Derivation of the drift-kinetic equation in a magnetized plasma

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1 Introduction

1.1 General framework

It is a complex problem the possibility to describe the collective behavior of an ensemble of charged particles in electromagnetic field. It is easier to understand that we need a theory of probability for every particle to be found in a certain infinitesimal volume of the phase space, but this is hardly an effective instrument of work, when the number of particles is $\sim 10^{20}\ m^{-3}$. Coming down, from this complete but inaccessible description to an effective instrument, the distribution function $f(\mathbf{x}, \mathbf{v}, t)$, represents a complicated theoretical construction, which we do not attampt, - it is a matter belonging to fundamental treaties on physics.

We will show how one can derive the equation verified by the distribution function $f(\mathbf{x}, \mathbf{v}, t)$ in a plasma embedded in a strong magnetic field.

1.1.1 The Liouville theorem

The phase space distribution function is constant along the trajectories of the system.

The meaning is as follows. The state at time t of a system with n degrees of freedom, can be represented as a point in the phase space of dimension $2n$, with coordinates $(q_i, p_i)_{i=1,n}$. From an initial state ($t = 0$) the system evolves according to the specific dynamics and its successive states form a line in the space of $2n$ coordinates. The ensemble of states at time t is a new set of points. Relative to the set of initial points, the shape of the set at time t will change but not the volume (simply: no dissipation).

The system is said to be incompressible in the phase space.

One can define the *probability to find the system state* (i.e. the current point of the trajectory) in an infinitesimal volume around some point, at time t

$$\rho\left(p,q,t\right)d^{2}pd^{n}q$$

The Liouville equation is

$$\frac{\partial\rho}{\partial t}+\sum_{i=1}^{n}\left(\frac{\partial\rho}{\partial q_{i}}\frac{dq_{i}}{dt}+\frac{\partial\rho}{\partial p_{i}}\frac{dp_{i}}{dt}\right)=0$$

is just the equation of continuity in the phase space, $d\rho/dt=0$.

conservation of density in the phase space

1.1.2 The Chapman Kolmogorov equation

The definition of *conditional probability*

$$p_{1,1}\left(y_{2},t_{2}|y_{1},t_{1}\right)$$

is the probability that a *stochastic process* $Y\left(t\right)$ takes the value y_2 at t_2 given that its value at t_1 is y_1. The indice $1,1$ means that there is one initial state and there is one final state.

A Markov process is a stochastic process with the property that for any set of n successive times

$$t_{1}<t_{2}<...<t_{n}$$

one has

$$\begin{aligned}&p_{1,n-1}\left(y_{n},t_{n}\mid y_{1},t_{1},...,y_{n-1},t_{n-1}\right)\\=\;&p_{1,1}\left(y_{n},t_{n}\mid y_{n-1},t_{n-1}\right)\end{aligned}$$

i.e. the probability density at t_n is fully determine by the preceding "event", y_{n-1} at t_{n-1}, and is independent of the previous steps. A Markov process is determined by the probability of the first "event" y_1 value taken at t_1 and by the transition probability $p_{1,1}\left(y_{2}t_{2}\mid y_{1},t_{1}\right)$. The transition probability connects two successive states ("events") and is applied repetitively to construct the series of values y_i taken at t_i. An example of Markovian process is the *Brownian* motion.

Usually one says that such processes have no memory. Many processes in physics *have* memory.

Consider the moments of time

$$t_{1}<t_{2}<t_{3}$$

The Chapman Kolmogorov equation in its simplest form is

$$p_{1,1}(y_3, t_3 \mid y_1, t_1) = \int p_{1,1}(y_3 t_3 \mid y_2, t_2)\, p_{1,1}(y_2, t_2 \mid y_1 t_1)\ dy_2$$

(N.G. van Kampen notations)

Consider a set of random variables, to which we assign the labels

$$(i_1, ..., i_n)$$

which take values like

$$(f_1, f_2, ..., f_n)$$

This is a *stochastic process*. [like a time series of a stochastic variable]

Consider the joint probability (density) that the random variable i_1 takes the value f_1, that the random variable i_2 takes the value f_2, ... :

$$p_{i_1,i_2,...,i_n}$$

The Chapman Kolmogorov equation is

$$\begin{aligned} & p_{i_1,i_2,...,i_{n-1}}(f_1, f_2, ..., f_{n-1}) \\ = & \int_{-\infty}^{\infty} p_{i_1,i_2,...,i_n}(f_1, f_2, ..., f_{n-1}, f_n)\ df_n \end{aligned}$$

When the process is Markovian, the CK equation provides the relation between *transition probabilities.*

The property of the process to be Markovian results in expressing the *joint probability* as a product of transition probabilities

$$\begin{aligned} & p_{i_1,i_2,...,i_{n-1}}(f_1, f_2, ..., f_{n-1}) \\ = & p_{i_1}(f_1) \\ & \times p_{i_2,i_1}(f_2 \mid f_1) \\ & \times p_{i_3,i_2}(f_3 \mid f_2) \\ & ... \\ & \times p_{i_n,i_{n-1}}\left(f_{i_n} \mid f_{i_{n-1}}\right) \end{aligned}$$

Define the transition probabilities between two times ordered as

$$t_i > t_j$$

$$p_{i;j}(f_i \mid f_j)$$

The CK equation gives

$$p_{i_3,i_1}\left(f_3 \mid f_1\right) = \int_{-\infty}^{\infty} p_{i_3,i_2}\left(f_3 \mid f_2\right)\ p_{i_2,i_1}\left(f_2 \mid\ f_{i_1}\right)\ df_2$$

Integration over all intermediate states (f_2) that can participate to the transition of the system from $i_1 = f_1$ to $i_3 = f_3$.

Examples

- the *Wiener* or *Wiener-Levy* process,

$$p_{1,1}\left(y_2, t_2 \mid y_1, t_1\right) = \frac{1}{\sqrt{2\pi\left(t_2 - t_1\right)}} \exp\left[-\frac{\left(y_2 - y_1\right)^2}{2\left(t_2 - t_1\right)}\right]$$

(diffusive behavior of the Brownian motion).

- the *Poisson* process, for discrete values

$$p_{1,1}\left(n_2, t_2 \mid n_1, t_1\right) = \frac{\left(t_2 - t_1\right)^{\left(n_2 - n_1\right)}}{\left(n_2 - n_1\right)!} \exp\left[-\left(t_2 - t_1\right)\right]$$

The particular case of a stationary Markovian stochastic process where the *transition probability*

$$p_{1,1}\left(y_2, t_2 \mid y_1, t_1\right)$$

only depends not on t_1 and t_2 but on the difference

$$t_2 - t_1 \equiv \tau$$

The notation is adopted

$$p_{1,1}\left(y_2, t_2 \mid y_1, t_1\right) \equiv T_\tau\left(y_2 \mid y_1\right)$$

with the Chapman Kolmogorov equation taking the form

$$T_{\tau+\tau'}\left(y_3 \mid y_1\right) = \int T_{\tau'}\left(y_3 \mid y_2\right)\ T_\tau\left(y_2 \mid y_1\right)\ dy_2$$

The differential form of the CK equation is the *master equation.* It is obtained by taking the limit $\tau' \to 0$ and using the *transition probability per unit time*

$$W\left(y_2 \mid y_1\right)$$

The CK eq. takes the form

$$\frac{\partial}{\partial\tau} T_\tau\left(y_3 \mid y_1\right) = \int \left\{W\left(y_3 \mid y_2\right)\ T_\tau\left(y_2 \mid y_1\right) - W\left(y_2 \mid y_3\right) T_\tau\left(y_3 \mid y_1\right)\right\} dy_2$$

This is the *master equation.*

The discrete form

$$\frac{dp_n(t)}{dt} = \sum_{n'} \{W_{nn'}\, p_{n'}(t) - W_{n'n}\, p_n(t)\}$$

the master equation is a gain-loss equation for the probabilities of the discrete states n

The first term $W_{nn'}\, p_{n'}(t)$ is the gain of the state n by transitions from the other states n' and the second term $-W_{n'n}\, p_n(t)$ is the loss from the state n by transitions to other states n'.

One can introduce a *matrix* that connects the discrete states n, n'

$$\mathfrak{W}_{nn'} = W_{nn'} - \delta_{nn'} \left(\sum_{n''} W_{n''n} \right)$$

such that the above form of the *master* equation can be written

$$\frac{d}{dt} p_n(t) = \sum_{n'} \mathfrak{W}_{nn'}\, p_{n'}(t)$$

1.1.3 The Fokker Planck equation

It is a *master equation* where $\mathfrak{W}$ has been taken as a second order differential operator

$$\frac{\partial P(y,t)}{\partial t} = -\frac{\partial}{\partial y} [A(y) P] + \frac{1}{2} \frac{\partial^2}{\partial y^2} [B(y) P]$$

It is an equation of conservation of the density of probability

$$\frac{\partial P}{\partial t} + \frac{\partial}{\partial y} J = 0$$

where the *current (or flux) of probability* is

$$J(y,t) = A(y) P - \frac{1}{2} \frac{\partial}{\partial y} [B(y) P]$$

Consider $1D$.

A process X_t

The Wiener process

$$W_t$$

The drift is a deterministic part of the dynamics

$$\mu\left(X_t, t\right)$$

The coefficient of the Wiener process W_t is

$$\sigma\left(X_t, t\right)$$

with the property that

$$D\left(X_t, t\right) = \frac{1}{2}\sigma^2\left(X_t, t\right)$$

becomes the *diffusion coefficient* (e.g. the mean square displacement of a particle increases linearly in time, with the coefficient given by $2D$)

The *stochastic equation*

$$\begin{aligned} dX_t &= \mu\left(X_t, t\right)\ dt \\ &\quad +\sigma\left(X_t, t\right)\ dW_t \end{aligned}$$

This is **Ito** stochastic differential equation.

Then it is defined the *probability density* of the stochastic process X_t

$$p\left(x, t\right)$$

as solution of

$$\frac{\partial}{\partial t}p\left(x, t\right) = -\frac{\partial}{\partial x}\left[\mu\left(x, t\right)p\left(x, t\right)\right] + \frac{\partial^2}{\partial x^2}\left[D\left(x, t\right)p\left(x, t\right)\right]$$

This is the **Fokker Planck** equation.

For plasma, the probability density function for species s population is the *distribution function*

$$p_s\left(\mathbf{x}, \mathbf{v}, t\right)$$

which verifies the equation

$$\begin{aligned} & \frac{\partial}{\partial t}p_s + \left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)p_s + \frac{e}{m}\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)\cdot\frac{\partial}{\partial \mathbf{v}}p_s \\ = & -\frac{\partial}{\partial v_i}\left(p_s\left\langle \Delta v_i\right\rangle\right) + \frac{1}{2}\frac{\partial^2}{\partial v_i \partial v_j}\left(p_s\ \left\langle \Delta v_i\ \Delta v_j\right\rangle\right) \end{aligned}$$

The Fokker Planck equation which is behind this equation has contributed with the RHS terms, where change of velocity Δv due to collisions have been retained.

With appropriate representation of the collisions, this is the **Boltzmann equation**. The Boltzmann equation has been written to describe a particular problem: the time evolution in the $6-$dimensional phase space $(\mathbf{x}, \mathbf{p})$ of the density of probability to find a particle in an infinitesimal volume around a point.

Without collision it is the **Vlasov equation**. Fields are created by the charged particles themselves (plus the externally imposed) and the particles interact between them via these fields.

1.1.4 Drift kinetic equations

The drift kinetic equation is derived from the basic definition

$$\frac{df}{dt} = C[f]$$

where d/dt is the operator of derivation along the particle trajectory in the space $(\mathbf{x}, \mathbf{v})$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} + \frac{\mathbf{F}}{m} \cdot \frac{\partial}{\partial \mathbf{v}}$$

where $\mathbf{F}$ is the force acting on the particle of mass m. $C[f]$ is the operator of collisions.

From this general equation, in developing a form that is more convenient to a particular application, we must introduce what we know about the trajectory of the particle $[\mathbf{x}(t), \mathbf{v}(t)]$. It is necessary to use variables that are adapted to the geometry where the system evolves, for example, for a toroidal geometry, a change of space variables is useful

$$\mathbf{x}(t) \rightarrow [r(t), \theta(t), \varphi(t)]$$

and, for velocity variables

$$\mathbf{v}(t) \rightarrow \left[v_{\parallel}, \lambda, \zeta\right]$$

as described below. It is important to identify the *invariants* of motion, $\epsilon \equiv$energy, $\mu \equiv$ the magnetic moment, $J \equiv$ the longitudinal invariant.

We must take into account the separation of scales

$$\begin{aligned} \rho_L &\equiv \text{Larmor gyration radius} \\ &\sim 0.001 \;\; (m) \;\; \text{order of magnitude} \end{aligned}$$

$$\begin{aligned} a \text{ or } qR & \text{ some large scale} \\ \sim & \; 1 \; (m) \ldots 10 \; (m) \quad \text{O-of-M} \end{aligned}$$

This provides the first small parameter that will allow us to make a perturbative expansion

$$\delta = \frac{\rho_L}{a} \ll 1$$

Another separation of scales

$$\begin{aligned} \Omega_c & \equiv \frac{eB}{m} \text{frequency of the cyclotron gyration} \\ & \sim 10^8 \quad \left(s^{-1}\right) \end{aligned}$$

$$\begin{aligned} \nu_t & \equiv \text{ frequency associated to transit motions} \\ & \sim 10^3 \; \left(s^{-1}\right) \quad (\sim \text{a millisecond}) \end{aligned}$$

The different type of particle trajectories (trapped or circulating) and the changes between the populations, induced by collisions,

$$\begin{aligned} \nu_c & \equiv \text{ frequency of collisions} \\ & \sim 10^5 \; \left(s^{-1}\right) \end{aligned}$$

introduce a new parameter

$$\frac{\nu_t}{\nu_c}$$

that can be small.

The large separation of scales suggests averages over the very fast gyration. The result is the drift kinetic equation.

The neoclassical transport is decribed by a drift kinetic equation which provides the time-dependence of the *guiding center* distribution function, under the influence of the static (inhomogeneous) magnetic field and two-body Coulombian collisions

The drift kinetic equation takes different forms according to the regime or the objective of the calculations.

The coordinates are

$$\begin{aligned} r & \equiv \text{ radial} \\ \theta & \equiv \text{ poloidal} \end{aligned}$$

and in velocity space

$$\lambda \equiv \frac{v_\perp^2}{v^2} h \qquad \text{where} \qquad h = 1 + \frac{r}{R}\cos\theta$$

$$v_\parallel = \sigma\sqrt{\epsilon - \mu B - e\phi}$$

The equations in phase space for a particle (a small variation of the electrostatic potential, $\phi_1(r,\theta)$ has been included, beside $\phi_0(r)$)

$$\begin{aligned}
\frac{dx_r}{dt} &= -\frac{1}{\Omega_0}\frac{v_\parallel^2+\lambda}{R}\sin\theta + \frac{1}{B_0}\left(-\frac{\partial\phi_1}{r\partial\theta}\right) \\
\frac{dx_\theta}{dt} &= v_\parallel\frac{B_\theta}{B_T} - \frac{1}{B_0}\left(-\frac{d\phi_0}{dr}\right) \\
\frac{d\lambda}{dt} &= \lambda\, v_\parallel\frac{B_\theta}{B_T}\,\frac{1}{R}\sin\theta - \lambda\frac{1}{B_0}\left(-\frac{d\phi_0}{dr}\right)\,\frac{1}{R}\sin\theta \\
\frac{dv_\parallel}{dt} &= -\lambda\,\frac{B_\theta}{B_T}\,\frac{1}{R}\sin\theta - v_\parallel\frac{1}{B_0}\left(-\frac{d\phi_0}{dr}\right)\,\frac{1}{R}\sin\theta + \frac{e}{m}\frac{B_\theta}{B_T}\left(-\frac{\partial\phi_1}{r\partial\theta}\right)
\end{aligned}$$

For the approximate drift kinetic equation

$$\left(\frac{d\mathbf{x}}{dt}\cdot\boldsymbol{\nabla}\right) f + \frac{dv_\parallel}{dt}\frac{\partial f}{\partial v_\parallel} + \frac{d\lambda}{dt}\frac{\partial f}{\partial\lambda} = C(f,f)$$

2 Change of variables in the Fokker-Planck equation

The equation is

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial\mathbf{x}} + \frac{\mathbf{F}}{m}\cdot\frac{\partial f}{\partial\mathbf{v}} = C(f) \tag{1}$$

The new variables are

$$(\mathbf{x},\epsilon,\mu,\zeta,t) \tag{2}$$

with the meaning:

$$\epsilon \equiv \frac{1}{2}\mathbf{v}^2 + \frac{e}{m}\phi \text{ the total particle energy} \tag{3}$$

per unit of mass

$$\mu \equiv \frac{v_\perp^2}{2B} \text{ the magnetic moment} \tag{4}$$

per unit of mass

$$\zeta \equiv \text{ the angle in the Larmor gyration plane} \tag{5}$$

The connection with the particle velocity

$$\begin{aligned}
\mathbf{v} =& v_{\parallel}\widehat{\mathbf{n}} + \mathbf{v}_{\perp} \\
v_{\parallel} =& \pm\sqrt{2\left(\epsilon - \frac{e}{m}\phi - \mu B\right)} \\
\mathbf{v}_{\perp} =& v_{\perp}\left(\widehat{\mathbf{e}}_1 \cos\zeta + \widehat{\mathbf{e}}_2 \sin\zeta\right)
\end{aligned}$$

The change of variables is

$$(\mathbf{x}, \mathbf{v}, t) \rightarrow (\mathbf{x}, \epsilon, \mu, \zeta, t)$$

and we will calculate the Jacobian.

2.1 The Jacobian in velocity space

In the space of velocity the element of volume is

$$dv_x dv_y dv_z = \frac{\partial\left(v_x, v_y, v_z\right)}{\partial\left(\epsilon, \mu, \zeta\right)} d\epsilon d\mu d\zeta$$

The entries of the matrix that multiplies the elemnt of volume in the new variables will be calculated as follows.

For later use we calculate

$$\begin{aligned}
\frac{\partial v_{\parallel}}{\partial \epsilon} &= \frac{\partial}{\partial \epsilon}\sqrt{2\left(\epsilon - \frac{e}{m}\phi - \mu B\right)} \\
&= \frac{1}{2}\frac{1}{\sqrt{2\left(\epsilon - \frac{e}{m}\phi - \mu B\right)}}2 \\
&= \frac{1}{v_{\parallel}}
\end{aligned}$$

For

$$\frac{\partial v_x}{\partial \epsilon}$$

We have

$$\begin{aligned} v_x &= v_\perp \left(\widehat{e}_{1x} \cos\zeta + \widehat{e}_{2x} \sin\zeta\right) + v_\parallel n_x \\ \frac{\partial v_x}{\partial \epsilon} &= 0 + \frac{\partial v_\parallel}{\partial \epsilon} n_x \\ &= \frac{1}{v_\parallel} n_x \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\partial v_x}{\partial \epsilon} &= \frac{1}{v_\parallel} n_x \\ \frac{\partial v_y}{\partial \epsilon} &= \frac{1}{v_\parallel} n_y \\ \frac{\partial v_z}{\partial \epsilon} &= \frac{1}{v_\parallel} n_z \end{aligned}$$

For

$$\frac{\partial v_x}{\partial \mu}$$

the expression of the x component of the velocity is

$$v_x = v_\perp \left(\widehat{e}_{1x} \cos\zeta + \widehat{e}_{2x} \sin\zeta\right) + v_\parallel n_x$$

We have

$$\mu = \frac{v_\perp^2}{2B} \ , \ v_\perp = \sqrt{2\mu B} \ , \ \frac{\partial v_\perp}{\partial \mu} = \frac{v_\perp}{2\mu}$$

and

$$\begin{aligned} v_\parallel^2 &= 2\left(\epsilon - \frac{e}{m}\phi - \mu B\right) \\ \frac{\partial v_\parallel}{\partial \mu} &= -\frac{B}{\left|v_\parallel\right|} \end{aligned}$$

The absolute value taken in the denominator means that when μ increases the parallel velocity $v_\parallel$ decreases at fixed energy ϵ. Together with the other components we have

$$\begin{aligned} \frac{\partial v_x}{\partial \mu} &= \frac{v_\perp}{2\mu} \left(\widehat{e}_{1x} \cos\zeta + \widehat{e}_{2x} \sin\zeta\right) - \frac{B}{\left|v_\parallel\right|} n_x \\ \frac{\partial v_y}{\partial \mu} &= \frac{v_\perp}{2\mu} \left(\widehat{e}_{1y} \cos\zeta + \widehat{e}_{2y} \sin\zeta\right) - \frac{B}{\left|v_\parallel\right|} n_y \\ \frac{\partial v_z}{\partial \mu} &= \frac{v_\perp}{2\mu} \left(\widehat{e}_{1z} \cos\zeta + \widehat{e}_{2z} \sin\zeta\right) - \frac{B}{\left|v_\parallel\right|} n_z \end{aligned}$$

For

$$\frac{\partial v_x}{\partial \zeta}$$

we have

$$\begin{aligned}
\frac{\partial v_x}{\partial \zeta} &= v_\perp \left(-\widehat{e}_{1x} \sin\zeta + \widehat{e}_{2x} \cos\zeta\right) \\
\frac{\partial v_y}{\partial \zeta} &= v_\perp \left(-\widehat{e}_{1y} \sin\zeta + \widehat{e}_{2y} \cos\zeta\right) \\
\frac{\partial v_z}{\partial \zeta} &= v_\perp \left(-\widehat{e}_{1z} \sin\zeta + \widehat{e}_{2z} \cos\zeta\right)
\end{aligned}$$

Now the matrix of the change of variables is

$$\frac{\partial \left(v_x, v_y, v_z\right)}{\partial \left(\epsilon, \mu, \zeta\right)}$$

$$\begin{pmatrix}
\frac{1}{\left|v_\parallel\right|}\widehat{n}_x & \frac{1}{\left|v_\parallel\right|}\widehat{n}_y & \frac{1}{\left|v_\parallel\right|}\widehat{n}_z \\
\frac{v_\perp}{2\mu}\left(\widehat{e}_{1x}\cos\zeta + \widehat{e}_{2x}\sin\zeta\right) - \frac{B}{\left|v_\parallel\right|}\widehat{n}_x & \frac{v_\perp}{2\mu}\left(\widehat{e}_{1x}\cos\zeta + \widehat{e}_{2x}\sin\zeta\right) - \frac{B}{\left|v_\parallel\right|}\widehat{n}_x & \frac{v_\perp}{2\mu}\left(\widehat{e}_{1x}\cos\zeta + \widehat{e}_{2x}\sin\zeta\right) - \frac{B}{\left|v_\parallel\right|}\widehat{n}_x \\
v_\perp\left(-\widehat{e}_{1x}\sin\zeta + \widehat{e}_{2x}\cos\zeta\right) & v_\perp\left(-\widehat{e}_{1y}\sin\zeta + \widehat{e}_{2y}\cos\zeta\right) & v_\perp\left(-\widehat{e}_{1z}\sin\zeta + \widehat{e}_{2z}\cos\zeta\right)
\end{pmatrix}$$

The fact that we choose to take the absolute value of the parallel velocity is related with the introduction of a new variable in the space of velocity: this is a discrete variable,

$$\sigma = \pm 1$$

and takes care of the two signs of the parallel velocity relative to the versor $\widehat{\mathbf{n}}$ of the magnetic field.

