1 Solutions of the drift-kinetic equation. Introduction

Raw material for the third Meeting of the Work Sessions of Plasma Theory, Florin Spineanu and Madalina Vlad.

the first approximation to the distribution function is the Maxwellian. It is determined by the density of particles and the local temperature of the plasma.

The toroidality imposes the drift of charged particles and this inevitably means a change relative to the Maxwellian. This change is measured in neoclassical theory by two small parameters

$$\delta = \frac{\rho_\theta}{a}$$

and

$$\Delta = \frac{\omega_{bounce}}{\nu_{ei}}$$

In these definitions ρ_θ is the poloidal Larmor radius and a is the small radius of the torus. ω_{bounce} is the inverse of the time of bounce of a trapped particle in its motion on the banana orbit and ν_{ei} is the collision frequency. We then expect the correction which the neoclassical theory will add to the basic Maxwellian to be of the order δ and the role of trapped particles to become important when the collisions are not so frequent such as to prevent a banana orbit to be travelled. Of course, when the collisionality is high, even if the banana are less visible since they are not completed by the particle, the neoclassical physics is still manifested through the Pfirsch Schluter flows.

Basic fact

A good basic text is **collisional diffusion non-axisymmetric Frieman**

The basic content of the first order drift-kinetic equation is

$$v_\parallel \nabla_\parallel f^{(1)} + \mathbf{v}_D \cdot \boldsymbol{\nabla} f^{(0)} = 0$$

The short expression for this would be

parallel advection of the perturbation is balanced by the radial drift-advection of the equilibrium

NOTE

Since the variation in the parallel direction of $f^{(1)}$ is actually the variation in θ-angle poloidal direction - projected on parallel, it is possible to integrate this equation if one represents the drift velocity as

$$\mathbf{v}_D = -v_\parallel \widehat{\mathbf{n}} \times \boldsymbol{\nabla} \left(\frac{v_\parallel}{\Omega_c} \right)$$

The expression that is useful is

$$\mathbf{v}_D \cdot \boldsymbol{\nabla} \psi = v_\parallel \nabla_\parallel \left(I \frac{v_\parallel}{\Omega_c} \right)$$

since it will multiply

$$\frac{\partial f^{(0)}}{\partial \psi}$$

which is the assumption

$$\begin{aligned} f_0 &\sim \text{function of surface coordinate, } \psi \\ \text{then } \boldsymbol{\nabla} f_0 &= \frac{\partial f_0}{\partial \psi} \boldsymbol{\nabla} \psi \end{aligned}$$

But here one actually MUST include collisions

$$f^{(-1)} \sim \frac{1}{\nu}$$

and *energetic* effects, *i.e.* acceleration of particles in the electric field of the wave,

$$v_{\parallel}\ e \left(-\nabla_{\parallel} \widetilde{\Phi}\right) \frac{\partial f^{(0)}}{\partial \epsilon}$$

An example is **Rosenbluth Hazeltine Hinton 1972**

$$f = f_0 + \widehat{f}\ f_0$$

$$\begin{aligned} & v_{\parallel} \frac{B_\theta}{B} \frac{\partial \widehat{f}}{r\partial \theta}\ f_0 \quad \left(\text{parallel} \rightarrow \text{poloidal convection } v_{\parallel}\nabla_{\parallel} \sim \frac{\partial}{r\partial\theta} \text{ of the perturbed } f\right) \\ & + \left. v_D \right|_r \frac{\partial f_0}{\partial r} \ \text{(drift convection of the equilibrium, only radial)} \\ & + v_{\parallel}\ eE_{\parallel} \frac{\partial f_0}{\partial \epsilon} \quad \left(\text{energetic effect of the neoclassic } \widetilde{\Phi} \text{ or of the wave}\right) \\ = \ & C(f) \qquad \text{(collision)} \end{aligned}$$

After substituting the Maxwellian distribution

$$f_M \left\{ \left. v_D \right|_r \ \left[A_1 + A_2 (\epsilon - e\Phi)\right] + v_{\parallel} \left(\frac{B_\theta}{B} \frac{\partial \widehat{f}}{r\partial\theta} - \frac{eE_{\parallel}}{T} \right) \right\} = C(f)$$

where the *forces* are, for species a

$$A_{1a} = \frac{d}{dr} \ln n_a - \frac{3}{2} \frac{d}{dr} \ln T_a + \frac{e_a}{T_a} \frac{d\Phi}{dr}$$

$$A_{2a} = \frac{1}{T} \frac{d}{dr} \ln T$$

$$a \equiv e, i$$

(see further:

This problem is only an extension (gradients, drifts) of a simpler old problem, the distribution function in the presence of a parallel electric field, balanced by collisions, the Spitzer problem.

Then the electric term is extracted by shifting the distribution function with the Spitzer function

$$-v_{\parallel}\ eE_{\parallel}\frac{1}{T_e}f_M = C\left(v_{\parallel}E_{\parallel}\ f^{Spitzer}\right)$$

and

$$f \to f - v_{\parallel}E_{\parallel}\ f^{Spitzer}$$

)

Note
From **Hirshman Sigmar Clarke** (ions plus impurities)
$E^{(A)}$ is the electric field induced by the transformer

$$v_{\parallel}\nabla_{\parallel}\left(f_{a1} + I\frac{v_{\parallel}}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial\psi}\right) = v_{\parallel}\frac{e_a E^{(A)}}{T_a}f_{a0} + \sum_b C_{ab}\left(f_{a1}, f_{b1}\right)$$

The periodicity cancels the LHS and leads to (more explanation in *bootstrap.tex*)

$$\left\langle\frac{B}{v_{\parallel}}\left(v_{\parallel}\frac{e_a E^{(A)}}{T_a}f_{a0} + \sum_b C_{ab}\left(f_{a1}, f_{b1}\right)\right)\right\rangle = 0$$

Note
In **Helander ECRH** the bootstrap current is found from the distribution function $f^{(1)}$ that verifies

$$\left\langle\frac{B}{v_{\parallel}}C\left(f\right)\right\rangle = 0$$

The neoclassical correction exists for electrons (it is small) for ions and for NBI fast ions.

In the treatment fluid, see **Terry Diamond 1985 renormalization**, we have

$$\frac{dv_{\parallel e}}{dt} = v_{th,e}^2\nabla_{\parallel}\left(\frac{e\varphi}{T_e} - \widehat{n}\right) - \nu_{ei}v_{\parallel,e}$$

where

$$n = n_0 + \widehat{n}$$

and after defining

$$H_e \equiv \widehat{n} - \frac{|e|\,\varphi}{T_e}$$

we obtain

$$\begin{aligned}
& \frac{\partial H_e}{\partial t} \\
& + \left[\frac{1}{B_0} \left(-\boldsymbol{\nabla}\varphi \right) \times \widehat{\mathbf{n}} \right] \cdot \boldsymbol{\nabla} H_e + \mathbf{v}_D \cdot \boldsymbol{\nabla} H_e \quad \text{electric plus neo-drift} \\
= \quad & -\frac{|e|}{T_e} \left(\frac{\partial \varphi}{\partial t} + v_{dia} \nabla_\theta \varphi \right) \quad \text{convection of the potential} \\
& + \frac{v_{th,e}^2}{\nu_{ei}} \nabla_{\|}^2 H_e \quad \text{balance of forces in parallel direction}
\end{aligned}$$

The balance of forces along the line (parallel) we have dynamic equality between the parallel gradient of pressure (assuming that T_e is constant) - tehrefore of the density H_e with the electric field of the wave and with the *friction* $\sim \nu_{ei}$. This fixes $v_{\|}$.

NOTE
In an area which is common for

- NBI fast ions
- bootstrap current
- impurity friction

it is located a problem of discerning the role of

1. diamagnetic flow, seen as *unbalanced fluxes*
   - Larmor gyration for poloidal flow AND
   - banana-trapped for toroidal flow
2. radial gradient of pressure, which is a force, combined with toroidal magnetic field respectively poloidal magnetic field $\mathbf{F} \times \mathbf{B}$.

It is seen that the second is sufficient to get a parallel flow (which is attributed to the gradient of pressure of all particles NOT ONLY trapped ones and the poloidal magnetic field)

Further, it is noted that this flow is different for different species.

Then there is collisional *friction*.

Hirshman Sigmar Clarke for impurities.

Hsu Catto Sigmar for NBI fast ions isotropic.

Helander 3999 for ion-impurity parallel friction.

The friction tends to equalize the flows and on a slow time scale this modifies the gradients.

In **Hirshman Sigmar Clarke**

$$\begin{aligned} f_{a1} &= -I\frac{v_{\|}}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial \psi} \\ &\quad +g_a\left(\epsilon,\mu,\psi\right) \end{aligned}$$

where

$$RB_T\ \frac{v_{\|}}{eB/m}\frac{1}{RB_\theta}\frac{\partial f_{a0}}{\partial r}=\frac{v_{\|}}{\Omega_{a\theta}}\frac{\partial f_{a0}}{\partial r}=\rho_{a\theta}\frac{\partial f_{a0}}{\partial r}$$

$$\begin{aligned} -I\frac{v_{\|}}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial \psi} &\equiv \text{neoclassical correction for the species } a \\ g_a &\equiv \text{collisional response of the species } a \end{aligned}$$

This is the equivalent of the *diamagnetic unbalance of flows* in the poloidal plane.

In that case, the fluxes are produced by the Larmor gyration of the particles.

In that case the final flow is poloidal and results from the combination of radial gradient and the *toroidal* magnetic field

$$\frac{1}{B_T}\times\frac{\partial f}{\partial r}$$

This would suggest that for the toroidal diamagnetic flows we have to do with fluxes of trapped particles (since they are similar to Larmor gyration). We would expect that only part of the density participates, the *trapped* particles. Actually the gradient of the equilibrium pressure is here combined with B_θ field. And the direction is toroidal, but it is again due to the

$$(\text{radial gradient of pressure})\times(\text{magnetic field } B_\theta\ )$$

The similarity with the poloidal diamagnetic flow and looking for parallel:

$$(\text{Larmor gyration})\rightarrow(\text{trapped bananas})$$

is NOT relevant.

NO particle is excluded from the calculation of the "toroidal diamagnetic" flow (*i.e.* the untrapped particles are NOT excluded).

Then the so-called "toroidal diamagnetic" flow is simply a flow in the toroidal direction.

The comments (**Hirshman Sigmar Clarke**) can now go further.

The different species have different parallel flows and the collisional friction attempts to suppress the relative motion, *i.e.* to reduce it to a single common velocity. This is only possible if *poloidal* flows are generated. The poloidal flows are necessary because by their projection on the parallel direction, they can compensate for the differences of the parallel flow velocities. Bringing all parallel velocities to one single value (as collisional friction intends to do) means to generate poloidal flows. Of course these are different for different species.

However, the *trapped* particles cannot move poloidally, so the poloidal flows cannot be created to the magnitude of their velocity as requested by the frictional suppression of the differences in the parallel flows. Then the frictional suppression of the differences between the parallel velocities will remain *incomplete*.

There will still be a relative motion between species, in the parallel direction.

Hsu Gormley Shaing Sigmar consider that this is the source of the *bootstrap* current.

The function in **Rewoldt Tang Frieman trapped electron mode**

The function for *trapped electrons*

$$f_j^{(0)} = f_{Me} + f_{Me}\ \widehat{f}_e$$

where

$$f_{Me} = \frac{n(r)}{[2\pi T_e/m_e]^{3/2}} \exp\left[-\frac{\epsilon}{T_e(r)}\right]$$

$$\widehat{f}_e = -\frac{v_\varphi}{\Omega_{\theta e}}\left[\frac{d}{dr}\ln n + \frac{d}{dr}\ln T_e\left(\frac{\epsilon}{T_e} - \frac{3}{2}\right)\right]$$

$$\begin{aligned}
\epsilon &= \frac{m_e v^2}{2} \quad \text{[no static potential } \phi_0(r)\text{]} \\
\lambda &= \frac{\mu B_0}{\epsilon} = \frac{v_\perp^2}{v^2} h \\
\mu &= \frac{m_e v_\perp^2}{2B} \\
v_\parallel &= \sigma\ v\sqrt{1 - \frac{\lambda}{h}}
\end{aligned}$$

$$\begin{aligned}
\text{circulating } 0 &\leq \lambda < 1 - \varepsilon_0 \\
\text{trapped } 1 - \varepsilon_0 &< \lambda \leq 1 + \varepsilon_0
\end{aligned}$$

For trapped

$$\theta_0 = \pm\arccos\left(\frac{\lambda - 1}{\varepsilon_0}\right)$$

The function for circulating particles

$$f_e^C = f_{Me}^C + f_{Me}^C\ \widehat{f}_e$$

where

$$
\begin{aligned}
f_{Me}^{C} &= \frac{n\left(r\right)}{\left[2\pi T_e/m_e\right]^{3/2}} \exp\left\{-\frac{\left(v_{\parallel}-u_0\right)^2-v_{\perp}^2}{2T_e/m_e}\right\} \\
&\approx \left(1+\frac{2u_0 v_{\parallel}}{v_{th,e}^2}\right) f_{Me} \\
\text{for } \frac{u_0}{v_{th,e}} &\ll 1
\end{aligned}
$$

The derivatives for *circulating* electrons

$$
\frac{\partial f_{Me}^{C}}{\partial \epsilon} = -\frac{1}{T_e}\left(1-\frac{u_0}{v_{\parallel}^{(0)}}\right) f_{Me}^{C}
$$

where

$$
\begin{aligned}
v_{\parallel}^{(0)} &= qR_0\omega_t \quad , \; t \equiv \text{transit of circulating} \\
\omega_t &= \frac{2\pi}{\tau_t} \\
\tau_t &= qR_0 \int_{-\pi}^{\pi} \frac{d\theta}{v_{\parallel}\left(\theta\right)}
\end{aligned}
$$

The variable $v_{\parallel}^{(0)}$ is the time average of the parallel velocity along the orbit of *circulating* electron.

And

$$
\begin{aligned}
\boldsymbol{\nabla} f_{Me}^{C} &= \widehat{\mathbf{e}}_r \frac{\partial f_{Me}^{C}}{\partial r} \\
&= \widehat{\mathbf{e}}_r \left[\frac{d}{dr}\ln n + \frac{d}{dr}\ln T_e \left(\frac{\epsilon - m_e u_0 v_{\parallel}^{(0)}}{T_e} - \frac{3}{2}\right)\right]
\end{aligned}
$$

for the conditions

$$
\begin{aligned}
\frac{u_0}{v_{th,e}} &\ll 1 \\
\frac{\partial u_0}{\partial r} &\approx 0
\end{aligned}
$$

The follwoing quantities are also used for trapped particles

$$
\omega_{*e} = \frac{m_0}{r_0} \frac{T_e/m_e}{\left|e\right| B_0/m_e} \frac{d}{dr}\ln n
$$

$$
\eta_e = \frac{\frac{d}{dr}\ln T_e}{\frac{d}{dr}\ln n} = \frac{L_n}{L_T}
$$

$$\omega_{*e}^{T}=\omega_{*e}\left[1+\eta_{e}\left(\frac{\epsilon-m_{e}u_{0}v_{\parallel}^{(0)}}{T_{e}}-\frac{3}{2}\right)\right]$$

The critical parameter λ

$$\lambda_{c}=1-\varepsilon_{0}$$

separates trapped from circulating particles.

$$\begin{aligned}\omega_{b} &= \text{freq. of bounce of trapped}\\ &= \frac{2\pi}{\tau_{b}}\end{aligned}$$

$$\begin{aligned}\omega_{t} &= \text{freq. of transit of circulating}\\ &= \frac{2\pi}{\tau_{t}}\end{aligned}$$

Change of variable "time"

$$\widehat{t}\,(\theta)=qR_{0}\int_{0}^{\theta}\frac{d\theta'}{v_{\parallel}\left(\theta'\right)}+\text{const}$$

We **note** that the time of transit is $\left(\frac{v_{\parallel}}{qR}\right)^{-1}$. This is variable with θ and depends on θ due to the *mirror* effect of the magnetic field $\delta B/B$ along the line. This new variable $\widehat{t}\,(\theta)$ just reflects the fact that the "time" is NOT uniform. **End.**

$$\overline{\omega}_{De}=\omega_{*e}\frac{L_{n}}{R_{0}}$$

$$\omega_{De}=\overline{\omega}_{De}\left(\frac{\epsilon}{T_{e}}\right)\left[\left(1+\frac{v_{\parallel}^{2}}{v^{2}}\right)\cos\theta+\frac{q'r}{q}\,\frac{2}{\varepsilon_{0}}\frac{v_{\parallel}\left(v_{\parallel}-v_{\parallel}^{(0)}\right)}{v^{2}}\right]$$

The time average of ω_{De} is

$$\begin{aligned}\omega_{De}^{(0)} &= \overline{\omega}_{De}\left(\frac{\epsilon}{T_{e}}\right)\,G\left(\kappa\right)\qquad\text{for trapped}\\ \kappa &= \sqrt{\frac{1}{2}-\frac{1-\lambda}{2\varepsilon_{0}}}\\ G\left(\kappa\right) &= \left(2\frac{\mathbf{E}\left(\kappa\right)}{\mathbf{K}\left(\kappa\right)}-1\right)+\frac{rq'}{q}\left[4\frac{\mathbf{E}\left(\kappa\right)}{\mathbf{K}\left(\kappa\right)}+4\left(\kappa^{2}-1\right)\right]\end{aligned}$$

with the elliptic functions

$$\begin{aligned}\mathbf{E}\left(\kappa\right) &= \int_{0}^{\pi/2}d\xi\sqrt{1-\kappa^{2}\sin^{2}\xi}\\ \mathbf{K}\left(\kappa\right) &= \int_{0}^{\pi/2}\frac{d\xi}{\sqrt{1-\kappa^{2}\sin^{2}\xi}}\end{aligned}$$

For circulating particles,

$$\omega_{De}^{(0)} \sim \text{negligible}$$

Volume element

$$\begin{aligned}
& \int d^3v \;\; (...) \\
= \;\; & \int_0^\infty 2\pi v_\perp dv_\perp \int_{-\infty}^{\infty} dv_\parallel \;\; (...) \\
= \;\; & \frac{\pi}{2}\left(\frac{2}{m_e}\right)^{3/2} \sum_{\sigma=\pm 1} \int_0^\infty d\epsilon \; \sqrt{\epsilon} \; \int_0^{h(\theta)} \frac{d\lambda}{h(\theta)\sqrt{1-\frac{\lambda}{h(\theta)}}} \;\; (...)
\end{aligned}$$

The distance to the resonant surface expressed in terms of $S(r)$ is

$$\widetilde{S} = \frac{S - S^{(0)}}{\Delta r_s} \approx \frac{r - r_0}{\Delta r_s} \approx \frac{v_\parallel - v_\parallel^{(0)}}{\Omega_\theta^0 \; \Delta r_s}$$

where

$$S^{(0)} = l\left[q\left(r^{(0)}\right) - q(r_0)\right]$$

where

$$\begin{aligned}
r^{(0)} \;\; & \equiv \;\; \oint \frac{dt}{\tau_{b,t}} r \\
& \equiv \;\; \text{the average position of the particle} \\
& \;\;\;\;\;\; \text{along the bounce/transit orbit}
\end{aligned}$$

Then $S^{(0)}$ is the effective spatial (radial) separation between the average position of the particle during the orbit and the reference (mode) surface, - but expressed in terms of the $q(r)$ function.

$$\begin{aligned}
\widetilde{S} \;\; & = \;\; \frac{\sigma \; v\sqrt{1-\frac{\lambda}{h(\theta)}} - qR_0\omega_t}{\Omega_\theta^0 \; \Delta r_s} \\
& = \;\; \sigma \; \sqrt{\frac{\epsilon}{T_i}} \; \frac{\rho_i}{\Delta r_s} \frac{q}{\varepsilon_0} \left(\sqrt{1 - \frac{\lambda}{h(\theta)}} - \frac{\sqrt{\varepsilon_0}}{L_t(\lambda)}\right)
\end{aligned}$$

where

$$L_t(\lambda) = \frac{\sqrt{\varepsilon_0}}{2\pi} \int_{-\pi}^{\pi} d\theta \frac{1}{\sqrt{1-\frac{\lambda}{h(\theta)}}}$$

NOTE

From **Frieman Rewoldt Tang Glasser balloonning mode 1980**
This will be useful
The volume in velocity space for the *circulating* electrons

$$\int_{circ} d^3v = \frac{\pi}{2}\left(\frac{2}{m_e}\right)^{3/2} \sum_{\sigma=\pm 1} \int_0^{\infty} d\epsilon\ \sqrt{\epsilon}\ \int_0^{1-\varepsilon_0} d\Lambda \frac{1}{h(\theta)} \frac{1}{\sqrt{1-\frac{\Lambda}{h(\theta)}}}$$

The volume in the velocity space of the *trapped* electrons

$$\int_{trapped} d^3v = \frac{\pi}{2}\left(\frac{2}{m_e}\right)^{3/2} \sum_{\sigma=\pm 1} \int_0^{\infty} d\epsilon\ \sqrt{\epsilon}\ \int_{1-\varepsilon_0}^{h(\theta)} d\Lambda \frac{1}{h(\theta)} \frac{1}{\sqrt{1-\frac{\Lambda}{h(\theta)}}}$$

The volume in velocity space for the *ions*

$$\int d^3v = 2\pi \int dv_\perp\ v_\perp \int_{-\infty}^{\infty} dv_\parallel$$

END

2 The drift kinetic equation according to Galeev 1971

It is **Galeev JETP 32 (1971) 752**.
The equation in the collisional regime

$$\frac{\Theta v_{th,j}\ \varepsilon^{3/2}}{r} \ll \nu_j$$

The bounce frequency is much smaller than the collision frequency.
Adopting a Batnagar Gross Krook (BGK) collision term

$$\begin{aligned} & \frac{\partial f_j}{\partial t} + \left(\Theta v_\parallel + v_E\right) \frac{\partial f_j}{r\partial\theta} \\ & - \frac{\mu B_0/m_j + v_\parallel^2}{R} \sin\theta \left(\Theta \frac{\partial}{\partial v_\parallel} + \frac{1}{\Omega_{cj}} \frac{\partial}{\partial r}\right) f_j \\ & + \frac{e_j E_{0\varphi}}{m_j} \frac{\partial f_j}{\partial v_\parallel} \\ = & -\nu_j \left(\delta f_j - \frac{\delta n_j}{n} f_{0j}\right) + \sum_k \nu_{jk} \frac{m_j}{T_{0j}} \mathbf{v}\cdot\mathbf{u}_k\ f_{0j} \end{aligned}$$

where

$$\delta f_j \equiv f_j - f_{0j}$$

deviation of the distribution function from a Maxwellian distribution with (n_j, T_{0j}).

$$\delta n_j = \int d^3v \; \delta f_j$$

$$\mathbf{u}_j = \int d^3v \; \mathbf{v} \; \delta f_j$$

NOTE

that in **Connor 1973** the same velocity is weighted in the integration by the collision frequency ν_{kj} of species k with species j.

END.

Collision frequencies

$$\nu_{ei} = \frac{m_i}{m_e} \nu_{ee}$$

$$\nu_{ee} = \frac{16\sqrt{\pi}}{3} \frac{e^4}{m_j^2} \Lambda \; \frac{n}{v_{th,j}^3}$$

The zeroth order.

This is a Maxwellian function with a parallel flow included u_{0j}.

It is not a shifted Maxwellian.

It is represented as a series of Laguerre 3/2 polynomials (Sonine polynomials)

$$\begin{aligned} f_j^{(0)} &= \frac{n}{\left(\sqrt{\pi} v_{th,j}\right)^3} \exp\left[-\frac{\left(v_\parallel - u_{0j}\right)^2}{2T_{0j}/m_j} - \frac{\mu B_0}{T_{0j}} \right] \\ &\quad \times \left\{ 1 - \frac{m_j}{T_{0j}} v_\parallel u_{0j} \sum_{l=1}^{\infty} a_l \; L_l^{(3/2)} \left(\frac{v^2}{2T_{0j}/m_j} \right) \right\} \end{aligned}$$

for

$$\begin{aligned} u_{0i} &= 0 \;\; \text{no flow for ions} \\ u_{0e} &= u_0 \; \text{electrons have a flow} \end{aligned}$$

Here there is yet NO neoclassical correction.

A similar form is adopted by **Hirshman 1977.**

The next order

$$\begin{aligned} f_j^{(1)} &= -\frac{\mu B_0/m_j + v_\parallel^2}{R} \left[\Theta \frac{\partial}{\partial v_\parallel} + \frac{1}{\Omega_{cj}} \frac{\partial}{\partial r} \right] \; f_j^{(0)} \\ &\quad \times \left\{ \mathbf{P} \frac{r\cos\theta}{\Theta v_\parallel + v_E} - \pi r \delta \left(\Theta v_\parallel + v_E \right) \sin\theta \right\} \end{aligned}$$

The term that calculates the function $f_j^{(1)}$ is

$$\left(\Theta v_\parallel + v_E \right) \frac{\partial f_j^{(1)}}{r\partial\theta}$$

and the inverse of the coefficient

$$\frac{1}{\Theta v_{\parallel}+v_{E}}=\mathbf{P}\frac{1}{\Theta v_{\parallel}+v_{E}}+i\pi\delta\left(\Theta v_{\parallel}+v_{E}\right)$$

and it is multiplied with the term

$$-\frac{\mu B_{0}/m_{j}+v_{\parallel}^{2}}{R}\sin\theta\left(\Theta\frac{\partial}{\partial v_{\parallel}}+\frac{1}{\Omega_{cj}}\frac{\partial}{\partial r}\right)f_{j}$$

Here the trigonometric function $\sin\theta$ is integrated as requested by the operator $\frac{\partial}{r\partial\theta}$ in LHS.

