left[\frac{dB_0}{dr}\langle f\rangle+\frac{1}{B_0}\frac{dp}{dr}\left\langle fh^2\right\rangle\right] \\
& +\eta\left(\frac{q}{\varepsilon}\right)^2\frac{1}{B_0}\frac{\left\langle\frac{dB_0}{dr}\frac{1}{h^2}\left(1+\frac{\varepsilon^2}{q^2}\right)+\frac{1}{B_0}\frac{dp}{dr}\right\rangle\langle f\rangle}{\left\langle\frac{1}{h^2}\right\rangle}
\end{aligned}$$

or

$$\begin{aligned}
& \langle fv_r\rangle \\
= \; & \eta\left(\frac{q}{\varepsilon}\right)^2\frac{1}{B_0}\left\{-\left[\frac{dB_0}{dr}\langle f\rangle+\frac{1}{B_0}\frac{dp}{dr}\left\langle fh^2\right\rangle\right]+\frac{\left\langle\frac{dB_0}{dr}\frac{1}{h^2}\left(1+\frac{\varepsilon^2}{q^2}\right)+\frac{1}{B_0}\frac{dp}{dr}\right\rangle\langle f\rangle}{\left\langle\frac{1}{h^2}\right\rangle}\right\}
\end{aligned}$$

Let us **NOTE** right here (before other calculations) that, taking as usual $\frac{dB_0(r)}{dr}=0$ we remain with two terms containing the same factor, the *diamagnetic current*, $\frac{1}{B_0}\frac{dp}{dr}\sim env_{dia}$. The factors are

$$-\left\langle fh^2\right\rangle+\frac{\langle f\rangle}{\left\langle\frac{1}{h^2}\right\rangle}$$

and this is the neaoclassical factor we were looking for.

The first part $\left\langle fh^2\right\rangle$ comes from a determination of the radial velocity v_r which is generated by the diamagnetic current $\frac{1}{B_0}\frac{dp}{dr} = env_{dia}$ which multiplied by the resistivity gives an *electric field* and divided by B_0 gives a radial velocity $\sim v_r$.

The second part comes from the balance of the two contributions in the order in which the average radial flux of f is zero.

END

This must be taken as a basis for the averages that will involve a function f.

$$\begin{aligned}
&\left\langle fv_r\right\rangle \\
= \; & \eta\left(\frac{q}{\varepsilon}\right)^2\frac{1}{B_0}\left\{\frac{dB_0}{dr}\left(-\left\langle f\right\rangle + \frac{\left\langle\frac{1}{h^2}\left(1+\frac{\varepsilon^2}{q^2}\right)\right\rangle\left\langle f\right\rangle}{\left\langle\frac{1}{h^2}\right\rangle}\right)\right. \\
& \left. + \frac{1}{B_0}\frac{dp}{dr}\left(-\left\langle fh^2\right\rangle + \frac{\left\langle f\right\rangle}{\left\langle\frac{1}{h^2}\right\rangle}\right)\right\}
\end{aligned}$$

The first round paranthesis is

$$\begin{aligned}
-\left\langle f\right\rangle + \frac{\left\langle\frac{1}{h^2}\left(1+\frac{\varepsilon^2}{q^2}\right)\right\rangle\left\langle f\right\rangle}{\left\langle\frac{1}{h^2}\right\rangle} &= \left\langle f\right\rangle\left(-1+1+\frac{\varepsilon^2}{q^2}\right) \\
&= \left\langle f\right\rangle\frac{\varepsilon^2}{q^2}
\end{aligned}$$

and the expression becomes

$$\begin{aligned}
&\left\langle fv_r\right\rangle \\
= \; & \eta\left(\frac{q}{\varepsilon}\right)^2\frac{1}{B_0}\left\{\frac{dB_0}{dr}\left\langle f\right\rangle\frac{\varepsilon^2}{q^2} + \frac{1}{B_0}\frac{dp}{dr}\left(-\left\langle fh^2\right\rangle + \frac{\left\langle f\right\rangle}{\left\langle\frac{1}{h^2}\right\rangle}\right)\right\} \\
= \; & \eta\frac{1}{B_0}\left\{\frac{dB_0}{dr}\left\langle f\right\rangle + \frac{q^2}{\varepsilon^2}\frac{1}{B_0}\frac{dp}{dr}\left(-\left\langle fh^2\right\rangle + \frac{\left\langle f\right\rangle}{\left\langle\frac{1}{h^2}\right\rangle}\right)\right\}
\end{aligned}$$

Rearranging and factoring the sign

$$\begin{aligned}
\left\langle fv_r\right\rangle &= -\eta\frac{dp}{dr}\frac{1}{B_0}\frac{1}{B_0}\left\{-\frac{B_0\frac{dB_0}{dr}}{\frac{dp}{dr}}\left\langle f\right\rangle\right. \\
&\quad \left. + \left(\frac{q}{\varepsilon}\right)^2\left(\left\langle fh^2\right\rangle - \frac{\left\langle f\right\rangle}{\left\langle\frac{1}{h^2}\right\rangle}\right)\right\}
\end{aligned}$$

The reason to factor out the gradient of the pressure comes from the definition of the average

$$\frac{\int \mathbf{dS}\cdot\mathbf{v}\; f}{\int\frac{\mathbf{dS}}{|\boldsymbol{\nabla} p|}} = \frac{dp}{dr}\left\langle fv_r\right\rangle$$

We introduce the notations

$$E_D^{rez} \equiv -\eta \frac{1}{B_0} \frac{dp}{dr}$$

It has similar parametric dependence as the diamagnetic velocity but contains the resistivity η. It is the *resistive classical flow.* This is because

$$\frac{1}{B_0} \frac{dp}{dr}$$

is a *current*,

$$env_{dia} = \frac{1}{B_0} \frac{dp}{dr}$$

and if this is multiplied by η we have an electric field associated to the diamagnetic current

$$\eta \left(env_{dia} \right) \sim \text{electric field}$$

The factor becomes

$$-\eta \frac{dp}{dr} \frac{1}{B_0} \frac{1}{B_0} = \frac{E_D^{rez}}{B_0}$$

Notation

$$D \equiv -B_0 \frac{\frac{dB_0}{dr}}{\frac{dp}{dr}}$$

It is possible to replace E_D^{rez} by another notation,

$$v_D^{rez} = \frac{E_D^{rez}}{B_0} \sim \text{velocity}$$

Then

$$\langle f v_r \rangle = v_D^{rez} \left[D \langle f \rangle + \left(\frac{q}{\varepsilon} \right)^2 \left(\langle f h^2 \rangle - \frac{\langle f \rangle}{\left\langle \frac{1}{h^2} \right\rangle} \right) \right]$$

NOTE the presence of the *resistivity* as FACTOR to the entire expression, *i.e.* again we see that v_r owes its existence to the resistivity that, in the Ohm's law, introduces the imperfect neutralization via parallel currents of the charge separation induced by the different drifts of electrons and ions. (Stringer PRL). **END.**

The equation of continuity

$$\frac{\partial}{\partial t} \langle n \rangle + \frac{1}{\int \frac{dS}{|\nabla p|}} \frac{\partial}{\partial p} \int \mathbf{dS} \cdot n\mathbf{v} = 0$$

means

$$\frac{1}{\int \frac{dS}{|\nabla p|}} \int \mathbf{dS} \cdot n\mathbf{v} = \frac{dp}{dr} \langle n v_r \rangle$$

$$\frac{\partial n}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left(rn\left\langle v_{r}\right\rangle\right)=0$$

To calculate this averaged over the surface we use the prefiously derived equation for

$$f=1$$

and obtain

$$\left\langle v_{r}\right\rangle=v_{D}^{rez}\left[D+q^{2}\frac{1}{\varepsilon^{2}}\left(\left\langle h^{2}\right\rangle-\frac{1}{\left\langle\frac{1}{h^{2}}\right\rangle}\right)\right]$$

and introduce the notation

$$\alpha_{1}\equiv\frac{1}{2\varepsilon^{2}}\left(\left\langle h^{2}\right\rangle-\frac{1}{\left\langle\frac{1}{h^{2}}\right\rangle}\right)$$

then

$$\left\langle v_{r}\right\rangle=v_{D}^{rez}\left(D+2q^{2}\alpha_{1}\right)$$

The equation of continuity becomes

$$\frac{\partial n}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left[rnv_{D}^{rez}\left(D+2q^{2}\alpha_{1}\right)\right]=0$$

We note that α_1 is close to 1.

Another notation

$$\alpha_{3}=\left\langle\frac{1}{h^{2}}\right\rangle$$

In a similar way, it is obtained the time evolution of the toroidal component of the flow. Define

$$v_{t}\equiv\left\langle hv_{\varphi}\right\rangle$$

We must repeat the calculation made for v_r . The velocity v_φ is obtained from the φ projection of the Ohm's law, *i.e.* after multiplying it with $\widehat{\mathbf{e}}_\varphi$ we take the average. We will need the component j_φ of the current, already derived. Finally

$$\begin{aligned} v_{t} &\equiv \left\langle hv_{\varphi}\right\rangle \\ &= \frac{q}{\varepsilon}\left(v_{p}-v_{E}\left\langle h^{2}\right\rangle\right) \end{aligned}$$

where

$$v_{E}=\frac{1}{B_{0}}\frac{\partial\phi}{r\partial\theta}$$

and the poloidal rotation velocity v_θ is expressed through the function v_p that only depends on the magnetic surface (ψ)

$$v_{\theta}=\frac{v_{p}\left(r\right)}{h}$$

The projection of the rotation velocity perpendicular on the magnetic line is

$$\begin{aligned} v_{\perp} &= v_E \frac{h}{\sqrt{1+\frac{\varepsilon^2}{q^2}}} \\ &= \frac{1}{B_0/h} \frac{\partial \phi}{r\partial \theta} \frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}} = \frac{\partial \phi}{r\partial \theta} \frac{1}{B} \end{aligned}$$

As before, together with (v_θ, v_φ) it is possible to work with $\left(v_\parallel, v_\perp\right)$.

Consider the factor

$$\begin{aligned} & \alpha_3 + 2q^2\alpha_1 \\ = & \left\langle \frac{1}{h^2} \right\rangle + \frac{q^2}{\varepsilon^2} \left(\left\langle h^2 \right\rangle - \frac{1}{\left\langle \frac{1}{h^2} \right\rangle} \right) \end{aligned}$$

This factor, multiplied by $\frac{\varepsilon}{q} v_p$ gives a component of the *toroidal* velocity

$$\begin{aligned} & \left[\frac{\varepsilon}{q} \left\langle \frac{1}{h^2} \right\rangle + \frac{q}{\varepsilon} \left(\left\langle h^2 \right\rangle - \frac{1}{\left\langle \frac{1}{h^2} \right\rangle} \right) \right] \\ & \times v_{pol} \\ \sim & \ v_{tor} \end{aligned}$$

The equation for a combination of v_p and v_t has been derived from the average of the equation for the *circulation* $\mathbf{v} \cdot \mathbf{B}$ by Hassam and Drake.

$$\begin{aligned} & \frac{\partial}{\partial t} \left[v_t + \left(\alpha_3 + 2q^2\alpha_1 \right) \frac{\varepsilon}{q} v_p \right] \\ & + \frac{1}{r} \frac{\partial}{\partial r} \left\{ r \left[v_D^{rez} \left(D + 2q^2\alpha_1 \right) \left(v_t + \left(\alpha_3 + 2q^2\alpha_1 \right) \frac{\varepsilon}{q} v_p \right) \right] - 2q^2\alpha_1 \frac{v_p}{\varepsilon/q} \right\} \\ = & -\frac{3}{2} \alpha_4 \frac{\eta_0}{n m_i R^2} \frac{\varepsilon}{q} v_p \\ & + \Xi \end{aligned}$$