Multiplying the first line by B and adding to the second line, the second terms in each entry of the latter disappear. Reintroducing the notation for the terms of the last line, we have

$$\begin{aligned}
& \det \frac{1}{\left|v_\parallel\right|}\frac{1}{2\mu}\begin{pmatrix}
\widehat{n}_x & \widehat{n}_y & \widehat{n}_z \\
v_{\perp x} & v_{\perp y} & v_{\perp z} \\
\frac{\partial v_{\perp x}}{\partial \zeta} & \frac{\partial v_{\perp y}}{\partial \zeta} & \frac{\partial v_{\perp z}}{\partial \zeta}
\end{pmatrix} \\
= & \frac{1}{\left|v_\parallel\right|}\frac{1}{2\mu}\widehat{\mathbf{n}} \cdot \left(\mathbf{v}_\perp \times \frac{\partial \mathbf{v}_\perp}{\partial \zeta}\right)
\end{aligned}$$

We note the relation

$$\widehat{\mathbf{n}} \times \mathbf{v}_\perp = v_\perp \left[\left(\widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_1\right)\cos\zeta + \left(\widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_2\right)\sin\zeta\right]$$

with the relationships between the versors of the local system along the trajectory

$$\begin{aligned}
\widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_1 &= \widehat{\mathbf{e}}_2 \\
\widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_2 &= -\widehat{\mathbf{e}}_1
\end{aligned}$$

therefore

$$\begin{aligned}
\widehat{\mathbf{n}} \times \mathbf{v}_\perp &= v_\perp \left[\left(\widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_1 \right) \cos\zeta + \left(\widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_2 \right) \sin\zeta \right] \\
&= v_\perp \left(-\widehat{\mathbf{e}}_1 \sin\zeta + \widehat{\mathbf{e}}_2 \cos\zeta \right) \\
&= \frac{\partial \mathbf{v}_\perp}{\partial \zeta}
\end{aligned}$$

$$\frac{\partial \mathbf{v}_\perp}{\partial \zeta} = \widehat{\mathbf{n}} \times \mathbf{v}_\perp$$

and the determinant of the Jacobian becomes

$$\begin{aligned}
\det \frac{\partial \left(v_x, v_y, v_z \right)}{\partial \left(\epsilon, \mu, \zeta \right)} &= \frac{1}{\left| v_\parallel \right|} \frac{1}{2\mu} \widehat{\mathbf{n}} \cdot \left(\mathbf{v}_\perp \times \frac{\partial \mathbf{v}_\perp}{\partial \zeta} \right) \\
&= \frac{1}{\left| v_\parallel \right|} \frac{1}{2\mu} \widehat{\mathbf{n}} \cdot \left[\mathbf{v}_\perp \times \left(\widehat{\mathbf{n}} \times \mathbf{v}_\perp \right) \right] \\
&= \frac{1}{\left| v_\parallel \right|} \frac{1}{2\mu} \widehat{\mathbf{n}} \cdot \left[\widehat{\mathbf{n}} v_\perp^2 - \mathbf{v}_\perp \left(\widehat{\mathbf{n}} \cdot \mathbf{v}_\perp \right) \right] \\
&= \frac{v_\perp^2}{2\mu \left| v_\parallel \right|} \\
&= \frac{B}{\left| v_\parallel \right|}
\end{aligned}$$

This is the final result

$$\det \frac{\partial \left(v_x, v_y, v_z \right)}{\partial \left(\epsilon, \mu, \zeta \right)} = \frac{B}{\left| v_\parallel \right|}$$

and the element of volume includes the summation over the two signs of the parallel velocity

$$dv_x dv_y dv_z = \sum_{\sigma = \pm 1} \sigma \frac{B}{\left| v_\parallel \right|} d\epsilon d\mu d\zeta$$

2.2 The change of variables for the derivatives in the velocity space

The change is

$$\left(\frac{\partial}{\partial v_x},\frac{\partial}{\partial v_y},\frac{\partial}{\partial v_z}\right)\rightarrow\left(\frac{\partial}{\partial \epsilon},\frac{\partial}{\partial \mu},\frac{\partial}{\partial \zeta}\right)$$

The expressions look like

$$\begin{aligned}
\frac{\partial}{\partial \epsilon} &= \frac{\partial v_x}{\partial \epsilon}\frac{\partial}{\partial v_x}+\frac{\partial v_y}{\partial \epsilon}\frac{\partial}{\partial v_y}+\frac{\partial v_z}{\partial \epsilon}\frac{\partial}{\partial v_z} \\
\frac{\partial}{\partial \mu} &= ... \\
\frac{\partial}{\partial \zeta} &= ...
\end{aligned}$$

or, symbolically,

$$\begin{pmatrix} \partial/\partial\epsilon \\ \partial/\partial\mu \\ \partial/\partial\zeta \end{pmatrix} = \frac{\partial\left(v_x,v_y,v_z\right)}{\partial\left(\epsilon,\mu,\zeta\right)}\begin{pmatrix} \partial/\partial v_x \\ \partial/\partial v_y \\ \partial/\partial v_z \end{pmatrix}$$

Let us introduce the notation

$$Q\equiv\frac{\partial\left(v_x,v_y,v_z\right)}{\partial\left(\epsilon,\mu,\zeta\right)}$$

and the transformation of variables requires to calculate

$$\begin{pmatrix} \partial/\partial v_x \\ \partial/\partial v_y \\ \partial/\partial v_z \end{pmatrix} = Q^{-1}\begin{pmatrix} \partial/\partial\epsilon \\ \partial/\partial\mu \\ \partial/\partial\zeta \end{pmatrix}$$

i.e. the inverse of the matrix Q:

$$Q=\begin{pmatrix} \frac{\widehat{n}_x}{\left|v_\parallel\right|} & \frac{\widehat{n}_y}{\left|v_\parallel\right|} & \frac{\widehat{n}_z}{\left|v_\parallel\right|} \\ \frac{v_{\perp x}}{2\mu}-\frac{B}{\left|v_\parallel\right|}\widehat{n}_x & \frac{v_{\perp y}}{2\mu}-\frac{B}{\left|v_\parallel\right|}\widehat{n}_y & \frac{v_{\perp z}}{2\mu}-\frac{B}{\left|v_\parallel\right|}\widehat{n}_z \\ \left(\widehat{\mathbf{n}}\times\mathbf{v}_\perp\right)_x & \left(\widehat{\mathbf{n}}\times\mathbf{v}_\perp\right)_y & \left(\widehat{\mathbf{n}}\times\mathbf{v}_\perp\right)_z \end{pmatrix}$$

We calculate now the transpose Q^T then the adjoint matrix $Q^\dagger$.

$$Q^T=\begin{pmatrix} \frac{\widehat{n}_x}{\left|v_\parallel\right|} & \frac{v_{\perp x}}{2\mu}-\frac{B}{\left|v_\parallel\right|}\widehat{n}_x & \left(\widehat{\mathbf{n}}\times\mathbf{v}_\perp\right)_x \\ \frac{\widehat{n}_y}{\left|v_\parallel\right|} & \frac{v_{\perp y}}{2\mu}-\frac{B}{\left|v_\parallel\right|}\widehat{n}_y & \left(\widehat{\mathbf{n}}\times\mathbf{v}_\perp\right)_y \\ \frac{\widehat{n}_z}{\left|v_\parallel\right|} & \frac{v_{\perp z}}{2\mu}-\frac{B}{\left|v_\parallel\right|}\widehat{n}_z & \left(\widehat{\mathbf{n}}\times\mathbf{v}_\perp\right)_z \end{pmatrix}$$

Then

$$Q^{-1} = \frac{1}{\det Q} Q^{\dagger} =$$

$$\frac{\left|v_{\|}\right|}{B}\begin{pmatrix} \left.\left(\frac{\mathbf{v}_\perp}{2\mu} - \frac{\mathbf{B}}{\left|v_{\|}\right|}\right) \times \left(\widehat{\mathbf{n}} \times \mathbf{v}_\perp\right)\right|_x & \left.\left(\widehat{\mathbf{n}} \times \mathbf{v}_\perp\right) \times \frac{\widehat{\mathbf{n}}}{\left|v_{\|}\right|}\right|_x & \left.\frac{\widehat{\mathbf{n}}}{\left|v_{\|}\right|} \times \left(\frac{\mathbf{v}_\perp}{2\mu} - \frac{\mathbf{B}}{\left|v_{\|}\right|}\right)\right|_x \\ \left.\left(\frac{\mathbf{v}_\perp}{2\mu} - \frac{\mathbf{B}}{\left|v_{\|}\right|}\right) \times \left(\widehat{\mathbf{n}} \times \mathbf{v}_\perp\right)\right|_y & \left.\left(\widehat{\mathbf{n}} \times \mathbf{v}_\perp\right) \times \frac{\widehat{\mathbf{n}}}{\left|v_{\|}\right|}\right|_y & \left.\frac{\widehat{\mathbf{n}}}{\left|v_{\|}\right|} \times \left(\frac{\mathbf{v}_\perp}{2\mu} - \frac{\mathbf{B}}{\left|v_{\|}\right|}\right)\right|_y \\ \left.\left(\frac{\mathbf{v}_\perp}{2\mu} - \frac{\mathbf{B}}{\left|v_{\|}\right|}\right) \times \left(\widehat{\mathbf{n}} \times \mathbf{v}_\perp\right)\right|_z & \left.\left(\widehat{\mathbf{n}} \times \mathbf{v}_\perp\right) \times \frac{\widehat{\mathbf{n}}}{\left|v_{\|}\right|}\right|_z & \left.\frac{\widehat{\mathbf{n}}}{\left|v_{\|}\right|} \times \left(\frac{\mathbf{v}_\perp}{2\mu} - \frac{\mathbf{B}}{\left|v_{\|}\right|}\right)\right|_z \end{pmatrix}$$

From here we calculate term by term the new operators of derivation

$$\begin{aligned} & \frac{\partial}{\partial v_x} \\ = & \frac{\left|v_{\|}\right|}{B}\left\{\left.\left(\frac{\mathbf{v}_\perp}{2\mu} - \frac{\mathbf{B}}{\left|v_{\|}\right|}\right) \times \left(\widehat{\mathbf{n}} \times \mathbf{v}_\perp\right)\right|_x \frac{\partial}{\partial \epsilon}\right. \\ & + \left.\left(\widehat{\mathbf{n}} \times \mathbf{v}_\perp\right) \times \frac{\widehat{\mathbf{n}}}{\left|v_{\|}\right|}\right|_x \frac{\partial}{\partial \mu} \\ & \left.+ \left.\frac{\widehat{\mathbf{n}}}{\left|v_{\|}\right|} \times \left(\frac{\mathbf{v}_\perp}{2\mu} - \frac{\mathbf{B}}{\left|v_{\|}\right|}\right)\right|_x \frac{\partial}{\partial \zeta}\right\} \end{aligned}$$

and similar formulas for y and z. We collect these terms in the vector form

$$\begin{aligned} & \frac{\partial}{\partial \mathbf{v}} \\ = & \left(\frac{\mathbf{v}_\perp}{v_\perp^2}\left|v_{\|}\right| - \widehat{\mathbf{n}}\right) \times \left(\widehat{\mathbf{n}} \times \mathbf{v}_\perp\right) \frac{\partial}{\partial \epsilon} \\ & + \left(\widehat{\mathbf{n}} \times \mathbf{v}_\perp\right) \times \widehat{\mathbf{n}} \frac{\partial}{B \partial \mu} \\ & + \widehat{\mathbf{n}} \times \left(\frac{\mathbf{v}_\perp}{v_\perp^2} - \frac{\widehat{\mathbf{n}}}{\left|v_{\|}\right|}\right) \frac{\partial}{\partial \zeta} \end{aligned}$$

We take separately the coefficient of $\partial/\partial\epsilon$:

$$\begin{aligned} & \frac{\left|v_{\|}\right|}{v_\perp^2} \mathbf{v}_\perp \times \left(\widehat{\mathbf{n}} \times \mathbf{v}_\perp\right) - \widehat{\mathbf{n}} \times \left(\widehat{\mathbf{n}} \times \mathbf{v}_\perp\right) \\ = & \frac{\left|v_{\|}\right|}{v_\perp^2}\left[\widehat{\mathbf{n}} v_\perp^2 - \mathbf{v}_\perp\left(\widehat{\mathbf{n}} \cdot \mathbf{v}_\perp\right)\right] \\ & - \widehat{\mathbf{n}}\left(\widehat{\mathbf{n}} \cdot \mathbf{v}_\perp\right) + \mathbf{v}_\perp \\ = & \widehat{\mathbf{n}}\left|v_{\|}\right| + \mathbf{v}_\perp \\ = & \mathbf{v} \end{aligned}$$

The coefficient of $\partial / B\partial\mu$ is

$$\begin{aligned} & (\widehat{\mathbf{n}} \times \mathbf{v}_\perp) \times \widehat{\mathbf{n}} \\ = & -\widehat{\mathbf{n}} \left(\widehat{\mathbf{n}} \cdot \mathbf{v}_\perp\right) + \mathbf{v}_\perp \\ = & \mathbf{v}_\perp \end{aligned}$$

and the coefficient of $\partial/\partial\zeta is$

$$\begin{aligned} & \widehat{\mathbf{n}} \times \left(\frac{\mathbf{v}_\perp}{v_\perp^2} - \frac{\widehat{\mathbf{n}}}{\left|v_\parallel\right|} \right) \\ = & \frac{\widehat{\mathbf{n}} \times \mathbf{v}_\perp}{v_\perp^2} \end{aligned}$$

This allows to write the operator

$$\frac{\partial}{\partial \mathbf{v}} = \mathbf{v} \frac{\partial}{\partial \epsilon} + \mathbf{v}_\perp \frac{\partial}{B\partial\mu} + \frac{\widehat{\mathbf{n}} \times \mathbf{v}_\perp}{v_\perp^2} \frac{\partial}{\partial\zeta}$$

which is the final expression.

2.3 The change of variables in the expression of the gradient in the coordinate space

This is $\boldsymbol{\nabla}$.

$$\frac{\partial f}{\partial x} \rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial \epsilon}\frac{\partial \epsilon}{\partial x} + \frac{\partial f}{\partial \mu}\frac{\partial \mu}{\partial x} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial x}$$

We have to calculate separately the contributions

$$\frac{\partial \epsilon}{\partial x} = \frac{e}{m}\frac{\partial \phi}{\partial x}$$

$$\begin{aligned} \frac{\partial \mu}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{v_\perp^2}{2B}\right) \\ &= -\frac{v_\perp^2}{2B^2}\frac{\partial B}{\partial x} + \frac{v_\perp}{B}\frac{\partial v_\perp}{\partial x} \end{aligned}$$

where

$$\begin{aligned} v_\perp^2 &= v^2 - v_\parallel^2 \\ v_\perp \frac{\partial v_\perp}{\partial x} &= -v_\parallel \frac{\partial v_\parallel}{\partial x} \end{aligned}$$

and

$$\begin{aligned}
\frac{\partial v_{\|}}{\partial x} &= \frac{\partial}{\partial x}\left(\widehat{\mathbf{n}}\cdot\mathbf{v}\right) \\
&= \mathbf{v}\cdot\frac{\partial\widehat{\mathbf{n}}}{\partial x}
\end{aligned}$$

which gives

$$v_{\perp}\frac{\partial v_{\perp}}{\partial x} = -v_{\|}\mathbf{v}\cdot\frac{\partial\widehat{\mathbf{n}}}{\partial x}$$

The space derivative of the magnetic momentum

$$\frac{\partial\mu}{\partial x} = -\frac{\mu}{B}\frac{\partial B}{\partial x} - \frac{v_{\|}}{B}\mathbf{v}\cdot\frac{\partial\widehat{\mathbf{n}}}{\partial x}$$

The derivatives implying the gyration angle ζ are

$$\begin{aligned}
\frac{\partial\mathbf{v}_{\perp}}{\partial x} &= \frac{\partial v_{\perp}}{\partial x}\left(\widehat{\mathbf{e}}_1\cos\zeta + \widehat{\mathbf{e}}_2\sin\zeta\right) \\
&\quad + v_{\perp}\left(\frac{\partial\widehat{\mathbf{e}}_1}{\partial x}\cos\zeta + \frac{\partial\widehat{\mathbf{e}}_2}{\partial x}\sin\zeta\right) \\
&\quad + v_{\perp}\left[\widehat{\mathbf{e}}_1\left(-\sin\zeta\right) + \widehat{\mathbf{e}}_2\left(\cos\zeta\right)\right]\frac{\partial\zeta}{\partial x}
\end{aligned}$$

We calculate the same derivative in a different way

$$\begin{aligned}
\frac{\partial\mathbf{v}_{\perp}}{\partial x} &= \frac{\partial}{\partial x}\left(\mathbf{v}-v_{\|}\widehat{\mathbf{n}}\right) \\
&= -\frac{\partial v_{\|}}{\partial x}\widehat{\mathbf{n}} - v_{\|}\frac{\partial\widehat{\mathbf{n}}}{\partial x}
\end{aligned}$$

and in the first term we use the fact that $\mathbf{v}$ is an independent variable

$$\begin{aligned}
\frac{\partial v_{\|}}{\partial x} &= \frac{\partial}{\partial x}\left(\mathbf{v}\cdot\widehat{\mathbf{n}}\right) \\
&= \mathbf{v}\cdot\frac{\partial\widehat{\mathbf{n}}}{\partial x}
\end{aligned}$$

Then the other formula is

$$\frac{\partial\mathbf{v}_{\perp}}{\partial x} = -\widehat{\mathbf{n}}\left(\mathbf{v}\cdot\frac{\partial\widehat{\mathbf{n}}}{\partial x}\right) - v_{\|}\frac{\partial\widehat{\mathbf{n}}}{\partial x}$$

and this is equal to the previous expression

$$
\begin{aligned}
& -\widehat{\mathbf{n}}\left(\mathbf{v}\cdot\frac{\partial\widehat{\mathbf{n}}}{\partial x}\right)-v_{\|}\frac{\partial\widehat{\mathbf{n}}}{\partial x} \\
= \; & \frac{\partial v_{\perp}}{\partial x}\left(\widehat{\mathbf{e}}_1\cos\zeta+\widehat{\mathbf{e}}_2\sin\zeta\right) \\
& +v_{\perp}\left(\frac{\partial\widehat{\mathbf{e}}_1}{\partial x}\cos\zeta+\frac{\partial\widehat{\mathbf{e}}_2}{\partial x}\sin\zeta\right) \\
& +v_{\perp}\left[\widehat{\mathbf{e}}_1\left(-\sin\zeta\right)+\widehat{\mathbf{e}}_2\left(\cos\zeta\right)\right]\frac{\partial\zeta}{\partial x}
\end{aligned}
$$

This equality is multiplied with $\widehat{\mathbf{e}}_1$ and with $\widehat{\mathbf{e}}_2$ and the we separate the terms that contain the derivatives of ζ :

$$
\begin{aligned}
-\sin\zeta\frac{\partial\zeta}{\partial x} & = \\
& = -\frac{v_{\|}}{v_{\perp}}\widehat{\mathbf{e}}_1\cdot\frac{\partial\widehat{\mathbf{n}}}{\partial x}+\frac{v_{\|}}{v_{\perp}^2}\left(\mathbf{v}\cdot\frac{\partial\widehat{\mathbf{n}}}{\partial x}\right)\cos\zeta \\
& \quad -\widehat{\mathbf{e}}_1\cdot\frac{\partial\widehat{\mathbf{e}}_1}{\partial x}\cos\zeta-\widehat{\mathbf{e}}_1\cdot\frac{\partial\widehat{\mathbf{e}}_2}{\partial x}\sin\zeta
\end{aligned}
$$

$$
\begin{aligned}
\cos\zeta\frac{\partial\zeta}{\partial x} & = \\
& = -\frac{v_{\|}}{v_{\perp}}\widehat{\mathbf{e}}_2\cdot\frac{\partial\widehat{\mathbf{n}}}{\partial x}+\frac{v_{\|}}{v_{\perp}^2}\left(\mathbf{v}\cdot\frac{\partial\widehat{\mathbf{n}}}{\partial x}\right)\sin\zeta \\
& \quad -\widehat{\mathbf{e}}_2\cdot\frac{\partial\widehat{\mathbf{e}}_1}{\partial x}\cos\zeta-\widehat{\mathbf{e}}_2\cdot\frac{\partial\widehat{\mathbf{e}}_2}{\partial x}\sin\zeta
\end{aligned}
$$

The first equation is multiplied by $-\sin\zeta$ and the second by $\cos\zeta$ and are added

$$
\begin{aligned}
\frac{\partial\zeta}{\partial x} = \; & -\frac{v_{\|}}{v_{\perp}}\frac{\partial\widehat{\mathbf{n}}}{\partial x}\cdot\left(-\widehat{\mathbf{e}}_1\sin\zeta+\widehat{\mathbf{e}}_2\cos\zeta\right) \\
& +\sin\zeta\cos\zeta\left(\widehat{\mathbf{e}}_1\cdot\frac{\partial\widehat{\mathbf{e}}_1}{\partial x}-\widehat{\mathbf{e}}_2\cdot\frac{\partial\widehat{\mathbf{e}}_2}{\partial x}\right) \\
& +\widehat{\mathbf{e}}_1\cdot\frac{\partial\widehat{\mathbf{e}}_2}{\partial x}\sin^2\zeta-\widehat{\mathbf{e}}_2\cdot\frac{\partial\widehat{\mathbf{e}}_1}{\partial x}\cos^2\zeta
\end{aligned}
$$

Since we have for the versors

$$
\left(\widehat{\mathbf{e}}_1\right)^2=1\rightarrow\frac{\partial}{\partial x}\left(\widehat{\mathbf{e}}_1\right)^2=0 \text{ or } \widehat{\mathbf{e}}_1\cdot\frac{\partial\widehat{\mathbf{e}}_1}{\partial x}=0
$$

$$\widehat{\mathbf{e}}_1 \cdot \widehat{\mathbf{e}}_2 = 0 \rightarrow \widehat{\mathbf{e}}_1 \cdot \frac{\partial \widehat{\mathbf{e}}_2}{\partial x} = -\widehat{\mathbf{e}}_2 \cdot \frac{\partial \widehat{\mathbf{e}}_1}{\partial x}$$

Using these formulas we get

$$\frac{\partial \zeta}{\partial x} = -\frac{v_\parallel}{v_\perp^2} \frac{\partial \mathbf{v}}{\partial \zeta} \frac{\partial \widehat{\mathbf{n}}}{\partial x} + \widehat{\mathbf{e}}_1 \cdot \frac{\partial \widehat{\mathbf{e}}_2}{\partial x}$$

or

$$\frac{\partial \zeta}{\partial x} = -\frac{v_\parallel}{v_\perp^2} \left(\widehat{\mathbf{n}} \times \mathbf{v}_\perp\right) \cdot \frac{\partial \widehat{\mathbf{n}}}{\partial x} + \widehat{\mathbf{e}}_1 \cdot \frac{\partial \widehat{\mathbf{e}}_2}{\partial x}$$

Then, collecting the all terms we find

$$\begin{aligned}
\partial_k \quad \rightarrow \quad & \partial_k + \frac{e}{m} \partial_k \phi \frac{\partial}{\partial \epsilon} \\
& - \left[\mu \partial_k B + v_\parallel \mathbf{v} \cdot \left(\partial_k \widehat{\mathbf{n}}\right)\right] \frac{\partial}{B \partial \mu} \\
& + \left[\left(\partial_k \widehat{\mathbf{e}}_2\right) \cdot \widehat{\mathbf{e}}_1 - \frac{v_\parallel}{v_\perp^2} \left(\partial_k \widehat{\mathbf{n}}\right) \cdot \left(\widehat{\mathbf{n}} \times \mathbf{v}_\perp\right)\right] \frac{\partial}{\partial \zeta}
\end{aligned}$$

NOTE

In **Tang review instabilities** the change is

$$\begin{aligned}
\boldsymbol{\nabla} \quad = \quad & \frac{\partial}{\partial \mathbf{x}} + \frac{e}{m} \boldsymbol{\nabla} \phi_0 \frac{\partial}{\partial \epsilon} \\
& - \left(\mu \boldsymbol{\nabla} B + v_\parallel \boldsymbol{\nabla}\ \widehat{\mathbf{n}} \cdot \mathbf{v}_\perp\right) \frac{1}{B} \frac{\partial}{\partial \mu} \\
& + \left[\boldsymbol{\nabla}\ \widehat{\mathbf{e}}_2 \cdot \widehat{\mathbf{e}}_1 + \frac{v_\parallel}{v_\perp^2}\ \boldsymbol{\nabla}\ \widehat{\mathbf{n}}\ \cdot \left(\mathbf{v}_\perp \times \widehat{\mathbf{n}}\right)\right] \frac{\partial}{\partial \zeta}
\end{aligned}$$

and for velocity

$$\boldsymbol{\nabla}_{\mathbf{v}} = \mathbf{v} \frac{\partial}{\partial \epsilon} + \mathbf{v}_\perp \frac{1}{B} \frac{\partial}{\partial \mu} + \frac{1}{v_\perp^2} \left(\widehat{\mathbf{n}} \times \mathbf{v}_\perp\right) \frac{\partial}{\partial \zeta}$$

END

2.4 Identities between vectors in toroidal geometry

In the geometry which is particular to the drift of particles we will invoke several identities that we have to examine separately.