For the first part there is a simple integration

$$\begin{aligned}&\int^{\theta}d\theta'\ r\sin\theta'\ \mathbf{P}\frac{1}{\Theta v_{\parallel}+v_{E}}\\ =\ &\mathbf{P}\frac{1}{\Theta v_{\parallel}+v_{E}}r\cos\theta\end{aligned}$$

and for the second part

$$\begin{aligned}&\int^{\theta}d\theta'\ r\sin\theta'\ i\pi\delta\left(\Theta v_{\parallel}+v_{E}\right)\\ \rightarrow\ &-r\sin\theta\ \pi\delta\left(\Theta v_{\parallel}+v_{E}\right)\end{aligned}$$

Solution of the drift kinetic equation at low collisionality
The trapped particles are important
The distribution function is an expression of the invariants

- the energy
- the magnetic momentum (transverse adiabatic invariant)
- the generalized momentm (longitudinal invariant)

$$J=mv_{\parallel}\left(1+\varepsilon\cos\theta\right)-e\int_{0}^{r}dr'\ B_{\theta}\left(r'\right)+eA_{0z}\left(t\right)$$

- [for circulating particles: σ the sign of $v_{\parallel}$]

Without collisions and without *induced electric field* $A_{0z}(t)$,
for trapped particles

$$\begin{aligned}f_{j,t}^{(0)}&=\frac{n_{0}}{\left(\sqrt{\pi}v_{th,j}\right)^{3}}\exp\left[-\frac{\mu B_{0}/m_{j}}{T_{j}/m_{j}}-2\varepsilon\ \kappa^{2}\ \frac{\mu B_{0}/m_{j}}{T_{j}/m_{j}}-\frac{u_{j}^{2}}{v_{th,j}^{2}}\right]\\ \text{for }\kappa^{2}&<1\quad\text{(trapped)}\end{aligned}$$

for circulating particles

$$
\begin{aligned}
f_{j,c}^{(0)} &= \frac{n_0}{\left(\sqrt{\pi} v_{th,j}\right)^3} \exp\left[-\frac{\mu B_0/m_j}{T_j/m_j} - 2\varepsilon\ \kappa^2\ \frac{\mu B_0/m_j}{T_j/m_j} - \frac{u_j^2}{v_{th,j}^2} \right. \\
&\quad \left. +0.5\pi\ \sigma\ \sqrt{2\frac{\mu B_0/m_j}{T_j/m_j}\varepsilon}\ \frac{u_j}{v_{th,j}} \int_1^{\kappa^2} \frac{dt}{\sqrt{t} E\left(\frac{1}{\sqrt{t}}\right)} \right] \\
&\text{for } \kappa^2 \quad > \quad 1 \quad \text{(circulating)}
\end{aligned}
$$

with

$$
u_j = u_0 \delta_{je} + \frac{v_E}{\Theta}
$$

$$
v_{\parallel} = -\frac{v_E}{\Theta} + \sigma v_{th,j} \sqrt{2\frac{\mu B_0/m_j}{T_j/m_j}\varepsilon \left(\kappa^2 - \sin^2 \frac{\theta}{2}\right)}
$$

First it is examined the problem

- without collisions, and
- without electric field

The solutions for the distribution function is in this case of lowest order, if we will consider later that the collisions are small, $\nu_{ei}/\omega_{t,b} \ll 1$.

For trapped particles

$$
\begin{aligned}
f_{tj}^{(0)} &= \frac{n_0}{\left[\sqrt{\pi} v_{th,j}\right]^3} \exp\left[-x_j - 2\varepsilon\ \kappa^2 x_j - \frac{u_j^2}{v_{th,j}^2} \right] \\
\text{for} \quad \kappa^2 &< 1 \qquad \text{trapped}
\end{aligned}
$$

For circulating particles

$$
\begin{aligned}
f_{cj}^{(0)} &= \frac{n_0}{\left[\sqrt{\pi} v_{th,j}\right]^3} \exp\left[-x_j - 2\varepsilon\ \kappa^2 x_j - \frac{u_j^2}{v_{th,j}^2} \right. \\
&\qquad \left. + \frac{1}{2}\pi\sigma\sqrt{2\varepsilon x_j}\ \frac{u_j}{v_{th,j}} \int_1^{\kappa^2} \frac{dt}{\sqrt{t}\mathbf{E}\left(\frac{1}{\sqrt{t}}\right)} \right] \\
\kappa^2 &> 1 \qquad \text{circulating}
\end{aligned}
$$

where

$$
x_j \equiv \frac{\mu B_0/m_j}{T_j/m_j} = \frac{v_\perp^2}{v_{th,j}^2} h
$$

$$
u_j = u_0 \delta_{je} + \frac{v_E}{\Theta}
$$

$$v_\parallel = -\frac{v_E}{\Theta} + \sigma v_{th,j} \sqrt{2\ \varepsilon x_j \left(\kappa^2 - \sin^2 \frac{\theta}{2} \right)}$$

In the next step one includes the

- collisions

But the problem is solved in velocity space *far from the boundary trapped/circulating.*

The distribution function

$$\begin{aligned} f_j &= f_j^{(0)}(\mu, \epsilon; J, \sigma) \left[1 - W_j^{(0)}\ u_0 \right] \\ &\quad + f_j^{(1)}(\mu, \epsilon; J, \sigma; \theta) \end{aligned}$$

The correction

$$f_j^{(0)} W_j^{(0)} \quad \text{comes from collisions}$$

The correction factor $W_j^{(0)}$ is expressed in terms of Laguerre polynomials with coefficients that become the unknown quantititis to be determined by the drift kinetic equation.

First it is studied the case with uniform magnetic field, *i.e.* without *trapped* particles

$$W_e^{(0)}\left(v^2, v_\parallel\right) = \frac{m_e}{T_{0e}} \sum_{n=1}^{\infty} a_n\ L_n^{(3/2)} \left(\frac{v^2}{2T_e/m_e} \right)\ v_\parallel$$

and for ions

$$W_i^{(0)} \equiv 0$$

The drift kinetic equation becomes a system for a_n

$$\sum_{n=1}^{\infty} \left(\alpha_{nm} + \alpha'_{nm} \right) a_n = \alpha'_{0m}$$

where $\alpha' s$ are integrals over the collision operators in which it is inserted the distribution function expressed as a series of Sonine polynomials.

$$\alpha_{nm} \quad , \quad \alpha'_{nm} \quad \sim \frac{1}{\nu_{ee}}$$

$$\begin{aligned} \alpha_{nm} &= -\frac{1}{3} \frac{1}{\nu_{ee}} \frac{m_e}{T_e} \int d^3v\ L^{(3/2)} \\ &\quad \times \left\{ C \left[f_{Me}^{shifted} L_m^{(3/2)} v_\parallel \ ,\ f_{Me}^{shifted} \right] \right. \\ &\qquad \left. + C \left[f_{Me}^{shifted} \ ,\ f_{Me}^{shifted} L_m^{(3/2)} v_\parallel \right] \right\} \end{aligned}$$

$$\alpha'_{nm} = -\frac{1}{3}\frac{1}{\nu_{ee}}\frac{m_e}{T_e}\int d^3v \;\; v_\parallel L_n^{(3/2)} \;\; C\left[f_{Me}^{shifted} L_m^{(3/2)} v_\parallel \;\; , \;\; f_{Mi}^{shifted}\right]$$

The upperscript $shifted$ means that the maxwellian is shifted with the velocity u_{0e} of the flow of the electrons.

In the next step, the problem is solved in the region of the boundary between trapped and circulating particles.

"*The friction of the transiting particles, which transport current, against the particles from the transition layer causes an additional deceleration of the current*"

"to write the Ohm's law it is necessary to take into account (in the equation for a_q) the additional term describing the action of the particles of the transition layer on the transiting particle"

The coefficients α_{nm} are now expanded in series of

$$\varepsilon^{1/2} \quad = \text{ fraction of trapped particles}$$

In this case the correction (multiplying the Maxwellian in the part that does not come from collisions) is subject to the constraint that results from average over surface

$$W_e^{untrap} = \frac{1}{v_{th,e}}\sum_{n=1}^{\infty} a_{n,e}^{(0)} L_n^{(3/2)}(x_e) \;\; \sigma\sqrt{2x_e\varepsilon}\left(\frac{\pi}{2}\right)\int_1^{\kappa^2}\frac{dt}{\sqrt{t}\mathbf{E}\left(\frac{1}{\sqrt{t}}\right)}$$

$$W_e^{trapped} = 0$$

The unknown coefficients

$$a_{n,e}$$

are calculated as series of powers of the small number of particles $\sqrt{\varepsilon}$ in the *transition region* between trapped and circulating.

$$a_{n,e}^{(0)} + a_{n,e}^{(1)} + ...$$

where the zero order coincides with the previous result, where the toroidality (inhomogeneity of $\mathbf{B}$) was neglected.

$$\begin{aligned}
\nu_{ee}\; I \;\; & \left\{\alpha_{p0} - \sum_{q=1}^{\infty}\left(\alpha_{pq} + \alpha'_{pq}\right)\left(a_{q,e}^{(0)} + a_{q,e}^{(1)}\right)\right. \\
& \left. \qquad - \sqrt{\varepsilon}\left[\alpha_{p0}^{ue,te} - \sum_{q=1}^{\infty}\alpha_{pq}^{ue,te}\; a_{qe}^{(0)}\right]\right\} \\
= \;\; & \frac{n_* e^2}{m_e} E_{0z}\delta_{p0}
\end{aligned}$$

where

$$
\begin{aligned}
I &= -n_* e u_{0e} \\
&\quad \text{current carried by circulating electrons}
\end{aligned}
$$

$$
\begin{aligned}
n_* &= n_0 \left(1 - 2\sqrt{\frac{2\varepsilon}{\pi}}\right) \\
&= \text{density of circulating electrons} \\
&\quad \text{averaged over surface}
\end{aligned}
$$

$$\alpha_{pq} = \text{calculated}$$

The element of volume in the coordinates

$$\left(\mu, \kappa^2\right)$$

$$
\begin{aligned}
\langle n v_r \rangle_j &= \frac{-(2\varepsilon)^{3/2}}{\Omega_{cj}} \int_0^{2\pi} d\theta \int \left(\frac{\mu B_0}{m_j}\right)^{3/2} d\left(\frac{\mu B_0}{m_j}\right) \\
&\quad \times \int_{\sin^2(\theta/2)}^{\infty} \frac{d\kappa^2}{2\sqrt{2}\sqrt{\kappa^2 - \sin^2(\theta/2)}} \frac{\sin\theta}{r} \left[f_j^{(0)} + f_j^{(1)}\right]
\end{aligned}
$$

This is to calculate the radial flux

$$\langle n v_r \rangle = -\int_0^{2\pi} \frac{d\theta}{2\pi} \int d^3 v \quad \frac{\mu B_0/m_j + v_\parallel^2}{\Omega_{cj} R} \sin\theta \quad f_j^{(1)}$$

3 Solutions circulating - trapped (Galeev Sagdeev 1968)

For the trajectories of the particles see *particle equations of motion*.

The equation

$$\frac{\partial f_j}{\partial t} + [H, f_j] = St\,(f_j)$$

where

$$
\begin{aligned}
& [H, f_j] \\
= \quad & \left\{ -\frac{1}{\Omega_j} \frac{\frac{v_\perp^2}{2} + v_\parallel^2}{R} \sin\theta \; \frac{\partial}{\partial r} \right. \\
& + \left(-\Theta v_\parallel + \frac{1}{B_0}\frac{d\Phi}{dr} - \frac{1}{\Omega_j}\frac{\frac{v_\perp^2}{2} + v_\parallel^2}{R}\cos\theta \right) \frac{\partial}{r\partial\theta} \\
& \left. + \Theta \frac{\frac{v_\perp^2}{2} + v_\parallel^2}{R} \sin\theta \; \frac{\partial}{\partial v_\parallel} \right\} f_j
\end{aligned}
$$

We recognize the terms

$$
\begin{aligned}
-\frac{1}{\Omega_j} \frac{\frac{v_\perp^2}{2} + v_\parallel^2}{R} \sin\theta \; \frac{\partial}{\partial r} f_j \quad \rightarrow \quad & v_{D,r} \frac{\partial}{\partial r} f_j \\
& \text{the radial component} \\
& \text{of the DRIFT velocity } v_D
\end{aligned}
$$

$$
\begin{aligned}
& -\Theta v_\parallel + \frac{1}{B_0}\frac{d\Phi}{dr} - \frac{1}{\Omega_j}\frac{\frac{v_\perp^2}{2} + v_\parallel^2}{R}\cos\theta \\
= \quad & \text{the poloidal velocity, composed of} \\
& \text{- poloidal projection of the parallel velocity } \frac{B_\theta}{B_T} v_\parallel \\
& \text{- poloidal velocity due to radial electric field } V_E \\
& \text{- poloidal projection of the drift velocity, } v_{D,\theta}
\end{aligned}
$$

this multiplies $\frac{\partial f_j}{r\partial\theta}$.

The last term is *energetic* and is the transfer of energy from the *time variation* of the parallel projection of the drift velocity multiplied by the derivative of the distribution function to the parallel velocity

$$
\frac{dv_\parallel}{dt}\frac{\partial f_j}{\partial v_\parallel}
$$

where we find above (**Burrell Wong**, **Berk Galeev**)

$$
\begin{aligned}
\frac{dV_{\parallel j}}{dt} \quad = \quad & \frac{\varepsilon}{q}\left(-\frac{e_j}{m_j}\frac{\partial \phi}{r\partial\theta} \right) \\
& -\frac{\varepsilon}{q} V_{\perp j}^2 \frac{\varepsilon \sin\theta}{2r} \\
& + V_0 V_\parallel \frac{\varepsilon\sin\theta}{r}
\end{aligned}
$$

Galeev Sagdeev consider the middle term. Here one replaces the definition

$$\begin{aligned}
\frac{\varepsilon}{q} &= \Theta \\
&= \frac{B_\theta}{B_T} \ll 1 \\
& -\frac{\varepsilon}{q} V_{\perp j}^2 \frac{\varepsilon \sin\theta}{2r} \\
&= -\Theta \frac{\frac{v_\perp^2}{2}}{R} \sin\theta
\end{aligned}$$

NOTE

The velocity V_0 which occurs in the **Berk Galeev**, **Wang Burrell** system is, for **GS**, produced by an electric field in the radial direction, resulting from a distribution of constant-on-surface electric potential $\Phi(r)$. Then the velocity is mainly poloidal

$$V_0 \equiv V^{poloidal} \approx \frac{E_r}{B} = \frac{1}{B}\left(-\frac{\partial\Phi}{\partial r}\right)$$

The poloidal velocity is the projection of the velocity of the parallel direction since the following combination must be close to zero

$$V_0 + \Theta V_\parallel \approx 0$$

and this transforms the last term $V_0 V_\parallel \frac{\varepsilon \sin\theta}{r}$ in **Berk Galeev** in

$$V_0 V_\parallel \frac{\varepsilon \sin\theta}{r} \rightarrow -\Theta V_\parallel^2 \frac{\varepsilon \sin\theta}{r}$$

Together with the first term this gives

$$\begin{aligned}
\frac{dV_{\parallel j}}{dt} &= -\Theta \frac{\frac{v_\perp^2}{2}}{R} \sin\theta - \Theta V_\parallel^2 \frac{\varepsilon \sin\theta}{r} \\
&= -\Theta \frac{\frac{v_\perp^2}{2} + v_\parallel^2}{R} \sin\theta
\end{aligned}$$

which is the energy term in the equation.

END

NOTE.

Then there is an error in **GS** 1968, in that the last term, which is energetic, contains in the denominator a supplementary Ω_j which must not be there

$$\begin{aligned}
& \Theta \frac{1}{\Omega_j} \frac{\frac{v_\perp^2}{2} + v_\parallel^2}{R} \sin\theta \ \frac{\partial}{\partial v_\parallel} \\
& \text{has dimensions } \frac{m^2 s^{-2}}{s^{-1} m} \left(\frac{\partial}{\partial v_\parallel}\right) \\
= \ & \frac{m}{s}\left(\frac{\partial}{\partial v_\parallel}\right) \sim 1 \text{ non-dimensional, which is NOT correct}
\end{aligned}$$

The cause is the presence of Ω_j in the denominator. We would need

$$\Theta \frac{\frac{v_\perp^2}{2} + v_\parallel^2}{R} \sin\theta \ \frac{\partial}{\partial v_\parallel}$$

$$\text{with dimensions } \frac{m^2 s^{-2}}{m} \left(\frac{\partial}{\partial v_\parallel} \right)$$

$$\sim \quad \frac{1}{s} \quad \text{(correct)}$$

since in this way the dimensions are correct.

Actually the correct term should be

$$\Theta \frac{\mu B_0 / m + v_\parallel^2}{R} \sin\theta \ \frac{\partial}{\partial v_\parallel}$$

where we **note** that it is B_0 instead of B as would be needed to connect μ with $v_\perp^2/2$.

This is indeed the form adopted by **Galeev 1971**,

$$\begin{aligned} & \frac{\partial f_j}{\partial t} + \left(\Theta v_\parallel + v_E \right) \frac{\partial f_j}{r \partial \theta} - \frac{1}{\Omega_j} \frac{\mu B_0 / m_j + v_\parallel^2}{R} \sin\theta \frac{\partial f_j}{\partial r} \\ & - \Theta \frac{\mu B_0 / m_j + v_\parallel^2}{R} \sin\theta \frac{\partial f_j}{\partial v_\parallel} + \frac{e_j E_{0z}}{m_j} \frac{\partial f_j}{\partial v_\parallel} \\ = & - \nu_j \left(\delta f_j - \frac{\delta n_j}{n} f_{0j} \right) + \sum_k \nu_{jk} \frac{m_j}{T_{0j}} \mathbf{v} \cdot \mathbf{u}_k \ f_{0k} \end{aligned}$$

where

$$\delta f_j = f_j - f_{0j} \quad \text{deviation from Maxwellian}$$

$$\delta n_j = \int d^3 v \ \delta f_j = \text{perturbation of density}$$

$$\mathbf{u}_j = \int d^3 v \ \mathbf{v} \ \left(f_j - f_{0j} \right)$$

is the mass velocity, the integration of the term $f_{0j} = f_{Mj}$ is zero.

$$\nu_j = \nu_{jj} + \nu_{jk}$$

See below **Galeev 1971**.

The last term

$$\sum_k \nu_{jk} \frac{m_j}{T_{0j}} \mathbf{v} \cdot \mathbf{u}_k \ f_{0k}$$

usually arises together with the Lorentz collision operator.

END

This is however a new and important variable.

In **Galeev Sagdeev** the expansion of the invariant J to second order in the distance relative to the magnetic surface

$$r - r_0$$

it si found

$$r - r_0 = \frac{1}{\Omega_c \Theta} \left\{ \Delta v \pm \sqrt{(\Delta v)^2 + 2 r_0 \Omega_c v_g (\cos\theta - 1)} \right\}$$

where

$$\Delta v (r_0, 0) = v_{\parallel} (r_0, 0) - \frac{v_E}{\Theta}$$

$$v_g \equiv \frac{1}{\Omega_c} \frac{\frac{\mu B_0}{m} + \frac{v_E^2}{\Theta^2}}{R}$$

(note that v_g looks similar to v_D but has a different origin).

END

Comment. Returning to the previous argument that **GS** take, for the energetic term

$$\begin{aligned} \frac{dv_{\parallel}}{dt} & : \\ & \frac{\varepsilon}{q} \left(-\frac{e_j}{m_j} \frac{\partial \phi}{r \partial \theta} \right) \\ & + V_0 V_{\parallel} \frac{\varepsilon \sin\theta}{r} \end{aligned}$$

and in addition the presence of

$$v_{\parallel}^2$$

at the numerator, making this expression to have the same velocity-dependence as v_D.

The reason may be that

$$V_E \approx -\Theta v_{\parallel}$$

Then the last term is the one that produces $v_{\parallel}^2$

$$\begin{aligned} & V_0 V_{\parallel} \frac{\varepsilon \sin\theta}{r} \\ \approx & \quad -\Theta V_{\parallel} \times V_{\parallel} \frac{\sin\theta}{R} \end{aligned}$$

This should be added to the term derived previously

$$\begin{aligned} & -\Theta \frac{\frac{v_{\perp}^2}{2}}{R} \sin\theta \\ \text{leading to} \quad \downarrow & \\ & -\Theta \frac{\frac{v_{\perp}^2}{2} + V_{\parallel}^2}{R} \sin\theta \end{aligned}$$

The collision operator is

$$
\begin{aligned}
& St\left(f_a\right) \\
= \; & -\sum_{\text{species } b} \frac{\partial}{\partial v_\alpha} 2\pi \ln \Lambda \frac{\left(e_a e_b\right)^2}{m_a} \\
& \times \int d\mathbf{v}' \left[\frac{\delta_{\alpha\beta}}{u} - \frac{u_\alpha u_\beta}{u^3}\right] \\
& \times \left(f_a\left(\mathbf{v}\right) \frac{1}{m_b} \frac{\partial f_b\left(\mathbf{v}'\right)}{\partial v_\beta} - f_b\left(\mathbf{v}'\right) \frac{1}{m_a} \frac{\partial f_a\left(\mathbf{v}\right)}{\partial v_\beta}\right)
\end{aligned}
$$

where

$$u_\alpha = v_\alpha - v'_\alpha$$

(same as in **Rutherford 1970**)

The expansion

$$f_a\left(\mu, v_\parallel; r, \theta\right) = f_a^{(0)}\left(\mu, v_\parallel; r\right) + f_a^{(1)}\left(\mu, v_\parallel; r, \theta\right)$$

The local maxwellian function

$$f_a^{(0)} = \frac{n_a\left(r\right)}{\left(\pi v_{th,a}^2\right)^{3/2}} \exp\left[-\frac{2\mu B/m_a + v_\parallel^2}{v_{th,a}^2} - \frac{e_a \phi\left(r\right)}{T_a}\right]$$

and the equation for the first order

$$
\begin{aligned}
\left(-\Theta v_\parallel + v_E\right) \frac{\partial f_a^{(1)}}{r\partial\theta} \;=\; & -\nu_a f_a^{(1)} \\
& + \frac{\mu B_0/m_a + v_\parallel^2}{R} \sin\theta \left[-\Theta \frac{\partial}{\partial v_\parallel} + \frac{1}{\Omega_{ca}} \frac{\partial}{\partial r}\right] f_a^{(0)}
\end{aligned}
$$

Note in the left we recognize the *convection* of the poloidal variation of the distribution function by the projection of the parallel velocity on the poloidal direction. In the right we recognize $v_{Dr} \frac{\partial f_M}{\partial r}$. There is however in the right an energetic term, $\partial f_M / \partial v_\parallel$, the advection in *velocity space* due to the energetic effect of the particle's drift velocity. The poloidal projection of the *drift* velocity produces an acceleration that has effect on the parallel velocity.

For a trapped particle, the perpendicular velocity (through the invariant μ) and the parallel velocity are changing periodically, up to full suppression of $v_\parallel$ at the tips of the bananas. Then one would expect a periodic change of the distribution function, if this is expressed in terms of

$$f\left(v_\perp, v_\parallel; t, \mathbf{x}\right)$$

But actually there is NO change of f when it is expressed in terms of

$$f\left(\mu, \epsilon; t, \mathbf{x}\right)$$

invariants.

For a passing particle, the parallel velocity is also variable along the orbit and correspondingly $v_\perp$ is also variable.

End.

The solution

$$f_a^{(1)} = \sum_{\pm} \frac{\mu B_0/m_a + v_\parallel^2}{R} \left[\Theta \frac{\partial}{\partial v_\parallel} - \frac{1}{\Omega_{ca}} \frac{\partial}{\partial r} \right] f_a^{(0)} \frac{1}{\frac{v_E}{r} - \Theta \frac{v_\parallel}{r} \mp i\nu_a} \exp\left(\pm i\theta\right)$$

(note that $1/r$ has the role of wavenumber k in the expression at the propagator $kv + i\nu$).

The weak collisionality (banana regime)

$$f_a = f_a^{(0)}\left(\epsilon, \mu, J; \sigma\right) + f_a^{(1)}\left(\epsilon, \mu, J; \sigma; \theta\right)$$

The collision term is linearized

$$\begin{aligned}
& \left(-\Theta v_\parallel + v_E\right) \frac{\partial f_a^{(1)}}{r\partial\theta} + \Theta \frac{\frac{\mu B_0}{m_a} + \frac{v_E^2}{\Theta^2}}{R} \sin\theta \frac{\partial f_a^{(1)}}{\partial v_\parallel} \\
= & \sum_b 2\pi \ln\Lambda \frac{\left(e_a e_b\right)^2}{m_a} \\
& \times \frac{\partial}{\partial v_\alpha} \left\{ \left(\eta_b + \frac{d\eta_b}{dx_b} - \frac{\eta_b}{2x_b} \right) \left[\frac{\delta_{\alpha\beta}}{v} - \frac{v_\alpha v_\beta}{v^3} \right] + \frac{v_\alpha v_\beta}{v^3} \frac{\eta_b}{x_b} \right\} \\
& \times \left(\frac{1}{m_a} \frac{\partial f_a^{(0)}}{\partial v_\beta} + \frac{1}{m_b} \frac{2v_\beta}{v_{th,b}^2} f_a^{(0)} \right)
\end{aligned}$$

where

$$\eta_b \equiv \eta\left(x_b\right) = \frac{2}{\sqrt{\pi}} \int_0^{x_b} dt \ \exp\left(-t\right) \sqrt{t}$$

$$\begin{aligned}
x_b & \equiv \frac{2\mu B_0}{m_b v_{th,b}^2} \\
& = \frac{v_\perp^2}{v_{th,b}^2} h
\end{aligned}$$

Now, to solve the equation in the two regions of the phase space: circulating and trapped, it is adopted

- neglect of all derivatives to other velocity variables except *parallel velocity*. This is because the distribution function is *most sensible to the variations of* $v_\parallel$;

- neglect order-2 powers of electric velocity
- adopt *ambipolar assumption*

$$\frac{e\phi\left(r\right)}{T_i}\sim\ln n\left(r\right)$$

then

$$v_E-\Theta v_{th,i}\ll 0$$

and one substitutes

$$v_{\parallel}=\frac{v_E}{\Theta}+2\sigma\sqrt{\frac{\mu B_0}{m}\varepsilon\left(\kappa^2-\sin^2\frac{\theta}{2}\right)}$$

NOTE

This expression for the parallel velocity is simliar to the expression for the *angular poloidal velocity (* **or** frequency) ω

$$\omega=\left(u\widehat{\mathbf{n}}+\mathbf{V}_E\right)\cdot\boldsymbol{\nabla}\theta$$

from **shaing hsu dominguez resonance, and from orbit squeezing**.

there we have

$$\widehat{\omega}\equiv\left|\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta\right|\left(u_0+V_{E0}\right)$$

then

$$\omega=\widehat{\omega}\left[1-k\sin^2\frac{\theta}{2}\right]^{1/2}$$

and

$$k=4S\varepsilon\frac{\left|u_0^2+\mu B_0\right|}{\left(\frac{\widehat{\omega}}{\left|\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta\right|}\right)^2}$$

END

The kinetic equation becomes

$$\begin{aligned}&-\Theta v_{th,a}\sqrt{2x_a\varepsilon}\frac{\partial f_a^{(1)}}{r\partial\theta}\\=\quad&\frac{1}{\varepsilon}\nu_a\ A\left(x_a\right)\\&\times\frac{\partial^2}{\partial\kappa^2}\left\{\sigma\sqrt{\kappa^2-\sin^2\frac{\theta}{2}}\ \left(\frac{\partial f_a^{(0)}}{\partial\kappa^2}+2x_a\varepsilon f_a^{(0)}\right)+c_a\sqrt{2x_a\varepsilon}f_a^{(0)}\right\}\end{aligned}$$

NOTE

The presence of the second derivative

$$\frac{\partial^2}{\partial\kappa^2}\left[...\right]$$

is connected with the simpler form of the collision operator, which is a transformation of the *pitch angle.*

It is described in **shaing hsu dominguez resonance.**

There however it is different second order derivation

$$C\left(f\right)=\nu_{D}\frac{v^{2}}{2R^{2}q^{2}}\frac{\partial^{2}f}{\partial\omega^{2}}$$

END

where

$$A_{a}\left(x_{a}\right)=\frac{3\sqrt{\pi}}{4}\sum_{\text{species } b}\left(\eta_{b}+\frac{d\eta_{b}}{dx_{b}}-\frac{\eta_{b}}{2x_{b}}\right)x_{a}^{-3/2}$$

and

$$c_{a}\equiv\frac{v_{E}}{\Theta v_{th,a}}$$

The main variation of the perturbation $f_j^{(1)}$ is on the poloidal direction θ, not on r. Then

$$\sqrt{\varepsilon}\frac{\partial f_{j}^{(1)}}{\partial r}\ll\frac{\partial f_{j}^{(1)}}{r\partial\theta}$$

To find the distribution function of order *zero* we use the procedure of creating a constraint for it. This means that we will integrate the equation for $f_j^{(1)}\left(r,\theta\right)$ over poloidal angle, using the *periodicity.*

The interval of integration is $[0,2\pi]$.