By Ξ we note the terms related to thermal diffusion of the adiabatic species of particles (electrons). The notations are

$$\begin{aligned} \alpha_3 &= \left\langle \frac{1}{h^2} \right\rangle \\ \alpha_4 &= \left(2R^2 \right) \left\langle \frac{1}{h^2} \left(\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}} \frac{\partial}{r\partial \theta} \ln h \right)^2 \right\rangle \end{aligned}$$

An equation for v_t results from the momentum conservation projected along the toroidal direction then averaged

$$\frac{\partial}{\partial t}\left\langle n v_t \right\rangle + \frac{1}{r}\frac{\partial}{\partial r}\left(r n v_D^{rez}\left[\left(D + 2q^2\alpha_1\right) v_t + 2q^2\alpha_2\left(v_t - v_p\frac{q}{\varepsilon}\right)\right]\right) = 0$$

where

$$\alpha_2 = \frac{1}{2\varepsilon^2}\left[\left\langle h^4\right\rangle \frac{1}{\left\langle h^2\right\rangle} - \left\langle h^2\right\rangle\right]$$

8.3 Equation for poloidal rotation

The previous calculations allow to write down the equation for the evolution of the poloidaal velocity

$$\begin{aligned}\left(1+\frac{1}{2q^2}\right)\frac{\partial \ln v_p}{\partial t} &= -\frac{q^2}{\varepsilon^2} v_D^{rez}\frac{1}{n}\frac{dn}{dr} \\ &\quad -\frac{3}{4}\frac{\eta_0}{n m_i q^2 R^2} + \Xi'\end{aligned}$$

The symbol Ξ' is introduced to represent the effect of the thermal conductivity of the electrons

$$\Xi' \sim \chi_e$$

Since v_t is connected with v_p we have the equation for it

$$\frac{\partial v_t}{\partial t} = \frac{1}{2n}\frac{1}{r}\frac{\partial}{\partial r}\left(r n q^2 v_D^{rez}\frac{q}{\varepsilon} v_p\right)$$

9 Spontaneous poloidal rotation (instability) Hassam Antonsen

9.1 Detailed treatment

The equations

$$\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{n v_\parallel}{B}\right) = -\boldsymbol{\nabla}\cdot\left(n\mathbf{v}_\perp\right) - \frac{\partial n}{\partial t}$$

This arises from

$$\begin{aligned}\boldsymbol{\nabla}\cdot\left(n\mathbf{v}\right) &= \boldsymbol{\nabla}\cdot\left[n\left(\mathbf{v}_\parallel + \mathbf{v}_\perp\right)\right] = \boldsymbol{\nabla}\cdot\left(n v_\parallel \frac{\mathbf{B}}{B} + n\mathbf{v}_\perp\right) \\ &= \left[\boldsymbol{\nabla}\left(\frac{n v_\parallel}{B}\right)\cdot\mathbf{B} + \frac{n v_\parallel}{B}\boldsymbol{\nabla}\cdot\mathbf{B}\right] + \boldsymbol{\nabla}\cdot\left(n\mathbf{v}_\perp\right) \\ &= \mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{n v_\parallel}{B}\right) + \boldsymbol{\nabla}\cdot\left(n\mathbf{v}_\perp\right)\end{aligned}$$

The equation of continuity

$$\begin{aligned}\frac{\partial n}{\partial t}+\boldsymbol{\nabla}\cdot(n\mathbf{v}) &= 0\\ \frac{\partial n}{\partial t}+\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{nv_{\|}}{B}\right)+\boldsymbol{\nabla}\cdot(n\mathbf{v}_{\perp}) &= 0\end{aligned}$$

As it is written it shows that we will calculate the parallel gradient of the parallel velocity.

The second equation

$$\begin{aligned}T\mathbf{B}\cdot\boldsymbol{\nabla}n &= -nm_i\mathbf{B}\cdot\mathbf{v}:\boldsymbol{\nabla}\mathbf{v}\\ -\mathbf{B}\boldsymbol{\nabla}:\boldsymbol{\Pi} & \\ & -nm_i\frac{\partial(\mathbf{B}\cdot\mathbf{v})}{\partial t}\end{aligned}$$

We recognize here the momentum equation

$$nm_i\frac{\partial\mathbf{v}}{\partial t}+nm_i(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}=-\boldsymbol{\nabla}(nT)-\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}$$

considering T constant, multiply by $\mathbf{B}$. This corresponds to the equation for the *circulation* $\mathbf{v}\cdot\mathbf{B}$.

The equation of Ohm, in the absence of resistivity

$$\eta\mathbf{j}=-\boldsymbol{\nabla}\phi+\mathbf{v}\times\mathbf{B}$$

$\eta=0$, multiplied by $\mathbf{B}$ is

$$\mathbf{B}\cdot\boldsymbol{\nabla}\phi=0$$

the potential is constant on magnetic surfaces.

$$\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{j_{\|}}{B}\right)=-\boldsymbol{\nabla}_{\perp}\cdot\mathbf{j}_{\perp}$$

The perpendicular current is extracted from the equation of momentum. For this, in contrast to previous multiplication by $\mathbf{B}$ we multiply vectorially by $\mathbf{B}$

$$\mathbf{j}_{\perp}=\frac{1}{B^2}\mathbf{B}\times\left(T\boldsymbol{\nabla}n+\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}+nm_i\frac{d\mathbf{v}}{dt}\right)$$

This is essentially the *diamagnetic current.*

the equilibrium is defined by the functions that are *flux-functions*

$$\begin{aligned}& n(r)\\ V_p(r) &= \langle v_\theta h\rangle\\ V_t &= \langle v_\varphi h\rangle\end{aligned}$$

The average over the flux surface is

$$\langle f \rangle = \int \frac{d\theta}{2\pi} \ h \ f$$

the equilibrium state means

$$\begin{aligned} \frac{\partial}{\partial t} &\equiv 0 \\ \mathbf{\Pi} &= 0 \ \ \text{pressure is isotropic} \\ \mathbf{R}_{\perp} &= 0 \ \ \text{no friction} \end{aligned}$$

the equations under this equilibrium assumption lead to

$$v_\theta = \frac{V_p(r)}{h}$$

$$\begin{aligned} v_\varphi &\approx V_t - 2qV_p\cos\theta \\ &+\varepsilon\left[V_t\cos\theta + 2qV_p\left(1+\frac{1}{4}\cos 2\theta\right)\right] \end{aligned}$$

The first equation says that the poloidal rotation is the rotation uniform on surface V_p modulated by

$$h = 1 + \varepsilon\cos\theta$$

Then

$$\langle nRv_\varphi \rangle = nR_0V_t$$

$$\left\langle v_\parallel \frac{B}{B_0} \right\rangle = V_t + \frac{\varepsilon}{q}\left(1+2q^2\right)V_p$$

The equations

$$\frac{\partial n}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(rn\overline{v}_r\right) = 0$$

$$\frac{\partial}{\partial t}\left[nV_t\right] + \frac{1}{r}\frac{\partial}{\partial r}\left(r \ n\left[V_t\overline{v}_r - qV_p\widetilde{v}_r\right]\right) = 0$$

$$\begin{aligned} & \frac{\partial}{\partial t}\left[V_t + \frac{\varepsilon}{q}\left(1+2q^2\right)V_p\right] \\ & +\overline{v}_r\frac{\partial V_t}{\partial r} - \widetilde{v}_r\frac{\partial}{\partial r}\left[qV_p\right] \\ & + \ \text{(magnetic pumping)} \\ = & \ \ 0 \end{aligned}$$

The velocity is associated to a flux of transport of particles, across the surfaces (in radial direction). The flux is generated by the collisional friction that acts perpendicular to the magnetic field line

$$nv_r = \frac{R_\perp}{|e|\,B}$$

Then the average and the variable part

$$\overline{v}_r = \langle v_r \rangle$$

and

$$\widetilde{v}_r = \langle 2\cos\theta \; v_r \rangle$$

The origin of the poloidal spin-up : *the existence of the variation of the flux of transport with poloidal angle*

Equivalently,

$$\widetilde{v}_r \neq 0$$

The equation

$$\begin{aligned} & \frac{\varepsilon}{q}\left(1+2q^2\right)\left(\frac{\partial V_p}{\partial t}+\gamma_{MP}V_p\right) \\ & +qV_p\frac{\partial}{\partial r}\left(r\; n\widetilde{v}_r\right) \\ = \;& 0 \end{aligned}$$

the logic of the instability that consists of poloidal spin-up

Assume there is a poloidal velocity.

Due to the toroidality and

$$\boldsymbol{\nabla}\cdot(n\mathbf{v}) = 0$$

the poloidal rotation (with compression - distension of volume alternatively in low-field and high-field sides) necessarily is accompanied by toroidal flows that ensure the preservation of the incompressibility.

the toroidal flows have a spatial distribution which is harmonic $\cos\theta$ in the poloidal section. It is Pfirsch Schluter flow and current.

The friction $R_\perp$ is modulated in the surface by these flows.

The friction generates transport fluxes Γ_r which are themselves modulated in the surface but for reasons that are independent of the Pfirsch-Schluter harmonic flows. The radial velocity they induce is also modulated, it is

$$\begin{aligned} \Gamma_r &= nv_r \\ v_r &= -D\left(1+\delta\cos\theta\right) \\ & \quad -v_0\frac{r}{a} \end{aligned}$$

From the combination between the two independent poloidal modulations

$$\begin{aligned} \Gamma_r &\sim f(\theta) \\ \text{Pfrirsch-Schluter flow} &\sim g(\theta) \end{aligned}$$

it is induced a variation of the radial velocity

$$\widetilde{v_r} = \langle 2\cos\theta \ v_r \rangle$$

This combination acts like a drive (a torque) in the equation for the poloidal velocity V_p (function of surface ψ).

The higher the angular matching between the poloidal variation of transport rate $\Gamma_r(\theta)$ with the harmonic Pfirsch Schluter flow $\cos\theta$, the higher the drive of poloidal rotation.

If the poloidal rotation is enhanced by this drive $qV_p \frac{\partial}{\partial r}(r \ n\widetilde{v_r})$ then the amplitude of the harmonic compensatory Pfirsch Schluter flows increases

then the poloidal drive is still higher.

9.2 The spontaneous poloidal spin-up due to poloidal asymmetry of particle fluxes (Hassam Antonsen preprint)

The equations for the *equilibrium state*.

The continuity ignores the time variation of the density $\partial n/\partial t$.

$$\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{nv_\parallel}{B}\right) = -\boldsymbol{\nabla}\cdot(n\mathbf{v}_\perp) \tag{1}$$

The plasma momentum conservation projected along $\mathbf{B}$

$$\mathbf{B}\ \mathbf{u} : \boldsymbol{\nabla}\mathbf{u} = -\mathbf{B}\cdot\boldsymbol{\nabla}\ln n \tag{2}$$

(one still needs a temperature in the pressure term). The static convection term $(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}$ balances the variation of the density (pressure) along the magnetic field line.

Ohm's law

$$\mathbf{B}\cdot\boldsymbol{\nabla}\phi = 0 \tag{3}$$

Current conservation

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{j} &= 0 \\ \mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{j_\parallel}{B}\right) &= -\boldsymbol{\nabla}\cdot\mathbf{j}_\perp \end{aligned} \tag{4}$$

At equilibrium

$$\boldsymbol{\nabla}\cdot(n\mathbf{v}_\perp) = -\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{In}{B^2}\frac{d\phi}{d\psi}\right) \tag{5}$$

and

$$\mathbf{B}\ \mathbf{u}:\boldsymbol{\nabla}\mathbf{u}=\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{u_{\parallel}^{2}}{2}-\frac{\left|\mathbf{u}_{\perp}\right|^{2}}{2}-\frac{I}{B}\frac{d\phi}{d\psi}u_{\parallel}\right) \tag{6}$$

Now we have expression for the divergence of the perpendicular flux

$$\begin{aligned}\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{nv_{\parallel}}{B}\right) &= -\boldsymbol{\nabla}\cdot\left(n\mathbf{v}_{\perp}\right)\\ &= \mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{In}{B^{2}}\frac{d\phi}{d\psi}\right)\end{aligned}$$

and we know that the operator is

$$\mathbf{B}\cdot\boldsymbol{\nabla}=B\nabla_{\parallel}=B\frac{1}{qR}\frac{\partial}{\partial\theta}$$

and since this operator is the same in the left and in the right side the equation can be integrated with introduction of a function that does not depend on θ,

$$\frac{nv_{\parallel}}{B}=\frac{In}{B^{2}}\frac{d\phi}{d\psi}+f\left(r\right)$$

10

11 Spontaneous spin-up (Hassam Drake)

The equations.