We have adopted a system of three versors

$$(\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2, \widehat{\mathbf{n}}) \tag{6}$$

such as to be a right system like (x, y, z). The perpendicular velocity is in the plane $(\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2)$ and makes an angle ζ with $\widehat{\mathbf{e}}_1$, therefore the definition of the perpendicular velocity, as used until now, is

$$\mathbf{v}_\perp = v_\perp (\widehat{\mathbf{e}}_1 \cos\zeta + \widehat{\mathbf{e}}_2 \sin\zeta) \tag{7}$$

We have the basic vector relations

$$\begin{aligned} \widehat{\mathbf{n}} &= \widehat{\mathbf{e}}_1 \times \widehat{\mathbf{e}}_2 \\ \widehat{\mathbf{e}}_1 &= \widehat{\mathbf{e}}_2 \times \widehat{\mathbf{n}} \\ \widehat{\mathbf{e}}_2 &= \widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_1 \end{aligned} \tag{8}$$

An expression the arises frequently is

$$\begin{aligned} & -\widehat{\mathbf{e}}_2 \cos\zeta + \widehat{\mathbf{e}}_1 \sin\zeta \\ =\ & (\widehat{\mathbf{e}}_1 \times \widehat{\mathbf{n}}) \cos\zeta + (\widehat{\mathbf{e}}_2 \times \widehat{\mathbf{n}}) \sin\zeta \\ =\ & \widehat{\mathbf{v}}_\perp \times \widehat{\mathbf{n}} \\ =\ & \widehat{\boldsymbol{\rho}} \end{aligned} \tag{9}$$

Formula 1 We will use

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \rightarrow -\frac{(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) B}{B} = -\frac{\nabla_\parallel B}{B} = \boldsymbol{\nabla} \cdot \widehat{\mathbf{n}} \tag{10}$$

and

$$\boldsymbol{\nabla} = \widehat{\mathbf{e}}_1 (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}) + \widehat{\mathbf{e}}_2 (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}) + \widehat{\mathbf{n}} (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \tag{11}$$

In particular

$$\boldsymbol{\nabla} \cdot \widehat{\mathbf{n}} = \widehat{\mathbf{e}}_1 \cdot (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} + \widehat{\mathbf{e}}_2 \cdot (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} + \widehat{\mathbf{n}} \cdot (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} \tag{12}$$

The last term is

$$\begin{aligned} \widehat{\mathbf{n}} \cdot [(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}}] &= (\widehat{n}_i \partial_i) \left(\frac{1}{2} \widehat{n}_k^2 \right) = 0 \\ \widehat{\mathbf{n}} \cdot \boldsymbol{\kappa} &= 0 \end{aligned} \tag{13}$$

then

$$\begin{aligned}
& \widehat{\mathbf{e}}_1 \cdot [(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}] + \widehat{\mathbf{e}}_2 \cdot [(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}] \qquad (14) \\
= \; & \boldsymbol{\nabla} \cdot \widehat{\mathbf{n}} - \widehat{\mathbf{n}} \cdot [(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}] \\
= \; & \boldsymbol{\nabla} \cdot \widehat{\mathbf{n}}
\end{aligned}$$

This is

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \widehat{\mathbf{n}} &= \boldsymbol{\nabla} \cdot \left(\frac{\mathbf{B}}{B} \right) = \frac{1}{B} \boldsymbol{\nabla} \cdot \mathbf{B} + \mathbf{B} \cdot \boldsymbol{\nabla} \left(\frac{1}{B} \right) \\
&= B\, (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \left(\frac{1}{B} \right) = B \nabla_{\|} \left(\frac{1}{B} \right) = -\frac{1}{B} \nabla_{\|} B \\
&= -\nabla_{\|} \ln B
\end{aligned}$$

Formula 2 The mixed products of versors are

$$\widehat{\mathbf{n}} \cdot [\widehat{\mathbf{e}}_2 \times (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}] = (\widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_2) \cdot (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}} = -\widehat{\mathbf{e}}_1 \cdot (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}} \qquad (15)$$

$$\widehat{\mathbf{n}} \cdot [\widehat{\mathbf{e}}_1 \times (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}] = (\widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_1) \cdot (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}} = \widehat{\mathbf{e}}_2 \cdot (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}} \qquad (16)$$

$$\widehat{\mathbf{n}} \cdot [\widehat{\mathbf{e}}_1 \times (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}] = (\widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_1) \cdot (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}} = \widehat{\mathbf{e}}_2 \cdot (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}} \qquad (17)$$

$$\widehat{\mathbf{n}} \cdot [\widehat{\mathbf{e}}_2 \times (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}] = (\widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_2) \cdot (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}} = -\widehat{\mathbf{e}}_1 \cdot (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}} \qquad (18)$$

$$\widehat{\mathbf{n}} \cdot [\widehat{\mathbf{e}}_2 \times (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}] = (\widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_2) \cdot (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}} = -\widehat{\mathbf{e}}_1 \cdot (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}} \qquad (19)$$

$$\widehat{\mathbf{n}} \cdot [\widehat{\mathbf{e}}_1 \times (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}] = (\widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_1) \cdot (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}} = \widehat{\mathbf{e}}_2 \cdot (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}} \qquad (20)$$

Formula 3 Start from the identity

$$\widehat{\mathbf{e}}_1 \cdot \widehat{\mathbf{e}}_2 = 0 \qquad (21)$$

and apply the operator $(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})$,

$$\widehat{\mathbf{e}}_1 \cdot [(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{e}}_2] + \widehat{\mathbf{e}}_2 \cdot [(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{e}}_1] = 0 \qquad (22)$$

Analogously, using

$$\widehat{\mathbf{e}}_1 \cdot \widehat{\mathbf{e}}_1 = 1 \qquad (23)$$

and applying $(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})$ we find

$$\widehat{\mathbf{e}}_1 \cdot [(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{e}}_1] = 0 \qquad (24)$$

and, in a similar way

$$\widehat{\mathbf{e}}_2 \cdot [(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{e}}_2] = 0 \qquad (25)$$

Formula 4 We will consider later the expression (in the definition of B_1)

$$\begin{aligned} & \widehat{\mathbf{e}}_1 \cdot \left[\left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} \right] - \widehat{\mathbf{e}}_2 \cdot \left[\left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} \right] \\ = & \; e_{1k} e_{2i} \partial_i n_k - e_{2k} e_{1i} \partial_i n_k \\ = & \; \left(e_{1k} e_{2i} - e_{2k} e_{1i} \right) \partial_i n_k \end{aligned}$$

From Eq.(29) we have

$$\begin{aligned} e_{1k} e_{2i} - e_{2k} e_{1i} &= \varepsilon_{kil} \left(\widehat{\mathbf{e}}_1 \times \widehat{\mathbf{e}}_2 \right)_l \\ &= \varepsilon_{kil} n_l \end{aligned}$$

Then

$$\begin{aligned} \widehat{\mathbf{e}}_1 \cdot \left[\left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} \right] - \widehat{\mathbf{e}}_2 \cdot \left[\left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} \right] &= \varepsilon_{kil} n_l \partial_i n_k \\ &= -\widehat{\mathbf{n}} \cdot \left(\boldsymbol{\nabla} \times \widehat{\mathbf{n}} \right) \end{aligned}$$

Formula 5 We have the identity

$$- \left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{e}}_2 + \left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{e}}_1 = \boldsymbol{\nabla} \times \widehat{\mathbf{n}} \tag{26}$$

In **Hinton Waltz**

$$\boldsymbol{\nabla} \times \widehat{\mathbf{n}} = \widehat{\mathbf{n}} \left[\widehat{\mathbf{n}} \cdot \left(\boldsymbol{\nabla} \times \widehat{\mathbf{n}} \right) \right] + \widehat{\mathbf{n}} \times \boldsymbol{\kappa}$$

Formula 6 We look to the general formula

$$\alpha \equiv \left(\widehat{\mathbf{e}}_1 \cdot \mathbf{a} \right) \left(\widehat{\mathbf{e}}_2 \cdot \mathbf{b} \right) - \left(\widehat{\mathbf{e}}_2 \cdot \mathbf{b} \right) \left(\widehat{\mathbf{e}}_1 \cdot \mathbf{a} \right) \tag{27}$$

This can be expressed as

$$\begin{aligned} \alpha &= e_{1k} a_k e_{2i} b_i - e_{2k} a_k e_{1i} b_i \\ &= \left(e_{1k} e_{2i} - e_{2k} e_{1i} \right) a_k b_i \end{aligned} \tag{28}$$

We have

$$\begin{aligned} e_{1k} e_{2i} - e_{2k} e_{1i} &= \varepsilon_{kil} \left(\widehat{\mathbf{e}}_1 \times \widehat{\mathbf{e}}_2 \right)_l \\ &= \varepsilon_{kil} n_l \end{aligned} \tag{29}$$

Then

$$\alpha = \varepsilon_{kil} n_l a_k b_i$$

$$\left(\widehat{\mathbf{e}}_1 \cdot \mathbf{a} \right) \left(\widehat{\mathbf{e}}_2 \cdot \mathbf{b} \right) - \left(\widehat{\mathbf{e}}_2 \cdot \mathbf{b} \right) \left(\widehat{\mathbf{e}}_1 \cdot \mathbf{a} \right) = \widehat{\mathbf{n}} \cdot \left(\mathbf{a} \times \mathbf{b} \right) \tag{30}$$

Formula 7 We have (we only write the versors)

$$\widehat{\mathbf{n}} \times [(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{e}}_1 + (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{e}}_2] \tag{31}$$

Take separately the two terms

$$\widehat{\mathbf{n}} \times (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{e}}_1 = (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, (\widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_1) - [(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}] \times \widehat{\mathbf{e}}_1 \tag{32}$$

$$\widehat{\mathbf{n}} \times (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{e}}_2 = (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla})\, (\widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_2) - [(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}] \times \widehat{\mathbf{e}}_2 \tag{33}$$

In Eq.(32) the vector product in the first term is $\widehat{\mathbf{e}}_2$, then

$$\begin{aligned} & \widehat{\mathbf{n}} \times [(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{e}}_1 + (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{e}}_2] \\ = \; & (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{e}}_2 - (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{e}}_1 \\ & - \{[(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}] \times \widehat{\mathbf{e}}_1 + [(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}] \times \widehat{\mathbf{e}}_2\} \end{aligned} \tag{34}$$

The first two terms are

$$(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{e}}_2 - (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{e}}_1 = -\boldsymbol{\nabla} \times \widehat{\mathbf{n}} \tag{35}$$

and we now calculate separately the first term from the curly bracket, which we write as

$$[(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}] \times \widehat{\mathbf{e}}_1 = \beta \widehat{\mathbf{e}}_2 + \gamma \widehat{\mathbf{n}} \tag{36}$$

This is because this term is perpendicular on $\widehat{\mathbf{e}}_1$. We scalar multiply this identity with $\widehat{\mathbf{e}}_2$

$$\begin{aligned} \beta &= \widehat{\mathbf{e}}_2 \cdot \{[(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}] \times \widehat{\mathbf{e}}_1\} \\ &= [(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}] \cdot \{\widehat{\mathbf{e}}_1 \times \widehat{\mathbf{e}}_2\} \\ &= [(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}] \cdot \widehat{\mathbf{n}} \\ &= (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \frac{(\widehat{\mathbf{n}})^2}{2} \\ &= 0 \end{aligned} \tag{37}$$

Now we scalar multiply with $\widehat{\mathbf{n}}$

$$\begin{aligned} \gamma &= \widehat{\mathbf{n}} \cdot \{[(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}] \times \widehat{\mathbf{e}}_1\} \\ &= [(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}] \cdot (\widehat{\mathbf{e}}_1 \times \widehat{\mathbf{n}}) \\ &= [(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}] \cdot (-\widehat{\mathbf{e}}_2) \\ &= -\widehat{\mathbf{e}}_2 \cdot (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}} \end{aligned} \tag{38}$$

The second term is treated in an analogous way

$$[(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}] \times \widehat{\mathbf{e}}_2 = \beta' \widehat{\mathbf{e}}_1 + \gamma' \widehat{\mathbf{n}} \tag{39}$$

and we find as before

$$
\begin{aligned}
\beta' &= 0 \\
\gamma' &= \widehat{\mathbf{e}}_1 \cdot (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}
\end{aligned} \tag{40}
$$

For the two terms it results

$$
\begin{aligned}
& [(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}] \times \widehat{\mathbf{e}}_1 + [(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}] \times \widehat{\mathbf{e}}_2 \\
= \;& \{-\widehat{\mathbf{e}}_2 \cdot (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}} + \widehat{\mathbf{e}}_1 \cdot (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}\}\, \widehat{\mathbf{n}} \\
= \;& \left\{ (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}) \left(\widehat{\mathbf{e}}_2 \cdot \overset{\downarrow}{\widehat{\mathbf{n}}} \right) - (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}) \left(\widehat{\mathbf{e}}_1 \cdot \overset{\downarrow}{\widehat{\mathbf{n}}} \right) \right\} \widehat{\mathbf{n}} \\
= \;& [(\boldsymbol{\nabla} \times \widehat{\mathbf{n}}) \cdot \widehat{\mathbf{n}}]\, \widehat{\mathbf{n}}
\end{aligned} \tag{41}
$$

Returning to the Eq.(31)

$$
\begin{aligned}
& \widehat{\mathbf{n}} \times [(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{e}}_1 + (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{e}}_2] \\
= \;& -\boldsymbol{\nabla} \times \widehat{\mathbf{n}} + [(\boldsymbol{\nabla} \times \widehat{\mathbf{n}}) \cdot \widehat{\mathbf{n}}]\, \widehat{\mathbf{n}}
\end{aligned} \tag{42}
$$

Formula 8 This calculation can be continued by vector multiplying with $\widehat{\mathbf{n}}$

$$
\begin{aligned}
& \widehat{\mathbf{n}} \times \{\widehat{\mathbf{n}} \times [(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{e}}_1 + (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{e}}_2]\} \\
= \;& \widehat{\mathbf{n}} \times \{-\boldsymbol{\nabla} \times \widehat{\mathbf{n}} + [(\boldsymbol{\nabla} \times \widehat{\mathbf{n}}) \cdot \widehat{\mathbf{n}}]\, \widehat{\mathbf{n}}\}
\end{aligned} \tag{43}
$$

The left hand side is

$$
\begin{aligned}
& -(\widehat{\mathbf{n}} \cdot \widehat{\mathbf{n}})\, [(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{e}}_1 + (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{e}}_2] \\
& + \widehat{\mathbf{n}}\, \{\widehat{\mathbf{n}} \cdot [(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{e}}_1 + (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{e}}_2]\}
\end{aligned} \tag{44}
$$

The second line of the above expression contains the scalar product

$$
\widehat{\mathbf{n}} \cdot [(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{e}}_1 + (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{e}}_2] = -\boldsymbol{\nabla} \cdot \widehat{\mathbf{n}} \tag{45}
$$

This can be seen as follows.
The scalar operator $(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})$ applied to the scalar product $\widehat{\mathbf{n}} \cdot \widehat{\mathbf{e}}_1$ is

$$
(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\;\; \widehat{\mathbf{n}} \cdot \widehat{\mathbf{e}}_1 = \widehat{\mathbf{e}}_1 \cdot [(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}] + \widehat{\mathbf{n}} \cdot [(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{e}}_1]
$$

and since $\widehat{\mathbf{n}} \cdot \widehat{\mathbf{e}}_1 = 0$ we have

$$
\begin{aligned}
\widehat{\mathbf{n}} \cdot [(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{e}}_1] &= -\widehat{\mathbf{e}}_1 \cdot [(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}] \quad \text{and similarly} \\
\widehat{\mathbf{n}} \cdot [(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{e}}_2] &= -\widehat{\mathbf{e}}_2 \cdot [(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}]
\end{aligned}
$$

Then the factor in curly brackets in the second line in 44 becomes

$$\begin{aligned}
& \widehat{\mathbf{n}} \cdot [(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_1 + (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_2] \\
= & -\boldsymbol{\nabla} \cdot \widehat{\mathbf{n}} + \widehat{\mathbf{n}} \cdot (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} \\
= & -\boldsymbol{\nabla} \cdot \widehat{\mathbf{n}}
\end{aligned}$$

The full expression in 44 becomes

$$\begin{aligned}
& -[(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_1 + (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_2] \\
& +\widehat{\mathbf{n}} (-\boldsymbol{\nabla} \cdot \widehat{\mathbf{n}}) \\
= & \widehat{\mathbf{n}} \times \{-\boldsymbol{\nabla} \times \widehat{\mathbf{n}} + [(\boldsymbol{\nabla} \times \widehat{\mathbf{n}}) \cdot \widehat{\mathbf{n}}] \widehat{\mathbf{n}}\}
\end{aligned} \tag{46}$$

The last term in the right hand side is zero. Then

$$(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_1 + (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_2 = -\widehat{\mathbf{n}} (\boldsymbol{\nabla} \cdot \widehat{\mathbf{n}}) + \widehat{\mathbf{n}} \times (\boldsymbol{\nabla} \times \widehat{\mathbf{n}}) \tag{47}$$

The last term can still be modified

$$\begin{aligned}
\widehat{\mathbf{n}} \times (\boldsymbol{\nabla} \times \widehat{\mathbf{n}}) &= \varepsilon^{ijk} n_j \varepsilon^{klm} \partial_l n_m \\
&= \left(\varepsilon^{ijk} \varepsilon^{klm}\right) n_j \partial_l n_m \\
&= \left(\delta^{il} \delta^{jm} - \delta^{im} \delta^{lj}\right) n_j \partial_l n_m \\
&= n_m \partial_i n_m - n_l (\partial_l n_i) \\
&= \partial_i \left(\frac{\widehat{\mathbf{n}}^2}{2}\right) - (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}}
\end{aligned} \tag{48}$$

but the square of the versor is 1 and the first term vanishes

$$\widehat{\mathbf{n}} \times (\boldsymbol{\nabla} \times \widehat{\mathbf{n}}) = -(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} \tag{49}$$

This is actually the curvature term in the drift of the guiding centre

$$\boldsymbol{\kappa} = (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} \simeq \frac{1}{R} (-\widehat{\mathbf{e}}_R)$$

Returning to Eq.(47) we obtain

$$(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_1 + (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_2 = -\widehat{\mathbf{n}} (\boldsymbol{\nabla} \cdot \widehat{\mathbf{n}}) - (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} \tag{50}$$

or

$$\begin{aligned}
(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_1 + (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_2 &= -\widehat{\mathbf{n}} (\boldsymbol{\nabla} \cdot \widehat{\mathbf{n}}) + \widehat{\mathbf{n}} \times (\boldsymbol{\nabla} \times \widehat{\mathbf{n}}) \\
&= -\widehat{\mathbf{n}} (\boldsymbol{\nabla} \cdot \widehat{\mathbf{n}}) - (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} \\
&= \widehat{\mathbf{n}} \, \nabla_{\|} \ln B - \boldsymbol{\kappa}
\end{aligned}$$

For a check, we see that by taking the scalar product of this equation with $\widehat{\mathbf{n}}$ we obtain Eq.(45)

We also have

$$\widehat{\mathbf{n}} \cdot \boldsymbol{\kappa} = -\frac{1}{R} \widehat{\mathbf{e}}_R \cdot \widehat{\mathbf{n}} = 0$$

Formula 9 Let us consider

$$\begin{aligned} W &= \\ & \widehat{\mathbf{e}}_1 \sin\zeta \left[(\widehat{\mathbf{e}}_1 \cos\zeta + \widehat{\mathbf{e}}_2 \sin\zeta) \cdot \boldsymbol{\nabla} \right] \widehat{\mathbf{n}} + \\ & \widehat{\mathbf{e}}_1 \cos\zeta \left[(\widehat{\mathbf{e}}_1 \sin\zeta - \widehat{\mathbf{e}}_2 \cos\zeta) \cdot \boldsymbol{\nabla} \right] \widehat{\mathbf{n}} + \\ & \widehat{\mathbf{e}}_2 \sin\zeta \left[(-\widehat{\mathbf{e}}_2 \cos\zeta + \widehat{\mathbf{e}}_1 \sin\zeta) \cdot \boldsymbol{\nabla} \right] \widehat{\mathbf{n}} - \\ & \widehat{\mathbf{e}}_2 \cos\zeta \left[(\widehat{\mathbf{e}}_2 \sin\zeta + \widehat{\mathbf{e}}_1 \cos\zeta) \cdot \boldsymbol{\nabla} \right] \widehat{\mathbf{n}} \end{aligned} \tag{51}$$

The first and the fourth line can be combined and similarly the second and the third lines

$$\begin{aligned} W &= \\ & (\widehat{\mathbf{e}}_1 \sin\zeta - \widehat{\mathbf{e}}_2 \cos\zeta) \left[(\widehat{\mathbf{e}}_1 \cos\zeta + \widehat{\mathbf{e}}_2 \sin\zeta) \cdot \boldsymbol{\nabla} \right] \widehat{\mathbf{n}} + \\ & (\widehat{\mathbf{e}}_1 \cos\zeta + \widehat{\mathbf{e}}_2 \sin\zeta) \left[(\widehat{\mathbf{e}}_1 \sin\zeta - \widehat{\mathbf{e}}_2 \cos\zeta) \cdot \boldsymbol{\nabla} \right] \widehat{\mathbf{n}} \end{aligned} \tag{52}$$

We now express these formulas using the versors

$$W = -\widehat{\boldsymbol{\rho}} \cdot \left[(\widehat{\mathbf{v}}_\perp \cdot \boldsymbol{\nabla}) \, \widehat{\mathbf{n}} \right] - \widehat{\mathbf{v}}_\perp \cdot \left[(\widehat{\boldsymbol{\rho}} \cdot \boldsymbol{\nabla}) \, \widehat{\mathbf{n}} \right] \tag{53}$$