The constraint for $f_j^{(0)}$ is

$$\begin{aligned}&\frac{\partial}{\partial\kappa^{2}}\left\{\int_{0}^{2\pi}\left[\sigma\sqrt{\kappa^{2}-\sin^{2}\frac{\theta}{2}}\left(\frac{\partial f_{j}^{(0)}}{\partial\kappa^{2}}+2x_{j}\varepsilon\ f_{j}^{(0)}\right)+\frac{v_{E}}{v_{th,j}\Theta}\sqrt{2x_{j}\varepsilon}f_{j}^{(0)}\right]d\theta\right\}\\ =\ \ &0\end{aligned}$$

3.1 Transiting particles distribution function

The constants of motion

$$\begin{aligned}&J-2\sigma\frac{\sqrt{\frac{\mu B_{0}}{m}\frac{\varepsilon}{\kappa}}}{\Theta}\\&\mu\\&\kappa^{2}\end{aligned}$$

The zero function becomes a Maxwellian for

$$\kappa^{2}\rightarrow\infty$$

The distribution function for untrapped particles

$$
\begin{aligned}
& f_{j,untrapped}^{(0)} \\
= \quad & \frac{n_j(r)}{\left(\sqrt{\pi} v_{th,j}\right)^3} \exp\left[-\frac{e\Phi(r)}{T_j} - \frac{v^2}{v_{th,j}^2} - \frac{\left(v_E/\Theta\right)^2}{v_{th,j}^2} \right. \\
& \qquad \left. -2x_j \ \varepsilon \ \kappa^2 - \sigma \frac{\pi\sqrt{2x_j\varepsilon}}{2} \frac{v_E/\Theta}{v_{th,j}} \int_1^{\kappa^2} \frac{dt}{\sqrt{t}E\left(\frac{1}{\sqrt{t}}\right)} \right] \\
& \times \left\{ 1 + \sigma \frac{\sqrt{2x_j\varepsilon}}{\Theta} \rho_j \frac{1}{n(r)} \frac{dn(r)}{dr} \left(\sqrt{\kappa^2 - \sin^2\frac{\theta}{2}} - \frac{1}{2} \int_1^{\kappa^2} \frac{dt}{\sqrt{t}E\left(\frac{1}{\sqrt{t}}\right)} \right) \right\}
\end{aligned}
$$

In the curly bracket there are the first two terms of an expansion of the function of the third constants of motion, κ^2.

3.2 Trapped particles distribution function

he dependence on σ does not exist for them.

One starts again from the equation for $f_j^{(1)}$ which is integrated over the periodic variable θ obtaining a constraint for $f_j^{(0)}$. The equation for $f_j^{(0)}$ can have a solution like

$$
\begin{aligned}
f_{j,trapped}^{(0)} \quad = \quad & \frac{n(r)}{\left(\sqrt{\pi} v_{th,j}\right)^3} \exp\left(-x_j\right) \ F_j\left(\kappa^2\right) \\
& \times \left\{ 1 + \sigma \frac{\sqrt{2x_j\varepsilon}}{\Theta} \rho_j \frac{1}{n(r)} \frac{dn(r)}{dr} \sqrt{\kappa^2 - \sin^2\frac{\theta}{2}} \right\}
\end{aligned}
$$

and introducing this form in the constraint

$$
\frac{\partial F_j}{\partial \kappa^2} + 2x_j\varepsilon \ F_j\left(\kappa^2\right) = 0
$$

One joins to this equation the equation for $f_j^{(1)}$,

$$
-\Theta v_{th,j} \frac{\partial f_j^{(1)}}{r\partial\theta} = \nu_j A_j\left(x_j\right) \frac{\partial}{\partial\kappa^2} \left(c_j + \frac{1}{2\Theta}\rho_j \frac{1}{n(r)} \frac{dn(r)}{dr} \right) f_j^{(0)}
$$

The full distribution function for *trapped* particles

$$
\begin{aligned}
f_{j,trapped} \quad = \quad & \frac{n(r)}{\left(\sqrt{\pi} v_{th,j}\right)^3} \exp\left(-x_j - c_j^2 - 2x_j\varepsilon \ \kappa^2\right) \\
& \times \left\{ 1 + \sigma \frac{\sqrt{2x_j\varepsilon}}{\Theta} \rho_j \frac{1}{n(r)} \frac{dn(r)}{dr} \sqrt{\kappa^2 - \sin^2\frac{\theta}{2}} \right. \\
& \left. + \nu_j \frac{2A_j}{v_{th,j}} (r\theta) \frac{1}{\Theta} x_j\varepsilon \left(c_j + \frac{\rho_j}{2\Theta} \frac{1}{n(r)} \frac{dn(r)}{dr} \right) \right\}
\end{aligned}
$$

3.3 The radial diffusion resulting from the circulating, trapped and transition

These are calculated using the distribution function.

It is in this paper that **Galeev Sagdeev** have shown that the diffusion due to trapped particles is

$$\varepsilon^{-3/2}$$

higher than the Pfirsch Schluter (collisional) diffusion.

4 Spontaneous poloidal rotation in banana regime Galeev Sagdeev Liu Novakovskii

The work relies on the JETP paper **Galeev Sagdeev 1968** , discussed further in the present text.

The objective is to find the time variation of the poloidal rotation, which exists due to the non-ambipolarity of the radial transport fluxes of the *ions* and of the *electrons.*

The physical process is illustrated by the collision of two particles that move on large bananas. Head-on collision with reversal of direction of motion relative to the magnetic field line means that the banana that was fully inside the magnetic surface becomes fully outside and the radial "step" is very large.

The magnetic field

$$\begin{aligned}
\mathbf{B} &= \frac{B_0}{1+\varepsilon\cos\theta}\widehat{\mathbf{e}}_\varphi + \frac{B_0\Theta}{1+\varepsilon\cos\theta}\widehat{\mathbf{e}}_\theta \\
\text{where } \Theta &= \frac{B_\theta}{B_\varphi} \\
&\quad \text{note that } \Theta \text{ is function of only } r
\end{aligned}$$

The longitudinal invariant

$$J = m\Omega_c \int_0^r dr\ \Theta(r) - mv_\parallel (1+\varepsilon\cos\theta)$$

will be expanded to order two in the deviation of the orbit from the surface $r - r_0$.

The variables are

$$w = \frac{v^2}{2} \equiv \text{kinetic energy}$$

$$E = \frac{m}{2}v_\parallel^2 + \mu B(r,\theta) + e\phi(r,t)$$

We see the definition

$$\mu = \frac{mv_\perp^2}{2B}$$

The particle orbit is

$$r - r_0 = 2\sqrt{\frac{\mu B_0}{m}\varepsilon S}\,\frac{\kappa + \sigma\sqrt{\kappa^2 - \sin^2\frac{\theta}{2}}}{\Theta\Omega_c S}$$

where

$$\begin{aligned}
\kappa^2 &= \frac{m\left(\Delta v_{\|}\right)^2}{4\mu B_0 \varepsilon S} \\
\Delta v_{\|} &= v_{\|}\left(r_0, 0\right) + \frac{B_{\varphi}}{B_{\theta}} V_E\left(r_0, t\right) \\
V_E\left(r, t\right) &= \frac{1}{B_0}\frac{\partial \phi\left(r, t\right)}{\partial r} \\
S &= 1 + \frac{\partial V_E}{\partial r}\frac{1}{\Omega_c}\left(\frac{B_{\varphi}}{B_{\theta}}\right)^2 \text{ squeezing factor}
\end{aligned}$$

The equation for the distribution function

$$\left(v_{\|}\frac{B_{\theta}}{B_{\varphi}} + V_E\right)\frac{\partial f^{(1)}}{r\partial\theta} - \frac{\mu}{m}\nabla_{\|}B\,\frac{\partial f^{(1)}}{\partial v_{\|}} = C\left(f^{(0)}\right)$$

NOTE

As it is formulated, the equation looks like an intention to derive the *neo-classical drift* of the particles, the motion on banana orbits, in particular.In the convective term there is no v_D,

$$\left(v_{\|}\widehat{\mathbf{n}} + \mathbf{V}_E + \mathbf{v}_D\right)\cdot\boldsymbol{\nabla}$$

In our equation $\mathbf{v}_D$ is absent. Instead it is retained an energetic term that moves the problem in the velocity space, where the motion - for example - on banana, is a variation of the distribution function in the region of parallel velocity $v_{\|}$ AND of the perpendicular velocity $v_{\perp}$ due to the variation of the magnetic field $\mathbf{B}$ along the line.

The change is

$$\begin{aligned}
-\frac{\mu}{m}\nabla_{\|}B &= -\frac{v_{\perp}^2}{2}\nabla_{\|}\ln B = \frac{dv_{\|}}{dt} \\
&\quad\text{(from the eq. of motion)}
\end{aligned}$$

and the energetic term is

$$\frac{dv_{\|}}{dt}\frac{\partial f^{(1)}}{\partial v_{\|}}$$

In **Galeev Sagdeev 1968** and in other papers one finds

$$\begin{aligned}
\frac{dv_{\|}}{dt} &= -\frac{\varepsilon}{q}v_{\perp}^2\frac{\varepsilon\sin\theta}{2r} \\
&= -\frac{B_{\theta}}{B_{\varphi}}\frac{v_{\perp}^2}{2}\frac{\sin\theta}{R}
\end{aligned}$$

where

$$\frac{B_\theta}{B_\varphi}\frac{\sin\theta}{R} \rightarrow \nabla_\parallel \ln B$$

$$\begin{aligned}
\nabla_\parallel \ln B &= \frac{1}{qR}\frac{\partial}{\partial\theta}\ln B \\
&= \frac{1}{B}\frac{1}{B_\varphi/B_\theta}\frac{\partial}{r\partial\theta}\left(\frac{B_0}{1+\varepsilon\cos\theta}\right) \\
&= \frac{B_\theta}{B_\varphi}\frac{1}{B}\frac{1}{r}B_0\left[-\frac{-\varepsilon\sin\theta}{(1+\varepsilon\cos\theta)^2}\right] \\
&= \frac{B_\theta}{B_\varphi}\frac{1}{\frac{B_0}{h}}\frac{1}{R}B_0\frac{\sin\theta}{h^2} = \frac{B_\theta}{B_\varphi}\frac{\sin\theta}{R}\times\frac{1}{h}
\end{aligned}$$

An approximative equality with the above expression.

This happens on the time interval of the bounce on banana.

END

Here the parallel velocity is replaced by its expression

$$\begin{aligned}
v_\parallel &= -V_E\frac{B_\varphi}{B_\theta} \\
&\quad +2\sigma\sqrt{\frac{\mu B}{m}\varepsilon S}\;\sqrt{\kappa^2-\sin^2\frac{\theta}{2}}
\end{aligned}$$

The equation, after inserting $v_\parallel$, is

$$2\sigma\frac{B_\theta}{B_\varphi}\sqrt{\frac{\mu B_0}{m}\varepsilon S}\;\sqrt{\kappa^2-\sin^2\frac{\theta}{2}}\frac{\partial f^{(1)}}{r\partial\theta} = C\left(f^{(0)}\right)$$

NOTE

It may seem strange that replacing the expression of $v_\parallel$ obtained from the expansion of the longitudinal invariant to second order in $(r-r_0)$ in the kinetic equation we ignore the last term, $\frac{dv_\parallel}{dt}\frac{\partial f}{\partial v_\parallel}$.

Actually this term is the physical description of what happens in the distribution function when a particle moves along the magnetic field line and the parallel and perpendicular velocities change in order to preserve the energy and μ invariants. This is what leads to the *neoclassical drift* velocity v_D.

Since we have now included the neoclassical drift through the expression of $v_\parallel$ there is no need to keep the velocity space changes due to the periodic exchanges $v_\parallel \rightarrow v_\perp$.

END

The collision operator

$$
\begin{aligned}
C\left(f^{(0)}\right) &= \nu \, \frac{1}{S\varepsilon} A\left(x\right) \sigma \sqrt{\kappa^2 - \sin^2 \frac{\theta}{2}} \\
&\quad \times \frac{\partial}{\partial \kappa^2} \left\{ \sigma \sqrt{\kappa^2 - \sin^2 \frac{\theta}{2}} \, \left(\frac{\partial f^{(0)}}{\partial \kappa^2} + 2x\varepsilon S \, f^{(0)} \right) \right. \\
&\qquad \left. - \sqrt{2x\varepsilon S} \, \left(\frac{V_E \left(\frac{B_\varphi}{B_\theta} \right)}{v_{th}} + \frac{u_\parallel}{v_{th}} \right) f^{(0)} \right\}
\end{aligned}
$$

where

$$
\begin{aligned}
x &\equiv \frac{2\mu B_0}{v_{th}^2} = \frac{v_\perp^2}{v_{th}^2} \frac{B_0}{B} = \frac{v_\perp^2}{v_{th}^2} h \\
A\left(x\right) &= \frac{3\sqrt{\pi}}{4} \left(\eta - \frac{d\eta}{dx} - \frac{\eta}{2x} \right) \frac{1}{x^{3/2}} \\
\eta\left(x\right) &\equiv \frac{2}{\sqrt{\pi}} \int_0^x dx' \, \sqrt{x'} \exp\left(-x'\right)
\end{aligned}
$$

The parallel velocity of the plasma flow is denoted $U_\parallel$.

The usual constraint to ensure solution is obtained by exploiting periodicity on θ direction. In the general case, one has to divide by $v_\parallel$, multiply by B and apply surface averaging. Here the same condition is expressed as

$$
\int_0^{2\pi} d\theta \, \frac{1}{\sqrt{\kappa^2 - \sin^2 \frac{\theta}{2}}} C\left(f^{(0)}\right) = 0
$$

(**Note** this is $\int_0^{2\pi} \frac{d\theta}{2\pi} \, \frac{1}{v_\parallel} C = 0$).

The constraint permits the determination of the zeroth order distribution function $f^{(0)}$.

$$
\begin{aligned}
f^{(0)} &= \frac{n\left(r\right)}{\left(\sqrt{\pi} v_{th}\right)^3} \exp\left[-x\left(1 + 2\kappa^2 \varepsilon S\right) \right. \\
&\qquad \left. + \frac{\pi}{2} \frac{1}{v_{th}} \sigma \sqrt{2x\varepsilon S} \left(V_E \frac{B_\varphi}{B_\theta} + U_\parallel \right) \, \mathrm{H}\left(\kappa^2 - 1\right) \int_1^{\kappa^2} \frac{d\xi}{\sqrt{\xi} \mathbf{E}\left(\frac{1}{\xi}\right)} \right] \\
&\quad \times \left\{ 1 - \sigma \frac{\sqrt{2x\varepsilon}}{\left(B_\theta \Omega_c / B_\varphi\right) \sqrt{S}} v_{th} \, \left[\frac{d \ln n}{dr} + \left(x\left(1 + 2\kappa^2 \varepsilon S\right) - \frac{3}{2} \right) \frac{d \ln T}{dr} \right] \right. \\
&\qquad \left. \times \left[\sqrt{\kappa^2 - \sin^2 \frac{\theta}{2}} - \mathrm{H}\left(\kappa^2 - 1\right) \frac{\pi}{4} \int_1^{\kappa^2} \frac{d\xi}{\sqrt{\xi} \mathbf{E}\left(\frac{1}{\xi}\right)} \right] \right\}
\end{aligned}
$$

The Heaviside function $\mathrm{H}\left(\kappa^2 - 1\right) = 0$ for $\kappa^2 < 1$ *i.e.* for trapped particles.

There is *time dependence* of the distribution function $f^{(0)}$ through

$$V_E(r,t) \quad \text{and} \quad \kappa^2$$

We have to go to higher order, $f^{(2)}$.

$$\begin{aligned}\sigma \frac{B_\theta}{B_\varphi} \sqrt{2x\varepsilon S}\ v_{th} \sqrt{\kappa^2 - \sin^2 \frac{\theta}{2}}\ \frac{\partial f^{(2)}}{r\partial\theta} &= C(\delta f) \\ &\quad - \frac{\partial f^{(0)}}{\partial t}\end{aligned}$$

Here the time derivative of the zero-order $f^{(0)}$ uses the expression derived above where the time enters through κ^2 which depends on V_E and this has time variation. The time variation of V_E is due to the non-ambipolar radial fluxes and also due to the damping of the poloidal rotation by transit time magnetic pumping,

$$\begin{aligned}\frac{\partial f^{(0)}}{\partial t} &= -2\sigma \frac{B_\varphi}{B_\theta} \frac{1}{v_{th}} \left(\frac{\partial V_E}{\partial t} \right) \sqrt{2x\varepsilon S} \\ &\quad \times \left[\sqrt{\kappa^2 - \sin^2 \frac{\theta}{2}} - \frac{\pi}{4} \mathrm{H} \int_1^{\kappa^2} d\xi \frac{1}{\sqrt{\xi}\mathbf{E}\left(\frac{1}{\xi}\right)} \right] f^{(0)}\end{aligned}$$

Again, H means that the term exists only for $\kappa^2 > 1$, <u>circulating</u> particles.

The equation for $f^{(2)}$ contains the function δf.
To find δf the equation for $f^{(2)}$ is used as a constraint, imposing periodicity on θ.

$$\int_0^{2\pi} d\theta \frac{C(\delta f) - \frac{\partial f^{(0)}}{\partial t}}{\sqrt{\kappa^2 - \sin^2 \frac{\theta}{2}}} = 0$$

The following terms are small

$$\frac{V_E}{\frac{B_\theta}{B_\varphi} v_{th}} \ll 1$$

$$\frac{1}{\frac{\partial}{\partial t} \ln V_E} \ll \tau_R$$

The equation after neglecting terms quadratic in these parameters

$$\begin{aligned}&\frac{\nu}{\varepsilon S} A(x) \frac{\partial}{\partial \kappa^2} \left[\kappa \mathbf{E}\left(\frac{1}{\kappa^2}\right) \frac{\partial}{\partial \kappa^2} \delta f_u \right] \\ =\ &-4\pi\sigma \frac{V_E}{\frac{B_\theta}{B_\varphi} v_{th}} \sqrt{2x\varepsilon S} \left[1 - \mathbf{K}\left(\frac{1}{\kappa^2}\right) \frac{1}{2\kappa} \int_1^{\kappa^2} d\xi \frac{1}{\sqrt{\xi}\mathbf{E}\left(\frac{1}{\xi}\right)} \right] f^{(0)}\end{aligned}$$

This function δf_u for the *circulating* particles will be used to calculate the radial electric current.

The radial electric current of the electrons can be neglected.

The constraint: *the total radial electric current through a magnetic surface must be zero.*

The radial displacement is due to the *neoclassical drift* of the ions.

For the ions

$$\langle nV_r \rangle_D = -\sum_{\sigma=\pm 1} \int_0^{2\pi} \int_0^{\infty} \int_{\sin^2 \frac{\theta}{2}}^{\infty} x \left[C\left(f^{(0)}\right) - \frac{\partial f^{(0)}}{\partial t} + C\left(\delta f_u\right) \right] d\left(\kappa^2\right) dx d\theta$$

The first term is the neoclassical diffusion of the ions and can be expressed through the *parallel viscous force*

$$\begin{aligned} \langle nV_r \rangle_D &= -\frac{F_\parallel}{m\Omega_{ci} \frac{B_\theta}{B_\varphi}} \\ &= -\int \frac{d\theta}{2\pi} \frac{1}{m\Omega_{ci} \frac{B_\theta}{B_\varphi}\ S} \int d^3v\ mv_\parallel\ C\left(f^{(0)}\right) \end{aligned}$$

We note that this is

$$\langle j_r \rangle = -\left\langle \frac{\int d^3v\ mv_\parallel C\left(f^{(0)}\right)}{B_\theta} \right\rangle$$

$$\begin{aligned} F_\parallel &= \left(\boldsymbol{\nabla}\ \boldsymbol{\pi}\right)_\parallel \\ &= 0.37\ mn\ \nu \sqrt{\frac{\varepsilon}{S}} \left[V_E \frac{B_\varphi}{B_\theta} + U_\parallel + \frac{1}{\frac{B_\theta}{B_\varphi} S} \left(U_{pol} - 1.17\ U_T\right) \right] \end{aligned}$$

$$\begin{aligned} U_{pol} &= \frac{T}{eB_0} \frac{d \ln (nT)}{dr} \\ U_T &= \frac{T}{eB_0} \frac{d \ln T}{dr} \end{aligned}$$

The next two terms in the square paranthesis of the expression of $\langle nV_r \rangle_D$ are the *inertial* effect on the plasma rotation.

One has to insert $\frac{\partial f^{(0)}}{\partial t}$ and δf_u.

The calculations lead to

$$\begin{aligned}
& \langle nV_r \rangle_{inertial} \\
= \; & -\frac{8\sqrt{2}}{\pi^{3/2}}\Gamma\left(\frac{5}{2}\right)\frac{(\varepsilon S)^{3/2}}{\Omega_{ci}\frac{B_\theta}{B_\varphi}S}\;V_E \\
& \times\left[\frac{4}{9}+\int_1^\infty\left\{\kappa\mathbf{E}-\frac{\pi^2}{8}\int_1^{\kappa^2}\frac{d\xi}{\sqrt{\xi}\mathbf{E}}\right.\right. \\
& \left.\left.-\frac{\pi^2}{4}\sqrt{\kappa^2-\frac{1}{2}}\frac{\partial}{\partial\kappa^2}\left[\sqrt{\kappa^2-\frac{1}{2}}\frac{1}{\kappa\mathbf{E}}\int_1^{\kappa^2}\left(1-\frac{K}{2\kappa'}\int_1^{\kappa'^2}\frac{d\xi}{\sqrt{\xi}\mathbf{E}}\right)d\kappa'\right]\right\}d\kappa^2\right]
\end{aligned}$$

with

$$\Gamma\left(\frac{5}{2}\right)=\frac{3\sqrt{\pi}}{4}$$

The two contribution

- first $\frac{4}{9}$ is from trapped ion
- second, $\int_1^\infty(...)\,d\kappa^2$ is from untrapped ions.

The radial drift is composed of these two terms

$$\langle nV_r\rangle_D=-\sum_{\sigma=\pm1}\int_0^{2\pi}\int_0^\infty\int_{\sin^2\frac{\theta}{2}}^\infty x\left[C(f_0)-\frac{\partial f_0}{\partial t}+C(\delta f_u)\right]d\left(\kappa^2\right)dxd\theta$$

and $\langle nV_r\rangle_{inertial}$.

The calculation leads to

$$1.52\;nm_i\;q^2\sqrt{\frac{S}{\varepsilon}}\;\frac{\partial V_E}{\partial t}=-\frac{F_\parallel}{B_\theta/B_\varphi}$$

5 Radial electric field Novakovskii Liu Sagdeev Rosenbluth

The transport is non-ambipolar, trapped ions after collision with another trapped ion will depart significantly from the magnetic surface. Then the transport is high.

Derive the equation for the time evolution of the poloidal velocity.

The stationary state is Hazeltine.

5.1 Static

The dimensionless parameter of transition from plateau to banana

$$\nu_* = \frac{\nu_{ii} \times \tau_{bounce}}{\varepsilon^{3/2}}$$

with

$$\nu_* \leq 1 \quad \text{banana regime}$$

The result of **Hazeltine**

$$U_\theta = kV_T$$

where

$$U_\theta = \frac{B_\theta}{B_T}\overline{U}_\parallel + V_E + V_n + V_T$$

$$\begin{aligned} \overline{U}_\parallel &= \text{equilibrium parallel velocity} \\ & \quad \text{averaged over } \theta \end{aligned}$$

$$V_E = -\frac{E_r}{B}$$

$$V_n = \frac{T}{eB}\frac{1}{L_n} \quad \text{dia}$$

$$V_T = \frac{T}{eB}\frac{1}{L_T} \quad \text{T-dia}$$

$$k = \begin{cases} -2.1 & \rightarrow \quad \text{Pfirsch Schluter} \\ -0.5 & \rightarrow \quad \text{plateau} \\ 1.17 & \rightarrow \quad \text{banana} \end{cases}$$

5.2 Non-steady state neoclassical evolution

Hassam Kulsrud

$$\frac{\partial U_\theta}{\partial t} = -\nu_{MP}\left(U_\theta - kV_T\right)$$

$$k = k_{Pfirsch-Schluter} = -2.1$$

$$\begin{aligned} \nu_{MP} &= \nu_{Pfirsch-Schluter} \\ &\approx \frac{\nu_{bounce}^2}{\nu_{ii}} \end{aligned}$$

5.3 Fast processes

Plasma parallel viscosity.

A quasi-static viscosity cannot be useful.