The continuity

$$\frac{\partial n}{\partial t}+\boldsymbol{\nabla}\cdot\left(n\mathbf{u}_{\perp}\right)+\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{nu_{\parallel}}{B}\right)=S-\frac{1}{r}\frac{\partial}{\partial r}\left(r\Gamma_{r}\right)$$

This equation is important for the derivation of the expression for the Pfirsch Schluter current.

This is because it introduces the *divergence of the flux of particles*, of the flow. This is where the *geometrical* poloidal compression and dilation will enter the dynamics. In the term $\boldsymbol{\nabla}\cdot\left[\widehat{\mathbf{e}}_{\theta}\left(1+\varepsilon\cos\theta\right)\right]$.

The momentum for all plasma (the mass is taken m_i), isothermal

$$nm_{i}\left[\frac{\partial\mathbf{u}}{\partial t}+\left(\mathbf{u}\cdot\boldsymbol{\nabla}\right)\mathbf{u}\right]=-T\boldsymbol{\nabla}n+\mathbf{j}\times\mathbf{B}-m_{i}S\mathbf{u}$$

The current conservation is essential in connecting the perpendicular current (diamagnetic) with the parallel current (Pfirsch Schluter)

$$\boldsymbol{\nabla}\cdot\mathbf{j}=0$$

The Ohm's law. Here, without the resistivity. This means that the radial velocity v_r will be attributed to another reason for which the charge neutrality cannot be fully suppressed by the parallel currents. The reason may be the *Landau damping* which acts when the collisionality is low. This appears in **Stringer** where a kinetic treatment allows to calculate the variation of the density and of the potential on the magnetic surface, by integrating over the velocity space the distribution functions of electron and ions and imposing neutrality. During the integration, one has to traverse the singularity $v_\parallel - \frac{\varepsilon}{q} v_\theta^E = 0$.

Hewre is the Ohm's law, without resistivity

$$-\nabla\phi + \mathbf{u} \times \mathbf{B} = 0$$

The magnetic field is

$$\mathbf{B} = \nabla\psi \times \nabla\varphi + I(\psi)\nabla\varphi$$

Relative to the work **Hassam Kulsrud** here it is assumed that the electrons and ions are *isothermal*.

$$S(r,\theta) \equiv \text{particle source}$$

It is interesting to note how the *source* extracts from the momentum a part which is proportional with $m_i\mathbf{u}$ through S.

The radial flux

$$\begin{aligned} \Gamma_r &= \langle\langle \widetilde{n}\widetilde{v}_r \rangle\rangle \\ &= -D(r,\theta)\frac{\partial n}{\partial r} \end{aligned}$$

An object of study is the *circulation*.

This is obtained taking the projection in the *parallel* direction of the equation of momentum conservation. It is interesting that the variation of the density in the parallel direction (for isothermal plasma) gives the pressure that opposes to the geometrical advection of the flow, $\mathbf{B}\cdot(\mathbf{u}\cdot\nabla)\mathbf{u}$, which can be static. The imbalance gives $\frac{\partial}{\partial t}(\mathbf{u}\cdot\mathbf{B})$.

$$\begin{aligned} \frac{\partial}{\partial t}(\mathbf{u}\cdot\mathbf{B}) + \mathbf{B}\cdot(\mathbf{u}\cdot\nabla)\mathbf{u} &= -c_s^2\mathbf{B}\cdot\nabla\ln n \quad \text{(parallel pressure)} \\ &\quad -\frac{S}{n}\mathbf{u}\cdot\mathbf{B} \quad \text{(external source of momentum)} \end{aligned}$$

The poloidal component of the equation for the plasma momentum

$$\mathbf{B}_{pol}\cdot\left(nm_i\frac{d\mathbf{u}}{dt} + T\nabla n\right) = \frac{B_\varphi}{R}\mathbf{j}\cdot\nabla\psi \quad \text{(radial current)}$$

We note that

$$\mathbf{B}_{pol} = \nabla\psi \times \nabla\varphi$$

and the product $\mathbf{j} \cdot \boldsymbol{\nabla}\psi$ extracts the radial current.

Now comes the constraint that will provide the third equation: the total radial current traversing a magnetic surface must be zero.

Integrated over a magnetic surface the zero-divergence of the current density leads to

$$\langle j_r \rangle = 0$$

(basic constraint for **Novakovskii** damping rate $\partial V_E / \partial t$ with relevance of the inertia factor $1 + 2q^2$)

$$\int_{flux\_surf} \mathbf{ds} \cdot \mathbf{j} = \int \frac{ds}{|\boldsymbol{\nabla}\psi|} (\mathbf{j} \cdot \boldsymbol{\nabla}\psi) = 0$$

where

$$\begin{aligned} \mathbf{ds} &= ds\, \widehat{\mathbf{e}}_r \\ &= 2\pi R r d\theta\, \widehat{\mathbf{e}}_r \end{aligned}$$

The projection

$$\mathbf{j} \cdot \boldsymbol{\nabla}\psi$$

is already expressed through the *poloidal* balance of momenta. Then

$$\int_{flux\_surf} \frac{ds}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol} \cdot \left(n m_i \frac{d\mathbf{u}}{dt} + T \boldsymbol{\nabla} n \right) = 0$$

This equation, derived from current conservation

$$\boldsymbol{\nabla} \cdot \mathbf{j} = 0$$

will be used to derive the *time variation of the poloidal velocity*, $\partial V_E / \partial t$.

It is assumed first that the plasma velocity is smaller than the sound velocity.

The equilibrium, in zero order

$$\frac{\partial n_0}{\partial t} = 0$$

the equilibrium density is constant in time

$$\mathbf{u}_{\perp 0} \cdot \boldsymbol{\nabla} n_0 = 0$$

The perpendicular advection of the equilibrium density is zero: the density does not vary along *perpendicular* direction.

$$0 = T \mathbf{B}_0 \cdot \boldsymbol{\nabla} n_0$$

The equilibrium density does not vary *parallel* with the magnetic field line

$$\int d\theta \, \frac{1}{r} \frac{\partial n_0}{\partial \theta} = 0$$

If there is a poloidal variation of the density (in higher order) along the poloidal direction, the periodicity must be taken into account.

$$0 = -\boldsymbol{\nabla}\phi_0 + \mathbf{u}_0 \times \mathbf{B}_0$$

The Ohm's law without *resistivity*.

This means that the lowest order density is constant on the surfaces

$$n_0(r)$$

and the velocity which is perpendicular on the magnetic field is contained in the magnetic surface. It is the electric velocity

$$\begin{aligned}
\mathbf{u}_{\perp 0} &= V_E \widehat{\mathbf{e}}_\theta \\
&= \frac{1}{B}\frac{d\phi_0}{dr}\widehat{\mathbf{e}}_\theta
\end{aligned}$$

This velocity V_E is poloidal.

The first order in ε will reveal the presence of a perturbation of the density on the magnetic surface, $n_1(\theta)$.

Also we will have to work with the parallel velocity $u_\|$. The parallel velocity is modulated by the effect of magnetic mirror. There is a non-zero divergence of the parallel velocity, $\nabla_\| u_\| \neq 0$ which is part of the balance of density continuity eq.

$$\begin{aligned}
& \frac{\partial n_1}{\partial t} + V_E \frac{\partial n_1}{r\partial\theta} + n_0 V_E \left(-2\varepsilon\frac{\sin\theta}{r}\right) \\
& + n_0 \nabla_\| u_\| \\
= {} & S - \frac{1}{r}\frac{\partial}{\partial r}(r\Gamma_r)
\end{aligned}$$

We see here that the poloidal rotation velocity V_E is very active: it advects the perturbation of the density on the surface

$$V_E \frac{\partial n_1(\theta)}{r\partial\theta}$$

The poloidal velocity V_E also includes the diamagnetic velocity, $nv_{Dia} = 1/(m\Omega)\,dp/dr$. And V_E is equal with $-\boldsymbol{\nabla}\phi/B$.

To understand the third term we should remember

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\left[\widehat{\mathbf{e}}_\theta(1+\varepsilon\cos\theta)\right] &= \frac{1}{r(R_0 + r\cos\theta)}\frac{\partial}{\partial\theta}\left((R_0 + r\cos\theta)(1+\varepsilon\cos\theta)\right) \\
&= \varepsilon\frac{(-2\sin\theta)}{r}
\end{aligned}$$

The factor $h = 1 + \varepsilon\cos\theta$ comes from the magnitude of the magnetic field $B = B_0/h$. The divergence is calculated for the poloidal flow resulting from the electric velocity V_E that carries the density $n_0 + n_1$. Both quantities do not have variation in this order but the *geometry* is essential.

Termenul $\nabla_{\|} u_{\|}$

The parallel momentum

$$n_0 m_i \left(\frac{\partial u_{\|}}{\partial t} + V_E \frac{\partial u_{\|}}{r \partial \theta} \right) = -T \nabla_{\|} n_1$$

Note that it is here that the *parallel viscosity* $\mathbf{\Pi}$ should appear to introduce the *magnetic damping.* Shaing, etc. **End.**

The parallel gradient is

$$\nabla_{\|} = \frac{1}{qR} \frac{\partial}{\partial \theta}$$

Finally the condition of *periodicity*

$$\int d\theta \ B_{\theta 0} \frac{\partial n_1}{r \partial \theta} = 0$$

Since $B_{\theta 0}$ is actually constant on the surface and is taken out the integral the condition is trivially satisfied in this order.

The condition satisfied trivially at the first order must be recalculated in higher order, *i.e.* two, ε^2.

We will use the constraint

$$\langle j_r \rangle = 0$$

The equation to be used is

$$\begin{aligned} \text{or, the integral form } \quad \mathbf{\nabla} \cdot \mathbf{j} &= 0 \\ \int_{flux\_surf} \mathbf{ds} \cdot \mathbf{j} &= \mathbf{0} \end{aligned}$$

$$\int_{flux\_surf} \frac{ds}{|\mathbf{\nabla} \psi|} R^2 \mathbf{B}_{pol} \cdot \left(n m_i \frac{d\mathbf{u}}{dt} + T \mathbf{\nabla} n \right) = 0$$

derived from the condition of zero-divergence of the current.

The part

$$\int_{flux\_surf} \frac{ds}{|\mathbf{\nabla} \psi|} R^2 \mathbf{B}_{pol} \cdot (T \mathbf{\nabla} n)$$

will be calculated as

$$\begin{aligned} R^2 &\approx 1 \\ \frac{ds}{|\mathbf{\nabla} \psi|} &\sim \text{order } 1 \\ \mathbf{B}_{pol} \cdot \mathbf{\nabla} n &\sim \text{order } 1 \end{aligned}$$

An approximation

$$\mathbf{B}_{pol} \cdot \boldsymbol{\nabla} n \approx B_\theta \frac{\partial n_1}{r\partial\theta}$$

and

$$|\boldsymbol{\nabla}\psi| = 2\pi R B_\theta$$

In the first term of the integrand, there is the product

$$\mathbf{B}_{pol} \cdot \left(n m_i \frac{d\mathbf{u}}{dt} \right)$$

we only retain

$$\mathbf{B}_{pol} \cdot \left(n m_i \frac{\partial\mathbf{u}}{\partial t} \right)$$

since $\mathbf{B}_{pol} \cdot (\mathbf{u} \cdot \boldsymbol{\nabla})\,\mathbf{u}$ is of higher order. This term will provide the time variation of the *poloidal velocity* $V_E(r,t)$.