Consider separately the first expression

$$\begin{aligned} \widehat{\boldsymbol{\rho}} \cdot \left[(\widehat{\mathbf{v}}_\perp \cdot \boldsymbol{\nabla}) \, \widehat{\mathbf{n}} \right] &= \widehat{\boldsymbol{\rho}} \widehat{\mathbf{v}}_\perp : \boldsymbol{\nabla} \widehat{\mathbf{n}} \\ &= \widehat{\rho}_i \widehat{v}_{\perp j} \partial_j \widehat{n}_i \end{aligned} \tag{54}$$

We know that

$$\begin{aligned} \widehat{\boldsymbol{\rho}} \times \widehat{\mathbf{v}}_\perp &= \widehat{\mathbf{n}} \\ \widehat{\mathbf{n}} \times \widehat{\boldsymbol{\rho}} &= \widehat{\mathbf{v}}_\perp \end{aligned} \tag{55}$$

The first relationship

$$\varepsilon_{kij} \widehat{\rho}_i \widehat{v}_{\perp j} = n_k \tag{56}$$

is contracted withe the completely antisymmetric tensor

$$\begin{aligned} \varepsilon_{klm} \varepsilon_{kij} \widehat{\rho}_i \widehat{v}_{\perp j} &= \varepsilon_{klm} n_k \\ (\delta_{li} \delta_{mj} - \delta_{lj} \delta_{mi}) \, \widehat{\rho}_i \widehat{v}_{\perp j} &= \varepsilon_{klm} n_k \\ \widehat{\rho}_i \widehat{v}_{\perp j} - \widehat{\rho}_j \widehat{v}_{\perp i} &= \varepsilon_{kij} n_k \end{aligned} \tag{57}$$

(after a re-labelling). From here we separate the product that appears in Eq.(54)

$$\widehat{\rho}_i \widehat{v}_{\perp j} = \varepsilon_{ijk} n_k + \widehat{\rho}_j \widehat{v}_{\perp i} \tag{58}$$

Then in Eq.(53) we have, using Eq.(58)

$$\begin{aligned} W &= -\widehat{\boldsymbol{\rho}} \cdot \left[\left(\widehat{\mathbf{v}}_\perp \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} \right] - \widehat{\mathbf{v}}_\perp \cdot \left[\left(\widehat{\boldsymbol{\rho}} \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} \right] \\ &\equiv -\widehat{\rho}_i \widehat{v}_{\perp j} \partial_j \widehat{n}_i - \widehat{v}_{\perp j} \widehat{\rho}_i \partial_i \widehat{n}_j \\ &= -\left(\varepsilon_{ijk} n_k + \widehat{\rho}_j \widehat{v}_{\perp i} \right) \partial_j \widehat{n}_i - \widehat{v}_{\perp j} \widehat{\rho}_i \partial_i \widehat{n}_j \\ &= -2 \widehat{v}_{\perp j} \widehat{\rho}_i \partial_i \widehat{n}_j - \varepsilon_{ijk} n_k \partial_j \widehat{n}_i \end{aligned} \tag{59}$$

or

$$W = -2 \widehat{\mathbf{v}}_\perp \cdot \left(\widehat{\boldsymbol{\rho}} \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} + \widehat{\mathbf{n}} \cdot \left(\boldsymbol{\nabla} \times \widehat{\mathbf{n}} \right)$$

2.4.1 Drift ordering and expansion of the Fokker-Planck equation

After the change of variables we re-write the FP equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} f - \frac{e}{m} \boldsymbol{\nabla} \phi \cdot \frac{\partial f}{\partial \mathbf{v}} + \frac{e}{m} \left(\mathbf{v} \times \mathbf{B} \right) \cdot \frac{\partial f}{\partial \mathbf{v}} = C\left(f \right) \tag{60}$$

The change is

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} \tag{61}$$

$$\frac{\partial}{\partial \mathbf{v}} \rightarrow \mathbf{v} \frac{\partial}{\partial \epsilon} + \mathbf{v}_\perp \frac{\partial}{B \partial \mu} - \frac{\mathbf{v} \times \widehat{\mathbf{n}}}{v_\perp^2} \frac{\partial}{\partial \zeta} \tag{62}$$

and we introduce the separation of parallel and perpendicular components of the velocity: the dynamics on the two directions is largely different.

$$\mathbf{v} \cdot \boldsymbol{\nabla} f = \left(v_\parallel \widehat{\mathbf{n}} + \mathbf{v}_\perp \right) \cdot \boldsymbol{\nabla} f \tag{63}$$

where the spatial gradient is expressed in the new variables. We expand one after the other the two terms.

The first operator in Eq.(63) is

$$v_\parallel \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \tag{64}$$

The explicit expression is

$$\begin{aligned} & v_\parallel \widehat{n}_k \left\{ \partial_k + \frac{e}{m} \partial_k \phi \frac{\partial}{\partial \epsilon} \right. \\ & - \left(\mu \partial_k B + v_\parallel \mathbf{v} \cdot \partial_k \widehat{\mathbf{n}} \right) \frac{\partial}{B \partial \mu} \\ & \left. + \left[\widehat{\mathbf{e}}_2 \cdot \left(\partial_k \widehat{\mathbf{e}}_1 \right) - \frac{v_\parallel}{v_\perp^2} \left(\partial_k \widehat{\mathbf{n}} \right) \cdot \left(\widehat{\mathbf{n}} \times \mathbf{v}_\perp \right) \right] \frac{\partial}{\partial \zeta} \right\} \end{aligned} \tag{65}$$

In order to look for simplifications we take separately the second term in the second line of Eq.(65), coefficient of the term $\frac{\partial}{B\partial\mu}$:

$$\begin{aligned} & v_{\parallel}\widehat{n}_k v_{\parallel}\mathbf{v}\cdot\partial_k\widehat{\mathbf{n}} \\ = & v_{\parallel}^2\widehat{n}_k v_i\partial_k\widehat{n}_i \\ = & v_{\parallel}^2\widehat{n}_k\left(v_{\perp i}+v_{\parallel}\widehat{n}_i\right)\partial_k\widehat{n}_i \\ = & v_{\parallel}^2\widehat{n}_k v_{\perp i}\partial_k\widehat{n}_i \\ = & v_{\parallel}^2\mathbf{v}_{\perp}\cdot\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}} \end{aligned} \tag{66}$$

because the term $v_{\parallel}^3\widehat{n}_k\widehat{n}_i\partial_k\widehat{n}_i$ is zero since $|\widehat{\mathbf{n}}|=1$. Obviously this term will lead to the curvature drift.

The second term in the last line of the Eq.(65) (coefficient of $\partial/\partial\zeta$) is

$$\begin{aligned} & \frac{v_{\parallel}^2}{v_{\perp}^2}\left(n_k\partial_k\widehat{\mathbf{n}}\right)\left(\widehat{\mathbf{n}}\times\mathbf{v}_{\perp}\right) \\ = & \frac{v_{\parallel}^2}{v_{\perp}^2}\widehat{\mathbf{n}}\cdot\left[\mathbf{v}_{\perp}\times\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right] \end{aligned} \tag{67}$$

In the Eq.(63) the second operator is

$$\begin{aligned} \mathbf{v}_{\perp}\cdot\boldsymbol{\nabla} \quad \rightarrow \quad & v_{\perp k}\left\{\partial_k+\frac{e}{m}\partial_k\phi\frac{\partial}{\partial\epsilon}\right. \\ & -\left[\mu\partial_k B+v_{\parallel}\mathbf{v}\cdot\partial_k\widehat{\mathbf{n}}\right]\frac{\partial}{B\partial\mu} \\ & \left.+\left[\left(\partial_k\widehat{\mathbf{e}}_2\right)\cdot\widehat{\mathbf{e}}_1-\frac{v_{\parallel}}{v_{\perp}^2}\left(\partial_k\widehat{\mathbf{n}}\right)\cdot\left(\widehat{\mathbf{n}}\times\mathbf{v}_{\perp}\right)\right]\frac{\partial}{\partial\zeta}\right\} \end{aligned} \tag{68}$$

We take separately few terms from the product between the k-th component of the perpendicular velocity and various components of the gradient in the new variables. The second term in the coefficient of $\partial/B\partial\mu$ in Eq.(68) is

$$\begin{aligned} & v_{\perp k}v_{\parallel}\mathbf{v}\cdot\partial_k\widehat{\mathbf{n}} \\ = & v_{\perp k}v_{\parallel}\left(v_{\parallel}\widehat{\mathbf{n}}+\mathbf{v}_{\perp}\right)\cdot\partial_k\widehat{\mathbf{n}} \\ = & v_{\perp k}v_{\parallel}\left(v_{\parallel}\widehat{n}_i+v_{\perp i}\right)\partial_k\widehat{n}_i \\ = & v_{\parallel}v_{\perp k}v_{\perp i}\partial_k\widehat{n}_i+v_{\parallel}^2 v_{\perp k}\widehat{n}_i\partial_k\widehat{n}_i \end{aligned} \tag{69}$$

where the last term is zero since $\widehat{\mathbf{n}}$ has norm one

$$\widehat{n}_i\partial_k\widehat{n}_i=\frac{1}{2}\partial_k\left(\widehat{\mathbf{n}}^2\right)=0 \tag{70}$$

Then

$$v_{\perp k} v_\parallel \mathbf{v} \cdot \partial_k \widehat{\mathbf{n}} = v_\parallel v_{\perp k} v_{\perp i} \partial_k \widehat{n}_i \tag{71}$$
$$= v_\parallel \mathbf{v}_\perp \mathbf{v}_\perp : \boldsymbol{\nabla} \widehat{\mathbf{n}}$$

This is a tensorial product.

The second term in the coefficient of $\partial/\partial\zeta$ in Eq.(68) is

$$v_{\perp k} \frac{v_\parallel}{v_\perp^2} \left(\partial_k \widehat{\mathbf{n}}\right) \cdot \left(\widehat{\mathbf{n}} \times \mathbf{v}_\perp\right) = \frac{v_\parallel}{v_\perp^2} \left[\left(\mathbf{v}_\perp \cdot \boldsymbol{\nabla}\right) \widehat{\mathbf{n}}\right] \cdot \left(\widehat{\mathbf{n}} \times \mathbf{v}_\perp\right) \tag{72}$$
$$= \frac{v_\parallel}{v_\perp^2} \widehat{\mathbf{n}} \cdot \left[\mathbf{v}_\perp \times \left(\mathbf{v}_\perp \cdot \boldsymbol{\nabla}\right) \widehat{\mathbf{n}}\right]$$

Now we write again the Fokker-Planck equation and insert the velocity $\mathbf{v}$ with the two components: parallel and perpendicular. We only write the operator, acting on the distribution function f:

$$\begin{aligned}
& \frac{\partial}{\partial t} \\
& + v_\parallel n_k \partial_k + \frac{e}{m} v_\parallel \widehat{n}_k \left(\partial_k \phi\right) \frac{\partial}{\partial \epsilon} \\
& + v_{\perp k} \partial_k + \frac{e}{m} v_{\perp k} \left(\partial_k \phi\right) \frac{\partial}{\partial \epsilon} \\
& - \left(\mu v_{\perp k} \partial_k B + v_\parallel v_{\perp k} v_{\perp i} \partial_k n_i\right) \frac{\partial}{B \partial \mu} \\
& - \left(\mu v_\parallel n_k \partial_k B + v_\parallel^2 \mathbf{v}_\perp \left(\mathbf{n} \cdot \boldsymbol{\nabla}\right) \mathbf{n}\right) \frac{\partial}{B \partial \mu} \\
& + \left\{ \widehat{\mathbf{e}}_1 \cdot \left(v_{\perp k} \partial_k\right) \widehat{\mathbf{e}}_2 - \frac{v_\parallel}{v_\perp^2} \widehat{\mathbf{n}} \cdot \left[\mathbf{v}_\perp \times \left(v_{\perp k} \partial_k\right) \widehat{\mathbf{n}}\right] \right\} \frac{\partial}{\partial \zeta} \\
& + \left\{ \widehat{\mathbf{e}}_1 \cdot \left(v_\parallel n_k \partial_k\right) \widehat{\mathbf{e}}_2 - \frac{v_\parallel^2}{v_\perp^2} \left[\mathbf{v}_\perp \times \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right) \widehat{\mathbf{n}}\right] \right\} \frac{\partial}{\partial \zeta} \\
& - \frac{e}{m} \boldsymbol{\nabla} \phi \cdot \left(\mathbf{v} \frac{\partial}{\partial \epsilon} + \mathbf{v}_\perp \frac{\partial}{B \partial \mu} - \frac{\mathbf{v}_\perp \times \widehat{\mathbf{n}}}{v_\perp^2} \frac{\partial}{\partial \zeta} \right) \\
& + \frac{e}{m} \left(\mathbf{v} \times \mathbf{B}\right) \cdot \left(\mathbf{v} \frac{\partial}{\partial \epsilon} + \mathbf{v}_\perp \frac{\partial}{B \partial \mu} - \frac{\mathbf{v}_\perp \times \widehat{\mathbf{n}}}{v_\perp^2} \frac{\partial}{\partial \zeta} \right) \\
= \; & C
\end{aligned} \tag{73}$$

We note that the term with the operator $\partial/\partial\epsilon$ in the line containing $\boldsymbol{\nabla}\phi$ (the eighth line) can be combined with the second term from the second line and with the second term from the third line and all cancel (these two also contain the operator $\partial/\partial\epsilon$) and this gives

$$\begin{aligned} & -\frac{e}{m}\boldsymbol{\nabla}\phi\cdot\mathbf{v}\frac{\partial}{\partial\epsilon} \\ & +\frac{e}{m}v_{\parallel}\widehat{n}_k\left(\partial_k\phi\right)\frac{\partial}{\partial\epsilon} \\ & +\frac{e}{m}v_{\perp k}\left(\partial_k\phi\right)\frac{\partial}{\partial\epsilon} \\ = & \ 0 \end{aligned} \tag{74}$$

We also note the obvious cancellation of the scalar products in the first two terms of the last line of the left hand side of the equation:

$$\begin{aligned} & \frac{e}{m}\left(\mathbf{v}\times\mathbf{B}\right)\cdot\left(\mathbf{v}\frac{\partial}{\partial\epsilon}+\mathbf{v}_{\perp}\frac{\partial}{B\partial\mu}-\frac{\mathbf{v}_{\perp}\times\widehat{\mathbf{n}}}{v_{\perp}^2}\frac{\partial}{\partial\zeta}\right) \\ = & -\frac{e}{m}\left(\mathbf{v}\times\mathbf{B}\right)\cdot\frac{\mathbf{v}_{\perp}\times\widehat{\mathbf{n}}}{v_{\perp}^2}\frac{\partial}{\partial\zeta} \end{aligned} \tag{75}$$

This term is

$$-\frac{e}{m}\left(\mathbf{v}\times\mathbf{B}\right)\cdot\frac{\mathbf{v}_{\perp}\times\widehat{\mathbf{n}}}{v_{\perp}^2}=-\frac{eB}{m}\equiv-\Omega \tag{76}$$

The equation becomes

$$\begin{aligned} & \frac{\partial}{\partial t}+v_{\parallel}\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right) \\ & +\mathbf{v}_{\perp}\cdot\left\{\boldsymbol{\nabla}-\left[\mu\boldsymbol{\nabla}B+v_{\parallel}\left(\mathbf{v}_{\perp}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\frac{\partial}{B\partial\mu}\right\} \\ & -\left[\mu v_{\parallel}\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)B+v_{\parallel}^2\mathbf{v}_{\perp}\cdot\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\frac{\partial}{B\partial\mu} \\ & -\frac{e}{m}\mathbf{v}_{\perp}\cdot\boldsymbol{\nabla}\phi\frac{\partial}{B\partial\mu} \\ & +\left[\widehat{\mathbf{e}}_1\cdot\left(\mathbf{v}_{\perp}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{e}}_2+v_{\parallel}\widehat{\mathbf{e}}_1\cdot\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{e}}_2\right]\frac{\partial}{\partial\zeta} \\ & -\frac{v_{\parallel}}{v_{\perp}^2}\widehat{\mathbf{n}}\cdot\left\{\mathbf{v}_{\perp}\times\left[\left(\mathbf{v}_{\perp}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}+v_{\parallel}\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\right\}\frac{\partial}{\partial\zeta} \\ & +\frac{e}{m}\boldsymbol{\nabla}\phi\cdot\frac{\mathbf{v}_{\perp}\times\widehat{\mathbf{n}}}{v_{\perp}^2}\frac{\partial}{\partial\zeta}-\Omega\frac{\partial}{\partial\zeta} \\ = & \ 0 \end{aligned} \tag{77}$$

The Eq.(77) is now prepared for the explicit representation of $\mathbf{v}_{\perp}$ in terms of the gyration angle ζ.

For the beginning we consider the second line in Eq.(77). The third term of the second line is

$$-\mathbf{v}_\perp \cdot v_\parallel \left(\mathbf{v}_\perp \cdot \boldsymbol{\nabla}\right) \widehat{\mathbf{n}} \frac{\partial}{B\partial\mu} \tag{78}$$

This term contains the tensorial product

$$\mathbf{v}_\perp \cdot \left(\mathbf{v}_\perp \cdot \boldsymbol{\nabla}\right) \widehat{\mathbf{n}} = \mathbf{v}_\perp \mathbf{v}_\perp : \boldsymbol{\nabla}\widehat{\mathbf{n}} = v_{\perp k} v_{\perp i} \partial_i \widehat{n}_k \tag{79}$$

where we make explicit the trigonometric functions of the gyration angle ζ

$$\begin{aligned} & v_{\perp k} v_{\perp i} \partial_i \widehat{n}_k \\ = \; & v_\perp^2 \left(\widehat{\mathbf{e}}_1 \cos\zeta + \widehat{\mathbf{e}}_2 \sin\zeta\right)_k \left(\widehat{\mathbf{e}}_1 \cos\zeta + \widehat{\mathbf{e}}_2 \sin\zeta\right)_i \partial_i \widehat{n}_k \\ = \; & v_\perp^2 \left\{ \widehat{e}_{1k} \widehat{e}_{1i} \partial_i \widehat{n}_k \cos^2\zeta + \widehat{e}_{1k} \widehat{e}_{2i} \partial_i \widehat{n}_k \cos\zeta \sin\zeta \right. \\ & \left. + \widehat{e}_{2k} \widehat{e}_{1i} \partial_i \widehat{n}_k \sin\zeta \cos\zeta + \widehat{e}_{2k} \widehat{e}_{2i} \partial_i \widehat{n}_k \sin^2\zeta \right\} \end{aligned} \tag{80}$$

Replacing the products with the trigonometric functions of double argument

$$\begin{aligned} \mathbf{v}_\perp \mathbf{v}_\perp & : \boldsymbol{\nabla}\widehat{\mathbf{n}} \\ & = \frac{1}{2} v_\perp^2 \left\{ \widehat{\mathbf{e}}_1 \cdot \left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}\right) \widehat{\mathbf{n}} + \widehat{\mathbf{e}}_2 \cdot \left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}\right) \widehat{\mathbf{n}} \right. \\ & \quad + \left[\widehat{\mathbf{e}}_1 \cdot \left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}\right) \widehat{\mathbf{n}} - \widehat{\mathbf{e}}_2 \cdot \left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}\right) \widehat{\mathbf{n}}\right] \cos 2\zeta \\ & \quad \left. + \left[\widehat{\mathbf{e}}_1 \cdot \left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}\right) \widehat{\mathbf{n}} + \widehat{\mathbf{e}}_2 \cdot \left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}\right) \widehat{\mathbf{n}}\right] \sin 2\zeta \right\} \end{aligned} \tag{81}$$

Using Eq.(14) we replace the first term in the curly bracket. At this point we have obtained

$$\begin{aligned} \mathbf{v}_\perp \mathbf{v}_\perp & : \boldsymbol{\nabla}\widehat{\mathbf{n}} \\ & = \frac{1}{2} v_\perp^2 \left\{ \boldsymbol{\nabla}\cdot\widehat{\mathbf{n}} \right. \\ & \quad + \left[\widehat{\mathbf{e}}_1 \cdot \left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}\right) \widehat{\mathbf{n}} - \widehat{\mathbf{e}}_2 \cdot \left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}\right) \widehat{\mathbf{n}}\right] \cos 2\zeta \\ & \quad \left. + \left[\widehat{\mathbf{e}}_1 \cdot \left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}\right) \widehat{\mathbf{n}} + \widehat{\mathbf{e}}_2 \cdot \left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}\right) \widehat{\mathbf{n}}\right] \sin 2\zeta \right\} \end{aligned} \tag{82}$$

In Eq.(82) the first term, not containing functions of ζ is multiplied with $-v_\parallel$, as shown in Eq.(78)

$$\begin{aligned} & -v_\parallel \\ & \times \frac{1}{2} v_\perp^2 \boldsymbol{\nabla}\cdot\widehat{\mathbf{n}} \\ & \times \frac{\partial}{B\partial\mu} \\ = \; & -v_\parallel \mu \left(\boldsymbol{\nabla}\cdot\widehat{\mathbf{n}}\right) \frac{\partial}{\partial\mu} \end{aligned} \tag{83}$$

Now, from the third line in Eq.(77)

$$-\left[\mu v_{\|}\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)B+v_{\|}^{2}\mathbf{v}_{\perp}\cdot\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\frac{\partial}{B\partial\mu} \tag{84}$$

we consider the first term and use Eq.(10)

$$-\mu v_{\|}\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)B\frac{\partial}{B\partial\mu}=\mu v_{\|}B\left(\boldsymbol{\nabla}\cdot\widehat{\mathbf{n}}\right)\frac{\partial}{B\partial\mu} \tag{85}$$

and note that this cancels with the term we have obtained in Eq.(83). The rest of the expression of the third line in Eq.(77) is

$$v_{\|}^{2}v_{\perp}\left[\widehat{\mathbf{e}}_{1}\cdot\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\cos\zeta+\widehat{\mathbf{e}}_{2}\cdot\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\sin\zeta\right]\frac{\partial}{B\partial\mu} \tag{86}$$

Collecting the results up to this point

$$\begin{aligned}
&\frac{\partial f}{\partial t}+v_{\|}\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)f \\
&+v_{\perp}\left(\widehat{\mathbf{e}}_{1}\cos\zeta+\widehat{\mathbf{e}}_{2}\sin\zeta\right)\cdot\left[\boldsymbol{\nabla}f-\left(\mu\boldsymbol{\nabla}B+v_{\|}^{2}\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}+\frac{e}{m}\boldsymbol{\nabla}\phi\right)\frac{\partial f}{B\partial\mu}\right] \\
&-v_{\|}\mu\frac{\partial f}{\partial\mu}\left\{\left[\widehat{\mathbf{e}}_{1}\cdot\left(\widehat{\mathbf{e}}_{1}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}-\widehat{\mathbf{e}}_{2}\cdot\left(\widehat{\mathbf{e}}_{2}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\cos 2\zeta\right. \\
&\left.+\left[\widehat{\mathbf{e}}_{1}\cdot\left(\widehat{\mathbf{e}}_{2}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}+\widehat{\mathbf{e}}_{2}\cdot\left(\widehat{\mathbf{e}}_{1}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\sin 2\zeta\right\} \\
&+\Xi\left(\zeta\right)\frac{\partial f}{\partial\zeta} \\
=\;& C\left(f\right)
\end{aligned} \tag{87}$$