Because the viscosity that resuts from pressure anisotropy depend on non-stationary terms like

$$\frac{\partial V_E}{\partial t}$$

which is the polarization current.

$$\frac{\partial}{\partial t} \sim \omega_{bounce}$$

Then, *Geodesic Acoustic Modes* (Winsor Johnson Dawson): oscillations of plasma column in verical direction with characteristic frequency

$$\omega_{GAM} \sim \frac{V_T}{R}$$

Solution by **Hassam Drake**

$$\begin{aligned} V_E &= V_E^{\infty} \\ &+A\exp\left(-\gamma_{MP}t\right) \\ &+B\cos\left(\omega_{GAM}\ t+\varphi\right)\ \exp\left(-\gamma_{GAM}\ t\right) \end{aligned}$$

To discuss the time variation of the distribution function on time scales of

- polarization current
- bounce on banana
- GAM
- Magnetic Pumping decay

one takes

$$\mathbf{B} = \Theta\frac{B_0}{h}\widehat{\mathbf{e}}_\theta + \frac{B_0}{h}\widehat{\mathbf{e}}_\varphi$$

$$\begin{aligned} h &= 1+\varepsilon\cos\theta \\ \varepsilon &= \frac{r}{R} \\ R &= R_0 + r\cos\theta \end{aligned}$$

$$\phi\left(r,\theta,t\right) = \phi_0\left(r,t\right) + \widetilde{\phi}\left(r,\theta,t\right)$$

$$E_r\left(r,t\right) = -\frac{d\phi_0\left(r,t\right)}{dr}$$

The distribution function

$$f\left(r,\theta,v,v_{\parallel}\right)$$

The drift kinetic equation

$$\begin{aligned} & \frac{\partial f}{\partial t}+ \\ & +V_r\frac{\partial f}{\partial r}+\left(\frac{B_\theta}{B_T}v_{\parallel}+V_E\right)\frac{\partial f}{r\partial\theta} \\ & +\frac{dv_{\parallel}}{dt}\frac{\partial f}{\partial v_{\parallel}}+\frac{d\left(v_{\perp}^2/2\right)}{dt}\frac{\partial f}{\partial\left(v_{\perp}^2/2\right)} \\ = & \; C\left(f\right) \end{aligned}$$

The radial drift velocity

$$V_r=-\frac{1}{\Omega}\frac{v_{\parallel}^2+\frac{v_{\perp}^2}{2}}{R}\sin\theta-\frac{1}{\Omega}\frac{\partial V_E}{\partial t}$$

for

$$V_E=-\frac{E_r}{B}$$

In the expression of V_r it is included the polarization drift (iterated $E\times B$ velocity in the momentum equation).

$$\frac{dv_{\parallel}}{dt}=-\varepsilon\frac{v_{\perp}^2}{2qR}\sin\theta+\frac{v_{\parallel}V_E\sin\theta}{R}$$

$$\frac{d\left(v^2/2\right)}{dt}=-\frac{v_{\parallel}^2+\frac{v_{\perp}^2}{2}}{R}\sin\theta\;V_E$$

NOTE on the equation for

$$\frac{v^2}{2}$$

The equation in **Wong Burrell** is

$$\frac{d}{dt}\left(\frac{v_{\perp}^2}{2}\right)=\left(\frac{v_{\perp}^2}{2}\right)v_{\parallel}\frac{B_\theta}{B_T}\frac{\sin\theta}{R}+\left(\frac{v_{\perp}^2}{2}\right)\frac{1}{B_0}\left(\frac{d\phi_0}{dr}\right)\frac{\sin\theta}{R}$$

Consider the equation for $v_{\parallel}$ and multiply with $v_{\parallel}$

$$\frac{d\left(\frac{v_{\parallel}^2}{2}\right)}{dt}=-\frac{\varepsilon}{q}\frac{\frac{v_{\perp}^2}{2}}{R}v_{\parallel}\sin\theta+v_{\parallel}^2V_E\frac{1}{R}\sin\theta$$

Add the two equations, for $v_{\perp}^2/2$ and for $v_{\parallel}^2/2$. We note that the first term is cancelled. The electric velocity is

$$V_E=-\frac{E_r}{B}=-\frac{1}{B}\left(-\frac{d\phi_0}{dr}\right)=\frac{1}{B}\frac{d\phi_0}{dr}$$

$$\frac{d}{dt}\left(\frac{v^2}{2}\right)=\frac{v_\parallel^2+\frac{v_\perp^2}{2}}{R}\sin\theta\;V_E$$

We then have a problem of sign.
END.

The expansion of the distribution function

$$f=f_M+\widetilde{f}$$

$$\begin{aligned}
& \frac{\partial\widetilde{f}}{\partial t}+\frac{v_\parallel}{qR}\frac{\partial\widetilde{f}}{r\partial\theta} \\
& -\frac{\varepsilon}{q}\frac{\frac{v_\perp^2}{2}}{R}\sin\theta\;\frac{\partial\widetilde{f}}{\partial v_\parallel}-C(f) \\
= \; & \frac{v_\parallel^2+\frac{v_\perp^2}{2}}{R}\sin\theta\;\frac{m}{T}\left[V_E+V_n+\left(\frac{v^2}{2T/m}-\frac{3}{2}\right)V_T\right]f_M
\end{aligned}$$

In this equation *polarization drift* $\partial V_E/\partial t$ has been neglected.

Change of variable

$$\begin{aligned}
\left(v,v_\parallel\right) & \rightarrow & (v,\xi) \\
\xi & \equiv & \frac{v_\parallel}{v}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial\widetilde{f}}{\partial t}+\frac{\xi v}{qR}\frac{\partial\widetilde{f}}{\partial\theta}-\frac{\varepsilon}{q}\frac{v^2\left(1-\xi^2\right)}{2}\frac{1}{R}\sin\theta\frac{\partial\widetilde{f}}{\partial\xi}-C\left(\widetilde{f}\right) \\
= \; & \frac{v^2\left(1+\xi^2\right)}{2}\frac{1}{R}\sin\theta\;\frac{m}{T}\left[V_E+V_n+\left(\frac{v^2}{2T/m}-\frac{3}{2}\right)V_T\right]f_M
\end{aligned}$$

The collision operator

$$\begin{aligned}
C(f) & = & \nu_c(x)\frac{\partial}{\partial\xi}\left(1-\xi^2\right)\frac{\partial f}{\partial\xi} \\
& & +\xi\;\widehat{S}_a\;f
\end{aligned}$$

$$x^2=\frac{v^2}{2T/m}$$

$$\nu_c(x)=\nu_{ii}\frac{3\sqrt{2\pi}}{4}\frac{1}{\xi^3}\left[\left(1-\frac{1}{2x^2}\right)\operatorname{erf}(x)+\frac{\exp\left(-x^2\right)}{\sqrt{\pi}x}\right]$$

$$\nu_{ii}=\frac{e^4}{\left(4\pi\varepsilon_0\right)^2}\frac{1}{m^2}\ln\Lambda\frac{n}{v_{th}^3}$$

The operator **Hirshman Sigmar Clarke**

$$\begin{aligned} \widehat{S}_1 f &= 3\left[\nu_c(x) - \nu_S(x)\right] \int_{-1}^{1} d\xi\ \xi f \\ &+3x\ \nu_S(x)\ \ f_M \frac{\int dx'\ \nu_C(x')\, x'^3 \left(\int_{-1}^{1} d\xi\ \xi f\right)}{\int dx'\ \nu_S(x')\, x'^4 f_M} \end{aligned}$$

The frequency $\nu_S(x)$ is the *slowing down* frequency

$$\nu_S(x) = \nu_{ii} \frac{2}{x^3} \left[\mathrm{erf}(x) - \frac{2x\exp\left(-x^2\right)}{\sqrt{\pi}}\right]$$

Normalization
The frequency

$$\widehat{\nu}_c = \frac{\nu_c(x)}{v_{th}/(qR)}$$

$$\widehat{t} = \frac{t}{v_{th}/(qR)}$$

$$\widehat{f} = \frac{\widetilde{f}}{f_M}$$

$$\widehat{V}_n = q\frac{V_n}{v_{th}}$$

$$\widehat{V}_T = q\frac{V_T}{v_{th}}$$

$$\widehat{V}_E = q\frac{V_E}{v_{th}}$$

then

$$\begin{aligned} & \frac{\partial \widehat{f}}{\partial \widehat{t}} + \xi x \frac{\partial \widehat{f}}{\partial \theta} - \varepsilon \frac{x\left(1-\xi^2\right)}{2} \sin\theta \frac{\partial \widehat{f}}{\partial \xi} - C\left(\widehat{f}\right) \\ =\ & \sin\theta\ x^2\left(1+\xi^2\right)\left[\widehat{V}_n + \widehat{V}_T + \left(x^2 - \frac{3}{2}\right)\widehat{V}_T\right] \end{aligned}$$

The condition of neutrality

$$\langle j_r \rangle = \int R d\theta\ d^3v\ V_r\ f$$

$$d^3v = 2\pi v^2 dv\ d\xi$$

and

$$\int_0^\infty dx \; ...$$

$$\int_{-1}^1 d\xi \; ...$$

The linearized version of this equation (*quasineutrality condition*) is

$$\frac{\partial \widehat{V}_E}{\partial \widehat{t}} + \frac{q^2}{2\pi^{3/2}} \int d\theta \; dx \; d\xi \; \left(1+\xi^2\right) x^4 \; \exp\left(-x^2\right) \widehat{f} \; \sin\theta = 0$$

NOTE

In *impurity accumulation, notes* it is detailed the model of **Hassam Kleva Drake** and this equation is the third of the set for the Stringer equation.

END

The expansion

$$\widehat{f}\left(\theta,\xi,x\right) = \sum F_n\left(x,\theta\right) \; P_n\left(\xi\right)$$

Some comments are in *polarization.tex.*

6 Solution of the drift-kinetic equation: Wong Burrell 1982

The paper is discussed in *derivation of drift kinetic equation.tex.*

The equations of motion are written for the variables

$$\lambda \equiv \frac{v_\perp^2}{2}$$

with

$$E = \frac{1}{2} v_\parallel^2 + \lambda + \frac{e\phi}{m}$$

and

$$\mu = \frac{\lambda}{B} = \frac{v_\perp^2}{2B}$$

In previous calculations it has been obtained (described in detail above)

$$\frac{dv_\parallel}{dt} = -\left(\frac{v_\perp^2}{2}\right) \nabla_\parallel \ln B + v_\parallel \frac{-\boldsymbol{\nabla}\phi \times \widehat{\mathbf{n}}}{B} \cdot \boldsymbol{\nabla} \ln B - \frac{e}{m} \nabla_\parallel \phi$$

Then the equations are

$$\frac{rd\theta}{dt} = \Theta v_\parallel + \frac{1}{B_0}\frac{d\phi_0}{dr}$$

$$\frac{dr}{dt} = -\frac{1}{\Omega_c}\frac{\lambda + v_\parallel^2}{R}\sin\theta$$

$$\frac{d\lambda}{dt} = \lambda v_\parallel \Theta \sin\theta + \lambda\left(\frac{1}{B_0}\frac{d\phi_0}{dr}\right)\frac{1}{R}\sin\theta$$

$$\frac{dv_\parallel}{dt} = -\lambda\Theta\frac{1}{R}\sin\theta + v_\parallel\left(\frac{1}{B_0}\frac{d\phi_0}{dr}\right)\frac{1}{R}\sin\theta - \Theta\frac{e}{m}\frac{\partial\phi_1}{r\partial\theta}$$

Note the presence of the *variation of the potential in the magnetic surface.*

The equation to be solved (drift-kinetic) is

$$\frac{d\mathbf{x}}{dt}\cdot\boldsymbol{\nabla}f + \frac{dv_\parallel}{dt}\frac{\partial f}{\partial v_\parallel} + \frac{d\lambda}{dt}\frac{\partial f}{\partial\lambda} = 0$$

Wong Burrell estimate

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{v}_D &= \boldsymbol{\nabla}\cdot\left(\frac{1}{\Omega_c}\frac{\lambda + v_\parallel^2}{R}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\ln B\right) + O\left(\varepsilon^2\right) \\
&\approx \frac{1}{\Omega_c}\frac{\lambda + v_\parallel^2}{R}\;\frac{1}{B^2}\mathbf{J}\cdot\boldsymbol{\nabla}\ln B \\
&\sim \varepsilon^3\frac{v_D}{r} \\
&\quad \text{can be neglected}
\end{aligned}$$

This is strange since the divergence of the *diamagnetic* drift is the origin of the Pfirsch-Schluter *return* current.

The solution is expanded reflecting the existence of different, largely separated, frequency scales

$$f = f_0 + f_1 + ...$$

The first order of these equations

$$\frac{dx_\theta}{dt}\frac{\partial f_0}{\partial\theta} = C\left(f_0, f_0\right)$$

$$f_0 = \frac{n}{\left(\pi\ 2T/m\right)^{3/2}}\exp\left[-\frac{\left(v_\parallel - U\right)^2}{2T/m} - \frac{v_\perp^2}{2T/m}\right]$$

with

$$\begin{aligned}
U &\equiv U\left(r\right) \\
&\quad \text{parallel flow}
\end{aligned}$$

The next order linearized equation

$$
\begin{aligned}
& \left(v_{\|}+\frac{1}{B_\theta}\frac{d\phi_0}{dr}\right)\Theta\frac{\partial f_1}{r\partial\theta} \\
& +\left(v_{\|}-U\right)\Theta\frac{\partial}{r\partial\theta}\left(\frac{e\phi_1}{T}\right)\ f_0 \\
& -C_l\left(f_1\right) \\
= \quad & \Theta\left[\frac{v_\perp^2}{2T/m}\left(U+\frac{1}{B_\theta}\frac{d\phi_0}{dr}\right)+\frac{2v_{\|}\left(v_{\|}-U\right)}{2T/m}\frac{1}{B_\theta}\frac{d\phi_0}{dr}\right]\frac{\sin\theta}{R}\ f_0 \\
& +\frac{T}{eB_\theta}\left[\Theta\frac{m}{T}\left(v_{\|}^2+\frac{v_\perp^2}{2}\right)\frac{\sin\theta}{R}+\Theta\frac{\partial}{r\partial\theta}\left(\frac{e\phi_1}{T}\right)\right]\times \\
& \times\left[\frac{d}{dr}\ln n+\left(\frac{\left(v_{\|}-U\right)^2}{2T/m}+\frac{v_\perp^2}{2T/m}-\frac{3}{2}\right)\frac{d}{dr}\ln T+\frac{2\left(v_{\|}-U\right)\frac{dU}{dr}}{2T/m}\right]\ f_0
\end{aligned}
$$

We interpret this equation as follows
the first term refers to the first-order correction f_1 .

This first order function f_1 is assumed to correspond to a small variation of the distribution function in the magnetic surface. The reason of this variation is contained in the *drift* trajectories of the particles, which contain the trigonometric functions $\sin\theta$ and $\cos\theta$ and these occur because of the projection of the vertical drift velocity. Another reason for dependence of the full f with θ (*i.e.* in the magnetic surface) is the trapped particles, located in general in the low-field side.

With a radial variation of the potential $\phi_0(r)$ the expression $v_{\|}+\frac{1}{B_\theta}\frac{d\phi_0}{dr}$ is approximatively the parallel velocity, due to $v_{\|}$ and the $E\times B$ velocity. Then the factor

$$\left(v_{\|}+\frac{1}{B_\theta}\frac{d\phi_0}{dr}\right)\Theta$$

is the projection onto the poloidal direction of the parallel velocity. This will "move" the gradient along the poloidal direction, of f_1: $\mathbf{v}_\theta\cdot\boldsymbol{\nabla}_\theta f_1$.

The term

$$\left(v_{\|}-U\right)\Theta\frac{\partial}{r\partial\theta}\left(\frac{e\phi_1}{T}\right)\ f_0$$

does the same thing, but here the variation at this order comes from the potential

$$\phi=\phi_0(r)+\phi_1(r,\theta)$$

and the parallel velocity is

$$v_{\|}-U$$

which is further projected onto the poloidal direction

$$\left(v_{\|}-U\right)\Theta$$

The first line in the right hand side is

$$\Theta\left[\frac{v_\perp^2}{2T/m}\left(U+\frac{1}{B_\theta}\frac{d\phi_0}{dr}\right)+\frac{2v_\parallel\left(v_\parallel-U\right)}{2T/m}\frac{1}{B_\theta}\frac{d\phi_0}{dr}\right]\frac{\sin\theta}{R}\ f_0$$

and comes from the derivation of f_0 to the velocity.

This term reveals the energetic aspect of the *drift* motion of the particles. The drift motion is characterized by the variation of the parallel and of the perpendicular velocities.

The last two lines

$$\begin{aligned}&+\frac{T}{eB_\theta}\left[\Theta\frac{m}{T}\left(v_\parallel^2+\frac{v_\perp^2}{2}\right)\frac{\sin\theta}{R}+\Theta\frac{\partial}{r\partial\theta}\left(\frac{e\phi_1}{T}\right)\right]\times\\&\times\left[\frac{d}{dr}\ln n+\left(\frac{\left(v_\parallel-U\right)^2}{2T/m}+\frac{v_\perp^2}{2T/m}-\frac{3}{2}\right)\frac{d}{dr}\ln T+\frac{2\left(v_\parallel-U\right)\frac{dU}{dr}}{2T/m}\right]\ f_0\end{aligned}$$

come from

$$\frac{df_0}{dr}$$

which is from

$$\mathbf{v}_r\cdot\boldsymbol{\nabla}f_0$$

and we see that

$$\begin{aligned}&\frac{T}{eB_\theta}\times\Theta\frac{m}{T}\left(v_\parallel^2+\frac{v_\perp^2}{2}\right)\frac{\sin\theta}{R}\\=&\ \frac{1}{\Omega_c}\left(v_\parallel^2+\frac{v_\perp^2}{2}\right)\frac{\sin\theta}{R}\\=&\ \text{radial projection of } \mathbf{v}_D\end{aligned}$$

and

$$\begin{aligned}&\frac{T}{eB_\theta}\times\Theta\frac{\partial}{r\partial\theta}\left(\frac{e\phi_1}{T}\right)\\=&\ \frac{1}{B_0}\frac{\partial\phi_1}{r\partial\theta}\\=&\ \text{radial velocity induced by poloidal}\\&\ \text{variation of the potential in the}\\&\ \text{magnetic surface}\end{aligned}$$

NOTE

The developments in this approach reveal the *variation of the distribution function and of the electric potential* in the magnetic surface.

END

7 Drift kinetic equation and solution for collisional diffusion Rutherford 1970

The basic expansions and the basic analytical operations to obtain the solution to the drift kinetic equation.

Useful to study the parallel with **Helander ECRH**

The field

$$\begin{aligned}
\mathbf{B}_T &= B_T \widehat{\mathbf{e}}_\varphi \\
\mathbf{B}_\theta &= \boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\varphi = \boldsymbol{\nabla}\chi
\end{aligned}$$

The drift kinetic equation

$$\frac{\partial f_j}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} f_j + \frac{e}{m}\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)\frac{\partial f_j}{\partial \mathbf{v}} = \sum_k C_{jk}\left(f,f\right)$$

where

$$C_{jk}\left(f,f\right) = c_{jk}\frac{\partial}{\partial \mathbf{v}}\cdot\int d^3v' \, \frac{\partial^2 \mathbf{u}}{\partial \mathbf{v}\,\partial \mathbf{v}'}\cdot\left(f_k\left(\mathbf{v}'\right)\frac{\partial f_j\left(\mathbf{v}\right)}{\partial \mathbf{v}} - \frac{m_j}{m_k} f_j\left(\mathbf{v}\right)\frac{\partial f_k\left(\mathbf{v}'\right)}{\partial \mathbf{v}'}\right)$$

$$c_{jk} = 2\pi \frac{e_j^2 e_k^2}{m_j^2}\ln\Lambda$$

and

$$\mathbf{u} = \mathbf{v} - \mathbf{v}'$$

$$\frac{\partial^2 \mathbf{u}}{\partial \mathbf{v}\,\partial \mathbf{v}'} = \frac{1}{u^3}\left(u^2\mathbf{I} - \mathbf{u}\,\mathbf{u}\right)$$

The first step is to invoke the axisymmetry and its consequence *the conservation of the canonical angular momentum*

$$\frac{d}{dt}\left[Rmv_\varphi + R\, eA_\varphi\right] = 0$$

Now we have the connection

$$\psi = R\, A_\varphi$$

and from this it results that

$$\begin{aligned}
\psi' &= \psi + \frac{mRv_\varphi}{e} \\
&= \text{dynamical constant}
\end{aligned}$$

Another dynamical constant is the energy

$$\epsilon = \frac{v^2}{2} + \frac{e\phi}{m}$$

The distribution function f_j will be function of these invariants.

The distribution function for simply moving in external fields, in the absence of collisions is

$$f = f\left(\epsilon, \psi'\right)$$

And, when collisions are considered, it is assumed that the perturbation is additive

$$f = f\left(\epsilon, \psi'\right) + g$$

Next step is to fix the scales.

The parameter is

$$\rho/L$$

There are expansions of both the function $f\left(\epsilon, \psi'\right)$ in the absence of collisions and of the correction g due to the collisions

$$f\left(\epsilon, \psi'\right) = f^{(0)} + f^{(1)} + f^{(2)} + ...$$

$$g = g^{(1)} + g^{(2)} + ...$$

NOTE

a question of notation.

Usually the first order distribution function $f^{(1)}$ in the *neoclassical* drift kinetic equation is due to the drift of the particles, $\mathbf{v}_D$ and consists of the neoclassical term $\sim \rho_\theta/L$ and a function g that is determined by pitch angle scattering in velocity space, with $\partial g/\partial \mu$ calculated for *trapped* and for *passing* particles.

END

The first two lowest order terms for the distribution f (in the absence of collisions) are

$$f^{(0)} = f^{(0)}\left(\epsilon, \psi\right)$$

and

$$f^{(1)} = R \frac{1}{\left(\frac{e}{m}\right)} v_\varphi \frac{\partial f^{(0)}}{\partial \psi}$$

The lowest order is Maxwellian

$$f^{(0)} = \frac{N\left(\psi\right)}{\left(\sqrt{2\pi} v_{th}\right)^3} \exp\left(-\frac{\epsilon}{v_{th}^2}\right)$$

with the definition (**NOTE**)

$$v_{th}^2 = \frac{\kappa T}{m}$$

The density is

$$n = N \exp\left(-\frac{e\phi}{\kappa T}\right)$$

Another step in defining the structure of the problem: *neutrality*

$$n_e = n_i$$

The potential is constant on magnetic surfaces, $\phi^{(0)}$ =function of only ψ.

Orders

$$\begin{aligned}
\frac{e}{m}\mathbf{v}\times\mathbf{B}\cdot\frac{\partial}{\partial\mathbf{v}} &\rightarrow \text{zeroth order in } \frac{\rho}{L} \\
\mathbf{v}\cdot\boldsymbol{\nabla} &\rightarrow \text{first order} \\
\frac{e}{m}\left(-\boldsymbol{\nabla}\phi\right)\cdot\frac{\partial}{\partial\mathbf{v}} &\rightarrow \text{first order} \\
\frac{\partial}{\partial t} &\rightarrow \text{third order}
\end{aligned}$$

The third order is the scale where the *collisions* succeed to induce changes in the zeroth order plasma quantities.

According to **Hazeltine Hinton review** the third order is the time-variation of the magnetic surfaces due to diffusion.

Result of this scaling

$$\frac{e}{m}\mathbf{v}\times\mathbf{B}\cdot\frac{\partial g^{(1)}}{\partial\mathbf{v}} = C\left(f^{(0)}, f^{(0)}\right) = 0$$

and

$$\begin{aligned}
& \frac{e}{m}\mathbf{v}\times\mathbf{B}\cdot\frac{\partial g^{(2)}}{\partial\mathbf{v}} \\
& +\mathbf{v}\cdot\boldsymbol{\nabla}g^{(1)} + \frac{e}{m}\left(-\boldsymbol{\nabla}\phi\right)\cdot\frac{\partial g^{(1)}}{\partial\mathbf{v}} \\
= \; & C\left(f^{(0)}, f^{(1)}\right) + C\left(f^{(0)}, g^{(1)}\right)
\end{aligned}$$

The collision operator can be linearized

$$\begin{aligned}
& C_{jk}\left(f^{(0)}, f^{(1)}\right) \\
= \; & c_{jk}\frac{\partial}{\partial\mathbf{v}}\cdot\int d^3v' \; \frac{\partial^2\mathbf{u}}{\partial\mathbf{v}\,\partial\mathbf{v}'}\cdot\left[f_k^{(0)}\left(\mathbf{v}\frac{f_j^{(1)}}{v_{th,j}^2}+\frac{\partial f_j^{(1)}}{\partial\mathbf{v}}\right)-\frac{m_j}{m_k}f_j^{(0)}\left(\mathbf{v}'\frac{f_k^{(1)}}{v_{th,k}^2}+\frac{\partial f_k^{(1)}}{\partial\mathbf{v}'}\right)\right]
\end{aligned}$$

Chnage of coordinates

$$(t,\mathbf{x},\mathbf{v}) \rightarrow (t,\mathbf{x},\epsilon,\mu,\zeta)$$

and can be found in the file *plasma general theory, derivation drift kinetic.*

The equation for $g^{(2)}$ is

$$
\begin{aligned}
g^{(2)} &= -\frac{1}{\Omega}\widehat{\mathbf{n}}\times\mathbf{v}\cdot\boldsymbol{\nabla}g^{(1)} \\
&\quad -\left[\mathbf{v}\cdot\mathbf{v}_D+\frac{v_\|}{\Omega}\int^{\zeta}d\zeta\left(\mathbf{v}_\perp\ \mathbf{v}_\perp:\boldsymbol{\nabla}\widehat{\mathbf{n}}-\frac{v_\perp^2}{2}\boldsymbol{\nabla}\cdot\widehat{\mathbf{n}}\right)\right]\frac{\partial g^{(1)}}{B\partial\mu} \\
&\quad +g^{(2)\prime}
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{v}_D &= \frac{1}{\Omega}\widehat{\mathbf{n}}\times\left(\mu\boldsymbol{\nabla}B+v_\|^2\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}+\frac{e}{m}\boldsymbol{\nabla}\phi\right) \\
&= -v_\|\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\left(\frac{v_\|}{\Omega}\right)
\end{aligned}
$$

By introducing a new function $g^{(2)\prime}$ the equation at order 1 (for $g^{(1)}$) becomes

$$
\begin{aligned}
& v_\|\nabla_\|g^{(1)}-\Omega\frac{\partial g^{(2)\prime}}{\partial\zeta} \\
&= C\left(f^{(0)},f^{(1)}\right)+C\left(f^{(0)},g^{(1)}\right)
\end{aligned}
$$

Now for the functions $g^{(0)}$, $g^{(1)}$, $g^{(2)\prime}$, etc, it is explicitely separated a part that depends on ζ and a part that is averaged over ζ

$$g=\overline{g}+\widetilde{g}$$

where

$$\overline{g}=\frac{1}{2\pi}\int_0^{2\pi}d\zeta\ g$$

Using the *averaging* the equation is splitted into two equations

$$v_\|\nabla_\|\overline{g}^{(1)}=\overline{C}\left(f^{(0)},f^{(1)}\right)+\overline{C}\left(f^{(0)},\overline{g}^{(1)}\right)$$

since

$$
\begin{aligned}
\widetilde{g}^{(1)} &= 0 \ \text{ because} \\
-\Omega\frac{\partial g^{(1)}}{\partial\zeta} &= 0 \ \text{ (the first order equation)}
\end{aligned}
$$

The second equation is

$$-\Omega\frac{\partial\widetilde{g}^{(2)}}{\partial\zeta}=\widetilde{C}\left(f^{(0)},f^{(1)}\right)+\widetilde{C}\left(f^{(0)},g^{(1)}\right)$$

The equations are rewritten after showing that the collision operator acting on gyrophase dependent parts are zero.

$$v_\|\nabla_\|\overline{g}^{(1)}=C\left(f^{(0)},\overline{f}^{(1)}\right)+C\left(f^{(0)},\overline{g}^{(1)}\right)$$

and

$$-\Omega \frac{\partial \widetilde{g}^{(2)\prime}}{\partial \zeta} = C\left(f^{(0)}, \widetilde{f}^{(1)}\right)$$

The averaged part is

$$\overline{f}^{(1)} = RB_T \frac{1}{\left(\frac{eB_T}{m}\right)} v_\parallel \frac{\partial f^{(0)}}{\partial \psi}$$

and the gyrophase dependent part

$$\widetilde{f}^{(1)} = \frac{1}{\Omega} \left(\widehat{\mathbf{n}} \times \boldsymbol{\nabla}\psi\right) \cdot \mathbf{v} \; \frac{\partial f^{(0)}}{\partial \psi}$$

Equation for g is

$$\left\langle \frac{\partial}{\partial \lambda} \lambda v_\parallel \frac{\partial}{\partial \lambda} \left(g - I \frac{v_\parallel}{\Omega} \frac{\partial f_0}{\partial \psi} \right) \right\rangle = 0$$

This formula takes the following form in **Rutherford collisional diffusion 1970**

$$\begin{aligned} \frac{\partial}{\partial \mu} \left(\mu \frac{\partial g_{1e}^{(0)}}{\partial \mu} \oint \frac{v_\parallel \, d\chi}{B_\perp^2} \right) + 2RB_T \frac{1}{\left(\frac{e}{m_e}\right)} \frac{1}{n} \frac{\partial n}{\partial \psi} \oint \frac{d\chi}{B_\perp^2} &= 0 \\ \text{particles} &= \text{circulating} \end{aligned}$$

where (using a change of notation here $B_\perp^{Ruth} \equiv B_\theta^{here}$)

$$\begin{aligned} \mathbf{B}_\theta &= \boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\varphi \;\; \text{poloidal component} \\ &= \boldsymbol{\nabla}\chi \;\; \text{derived from a potential } \chi \end{aligned}$$

The annihilator is

$$\oint \frac{dl}{B}$$

and in axisymmetric system

$$\oint \frac{d\chi}{B_\theta^2}$$

Integrating the drift kinetic equation for $g_{1e}^{(0)}$ in μ, it is obtained for CIRCULATING particles

$$\begin{aligned} \frac{\partial g_{1e}^{(0)}}{\partial \mu} &= -2RB_T \frac{1}{\left(\frac{e}{m_e}\right)} f_{0e} \frac{1}{n} \frac{\partial n}{\partial \psi} \; \frac{\oint \frac{d\chi}{B_\theta^2}}{\oint \frac{d\chi}{B_\theta^2} v_\parallel} \\ & \text{passing} \end{aligned}$$

with the condition that $g_{1e}^{(0)}$ is finite at $\mu = 0$.