We also have

$$ds = 2\pi R r d\theta$$

The integration of the first part is

$$\begin{aligned}
& \int_{flux_surf} \frac{ds}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol} \cdot \left(n m_i \frac{d\mathbf{u}}{dt} \right) \\
\approx & \int_{flux_surf} \frac{2\pi R r d\theta}{2\pi R B_\theta} B_\theta n m_i \frac{\partial V_E}{\partial t} \\
= & \ (2\pi)\, r n m_i \frac{\partial V_E}{\partial t}
\end{aligned}$$

the integration of the second term

$$\begin{aligned}
& \int_{flux_surf} \frac{ds}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol} \cdot (T \boldsymbol{\nabla} n) \\
= & \ T \int_{flux_surf} \frac{ds}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol} \cdot \boldsymbol{\nabla} \left[n_0(r) + n_1(r,\theta) \right] \\
= & \ T \int_{flux_surf} \frac{ds}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol} \cdot \boldsymbol{\nabla} n_1(r,\theta)
\end{aligned}$$

we make an integration by parts and take into account the periodicity

$$\begin{aligned}
& []_0^{2\pi} - T \int_{flux_surf} n_1 \boldsymbol{\nabla} \cdot \left(\frac{ds}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol} \right) \\
= & \ -T \int_{flux_surf} n_1 \boldsymbol{\nabla} \cdot \left(\frac{2\pi R r d\theta}{2\pi R B_\theta} R^2 B_\theta \widehat{\mathbf{e}}_\theta \right)
\end{aligned}$$

we must calculate

$$
\begin{aligned}
& \int_{flux\_surf} \frac{ds}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol} \cdot (T \boldsymbol{\nabla} n) \\
= & -T \int_{flux\_surf} n_1 \boldsymbol{\nabla} \cdot \left(\frac{ds}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol} \right) \qquad \text{(integration by parts)} \\
= & -Tr \int n_1 d\theta \boldsymbol{\nabla} \cdot (h \widehat{\mathbf{e}}_\theta) \\
= & -Tr \int d\theta \left(\varepsilon \frac{-2 \sin \theta}{r} \right) n_1
\end{aligned}
$$

Then the constraint which is the starting point of these calculations

$$
\langle j_r \rangle = 0
$$

becomes

$$
\int_{flux\_surf} \frac{ds}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol} \cdot \left(n m_i \frac{d\mathbf{u}}{dt} + T \boldsymbol{\nabla} n \right) = 0
$$

$$
\begin{aligned}
(2\pi)\, r n m_i \frac{\partial V_E}{\partial t} + \left\{ -Tr \int d\theta \left(\varepsilon \frac{-2 \sin \theta}{r} \right) n_1 \right\} &= 0 \\
(2\pi)\, r n m_i \frac{\partial V_E}{\partial t} &= -2T\varepsilon \int d\theta \sin \theta \; n_1 \\
\frac{\partial V_E}{\partial t} &= -\frac{1}{r} \varepsilon \frac{c_s^2}{n_0} \int \frac{d\theta}{2\pi} \sin \theta \; n_1
\end{aligned}
$$

Comment

This is a very interesting relation

The rate of damping, $\partial V_E / \partial t$ is determined as the surface integral over a $\sin \theta$-modulation of the first order density, which has a θ variation

$$
n_1 (r, \theta)
$$

End

New notation

$$
N \equiv \frac{n_1}{n_0}
$$

The equation of continuity

$$
\begin{aligned}
& \frac{\partial N}{\partial t} + V_E \frac{\partial N}{r \partial \theta} + V_E \left(-2\varepsilon \frac{\sin \theta}{r} \right) \\
& + \nabla_{\|} u_{\|} \\
= & \frac{S}{n_0} - \frac{1}{n_0} \frac{1}{r} \frac{\partial}{\partial r} (r \Gamma_r)
\end{aligned}
$$

The parallel momentum

$$\frac{\partial u_{\|}}{\partial t} + V_E \frac{\partial u_{\|}}{r\partial \theta} = -c_s^2 \nabla_{\|} N$$

The constraint $\langle j_r \rangle = 0$

$$\frac{\partial V_E}{\partial t} = c_s^2 \int \frac{d\theta}{2\pi} N \left(-2\varepsilon \frac{\sin\theta}{r} \right)$$

NOTE

Let us stop to make a comparison between this (**Hassam Drake**) system prepared for the spontaneous spin-up and the **Stringer PRL** system.

We note that the *time variation* in the equation for

- the density, $\partial n_1 / \partial t$, and
- the velocity

$$n m_i V_{E\theta}^{(0)} \frac{\partial v_{i\|}}{r\partial \theta} = -\left(T_e + T_i\right) \frac{\varepsilon}{q} \frac{\partial n_1}{r\partial \theta}$$

  this equation shows the balance of momentum carried by the "static advected" velocity (*i.e.* space variation of the velocity, $(\mathbf{v} \cdot \boldsymbol{\nabla}) \mathbf{v}$) with the *pressure*. The projection is made on poloidal direction.

is *absent* at **Stringer**.

Since Hassam Drake work for spin-up driven by external source (poloidally asymmetric) the explicit time variation must be retained.

The main term however in the formulation Hassam Drake is still $V_E \frac{\partial u_{\|}}{r\partial\theta}$ (later $\widehat{u}_E \frac{\partial \widetilde{u}_{\|}}{r\partial\theta}$) [partly balanced by parallel gradient of pressure] which is the same as in Stringer. This term will be the main part of the expansion around the equilibrium static state.

The equilibrium static state at Hassam Drake is

$$\begin{aligned} \nabla_{\|} \widetilde{u}_{\|} &= \widetilde{F} \\ \nabla_{\|} \widetilde{N} &= 0 \end{aligned}$$

and the expansion introduces new, small, quantities

$$\widehat{u}_E \ , \ \widehat{u}_{\|} \ , \ \widehat{N}$$

with the system

$$\begin{aligned} -2\varepsilon \widehat{u}_E \frac{\sin\theta}{r} + \nabla_{\|} \widehat{u}_{\|} &= 0 \\ \frac{\partial \widehat{u}_{\|}}{\partial t} + \widehat{u}_E \frac{\partial \widetilde{u}_{\|}}{r\partial\theta} &= -c_s^2 \nabla_{\|} \widehat{N} \\ \frac{\partial \widehat{u}_E}{\partial t} &= -\frac{2\varepsilon c_s^2}{r} \int \frac{d\theta}{2\pi} \widehat{N} \sin\theta \end{aligned}$$

See the explanations below.
END

The functions that must be determined

$$N(r,\theta,t) \quad , \quad u_{\parallel}(r,\theta,t) \quad , \quad V_E(r,t)$$

The global balance is obtained by integrating over the surface $\int \frac{d\theta}{2\pi}(...)$.

$$\begin{aligned}
\frac{\partial \overline{N}}{\partial t} &= \frac{\overline{S}}{n_0} - \frac{1}{n_0}\frac{\partial}{r\partial\theta}\left(r\overline{\Gamma}_r\right) \\
\frac{\partial \overline{u}_{\parallel}}{\partial t} &= 0
\end{aligned}$$

After introducing the average over surfaces, the new variables are the differences that have variations in the surfaces

$$\widetilde{f} = f - \overline{f}$$

The source in the surface is

$$F \equiv \frac{S - \frac{1}{r}\frac{\partial}{\partial r}(r\Gamma_r)}{n_0}$$

The state that is taken as reference is the absence of the poloidal rotation

$$V_E^{ref} = 0$$

and this reduces the equations to

$$\begin{aligned}
N^{ref} &= \overline{N} \\
\text{i.e.} \quad \widetilde{N}^{ref} &= 0 \quad \text{(no variation of the density in the surface)}
\end{aligned}$$

$$\nabla_{\parallel}\widetilde{u}_{\parallel}^{ref} = \widetilde{F}$$

$$\nabla_{\parallel}\widetilde{N}^{ref} = 0 \;\; \text{from where } \widetilde{N} = 0$$

The variation of the parallel velocity along the magnetic line (equivalently, in the magnetic surface) is obtained in terms of the source

$$\begin{aligned}
\nabla_{\parallel}\widetilde{u}_{\parallel}^{ref} &= \widetilde{F} \\
\text{or} \quad \frac{1}{qR}\frac{\partial}{\partial\theta}\widetilde{u}_{\parallel}^{ref} &= \widetilde{F} \\
\text{and after integration} \quad \widetilde{u}_{\parallel}^{ref} &= qR\int d\theta' \; \widetilde{F}
\end{aligned}$$

Consider a perturbation of this reference state

$$\begin{aligned}
V_E &= V_E^{ref} + \widehat{V}_E \\
u_\parallel &= \widetilde{u}_\parallel^{ref} + \widehat{u}_\parallel \\
N &= \overline{N} + \widetilde{N}^{ref} + \widehat{N}
\end{aligned}$$

This will induce a time variation of the poloidal (electric) velocity and of the density N and of the parallel velocity.

However the time variation is assumed to be slower than the sound speed

$$\frac{\partial}{\partial t} \ll \frac{c_s}{qR}$$

In the equation of continuity the time variation for N is neglected and the equation for density becomes a balance

$$\widehat{V}_E \left(-2\varepsilon \frac{\sin\theta}{r}\right) + \nabla_\parallel \widehat{u}_\parallel = 0$$

$$\frac{\partial \widehat{u}_\parallel}{\partial t} + \widehat{V}_E \frac{\partial \widetilde{u}_\parallel^{ref}}{r\partial\theta} = -c_s^2 \nabla_\parallel \widehat{N}$$

$$\frac{\partial \widehat{V}_E}{\partial t} = c_s^2 \int \frac{d\theta}{2\pi} \widehat{N} \left(-2\varepsilon \frac{\sin\theta}{r}\right)$$

Note the preservation of the poloidal derivative of the *reference* parallel velocity in the second equation. This reference value of the paralel velocity is fixed by the radial flux and the source of particles. It exists only because these sources and fluxes are *NOT constant on the poloidal circumference.*

This set of equations can be integrated.

The operator that must be made explicit is

$$\nabla_\parallel = \frac{1}{qR} \frac{\partial}{\partial\theta}$$

Then, since $\widehat{V}_E$ is constant on magnetic surfaces, the first equation is

$$\begin{aligned}
\widehat{V}_E \left(-2\varepsilon \frac{\sin\theta}{r}\right) + \nabla_\parallel \widehat{u}_\parallel &= 0 \quad \text{or} \\
\frac{1}{qR} \frac{\partial}{\partial\theta} \widehat{u}_\parallel &= \widehat{V}_E \left(2\varepsilon \frac{\sin\theta}{r}\right) \\
\text{(after integration on } \theta\text{)} \ \widehat{u}_\parallel &= -2qR\varepsilon \frac{\cos\theta}{r} \widehat{V}_E
\end{aligned}$$

The parallel velocity so calculated $\widehat{u}_\parallel$ is introduced in the second equation

$$\begin{aligned}
\frac{\partial}{\partial t} \left[-2qR\varepsilon \frac{\cos\theta}{r} \widehat{V}_E\right] + \widehat{V}_E \frac{\partial \widetilde{u}_\parallel^{ref}}{r\partial\theta} &= -c_s^2 \nabla_\parallel \widehat{N} \\
&= -c_s^2 \frac{1}{qR} \frac{\partial \widehat{N}}{\partial\theta}
\end{aligned}$$

First it is multiplied by (qR) from the denominator of the term with θ variation of the density and then is integrated up to a current θ

$$
\begin{aligned}
-c_s^2 \widehat{N}(\theta) &= -2(qR)^2 \varepsilon \frac{1}{r} \frac{\partial \widehat{V}_E}{\partial t} \int^{\theta} d\theta' \cos\theta' \\
&\quad + \widehat{V}_E \frac{qR}{r} \widetilde{u}_{\|}^{ref}
\end{aligned}
$$

we ignore for the moment the constant of integration which should be a function of surface.

NOTE

This is an essential step.