The detailed form of the factor $\Xi\left(\zeta\right)$ is

$$\begin{aligned}
&\Xi\left(\zeta\right) \\
\equiv\;&\left[\widehat{\mathbf{e}}_{1}\cdot\left(\mathbf{v}_{\perp}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{e}}_{2}+v_{\|}\widehat{\mathbf{e}}_{1}\cdot\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{e}}_{2}\right] \\
&-\frac{v_{\|}}{v_{\perp}^{2}}\widehat{\mathbf{n}}\cdot\left\{\mathbf{v}_{\perp}\times\left[\left(\mathbf{v}_{\perp}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}+v_{\|}\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\right\} \\
&+\frac{e}{m}\boldsymbol{\nabla}\phi\cdot\frac{\mathbf{v}\times\widehat{\mathbf{n}}}{v_{\perp}^{2}}-\Omega
\end{aligned} \tag{88}$$

We will now make explicit the presence of trigonometric functions

$$\begin{aligned}
\Xi(\zeta) \; = \; & \\
& v_\perp \left[\widehat{\mathbf{e}}_1 \cdot (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_2 \cos\zeta + \widehat{\mathbf{e}}_1 \cdot (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_2 \sin\zeta\right] \\
& + v_\parallel \widehat{\mathbf{e}}_1 \cdot (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_2 \\
& - v_\parallel \widehat{\mathbf{n}} \cdot \big[\widehat{\mathbf{e}}_1 \times (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} \cos^2\zeta + \widehat{\mathbf{e}}_2 \times (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} \sin^2\zeta \\
& + \widehat{\mathbf{e}}_2 \times (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} \cos\zeta \sin\zeta + \widehat{\mathbf{e}}_1 \times (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} \sin\zeta \cos\zeta\big] \\
& - \frac{v_\parallel^2}{v_\perp} \widehat{\mathbf{n}} \cdot \left[\widehat{\mathbf{e}}_1 \times (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} \cos\zeta + \widehat{\mathbf{e}}_2 \times (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} \sin\zeta\right] \\
& + \frac{e}{m} \boldsymbol{\nabla}\phi \cdot \frac{1}{v_\perp} \left(-\widehat{\mathbf{e}}_2 \cos\zeta + \widehat{\mathbf{e}}_1 \sin\zeta\right) \\
& - \Omega
\end{aligned} \tag{89}$$

All this expression multiplies $\partial f / \partial \zeta$. We collect together the terms that contains the same trigonometric functions

$$\begin{aligned}
\Xi(\zeta) \; = \; & \\
& \left[v_\perp \widehat{\mathbf{e}}_1 \cdot (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_2 - \frac{v_\parallel^2}{v_\perp} \widehat{\mathbf{n}} \cdot \widehat{\mathbf{e}}_1 \times (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} - \frac{1}{v_\perp} \frac{e}{m} \boldsymbol{\nabla}\phi \cdot \widehat{\mathbf{e}}_2\right] \cos\zeta \\
& + \left[v_\perp \widehat{\mathbf{e}}_1 \cdot (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_2 - \frac{v_\parallel^2}{v_\perp} \widehat{\mathbf{n}} \cdot \widehat{\mathbf{e}}_2 \times (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} + \frac{1}{v_\perp} \frac{e}{m} \boldsymbol{\nabla}\phi \cdot \widehat{\mathbf{e}}_1\right] \sin\zeta \\
& - \frac{1}{2} v_\parallel \widehat{\mathbf{n}} \cdot \left[\widehat{\mathbf{e}}_1 \times (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} - \widehat{\mathbf{e}}_2 \times (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}}\right] \cos(2\zeta) \\
& - \frac{1}{2} v_\parallel \widehat{\mathbf{n}} \cdot \left[\widehat{\mathbf{e}}_2 \times (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} + \widehat{\mathbf{e}}_1 \times (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}}\right] \sin(2\zeta) \\
& + v_\parallel \widehat{\mathbf{e}}_1 \cdot (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_2 \\
& + \frac{1}{2} v_\parallel \widehat{\mathbf{n}} \cdot \left[\widehat{\mathbf{e}}_1 \times (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} + \widehat{\mathbf{e}}_2 \times (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}}\right] \\
& - \Omega
\end{aligned} \tag{90}$$

Take separately the first terms of the first (coefficient of $\cos\zeta$) and second (coefficient if $\sin\zeta$) lines

$$v_\perp \left[\widehat{\mathbf{e}}_1 \cdot (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_2 \cos\zeta + \widehat{\mathbf{e}}_1 \cdot (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_2 \sin\zeta\right] \tag{91}$$

Few transformations are possible (we do not write for the moemnt the coefficient $v_\perp$)

$$\begin{aligned}
& \widehat{\mathbf{e}}_1 \cdot (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_2 \cos\zeta + \widehat{\mathbf{e}}_1 \cdot (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_2 \sin\zeta \\
= \; & (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_1 \cdot (-\widehat{\mathbf{e}}_2 \cos\zeta) + (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_2 \cdot (\widehat{\mathbf{e}}_1 \sin\zeta) \\
& + (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_2 \cdot (-\widehat{\mathbf{e}}_2 \cos\zeta) + (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_1 \cdot (\widehat{\mathbf{e}}_1 \sin\zeta)
\end{aligned} \tag{92}$$

but the last two terms vanish according to the equations derived above. The result obtained by grouping the first terms in the first two lines of Ξ in Eq.(90) is

$$\left[\left(\widehat{\mathbf{e}}_1\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{e}}_1+\left(\widehat{\mathbf{e}}_2\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{e}}_2\right]\left(-\widehat{\mathbf{e}}_2\cos\zeta+\widehat{\mathbf{e}}_1\sin\zeta\right) \tag{93}$$

We take together the second terms of the first two lines of Ξ in Eq.(90) and write

$$\begin{aligned} & -\frac{v_\parallel^2}{v_\perp}\widehat{\mathbf{n}}\cdot\widehat{\mathbf{e}}_1\times\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\cos\zeta-\frac{v_\parallel^2}{v_\perp}\widehat{\mathbf{n}}\cdot\widehat{\mathbf{e}}_2\times\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\sin\zeta \\ = & \frac{v_\parallel^2}{v_\perp}\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\cdot\left(-\widehat{\mathbf{e}}_2\cos\zeta+\widehat{\mathbf{e}}_1\sin\zeta\right)\end{aligned} \tag{94}$$

Finally we put together the third terms of the first two lines

$$-\frac{1}{v_\perp}\frac{e}{m}\boldsymbol{\nabla}\phi\cdot\widehat{\mathbf{e}}_2\cos\zeta+\frac{1}{v_\perp}\frac{e}{m}\boldsymbol{\nabla}\phi\cdot\widehat{\mathbf{e}}_1\sin\zeta \tag{95}$$

Now we collect all these three results, obtaining a new form for the first two lines in Eq.(90)

$$\begin{aligned} & \left[\frac{v_\parallel^2}{v_\perp}\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}+v_\perp\left(\widehat{\mathbf{e}}_1\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{e}}_1+v_\perp\left(\widehat{\mathbf{e}}_2\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{e}}_2\right]\cdot\left(-\widehat{\mathbf{e}}_2\cos\zeta+\widehat{\mathbf{e}}_1\sin\zeta\right) \\ & +\frac{1}{v_\perp}\frac{e}{m}\boldsymbol{\nabla}\phi\cdot\left(-\widehat{\mathbf{e}}_2\cos\zeta+\widehat{\mathbf{e}}_1\sin\zeta\right)\end{aligned} \tag{96}$$

Then at this moment $\Xi(\zeta)$ has the following expression

$$\begin{aligned} & \Xi(\zeta)= \\ & \left[\frac{v_\parallel^2}{v_\perp}\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}+v_\perp\left(\widehat{\mathbf{e}}_1\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{e}}_1+v_\perp\left(\widehat{\mathbf{e}}_2\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{e}}_2\right]\cdot\left(-\widehat{\mathbf{e}}_2\cos\zeta+\widehat{\mathbf{e}}_1\sin\zeta\right) \\ & -\frac{v_\parallel}{2}\left\{\left[\widehat{\mathbf{e}}_2\cdot\left(\widehat{\mathbf{e}}_1\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}+\widehat{\mathbf{e}}_1\cdot\left(\widehat{\mathbf{e}}_2\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\cos(2\zeta)\right. \\ & \left.+\left[-\widehat{\mathbf{e}}_1\cdot\left(\widehat{\mathbf{e}}_1\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}+\widehat{\mathbf{e}}_2\cdot\left(\widehat{\mathbf{e}}_2\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\sin(2\zeta)\right\} \\ & +\frac{1}{v_\perp}\frac{e}{m}\boldsymbol{\nabla}\phi\cdot\left(-\widehat{\mathbf{e}}_2\cos\zeta+\widehat{\mathbf{e}}_1\sin\zeta\right) \\ & -v_\parallel\left\{\widehat{\mathbf{e}}_2\cdot\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{e}}_1+\frac{1}{2}\left[\widehat{\mathbf{e}}_1\cdot\left(\widehat{\mathbf{e}}_2\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}-\widehat{\mathbf{e}}_2\cdot\left(\widehat{\mathbf{e}}_1\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\right\} \\ & -\Omega\end{aligned} \tag{97}$$

Although is a lengthy expression, we write here the form of the Fokker-Planck equation at this moment.

$$\frac{\partial f}{\partial t}+v_{\parallel}\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)f$$
$$+v_{\perp}\left(\widehat{\mathbf{e}}_1\cos\zeta+\widehat{\mathbf{e}}_2\sin\zeta\right)\cdot\left[\boldsymbol{\nabla}f-\left(\mu\boldsymbol{\nabla}B+v_{\parallel}^2\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}+\frac{e}{m}\boldsymbol{\nabla}\phi\right)\frac{\partial f}{B\partial\mu}\right]$$
$$-v_{\parallel}\left\{\left[\widehat{\mathbf{e}}_1\cdot\left(\widehat{\mathbf{e}}_1\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}-\widehat{\mathbf{e}}_2\cdot\left(\widehat{\mathbf{e}}_2\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\cos 2\zeta\right.$$
$$\left.+\left[\widehat{\mathbf{e}}_1\cdot\left(\widehat{\mathbf{e}}_2\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}+\widehat{\mathbf{e}}_2\cdot\left(\widehat{\mathbf{e}}_1\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\sin 2\zeta\right\}\mu\frac{\partial f}{\partial\mu}$$
$$+\left\{-v_{\parallel}\left[\widehat{\mathbf{e}}_2\cdot\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{e}}_1+\frac{1}{2}\left[\widehat{\mathbf{e}}_1\cdot\left(\widehat{\mathbf{e}}_2\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}-\widehat{\mathbf{e}}_2\cdot\left(\widehat{\mathbf{e}}_1\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\right]\right.$$
$$+\left[\frac{v_{\parallel}^2}{v_{\perp}}\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}+v_{\perp}\left(\widehat{\mathbf{e}}_1\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{e}}_1+v_{\perp}\left(\widehat{\mathbf{e}}_2\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{e}}_2\right]\cdot\left(-\widehat{\mathbf{e}}_2\cos\zeta+\widehat{\mathbf{e}}_1\sin\zeta\right)$$
$$-\frac{v_{\parallel}}{2}\left[\left[\widehat{\mathbf{e}}_2\cdot\left(\widehat{\mathbf{e}}_1\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}+\widehat{\mathbf{e}}_1\cdot\left(\widehat{\mathbf{e}}_2\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\cos\left(2\zeta\right)\right.$$
$$\left.\left.+\left[-\widehat{\mathbf{e}}_1\cdot\left(\widehat{\mathbf{e}}_1\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}+\widehat{\mathbf{e}}_2\cdot\left(\widehat{\mathbf{e}}_2\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\sin\left(2\zeta\right)\right]\right\}\frac{\partial f}{\partial\zeta}$$
$$+\frac{1}{v_{\perp}}\frac{e}{m}\boldsymbol{\nabla}\phi\cdot\left(-\widehat{\mathbf{e}}_2\cos\zeta+\widehat{\mathbf{e}}_1\sin\zeta\right)\frac{\partial f}{\partial\zeta}$$
$$-\Omega\frac{\partial f}{\partial\zeta}=C\left(f\right)$$

This is #eq428

We introduce the notation for the drift velocity

$$\mathbf{v}_D\equiv\frac{1}{\Omega}\widehat{\mathbf{n}}\times\left(\mu\boldsymbol{\nabla}B+v_{\parallel}\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}+\frac{e}{m}\boldsymbol{\nabla}\phi\right)\tag{98}$$

Later we will use the following shorter notations

Notation A Let

$$\begin{aligned}A\ \equiv\ &\left[\widehat{\mathbf{e}}_1\cdot\left(\widehat{\mathbf{e}}_1\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}-\widehat{\mathbf{e}}_2\cdot\left(\widehat{\mathbf{e}}_2\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\cos 2\zeta\\&+\left[\widehat{\mathbf{e}}_1\cdot\left(\widehat{\mathbf{e}}_2\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}+\widehat{\mathbf{e}}_2\cdot\left(\widehat{\mathbf{e}}_1\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\sin 2\zeta\end{aligned}\tag{99}$$

For A, from Eq.(82) we have

$$A=\frac{2}{v_{\perp}^2}\left(\mathbf{v}_{\perp}\mathbf{v}_{\perp}:\boldsymbol{\nabla}\widehat{\mathbf{n}}-\frac{1}{2}v_{\perp}^2\boldsymbol{\nabla}\cdot\widehat{\mathbf{n}}\right)$$

or

$$A=2\widehat{\mathbf{v}}_{\perp}\widehat{\mathbf{v}}_{\perp}:\boldsymbol{\nabla}\widehat{\mathbf{n}}-\boldsymbol{\nabla}\cdot\widehat{\mathbf{n}}$$

Notation B_1 We introduce the following notation:

$$B_1 \equiv -v_\parallel \left[\widehat{\mathbf{e}}_2 \cdot (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_1 + \frac{1}{2} \left[\widehat{\mathbf{e}}_1 \cdot (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} - \widehat{\mathbf{e}}_2 \cdot (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} \right] \right] \tag{100}$$

For B_1, we calculate separately the second part. The formula for B_1 becomes

$$\begin{aligned} B_1 &\equiv -v_\parallel \left[\widehat{\mathbf{e}}_2 \cdot (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_1 + \frac{1}{2} \left[\widehat{\mathbf{e}}_1 \cdot (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} - \widehat{\mathbf{e}}_2 \cdot (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} \right] \right] \\ &= -v_\parallel \widehat{\mathbf{e}}_2 \cdot (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_1 + \frac{v_\parallel}{2} \widehat{\mathbf{n}} \cdot (\boldsymbol{\nabla} \times \widehat{\mathbf{n}}) \end{aligned}$$

Notation B_2 Let us introduce the notation

$$B_2 \equiv \left[\frac{v_\parallel^2}{v_\perp} (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} + v_\perp (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_1 + v_\perp (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_2 \right] \cdot (-\widehat{\mathbf{e}}_2 \cos\zeta + \widehat{\mathbf{e}}_1 \sin\zeta) \tag{101}$$

According to Eq.(47) the two terms in the square bracket can be written

$$(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_1 + (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{e}}_2 = -\widehat{\mathbf{n}} (\boldsymbol{\nabla} \cdot \widehat{\mathbf{n}}) + \widehat{\mathbf{n}} \times (\boldsymbol{\nabla} \times \widehat{\mathbf{n}})$$

and the factor containing trigonometric functions is

$$-\widehat{\mathbf{e}}_2 \cos\zeta + \widehat{\mathbf{e}}_1 \sin\zeta = \widehat{\boldsymbol{\rho}}$$

Then

$$B_2 = \widehat{\boldsymbol{\rho}} \cdot \left\{ \frac{v_\parallel^2}{v_\perp} (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} + v_\perp \left[-\widehat{\mathbf{n}} (\boldsymbol{\nabla} \cdot \widehat{\mathbf{n}}) + \widehat{\mathbf{n}} \times (\boldsymbol{\nabla} \times \widehat{\mathbf{n}}) \right] \right\}$$

and according to Eq.(49) we have

$$B_2 = \widehat{\boldsymbol{\rho}} \cdot \left\{ \frac{v_\parallel^2}{v_\perp} (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} + v_\perp \left[-\widehat{\mathbf{n}} (\boldsymbol{\nabla} \cdot \widehat{\mathbf{n}}) - (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} \right] \right\}$$

Since $\widehat{\boldsymbol{\rho}}$ is perpendicular on $\widehat{\mathbf{n}}$, the first term multiplying $v_\perp$ is suppressed.

$$\begin{aligned} B_2 &= \widehat{\boldsymbol{\rho}} \cdot \left\{ \frac{v_\parallel^2}{v_\perp} (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} - v_\perp (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} \right\} \\ &= \frac{1}{v_\perp} \left(v_\parallel^2 - v_\perp^2 \right) \widehat{\boldsymbol{\rho}} \cdot (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} \end{aligned}$$

Notation B_3 Let us introduce the notation

$$B_3 \equiv -\frac{v_\parallel}{2}\left[\widehat{\mathbf{e}}_2 \cdot \left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{n}} + \widehat{\mathbf{e}}_1 \cdot \left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\cos\left(2\zeta\right) -\frac{v_\parallel}{2}\left[-\widehat{\mathbf{e}}_1 \cdot \left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{n}} + \widehat{\mathbf{e}}_2 \cdot \left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\sin\left(2\zeta\right) \tag{102}$$

These notations will help when we have to perform the average over the gyroangle ζ, to simplify the writting. We should recall, wherever is necessary, the origin of these notations.

With these notations the kinetic equation reads

$$\begin{aligned}
&\frac{\partial f}{\partial t} + v_\parallel \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right) f \\
&+\mathbf{v}_\perp \cdot \left[\boldsymbol{\nabla} f - \left(\mu \boldsymbol{\nabla} B + v_\parallel^2 \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{n}} + \frac{e}{m}\boldsymbol{\nabla}\phi\right)\frac{\partial f}{B\partial \mu}\right] - v_\parallel \left[A\right]\mu\frac{\partial f}{\partial \mu} \\
&+\left[B_1 + B_2 + B_3\right]\frac{\partial f}{\partial \zeta} + \frac{1}{v_\perp}\frac{e}{m}\boldsymbol{\nabla}\phi \cdot \widehat{\boldsymbol{\rho}}\frac{\partial f}{\partial \zeta} - \Omega\frac{\partial f}{\partial \zeta} \\
= \; & C
\end{aligned} \tag{103}$$

or, using the expressions for the notations

$$\begin{aligned}
&\frac{\partial f}{\partial t} + v_\parallel \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right) f \\
&+\mathbf{v}_\perp \cdot \left[\boldsymbol{\nabla} f - \left(\mu \boldsymbol{\nabla} B + v_\parallel^2 \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{n}} + \frac{e}{m}\boldsymbol{\nabla}\phi\right)\frac{\partial f}{B\partial \mu}\right] \\
&-v_\parallel \left(2\widehat{\mathbf{v}}_\perp \widehat{\mathbf{v}}_\perp : \boldsymbol{\nabla}\widehat{\mathbf{n}} - \boldsymbol{\nabla}\cdot\widehat{\mathbf{n}}\right)\mu\frac{\partial f}{\partial \mu} \\
&+\left\{-v_\parallel \widehat{\mathbf{e}}_2 \cdot \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{e}}_1 + \frac{v_\parallel}{2}\widehat{\mathbf{n}} \cdot \left(\boldsymbol{\nabla}\times\widehat{\mathbf{n}}\right)\right. \\
&+\frac{1}{v_\perp}\left(v_\parallel^2 - v_\perp^2\right)\widehat{\boldsymbol{\rho}} \cdot \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{n}} \\
&-\frac{v_\parallel}{2}\left[\widehat{\mathbf{e}}_2 \cdot \left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{n}} + \widehat{\mathbf{e}}_1 \cdot \left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\cos\left(2\zeta\right) \\
&\left.-\frac{v_\parallel}{2}\left[-\widehat{\mathbf{e}}_1 \cdot \left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{n}} + \widehat{\mathbf{e}}_2 \cdot \left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\sin\left(2\zeta\right)\right\}\frac{\partial f}{\partial \zeta} \\
&+\frac{1}{v_\perp}\frac{e}{m}\boldsymbol{\nabla}\phi \cdot \widehat{\boldsymbol{\rho}}\frac{\partial f}{\partial \zeta} \\
&-\Omega\frac{\partial f}{\partial \zeta} \\
= \; & C
\end{aligned} \tag{104}$$

With Eq.(103) or Eq.(104) we further proceed to series expansion in terms of a small parameter.

2.4.2 Series expansion of the distribution function

We consider the folowing ordering (see also **Rutherford**)

$$\begin{array}{ccc} \frac{\partial}{\partial t} & \rightarrow & \text{order } 3 \\ \mathbf{v}\cdot\boldsymbol{\nabla} & \rightarrow & \text{order } 1 \\ C\left(f_0, f_0\right) & \rightarrow & \text{order } 1 \end{array} \tag{105}$$

in the small parameter

$$\varepsilon \equiv \frac{\rho}{L} = \frac{\text{Larmor radius}}{\text{a characteristic length}} \tag{106}$$

The distribution function is

$$f = f_0 + f_1 + f_2 + ... \tag{107}$$

and every term consists of a part that is averaged over gyration $\int_0^{2\pi} \frac{d\zeta}{2\pi}(...)$ and a part that depends on ζ.

Introducing this expansion in the Fokker-Planck equation and retaining successive orders, we obtain a chain of equations for f_i.

Order 0:

$$-\Omega\frac{\partial f_0}{\partial \zeta} = 0 \tag{108}$$

and this means that the zeroth-order distribution function does not depend on the gyroangle

$$\frac{\partial f_0}{\partial \zeta} = 0 \tag{109}$$

Order 1. We insert in the Eq.(103) the averaged function $\overline{f}$. The terms that contain derivatives at ζ, $\partial\overline{f}/\partial\zeta$, (the term with coefficient $B_1+B_2+B_3$) cancel

$$\begin{aligned} & v_{\parallel}\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\overline{f} \\ & +\mathbf{v}_{\perp}\cdot\left[\boldsymbol{\nabla}\overline{f} - \left(\mu\boldsymbol{\nabla}B + v_{\parallel}^2\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}} + \frac{e}{m}\boldsymbol{\nabla}\phi\right)\frac{\partial\overline{f}}{B\partial\mu}\right] \\ & -v_{\parallel}\mu\frac{\partial\overline{f}}{\partial\mu}\left[A\right] \\ & -\Omega\frac{\partial f_1}{\partial\zeta} \\ = \; & C\left(\overline{f},\overline{f}\right) \end{aligned} \tag{110}$$

Order 1 The equation in order 1 is averaged over the gyrophase ζ:

$$v_{\|} \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right) \overline{f} - \Omega \frac{1}{2\pi} \int_0^{2\pi} d\zeta \frac{\partial f}{\partial \zeta} = \overline{C}\left(f, f\right) \tag{111}$$

The first order distribution function f_1 contains a part that does not depend on the gyrophase $\overline{f}_1$ and a part that depend on it and is periodic on ζ, $\widetilde{f}_1$,

$$f_1 = \overline{f}_1 + \widetilde{f}_1 \tag{112}$$

The equation becomes

$$v_{\|} \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right) \overline{f} = C\left(\overline{f}, \overline{f}\right) \tag{113}$$

Which is an equation for the zeroth order distribution function

$$v_{\|} \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right) f_0 = C\left(f_0, f_0\right)$$

and whose solution is the Maxwellian function

$$f_0 = n \left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\frac{mv^2}{2T}\right) \tag{114}$$

where n is the density of particles of mass m and T is the temperature in units of electronvolts. We note that f_0 does not depend of the independent variable μ,

$$\frac{\partial f_0}{\partial \mu} = 0 \tag{115}$$

The function f_0 is the zero order component in $\overline{f}$. We return to the Fokker-Planck equation in order 1, and take into account the results already obtained for $\overline{f}$:

$$\mathbf{v}_{\perp} \cdot \boldsymbol{\nabla} \overline{f} - \Omega \frac{\partial}{\partial \zeta} \left(\overline{f}_1 + \widetilde{f}_1\right) = 0 \tag{116}$$

The equation

$$\frac{\partial \widetilde{f}_1}{\partial \zeta} = \frac{v_{\perp}}{\Omega} \left(\widehat{\mathbf{e}}_1 \cos\zeta + \widehat{\mathbf{e}}_2 \sin\zeta\right) \boldsymbol{\nabla} \overline{f} \tag{117}$$

is now integrated over ζ and we get the part of the distribution function in order 1 which depends on the gyrophase

$$\widetilde{f}_1 = \frac{v_{\perp}}{\Omega} \left(\widehat{\mathbf{e}}_1 \sin\zeta - \widehat{\mathbf{e}}_2 \cos\zeta\right) \boldsymbol{\nabla} \overline{f} \tag{118}$$

which conicides with Eq.(3.4) from **Hazeltine and Hinton 1976 (HH1976)** if our ζ is $\pi/2 - \zeta$ in their notation.