For *TRAPPED* particles

$$\begin{aligned}\frac{\partial g_{1e}^{(0)}}{\partial \mu} &= 0 \\ g_{1e}^{(0)} &= g_T = \text{const}\end{aligned}$$

Now for *passing* particles, integrating in μ

$$g_{1e}^{(0)} = g_T - \int_{\mu}^{\mu_0} d\mu \, \frac{\partial g_{1e}^{(0)}}{\partial \mu}$$

So there is *continuity* of the function.

But NOT of the derivative to μ.

8 Solution of the drift - kinetic equation Rosenbluth Hazeltine Hinton

This is **PF15, 1972, 116**.

8.1 Basic aproach with entropy functional

The starting equation is

$$\frac{d}{dt}\left(R_0 h m v_{\varphi} - R_0 \left|e\right| \int^r b\left(r\right) dr\right) = 0$$

we note $R = R_0 h$.

(since this is

$$R\left(p - eA\right) = ct$$

where

$$h = 1 + \varepsilon \cos\theta$$

$$\int^r b\left(r\right) dr = A_{\varphi}$$

From the first term, retaining only static convection part

$$\begin{aligned}\frac{d}{dt}\left(h m v_{\varphi}\right) &\approx m\mathbf{v}\cdot\boldsymbol{\nabla}\left(h v_{\|}\right) \\ &\approx m v_{\|} \frac{B_{\theta}}{B} \frac{\partial}{r\partial\theta}\left(h v_{\|}\right)\end{aligned}$$

where

$$B \equiv \left|B\right| \approx \frac{B_0}{h}$$

$$v_{\parallel} = \sqrt{\frac{2}{m}\left(\epsilon - |e|\,\phi - \mu B\right)}$$

$$\mu = \frac{m v_{\perp}^{2}}{2B}$$

Note that we can recognize

$$\frac{B_{\theta}}{B} = \Theta$$

and the similarity

$$v_{\parallel}\frac{\partial}{r\partial\theta}\left(h v_{\parallel}\right) \rightarrow v_{D}$$

End.

Now the derivation to time (convective) of the second term

$$\begin{aligned} R_0 \frac{d}{dt}\left(|e| \int^{r} b(r)\, dr\right) &= R_0 \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)\left(|e| \int^{r} b(r)\, dr\right) \\ &= R_0 \,|e|\, \frac{\partial A_{\varphi}}{\partial t} \\ &\quad + R_0\,|e|\,\frac{dr}{dt} b(r) \end{aligned}$$

We replace

$$\frac{dr}{dt} = v_{Dr}$$

and factorize the constant R_0, the seocnd term is

$$R_0 \left(|e|\, \frac{\partial A_{\varphi}}{\partial t} + |e|\, v_{Dr} b(r)\right)$$

and the result

$$\begin{aligned} \frac{d}{dt}\left(R_0 h m v_{\varphi} - R_0\,|e| \int^{r} b(r)\, dr\right) &= 0 \\ R_0 \left(m v_{\parallel} \frac{B_{\theta}}{B} \frac{\partial}{r\partial\theta}\left(h v_{\parallel}\right)\right) - R_0 \left(|e|\, \frac{\partial A_{\varphi}}{\partial t} + |e|\, v_{Dr} b(r)\right) &= 0 \end{aligned}$$

here we replace

$$E_{\varphi} = -\frac{\partial A_{\varphi}}{\partial t}$$

$$m v_{\parallel} \frac{B_{\theta}}{B} \frac{\partial}{r\partial\theta}\left(h v_{\parallel}\right) + |e|\, E_{\varphi} - |e|\, v_{Dr} b(r) = 0$$

In the absence of the electric field $E_{\varphi} = 0$, and taking into account that

$$\begin{aligned} B_{\theta} &= \frac{b(r)}{h} \\ B &= \frac{B_0}{h} \end{aligned}$$

we have

$$
\begin{aligned}
v_{Dr} &= \frac{1}{|e|\,b(r)} m v_\parallel \frac{B_\theta}{B} \frac{\partial}{r\partial\theta}\left(h v_\parallel\right) \\
&= \frac{m}{|e|\,B_0} v_\parallel \frac{\partial}{r\partial\theta}\left(h v_\parallel\right) \\
&= \frac{1}{\Omega_c} v_\parallel \frac{\partial}{r\partial\theta}\left(h v_\parallel\right)
\end{aligned}
$$

with the definition

$$\Omega_c = \frac{|e|\,B_0}{m}$$

Other parameters

$$
\begin{aligned}
\lambda &\equiv \frac{\mu B_0}{\epsilon - |e|\,\phi} = \frac{m v_\perp^2}{2B} B_0 \frac{1}{m v^2/2} \\
&= h \frac{v_\perp^2}{v^2}
\end{aligned}
$$

Next one introduces

$$\frac{v_\perp^2}{v^2} \equiv \sin^2\zeta$$

$$\lambda = h \sin^2\zeta$$

The parallel velocity

$$v_\parallel = |v| \sqrt{1 - \frac{\lambda}{h}}$$

For *trapping* it is necessary that $v_\parallel$ becomes zero. This is possible only if

$$
\begin{aligned}
1 + \varepsilon\cos\theta - \lambda &= 0 \\
\cos\theta &= \frac{\lambda - 1}{\varepsilon}
\end{aligned}
$$

but

$$
\begin{aligned}
-1 &< \cos\theta < 1 \\
-1 &< \frac{\lambda - 1}{\varepsilon} < 1
\end{aligned}
$$

or

$$
\begin{aligned}
1 - \frac{r}{R} &< \lambda < 1 + \frac{r}{R} \\
& \quad \text{circulating}
\end{aligned}
$$

This is the CIRCULATING region.

The distribution function is expanded

$$\begin{aligned} f &= f_0\left(1+\widehat{f}\right) = f_M + f_M\ \widehat{f} \\ \widehat{f} &\sim O\left(\frac{\rho}{l}\right) \\ f_0 &= \frac{n}{\left(\pi\ 2T/m\right)^{3/2}} \exp\left[-\frac{\epsilon - |e|\,\phi}{T}\right] \end{aligned}$$

the drift-kinetic equation is

$$v_\parallel \frac{B_\theta}{B} f_0 \frac{\partial \widehat{f}}{r\partial\theta} + v_{Dr}\frac{\partial f_0}{\partial r} + |e|\, E_\parallel v_\parallel \frac{\partial f_0}{\partial \epsilon} = 0$$

since

$$\frac{\partial \epsilon}{\partial t} = |e|\, E_\parallel v_\parallel$$

$$\begin{aligned} \frac{dr}{dt} &= v_{Dr} \\ \frac{rd\theta}{dt} &= v_{\parallel\theta} + v_{D\theta} \end{aligned}$$

Replacing the Maxwellian

$$f_0 \rightarrow f_M$$

$$f_M \left\{ v_{Dr}\left[A_1 + A_2\left(\epsilon - |e|\,\phi\right)\right] + v_\parallel \left(\frac{B_\theta}{B}\frac{\partial \widehat{f}}{r\partial\theta} - \frac{|e|\, E_\parallel}{T}\right)\right\} = C\left(f\right)$$

where

$$\begin{aligned} A_1 &= \frac{n'}{n} - \frac{3}{2}\frac{T'}{T} + \frac{|e|\,\phi'}{T} \\ A_2 &= \frac{T'}{T^2} \end{aligned}$$

with $' \equiv \frac{d}{dr}$. These are the "forces" that drive the "fluxes".

To solve this equation, the first step consists of *extracting* a Spitzer function f_s from $\widehat{f}$. This means to consider the definition of a function f_s,

$$-\frac{|e|}{T} E_\parallel v_\parallel f_M = C\left(E_\parallel v_\parallel f_s\right)$$

Then we may replace the last term in the equation for $\widehat{f}$ above,

$$-f_M v_\parallel \frac{|e|\, E_\parallel}{T}$$

by

$$C\left(E_\parallel v_\parallel f_s\right)$$

On the other hand the collision operator is *linearized* and then the arguments can be combined. We have

$$\begin{aligned} & C(f) - C\left(E_{\|} v_{\|} f_s\right) \\ = \; & C\left(f - E_{\|} v_{\|} f_s\right) \end{aligned}$$

and the equation is

$$f_M \left\{ v_{Dr} \left[A_1 + A_2 \left(\epsilon - |e| \phi \right) \right] + v_{\|} \frac{B_\theta}{B} \frac{\partial \widehat{f}}{r \partial \theta} \right\} = C\left(f - E_{\|} v_{\|} f_s\right)$$

It is an equation for $\widehat{f}$.

Now it occurs the possibility to exploit the separation of time scales: collisions ν_{eff} are rare relatively to the bounce frequency ω_B. This should be reflected in an expansion

$$\widehat{f} = \widehat{f}^{(0)} + \widehat{f}^{(1)} + ...$$

where the first order allows to neglect collisions.

$$v_{\|} \frac{B_\theta}{B} \frac{\partial \widehat{f}^{(0)}}{r \partial \theta} = -v_{Dr} \left[A_1 + A_2 \left(\epsilon - |e| \phi \right) \right] f_M$$

and we use

$$v_{Dr} = \frac{1}{\Omega_c} v_{\|} \frac{\partial}{r \partial \theta} \left(h v_{\|} \right)$$

then

$$v_{\|} \frac{B_\theta}{B} \frac{\partial \widehat{f}^{(0)}}{r \partial \theta} = -\frac{1}{\Omega_c} v_{\|} \frac{\partial}{r \partial \theta} \left(h v_{\|} \right) \left[A_1 + A_2 \left(\epsilon - |e| \phi \right) \right] f_M$$

with $B_\theta / B = b(r) / B_0$ independent of θ, as it is Ω_c.

Then, since the differential operator $v_{\|} \frac{\partial}{r \partial \theta}$ is the same in both terms, we factorize it and integrate on θ and obtain

$$\begin{aligned} \widehat{f}^{(0)} \; = \; & \frac{m}{|e| b} h v_{\|} \left[A_1 + A_2 \left(\epsilon - |e| \phi \right) \right] f_M \\ & + g\left(\mu, \epsilon, \sigma\right) \end{aligned}$$

where g is an "constant of integration" to the θ integration.

The method of solving this equation goes through the *extremization of the rate of generation of entropy*.

For this a particular operator, dependent on four distribution function, is prepared.

This will serve for representation of the collisional operator. The later is the source of generation of *entropy*.

The functions are denoted by two symbols, f and g (no connection with the previous g in $\widehat{f}$).

$$\begin{aligned} f_a &= f_{Ma}\left(1+\widehat{f}_a\right) \\ g_a &= f_{Ma}\left(1+\widehat{g}_a\right) \\ a &\equiv \text{species, } e, i \end{aligned}$$

and with them we define

$$K(f,g) = -\sum_a \int d^3v \ \widehat{f}_a C_a(g_a)$$

with the definition

$$\begin{aligned} C_a(g) &= C_{aa}(\widehat{g}_a, \widehat{g}_a) \\ &\quad + C_{ab}(\widehat{g}_a, \widehat{g}_b) \end{aligned}$$

K is the *local rate of entropy production.*

$$\frac{\partial S}{\partial t} = K(f,f)$$

$$\frac{\partial S}{\partial t} = -\int \frac{hd\theta}{2\pi} \sum_a \int d^3v \left(\widehat{f}_a - v_\parallel E_\parallel \widehat{f}_{sa}\right) \ C_a\left(\widehat{f} - v_\parallel E_\parallel \widehat{f}_s\right)$$

The operator of collisions is of *Fokker-Planck* form

$$\begin{aligned} C_{ab}(f) &= -\int d^3\mathbf{v}_b' \int d\Omega \ \sigma_{ab}(\Omega) \\ &\quad \times \left|\mathbf{v}_a - \mathbf{v}_b'\right| \\ &\quad \times \left[f_a(\mathbf{v}_a) f_b(\mathbf{v}_b) - f_a(\mathbf{v}_a'') f_b(\mathbf{v}_b'')\right] \end{aligned}$$

where

$$\begin{aligned} \mathbf{v}_a, \mathbf{v}_b' &\equiv \text{velocities before collisions} \\ \mathbf{v}_a'', \mathbf{v}_b'' &\equiv \text{velocities after collisions} \end{aligned}$$

The *prime* on $\mathbf{v}_b \rightarrow \mathbf{v}_b'$ just shows that it is a variable of integration.

$$\sigma_{ab} \equiv \text{cross section}$$

To *linearize* the collision operator, one uses the linear expansions of f and g.

Equal temperatures

For equilibium, at equal temperatures, one has

$$\begin{aligned} T_a &= T_b \\ f_{Ma}(\mathbf{v}_a) f_{Mb}(\mathbf{v}_b') &= f_{Ma}(\mathbf{v}_a'') f_{Mb}(\mathbf{v}_b'') \end{aligned}$$

The exponential expression of f_M allows a direct application of the *energy conservation*.

The *linearized* collision operator

$$\begin{aligned} C_{ab} &= -\int d^3v_b' \int d\Omega \ \sigma_{ab}(\Omega) \\ &\quad \times |\mathbf{v}_a - \mathbf{v}_b'| \\ &\quad \times f_{Ma}(\mathbf{v}_a) f_{Mb}(\mathbf{v}_b') \\ &\quad \times \left[\widehat{f}_a(\mathbf{v}_a) + \widehat{f}_b(\mathbf{v}_b') - \widehat{f}_a(\mathbf{v}_a'') - \widehat{f}_b(\mathbf{v}_b'')\right] \end{aligned}$$

On this basis one derives the property of *self-adjointness* of the operator

$$K(f,g) = K(g,f)$$

The expression is

$$\begin{aligned} & K(f,g) \\ = & \sum_{a,b} \int d^3v_a \int d^3v_b' \\ & \times \int d\Omega \ \sigma_{ab}(\Omega) \\ & \times |\mathbf{v}_a - \mathbf{v}_b'| \\ & \times f_{Ma}(\mathbf{v}_a) f_{Mb}(\mathbf{v}_b') \\ & \times \widehat{f}_a \left(\widehat{g}_a + \widehat{g}_b' - \widehat{g}_a'' - \widehat{g}_b''\right) \end{aligned}$$

with the short notation

$$\widehat{g}_a'' \equiv \widehat{g}_a(\mathbf{v}_a'')$$

Due to the self-adjointness, one will rewrite the expression as

$$\begin{aligned} & K(f,g) \\ = & \frac{1}{4}\sum_{a,b} \int d^3v_a \int d^3v_b' \\ & \times \int d\Omega \ \sigma_{ab}(\Omega) \\ & \times |\mathbf{v}_a - \mathbf{v}_b'| \\ & \times f_{Ma}(\mathbf{v}_a) f_{Mb}(\mathbf{v}_b') \\ & \times \left(\widehat{f}_a + \widehat{f}_b' - \widehat{f}_a'' - \widehat{f}_b''\right)\left(\widehat{g}_a + \widehat{g}_b' - \widehat{g}_a'' - \widehat{g}_b''\right) \end{aligned}$$

How to deal with the collision operator.

We take into account that the collisions are mainly *small-angle*.

One introduces the *relative velocity*

$$\mathbf{v} \equiv \mathbf{v}_a - \mathbf{v}_b$$

and the change $\mathbf{\Delta v}$ with which one calculates

$$\begin{aligned}
\Delta \mathbf{v}_a &\equiv \mathbf{v}_a' - \mathbf{v}_a = \frac{m_b}{m_a + m_b} \Delta \mathbf{v} \\
\Delta \mathbf{v}_b &= -\frac{m_a}{m_a + m_b} \Delta \mathbf{v}
\end{aligned}$$

The operator is expanded in *small quantity* $|\Delta \mathbf{v}|$ as

$$\begin{aligned}
K(f,g) &= \frac{1}{4} \sum_{a,b} \frac{1}{(m_a + m_b)^2} \\
&\times \int d^3 v_a \int d^3 v_b' \\
&\times |\mathbf{v}| \\
&\times f_{Ma}(\mathbf{v}_a) f_{Mb}(\mathbf{v}_b') \\
&\times \overline{\Delta v_\alpha \Delta v_\beta} \\
&\times \left(m_b \frac{\partial \widehat{f}_a}{\partial v_{a\beta}} - m_a \frac{\partial \widehat{f}_b}{\partial v_{b\beta}'} \right) \left(m_b \frac{\partial \widehat{g}_a}{\partial v_{a\alpha}} - m_a \frac{\partial \widehat{g}_b}{\partial v_{b\alpha}'} \right)
\end{aligned}$$

where

$$\overline{\Delta v_\alpha \Delta v_\beta} = \int d\Omega \; \sigma_{ab}(\Omega) \; \Delta v_\alpha \Delta v_\beta$$

with the Coulomb cross section

$$\sigma_{ab} = \frac{e^4}{4 m_{ab}^2 v^4} \frac{1}{\sin^4\left(\frac{1}{4}\xi\right)}$$

where

$$\xi \equiv \text{angle over which } \mathbf{v} \text{ turns in a collision}$$

NOTE

that at this moment one introduces *derivatives in the velocity space.*

This is because we have made an *expansion* in the small parameter "angle of scattering", or $\mathbf{\Delta v}$.

Then what matters is the variation of the distribution function in the velocity space.

The relative differences: trapped - circulating become important.

But *spatial* variations (gradients of n and T) are not important. It still is included in Maxwellian functions f_M.

END

Then

$$\begin{aligned}
\overline{\Delta v_\alpha \Delta v_\beta} &= \frac{4\pi e^4 \ln \Lambda}{m_{ab}^2 v^4} \\
&\times \left[\delta_{\alpha\beta} - \frac{v_\alpha v_\beta}{v^2} \right]
\end{aligned}$$

It is introduced the notation

$$V_{\alpha\beta} = \frac{1}{v^3}\left(v^2\delta_{\alpha\beta} - v_\alpha v_\beta\right)$$

We **Note** that by this factorization of powers of v at the denominator

$$\begin{aligned} & \frac{1}{v^4\,(\text{Coulomb cross-section})}\left[\delta_{\alpha\beta} - \frac{v_\alpha v_\beta}{v^2}\right] \\ \rightarrow \quad & \frac{1}{v^3} \times \frac{1}{v}\left[\delta_{\alpha\beta} - \frac{v_\alpha v_\beta}{v^2}\right] \\ = \quad & \frac{1}{v^3}\; V_{\alpha\beta} \end{aligned}$$

the tensor $V_{\alpha\beta}$ gets a $1/v$ factor and it remains in the main coefficient a dependence of the order

$$\frac{1}{v^3}$$

which is the typical dependence of the frequency of collisions. **End.**

And

$$\begin{aligned} & K\left(f,g\right) \\ = \quad & \pi e^4 \ln\Lambda \\ & \times \sum_{a,b} \int d^3 v_a \int d^3 v'_b \\ & \times f_{Ma}\left(\mathbf{v}_a\right) f_{Mb}\left(\mathbf{v}'_b\right) \\ & \times V_{\alpha\beta} \\ & \times \left(\frac{1}{m_a}\frac{\partial \widehat{f}_a}{\partial v'_{a\beta}} - \frac{1}{m_b}\frac{\partial \widehat{f}_b}{\partial v'_{b\beta}}\right)\left(\frac{1}{m_a}\frac{\partial \widehat{g}_a}{\partial v_{a\alpha}} - \frac{1}{m_b}\frac{\partial \widehat{g}_b}{\partial v'_{b\alpha}}\right) \end{aligned}$$

The rate of entropy production can be expressed as

$$\frac{dS}{dt} = \int \frac{h d\theta}{2\pi} K\left(f_1, f_1\right)$$

where one has defined

$$f_{1a} \equiv \widehat{f}_a - v_\parallel E_\parallel \widehat{f}_{sa} - D m_a v_\parallel$$

so there is a shift that we assume proportional with the parallel momentum of the particle, $m_a v_\parallel$ with a coefficient to be determined D from the condition of extremum of $\dot{S}$.

The function f_{1a} must be localised in the velocity space on a *small support*, which corresponds to the region close to the trapped-circulating transition.

Every velocity integration in the expression of $K(f_1, f_1)$ will reduce the amplitude of the answer with a factor $(r/R)^{1/2}$, which is the fraction of the velocity space occupied by the trapped particles. Then higher orders terms in the small parameter $(r/R)^{1/2}$ will be neglected.

$$\begin{aligned}
& K(f_1, f_1) \\
\approx \; & \pi e^4 \ln \Lambda \\
& \times \sum_{a,b} \int d^3 v_a \int d^3 v_b' \\
& \times f_{Ma}(\mathbf{v}_a) f_{Mb}(\mathbf{v}_b') \\
& \times V_{\alpha\beta} \\
& \times \left(\frac{1}{m_a^2} \frac{\partial f_{1a}}{\partial v_{a\beta}} \frac{\partial f_{1a}}{\partial v_{a\alpha}} + \frac{1}{m_b^2} \frac{\partial f_{1b}}{\partial v_{b\beta}'} \frac{\partial f_{1b}}{\partial v_{b\alpha}'} \right)
\end{aligned}$$

which is

$$K(f_1, f_1) \approx 2\pi e^4 \ln \Lambda \sum_{a,b} \frac{1}{m_a^2} \int d^3 v_a f_{Ma} \frac{\partial f_{1a}}{\partial v_{a\alpha}} \frac{\partial f_{1a}}{\partial v_{a\beta}} \int d^3 v_b f_{Mb} V_{\alpha\beta}$$

Now we change to the new variables in the velocity space

$$\begin{aligned}
w &\equiv \frac{v_a^2}{2} \\
&= \frac{1}{m_a} (\epsilon - |e| \phi)
\end{aligned}$$

$$\begin{aligned}
\lambda &\equiv h \frac{v_\perp^2}{v^2} \\
&= \frac{\mu B_0}{m_a w}
\end{aligned}$$

and the volume integral

$$\int d^3 v = \sum_\sigma \int \pi \frac{\sqrt{2w}}{h\sqrt{1 - \lambda/h}} dw d\lambda$$

Use the locality in the velocity space, which suggests that the most important variation takes place around the *trapping/ passing region* in *velocity space*; the variation with the energy is much weaker

$$\frac{\partial f_{1a}}{\partial \lambda} \gg \frac{\partial f_{1a}}{\partial w}$$

This allows to change the variable of derivation, from $v_{a,\alpha}$ to λ.

$$\begin{aligned}\frac{\partial f_{1a}}{\partial v_{a\alpha}} &\approx -\sigma\sqrt{\frac{2}{w}\left(1-\frac{\lambda}{h}\right)}n_\alpha\frac{\partial f_{1a}}{\partial\lambda}\\ &= -\sigma 2\frac{v_\parallel}{v^2}n_\alpha\frac{\partial f_{1a}}{\partial\lambda}=-\sigma\frac{2\xi}{v}n_\alpha\frac{\partial f_{1a}}{\partial\lambda}\end{aligned}$$

where

$$n_\alpha \equiv \text{component } \alpha \text{ of the versor } \widehat{\mathbf{n}}$$

and to replace in the expression of K.