We have found that the density $\widehat{N}(\theta)$ with poloidal $\sim \theta$ variation, must include a term proportional with $\partial V_E / \partial t$. The origin of the time derivative of V_E is the time derivative of the parallel velocity, $\partial \widehat{u}_{\|} / \partial t$. The time variation of the parallel flow $\widehat{u}_{\|}$ is the dynamic un-balance of the parallel momentum. The advection by V_E of the parallel velocity $\sim \theta$, like $(\mathbf{v} \cdot \boldsymbol{\nabla}) \mathbf{v}$, competes with the parallel gradient of the *pressure* (like in instabilities). Since the advective term contains V_E which we know that will have *time variation*, $V_E(t)$ the balance of parallel momenta will necessarily be time-dynamic. We must preserve $\partial \widehat{u}_{\|} / \partial t$.

And if we do that, the density $\widehat{N}(\theta)$ that will result from the balance advection-grad-of-pressure, will depend on the time derivative $\partial V_E / \partial t$.

END

This is introduced in the equation $\langle j_r \rangle = 0$ for the time variation of $\widehat{V}_E$, the third equation

$$
\begin{aligned}
\frac{\partial \widehat{V}_E}{\partial t} &= c_s^2 \int \frac{d\theta}{2\pi} \widehat{N} \left(-2\varepsilon \frac{\sin\theta}{r} \right) \quad \text{where we replace } \widehat{N}(\theta) \\
&= \int \frac{d\theta}{2\pi} \left(2\varepsilon \frac{\sin\theta}{r} \right) \left\{ -2(qR)^2 \varepsilon \frac{1}{r} \frac{\partial \widehat{V}_E}{\partial t} \int^{\theta} d\theta' \cos\theta' + \widehat{V}_E \frac{qR}{r} \widetilde{u}_{\|}^{ref} \right\} \\
&= -4(qR)^2 \varepsilon^2 \frac{\partial \widehat{V}_E}{\partial t} \frac{1}{r^2} \int \frac{d\theta}{2\pi} \sin\theta \sin\theta \\
&\quad + 2\varepsilon \frac{qR}{r^2} \widehat{V}_E \int \frac{d\theta}{2\pi} \sin\theta \widetilde{u}_{\|}^{ref}
\end{aligned}
$$

The first term

$$
-4(qR)^2 \varepsilon^2 \left(\frac{\partial \widehat{V}_E}{\partial t} \right) \frac{1}{r^2} \int \frac{d\theta}{2\pi} \sin\theta \sin\theta = -4q^2 R^2 \frac{r^2}{R^2} \left(\frac{\partial \widehat{V}_E}{\partial t} \right) \frac{1}{r^2} \frac{1}{2} = -2q^2 \left(\frac{\partial \widehat{V}_E}{\partial t} \right)
$$

and the second

$$
\begin{aligned}
2\varepsilon \frac{qR}{r^2} \widehat{V}_E &= 2 \frac{r}{R} q \frac{R}{r} \frac{1}{r} \widehat{V}_E \\
&= \frac{2q}{r} \widehat{V}_E
\end{aligned}
$$

$$\left(1+2q^{2}\right)\frac{\partial\widehat{V}_{E}}{\partial t}=\frac{2q}{r}\widehat{V}_{E}\int\frac{d\theta}{2\pi}\sin\theta\widetilde{u}_{\Vert}^{ref}$$

This is the way one finds the INERTIA factor

$$\left(1+2q^{2}\right)$$

of the plasma in the poloidal rotation.

In this equation we replace the reference state for the parallel velocity, which is fixed by the source and the flux, both these contributions being retained with their variation along the poloidal direction

$$\left(1+2q^{2}\right)\frac{\partial\widehat{V}_{E}}{\partial t}=\widehat{V}_{E}\quad\times\frac{1}{\varepsilon^{2}}2q^{2}\left[\frac{1}{n_{0}}\int\frac{d\theta}{2\pi}S\cos\theta-\frac{1}{n_{0}}\frac{1}{r}\frac{\partial}{\partial r}\left(r\int\frac{d\theta}{2\pi}\Gamma_{r}\cos\theta\right)\right]$$

we can easily recognize that an integration by parts have been made in the right hand side.

NOTE
How is generated this *inertia factor* $1+2q^{2}$.
We have seen that the first integration

$$\widehat{u}_{\Vert}=-2qR\varepsilon\frac{\cos\theta}{r}\widehat{V}_{E}$$

actually obtains the Pfirsch Schluter parallel current, when a poloidal flow is given by $\widehat{V}_{E}$. This PS flow has a coefficient q, as usual.

The second equation has the RHS term coming from the parallel gradient of pressure $-c_{s}^{2}\frac{1}{qR}\frac{\partial\widehat{N}}{\partial\theta}$ and this introduces the second q factor. The actual derivative was

$$\frac{\partial}{\partial l_{\Vert}}\widehat{N}\left(\theta\right)=\frac{1}{qR}\frac{\partial}{\partial\theta}\widehat{N}\left(\theta\right)$$

They now multiply the term $\partial\widehat{u}_{\Vert}/\partial t$.

The enhancement of the *radial diffusion* with a factor q^{2}, the known characteristics of the Pfirsch Schluter "regime" has the same origin. We note however that it is not yet clear what means PS regime.
END

11.0.1 Comments on the enhanced radial flux due to Pfirsch Schluter current

The radial velocity is determined above, with the functions

$$D\equiv-B_{0}\frac{\frac{dB_{0}}{dr}}{\frac{dp}{dr}}$$

$$v_D^{rez} \equiv -\eta \frac{1}{B_0} \frac{dp}{dr}$$

where η is the *resistivity*.

Then

$$\langle f v_r \rangle = v_D^{rez} \left[D \langle f \rangle + \left(\frac{q}{\varepsilon}\right)^2 \left(\langle f h^2 \rangle - \frac{\langle f \rangle}{\left\langle \frac{1}{h^2} \right\rangle} \right) \right]$$

In general

$$\frac{dB_0}{dr} = 0$$

and this means

$$D = 0$$

$$\langle v_r \rangle = v_D^{rez} \left(\frac{q}{\varepsilon}\right)^2 \left(\langle h^2 \rangle - \frac{1}{\left\langle \frac{1}{h^2} \right\rangle} \right)$$

12 Poloidal instability (Hazeltine, Lee, Rosenbluth 1970)

The most general equations, simplified, in the toroidal geometry.

$$0 = -\nabla p + \mathbf{j} \times \mathbf{B}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j}$$

$$\nabla \cdot (\rho \mathbf{v}) = 0$$

$$p = c_s^2 \rho$$

and the resulting equilibrium parameters (taking $\mu_0 = 1$)

$$p = p(r)$$

$$\begin{aligned} B_r &= 0 \\ B_\theta &= \frac{b(r)}{h} \\ B_\varphi &= \frac{B(r)}{h} \end{aligned}$$

$$\begin{aligned} j_r &= 0 \\ j_\theta &= -\frac{1}{h} \frac{dB(r)}{dr} \\ j_\varphi &= -\frac{1}{h} \frac{B(r)}{b(r)} \frac{dB(r)}{dr} - \frac{h}{b(r)} \frac{dp}{dr} \end{aligned}$$

The expressions for the current components is derived from the two equations $0 = -\boldsymbol{\nabla} p + \mathbf{j} \times \mathbf{B}$ and $\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j}$.

$$
\begin{aligned}
j_\varphi &= \frac{1}{r}\frac{\partial}{\partial r}\left(r B_\theta\right) \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{b\left(r\right)}{h}\right) \\
-\frac{1}{h}\frac{B\left(r\right)}{b\left(r\right)}\frac{dB\left(r\right)}{dr} - \frac{h}{b\left(r\right)}\frac{dp}{dr} &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{b\left(r\right)}{h}\right)
\end{aligned}
$$

The electric field has an externally induced component

$$
\mathbf{E} = -\boldsymbol{\nabla}\phi + \frac{\mathcal{E}}{h}\widehat{\mathbf{e}}_\varphi
$$

The next equation to be included is the Ohm's law

$$
\eta j_\varphi = \frac{\mathcal{E}}{h} + v_r B_\theta
$$

This results from taking the θ component of the Ohm's law

$$
\left(\mathbf{E}\right)_\theta + \left(\mathbf{v} \times \mathbf{B}\right)_\theta = \eta\left(\mathbf{J}\right)_\theta
$$

where

$$
\left(\mathbf{E}\right)_\theta = -\frac{\partial\phi}{r\partial\theta}
$$

and

$$
\left(\mathbf{v} \times \mathbf{B}\right)_\theta = \left(\begin{array}{ccc} \widehat{\mathbf{e}}_r & \widehat{\mathbf{e}}_\theta & \widehat{\mathbf{e}}_\varphi \\ v_r & v_\theta & v_\varphi \\ 0 & \frac{b}{h} & \frac{B}{h} \end{array}\right)_\theta = -v_r\frac{B}{h}
$$

then

$$
\begin{aligned}
-\frac{\partial\phi}{r\partial\theta} - v_r\frac{B}{h} &= \eta j_\theta \\
&= \eta\left[-\frac{1}{h}\frac{dB\left(r\right)}{dr}\right]
\end{aligned}
$$

we extract the derivative of the potential to θ,

$$
\frac{\partial\phi}{\partial\theta} = -r v_r\frac{B}{h} + \eta r\frac{1}{h}\frac{dB\left(r\right)}{dr}
$$

and integrate over $[0, 2\pi]$

$$
0 = \int_0^{2\pi} d\theta\frac{\partial\phi}{\partial\theta} = \int_0^{2\pi} d\theta\left[-r v_r\frac{B}{h} + \eta r\frac{1}{h}\frac{dB\left(r\right)}{dr}\right]
$$

From the integrand a result with partial determination is

$$v_r = \eta \frac{1}{B} \frac{dB}{dr} + [\text{function independent of } \theta]$$

which means a function that is constant on the surface, v^H.

Now we modify the equation

$$\eta j_\varphi = \frac{\mathcal{E}}{h} + v_r B_\theta$$

by taking $\frac{\mathcal{E}}{h}$ to be of higher order and obtaining

$$\begin{aligned} v_r^H &\approx \frac{\eta j_\varphi}{B_\theta} \\ &= \eta \frac{1}{B_\theta} \left(-\frac{1}{h} \frac{B}{b} \frac{dB}{dr} - \frac{h}{b} \frac{dp}{dr} \right) \\ &= \eta \frac{h}{b} \left(-\frac{1}{h} \frac{B}{b} \frac{dB}{dr} - \frac{h}{b} \frac{dp}{dr} \right) \\ &= -\eta \frac{B}{b^2} \frac{dB}{dr} - \eta \frac{h^2}{b^2} \frac{dp}{dr} \end{aligned}$$

and we **NOTE** that this expression is function of only r.

$$\begin{aligned} v_r &= \eta \frac{1}{B} \frac{dB}{dr} \\ &\quad -\eta \frac{B}{b^2} \frac{dB}{dr} - \eta \frac{h^2}{b^2} \frac{dp}{dr} \end{aligned}$$

The last term

$$\eta \frac{h^2}{b^2} \frac{dp}{dr} = \eta \frac{1}{b^2} \left(1 + 2\varepsilon \cos\theta + \varepsilon^2 \cos^2\theta\right) \frac{dp}{dr}$$

Now we have to use two equations. One is

$$-\frac{1}{h} \frac{B(r)}{b(r)} \frac{dB(r)}{dr} - \frac{h}{b(r)} \frac{dp}{dr} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{b(r)}{h} \right)$$

COMMENT

Before all calculations, we note that the radial velocity v_r only exists because there is a resistivity, which opposes the *parallel current*.

The presence of the resistivity imposes the activation of the term $\mathbf{v} \times \mathbf{B} \rightarrow v_r B_\theta$ in the Ohm's law.

This confirms the assertion of **Stringer** that the parallel current flows because it tries to neutralize the charge separation produced by the diamagnetic flow with non-zero divergence. And, when this is not possible in a complete way, due to a resistivity, then there is a *radial velocity*.