We have the relation

$$\begin{aligned}\boldsymbol{\rho} &= \widehat{\mathbf{n}}\times\mathbf{v}_{\perp}\frac{1}{\Omega_{ci}}\\ &= \frac{1}{\Omega_{ci}}\widehat{\mathbf{n}}\times v_{\perp}\left(\widehat{\mathbf{e}}_1\cos\zeta+\widehat{\mathbf{e}}_2\sin\zeta\right)\\ &= \frac{v_{\perp}}{\Omega_{ci}}\left(-\widehat{\mathbf{e}}_1\sin\zeta+\widehat{\mathbf{e}}_2\cos\zeta\right)\end{aligned}$$

Then we conculde from Eq.(118)

$$\widetilde{f_1}=-\boldsymbol{\rho}\cdot\boldsymbol{\nabla}\overline{f} \tag{119}$$

which is identical to Eq.(3.73) from **HH1976**.

It is of order 1 since

$$\frac{v_{\perp}}{\Omega}=\rho \tag{120}$$

The full expression for $\widetilde{f_1}$ will be obtained later, after we derive the equation for the averaged distribution function.

NOTE

In preparation for the drift wave theory (**Tang review 1978)** the equation is

$$\begin{aligned}&\mathbf{v}\cdot\boldsymbol{\nabla}F^{(0)}\\ &-\mathbf{v}\cdot\left(\mu\boldsymbol{\nabla}B+v_{\parallel}\boldsymbol{\nabla}\ \widehat{\mathbf{n}}\cdot\mathbf{v}_{\perp}\right)\frac{1}{B}\frac{\partial}{\partial\mu}F^{(0)}\\ &-\frac{e}{M}\boldsymbol{\nabla}\Phi_0\cdot\mathbf{v}_{\perp}\ \frac{1}{B}\frac{\partial}{\partial\mu}F^{(0)}\\ &-\Omega\frac{\partial}{\partial\zeta}F^{(1)}\\ =\ &C\left[F^{(0)},F^{(0)}\right]\end{aligned}$$

The second term in the second line is connected with what we have discussed above

$$\begin{aligned}v_{\perp k}v_{\parallel}\mathbf{v}\cdot\partial_k\widehat{\mathbf{n}} &= v_{\parallel}v_{\perp k}v_{\perp i}\partial_k\widehat{n}_i\\ &= v_{\parallel}\mathbf{v}_{\perp}\mathbf{v}_{\perp}:\boldsymbol{\nabla}\widehat{\mathbf{n}}\end{aligned}$$

and it is just a matter of notation of the operators.

The operator $\boldsymbol{\nabla}$ from **Tang** is acting on $\widehat{\mathbf{n}}$, but the contraction of the operator $\boldsymbol{\nabla}$ is with $\mathbf{v}_{\perp}$. The vectorial content of this term is still contained in $\widehat{\mathbf{n}}$ (afected by the operator) and this is scalarly multiplied with $\mathbf{v}$.

END

Order 2. The distribution function is

$$\begin{aligned} f &= f_0 \\ &\quad + \overline{f}_1 + \widetilde{f}_1 \\ &\quad + \overline{f}_2 + \widetilde{f}_2 \end{aligned} \tag{121}$$

where at this moment we know f_0 (Maxwell function) and $\widetilde{f}_1$. We insert this expansion in the Fokker-Planck equation where the ζ terms are all kept.

The higher order term in which we can keep f_2 is the derivative at the gyration angle ζ multiplying the gyrofrequency Ω

$$\begin{aligned} & v_{\parallel} \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \right) \left(\overline{f}_1 + \widetilde{f}_1 \right) \\ & + \mathbf{v}_{\perp} \cdot \left[\boldsymbol{\nabla} \left(\overline{f}_1 + \widetilde{f}_1 \right) - \left(\mu \boldsymbol{\nabla} B + v_{\parallel}^2 \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} + \frac{e}{m} \boldsymbol{\nabla} \phi \right) \frac{\partial}{B \partial \mu} \left(\overline{f}_1 + \widetilde{f}_1 \right) \right] \\ & - v_{\parallel} \left[A \right] \mu \frac{\partial}{\partial \mu} \left(\overline{f}_1 + \widetilde{f}_1 \right) \\ & + \left[B_1 + B_2 + B_3 \right] \frac{\partial}{\partial \zeta} \left(\overline{f}_1 + \widetilde{f}_1 \right) \\ & + \frac{1}{v_{\perp}} \frac{e}{m} \boldsymbol{\nabla} \phi \left(-\widehat{\mathbf{e}}_2 \cos \zeta + \widehat{\mathbf{e}}_1 \sin \zeta \right) \frac{\partial}{\partial \zeta} \left(\overline{f}_1 + \widetilde{f}_1 \right) \\ & - \Omega \frac{\partial f_2}{\partial \zeta} \\ = \; & 0 \end{aligned} \tag{122}$$

In this equation we make averaging over the gyroangle

$$\begin{aligned}
& v_{\parallel}\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\overline{f}_1 \\
& +\left\langle v_{\parallel}\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widetilde{f}_1\right\rangle \text{ (vanishes)} \\
& +\left\langle \mathbf{v}_{\perp}\cdot\boldsymbol{\nabla}\overline{f}_1\right\rangle \text{ (vanishes)} \\
& +\left\langle \mathbf{v}_{\perp}\cdot\boldsymbol{\nabla}\widetilde{f}_1\right\rangle \\
& -\left\langle \mathbf{v}_{\perp}\cdot\left(\mu\boldsymbol{\nabla}B+v_{\parallel}^2\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}+\frac{e}{m}\boldsymbol{\nabla}\phi\right)\frac{\partial\overline{f}_1}{B\partial\mu}\right\rangle \text{ (vanishes)} \\
& -\left\langle \mathbf{v}_{\perp}\cdot\left(\mu\boldsymbol{\nabla}B+v_{\parallel}^2\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}+\frac{e}{m}\boldsymbol{\nabla}\phi\right)\frac{\partial\widetilde{f}_1}{B\partial\mu}\right\rangle \\
& -\left\langle v_{\parallel}\left[A\right]\mu\frac{\partial\overline{f}_1}{\partial\mu}\right\rangle \text{ (vanishes)} \\
& -\left\langle v_{\parallel}\left[A\right]\mu\frac{\partial\widetilde{f}_1}{\partial\mu}\right\rangle \\
& +\left\langle \left[B_1\right]\frac{\partial\widetilde{f}_1}{\partial\zeta}\right\rangle \text{ (vanishes)} \\
& +\left\langle \left[B_2\right]\frac{\partial\widetilde{f}_1}{\partial\zeta}\right\rangle \\
& +\left\langle \left[B_3\right]\frac{\partial\widetilde{f}_1}{\partial\zeta}\right\rangle \\
& +\left\langle \frac{1}{v_{\perp}}\frac{e}{m}\boldsymbol{\nabla}\phi\left(-\widehat{\mathbf{e}}_2\cos\zeta+\widehat{\mathbf{e}}_1\sin\zeta\right)\frac{\partial\widetilde{f}_1}{\partial\zeta}\right\rangle \\
& -\left\langle \Omega\frac{\partial\widetilde{f}_2}{\partial\zeta}\right\rangle \text{ (vanishes)} \\
= & \left\langle C\left(f_1,f_0\right)\right\rangle
\end{aligned} \tag{123}$$

2.4.3 First nonvanishing term

This is unchanged by averaged

$$v_{\parallel}\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\overline{f}_1$$

2.4.4 Second nonvanishing term

Consider in the first term that remains after the gyrophase averaging

$$\left\langle \mathbf{v}_{\perp} \cdot \boldsymbol{\nabla} \widetilde{f}_1 \right\rangle \tag{124}$$

where we have

$$\widetilde{f}_1 = \left(\frac{\sqrt{2} m}{e} \right) \mu^{1/2} B^{-1/2} \left[\left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla} \overline{f} \right) \sin \zeta - \left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla} \overline{f} \right) \cos \zeta \right] \tag{125}$$

$$\begin{aligned} \partial_k \widetilde{f}_1 \;\; = \;\; & \left(\frac{\sqrt{2} m}{e} \right) \mu^{1/2} \left(-\frac{1}{2B} B^{-1/2} \right) \partial_k B \left[\left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla} \overline{f} \right) \sin \zeta - \left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla} \overline{f} \right) \cos \zeta \right] \\ & + \left(\frac{\sqrt{2} m}{e} \right) \mu^{1/2} B^{-1/2} \left[\left(\partial_k \widehat{e}_{1i} \partial_i \overline{f} \right) \sin \zeta - \left(\partial_k \widehat{e}_{2i} \partial_i \overline{f} \right) \cos \zeta \right] \\ & + \left(\frac{\sqrt{2} m}{e} \right) \mu^{1/2} B^{-1/2} \left[\left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla} \right) \left(\partial_k \overline{f} \right) \sin \zeta - \left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla} \right) \left(\partial_k \overline{f} \right) \cos \zeta \right] \end{aligned} \tag{126}$$

Then

$$\begin{aligned} \mathbf{v}_{\perp} \cdot \boldsymbol{\nabla} \widetilde{f}_1 \;\; = \;\; & v_{\perp k} \partial_k \widetilde{f}_1 \\ = \;\; & v_{\perp} \left(\widehat{e}_{1k} \cos \zeta + \widehat{e}_{2k} \sin \zeta \right) \\ & \times \left\{ -\frac{1}{2} \frac{\partial_k B}{B} \frac{v_{\perp}}{\Omega} \left[\left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla} \overline{f} \right) \sin \zeta - \left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla} \overline{f} \right) \cos \zeta \right] \right. \\ & + \frac{v_{\perp}}{\Omega} \left[\left(\partial_k \widehat{e}_{1i} \partial_i \overline{f} \right) \sin \zeta - \left(\partial_k \widehat{e}_{2i} \partial_i \overline{f} \right) \cos \zeta \right] \\ & \left. + \frac{v_{\perp}}{\Omega} \left[\left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla} \right) \left(\partial_k \overline{f} \right) \sin \zeta - \left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla} \right) \left(\partial_k \overline{f} \right) \cos \zeta \right] \right\} \end{aligned} \tag{127}$$

Averaging this product we obtain

$$
\begin{aligned}
&\left\langle \mathbf{v}_\perp \cdot \boldsymbol{\nabla} \widetilde{f}_1 \right\rangle \qquad (128)\\
= & -\frac{v_\perp^2}{2B}\frac{1}{\Omega}\left[\left(\widehat{e}_{2k}\partial_k B\right)\left(\widehat{e}_{1i}\partial_i \overline{f}\right)\frac{1}{2} - \left(\widehat{e}_{1k}\partial_k B\right)\left(\widehat{e}_{2i}\partial_i \overline{f}\right)\frac{1}{2}\right]\\
& + \frac{v_\perp^2}{2\Omega}\left[\widehat{e}_{2k}\left(\partial_k \widehat{e}_{1i}\right)\partial_i \overline{f} - \widehat{e}_{1k}\left(\partial_k \widehat{e}_{2i}\right)\partial_i \overline{f}\right]\\
& + \frac{v_\perp^2}{2\Omega}\left[-\widehat{e}_{1k}\left(\widehat{e}_{2i}\partial_i\right)\left(\partial_k \overline{f}\right) + \widehat{e}_{2k}\left(\widehat{e}_{1i}\partial_i\right)\left(\partial_k \overline{f}\right)\right]\\
= & \frac{1}{2}\frac{\mu}{\Omega}\left(\widehat{\mathbf{n}} \times \boldsymbol{\nabla} B\right)\cdot \boldsymbol{\nabla}\overline{f}\\
& + \frac{v_\perp^2}{2\Omega}\left\{\left[\left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{e}}_1\right]\cdot \boldsymbol{\nabla}\overline{f} - \left[\left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{e}}_2\right]\cdot \boldsymbol{\nabla}\overline{f}\right\}\\
& + \frac{v_\perp^2}{2\Omega}\left[-\widehat{e}_{1k}\widehat{e}_{2i} + \widehat{e}_{2k}\widehat{e}_{1i}\right]\left(\partial_i\partial_k \overline{f}\right)
\end{aligned}
$$

In the last line, the factor $[-\widehat{e}_{1k}\widehat{e}_{2i} + \widehat{e}_{2k}\widehat{e}_{1i}]$ is *antisymmetric*: changing $i \rightarrow k$ also changes the sign of the factor. The other factor $\partial_i\partial_k\overline{f}$ is symmetric in the indices (i,k). Therefore the contraction of the two terms is zero and the last line does not contribute to the expression.

$$
\begin{aligned}
&\left\langle \mathbf{v}_\perp \cdot \boldsymbol{\nabla} \widetilde{f}_1 \right\rangle \qquad (129)\\
= & \frac{1}{2}\frac{1}{\Omega}\left(\widehat{\mathbf{n}} \times \mu\boldsymbol{\nabla} B\right)\cdot \boldsymbol{\nabla}\overline{f}\\
& + \frac{v_\perp^2}{2\Omega}\left\{\left[\left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{e}}_1\right]\cdot \boldsymbol{\nabla}\overline{f} - \left[\left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{e}}_2\right]\cdot \boldsymbol{\nabla}\overline{f}\right\}
\end{aligned}
$$

We use Eq.(26) to obtain

$$
\begin{aligned}
\left\langle \mathbf{v}_\perp \cdot \boldsymbol{\nabla} \widetilde{f}_1 \right\rangle = & \frac{1}{2}\frac{1}{\Omega}\left(\widehat{\mathbf{n}} \times \mu\boldsymbol{\nabla} B\right)\cdot \boldsymbol{\nabla}\overline{f} \qquad (130)\\
& + \frac{v_\perp^2}{2\Omega}\left(\boldsymbol{\nabla}\times\widehat{\mathbf{n}}\right)\cdot \boldsymbol{\nabla}\overline{f}
\end{aligned}
$$

2.4.5 Third nonvanishing term

The next term whose average does not vanish identically is the fourth line in Eq.(123)

$$
\mathbf{u}_D \equiv \mu\boldsymbol{\nabla} B + v_\parallel^2\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}} + \frac{e}{m}\boldsymbol{\nabla}\phi
$$

$$
\begin{aligned}
& \left\langle \mathbf{v}_\perp \cdot \mathbf{u}_D \frac{\partial \widetilde{f}_1}{B\partial\mu} \right\rangle && (131)\\
= \;& \langle v_\perp \left[\left(\widehat{\mathbf{e}}_1 \cdot \mathbf{u}_D\right) \cos\zeta + \left(\widehat{\mathbf{e}}_2 \cdot \mathbf{u}_D\right) \sin\zeta \right] \\
& \times \frac{\partial}{B\partial\mu} \left\{ \frac{\sqrt{2}m}{e} \mu^{1/2} B^{-1/2} \left[\left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}\overline{f}\right) \sin\zeta - \left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla} f_0\right) \cos\zeta \right] \right\} \Big\rangle \\
= \;& \frac{1}{2\Omega} \left[\left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}\overline{f}\right) \left(\widehat{\mathbf{e}}_2 \cdot \mathbf{u}_D\right) - \left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}\overline{f}\right) \left(\widehat{\mathbf{e}}_1 \cdot \mathbf{u}_D\right) \right]
\end{aligned}
$$

Then Eq.(131) can be written, including also the sign as in Eq.(**??**)

$$
\begin{aligned}
-\left\langle \mathbf{v}_\perp \cdot \mathbf{u}_D \frac{\partial \widetilde{f}_1}{B\partial\mu} \right\rangle &= -\frac{1}{2\Omega} \left[\left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}\overline{f}\right) \left(\widehat{\mathbf{e}}_2 \cdot \mathbf{u}_D\right) - \left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}\overline{f}\right) \left(\widehat{\mathbf{e}}_1 \cdot \mathbf{u}_D\right) \right] \\
&= -\frac{1}{2\Omega} \widehat{\mathbf{n}} \cdot \left(\boldsymbol{\nabla}\overline{f} \times \mathbf{u}_D \right) \\
&= \frac{1}{2\Omega} \left(\widehat{\mathbf{n}} \times \mathbf{u}_D \right) \cdot \boldsymbol{\nabla}\overline{f}
\end{aligned}
$$

2.4.6 Fourth nonvanishing term

Consider now the sixth term in Eq.(123)

$$
v_\parallel \left[A\right] \mu \frac{\partial \widetilde{f}_1}{\partial\mu} \tag{132}
$$

where

$$
\begin{aligned}
\left[A\right] \equiv \; & \left\{ \left[\widehat{\mathbf{e}}_1 \cdot \left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}\right) \widehat{\mathbf{n}} - \widehat{\mathbf{e}}_2 \cdot \left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}\right) \widehat{\mathbf{n}} \right] \cos 2\zeta \right. && (133)\\
& \left. + \left[\widehat{\mathbf{e}}_1 \cdot \left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}\right) \widehat{\mathbf{n}} + \widehat{\mathbf{e}}_2 \cdot \left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}\right) \widehat{\mathbf{n}} \right] \sin 2\zeta \right\}
\end{aligned}
$$

and

$$
\frac{\partial \widetilde{f}_1}{\partial\mu} = \left(\frac{\sqrt{2}m}{e} \right) \frac{1}{B} \frac{1}{2} \mu^{-1/2} B^{-1/2} \left[\left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}\overline{f}\right) \sin\zeta - \left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}\overline{f}\right) \cos\zeta \right] \tag{134}
$$

By multiplication of the trigonometric function of arguments ζ and 2ζ the result will only contain trigonometric functions of odd multiples of ζ and this gives zero at averaging.

$$
\left\langle v_\parallel \left[A\right] \mu \frac{\partial \widetilde{f}_1}{\partial\mu} \right\rangle = 0 \tag{135}
$$

2.4.7 Fifth nonvanishing term

Further

$$[B_2] \frac{\partial \widetilde{f}_1}{\partial \zeta} \tag{136}$$

contains

$$[B_2] \equiv \left[\frac{v_\parallel^2}{v_\perp} (\widehat{\mathbf{n}} \cdot \mathbf{\nabla}) \widehat{\mathbf{n}} + v_\perp (\widehat{\mathbf{e}}_1 \cdot \mathbf{\nabla}) \widehat{\mathbf{e}}_1 + v_\perp (\widehat{\mathbf{e}}_2 \cdot \mathbf{\nabla}) \widehat{\mathbf{e}}_2 \right] \cdot (-\widehat{\mathbf{e}}_2 \cos\zeta + \widehat{\mathbf{e}}_1 \sin\zeta) \tag{137}$$

For later use we introduce yet another notation

$$\mathbf{u} \equiv \frac{v_\parallel^2}{v_\perp} (\widehat{\mathbf{n}} \cdot \mathbf{\nabla}) \widehat{\mathbf{n}} + v_\perp (\widehat{\mathbf{e}}_1 \cdot \mathbf{\nabla}) \widehat{\mathbf{e}}_1 + v_\perp (\widehat{\mathbf{e}}_2 \cdot \mathbf{\nabla}) \widehat{\mathbf{e}}_2 \tag{138}$$

The expression is

$$\begin{aligned} & [B_2] \frac{\partial \widetilde{f}_1}{\partial \zeta} \qquad (139) \\ = \; & [-(\widehat{\mathbf{e}}_2 \cdot \mathbf{u}) \cos\zeta + (\widehat{\mathbf{e}}_1 \cdot \mathbf{u}) \sin\zeta] \\ & \times \frac{v_\perp}{\Omega} \left[\left(\widehat{\mathbf{e}}_1 \cdot \mathbf{\nabla}\overline{f}\right) \cos\zeta + \left(\widehat{\mathbf{e}}_2 \cdot \mathbf{\nabla}\overline{f}\right) \sin\zeta \right] \end{aligned}$$

and after averaging

$$\begin{aligned} & \left\langle [B_2] \frac{\partial \widetilde{f}_1}{\partial \zeta} \right\rangle \qquad (140) \\ = \; & \frac{1}{2}\rho \left[(\widehat{\mathbf{e}}_1 \cdot \mathbf{u}) \left(\widehat{\mathbf{e}}_2 \cdot \mathbf{\nabla}\overline{f}\right) - (\widehat{\mathbf{e}}_2 \cdot \mathbf{u}) \left(\widehat{\mathbf{e}}_1 \cdot \mathbf{\nabla}\overline{f}\right) \right] \end{aligned}$$

From the general formula Eq.(30) we obtain for Eq.(140)

$$\begin{aligned} & \frac{1}{2}\rho \left[(\widehat{\mathbf{e}}_1 \cdot \mathbf{u}) \left(\widehat{\mathbf{e}}_2 \cdot \mathbf{\nabla}\overline{f}\right) - (\widehat{\mathbf{e}}_2 \cdot \mathbf{u}) \left(\widehat{\mathbf{e}}_1 \cdot \mathbf{\nabla}\overline{f}\right) \right] \qquad (141) \\ = \; & \frac{1}{2}\rho \widehat{\mathbf{n}} \cdot \left(\mathbf{u} \times \mathbf{\nabla}\overline{f}\right) \end{aligned}$$

and the final form is

$$\left\langle [B_2] \frac{\partial \widetilde{f}_1}{\partial \zeta} \right\rangle = \frac{1}{2}\rho (\widehat{\mathbf{n}} \times \mathbf{u}) \cdot \mathbf{\nabla}\overline{f} \tag{142}$$

This is the tenth line from Eq.(123)

2.4.8 Sixth nonvanishing term

The next term (the 11^{th} line from Eq.(123)) contains the trigonometric functions of double argument

$$[B_3] \frac{\partial \widetilde{f}_1}{\partial \zeta} \tag{143}$$

where

$$\begin{aligned} [B_3] &\equiv -\frac{v_\parallel}{2} \left[\widehat{\mathbf{e}}_2 \cdot (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} + \widehat{\mathbf{e}}_1 \cdot (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} \right] \cos(2\zeta) \\ &\quad -\frac{v_\parallel}{2} \left[-\widehat{\mathbf{e}}_1 \cdot (\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} + \widehat{\mathbf{e}}_2 \cdot (\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} \right] \sin(2\zeta) \end{aligned} \tag{144}$$

and

$$\frac{\partial \widetilde{f}_1}{\partial \zeta} = \frac{v_\perp}{\Omega} \left[\left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla} \overline{f} \right) \cos \zeta + \left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla} \overline{f} \right) \sin \zeta \right] \tag{145}$$

Again, the product of trigonometric functions will produce trigonometric functions of arguments ζ and 3ζ whose arevrage is zero

$$\left\langle [B_3] \frac{\partial \widetilde{f}_1}{\partial \zeta} \right\rangle = 0 \tag{146}$$

There is no contribution from the line 11.