The rate of production of entropy becomes

$$\begin{aligned}&K(f_1,f_1)\\ =\;&4\pi^2e^4\ln\Lambda\\ &\times\sum_a\frac{1}{m_a^2}\sum_\sigma\int\frac{dwd\lambda}{hw}\sqrt{2\omega}\sqrt{1-\frac{\lambda}{h}}f_{Ma}\left(\frac{\partial f_{1a}}{\partial\lambda}\right)^2\\ &\times\sum_b F_b(w)\end{aligned}$$

where

$$F_b(w)\equiv\int d^3v f_{Mb}n_\alpha n_\beta V_{\alpha\beta}$$

In **Rosenbluth Hazeltine Hinton** it is calculated

$$F_b(w)=\frac{\sqrt{2}}{4}\frac{nT}{m_b}w^{-3/2}\left\{\sqrt{\frac{m_bw}{T}}E'\left(\sqrt{\frac{m_bw}{T}}\right)+\left(\frac{2m_bw}{T}-1\right)E\left(\sqrt{\frac{m_bw}{T}}\right)\right\}$$

where

$$\begin{aligned}E(x) &\equiv \text{error function}\\ &= \frac{2}{\sqrt{\pi}}\int_0^x dy\exp\left(-y^2\right)\end{aligned}$$

Note the variable

$$\begin{aligned}\frac{m_bw}{T} &= \frac{v^2}{2}\frac{1}{T/m_b}\\ &= \frac{v^2}{v_{th,b}^2}\\ &\equiv x\end{aligned}$$

the notation appears sometimes. **End.**

Here one applies the condition of extremum of entropy production

$$\frac{\delta\dot{S}}{\delta\left(\frac{\partial f_{1a}}{\partial\lambda}\right)}=0$$

and it results

$$\sum_{\sigma}^{(trapped)} \int \frac{d\theta}{2\pi} \sqrt{1-\frac{\lambda}{h}} \frac{\partial f_{1a}}{\partial \lambda} = 0$$

The symbol $\sum_{\sigma}^{(trapped)}$ means that the sum over the directions of the parallel velocity σ only appears in the trapped region. The integrand is

$$\frac{v_{\parallel}}{v} \frac{\partial f_{1a}}{\partial \lambda} \equiv \xi \frac{\partial f_{1a}}{\partial \lambda}$$

Returning to the definition of f_{1a}, where the zero order (in ν/ω_{bounce}) has been found above

$$\begin{aligned} \widehat{f}_{1a}^{(0)} &= \frac{m}{|e|\, b} h v_{\parallel} \left[A_1 + A_2 \left(\epsilon - |e|\, \phi\right)\right] f_M \\ &\quad + g\left(\mu, \epsilon, \sigma\right) \end{aligned}$$

(**note** $\frac{b}{h} = B_\theta$) one finds for the first correction f_1

$$\begin{aligned} \frac{\partial f_{1a}}{\partial \lambda} &= -\sigma G_a\left(w\right) \frac{\sqrt{\frac{w}{2}}}{\sqrt{1-\frac{\lambda}{h}}} \\ &\quad + \frac{\partial g_a}{\partial \lambda} \end{aligned}$$

where G is only composed from the first term, neoclassic, $\sim A_{1,2,3}$. The factors appear due to the derivation to λ. For this factor,

$$\begin{aligned} v_{\parallel} &= v\sqrt{1-\frac{v_{\perp}^2}{v^2} h \times \frac{1}{h}} = v\sqrt{1-\frac{\lambda}{h}} \\ \frac{\partial v_{\parallel}}{\partial \lambda} &= v\left(-\frac{1}{2}\right) \frac{-\frac{1}{h}}{\sqrt{1-\frac{\lambda}{h}}} = \frac{v}{2h} \frac{1}{\sqrt{1-\frac{\lambda}{h}}} = \frac{1}{h} \frac{\sqrt{\frac{w}{2}}}{\sqrt{1-\frac{\lambda}{h}}} \end{aligned}$$

this is the factor in front of

$$\begin{aligned} G_a\left(w\right) &= -\frac{m_a}{e_a b\left(r\right)} \left(A_{1a} + A_2 m_a w\right) \\ &\quad - A_3 \widehat{f}_{sa} \quad \left(\text{Spitzer and } A_3 = E^{\parallel}\right) \\ &\quad - m_a D \quad \left(D \text{ is a parameter}\right) \end{aligned}$$

NOTE
that the function

$$\widehat{f}_{sa}$$

is *known* since it is the Spitzer-harm function, tabulated.

Then the function $G_a(w)$ is fully determined in terms of given profiles of parameters. It can be taken as known.

END.

The condition of extremum of the functional entropy production $\sum_{\sigma}^{(trapped)} \int \frac{d\theta}{2\pi}\sqrt{1-\frac{\lambda}{h}}\frac{\partial f_{1a}}{\partial\lambda} = 0$ becomes

$$\sum_{\sigma}^{(trapped)} \int \frac{d\theta}{2\pi}\sqrt{1-\frac{\lambda}{h}}\left(-\sigma G_a(w)\frac{\sqrt{\frac{w}{2}}}{\sqrt{1-\frac{\lambda}{h}}}+\frac{\partial g_a}{\partial\lambda}\right)=0$$

$$\sum_{\sigma}^{(trapped)}\left[-\sigma G_a(w)\sqrt{\frac{w}{2}}+\frac{\partial g_a}{\partial\lambda}\left\langle\left(1-\frac{\lambda}{h}\right)^{1/2}\right\rangle\right]=0$$

with the average

$$\langle f\rangle \equiv \int_0^{2\pi}\frac{d\theta}{2\pi}f$$

The solution is

$$\frac{\partial g_a}{\partial\lambda}=\frac{\sigma\sqrt{\frac{w}{2}}G_a(w)}{\left\langle\sqrt{1-\lambda/h}\right\rangle}\quad \text{untrapped } \lambda$$

$$\frac{\partial g_a}{\partial\lambda}=0\quad \text{trapped } \lambda$$

and

$$\begin{aligned}\frac{\partial f_{1a}}{\partial\lambda} &= \sigma\sqrt{\frac{w}{2}}G_a(w)\\ &\times\left[\frac{1}{\left\langle\sqrt{1-\lambda/h}\right\rangle_{untrapped}}-\frac{1}{\sqrt{1-\lambda/h}}\right]\end{aligned}$$

replacing the *untrapped* term by zero in the trapped region.

With this expression for $\frac{\partial f_{1a}}{\partial\lambda}$ the rate of entropy production $K(f_1,f_1)$ can be rewritten and

$$\begin{aligned}\frac{dS}{dt} &= 4\sqrt{2}\pi^2e^4\ \ln\Lambda\ \times I\\ &\times\sum_a\frac{1}{m_a^2}\int dw\ \sqrt{w}f_{Ma}\ G_a^2(w)\ \sum_b F_b(w)\end{aligned}$$

where

$$
\begin{aligned}
I &\equiv \int d\lambda \int \frac{d\theta}{2\pi}\sqrt{\frac{1-\lambda}{h}} \\
&\quad \times \left\{ \frac{1}{\left\langle \sqrt{1-\frac{\lambda}{h}} \right\rangle_{untrapped}} - \frac{1}{\sqrt{1-\frac{\lambda}{h}}} \right\}^2 \\
&= \int_0^{h_{\max}} d\lambda \left\langle \frac{1}{\sqrt{1-\frac{\lambda}{h}}} \right\rangle - \int_0^{h_{\min}} d\lambda \frac{1}{\left\langle \sqrt{1-\frac{\lambda}{h}} \right\rangle}
\end{aligned}
$$

where

$$
\begin{aligned}
h_{\min} &= 1 - \frac{r}{R} \\
h_{\max} &= 1 + \frac{r}{R}
\end{aligned}
$$

To evaluate this integrals
The change of variables

$$
\begin{aligned}
\lambda &\rightarrow \kappa \\
\kappa &\equiv \frac{2\lambda\varepsilon}{1-\lambda+\varepsilon\lambda}
\end{aligned}
$$

Then

$$
\xi = \sqrt{1-\frac{\lambda}{h}} = \sqrt{1-\lambda+\lambda\varepsilon}\,\sqrt{1-\kappa^2\sin^2\frac{\theta}{2}}
$$

The ordering is

$$
\sqrt{\varepsilon} \text{ small}
$$

Then

$$
d\lambda \approx 4\varepsilon\frac{d\kappa}{\kappa^3}
$$

and the integral becomes

$$
\begin{aligned}
I &\approx 4\sqrt{\frac{\varepsilon}{2}} \left\{ \int_0^{2/\varepsilon} \frac{d\kappa}{\kappa^2} \int \frac{d\theta}{2\pi} \frac{1}{\sqrt{1-\kappa^2\sin^2\frac{\theta}{2}}} \right. \\
&\qquad \left. - \int_0^1 \frac{d\kappa}{\kappa^2} \frac{1}{\int \frac{d\theta}{2\pi}\sqrt{1-\kappa^2\sin^2\frac{\theta}{2}}} \right\}
\end{aligned}
$$

$$
\approx 4\sqrt{\frac{\varepsilon}{2}} \left[\int_0^{\infty} \frac{d\kappa}{\kappa^2}\, \frac{2}{\pi}\mathbf{K}(\kappa) - \int_0^1 \frac{d\kappa}{\kappa^2} \frac{\pi}{2\mathbf{E}(\kappa)} \right]
$$

The well defined form is

$$I = 2\sqrt{2\varepsilon}\left[\int_0^1 \frac{d\kappa}{\kappa^2}\left(\frac{2}{\pi}\mathbf{K}(\kappa) - \frac{\pi}{2\mathbf{E}(\kappa)}\right) + \int_1^\infty \frac{d\kappa}{\kappa^2}\frac{2}{\pi}\mathbf{K}(\kappa)\right]$$

RHH use an identity

$$\int \frac{d\kappa}{\kappa^2}\frac{2}{\pi}\mathbf{K} = -\frac{\mathbf{E}}{\kappa}$$

$$\int_1^\infty \frac{d\kappa}{\kappa^2}\frac{2}{\pi}\mathbf{K} = \frac{2}{\pi}$$

Using these formulas, the first term is

$$\begin{aligned} & \int_0^1 \frac{d\kappa}{\kappa^2}\left(\frac{2}{\pi}\mathbf{K}(\kappa) - \frac{\pi}{2\mathbf{E}(\kappa)}\right) \\ = & \lim_{\varepsilon\to 0}\left[\int_\varepsilon^1 \frac{d\kappa}{\kappa^2}\left(\frac{2}{\pi}\mathbf{K} - 1\right) + \int_\varepsilon^1 \frac{d\kappa}{\kappa^2}\left(1 - \frac{\pi}{2\mathbf{E}}\right)\right] \end{aligned}$$

and here

$$\begin{aligned} & \lim_{\varepsilon\to 0}\int_\varepsilon^1 \frac{d\kappa}{\kappa^2}\left(\frac{2}{\pi}\mathbf{K} - 1\right) \\ = & \lim_{\varepsilon\to 0}\left(-\frac{2}{\pi}\left.\frac{\mathbf{E}(\kappa)}{\kappa}\right|_\varepsilon^1 + \left.\frac{1}{\kappa}\right|_\varepsilon^1\right) \\ = & 1 - \frac{2}{\pi} \end{aligned}$$

Then

$$\begin{aligned} I &= 2\sqrt{2}\varepsilon^{1/2}\left[1 + \int_0^1 \frac{d\kappa}{\kappa^2}\left(1 - \frac{\pi}{2\mathbf{E}(\kappa)}\right)\right] \\ &\approx 2\sqrt{2}\,(0.69)\ \sqrt{\varepsilon} \end{aligned}$$

Review of the main analytical structure of the response (extremum of the functional)

- the collision operator is linearized
- the collision operator is expanded in *small angle scattering*; integrate over $d\Omega$ in velocity space.
- change from $\mathbf{v}$ to (w, λ); calculate the functional $\dot{S}$.
- take the extremum and find an integral equation for $\partial f_{1a}/\partial\lambda$.
- expand $f_{1,a}$ in powers of ν/ω_{bounce}. $f_{1,a}^{(0)}$ is neoclassic $v_\parallel/\Omega_\theta$, plus Spitzer, plus correction D plus trapping-circulating correction g.

- surface-average and periodicity gives solution $\partial g/\partial\lambda \sim \frac{1}{\langle\xi\rangle}$.
- for the functional the integration I is made and approximated.

Compare with **Cordey NBI** where the expressions

$$\langle\xi\rangle \quad \text{and}$$
$$\left\langle\frac{1}{\xi}\right\rangle$$

are calculated. They occur in the expression of the collision operator (slowing-down plus pitch angle). The expansion in $\tau_{bounce}/\tau_s \ll 1$ of the distribution function is $f^{(0)} + f^{(1)}$... and the θ-periodicity constraint on the equation for $f^{(1)}$ leads to an equation for $f^{(0)}$ where the averages over θ written above appear.

Compare **Connor1973**
Here the equation

$$v_{\|}\nabla_{\|}f = Collision\,(\text{pitch} + \text{slowing down})$$

is solved by dividing with $v_{\|}$ and requesting θ-periodicity. This introduces both averages $\langle\xi\rangle$ and $\left\langle\frac{1}{\xi}\right\rangle$. The treatment is like in **mirrors Rosenbluth Hinton**.

8.2 Approach adapted to arbitrary aspect ratio (Hazeltine Hinton Rosenbluth)

The definition of the equilibrium state

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{B} &= 0\\ \boldsymbol{\nabla}\times\mathbf{B} &= \mu_0\mathbf{j}\\ 0 &= -\boldsymbol{\nabla}p + \mathbf{j}\times\mathbf{B}\end{aligned}$$

and

$$\begin{aligned}\mathbf{B} &= \boldsymbol{\nabla}\varphi\times\boldsymbol{\nabla}\psi + I\boldsymbol{\nabla}\varphi\\ p &= p(\psi)\\ R^2\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{\nabla}\psi}{R^2}\right) &= -II' - \mu_0R^2p'\end{aligned}$$

with variables on the magnetic surface

$$(\psi, \chi, \varphi)$$

and

$$()' \equiv \frac{d\,()}{d\psi}$$

The force balance of current field and pressure lead to

$$
\begin{aligned}
j_{\|} &= -\frac{Ip'}{B} \\
&\quad -\frac{1}{\mu_0} I' B
\end{aligned}
$$

Other formulas

$$d^3x = \frac{d\psi d\chi d\varphi}{\boldsymbol{\nabla}\chi \cdot \mathbf{B}}$$

$$V(\psi) = 2\pi \oint \frac{d\chi}{\boldsymbol{\nabla}\chi \cdot \mathbf{B}}$$

and the average

$$\langle A \rangle = \frac{2\pi}{V'(\psi)} \oint \frac{d\chi}{\boldsymbol{\nabla}\chi \cdot \mathbf{B}} A$$

$$\langle \boldsymbol{\nabla} \cdot \mathbf{A} \rangle = \frac{1}{V'} \frac{\partial}{\partial \psi} V' \langle \boldsymbol{\nabla}\psi \cdot \mathbf{A} \rangle$$

And

$$\psi = -RA_{\varphi}$$

such that

$$\frac{\partial \psi}{\partial t} = RE_{\varphi}$$

which must be interpreted as follows. The function ψ is the flux of the *poloidal* magnetic field. The toroidal electric field has a *circulation* around the torus, a non-zero rotational. Then there must exist a non-zero time variation of the magnetic field

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Then the magnetic flux has a time variation due to the toroidal electric field that has a non-zero rotational. the magnetic surfaces are inflated in time.

The drift kinetic equation

$$
\begin{aligned}
& \frac{\partial f}{\partial t} + \left(\mathbf{v}_{\|} + \mathbf{v}_D\right) \cdot \boldsymbol{\nabla} f \\
& + \frac{|e|}{m} \left(E_{\|} v_{\|} + \frac{\partial \phi}{\partial t} \right) \frac{\partial f}{\partial \epsilon} \\
= \; & C
\end{aligned}
$$

where $E_{\|} \equiv$ externally induced electric field.

$$\epsilon = \frac{v^2}{2} + \frac{|e|}{m} \phi$$

for ions.

NOTE that we keep the time variation of the electric potential ϕ, as if we would examine instabilities. Alternatively, this term allows us to study the spin-up or the decay of a rotation. **END.**

$$\mu = \frac{v^2}{2B}$$

$$f_M = \frac{n}{\left(\pi\ 2T/m\right)^{3/2}} \exp\left[-2\frac{\epsilon - |e|\,\phi}{v_{th}^2}\right]$$

the exponent is

$$-2\frac{\epsilon - |e|\,\phi}{v_{th}^2} = -\frac{v^2}{v_{th}^2}$$

$$\mathbf{v}_D = -v_\parallel \widehat{\mathbf{n}} \times \boldsymbol{\nabla}\left(\frac{v_\parallel}{\Omega_c}\right)$$

The equation

$$\begin{aligned}
&\frac{\partial f}{\partial t} + \frac{\partial \psi}{\partial t}\frac{\partial f}{\partial \psi} + \frac{\partial \chi}{\partial t}\frac{\partial f}{\partial \chi} + \\
&+v_\parallel \frac{\boldsymbol{\nabla}\chi \cdot \mathbf{B}}{B}\frac{\partial f}{\partial \chi} \\
&+v_\parallel \frac{\boldsymbol{\nabla}\chi \cdot \mathbf{B}}{B}\ I\ \left[\frac{\partial}{\partial \chi}\left(\frac{v_\parallel}{\Omega_c}\right)\frac{\partial f}{\partial \psi} - \frac{\partial}{\partial \psi}\left(\frac{v_\parallel}{\Omega_c}\right)\frac{\partial f}{\partial \chi}\right] \\
&+\frac{|e|}{m}\frac{\partial f}{\partial \epsilon}\left(E_\parallel v_\parallel + \frac{\partial \phi}{\partial t}\right)
\end{aligned}$$

the expansion

$$f = f_0\left(1 + \widehat{f}\right)$$

the equation becomes

$$\begin{aligned}
& v_\parallel \frac{\boldsymbol{\nabla}\chi \cdot \mathbf{B}}{B}\frac{\partial \widehat{f}}{\partial \chi} - \frac{1}{f_0}C\left(\widehat{f}\right) \\
= \quad & \frac{|e|\,E_\parallel v_\parallel}{T} \\
& -v_\parallel \frac{\boldsymbol{\nabla}\chi \cdot \mathbf{B}}{B}\ I\ \left(\frac{\partial}{\partial \chi}\frac{v_\parallel}{\Omega_c}\right)\frac{d}{d\psi}\ln f_0
\end{aligned}$$

The variables

$$w = \frac{v^2}{2} = \epsilon - \frac{|e|}{m}\phi$$

$$\lambda = \frac{\mu}{w}$$

$$v_\parallel = \sigma\sqrt{2w\left(1 - \lambda B\right)}$$

the limit for trapped

$$\lambda_c = \frac{1}{B_{\max}}$$

The boundary conditions

$$\begin{aligned} \widehat{f}_{\pm}(\chi = -\pi) &= \widehat{f}_{\pm}(\chi = \pi) \\ \text{for } \lambda &< \lambda_c \end{aligned}$$

$$f_{+}(\pm\chi_c) = f_{-}(\pm\chi_c)$$

$$\text{for } \lambda > \lambda_c$$

The banana regime

$$\begin{aligned} \widehat{f}_{\sigma} &= -\frac{I}{\Omega_c} v_{\parallel} \frac{d}{d\psi} \ln f_M \\ &\quad + g_{\sigma}(\psi, w, \lambda) \end{aligned}$$

where

$$\begin{aligned} g_{\sigma} &\equiv \text{independent on } \chi \\ g_{\sigma} &\equiv \text{even in } \sigma \text{ for trapped region} \end{aligned}$$

This result corresponds to the *zero-th* order in the expansion in ratio

$$\Delta \equiv \frac{\nu_{ei}}{\omega_{bounce}}$$

The solubility of the next order

$$\oint \frac{d\chi}{\boldsymbol{\nabla}\chi \cdot \mathbf{B}} \frac{B}{v_{\parallel}} \left[C(f) + \frac{|e| E_{\parallel} v_{\parallel} f_M}{T} \right] = 0$$

the interal over the periodic particle orbit. It results

$$g_{\sigma} = 0 \ \text{ in the trapped region}$$

NOTE this equation, without $E_{\parallel}$ which is external, is identical to the equation for *bootstrap* of **Helander ECRH**.
END.

The parallel current

$$j_{\parallel} = -I\frac{p'}{B} + |e| B (ZK_i - K_e)$$

and it is found that

$$I' = -\mu_0 \left|e\right| \left(ZK_i - K_e\right)$$

the current is written as

$$j_{\|} = j_{ps} + j_{NC}$$

$$\begin{aligned} j_{ps} &= \text{Pfirsch - Schluter} \\ &= ip'\mathbf{B}\left(\frac{1}{B_0^2} - \frac{1}{B^2}\right) \end{aligned}$$

with

$$\boldsymbol{\nabla}\cdot\mathbf{j}_{ps} = -\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{B}\times\boldsymbol{\nabla}p}{B^2}\right)$$

which means that j_{ps} is the return current necessary to compensate the non-zero divergence of the diamagnetic current;

and

$$\begin{aligned} j_{NC} &= \text{bootstrap} \\ &= \mathbf{B}\left[\left|e\right|\left(ZK_i - K_e\right) - \frac{Ip'}{B_0^2}\right] \end{aligned}$$

NOTE that here it is introduced the new variable function

$$K_j \ , \ j = e, i$$

This function is also present in **Hazeltine Hinton**.

It is a formal "constant of integration" - like function, to express the fact that there is also a variable that is constant on surface.

This is what it is: a constant on surface.

8.3 The parallel flow of ions at equilibrium

In **Hinton Rosenbluth low to intermediate collisions.**

The parallel flow of ions

$$u_{\|i}^{(0)} = -\frac{v_{th,i}^2}{\Omega_{ci}\Theta}\left[\frac{\partial}{\partial r}\ln n_{0i} + \frac{Z\left|e\right|}{T_i}\frac{\partial\left\langle\phi\right\rangle}{\partial r} + \left(y - \frac{3}{2}\right)\frac{\partial}{\partial r}\ln T_i\right]$$

we note that the ratio

$$\frac{v_{th,i}}{\Omega_{ci}\Theta} = \rho_{i\theta}$$

and the parallel flow of ions is the thermal velocity $\times$ the first order of spatial change

$$\begin{aligned} & \rho_{\theta i}\left[\frac{\partial}{\partial r}\ln n_{0i} + ...\right] \\ \sim\ & \rho_{\theta i}\frac{1}{L_{n_{0i}}} + ... \end{aligned}$$

This term is typical for order zero neoclassical correction averaged over the gyration.

$$\overline{f}_i^{(1)} \sim \rho_{\theta i} \frac{\partial f_M}{\partial r}$$

Then

$$u_{\parallel i}^{(0)} = v_{th,i} \times \rho_{\theta i} \left[\frac{\partial}{\partial r} \ln n_{0i} + ... \right]$$

and it is a form of the *diamagnetic* flow velocity

$$v_i^{dia} = \frac{c_s \rho_s}{L_n}$$

projected on the parallel direction

$$u_{\parallel i}^{(0)} = v_i^{dia} \times \Theta$$

9 Banana regime Taguchi

In **Taguchi 1992 poloidal**

$$V_r = \frac{v_\parallel}{(e_a B_0/m_a)} \frac{\partial}{r \partial \theta} \left(h v_\parallel \right)$$

This is for the kinetic equation

$$\begin{aligned} & v_\parallel \nabla_\parallel f_a^{(1)} - C_a \left[f_a^{(1)} \right] \\ = \quad & -V_r \frac{\partial f_a^{(0)}}{\partial r} - v_\parallel \nabla_\parallel \left(\frac{e_a \Phi^{(1)}}{T_a} \right) \ f_a^{(0)} \end{aligned}$$

after which the substitution is made

$$\begin{aligned} f_a^{(1)} \quad = \quad & -\left(\frac{e_a \Phi^{(1)}}{T_a} \right) f_a^{(0)} \\ & + g_a \end{aligned}$$

The first part in the content of $f_a^{(1)}$ is determined by the electrostatic potential with poloidal variation, $\Phi^{(1)} \sim \theta$.