A pure radial velocity v_r cannot exist.

There is also a poloidal velocity v_θ,

$$v_\theta = \frac{u(r)}{h}$$

and

$$\begin{aligned} v_\varphi &= \frac{B}{b}\frac{u(r)}{h} - \frac{h}{b}\frac{d\phi}{dr} \\ &\quad +O(\eta) \end{aligned}$$

$$\frac{d\phi}{dr} = Bu(r) - b\overline{v}_\varphi$$

13 Numerical study of the Stringer rotation

We have developed a numerical framework that incorporates the effect of poloidal variation of the rate of particle and energy losses. The equation for poloidal rotation under the Stringer effect is solved and the rate of rotation is compared with the *transit time magnetic pumping* damping.

14 System of equation for spontaneous poloidal rotation and <u>shock</u> formation (Rosenbluth Lee Hazeltine 2) hose-like

This part is also in **rotation.tex**.

The paper **Resistive plasma rotation shock formation Rosenbluth Lee Hazeltine 1971**.

The physical picture: in zeroth order in the dissipative mechanism (here the resistivity η) the quantities that are θ-averaged over the magnetic surface

$\overline{\rho}$ density
$\overline{u}$ poloidal rotation speed
$\overline{v}$ toroidal velocity

can be prescribed independently from surface to surface, by arbitrary functions.

The zeroth-order (in η) equations uniquely determine the *azimuthally-varying* parts of the full functions

$$\overline{\rho} + \delta\rho(\theta)$$
$$\overline{u} + \delta u(\theta)$$
$$\overline{v} + \delta v(\theta)$$

When there is an *interaction* due to the presence of a *resistivity* η, there is a slow

$$\tau \sim \frac{1}{\eta}$$

transition between the steady states found at the zeroth-order. The time-dependent equations resulted from the inclusion of the small η show that the rotations with small poloidal speeds have the tenddency to accelerate.

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\,\mathbf{v} = -\frac{c_s^2}{\rho}\boldsymbol{\nabla}\rho + \frac{1}{\rho}\mathbf{J} \times \mathbf{B} \tag{7}$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho \mathbf{v}) = 0 \tag{8}$$

$$\boldsymbol{\nabla} \cdot \mathbf{J} = 0 \tag{9}$$

$$-\boldsymbol{\nabla}\phi + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J} \tag{10}$$

Note the absence of anisotropy of the pressure tensor in $\|$ relative to $\perp$ directions, that are usually invoked to represent the magnetic pumping dumping of poloidal rotation.

End.

Note the absence of the Ampere's law

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}$$

from the system of equations. They asy that the fields are produced by external coils.

End.

The magnetic field

$$\begin{aligned} \mathbf{B} &= \left(0, \frac{\varepsilon}{q}\frac{B_0}{h}, \frac{B_0}{h}\right) \\ h &\equiv 1 + \varepsilon \cos\theta \end{aligned}$$

Note that looks like **Novakovskii**. **End.**

Axial symmetry

$$\frac{\partial}{\partial z} = 0$$

The zeroth-order equations are obtained

$$\text{zero-th order} \left\{ \begin{array}{l} \eta \simeq 0 \\ \frac{\partial}{\partial t} \rightarrow 0 \end{array} \right.$$

NOTE. The fact that the zero-order is obtained taking the *time-variation* as zero means that this result cannot be applied to the fast time increase of the poloidal flow at the onset of the cells of convection in **Inverse RH** problem.

Also there is no decay of an initial poloidal rotation, at relaxation, as treated in **Taguchi**, etc.

End.

It results

$$v_r = 0$$

and the Ohm's law (10) gives

$$\frac{\partial \phi}{\partial \theta} = 0$$

In the zeroth order the potential does not vary on the magnetic surface. We draw conclusion that **the variation of the electrostatic potential on a magnetic surface is connected with a dissipative mechanism: diffusion, resistivity.**

This is explained in **Stringer, Rosenbluth**, etc.

It also results from the continuity equation

$$\boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial}{\partial \theta} (h \rho v_\theta) = 0$$

(**Note** that this comes from, see **Morse Feshbach I pages 25-35** and **Geometry.tex**

$$\boldsymbol{\nabla} \cdot \mathbf{a} = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial r} (h_2 h_3 a_1) + \frac{\partial}{\partial \theta} (h_1 h_3 a_2) + \frac{\partial}{\partial \varphi} (h_1 h_2 a_3) \right)$$

with

$$\begin{aligned} h_r &= 1 \\ h_\theta &= r \\ h_\varphi &= R + r \cos \theta \end{aligned}$$

).

We multiply the equation of conservation of momentum (7) with $h\mathbf{B}$, at stationarity $\partial / \partial t = 0$,

$$h\mathbf{B} \cdot [(\mathbf{v} \cdot \boldsymbol{\nabla}) \mathbf{v}] = -\frac{c_s^2}{\rho} h\mathbf{B} \cdot \boldsymbol{\nabla} \rho + \frac{1}{\rho} h\mathbf{B} \cdot (\mathbf{J} \times \mathbf{B})$$

The last term is identically zero, since $\mathbf{J} \times \mathbf{B} = -\boldsymbol{\nabla} p$ and $\boldsymbol{\nabla} p$ is perpendicular on $\mathbf{B}$. Note the absence of *friction* that helps to reach balance of forces along the magnetic field line.

Note that the operator

$$(\mathbf{v} \cdot \boldsymbol{\nabla})$$

appreas also in **Stringer** to calculate the time variation of the parallel velocity $v_\parallel$ in view of writting the drift-kinetic equation for $f_j^{(1)}$ then $n_j^{(1)}$ and neutrality, to find the nonuniform potential $\Phi^{(1)}$ in the surface. **END.**

This equation will become a *Bernoulli-like law.*

It is a balance of forces along the magnetic field and must reflect the variation on the surface $f(\theta)$ of the physical parameters like density n (or ρ).

In the paper **poloidal rotation growth Rosenbluth Hazeltine Lee** it is written

$$\mathbf{B}\cdot\left[\left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{v}\right]=\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{v^2}{2}\right)+\left(\boldsymbol{\nabla}\times\mathbf{v}\right)\cdot\left(\mathbf{v}\times\mathbf{B}\right)$$

Note that the last term is

$$-\boldsymbol{\omega}\cdot\mathbf{E}$$

or

$$\eta\boldsymbol{\omega}\cdot\mathbf{J}$$

To calculate $\left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{v}$ we use the formulas from **geometry.tex** notes, or from **Morse and Feshbach**.

At this stage there will be no approximation yet.

The velocity is introduced with the non-zero components

$$\left(v_r,v_\theta,v_\varphi\right)=\left(0,v_\theta,v_\varphi\right)$$

Consider two vectors

$$\mathbf{A}\ \ \text{and}\ \ \mathbf{B}$$

$$\begin{aligned}
&\text{component of }\ \left(\mathbf{B}\cdot\boldsymbol{\nabla}\right)\mathbf{A}\text{ along }\widehat{\mathbf{e}}_1\\
=\quad&\left(B_1\frac{1}{h_1}\frac{\partial}{\partial\xi_1}+B_2\frac{1}{h_2}\frac{\partial}{\partial\xi_2}+B_3\frac{1}{h_3}\frac{\partial}{\partial\xi_3}\right)A_1\\
&+\frac{A_2}{h_1h_2}\left(B_1\frac{\partial h_1}{\partial\xi_2}-B_2\frac{\partial h_2}{\partial\xi_1}\right)+\frac{A_3}{h_1h_3}\left(B_1\frac{\partial h_1}{\partial\xi_3}-B_3\frac{\partial h_3}{\partial\xi_1}\right)
\end{aligned}$$

$$\begin{aligned}
&\text{component of }\ \left(\mathbf{B}\cdot\boldsymbol{\nabla}\right)\mathbf{A}\text{ along }\widehat{\mathbf{e}}_2\\
=\quad&\left(B_1\frac{1}{h_1}\frac{\partial}{\partial\xi_1}+B_2\frac{1}{h_2}\frac{\partial}{\partial\xi_2}+B_3\frac{1}{h_3}\frac{\partial}{\partial\xi_3}\right)A_2\\
&+\frac{A_3}{h_2h_3}\left(B_2\frac{\partial h_2}{\partial\xi_3}-B_3\frac{\partial h_3}{\partial\xi_2}\right)+\frac{A_1}{h_2h_1}\left(B_2\frac{\partial h_2}{\partial\xi_1}-B_1\frac{\partial h_1}{\partial\xi_3}\right)
\end{aligned}$$

$$\begin{aligned}
&\text{component of }\ \left(\mathbf{B}\cdot\boldsymbol{\nabla}\right)\mathbf{A}\text{ along }\widehat{\mathbf{e}}_3\\
=\quad&\left(B_1\frac{1}{h_1}\frac{\partial}{\partial\xi_1}+B_2\frac{1}{h_2}\frac{\partial}{\partial\xi_2}+B_3\frac{1}{h_3}\frac{\partial}{\partial\xi_3}\right)A_3\\
&+\frac{A_1}{h_3h_1}\left(B_3\frac{\partial h_3}{\partial\xi_1}-B_1\frac{\partial h_1}{\partial\xi_3}\right)+\frac{A_2}{h_3h_2}\left(B_3\frac{\partial h_3}{\partial\xi_2}-B_2\frac{\partial h_2}{\partial\xi_3}\right)
\end{aligned}$$

$$\text{component of }\ \left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{v}\text{ along }\widehat{\mathbf{e}}_r\ \ \text{is zero}$$

$$\begin{aligned} & \text{component of } \left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{v} \text{ along } \widehat{\mathbf{e}}_\theta \\ = & \left(v_\theta \frac{1}{h_\theta}\frac{\partial}{\partial\theta} + v_\varphi \frac{1}{h_\varphi}\frac{\partial}{\partial\varphi}\right) v_\theta \\ & + \frac{v_\varphi}{h_\theta h_\varphi}\left(v_\theta \frac{\partial h_\theta}{\partial\varphi} - v_\varphi \frac{\partial h_\varphi}{\partial\theta}\right) \end{aligned}$$

We assume that there is no variation along φ. We insert

$$\frac{\partial h_\varphi}{\partial\theta} = \frac{\partial}{\partial\theta}\left(R + r\cos\theta\right) = -r\sin\theta$$

or

$$-\sin\theta = \frac{1}{\varepsilon}\frac{\partial h}{\partial\theta}$$

and obtain

$$\begin{aligned} & \text{component of } \left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{v} \text{ along } \widehat{\mathbf{e}}_\theta \\ = & v_\theta \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} - \frac{v_\varphi^2}{r\left(R + r\cos\theta\right)}\left(-r\sin\theta\right) \end{aligned}$$

Further

$$\begin{aligned} & \text{component of } \left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{v} \text{ along } \widehat{\mathbf{e}}_\varphi \\ = & \left(v_\theta \frac{1}{h_\theta}\frac{\partial}{\partial\theta} + v_\varphi \frac{1}{h_\varphi}\frac{\partial}{\partial\varphi}\right) v_\varphi \\ & + \frac{v_\theta}{h_\varphi h_\theta}\left(v_\varphi \frac{\partial h_\varphi}{\partial\theta} - v_\theta \frac{\partial h_\theta}{\partial\varphi}\right) \end{aligned}$$

or

$$\begin{aligned} & \text{component of } \left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{v} \text{ along } \widehat{\mathbf{e}}_\varphi \\ = & v_\theta \frac{1}{r}\frac{\partial v_\varphi}{\partial\theta} + \frac{v_\theta v_\varphi}{r(R + r\cos\theta)}\left(-r\sin\theta\right) \end{aligned}$$