2.4.9 The seventh nonvanishing term

The next to last term in Eq.(123) is

$$\begin{aligned} &\left\langle \frac{1}{v_\perp} \frac{e}{m} \boldsymbol{\nabla} \phi \left(-\widehat{\mathbf{e}}_2 \cos \zeta + \widehat{\mathbf{e}}_1 \sin \zeta \right) \frac{\partial \widetilde{f}_1}{\partial \zeta} \right\rangle \\ &= \frac{1}{v_\perp} \frac{e}{m} \left[-\left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla} \phi \right) \cos \zeta + \left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla} \phi \right) \sin \zeta \right] \\ &\quad \times \frac{v_\perp}{\Omega} \left[\left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla} \overline{f} \right) \cos \zeta + \left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla} \overline{f} \right) \sin \zeta \right] \\ &= \frac{e}{m} \frac{1}{\Omega} \frac{1}{2} \left[-\left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla} \phi \right) \left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla} \overline{f} \right) + \left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla} \phi \right) \left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla} \overline{f} \right) \right] \end{aligned} \tag{147}$$

Using Eq.(30) we have

$$\begin{aligned} &\left\langle \frac{1}{v_\perp} \frac{e}{m} \boldsymbol{\nabla} \phi \left(-\widehat{\mathbf{e}}_2 \cos \zeta + \widehat{\mathbf{e}}_1 \sin \zeta \right) \frac{\partial \widetilde{f}_1}{\partial \zeta} \right\rangle \\ &= \frac{e}{m} \frac{1}{\Omega} \frac{1}{2} \widehat{\mathbf{n}} \cdot \left(\boldsymbol{\nabla} \phi \times \boldsymbol{\nabla} \overline{f} \right) \\ &= \frac{1}{\Omega} \frac{1}{2} \left(\widehat{\mathbf{n}} \times \frac{e}{m} \boldsymbol{\nabla} \phi \right) \cdot \boldsymbol{\nabla} \overline{f} \end{aligned} \tag{148}$$

2.4.10 Result of the averaging over the gyrophase in the first order equation

Now collecting the terms from the Eq.(123):

$$\begin{aligned}
& v_\parallel \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \overline{f}_1 \\
& + \frac{1}{2} \frac{1}{\Omega} \left(\widehat{\mathbf{n}} \times \mu \boldsymbol{\nabla} B \right) \cdot \boldsymbol{\nabla} f_0 + \frac{v_\perp^2}{2\Omega} \left(\boldsymbol{\nabla} \times \widehat{\mathbf{n}} \right) \cdot \boldsymbol{\nabla} \overline{f}_1 \\
& + \frac{1}{2\Omega} \left(\widehat{\mathbf{n}} \times \mathbf{u}_D \right) \cdot \boldsymbol{\nabla} \overline{f}_1 \\
& + \frac{1}{2} \rho \left[\widehat{\mathbf{n}} \times \left(\frac{v_\parallel^2}{v_\perp} \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} + v_\perp \left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{e}}_1 + v_\perp \left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{e}}_2 \right) \right] \cdot \boldsymbol{\nabla} \overline{f}_1 \\
& + \frac{1}{\Omega} \frac{1}{2} \left(\widehat{\mathbf{n}} \times \frac{e}{m} \boldsymbol{\nabla} \phi \right) \cdot \boldsymbol{\nabla} \overline{f}_1 \\
= \; & C \left[f_0, f_1 \right]
\end{aligned} \tag{149}$$

We collect together the first term from the second line, the first term in the fourth line and the term of the fifth line

$$\begin{aligned}
& \frac{1}{2} \frac{1}{\Omega} \left(\widehat{\mathbf{n}} \times \mu \boldsymbol{\nabla} B \right) \cdot \boldsymbol{\nabla} f_0 \\
& + \frac{1}{2} \rho \left(\widehat{\mathbf{n}} \times \frac{v_\parallel^2}{v_\perp} \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} \right) \cdot \boldsymbol{\nabla} f_0 \\
& + \frac{1}{\Omega} \frac{1}{2} \left(\widehat{\mathbf{n}} \times \frac{e}{m} \boldsymbol{\nabla} \phi \right) \cdot \boldsymbol{\nabla} f_0 \\
= \; & \frac{1}{2\Omega} \widehat{\mathbf{n}} \times \left(\mu \boldsymbol{\nabla} B + v_\parallel^2 \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} + \frac{e}{m} \boldsymbol{\nabla} \phi \right) \cdot \boldsymbol{\nabla} f_0 \\
= \; & \frac{1}{2} \mathbf{v}_D \cdot \boldsymbol{\nabla} f_0
\end{aligned} \tag{150}$$

Then it remains

$$\begin{aligned}
& v_\parallel \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \overline{f}_1 \\
& + \frac{v_\perp^2}{2\Omega} \left(\boldsymbol{\nabla} \times \widehat{\mathbf{n}} \right) \cdot \boldsymbol{\nabla} f_0 \\
& + \frac{1}{2\Omega} \left(\widehat{\mathbf{n}} \times \mathbf{u}_D \right) \cdot \boldsymbol{\nabla} f_0 + \frac{1}{2} \mathbf{v}_D \cdot \boldsymbol{\nabla} f_0 \\
& + \frac{1}{2} \rho \left\{ \widehat{\mathbf{n}} \times \left[v_\perp \left(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{e}}_1 + v_\perp \left(\widehat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{e}}_2 \right] \right\} \cdot \boldsymbol{\nabla} f_0 \\
= \; & C \left[f_0, f_1 \right]
\end{aligned} \tag{151}$$

We can still find another expression in the fourth line (coming from the term with B_2). For this we use the Eq.(42).

Then the fourth line becomes

$$\begin{aligned} & \frac{1}{2}\rho\left\{\widehat{\mathbf{n}}\times\left[v_{\perp}\left(\widehat{\mathbf{e}}_1\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{e}}_1+v_{\perp}\left(\widehat{\mathbf{e}}_2\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{e}}_2\right]\right\}\cdot\boldsymbol{\nabla}f_0 \\ = & \frac{v_{\perp}^2}{2\Omega}\left\{-\boldsymbol{\nabla}\times\widehat{\mathbf{n}}+\widehat{\mathbf{n}}\left[\widehat{\mathbf{n}}\cdot\left(\boldsymbol{\nabla}\times\widehat{\mathbf{n}}\right)\right]\right\}\cdot\boldsymbol{\nabla}f_0 \end{aligned}$$

We note that the first term in the curly bracket cancels the term from the second line in Eq.(151) and this equation becomes

$$\begin{aligned} & v_{\parallel}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\overline{f}_1 \\ & +\frac{v_{\perp}^2}{2\Omega}\left(\boldsymbol{\nabla}\times\widehat{\mathbf{n}}\right)\cdot\boldsymbol{\nabla}f_0 \\ & +\frac{1}{2\Omega}\left(\widehat{\mathbf{n}}\times\mathbf{v}_D\right)\cdot\boldsymbol{\nabla}f_0+\frac{1}{2}\mathbf{v}_D\cdot\boldsymbol{\nabla}f_0 \\ & +\frac{v_{\perp}^2}{2\Omega}\left\{-\boldsymbol{\nabla}\times\widehat{\mathbf{n}}+\widehat{\mathbf{n}}\left[\widehat{\mathbf{n}}\cdot\left(\boldsymbol{\nabla}\times\widehat{\mathbf{n}}\right)\right]\right\}\cdot\boldsymbol{\nabla}f_0 \\ = & C\left[f_0,f_1\right] \end{aligned}$$

$$v_{\parallel}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\overline{f}_1+\frac{1}{2\Omega}\left(\widehat{\mathbf{n}}\times\mathbf{u}_D\right)\cdot\boldsymbol{\nabla}f_0+\frac{1}{2}\mathbf{v}_D\cdot\boldsymbol{\nabla}f_0+\frac{v_{\perp}^2}{2\Omega}\widehat{\mathbf{n}}\left[\widehat{\mathbf{n}}\cdot\left(\boldsymbol{\nabla}\times\widehat{\mathbf{n}}\right)\right]\cdot\boldsymbol{\nabla}f_0=C\left[f_0,f_1\right]$$

Since

$$\frac{1}{2\Omega}\left(\widehat{\mathbf{n}}\times\mathbf{u}_D\right)=\mathbf{v}_D$$

we can write the final form

$$v_{\parallel}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\overline{f}_1+\frac{v_{\perp}^2}{2\Omega}\left[\widehat{\mathbf{n}}\cdot\left(\boldsymbol{\nabla}\times\widehat{\mathbf{n}}\right)\right]\ \widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}f_0+\mathbf{v}_D\cdot\boldsymbol{\nabla}f_0=C\left[f_0,f_1\right] \tag{152}$$

This is the equation of the averaged distribution function.

2.4.11 An alternative expression for the drift velocity

We have identified the full velocity of the particle after gyrophase averaging, as

$$\mathbf{v}=\mathbf{v}_{\parallel}+\underbrace{\frac{v_{\perp}^2}{2\Omega}\left[\widehat{\mathbf{n}}\cdot\left(\boldsymbol{\nabla}\times\widehat{\mathbf{n}}\right)\right]\widehat{\mathbf{n}}+\mathbf{v}_D}$$

The *drift* velocity is

$$\mathbf{v}_{drift}\equiv\frac{v_{\perp}^2}{2\Omega}\left[\widehat{\mathbf{n}}\cdot\left(\boldsymbol{\nabla}\times\widehat{\mathbf{n}}\right)\right]\widehat{\mathbf{n}}+\mathbf{v}_D$$

and it can shown that it has the approximative expression

$$\mathbf{v}_{drift}=-v_{\parallel}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\left(\frac{v_{\parallel}}{\Omega}\right)+O\left(\delta^2\right)$$

where δ is the ratio of the Larmor radius on the geometric length, $a =$ minor radius of the tokamak.

To show this we calculate

$$\begin{aligned}\nabla\left(\frac{v_\parallel}{\Omega}\right) &= \frac{m}{e}\nabla\left(\frac{v_\parallel}{B}\right)\\ &= \frac{1}{\Omega}\nabla v_\parallel + v_\parallel\left(-\frac{1}{\Omega}\right)\left(\frac{\nabla B}{B}\right)\end{aligned}$$

and

$$\begin{aligned}\nabla v_\parallel &= \frac{\sqrt{2}}{2\left(\epsilon-\mu B - e\phi/m\right)^{1/2}}\left(-\mu\nabla B - \frac{e}{m}\nabla\phi\right)\\ &= -\frac{1}{v_\parallel}\left(\mu\nabla B + \frac{e}{m}\nabla\phi\right)\end{aligned}$$

Then

$$\begin{aligned}-v_\parallel\widehat{\mathbf{n}}\times\nabla\left(\frac{v_\parallel}{\Omega}\right) &= -v_\parallel\widehat{\mathbf{n}}\times\left[-\frac{1}{v_\parallel\Omega}\left(\mu\nabla B + \frac{e}{m}\nabla\phi\right) - \frac{v_\parallel}{\Omega}\frac{\nabla B}{B}\right]\\ &= \frac{1}{\Omega}\widehat{\mathbf{n}}\times\left(\mu\nabla B + v_\parallel^2\left(\widehat{\mathbf{n}}\cdot\nabla\right)\widehat{\mathbf{n}} + \frac{e}{m}\nabla\phi\right)\\ &= \mathbf{v}_D\end{aligned}$$

So we have proved that

$$\begin{aligned}\mathbf{v}_D &= \frac{1}{\Omega}\widehat{\mathbf{n}}\times\left(\mu\nabla B + v_\parallel^2\left(\widehat{\mathbf{n}}\cdot\nabla\right)\widehat{\mathbf{n}} + \frac{e}{m}\nabla\phi\right)\\ &\approx -v_\parallel\widehat{\mathbf{n}}\times\nabla\left(\frac{v_\parallel}{\Omega}\right)\end{aligned}$$

to an approximation which is δ^2. This is also useful in the drift-kinetic theory of plasma rotation due to the alpha particles or due to NBI (**Rosenbluth Hinton**).

2.4.12 The full expression for $\widetilde{f}_1$

The complete expression for $\widetilde{f}_1$ is obtained from Eq.(110) taking into account Eq.(152). However it is easier to start from Eq.(103) with an expanded form of the symbol $[A]$. This will make it easier to integrate over the gyration angle ζ.

The highest order term is the term containing Ω. In all the others we replace f by $\overline{f}$.

$$\begin{aligned}
& \mathbf{v}_\perp \cdot \boldsymbol{\nabla}\overline{f} - \frac{v_\perp^2}{2\Omega}\left[\widehat{\mathbf{n}}\cdot(\boldsymbol{\nabla}\times\widehat{\mathbf{n}})\right]\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\overline{f} - \mathbf{v}_D\cdot\boldsymbol{\nabla}\overline{f} \\
& -\mathbf{v}_\perp\cdot\left(\mu\boldsymbol{\nabla}B + v_\parallel^2\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}} + \frac{e}{m}\boldsymbol{\nabla}\phi\right)\frac{\partial\overline{f}}{B\partial\mu} \\
& -v_\parallel\mu\frac{\partial\overline{f}}{\partial\mu}\left[A\right] \\
= \; & \Omega\frac{\partial\widetilde{f}_1}{\partial\zeta}
\end{aligned} \tag{153}$$

We have to calculate the integral over ζ, and we have

$$\begin{aligned}
\widetilde{f}_1 \; = \; & -\boldsymbol{\rho}\cdot\left[\boldsymbol{\nabla}\overline{f} - \left(\mu\boldsymbol{\nabla}B + v_\parallel^2\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}} + \frac{e}{m}\boldsymbol{\nabla}\phi\right)\frac{\partial\overline{f}}{B\partial\mu}\right] \\
& -\frac{1}{\Omega}v_\parallel\mu\frac{\partial\overline{f}}{\partial\mu}\int d\zeta\left\{\left[\widehat{\mathbf{e}}_1\cdot\left(\widehat{\mathbf{e}}_1\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}} - \widehat{\mathbf{e}}_2\cdot\left(\widehat{\mathbf{e}}_2\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\cos 2\zeta\right. \\
& \left. + \left[\widehat{\mathbf{e}}_1\cdot\left(\widehat{\mathbf{e}}_2\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}} + \widehat{\mathbf{e}}_2\cdot\left(\widehat{\mathbf{e}}_1\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\sin 2\zeta\right\}
\end{aligned} \tag{154}$$

The integration gives

$$\begin{aligned}
& \left[\widehat{\mathbf{e}}_1\cdot\left(\widehat{\mathbf{e}}_1\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}} - \widehat{\mathbf{e}}_2\cdot\left(\widehat{\mathbf{e}}_2\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\frac{\sin\left(2\zeta\right)}{2} \\
& - \left[\widehat{\mathbf{e}}_1\cdot\left(\widehat{\mathbf{e}}_2\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}} + \widehat{\mathbf{e}}_2\cdot\left(\widehat{\mathbf{e}}_1\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\frac{\cos\left(2\zeta\right)}{2} \\
= \; & \left[\widehat{\mathbf{e}}_1\cdot\left(\widehat{\mathbf{e}}_1\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}} - \widehat{\mathbf{e}}_2\cdot\left(\widehat{\mathbf{e}}_2\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\sin\zeta\cos\zeta \\
& - \left[\widehat{\mathbf{e}}_1\cdot\left(\widehat{\mathbf{e}}_2\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}} + \widehat{\mathbf{e}}_2\cdot\left(\widehat{\mathbf{e}}_1\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\frac{\cos^2\zeta - \sin^2\zeta}{2} \\
\equiv \; & Y
\end{aligned} \tag{155}$$

We have

$$\begin{aligned}
2Y \; = \; & 2\widehat{\mathbf{e}}_1\sin\zeta\cdot\left[\left(\widehat{\mathbf{e}}_1\cos\zeta\right)\cdot\boldsymbol{\nabla}\right]\widehat{\mathbf{n}} \\
& -2\widehat{\mathbf{e}}_2\sin\zeta\left[\left(\widehat{\mathbf{e}}_2\cos\zeta\right)\cdot\boldsymbol{\nabla}\right]\widehat{\mathbf{n}} \\
& -\widehat{\mathbf{e}}_1\cos\zeta\left[\left(\widehat{\mathbf{e}}_2\cos\zeta\right)\cdot\boldsymbol{\nabla}\right]\widehat{\mathbf{n}} \\
& -\widehat{\mathbf{e}}_2\cos\zeta\left[\left(\widehat{\mathbf{e}}_1\cos\zeta\right)\cdot\boldsymbol{\nabla}\right]\widehat{\mathbf{n}} \\
& +\widehat{\mathbf{e}}_1\sin\zeta\left[\left(\widehat{\mathbf{e}}_2\sin\zeta\right)\cdot\boldsymbol{\nabla}\right]\widehat{\mathbf{n}} \\
& +\widehat{\mathbf{e}}_2\sin\zeta\left[\left(\widehat{\mathbf{e}}_1\sin\zeta\right)\cdot\boldsymbol{\nabla}\right]\widehat{\mathbf{n}}
\end{aligned} \tag{156}$$

After grouping and factorization we arrive at the expression

$$\begin{aligned}
2Y &= W = \\
&\widehat{\mathbf{e}}_1 \sin\zeta \left[\left(\widehat{\mathbf{e}}_1 \cos\zeta + \widehat{\mathbf{e}}_2 \sin\zeta \right) \cdot \boldsymbol{\nabla} \right] \widehat{\mathbf{n}} + \\
&\widehat{\mathbf{e}}_1 \cos\zeta \left[\left(\widehat{\mathbf{e}}_1 \sin\zeta - \widehat{\mathbf{e}}_2 \cos\zeta \right) \cdot \boldsymbol{\nabla} \right] \widehat{\mathbf{n}} + \\
&\widehat{\mathbf{e}}_2 \sin\zeta \left[\left(-\widehat{\mathbf{e}}_2 \cos\zeta + \widehat{\mathbf{e}}_1 \sin\zeta \right) \cdot \boldsymbol{\nabla} \right] \widehat{\mathbf{n}} - \\
&\widehat{\mathbf{e}}_2 \cos\zeta \left[\left(\widehat{\mathbf{e}}_2 \sin\zeta + \widehat{\mathbf{e}}_1 \cos\zeta \right) \cdot \boldsymbol{\nabla} \right] \widehat{\mathbf{n}}
\end{aligned} \tag{157}$$

and this has been shown in Eq.(51) to be

$$Y = -\widehat{\mathbf{v}}_\perp \cdot \left(\widehat{\boldsymbol{\rho}} \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} + \frac{1}{2} \widehat{\mathbf{n}} \cdot \left(\boldsymbol{\nabla} \times \widehat{\mathbf{n}} \right) \tag{158}$$

Using this result we obtain

$$\begin{aligned}
\widetilde{f}_1 &= -\boldsymbol{\rho} \cdot \left[\boldsymbol{\nabla} \overline{f} - \left(\mu \boldsymbol{\nabla} B + v_\parallel^2 \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} + \frac{e}{m} \boldsymbol{\nabla} \phi \right) \frac{\partial \overline{f}}{B \partial \mu} \right] \\
&\quad - \frac{1}{\Omega} v_\parallel \mu \frac{\partial \overline{f}}{\partial \mu} \left[-\widehat{\mathbf{v}}_\perp \cdot \left(\widehat{\boldsymbol{\rho}} \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} + \frac{1}{2} \widehat{\mathbf{n}} \cdot \left(\boldsymbol{\nabla} \times \widehat{\mathbf{n}} \right) \right]
\end{aligned} \tag{159}$$

or

$$\begin{aligned}
\widetilde{f}_1 &= -\boldsymbol{\rho} \cdot \boldsymbol{\nabla} \overline{f} + \boldsymbol{\rho} \cdot \left(\mu \boldsymbol{\nabla} B + v_\parallel^2 \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} + \frac{e}{m} \boldsymbol{\nabla} \phi \right) \frac{\partial \overline{f}}{B \partial \mu} \\
&\quad + \frac{1}{\Omega} v_\parallel \mu \frac{\partial \overline{f}}{\partial \mu} \left[\widehat{\boldsymbol{\rho}} \widehat{\mathbf{v}}_\perp : \boldsymbol{\nabla} \widehat{\mathbf{n}} - \frac{1}{2} \widehat{\mathbf{n}} \cdot \left(\boldsymbol{\nabla} \times \widehat{\mathbf{n}} \right) \right]
\end{aligned} \tag{160}$$

After the consideration of the time-dependent terms, this formula becomes

$$\begin{aligned}
\widetilde{f}_1 &= -\boldsymbol{\rho} \cdot \boldsymbol{\nabla} \overline{f} + \boldsymbol{\rho} \cdot \left(\mu \boldsymbol{\nabla} B + v_\parallel^2 \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} + \frac{e}{m} \boldsymbol{\nabla} \phi \right) \frac{\partial \overline{f}}{B \partial \mu} \\
&\quad + \boldsymbol{\rho} \cdot \left[\frac{\partial \overline{f}}{B \partial \mu} \left(\mu \boldsymbol{\nabla} B + v_\parallel^2 \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} + \frac{e}{m} \boldsymbol{\nabla} \phi - v_\parallel \frac{\partial \widehat{\mathbf{n}}}{\partial t} \right) \right. \\
&\quad \left. + \frac{\partial \overline{f}}{\partial \epsilon} \frac{e}{m} \frac{\partial \mathbf{A}}{\partial t} \right] \\
&\quad + \frac{1}{\Omega} v_\parallel \mu \frac{\partial \overline{f}}{\partial \mu} \left[\widehat{\boldsymbol{\rho}} \widehat{\mathbf{v}}_\perp : \boldsymbol{\nabla} \widehat{\mathbf{n}} - \frac{1}{2} \widehat{\mathbf{n}} \cdot \left(\boldsymbol{\nabla} \times \widehat{\mathbf{n}} \right) \right]
\end{aligned} \tag{161}$$

NOTE

For comparison with Eq.(161) we mention the formula from **Catto Electric field tokamak** where we find (in the notations of **Catto**)

$$\begin{aligned}
\widetilde{f} &= \mathbf{v}\cdot\left\{\frac{1}{\Omega}\widehat{\mathbf{n}}\times\left(\boldsymbol{\nabla}|_{\epsilon,\mu,\zeta}\overline{f}\right)\right. \\
&\quad -\mathbf{v}_E\left(\frac{\partial\overline{f}}{\partial\epsilon}+\frac{1}{B}\frac{\partial\overline{f}}{\partial\mu}\right) \\
&\quad \left.-\mathbf{v}_M\frac{1}{B}\frac{\partial\overline{f}}{\partial\mu}\right\} \\
-\left(\mathbf{v}_\perp\ \mathbf{v}\times\widehat{\mathbf{n}}+\mathbf{v}\times\widehat{\mathbf{n}}\ \mathbf{v}_\perp\right) &: \boldsymbol{\nabla}\widehat{\mathbf{n}}\frac{v_\parallel}{4\Omega B}\frac{\partial\overline{f}}{\partial\mu}
\end{aligned}$$

where

$$\mathbf{v}_M\equiv\frac{1}{\Omega}\widehat{\mathbf{n}}\times\left(\mu\boldsymbol{\nabla}B+v_\parallel^2\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}+v_\parallel\frac{\partial\widehat{\mathbf{n}}}{\partial t}\right)$$

And, the **radial flux of the toroidal angular momentum** is defined as

$$\Pi=\left\langle\int d^3v\widetilde{f}\ \left(R\boldsymbol{\nabla}\varphi\cdot Rm_i\mathbf{v}\right)\ \mathbf{v}\cdot\boldsymbol{\nabla}\psi\right\rangle_\theta$$

after inserting here the formula of the ζ-dependent distribution function, $\widetilde{f}$ from the above formula, we get

$$\Pi\rightarrow\left\langle m_i\frac{I}{B}\int d^3vv_\parallel\left[\left(\mathbf{v}_E+\mathbf{v}_M\right)\cdot\boldsymbol{\nabla}\psi\right]\overline{f}\right\rangle_\theta$$

where I is introduced by the definition of $\mathbf{B}$,

$$\mathbf{B}=I\boldsymbol{\nabla}\varphi+\boldsymbol{\nabla}\zeta\times\boldsymbol{\nabla}\psi$$

(then $I\simeq RB_\varphi$)

2.5 Alternative derivation of the change of variables (Catto Tsang*)*

This is the approach developed by *Catto and Tsang*.