Here

$$\begin{aligned} & v_\parallel \nabla_\parallel g_a - C_a \left[g_a \right] \\ = \quad & -V_r \frac{\partial f_a^{(0)}}{\partial r} \end{aligned}$$

When the radial flux is calculated averaged over the magnetic surface, one notice a cancelation of the terms that contain $\Phi^{(1)}$.

$$\begin{aligned}\Gamma_a &= \left\langle n_{a0}\left(\frac{-\boldsymbol{\nabla}\Phi^{(1)}\times\widehat{\mathbf{n}}}{B}\right)_{radial}\right\rangle + \left\langle \int d^3v\ V_r\ f_a^{(1)}\right\rangle \\ &= \left\langle \int d^3v\ V_r\ g_a\right\rangle\end{aligned}$$

The approximative variation of the electrostatic potential on the poloidal is

$$\frac{e\Phi^{(1)}}{T_e} \approx \frac{1}{1+Z\left(\frac{T_e}{T_i}\right)}\frac{1}{n_{a0}}\int d^3v\ g_a$$

In the banana regime, to find g_a one needs to expand in a small parameter of *collisionality*

$$g_a = g_{a0} + g_{a1} + ...$$

with the equations

$$v_\parallel \nabla_\parallel g_{i0} = -V_r \frac{\partial f_i^{(0)}}{\partial r}$$

$$v_\parallel \nabla_\parallel g_{i1} = C_{ii}\left[g_{i0}\right]$$

with the solution

$$\begin{aligned}g_{i0} &= -\frac{1}{(e_i B_0/m_i)}\left(\frac{B_\varphi}{B_\theta}\right)\ \sigma\ \left(hx\sqrt{1-\frac{\lambda}{h}}\ \frac{\partial f_i^{(0)}}{\partial r}\right. \\ &\qquad \left. +\frac{1}{2f_{circ}}\mathbf{H}\left(\lambda_c-\lambda\right)\ K\left(x\right)\int_\lambda^{\lambda_c} d\lambda \frac{1}{\left\langle\sqrt{1-\frac{\lambda}{h}}\right\rangle}\frac{1}{T_i}\frac{dT_i}{dr}\right)\end{aligned}$$

where

$$\lambda = \frac{2B_0\mu}{v^2} = \frac{v_\perp^2}{v^2}h$$

so that

$$\sqrt{1-\frac{\lambda}{h}} = \frac{v_\parallel}{v} \equiv \xi \quad \text{(pitch angle variable)}$$
$$\times\sigma$$

$$\lambda_c = 1-\varepsilon$$

$$x \equiv \frac{v}{v_{th,a}} = \frac{v}{\sqrt{\frac{2T_a}{m_a}}}$$

$$f_{circ} = \frac{3}{4}\left\langle \frac{1}{h^2} \right\rangle \int_0^{\lambda_c} d\lambda \frac{\lambda}{\left\langle \sqrt{1-\frac{\lambda}{h}} \right\rangle}$$

for small ε

$$f_{circ} \approx 1 - 1.035\sqrt{2\varepsilon}$$

The function $K(x)$ is solution of

$$\begin{aligned} & (1-f_{circ})\,\nu_{ii}^{defl}(x)\;\; K \\ & -f_{circ}\; C_{ii}^{(1)}[K] \\ = \;\; & f_{circ}\; C_{ii}^{(1)}\left[x\left(x^2-\frac{5}{2}\right)\; f_i^{(0)}\right] \end{aligned}$$

The collision operator has the form

$$C_{ii}[g_{i0}] = \nu_{ii}^{defl}\;\mathcal{L}[g_{i0}] + \xi\;\left(C_{ii}^1 + \nu_{ii}^{defl}(x)\right)\widehat{g}$$

$$\mathcal{L} = \frac{v_\parallel}{B}\frac{\partial}{\partial\mu}\mu v_\parallel \frac{\partial}{\partial\mu}$$

and

$$\widehat{g} \equiv \frac{3}{2}\int_{-1}^{1} d\xi\;\xi\; g_{i0}$$

$$\nu_{ii}^{defl}(x) = \frac{\nu_{ii}}{2x^5}\left[\left(2x^2-1\right)\operatorname{erf}(x) + \frac{2}{\sqrt{\pi}}x\exp\left(-x^2\right)\right]$$

$$\nu_{aa} = \frac{4\pi e_a^4}{m_a^2}\log\Lambda\frac{n_{a0}}{v_{th,a}^3}$$

There is the property of the collision operator

$$C_{ii}[\xi\;\varphi(x)] = \xi\; C_{ii}^{(1)}[\varphi(x)]$$

After solving the equation for g_{i1} with the collision operator as assumed, one finds the *radial heat flux* which is due to the poloidally varying electrostatic potential, for species $a = e, i$

$$\begin{aligned} Q_{rad,a}^{\Phi} \;\; = \;\; & -\frac{T_i}{1+\frac{T_i}{ZT_e}}\; n_{a0}\rho_{pol,i}^2\;\nu_{ii} \\ & \times G \\ & \times \frac{1}{L_{T_i}} \end{aligned}$$

where

$$G = \left(\frac{1}{\left\langle \frac{1}{h^2} \right\rangle} - \left\langle h^2 \right\rangle \right) I_2 + \left[\frac{1}{\left\langle \frac{1}{h^2} \right\rangle} \left(1 - \frac{4}{3} f_{circ} \right) + f_p \right] \frac{I_1}{f_{circ}}$$

$$f_p \equiv -\frac{1}{2} \int_0^{\lambda_c} d\lambda \frac{\lambda}{\left\langle \sqrt{1 - \frac{\lambda}{h}} \right\rangle} \left\langle \sqrt{1 - \frac{\lambda}{h}} \log \left(\frac{1 - \sqrt{1 - \frac{\lambda}{h}}}{1 + \sqrt{1 - \frac{\lambda}{h}}} \right) \right\rangle$$

with approximative value

$$f_p \approx \frac{1}{3} + O(\varepsilon)$$

$$I_1 = \frac{2}{\sqrt{\pi}} \int dx \, \frac{\nu_{ii}^{delf}(x)}{\nu_{ii}} \, \frac{K(x)}{f_i^{(0)}} \exp\left(-x^2\right)$$

with approximative value

$$I_1 \approx 0.62$$

$$I_2 = \frac{2}{\sqrt{\pi}} \int dx \, x \, \frac{C_{ii}^{(1)} \left[x \left(\frac{5}{2} - x^2 \right) \, \exp\left(-x^2\right) \right]}{\nu_{ii}} \approx -0.37$$

$$\frac{1}{\left\langle \frac{1}{h^2} \right\rangle} - \left\langle h^2 \right\rangle \approx O\left(\varepsilon^2\right)$$

It results

$$G \approx 1.21 \sqrt{\varepsilon}$$

The heat flux of the electrons induced by the poloidally varying electrostatic potential $\Phi^{(1)} \sim \theta$, is higher by a factor

$$\sqrt{\frac{m_i}{m_e}}$$

than the uniform-on-surface heat flux.

10 Numerical solution of the drift kinetic equation Santarius Hinton

This is the paper **NumericalSol DriftKinetic Santarius Hinton 1980 PFL000537.pdf**.

It treats the *trapped electron* drift instability with a more accurate representation of the drifts of the electrons. The neoclassical correction to the distribution function is NOT retained for electrons but it is for ions.

10.1 The Fokker Planck equation

The equation is Fokker Planck for *electrons*

$$\frac{\partial f_e}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} f_e + \frac{\partial f_e}{\partial \mu}\frac{d\mu}{dt} + \frac{\partial f_e}{\partial \epsilon}\frac{d\epsilon}{dt} + \frac{\partial f_e}{\partial \zeta}\frac{d\zeta}{dt} = C\left(f_e\right)$$

and after gyro-averaging one obtains the *drift-kinetic* equation

$$\frac{\partial \overline{f}_e}{\partial t} + \left(\mathbf{v}_\parallel + \mathbf{v}_D + \mathbf{v}_E\right) \cdot \boldsymbol{\nabla} \overline{f}_e - \frac{|e|}{m_e}\frac{\partial \phi}{\partial t}\frac{\partial \overline{f}_e}{\partial \epsilon} = C\left(f_e\right)$$

where for electrons

$$\begin{aligned} \epsilon &= \frac{v^2}{2} - \frac{e}{m_e}\phi \\ \text{since we take } e &> 0 \text{ and this is energy of electrons} \end{aligned}$$

The last term in the LHS is the energy effect of the time variations of the electric potential. There can be two kinds of time variation of ϕ.

- One, a global change consisting in generation of an electric field, as it occurs at the transition L to H. A radial electric field is produced and it is balanced by rotation. A similar situation appears at the formation of Internal Transport Barriers. And the damping by magnetic pumping of a poloidal rotation. Such processes are transitory and have a large spatial scale. (Ex. **Hassam**, **Novakovski**)
- The second kind of situations where the time variation of the electric field plays an essential role is the wave-like perturbation.

In the present investigation the energy term is considered for the *wave perturbations*, which means for instabilities, not for pure neoclassics. One would expect

$$|e|\, E_\parallel v_\parallel \frac{\partial \overline{f}_e}{\partial \epsilon} \quad \text{(not present)}$$

which allows to represent the effect of an electric field in tokamak. This is important when variations on the magnetic surface is considered, like in neoclassical effect Pfirsch Schluter, etc.

The energy terms occur everytime the particle during its motion must do a *work* against an electric field. Then it is acted upon by a *force* and there is an acceleration and its velocity will be changed. In consequence the distribution function $f\left(\mathbf{x}, \mathbf{v}, t\right)$ will change in *velocity space.* The energy term must be present in the drift-kinetic equation.

This happens - for example, in **electrostatic trapping of impurities Hazeltine Ware** *variation on surfaces.*

The term of convection by the *neoclassical* drift velocity of the equilibrium distribution function f_{Me}

$$\mathbf{v}_D \cdot \boldsymbol{\nabla} f_M = v_{Dr}\frac{\partial f_M}{\partial r}$$

is *absent*. This is the basic *neoclassical* change of f, of the order of a *poloidal Larmor radius* ρ_θ / L.

It is absent because for electrons the spatial departure from a magnetic surface, due to drift, is small.

Instead, it is retained the drift convection of the *perturbed* distribution function, in $(\mathbf{v}+\mathbf{v}_D)\cdot\boldsymbol{\nabla} f_m$ below. The reason is that we need at this point the *trajectory*, such as to invert the equation and define the *propagator* by integration along the trajectory.

In this paper for *ions* the term of neoclassical drift-velocity convection of the Maxwellian is however included. The deviation from a magnetic surface, for ions, is large.

We note at this point that the *source terms* in the equation for the distribution function in first order (after expansion in Maxwellian plus the wave-induced perturbation) depend, as is normal, on the equilibrium (Maxwellian) distribution

$$\frac{\partial f_1}{\partial t} + \left(v_\parallel \widehat{\mathbf{n}} + \mathbf{v}_D\right)\cdot\boldsymbol{\nabla} f_1 - C\left[f_1\right] = e\left(-\frac{1}{T} f_{Me}\right)\frac{\partial \phi}{\partial t} - \mathbf{v}_E\cdot\boldsymbol{\nabla} f_{Me}$$

We also have electric drift convection of the Maxwellian, with the electric potential being that of the *drift wave*

$$\begin{aligned}
\mathbf{v}_E\cdot\boldsymbol{\nabla} f_M &= \frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla} f_M \\
&= \frac{1}{B}\frac{\partial\phi}{r\partial\theta}\frac{\partial f_M}{\partial r}
\end{aligned}$$

which is again of the type of *wave perturbation* since we expect that the fluctuations of the potential along the poloidal direction will produce the fluctuating advection of the density-temperature gradients from the equilibrium profiles, along the radial direction.

Therefore the *source terms* contains

- a time derivative of the electric potential (the energy effect of the wave) and
- a spatial derivative of the electric potential (*i.e.* the electric field $\mathbf{E}$ of the wave) which produces radial $E\times B$ convection of the Maxwellian.

Their combination will produce

$$\omega - \omega_{Te}^*$$

The drift velocity (for electrons) is

$$\mathbf{v}_D = -\frac{1}{\Omega_e}\widehat{\mathbf{n}}\times\frac{v^2+v_\parallel^2}{2}\boldsymbol{\nabla}\ln B$$

and

$$\mathbf{v}_E = \frac{\mathbf{E}\times\mathbf{B}}{B^2}$$

$$f_{Me} = \frac{n}{\left(\sqrt{\pi}v_{th,e}\right)^3}\exp\left(-\frac{v^2}{v_{th,e}^2}\right)$$

$$v_{th,e} = \frac{2T_e}{m_e}$$

The first order

$$f_e = \frac{e\phi}{T_e}f_{Me} + f_e^{(1)}$$

Therefore

$$\begin{aligned} & -i\omega f_e^{(1)} + \left(\mathbf{v}_{\parallel}+\mathbf{v}_D\right)\cdot\boldsymbol{\nabla} f_e^{(1)} - C\left(f_e^{(1)}\right) \\ = \quad & \frac{|e|}{T_e}f_{Me}\left\{i\omega\phi + \frac{T_e}{|e|\,B}\frac{\partial\phi}{r\partial\theta}\frac{d\ln n}{dr}\left[1+\eta_e\left(\frac{v^2}{v_{th,e}^2}-\frac{3}{2}\right)\right]\right\}\end{aligned}$$

We notice the wave -like time variations, harmonic as $i\omega$, for the fluctuating potential ϕ and for the fluctuating distribution function $f_e^{(1)}$.

Naturally, it follows an expansion in the helical geometry

$$\begin{array}{ll} \text{poloidal} & \theta \\ \text{toroidal} & \exp\left(-il\varphi\right) \end{array}$$

The potential

$$\phi = \exp\left(-il\varphi\right)\sum_{m=-\infty}^{\infty} a_m\ \exp\left(im\theta\right)$$

The first order distribution function is expanded as a series of coefficients of the potential expansion

$$f_e^{(1)} = \sum_{m=-\infty}^{\infty} f_m a_m$$

This transforms the *drift-kinetic equation* into a system of equations for the coefficients f_m,

$$\begin{aligned} & -i\omega f_m + \left(v_{\parallel}\widehat{\mathbf{n}}+\mathbf{v}_D\right)\cdot\boldsymbol{\nabla} f_m - C\left[f_m\right] \\ = \quad & i\left(\omega-\omega_{Te,m}^*\right)\frac{e}{T_e}f_{Me}\ \exp\left(im\theta - il\varphi\right)\end{aligned}$$

where

$$\omega_{Te,m}^* = \omega_{e,m}^*\left[1+\eta_e\left(\frac{v^2}{v_{th,e}^2}-\frac{3}{2}\right)\right]$$

$$\omega_{e,m}^* = -\frac{m}{r}\frac{T_e}{eB}\frac{d}{dr}\ln n$$

$$\eta_e = \frac{d\ln T_e}{d\ln n}$$

In the treatment of **Santarius Hinton** the *neoclassical correction* to $f_e^{(1)}$ is *neglected*.

10.2 The collisions

The collisional operator.

Pitch angle scattering.

No slowing down, since there is no *beam* or fast ions, etc.

$$\begin{aligned} C[f] &= \frac{1}{2}\nu_{ei}(v)\frac{\partial}{\partial\xi}\left(1-\xi^2\right)\frac{\partial f}{\partial\xi} \\ \xi &\equiv \frac{v_\parallel}{v} \end{aligned}$$

$$\nu_{ei}(v) = \frac{3\sqrt{\pi}}{4}\frac{1}{\tau_e}\frac{v_{th,e}^3}{v^3}$$

$$\tau_e = \frac{3}{16\sqrt{\pi}}\frac{m_e^2}{e^4}\frac{1}{\ln\Lambda}\frac{v_{th,e}^3}{n_i}$$

10.3 The trajectory (for the propagator)

the equations of motion for *electrons*

$$\frac{d\mathbf{r}}{dt} = v_\parallel\widehat{\mathbf{n}} - \frac{1}{\Omega_e}\frac{v^2+v_\parallel^2}{2}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\ln B$$

or

$$\frac{dr}{dt} = \frac{1}{\Omega_c}\frac{v_\perp^2/2+v_\parallel^2}{R}\sin\theta$$

$$\frac{d\theta}{dt} = \frac{v_\parallel}{qR} + \frac{1}{r}\frac{1}{\Omega_c}\frac{v_\perp^2/2+v_\parallel^2}{R}\cos\theta$$

$$\frac{d\varphi}{dt} = \frac{v_\parallel}{R} - \frac{1}{R}\frac{r}{qR}\frac{1}{\Omega_c}\frac{v_\perp^2/2+v_\parallel^2}{R}\cos\theta$$

Remark that no energy ($v_\parallel$, $v_\perp^2/2$) equation is involved. No orbit in the velocity space. This is because there is no need for a particle to make a work when moves against a static electric field, like in *variation on surface*, where there is a nonuniform distribution of charges on θ, or when there is a radial electric field due to some $\Phi^{(0)}$ potential, like in Internal Transport Barriers.

From the function f_m one factors out a harmonic component

$$\exp\left(-ilq\theta\right)$$

as

$$\widetilde{f}_m\left(r,\theta\right) = f_m\left(r,\theta,\varphi\right)\exp\left[il\left(\varphi - q\theta\right)\right]$$

and the convective part with the new function $\widetilde{f}_m$ is

$$\begin{aligned}
& \exp\left[il\left(\varphi - q\theta\right)\right]\left(\mathbf{v}+\mathbf{v}_D\right)\cdot\boldsymbol{\nabla} f_m \\
= \quad & \exp\left[il\left(\varphi - q\theta\right)\right]\left(\frac{d\theta}{dt}\frac{\partial f_m}{\partial\theta} + \frac{d\varphi}{dt}\frac{\partial f_m}{\partial\varphi} + \frac{dr}{dt}\frac{\partial f_m}{\partial r}\right) \\
= \quad & \left[\frac{\xi v}{qR} + \frac{1}{\Omega_0}\frac{1}{R}\frac{v^2\left(1+\xi^2\right)}{2}\cos\theta\right]\frac{\partial\widetilde{f}_m}{r\partial\theta} \\
& + \frac{1}{\Omega_0}\frac{1}{R}\frac{v^2\left(1+\xi^2\right)}{2}\frac{il}{r}\left(q\cos\theta + \frac{\varepsilon^2}{q}\cos\theta + \frac{dq}{dr}r\ \theta\sin\theta\right)\widetilde{f}_m \\
& + \frac{1}{\Omega_0}\frac{1}{R}\frac{v^2\left(1+\xi^2\right)}{2}\sin\theta\frac{\partial\widetilde{f}_m}{\partial r}
\end{aligned}$$

This is replaced in the drift kinetic equation

$$\begin{aligned}
& -i\omega\widetilde{f}_m \\
& + \left[\frac{\xi v}{qR} + \frac{1}{\Omega_0}\frac{1}{R}\frac{v^2\left(1+\xi^2\right)}{2}\cos\theta\right]\frac{\partial\widetilde{f}_m}{r\partial\theta} \\
& + \frac{1}{\Omega_0}\frac{1}{R}\frac{v^2\left(1+\xi^2\right)}{2}\frac{il}{r}\left(q\cos\theta + \frac{dq}{dr}r\ \theta\sin\theta\right)\widetilde{f}_m \\
& - \frac{1}{qR}\frac{v\left(1-\xi^2\right)}{2}\frac{\varepsilon}{h}\sin\theta\frac{\partial\widetilde{f}_m}{\partial\xi} \\
& - \frac{1}{2}\nu_{ei}\left(v\right)\ \frac{\partial}{\partial\xi}\left(1-\xi^2\right)\frac{\partial\widetilde{f}_m}{\partial\xi} \\
= \quad & i\left(\omega - \omega^*_{Te,m}\right)e\frac{1}{T_e}f_{Me}\exp\left[i\left(m-lq\right)\theta\right]
\end{aligned}$$

The term

$$-\frac{1}{qR}\frac{v\left(1-\xi^2\right)}{2}\frac{\varepsilon}{h}\sin\theta\frac{\partial\widetilde{f}_m}{\partial\xi}$$

arises from the change of variable from μ to ξ.

10.4 Boundary conditions

At the points in velocity space where the perpendicular velocity is zero

$$\left(1-\xi^2\right)\frac{\partial\widetilde{f}_m}{\partial\xi} \to 0 \quad \text{at } \xi \to \pm 1$$

We note that the limit $\xi = v_{\parallel}/v \to \pm 1$ means full passing, no trapping.

The periodicity

$$f_m(\theta + 2\pi) = f_m(\theta)$$

or

$$\widetilde{f}_m(\theta + 2\pi) = \exp(-2\pi i l q)\, \widetilde{f}_m(\theta)$$

The boundary between the trapped and untrapped regions

$$\xi = \frac{\varepsilon}{1 + \varepsilon \cos\theta} \cos^2\left(\frac{\theta}{2}\right)$$

No special condition at this transition point *trapped/untrapped*.

Somewhere the separatrix between trapped and passing has a more detailed form.

10.5 The distribution function for *ions*.

The equation

$$\frac{\partial f_i}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} f_i + \frac{e}{m_i}\left(-\boldsymbol{\nabla}\Phi + \mathbf{v} \times \mathbf{B}\right) \cdot \frac{\partial f_i}{\partial \mathbf{v}} = 0$$

Note that there are no collisions.

The equilibrium distribution function is neoclassic

$$f_i^{(0)} = f_{Mi}\left(1 - \frac{1}{\Omega_{\theta i}} v_{\varphi} \frac{d \ln n}{dr}\right)$$

and this is integrated along the orbit to obtain the perturbation

$$f_i^{(1)} = \int_{-\infty}^{t} dt' \; \frac{e}{m_i} \boldsymbol{\nabla}\Phi \cdot \frac{\partial f_i^{(0)}}{\partial \mathbf{v}}$$

It is assumed a time dependence

$$\exp(-i\omega t)$$

and for the potential it is assumed the harmonic expansion

$$\Phi = \exp(-il\varphi) \sum_{m=-\infty}^{\infty} a_m \; \exp(im\theta)$$

Then

$$\begin{aligned}
f_i^{(1)} &= -\frac{e}{T_i} f_{Mi} \; \exp(-il\varphi) \sum_m a_m \; \exp\left[i\left(m\theta - l\varphi - \omega t\right)\right] \\
&\quad -i\frac{e}{T_i}\left(\omega + \frac{T_i}{T_e}\omega_{e,m_0}^{*}\right) f_{Mi} \int_{-\infty}^{t} dt' \sum_m a_m \; \exp\left[i\left(m\theta' - l\varphi' - \omega t'\right)\right]
\end{aligned}$$

where

$$m_0 = \text{integer part closest to } lq$$

The orbits are simple, linear

$$\begin{aligned}\theta' &= \theta + \frac{v_\parallel}{qR}\left(t'-t\right)\\ \varphi' &= \varphi + \frac{v_\parallel}{R}\left(t'-t\right)\end{aligned}$$

The equality

$$n_e = n_i$$

leads to the dispersion relation.

Now, in view of imposing neutrality, we calculate the density

$$\begin{aligned}n_i &= \int d^3v\ f_i\\ &= -n\frac{e}{T_i}\exp\left(-il\varphi\right)\sum_m a_m \exp\left(im\theta\right)\left[1+\frac{\omega^*_{em_0}\ \frac{T_i}{T_e}}{k_{\parallel,m}\ v_{th,i}} Z\left(\frac{\omega}{k_{\parallel,m}\ v_{th,i}}\right)\right]\end{aligned}$$

where

$$k_{\parallel,m} = \frac{|m-lq|}{qR}$$

11 Ion distribution function in the banana regime (Taguchi)

It is the paper **ion thermal conductivity banana regime Taguchi**.

The equation

$$v_\parallel \frac{B_\theta}{B}\frac{\partial f_i^{(1)}}{\partial\theta} + \mathbf{v}_{Di}\cdot\boldsymbol{\nabla}\psi\ \left(x^2-\frac{3}{2}\right)\frac{1}{T_i}\frac{\partial T_i}{\partial\psi} f_{Mi}$$

where

$$\begin{aligned}x &\equiv \frac{v}{v_{th,i}}\\ v_{th,i} &= \sqrt{\frac{2T_i}{m_i}}\\ \xi &\equiv \frac{v_\parallel}{v}\end{aligned}$$

$$\lambda = \frac{v_\perp^2}{v^2}\frac{1}{B\left(\mathbf{x}\right)}$$

$$x\xi = \frac{v}{v_{th,i}} \frac{v_\parallel}{v} = \frac{v_\parallel}{v_{th,i}}$$

The first order perturbation of the *ion distribution function* $f_i^{(1)}$ is expanded

$$f_{i1} = f_i^{(0)} + \nu_{ii} f_i^{(1)} + ...$$

where the ion-ion collision freqquency is

$$\begin{aligned} \nu_{ii} &= \frac{4\pi e_i^4 \ln \Lambda}{m_i^2} \frac{1}{v_{th,i}^3} n_i \\ &\sim \frac{n_i}{T_i^{3/2}} \end{aligned}$$

The zeroth-order perturbed function is

$$f_i^{(0)} = -\frac{m_i}{e_i} I \; v_{th,i} \left[\frac{v_\parallel}{v_{th,i}} \frac{1}{B} \left(x^2 - \frac{3}{2} \right) f_{Mi} + G(x, \lambda, \sigma) \right] \frac{1}{T_i} \frac{\partial T_i}{\partial \psi}$$

$$\begin{aligned} G &\equiv 0 \quad \text{for trapped particles} \\ G &\neq 0 \quad \text{for untrapped} \end{aligned}$$

12 Drift kinetic eq. for the ion - impurity collsional coupling (Hirshman Sigmar Clarke)

this is also in *general, impurities.*

There is electrostatic variation on magnetic surfaces $\widetilde{\Phi}(r, \theta)$ due to $n_Z(\theta)$.
then, there is $E_\parallel$ which further requires an *energetic* term in the drift kinetic equation.
This work also includes a variation of the magnetic field *in time*

$$\mu \frac{\partial B}{\partial t} \times \frac{\partial}{\partial \epsilon} f \quad \text{energy term}$$

For $f_t \ll 1$ (very few trapped particles) the *circulating particles collisionally couple along the magnetic field to minimize the interspecies friction* and produces *a common diamagnetic toroidal flow of the plasma.*

For large aspect ratio limit there are two time scales.
The fast one. Collisions between ions and the toroidal diamagnetic flows of the individual plasma species couple to produce a unique common flow.
The slow one is transport. The gradients are slowly modified by transport such that

- the common parallel flow is maintained
- equalizing the individual diamagnetic responses, to minimize the parallel friction

The equation for the species a is

$$\begin{aligned}
& \frac{\partial f_a}{\partial t} + \left(v_\parallel \widehat{\mathbf{n}} + \mathbf{v}_{d,a}\right) \cdot \boldsymbol{\nabla} f_a + \left[\mu \frac{\partial B}{\partial t} + \frac{e_a}{m_a}\left(v_\parallel E_\parallel^{(A)} + \frac{\partial \phi}{\partial t}\right)\right] \frac{\partial f_a}{\partial \epsilon} \\
= & \sum_b C_{ab}\left(f_a, f_b\right)
\end{aligned}$$

where

$$\epsilon = \frac{1}{2} v^2 + \frac{e_a}{m_a}\phi$$

with

$$e_a = Z_a \left|e\right|$$

$$\mathbf{E}^{(A)} = E_{\varphi 0} \frac{R_0}{R} \widehat{\mathbf{e}}_\varphi$$

the electric field produced by induction by the tokamak transformer.

The drift velocity

$$\begin{aligned}
\mathbf{v}_{d,a} &= -v_\parallel \widehat{\mathbf{n}} \times \boldsymbol{\nabla}\left(\frac{v_\parallel}{\Omega_{ca}}\right) \\
& \quad + \frac{v_\parallel^2}{\Omega_{ca}} \frac{1}{B} \left(\boldsymbol{\nabla} \times \mathbf{B}\right)_\perp
\end{aligned}$$

The last term is

$$\sim \frac{1}{\Omega_{ca}} v_\parallel^2 \frac{J_\perp}{B}$$

and is a drift of particles which exists when there is a current perpendicular to the magnetic field.

The magnetic field

$$\mathbf{B} = I\left(\psi\right) \boldsymbol{\nabla}\varphi + \boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi$$

$$I = R B_T$$

$$\mathbf{B}_p = \boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi$$

Since the drift velocity will multiply the spatial gradient of the zero-order distribution function,

$$\mathbf{v}_{d,a} \cdot \boldsymbol{\nabla} f_{a0}$$

and since $f_{a0} = f_{Ma}$ has only variation in a direction perpendicular on the magnetic surface

$$\boldsymbol{\nabla} f_{Ma} = \boldsymbol{\nabla}\psi \frac{\partial f_{Ma}}{\partial \psi}$$

we will have to use

$$\begin{aligned}
\mathbf{v}_{d,a} \cdot \boldsymbol{\nabla}\psi &= v_{\parallel} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \left(I \frac{v_{\parallel}}{\Omega_{ca}} \right) \\
&= v_{\parallel} \nabla_{\parallel} \left(I \frac{v_{\parallel}}{\Omega_{ca}} \right)
\end{aligned}$$

(in this form the term $\mathbf{v}_{d,a} \cdot \boldsymbol{\nabla} f$ which actually is $\mathbf{v}_{d,a} \cdot \boldsymbol{\nabla}\psi \frac{\partial f_{Ma}}{\partial \psi}$ can be coupled to $v_{\parallel} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} f_{a1}$).