At this moment we have

$$\begin{aligned} & \left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{v} \\ = & \widehat{\mathbf{e}}_\theta \left[v_\theta \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} - \frac{v_\varphi^2}{r\left(R + r\cos\theta\right)}\left(-r\sin\theta\right)\right] \\ & + \widehat{\mathbf{e}}_\varphi \left[v_\theta \frac{1}{r}\frac{\partial v_\varphi}{\partial\theta} + \frac{v_\theta v_\varphi}{r(R + r\cos\theta)}\left(-r\sin\theta\right)\right] \end{aligned}$$

Returning to the equation of momentum conservation multiplied by $h\mathbf{B}$, we have

$$h\mathbf{B}\cdot\left[\left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{v}\right] = -\frac{c_s^2}{\rho} h\mathbf{B}\cdot\boldsymbol{\nabla}\rho$$

$$B_\theta h\left[v_\theta\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}-\frac{v_\varphi^2}{Rh}(-\sin\theta)\right]+B_\varphi h\left[v_\theta\frac{1}{r}\frac{\partial v_\varphi}{\partial\theta}+\frac{v_\theta v_\varphi}{Rh}(-\sin\theta)\right]=-\frac{c_s^2}{\rho}hB_\theta\frac{\partial\rho}{r\partial\theta}$$

NOTE that it is *here* that the diamagnetic velocity cannot be included since there was a projection along the magnetic field line, - by multiplying with $\mathbf{B}$. Then this equation just describe the variation of parameters like velocity and density on the magnetic surface. **END.**

The radial component of the Ohm's law

$$-\frac{\partial\phi}{\partial r}+\left.\begin{pmatrix}\widehat{\mathbf{e}}_r & \widehat{\mathbf{e}}_\theta & \widehat{\mathbf{e}}_\varphi\\ v_r & v_\theta & v_\varphi\\ 0 & B_\theta & B_\varphi\end{pmatrix}\right|_r=0$$

$$-\frac{\partial\phi}{\partial r}+v_\theta B_\varphi-v_\varphi B_\theta=0$$

NOTE that it is the classical formula to determine the radial electric field

$$E_r=B\frac{1}{n}\frac{1}{m\Omega_c}\frac{dp}{dr}+(\mathbf{v}\times\mathbf{B})_\theta$$

and we note that the diamagnetic velocity is neglected.
END

We can eliminate v_φ by expressing it through v_θ and the radial derivative of the potential

$$v_\varphi=\frac{1}{B_\theta}\left(-\frac{\partial\phi}{\partial r}\right)+\frac{B_\varphi}{B_\theta}v_\theta$$

we also have

$$\frac{B_\varphi}{B_\theta}=\frac{q}{\varepsilon}$$

From the first square bracket we take separately the second part. We obtain

$$\begin{aligned}&B_\theta\left[-\frac{v_\varphi^2}{R}(-\sin\theta)\right]\\ =\;&\frac{\varepsilon}{q}\frac{B_0}{h}\left(-\frac{1}{R}\right)(-\sin\theta)\left[\frac{1}{B_\theta}\left(-\frac{\partial\phi}{\partial r}\right)+\frac{B_\varphi}{B_\theta}v_\theta\right]^2\\ =\;&\frac{\varepsilon}{q}\frac{B_0}{h}\left(-\frac{1}{R}\right)(-\sin\theta)\left[\frac{1}{B_\theta^2}\left(\frac{\partial\phi}{\partial r}\right)^2+\frac{q^2}{\varepsilon^2}v_\theta^2-2\frac{1}{B_\theta}\left(\frac{\partial\phi}{\partial r}\right)\frac{q}{\varepsilon}v_\theta\right]\end{aligned}$$

The *first* term now becomes

$$
\begin{aligned}
& B_\theta h \left[v_\theta \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} - \frac{v_\varphi^2}{Rh} \left(-\sin\theta \right) \right] \\
= \quad & B_\theta h \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\theta^2}{2} \right) \\
& + \frac{\varepsilon}{q} \frac{B_0}{h} \left(-\frac{1}{R} \right) \left(-\sin\theta \right) \frac{1}{B_\theta^2} \left(\frac{\partial \phi}{\partial r} \right)^2 \\
& + \frac{\varepsilon}{q} \frac{B_0}{h} \left(-\frac{1}{R} \right) \left(-\sin\theta \right) \frac{q^2}{\varepsilon^2} v_\theta^2 \\
& + \frac{\varepsilon}{q} \frac{B_0}{h} \left(-\frac{1}{R} \right) \left(-\sin\theta \right) \left[-2 \frac{1}{B_\theta} \left(\frac{\partial \phi}{\partial r} \right) \frac{q}{\varepsilon} v_\theta \right]
\end{aligned}
$$

Few re-formulations for this *first* term,

$$
\begin{aligned}
& B_\theta h \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\theta^2}{2} \right) + \frac{qh}{B_0 \varepsilon} \left(-\frac{1}{R} \right) \frac{1}{\varepsilon} \frac{\partial h}{\partial \theta} \left(\frac{\partial \phi}{\partial r} \right)^2 \\
& + \frac{B_0}{h} \left(-\frac{1}{R} \right) \frac{1}{\varepsilon} \frac{\partial h}{\partial \theta} \frac{q}{\varepsilon} v_\theta^2 + \left(-\frac{1}{R} \right) \frac{1}{\varepsilon} \frac{\partial h}{\partial \theta} \frac{q}{\varepsilon} v_\theta \left[-2 \left(\frac{\partial \phi}{\partial r} \right) \right]
\end{aligned}
$$

and taking into account that

$$
\left(-\frac{1}{R} \right) \frac{1}{\varepsilon} = -\frac{1}{r}
$$

$$
\begin{aligned}
\text{first term} \quad = \quad & B_\theta h \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\theta^2}{2} \right) - \frac{h}{B_0} \frac{q}{\varepsilon} \frac{1}{r} \frac{\partial h}{\partial \theta} \left(\frac{\partial \phi}{\partial r} \right)^2 \\
& - \frac{B_0}{h} \frac{q}{\varepsilon} \frac{1}{r} \frac{\partial h}{\partial \theta} v_\theta^2 + 2 \frac{q}{\varepsilon} v_\theta \frac{1}{r} \frac{\partial h}{\partial \theta} \left(\frac{\partial \phi}{\partial r} \right)
\end{aligned}
$$

The next (*second*) term where we will substitute the expression of v_φ is

$$
\begin{aligned}
& B_\varphi h \left[v_\theta \frac{1}{r} \frac{\partial v_\varphi}{\partial \theta} + \frac{v_\theta v_\varphi}{Rh} \left(-\sin\theta \right) \right] \\
= \quad & \frac{B_0}{h} h v_\theta \frac{1}{r} \frac{\partial}{\partial \theta} \left[\frac{1}{B_\theta} \left(-\frac{\partial \phi}{\partial r} \right) + \frac{q}{\varepsilon} v_\theta \right] \\
& + \frac{B_0}{h} h \frac{v_\theta}{Rh} \left(-\sin\theta \right) \left[\frac{1}{B_\theta} \left(-\frac{\partial \phi}{\partial r} \right) + \frac{q}{\varepsilon} v_\theta \right]
\end{aligned}
$$

which we write

$$
\begin{aligned}
&\frac{B_0}{h}hv_\theta\frac{1}{r}\left(-\frac{\partial\phi}{\partial r}\right)\frac{\partial}{\partial\theta}\left(\frac{1}{\frac{\varepsilon}{q}\frac{B_0}{h}}\right)+\frac{B_0}{h}hv_\theta\frac{1}{r}\frac{1}{B_\theta}\left(-\frac{\partial^2\phi}{\partial\theta\partial r}\right)+\frac{B_0}{h}hv_\theta\frac{q}{\varepsilon}\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}\\
&+\frac{B_0}{h}h\frac{v_\theta}{Rh}\left(-\sin\theta\right)\frac{1}{B_\theta}\left(-\frac{\partial\phi}{\partial r}\right)\\
&+\frac{B_0}{h}h\frac{v_\theta}{Rh}\left(-\sin\theta\right)\frac{q}{\varepsilon}v_\theta
\end{aligned}
$$

The first contribution to this *second* term is

$$
\begin{aligned}
\frac{B_0}{h}hv_\theta\frac{1}{r}\left(-\frac{\partial\phi}{\partial r}\right)\frac{\partial}{\partial\theta}\left(\frac{1}{\frac{\varepsilon}{q}\frac{B_0}{h}}\right) &= \frac{B_0}{h}hv_\theta\frac{1}{r}\left(-\frac{\partial\phi}{\partial r}\right)\frac{q}{\varepsilon}\frac{1}{B_0}\frac{\partial h}{\partial\theta}\\
&= v_\theta\frac{q}{\varepsilon}\frac{1}{r}\frac{\partial h}{\partial\theta}\left(-\frac{\partial\phi}{\partial r}\right)
\end{aligned}
$$

The second contribution to this *second* term is

$$
\begin{aligned}
\frac{B_0}{h}hv_\theta\frac{1}{r}\frac{1}{B_\theta}\left(-\frac{\partial^2\phi}{\partial\theta\partial r}\right) &= B_0v_\theta\frac{qh}{B_0\varepsilon}\left(-\frac{1}{r}\frac{\partial^2\phi}{\partial\theta\partial r}\right)\\
&= v_\theta\frac{q}{\varepsilon}h\left(-\frac{1}{r}\frac{\partial^2\phi}{\partial\theta\partial r}\right)
\end{aligned}
$$

The third contribution to this *second* term is

$$
\frac{B_0}{h}hv_\theta\frac{q}{\varepsilon}\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}=B_0\frac{q}{\varepsilon}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\theta^2}{2}\right)
$$

The fourth contribution to this *second* term is

$$
\begin{aligned}
\frac{B_0}{h}h\frac{v_\theta}{Rh}\left(-\sin\theta\right)\frac{1}{B_\theta}\left(-\frac{\partial\phi}{\partial r}\right) &= B_0\frac{v_\theta}{Rh}\frac{1}{\varepsilon}\frac{\partial h}{\partial\theta}\frac{qh}{\varepsilon B_0}\left(-\frac{\partial\phi}{\partial r}\right)\\
&= \frac{v_\theta}{R\varepsilon}\frac{q}{\varepsilon}\frac{\partial h}{\partial\theta}\left(-\frac{\partial\phi}{\partial r}\right)\\
&= v_\theta\frac{q}{\varepsilon}\frac{1}{r}\frac{\partial h}{\partial\theta}\left(-\frac{\partial\phi}{\partial r}\right)
\end{aligned}
$$

The fifth contribution to this *second* term is

$$
\begin{aligned}
\frac{B_0}{h}h\frac{v_\theta}{Rh}\left(-\sin\theta\right)\frac{q}{\varepsilon}v_\theta &= B_0v_\theta^2\frac{q}{\varepsilon}\frac{1}{Rh}\frac{1}{\varepsilon}\frac{\partial h}{\partial\theta}\\
&= B_0v_\theta^2\frac{q}{\varepsilon}\frac{1}{h}\frac{1}{r}\frac{\partial h}{\partial\theta}
\end{aligned}
$$

Then the full *second* term is

$$
\begin{aligned}
& v_\theta \frac{q}{\varepsilon}\frac{1}{r}\frac{\partial h}{\partial \theta}\left(-\frac{\partial \phi}{\partial r}\right) \\
& + B_0 \frac{q}{\varepsilon}\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\theta^2}{2}\right) \\
& + v_\theta \frac{q}{\varepsilon} h \left(-\frac{1}{r}\frac{\partial^2 \phi}{\partial \theta \partial r}\right) \\
& + v_\theta \frac{q}{\varepsilon}\frac{1}{r}\frac{\partial h}{\partial \theta}\left(-\frac{\partial \phi}{\partial r}\right) \\
& + B_0 v_\theta^2 \frac{q}{\varepsilon}\frac{1}{h}\frac{1}{r}\frac{\partial h}{\partial \theta}
\end{aligned}
$$

the first and the fourth contributions are added together and the full *second* term is

$$
2 v_\theta \frac{q}{\varepsilon}\frac{1}{r}\frac{\partial h}{\partial \theta}\left(-\frac{\partial \phi}{\partial r}\right) + B_0 \frac{q}{\varepsilon}\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\theta^2}{2}\right) + v_\theta \frac{q}{\varepsilon} h \left(-\frac{1}{r}\frac{\partial^2 \phi}{\partial \theta \partial r}\right) + B_0 v_\theta^2 \frac{q}{\varepsilon}\frac{1}{h}\frac{1}{r}\frac{\partial h}{\partial \theta}
$$

Now we put together the *first* term and the *scond* term

$$
\begin{aligned}
& B_\theta h \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\theta^2}{2}\right) - \frac{h}{B_0}\frac{q}{\varepsilon}\frac{1}{r}\frac{\partial h}{\partial \theta}\left(\frac{\partial \phi}{\partial r}\right)^2 \\
& \underline{-\frac{B_0}{h}\frac{q}{\varepsilon}\frac{1}{r}\frac{\partial h}{\partial \theta} v_\theta^2} + \underline{\underline{2\frac{q}{\varepsilon} v_\theta \frac{1}{r}\frac{\partial h}{\partial \theta}\left(\frac{\partial \phi}{\partial r}\right)}} \\
& \underline{\underline{+2 v_\theta \frac{q}{\varepsilon}\frac{1}{r}\frac{\partial h}{\partial \theta}\left(-\frac{\partial \phi}{\partial r}\right)}} + B_0 \frac{q}{\varepsilon}\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\theta^2}{2}\right) + v_\theta \frac{q}{\varepsilon} h \left(-\frac{1}{r}\frac{\partial^2 \phi}{\partial \theta \partial r}\right) + \underline{B_0 v_\theta^2 \frac{q}{\varepsilon}\frac{1}{h}\frac{1}{r}\frac{\partial h}{\partial \theta}}
\end{aligned}
$$

We note that the underlined terms cancel. Then the left hand side of the equation of momentum conservation is

$$
B_\theta h \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\theta^2}{2}\right) - \frac{h}{B_0}\frac{q}{\varepsilon}\frac{1}{r}\frac{\partial h}{\partial \theta}\left(\frac{\partial \phi}{\partial r}\right)^2 + v_\theta \frac{q}{\varepsilon} h \left(-\frac{1}{r}\frac{\partial^2 \phi}{\partial \theta \partial r}\right)
$$

A last modification, we replace

$$
B_\theta h = \frac{\varepsilon}{q} B_0
$$

The right hand side of the equation of momentum conservation is

$$
-\frac{c_s^2}{\rho} h B_\theta \frac{\partial \rho}{r \partial \theta} = -c_s^2 \frac{\varepsilon}{q} B_0 \frac{\partial}{r \partial \theta} \ln \rho
$$

These two terms are equal

$$\begin{aligned}
& \frac{\varepsilon}{q} B_0 \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\theta^2}{2} \right) + B_0 \frac{q}{\varepsilon} \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\theta^2}{2} \right) \\
& - \frac{h}{B_0} \frac{q}{\varepsilon} \frac{1}{r} \frac{\partial h}{\partial \theta} \left(\frac{\partial \phi}{\partial r} \right)^2 + v_\theta \frac{q}{\varepsilon} h \left(- \frac{1}{r} \frac{\partial^2 \phi}{\partial \theta \partial r} \right) \\
= \quad & -c_s^2 \frac{\varepsilon}{q} B_0 \frac{\partial}{r \partial \theta} \ln \rho
\end{aligned}$$

We multiply everything by

$$\frac{\varepsilon}{q}$$

and divide everything by B_0.

$$\begin{aligned}
& \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\theta^2}{2} \right) \left(1 + \frac{\varepsilon^2}{q^2} \right) \\
& - \frac{1}{B_0^2} \left[\frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{h^2}{2} \right) \right] \left(\frac{\partial \phi}{\partial r} \right)^2 - \frac{v_\theta}{B_0 h} h^2 \left(\frac{1}{r} \frac{\partial^2 \phi}{\partial \theta \partial r} \right) \\
= \quad & -c_s^2 \left(\frac{\varepsilon}{q} \right)^2 \frac{\partial}{r \partial \theta} \ln \rho
\end{aligned}$$

Note. We need to substitute

$$v_\theta \to \frac{1}{B_0 / h} \left(\frac{\partial \phi}{\partial r} \right) = \frac{1}{B_T} \left(\frac{\partial \phi}{\partial r} \right)$$

such that the last term on the left becoms

$$- \frac{v_\theta}{B_0 h} h^2 \left(\frac{1}{r} \frac{\partial^2 \phi}{\partial \theta \partial r} \right) \to - \frac{1}{B_0^2} \left(\frac{\partial \phi}{\partial r} \right) h^2 \left(\frac{1}{r} \frac{\partial^2 \phi}{\partial \theta \partial r} \right)$$

and then we get

$$\begin{aligned}
& \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\theta^2}{2} \right) \left(1 + \frac{\varepsilon^2}{q^2} \right) - \frac{1}{B_0^2} \left[\frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{h^2}{2} \right) \right] \left(\frac{\partial \phi}{\partial r} \right)^2 - \frac{1}{B_0^2} \frac{h^2}{2} \frac{1}{r} \frac{\partial}{\partial \theta} \left[\left(\frac{\partial \phi}{\partial r} \right)^2 \right] \\
= \quad & -c_s^2 \left(\frac{\varepsilon}{q} \right)^2 \frac{\partial}{r \partial \theta} \ln \rho
\end{aligned}$$

or

$$\frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\theta^2}{2} \right) \left(1 + \frac{\varepsilon^2}{q^2} \right) - \frac{1}{r} \frac{\partial}{\partial \theta} \left[\frac{1}{B_0^2} \frac{h^2}{2} \left(\frac{\partial \phi}{\partial r} \right)^2 \right] + \frac{1}{r} \frac{\partial}{\partial \theta} \left[c_s^2 \left(\frac{\varepsilon}{q} \right)^2 \ln \rho \right] = 0$$

$$\frac{1}{r} \frac{\partial}{\partial \theta} \left[\left(1 + \frac{\varepsilon^2}{q^2} \right) \frac{v_\theta^2}{2} + c_s^2 \left(\frac{\varepsilon}{q} \right)^2 \ln \rho - \frac{1}{B_0^2} \frac{h^2}{2} \left(\frac{\partial \phi}{\partial r} \right)^2 \right] = 0$$

The second order term ε^2/q^2 from the paranthesis multiplying $v_\theta^2/2$ will be neglected, relative to 1.

A Bernoulli law

$$\frac{\partial}{\partial\theta}\left[\frac{v_\theta^2}{2}+\left(\frac{\varepsilon}{q}\right)^2 c_s^2\ln\rho-\left(\frac{\partial\phi}{\partial r}\right)^2\frac{h^2}{2B_0^2}\right]=0$$

Normalization

$$\begin{aligned}
v_\theta &\rightarrow u=\frac{q}{\varepsilon}\frac{v_\theta}{c_s}\\
v_z &\rightarrow v=\frac{v_z}{c_s}\\
E &= \frac{q}{\varepsilon}\frac{1}{c_s}\frac{h}{B_0}\frac{\partial\phi}{\partial r}
\end{aligned}$$

NOTE that the first normalization introduces the poloidal projection of the sound speed

$$c_s\frac{\varepsilon}{q}=c_s\frac{B_\theta}{B_T}=c_s^\theta$$

END.

NOTE that the last normalization introduces a new poloidal velocity

$$\begin{aligned}
E &\equiv \text{normalized poloidal velocity}\\
&= \frac{\frac{1}{B_0/h}\frac{\partial\phi}{\partial r}}{c_s^\theta}=\frac{v_E^\theta}{c_s^\theta}
\end{aligned}$$

END.

Defining

$$\overline{E}\equiv\frac{q}{\varepsilon}\frac{1}{c_s}\frac{1}{B_0}\frac{\partial\phi}{\partial r}$$

The *zeroth-order* equations are

$$\begin{aligned}
h\rho u &= L(r)\ \ \text{(arbitrary function of } r\text{)}\\
E &= \overline{E}h=\overline{E}(1+\varepsilon\cos\theta)\\
\frac{u^2}{2}-\frac{E^2}{2}+\ln\rho &= K(r)\ \ \text{(arbitrary function of } r\text{)}
\end{aligned}$$

the last equation means that $K(r)$ does not depend on the poloidal variable θ.

$$\frac{\partial}{\partial\theta}K(r)=0$$

Also **note** that the first equation comes from the *continuity*.

$$\boldsymbol{\nabla}\cdot(\rho\mathbf{v})=0$$

$$\frac{\partial}{\partial\theta}\left(h\rho v_\theta\right)=0$$

after normalization of v_θ.

The system is reduced by extracting ρ from the first equation and replacing it in the "Bernoulli "-like equation

$$\rho=\frac{L}{hu}$$

$$\frac{u^2}{2}-\frac{E^2}{2}+\ln\left(\frac{L}{hu}\right)=K\left(r\right)$$

$$\begin{aligned}
\frac{u^2}{2}-\frac{E^2}{2}-\ln\left(hu\right) &= K-\ln L \\
\frac{u^2}{2}-\frac{E^2}{2}-\ln\left(u\right)-\ln\left(1+\varepsilon\cos\theta\right) &= K-\ln L \\
\frac{u^2}{2}-\frac{\left(\overline{E}h\right)^2}{2}-\ln\left(u\right)-\varepsilon\cos\theta &\approx K-\ln L
\end{aligned}$$

It is defined the quantity

$$f\left(u\right)\equiv\frac{u^2}{2}-\ln\left|u\right|$$

and averaged over the surface

$$\overline{f}=\frac{1}{2\pi}\int_0^{2\pi}d\theta f$$

we have

$$f\left(u\right)=\overline{f}\left(u\right)+\left(1+\overline{E}^2\right)\varepsilon\cos\theta$$

15 Notes

The paper of **Stringer 1969** calculates the radial flux of particles taking into account the nonuniformity of the particle density $n_1\left(\theta\right)$ and of electric potential $\phi_1\left(\theta\right)$ on the magnetic surface. These result from the neoclassical drifts and the equations of continuity, the poloidal projected equation of momentum, in the presence of an equilibrium radial electric field represented by a potential $\phi_0\left(r\right)$. In the regime of low collisions instead of collisional resistivity that permits to use the Ohm law for the parallel current projected on poloidal direction, it is invoked the *kinetic* process of Landau damping.

The result is

$$nv_{Di} = -\frac{\sqrt{\pi}}{8}\varepsilon\frac{\rho_i^2 v_{th,i}}{r}\frac{1}{q}\left(1+\frac{v_0}{U_{in}}\right)\exp\left(-z_i^2\right)\left[1+\frac{S\left(S+\tau\right)}{F^2+L^2}\right]\frac{dn_0}{dr}$$

where

$$S \equiv 1+\tau+2z_i^2\left(1+\frac{U_{en}}{v_0}\right)$$

$$z_j \equiv -\frac{v_0}{v_{th,j}}\frac{q}{\varepsilon}$$

This is

$$\frac{1}{v_{th,j}}\frac{q}{\varepsilon} = \frac{1}{v_{th,j}}\frac{B_T}{B_\theta}$$

$$v_{th,j}\frac{B_\theta}{B_T} = v_{th,j}^\theta$$

the projection of the thermal (assumed parallel) velocity on the poloidal direction. And

$$v_0\widehat{\mathbf{e}}_\theta = \frac{-\boldsymbol{\nabla}\phi_0\times\widehat{\mathbf{n}}}{B}$$

$$z_j = -\frac{v_0}{v_{th,j}^\theta}$$

Later it is found that

$$\frac{v_0}{U_{ni}} = -1+\frac{1+\tau}{1+2z_i^2+2z_i^4}\left(\tau\frac{m_e}{m_i}\right)^{1/2}\exp\left(-z_i^2\right)$$

Then the diamagnetic and the electric rotations are almost equal in magnitude and opposite.

When

$$1+\frac{v_{dia}}{v_E}\approx 0$$

the neoclassical diffusion vanishes.