The expansion

$$f_a = f_{a0} + f_{a1} + ...$$

$$f_{a0} = f_{Ma} = n_a(\psi) \frac{1}{\left[\pi \frac{2T_a(\psi)}{m_a}\right]^{3/2}} \exp\left[-\frac{\epsilon}{\frac{T_a(\psi)}{m_a}} \right]$$

$$n_a(\psi) = n_{a0} \exp\left[\frac{e_a \phi}{T_{a0}(\psi)} \right]$$

The equation for the first order

$$\begin{aligned}
& v_{\parallel} \nabla_{\parallel} f_{a1} + v_{\parallel} \nabla_{\parallel} \left(I \frac{v_{\parallel}}{\Omega_{ca}} \frac{\partial f_{a0}}{\partial \psi} \right) \\
& -v_{\parallel} \frac{e_a E^{(A)}}{T_a} f_{a0} \quad \text{(external } E\text{)} \\
= & \sum_b C_{ab}(f_{a1}, f_{b1})
\end{aligned}$$

Note that here we have taken $\partial f_{a0}/\partial \psi$ inside the paranthesis that comes from the drift velocity. Then it is convenient to separate from f_{a1} the part resulting from the drift motion advection of the equilibrium f_{a0} distribution function.

$$\begin{aligned}
f_{a1} &= -I \frac{v_{\parallel}}{\Omega_{ca}} \frac{\partial f_{a0}}{\partial \psi} \\
&\quad + g_a(\epsilon, \mu, \psi)
\end{aligned}$$

where (**Hirshman Sigmar Clarke**)

$$\begin{aligned}
-I \frac{v_{\parallel}}{\Omega_{ca}} \frac{\partial f_{a0}}{\partial \psi} &\equiv \text{"diamagnetic" response of the species } a \\
g_a &\equiv \text{collisional response of the species } a
\end{aligned}$$

For a comment on the name *diamagnetic* in relation with *gyration/bananas* see **the text** *reference plasma.*

Consider the surface average

$$\langle A(\mathbf{x})\rangle = \frac{2\pi}{\frac{\partial V}{\partial \psi}} \oint d\chi \frac{A}{\boldsymbol{\nabla}\chi\cdot\mathbf{B}}$$

$$V' \equiv \frac{\partial V}{\partial \psi} = 2\pi \oint d\chi \frac{1}{\boldsymbol{\nabla}\chi\cdot\mathbf{B}}$$

Returning to the drift-kinetic equation, we have

$$v_{\|}\nabla_{\|}\left(f_{a1} + I\frac{v_{\|}}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial\psi}\right) = v_{\|}\frac{e_a E^{(A)}}{T_a} f_{a0} + \sum_b C_{ab}\left(f_{a1}, f_{b1}\right)$$

We divide by $v_{\|}$ and note that the surface average is applied to

$$\begin{aligned} & \oint d\chi \frac{1}{\mathbf{B}\cdot\boldsymbol{\nabla}\chi}\nabla_{\|}() \\ = & \oint d\theta \frac{1}{B_\theta \frac{1}{r}}\frac{d}{dl_{\|}}() \end{aligned}$$

but

$$dl_{\|} = \frac{B}{B_\theta} dl_\theta = r d\theta \frac{B}{B_\theta}$$

Then

$$\frac{d}{dl_{\|}} = \frac{B_\theta}{B}\frac{d}{rd\theta}$$

and

$$\begin{aligned} \oint d\theta \frac{1}{B_\theta \frac{1}{r}}\frac{d}{dl_{\|}}() &= \oint d\theta \frac{1}{B_\theta\frac{1}{r}}\frac{B_\theta}{B}\frac{d}{rd\theta}() \\ &= \oint d\theta \frac{1}{B}\frac{d}{d\theta}() \end{aligned}$$

The magnitude of B is

$$B \approx \frac{B_0}{h} = B_0 \frac{1}{1+\varepsilon\cos\theta}$$

and B is a function of θ. But it is of order ε and multiples a quantity which is also of order ε. Then $\frac{1}{B}$ can be inserted in the paranthesis

$$\begin{aligned} \nabla_{\|}\left(f_{a1} + I\frac{v_{\|}}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial\psi}\right) &= \frac{1}{v_{\|}}\left[v_{\|}\frac{e_a E^{(A)}}{T_a} f_{a0} + \sum_b C_{ab}\left(f_{a1}, f_{b1}\right)\right] \\ \frac{B_\theta}{B}\frac{d}{rd\theta}\left(f_{a1} + I\frac{v_{\|}}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial\psi}\right) &= \frac{1}{v_{\|}}\left[v_{\|}\frac{e_a E^{(A)}}{T_a} f_{a0} + \sum_b C_{ab}\left(f_{a1}, f_{b1}\right)\right] \\ B_\theta\frac{d}{rd\theta}\left(f_{a1} + I\frac{v_{\|}}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial\psi}\right) &= \frac{B}{v_{\|}}\left[v_{\|}\frac{e_a E^{(A)}}{T_a} f_{a0} + \sum_b C_{ab}\left(f_{a1}, f_{b1}\right)\right] \end{aligned}$$

Now we take the surface average

$$\left\langle B_\theta \frac{d}{rd\theta}\left(f_{a1} + I\frac{v_\parallel}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial\psi}\right)\right\rangle = \left\langle \frac{B}{v_\parallel}\left[v_\parallel \frac{e_a E^{(A)}}{T_a} f_{a0} + \sum_b C_{ab}\left(f_{a1}, f_{b1}\right)\right]\right\rangle$$

In the left hand side we replace

$$\begin{aligned}
\langle(...)\rangle &= \frac{2\pi}{V'}\oint d\theta \frac{1}{B_\theta \frac{1}{r}}(...) \\
\left\langle B_\theta \frac{d}{rd\theta}\left(f_{a1} + I\frac{v_\parallel}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial\psi}\right)\right\rangle &= \frac{2\pi}{V'}\oint d\theta \frac{1}{B_\theta \frac{1}{r}} B_\theta \frac{d}{rd\theta}\left(f_{a1} + I\frac{v_\parallel}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial\psi}\right) \\
&= 0
\end{aligned}$$

due to periodicity.

NOTE

As explained in *particle equations of motion* before averaging one must multiply the equation with B.

This will turn the averaging operation into an integration over the poloidal angle θ of a total derivative to θ. Then the periodicity on θ will vanish one of the terms.

The general form is

$$\left\langle \frac{B}{v_\parallel} C(f)\right\rangle = 0$$

END

This is what results

$$\left\langle \frac{B}{v_\parallel}\left[v_\parallel \frac{e_a E^{(A)}}{T_a} f_{a0} + \sum_b C_{ab}\left(f_{a1}, f_{b1}\right)\right]\right\rangle = 0$$

(remember the calculation of the *bootstrap* current in **Helander**; the equation is the same).

The expression of the collision operator.

HSC write

$$\begin{aligned}
C_{ab}\left(f_{a1}, f_{b1}\right) &= \nu_{ab}^{defl}\ \frac{v_\parallel}{B}\frac{\partial}{\partial\mu}\mu v_\parallel \frac{\partial}{\partial\mu} f_{a1} \\
&\quad + \left[\nu_{ab}^{defl} - \nu_{ab}^{slowing}\right]\frac{v_\parallel u_{a1}(v)}{v^2}\ f_{a0} \\
&\quad + \frac{2v_\parallel}{v_{th,a}^2} r_{ba}\ \nu_{ab}^{slowing}\ f_{a0}
\end{aligned}$$

The first is the *pitch-angle scattering*.

The second, with ν_{ab}^{defl} of the paranthesis, implies two relative velocities even inside a species with elements in relative motion.

The second term of the paranthesis is a friction tending to suppress the differences in velocities.

The last term contains r_{ab} $\equiv$momentum restoring coefficient.

The definitions

The *deflection* frequency

$$\nu_{ab}^{defl} = \nu_{ab} \frac{\Phi\left(\frac{v}{v_{th,b}}\right) - G\left(\frac{v}{v_{th,b}}\right)}{\left(\frac{v}{v_{th,a}}\right)^3}$$

The *slowing down* frequency

$$\nu_{ab}^{slowing} = 2\frac{T_{a0}}{T_{b0}}\left(1 + \frac{m_b}{m_a}\right)\ \nu_{ab}\ \frac{G\left(\frac{v}{v_{th,b}}\right)}{\left(\frac{v}{v_{th,a}}\right)}$$

The frequency of collisions (**Trubnikov**)

$$\nu_{ab} = \frac{4\pi}{2^{3/2}} \frac{e_a^2 e_b^2}{\sqrt{m_a}} \ln\Lambda \frac{n_{b0}}{T_a^{3/2}}$$

The function

$$G(x) = \frac{\Phi(x) - x\frac{d\Phi(x)}{dx}}{2x^2}$$
$$\text{the Chandrasekhar function}$$

where

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt \exp\left(-t^2\right)$$

In the formulas above it has been introduced

$$\begin{aligned} u_{a1}(v) &= \frac{1}{f_{a0}} \frac{3}{4\pi} \int v_{\|} f_{a1} d\Omega \\ d\Omega &\equiv \text{solid angle in velocity space} \end{aligned}$$

$$d\Omega = \pi \sum_{\sigma=\pm 1} \frac{B d\mu}{\epsilon} \frac{1}{\left|v_{\|}\right| / v}$$

NOTE

A similar object is introduced by **Helander 3999**

$$u \equiv \tau_{iZ} \frac{1}{n_i B} \int d^3v\ mv_{\|}\ C_{iZ}\left(f_i^{(1)}\right)$$

END

the *momentum restoring* coefficient

$$r_{ab} \equiv \frac{\int d^3v \ \nu_{ba}^{slowing} \quad m_b v_\parallel \quad f_{b1}}{m_a n_{a0} \left\{ \nu_{ab}^{slowing} \right\}}$$

This quantity is defined on the basis of the momentum exchange between species a and b.

The following operator is introduced, for any function that depends on two species indices, a and b

$$\{X_{ab}(v)\} \equiv 2 \int d^3v \ \left(\frac{v_\parallel}{v_{th,a}} \right)^2 \ \frac{f_{a0}}{n_{a0}} \ X_{ab}(v)$$

To solve the drift-kinetic equation

$$\lambda \equiv \mu \frac{\left\langle B^2 \right\rangle^{1/2}}{\epsilon - \frac{e_a \phi(\psi)}{m_a}}$$

and choose

$$f_{a1} = \widehat{f}_{a1} f_{a0}$$

results

$$\begin{aligned}
\widehat{f}_{a1} &= -I v_\parallel \frac{1}{\frac{e_a \langle B^2 \rangle^{1/2}}{m_a}} \frac{\left\langle B^2 \right\rangle^{1/2}}{B} \frac{\partial}{\partial \psi} \ln f_{a0} \\
&\quad + I \frac{1}{\frac{e_a \langle B^2 \rangle^{1/2}}{m_a}} \Theta \left[V_\parallel \frac{\partial \ln f_{a0}}{\partial \psi} \right] \quad \text{(Heaviside-like excludes the Trapped region)} \\
&\quad + \Theta \left[\frac{2V_\parallel}{v_{th,a}^2} \left(\frac{e_a \overline{E}_\parallel^{(A)}}{m_a} \frac{1}{\nu_a^{defl}} + \left(1 - \frac{\nu_a^{slow}}{\nu_a^{defl}} \right) \frac{1}{2x_a^2} \overline{u}_{a1}(v) + \sum_b \overline{r}_{ab} \frac{\nu_{ab}^{slow}}{\nu_{ab}^{defl}} \right) \right]
\end{aligned}$$

The terms that contain the Heaviside-like function Θ only exist for circulating particles.

For example the first two terms can be

$$\begin{aligned}
& h v_\parallel - V_\parallel \quad \text{for passing particles} \\
& h v_\parallel \quad \text{for trapped particles}
\end{aligned}$$

The notations are

$$\frac{\left\langle B^2 \right\rangle^{1/2}}{B} \equiv h$$

$$x_a = \frac{v}{v_{th,a}}$$

$$V_{\parallel}(\lambda) = \int_{\lambda}^{\lambda_c} d\lambda \, \frac{\frac{v^2}{2}}{\left\langle v_{\parallel} \right\rangle}$$

$$\lambda_c = \frac{\left\langle B^2 \right\rangle^{1/2}}{B_c}$$

As shown below for a different notation (and definition of λ) the integral that defines $V_{\parallel}(\lambda)$ is on the interval

$$\begin{aligned} & [\lambda, \lambda_c] \quad \text{which is passing-domain} \\ \lambda_c \quad \equiv & \quad \text{limit passing } \rightarrow \text{ trapped} \end{aligned}$$

so that the integration is for *passing* particles, up to the boundary passing/trapped.

$$\begin{aligned} B_c(\psi) \quad = & \quad \text{maximum of } B \text{ in a surface } \psi \\ & \quad \text{it is } B \text{ at the farthest equatorial point} \end{aligned}$$

and

$$\nu_a^{defl} = \sum_b \nu_{ab}^{defl}$$

$$\nu_a^{slow} = \sum_b \nu_{ab}^{slow}$$

a new avareging

$$\overline{A} \equiv \left\langle \frac{A}{h} \right\rangle$$

NOTE and Comment on parameters
This is in *particle equations of motion*
parameters

$$w = \frac{v^2}{2} = \epsilon - \frac{e\phi}{m}$$

$$\lambda = \frac{\mu}{w}$$

or

$$\lambda = \frac{v_{\perp}^2}{v^2} \frac{1}{B(\mathbf{x})}$$

We have

$$B(\mathbf{x}) = \frac{B_0}{h}$$

and

$$\lambda \rightarrow \lambda' = \frac{v_{\perp}^2}{v^2} h$$

then, since here $\lambda = \frac{v_{\perp}^2}{v^2} \frac{1}{B(\mathbf{x})}$ we have

$$v_{\parallel} = \sigma \sqrt{2w(1 - \lambda B)}$$

We define

$$\begin{aligned}
\lambda_m &= \frac{1}{B(\mathbf{x})} \\
&= \text{the largest } \lambda \text{ for which the function} \\
&\quad\ f(\mathbf{x},\lambda,w) \text{ is defined} \\
\text{corresponds to } v_\perp^2 &= v^2 \ \text{ (no parallel velocity, deep trapped)}
\end{aligned}$$

and

$$\begin{aligned}
\lambda_c &= \frac{1}{B_{\max}} \\
&\quad \text{where} \\
B_{\max} &= \text{the maximum of } |\mathbf{B}| \text{ along a field line}
\end{aligned}$$

or

$$B_{\max} = \frac{B_0}{h_{\min}} = \frac{B_0}{1-(r/R_0)}$$

then

$$\lambda_c = \frac{1-(r/R_0)}{B_0}$$

and λ_c is the critical λ for trapping.

The *trapped region* is

$$\lambda_c < \lambda < \lambda_m$$

the *untrapped* (passing, circulating) region

$$0 < \lambda < \lambda_c$$

All *circulating* particles have the perpendicular velocity sufficiently small (*i.e.* λ small) such that at the given energy the parallel velocity to be high enough for the particle to overcome the magnetic barrier along the line.

Then λ must be *small* for the particle to be *circulating*.

END of Note and Comment

To obtain an explicit expression for $\widehat{f}_{a1}$ we must eliminate u_{a1}. The present form of $\widehat{f}_{a1}$, which contains $\overline{u}_{a1}$ is introduced in the formula for u_{a1} ($u_{a1}(v) = \frac{1}{f_{a0}}\frac{3}{4\pi}\int v_\parallel f_{a1} d\Omega$) and the integration over the angular space $d\Omega$ is performed. The result is an equation for u_{a1}.

After flux averaging

$$\begin{aligned}
\frac{\overline{u}_{a1}(v)}{x_a^2} &= -f_T\, I\frac{v_{th,a}^2}{\widehat{\Omega}_a}\frac{\partial}{\partial\psi}\ln f_{a0} \times \frac{\nu_a^{defl}}{\nu_a} \\
&\quad +2f_c \left(\frac{e_a\overline{E}_\parallel^{(A)}}{m_a}\frac{1}{\nu_a} + \sum_b \overline{r}_{ab}\frac{\nu_{ab}^{slow}}{\nu_a}\right)
\end{aligned}$$

where

$$\frac{3}{4\pi}\int d\Omega \; v_{\parallel} \; \Theta\left[V_{\parallel}\right] = f_c \; v^2 \frac{1}{h}$$

Note that due to the presence of Heaviside-like Θ the integration is made over the circulating particle domain.

The proportions of circulating and trapped particles

$$\begin{aligned}
f_c &= \frac{3}{4}\int_0^{\lambda_c} \frac{\lambda d\lambda}{\left\langle \sqrt{1-\frac{\lambda}{h}} \right\rangle} \\
f_T &= 1 - f_c \\
&\approx 1.46 \times \sqrt{\varepsilon}
\end{aligned}$$

$$\nu_a = f_c \nu_a^{slow} + f_T \nu_a^{defl}$$

This frequency of collision combines the slowing-down collisions (friction) due to *circulating* particles and the deflection collsions due to the *trapped* particles.

The function $\overline{u}_{a1}(v)$ is replaced in the expression of $\widehat{f}_{a1}$.

$$\begin{aligned}
\widehat{f}_{a1} &= -I\frac{1}{\widehat{\Omega}_a} h v_{\parallel} \frac{\partial}{\partial \psi} \ln f_{a0} \\
&+ I\frac{1}{\widehat{\Omega}_a}\frac{\nu_{slow}^a}{\nu^a}\Theta\left[V_{\parallel}\right]\frac{\partial}{\partial \psi}\ln f_{a0} \\
&+ \Theta\left[\frac{2V_{\parallel}}{v_{th,a}^2}\left(\frac{e_a \overline{E}_{\parallel}^{(A)}}{m_a}\frac{1}{\nu_a} + \sum_b \frac{\nu_{ab}^{slow}}{\nu_a}\overline{r}_{ba}\right)\right]
\end{aligned}$$

The *restoring coefficients* must still be determined.

Transport

This is

$$\begin{aligned}
\Gamma_a &= \left\langle \int d^3v \; \left(\mathbf{v}_{Da}\cdot\boldsymbol{\nabla}\psi\right) f_a \right\rangle \\
&= -I(\psi)\left\langle \frac{\sum_b R_{ab}}{m_a \Omega_a} \right\rangle - I(\psi)\left\langle \frac{n_{a0}E_{\parallel}^{(A)}}{B} \right\rangle
\end{aligned}$$

where

$$\begin{aligned}
R_{ab} &= \int d^3v \; v_{\parallel} C_{ab}\left(f_{a1}, f_{ba}\right) \\
&= \text{parallel collisional friction} \\
&\quad\;\text{between species } a \text{ and } b
\end{aligned}$$

Note that, as usual, the parallel force (here *friction*) produces perpendicular flux.

There is a velocity V parallel with the field, which results from *neoclassical momentum balance*.

It results from the collisional coupling between passing particles among themselves and between passing particles and the trapped particles.

[**Observation**. In the absence of the component of trapped particles, *i.e.* for

$$f_T \to 0$$

the velocity V is *indeterminate*. This is because Galilean transformation is arbitrary and does not allow to fix a frame. Only if there is *trapping* which is equivalent to the existence of *modulation of the magnitude of the magnetic field* along the line, which is equivalent to provide a choice for the reference system and suppresses the arbitrary Galilean transformation.]

Pitch angle scattering in the region close to the separatrix *trapped/passing* λ_c produces a continuous flow of particles in velocity space, from trapped to passing.

Next the interspecies collisions will try to eliminate the differences between the velocities of the flows of *passing* particles.

First case

A plasma consisting of electrons and one species of ions.

In this simple plasma there is strong coupling between electrons and ions, since

$$\nu^{ee} \sim \nu^{ei}$$

and there is easily a common flow. It is defined by

$$R_{ii} = 0$$

where R_{ii} is the parallel friction force ion-ion.

Second case

A plasma with electrons and ions and impurity ions.

There is coupling of electrons to ions.

There is also coupling of ions to impurity ions.

For this latter case, assume

$$\nu^{ii} > \nu^{zi}$$

frequent ion-ion colisions

impurities are not coupled

Now assume

$$\nu^{zi} \geq \nu^{zz}$$

the impurity colides maily with ions

than between them

Then

$$R_{iz} \to 0$$

means that a flow will be established such that

$$u_{\|z} = u_{\|i}$$

ions and impurities flow together.

13 Solution in Bootstrap of fusion born ALPHA particles (Hsu Shaing Gormley Sigmar)

See *plasma, general theory, bootstrap* and in *models, diamagnetic bootstrap.*

The distribution function for α particles is written in the *ion rest frame.* First correction

$$\begin{aligned}
f_\alpha^{(1)} &= -I\frac{v_\|}{\Omega_\alpha}\frac{\partial f_\alpha^{(0)}}{\partial\psi} \quad \text{(neoclassical correction)} \\
&+v_\| V_{\|i}^* \frac{\partial f_\alpha^{(0)}}{\partial\epsilon} \\
&+P\left(\lambda,\epsilon,\psi\right)
\end{aligned}$$

and the function P

$$P\left(\lambda,\epsilon,\psi\right) = \sum_{j=1,2,3}\left(\sum_{n=1}^{\infty}\Lambda_n\left(\lambda,\psi\right)\; V_{nj}\left(\epsilon,\psi\right)\right)\; A_j\left(\epsilon,\psi\right)$$

where

$$\epsilon = \frac{v^2}{2}$$

$$\lambda = \frac{v_\perp^2}{v^2}h$$

$$\Omega_j = \frac{Z_j eB}{m_j}$$

$$I = \mathbf{B}\cdot R^2\boldsymbol{\nabla}\varphi$$

and the velocity

$$\begin{aligned}
V_{\|i}^* &= \frac{1}{n_i}\left(-I\frac{1}{m_i\Omega_i}\frac{\partial p_i}{\partial\psi}\right) \\
&+K_i B
\end{aligned}$$

This is the parallel flow from which it is subtracted the $\mathbf{E}\times\mathbf{B}$ drift induced return flow (*i.e.* it is absent in the formula, which only remains in terms of the gradient of pressure term)

Here

$$
\begin{aligned}
K_i &= \mathbf{V}_i \cdot \frac{\boldsymbol{\nabla}\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta} \\
& \quad \text{ion poloidal flow}
\end{aligned}
$$

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The paper uses the kinetic drift equation to study instabilities generated in a sheared parallel flow.

The equations of motion of the particle

$$
\begin{aligned}
\frac{d\mathbf{r}'}{dt} &= \mathbf{v}' \quad , \quad \text{with the initial condition } \mathbf{r}'(t'=t) = \mathbf{r} \\
\frac{d\mathbf{v}'}{dt} &= \frac{e}{m_i}\mathbf{v}' \times \mathbf{B}_0(x') \quad \text{with the condition } \mathbf{v}'(t'=t) = \mathbf{v}
\end{aligned}
$$

From these equations the following invariants result

energy

$$
|\mathbf{v}'|^2 = |\mathbf{v}|^2 = v^2
$$

and from conservation of the canonical angular momentum

$$
\begin{aligned}
\frac{d}{dt}\left(Rmv_\varphi + eRA_\varphi\right)|_\alpha &= 0 \\
\alpha &\equiv \text{species}
\end{aligned}
$$

one obtains

$$
x' + \frac{v_y'}{\Omega_i} = x + \frac{v_y}{\Omega}
$$

The x direction is radial. The y direction is poloidal.

$$
v_z' + \frac{x'}{L_s}v_y' + \frac{v_y'^2}{2\Omega_i L_s} = v_z + \frac{x}{L_s}v_y + \frac{v_y^2}{2\Omega_i L_s}
$$

The shear length is L_s.

The distribution function

$$
\begin{aligned}
f_0 = & \frac{n_0}{\left(2\pi v_{th,i}^2\right)^{3/2}} \exp\left\{ \frac{x + \frac{v_y}{\Omega_i}}{L_n} \right. \\
& -\frac{1}{2v_{th,i}^2}\left[v^2\right. \\
& \qquad -2\left(v_z + \frac{x}{L_s}v_y + \frac{v_y^2}{2\Omega_i L_s}\right)\left[u + \frac{dU}{dx}\left(x + \frac{v_y}{\Omega_i}\right)\right] \\
& \qquad \left. \left. + \left[u + \frac{dU}{dx}\left(x + \frac{v_y}{\Omega_i}\right)\right]^2 \right\} \right.
\end{aligned}
$$

It has been chosen

$$\begin{aligned} B_0 &> 0 \\ \frac{dU}{dx} &> 0 \end{aligned}$$

in the system $(\widehat{\mathbf{e}}_z, \widehat{\mathbf{e}}_x)$. The definition of the thermal velocity

$$v_{th,i} = \sqrt{\frac{k_{Boltzmann} T_i}{m_i}}$$

(remark no 2).

The velocities

$$\begin{aligned} u\,(\text{od ions}) &\approx 0 \\ u\,(\text{of electrons}) &= \text{finite, produces current} \end{aligned}$$

The sheared parallel velocity

$$\begin{aligned} \mathbf{u} &= \mathbf{U}\,(x) \\ &= \left(\frac{v_{th,i}^2}{\Omega_i L_n}\right) \widehat{\mathbf{e}}_y + \left(u + \frac{dU}{dx} x\right) \left[\widehat{\mathbf{e}}_z + \frac{x}{L_s} \widehat{\mathbf{e}}_y\right] \end{aligned}$$

If gyroradius corrections are neglected

$$f_0 = \frac{n_0\,(x)}{\left(2\pi v_{th,i}^2\right)^{3/2}} \exp\left(-\frac{v_x^2 + v_y^2 + \left[v_z - \left(u + \frac{dU}{dx} x\right)\right]^2}{2 v_{th,i}^2}\right)$$

Note the distinction between the need for the combination

$$x + \frac{v_y}{\Omega}$$

and the first correction to the distribution function

$$f = f_M - \frac{I v_\parallel}{\Omega} \frac{\partial f_M}{\partial \psi} + g$$

The first is due to the Larmor gyration. It will involve in the problem the Bessel expansion of argument

$$\frac{k_\perp v_\perp}{\Omega}$$

and will modify the dispersion relations such that to depend on the gyration radius.

The second is a manifestation of the *neoclassical* correction that involves an equivalent *poloidal Larmor radius,* ρ_θ which comes from the drift of the particles.

End.

This work is actually for instabilities (see *instabilities*). However the solution is interesting here too

$$f\left(\mathbf{r},\mathbf{v},t\right)=\frac{e}{m_i}\int_{-\infty}^{0}d\tau\ \boldsymbol{\nabla}'\Phi\left(\mathbf{r}',t'=\tau+t\right)\cdot\boldsymbol{\nabla}_{\mathbf{v}'}f_0\left(x',\mathbf{v}'\right)$$

After applying the operator of derivation to the velocity ($\boldsymbol{\nabla}_{\mathbf{v}'}$) one obtains from the Maxwellian a factor $\mathbf{v}'$. This gives

$$\begin{aligned}&\mathbf{v}'\cdot\boldsymbol{\nabla}'\Phi\left(\mathbf{r}',\tau+t\right)\\=\quad&\frac{d\Phi}{dt}-\frac{\partial\Phi}{\partial\tau}\end{aligned}$$