We have

$$\begin{aligned}
\frac{d\mu}{dt} &= -\frac{\mu}{B}\frac{dB}{dt}-v_\parallel\widehat{\mathbf{n}}\cdot\frac{d\widehat{\mathbf{n}}}{dt}+\frac{e}{mB}\mathbf{v}_\perp\cdot\mathbf{E} \\
\frac{d\epsilon}{dt} &= \frac{e}{m}\left(\frac{d\Phi}{dt}+\mathbf{v}\cdot\mathbf{E}\right) \\
\frac{d\zeta}{dt} &= \Omega_c+\widehat{\mathbf{e}}_3\cdot\frac{d\widehat{\mathbf{e}}_2}{dt}+\frac{\widehat{\mathbf{n}}\times\mathbf{v}_\perp}{v_\perp^2}\cdot\left(v_\parallel\frac{d\widehat{\mathbf{n}}}{dt}-\frac{e}{m}\mathbf{E}\right)
\end{aligned}$$

where the time derivative is convective

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla}$$

This is rather strange since we do not have to treat *fluid* quantities but *particle* quantities. However, as shown in **Wang Burrell** it is question of variation of the quantities along the orbit of particles.

The following operator is defined

$$\begin{aligned} \delta^* f \quad &\equiv \quad -\frac{1}{\Omega_c} \left\{ \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mu} \frac{d\mu}{dt} + \frac{\partial f}{\partial \epsilon} \frac{d\epsilon}{dt} - C(f,f) \right. \\ &\qquad \left. + \frac{\partial f}{\partial \zeta} \left(\frac{\partial \zeta}{\partial t} - \Omega_c \right) \right\} \end{aligned}$$

The Vlasov equation is written

$$\frac{\partial f}{\partial \zeta} = \delta^* f$$

3 Formulas for versors, etc.

From *impurities.tex*, **Ware Diamond**

From the divergence of the *electric velocity.*

The geometrical part is

$$\boldsymbol{\nabla} \times \left(\frac{\widehat{\mathbf{n}}}{B} \right) = \widehat{\mathbf{n}} \times \boldsymbol{\nabla} \ln B + \widehat{\mathbf{n}} \times (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \, \widehat{\mathbf{n}}$$

This is only geometry. The two terms need coefficients to become drift components

$$\begin{aligned} \widehat{\mathbf{n}} \times \boldsymbol{\nabla} \ln B \quad &\rightarrow \quad \widehat{\mathbf{n}} \times \mu \boldsymbol{\nabla} \ln B \\ \widehat{\mathbf{n}} \times (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \, \widehat{\mathbf{n}} \quad &\rightarrow \quad \widehat{\mathbf{n}} \times v_{\parallel}^2 \, (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \, \widehat{\mathbf{n}} \end{aligned}$$

In the same paper

$$\begin{aligned} &\boldsymbol{\nabla} \times \left(\frac{\widehat{\mathbf{n}}}{B} \right) \\ \approx \quad & 2 \frac{1}{R_0} \left(\widehat{\mathbf{e}}_\theta \ \cos\theta + \widehat{\mathbf{e}}_r \ \sin\theta \right) \\ = \quad & \widehat{\mathbf{n}} \times \boldsymbol{\nabla} \ln B + \widehat{\mathbf{n}} \times (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \, \widehat{\mathbf{n}} \end{aligned}$$

From **Hazeltine Hinton RMP**

Formulas

$$\begin{aligned}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\cdot\mathbf{P} &= \widehat{\mathbf{n}}\times\boldsymbol{\nabla}\ \mathbf{P}_{\perp} \\ &\quad+\left(\mathbf{P}_{\|}-\mathbf{P}_{\perp}\right)\ \widehat{\mathbf{n}}\times\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\end{aligned}$$

where $\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}$ is the curvature, $\sim\widehat{\mathbf{e}}_R/R$, $\widehat{\mathbf{n}}\times\frac{\widehat{\mathbf{e}}_R}{R}$ is a vector almost vertical and is the curvature drift after being multiplied by $\frac{1}{\Omega_c}v_{\|}^2$. This vector is contracted with the anisotropic part of $\mathbf{P}$.

And.

$$\begin{aligned}\boldsymbol{\nabla}\psi\cdot\widehat{\mathbf{n}}\times\mathbf{E} &= -I\ E_{\|} \\ &\quad+BR^2\boldsymbol{\nabla}\varphi\cdot\mathbf{E}\end{aligned}$$

The last term is the toroidal electric field $\times BR$ (but $B\sim B_0/R$). Then $\sim -IE_{\|}+E_{tor}$

Other formula

$$\begin{aligned}&\left(\boldsymbol{\nabla}\times\widehat{\mathbf{n}}\right)_{\perp} \\ =\ &\widehat{\mathbf{n}}\times\left[\left(\boldsymbol{\nabla}\times\widehat{\mathbf{n}}\right)\times\widehat{\mathbf{n}}\right] \\ =\ &\widehat{\mathbf{n}}\times\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\end{aligned}$$

This last form is the vectorial factor from the *curvature drift*

$$\frac{1}{\Omega}\widehat{\mathbf{n}}\times\left[v_{\|}^2\ \left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]$$

Then, up to the physical factor, this is the perpendicular (to $\widehat{\mathbf{n}}$) part of the *rotational* of $\widehat{\mathbf{n}}$.

4 Drift kinetic Hazeltine 1972

The derivation announces the future approaches.

The particle equations of motion

$$\frac{d\mathbf{v}}{dt}=\frac{e}{m}\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)$$

It is defined

$$\begin{aligned}\mathbf{v}_{\|} &= v_{\|}\widehat{\mathbf{n}} \\ \mathbf{v}_{\perp} &= \mathbf{v}-\mathbf{v}_{\|}=v_{\perp}\widehat{\mathbf{e}}_{\perp}\end{aligned}$$

The radius of Larmor gyration is a vector from the center of the circle

$$\begin{aligned}\boldsymbol{\rho} &= \frac{v_\perp}{\Omega}\widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_\perp \\ &\equiv \widehat{\mathbf{e}}_\rho \; \rho\end{aligned}$$

pointing to the current position of the particle on the circle.

$$\mu = \frac{v_\perp^2}{\Omega_c}$$

(note the factors, compare with $\mu = \frac{v_\perp^2}{2B}$)

$$\epsilon = \frac{v^2}{2} + \frac{e}{m}\phi$$

Here, for $(\widehat{\mathbf{n}}, \widehat{\mathbf{e}}_2, \widehat{\mathbf{e}}_3)$,

$$\begin{aligned}\widehat{\mathbf{e}}_\perp &= \widehat{\mathbf{e}}_2 \; \cos\zeta - \widehat{\mathbf{e}}_3 \; \sin\zeta \\ \widehat{\mathbf{e}}_\rho &= \widehat{\mathbf{e}}_2 \; \sin\zeta + \widehat{\mathbf{e}}_3 \; \cos\zeta\end{aligned}$$

Then

$$\frac{d}{dt}\left(\mathbf{v}_\parallel + \mathbf{v}_\perp\right) = \frac{e}{m}\mathbf{E} - \Omega_c \; v_\perp \; \widehat{\mathbf{e}}_\rho$$

The equation for the new variables are

$$\frac{d\mu}{dt} = -\frac{\mu}{B}\frac{dB}{dt} - \frac{v_\parallel}{B}\mathbf{v}_\perp \cdot \frac{d\widehat{\mathbf{n}}}{dt} + \frac{e}{m}\frac{1}{B} \; \mathbf{v}_\perp \cdot \mathbf{E}$$

$$\frac{d\epsilon}{dt} = \frac{e}{m}\left(\frac{d\phi}{dt} + \mathbf{v} \cdot \mathbf{E}\right)$$

$$\frac{d\zeta}{dt} = \Omega_c + \widehat{\mathbf{e}}_3 \cdot \frac{d\widehat{\mathbf{e}}_2}{dt} + \frac{v_\parallel}{v_\perp} + \widehat{\mathbf{e}}_\rho \cdot \frac{d\widehat{\mathbf{n}}}{dt} - \frac{e}{m}\frac{1}{v_\perp}\widehat{\mathbf{e}} \cdot \mathbf{E}$$

The operator is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla}$$

Now it is introduced the gyroaverage operator

$$\begin{aligned}\left\langle \frac{d\mu}{dt} \right\rangle_\zeta &= -\frac{\mu}{B}\frac{\partial B}{\partial t} \\ &= -\mu\frac{\partial}{\partial t}\ln B\end{aligned}$$

$$\left\langle \frac{d\epsilon}{dt} \right\rangle_\zeta = \frac{e}{m}\frac{\partial \phi}{\partial t} - \frac{e}{m} v_\parallel \widehat{\mathbf{n}} \cdot \frac{\partial \mathbf{A}}{\partial t}$$

Scales

$$L \sim \left| \frac{B}{\nabla B} \right|$$

$$\omega \sim \frac{v_{th}}{L}$$

A small parameter

$$\delta = \frac{\omega}{\Omega_c} \ll 1$$

When

$$\frac{\left|\frac{\partial B}{\partial t}\right|}{B} \sim \delta\omega$$

The following operator is defined

$$\mathcal{L}[f] = \frac{df}{dt} - \Omega_c \frac{\partial f}{\partial \zeta}$$

where the time derivative operator, $\frac{df}{dt}$, here, includes derivation with respect to variables of the phase space, $(\mathbf{x}, \mathbf{v})$ except for the gyro-angle ζ, since we intend to find the drift kinetic equation. We must include here the collisions, $C[f]$.

The equation is formally

$$\mathcal{L}[f] - C[f] = -\Omega_c \frac{\partial f}{\partial \zeta}$$

An order of magnitude

$$\mu \frac{\partial f}{\partial \mu} \sim \delta^{-1} f$$

The averages

$$\left\langle \frac{d\boldsymbol{\rho}}{dt} \right\rangle_\zeta = -\mathbf{v}_D - v_\perp^2 \frac{1}{2\Omega_c} \widehat{\mathbf{n}} \; [\widehat{\mathbf{n}} \cdot (\boldsymbol{\nabla} \times \widehat{\mathbf{n}})]$$

$$\left\langle \boldsymbol{\rho} \frac{d\mu}{dt} \right\rangle_\zeta = -\mu \mathbf{v}_D$$

$$\left\langle \boldsymbol{\rho} \frac{d\epsilon}{dt} \right\rangle_\zeta = -\mu \widehat{\mathbf{n}} \times \frac{\partial \mathbf{A}}{\partial t}$$

Define

$$\mathbf{h} \equiv -\boldsymbol{\nabla}\overline{f} - \frac{e}{m}\widehat{\mathbf{n}} \times \mathbf{v}_D \, \frac{\partial \overline{f}}{\partial \mu} + \frac{e}{m}\frac{\partial \mathbf{A}}{\partial t}\frac{\partial \overline{f}}{\partial \epsilon}$$

$$g \equiv \frac{v_\parallel}{\Omega_c}\mu\frac{\partial \overline{f}}{\partial \mu} \, \left[\left(\widehat{\mathbf{e}}_\rho \, \widehat{\mathbf{e}}_\perp \right) : \boldsymbol{\nabla} \, \widehat{\mathbf{n}} - \frac{1}{2}\widehat{\mathbf{n}} \, \cdot \left(\boldsymbol{\nabla} \times \widehat{\mathbf{n}} \right) \right]$$

This allows to write for the gyro-dependent part $\widetilde{f}_1$,

$$\widetilde{f}_1 = \boldsymbol{\rho} \cdot \mathbf{h} + g$$

In the calculation the following identity is used

$$\begin{aligned} \left\langle \widehat{\mathbf{e}}_\rho \, \widehat{\mathbf{e}}_\perp \right\rangle_\zeta \quad &: \quad \boldsymbol{\nabla} \left[-\boldsymbol{\nabla}\overline{f} - \frac{e}{m}\widehat{\mathbf{n}} \times \mathbf{v}_D \, \frac{\partial \overline{f}}{\partial \mu} + \frac{e}{m}\frac{\partial \mathbf{A}}{\partial t}\frac{\partial \overline{f}}{\partial \epsilon} \right] \\ &= \quad \frac{1}{2}\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \times \left[-\boldsymbol{\nabla}\overline{f} - \frac{e}{m}\widehat{\mathbf{n}} \times \mathbf{v}_D \, \frac{\partial \overline{f}}{\partial \mu} + \frac{e}{m}\frac{\partial \mathbf{A}}{\partial t}\frac{\partial \overline{f}}{\partial \epsilon} \right] \end{aligned}$$

The final form of the drift-kinetic equation is

$$\begin{aligned} & \frac{\partial \overline{f}}{\partial t} \\ & + \left(v_\parallel \widehat{\mathbf{n}} + \mathbf{v}_D + v_\perp^2 \frac{1}{2\Omega_c}\widehat{\mathbf{n}} \, \left[\widehat{\mathbf{n}} \cdot \left(\boldsymbol{\nabla} \times \widehat{\mathbf{n}} \right) \right] \right) \cdot \boldsymbol{\nabla}\overline{f} \\ & + \left\{ \frac{v_\parallel}{\Omega_c} \boldsymbol{\nabla} \cdot \left(\frac{\partial \widehat{\mathbf{n}}}{\partial t} \times \widehat{\mathbf{n}} \right) + \frac{\widehat{\mathbf{n}}}{B} \cdot \frac{\partial \mathbf{A}}{\partial t} \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \times \widehat{\mathbf{n}} \right) + \frac{v_\parallel}{\Omega_c} \mathbf{B} \cdot \boldsymbol{\nabla} \left(\frac{v_\parallel}{B} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \times \widehat{\mathbf{n}} \right) \right\} \mu \frac{\partial \overline{f}}{\partial \mu} \\ & - \left[v_\parallel \frac{\partial v_\parallel}{\partial t} + \frac{e}{m} \left(v_\parallel \widehat{\mathbf{n}} + \mathbf{v}_D + v_\perp^2 \frac{1}{2\Omega_c}\widehat{\mathbf{n}} \, \left[\widehat{\mathbf{n}} \cdot \left(\boldsymbol{\nabla} \times \widehat{\mathbf{n}} \right) \right] \right) \cdot \frac{\partial \mathbf{A}}{\partial t} \right] \frac{\partial \overline{f}}{\partial \epsilon} \\ = \quad & C \left[\overline{f}, \overline{f} \right] + \left\langle C \left[\widetilde{f}_1, \widetilde{f}_1 \right] \right\rangle \end{aligned}$$

Many terms result from retaining the time variation of $\mathbf{B}$.

5 Drift kinetic for trapped and untrapped particles Galeev Sagdeev

This is the equation

$$\frac{\partial f_j}{\partial t} + \left[H, f_j \right] = St \left(f_j \right)$$

where

$$
\begin{aligned}
& [H, f_j] \\
= \quad & \left\{ -\frac{1}{\Omega_j} \frac{\frac{v_\perp^2}{2} + v_\parallel^2}{R} \sin\theta \, \frac{\partial}{\partial r} \right. \\
& + \left(-\frac{B_\theta}{B_T} v_\parallel + \frac{1}{B_0} \frac{d\Phi}{dr} - \frac{1}{\Omega_j} \frac{\frac{v_\perp^2}{2} + v_\parallel^2}{R} \cos\theta \right) \frac{\partial}{r\partial\theta} \\
& \left. + \frac{B_\theta}{B_T} \frac{\frac{v_\perp^2}{2} + v_\parallel^2}{R} \sin\theta \, \frac{\partial}{\partial v_\parallel} \right\} f_j
\end{aligned}
$$

There is a radial electric field $-\frac{d\Phi}{dr}$. It is considered unavoidable due to the constraint of ambipolarity (the radial current produced by the ion diffusion - electrons can be neglected - averaged over surface is zero). The radial diffusion of electrons is different of that of ions, so an electric field is necessary.

The term $-\frac{1}{\Omega_j}\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{R}\sin\theta$ and the term $-\frac{1}{\Omega_j}\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{R}\cos\theta$ are components of the drift velocity, and they occur in the convective derivative of f. Sometimes one uses the factor

$$
\begin{aligned}
\Theta &= \frac{r}{qR} \\
&= \frac{B_\theta}{B_T} \ll 1
\end{aligned}
$$

Here $\Theta\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{R}\sin\theta$ is a convection in the space of velocity, along $v_\parallel$. It is an acceleration. It is the projection of $\mathbf{v}_D$ on the parallel direction

The solution is described in *drift kinetic solutions*.

6 Drift kinetic equation in a rotating plasma (Wong Burrell 1982)

Some comments are also in *solutions drift kinetic*.

This is also in *impurities.tex*.

It is related to the problem of *inward impurity flow*.

The intention is to derive a drift kinetic equation.

6.1 Equations of motion of the particles

The derivation of the equations of motion is in the text *particles equations of motion.*

For a circular geometry, when the potential can be separated into a main part (constant on magnetic surface) $\phi_0(r)$ and a part that depends on (r,θ), $|\phi_1(r,\theta)| \ll |\phi_0(r)|$,

$$\phi = \phi_0(r) + \phi_1(r,\theta)$$

$$\frac{dx_\theta}{dt} = v_\parallel \frac{B_\theta}{B_T} + \frac{1}{B_0}\frac{d\phi_0}{dr}$$

$$\frac{dx_r}{dt} = -\frac{1}{B_0}\frac{\partial\phi_1}{r\partial\theta} - \frac{1}{\Omega_{ci}}\frac{v_\perp^2/2 + v_\parallel^2}{R}\sin\theta$$

$$\frac{d}{dt}\left(\frac{v_\perp^2}{2}\right) = \left(\frac{v_\perp^2}{2}\right) v_\parallel \frac{B_\theta}{B_T}\frac{\sin\theta}{R} + \left(\frac{v_\perp^2}{2}\right)\frac{1}{B_0}\left(\frac{d\phi_0}{dr}\right)\frac{\sin\theta}{R}$$

$$\frac{dv_\parallel}{dt} = -\left(\frac{v_\perp^2}{2}\right)\frac{B_\theta}{B_T}\frac{\sin\theta}{R} + v_\parallel \frac{1}{B_0}\left(\frac{d\phi_0}{dr}\right)\frac{\sin\theta}{R} - \frac{e}{m}\frac{B_\theta}{B_T}\frac{\partial\phi_1}{r\partial\theta}$$

(we usually replace $dx_\theta = r d\theta$).
This is **Wong Burrell 1982**.

6.2 Drift kinetic equation

The drift-kinetic equation is written as

$$\frac{\partial f}{\partial t} + \boldsymbol{\nabla}\cdot\left(\frac{d\mathbf{x}}{dt} f\right) + \frac{\partial}{\partial v_\parallel}\left(\frac{dv_\parallel}{dt} f\right) + \frac{\partial}{\partial\left(v_\perp^2/2\right)}\left(\frac{d\left(v_\perp^2/2\right)}{dt} f\right) = C(f,f)$$

It is remarked that the volume in phase space is NOT preserved by the equations of motion

$$\begin{aligned} & \boldsymbol{\nabla}\cdot\frac{d\mathbf{x}}{dt} + \frac{\partial}{\partial v_\parallel}\left(\frac{dv_\parallel}{dt}\right) + \frac{\partial}{\partial\left(v_\perp^2/2\right)}\left(\frac{d\left(v_\perp^2/2\right)}{dt}\right) \\ = & \frac{1}{B^2}\left(\boldsymbol{\nabla}\times\mathbf{B}\right)\cdot\boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\cdot\mathbf{v}_D \end{aligned}$$

This is not so large and later it will be neglected.

From **Wong Burrell** we have

$$\mathbf{v}_D = \frac{1}{\Omega}\left(v_\parallel^2 + \frac{v_\perp^2}{2}\right)\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\ln B$$

$$\begin{aligned}\nabla \cdot \mathbf{v}_D &= \frac{m}{e}\left(v_\parallel^2 + \frac{v_\perp^2}{2}\right)\frac{\mathbf{J}\cdot\nabla B}{B^3} \\ &\approx \frac{\varepsilon^2 v_D}{r}\end{aligned}$$

and

$$\frac{J_\theta}{B} \sim \varepsilon^2 \frac{1}{r}$$

This divergence of the drift velocity is neglected.

It is also mentioned by **Hinton Waltz** with the intention to derive exact equation for the heating of plasma by instabilities.

Then the drift kinetic equation is

$$\frac{\partial f}{\partial t} + \frac{d\mathbf{x}}{dt}\cdot\nabla f + \frac{dv_\parallel}{dt}\frac{\partial f}{\partial v_\parallel} + \left[\frac{d}{dt}\left(\frac{v_\perp^2}{2}\right)\right]\frac{\partial f}{\partial\left(v_\perp^2/2\right)} = C\left(f,f\right)$$

and an expansion is made

$$\begin{aligned} f &= f_0 + f_1 + ... \\ &\quad \text{(multiple time-scale expansion)}\end{aligned}$$

$$\frac{dx_\theta}{dt}\frac{\partial f_0}{r\partial\theta} = C\left(f_0, f_0\right)$$

or

$$\frac{dx_\theta}{dt} = \frac{d\left(r\theta\right)}{dt}$$

$$f_0 = \frac{1}{\left[\pi\left(\frac{2T}{m}\right)\right]^{3/2}} n \ \exp\left[-\frac{\left(v_\parallel - U\right)^2}{(2T/m)} - \frac{v_\perp^2}{(2T/m)}\right]$$

where $U \equiv$parallel flow velocity

$$\begin{aligned} &\left(v_\parallel + \frac{1}{B_\theta}\frac{d\phi_0}{dr}\right)\frac{B_\theta}{B_T}\frac{\partial f_1}{r\partial\theta} \quad \text{(poloidal advection of } f_1\text{)} \\ &+ \left(v_\parallel - U\right)\frac{e}{T}\frac{B_\theta}{B_T}\frac{\partial\phi_1}{r\partial\theta} f_0 - C^{lin}\left(f_1\right) \\ = \ &\varepsilon\frac{B_\theta}{B_T}\frac{1}{r}\left\{\left[\frac{v_\perp^2}{(2T/m)}\left(U + \frac{1}{B_\theta}\frac{d\phi_0}{dr}\right) + 2\frac{v_\parallel\left(v_\parallel - U\right)}{(2T/m)}\frac{1}{B_\theta}\frac{d\phi_0}{dr}\right]\sin\theta\right. \\ &+ \frac{T}{eB_\theta}\left[\frac{v_\parallel^2 + \frac{v_\perp^2}{2}}{(T/m)}\sin\theta + \frac{1}{\varepsilon}\frac{e}{T}\frac{\partial\phi_1}{\partial\theta}\right]\times \\ &\quad \left.\times\left[\frac{d}{dr}\ln n + \left(\frac{\left(v_\parallel - U\right)^2}{(2T/m)} + \frac{v_\perp^2}{(2T/m)} - \frac{3}{2}\right)\frac{d}{dr}\ln T + \frac{\left(v_\parallel - U\right)\frac{dU}{dr}}{(T/m)}\right]\right\}\end{aligned}$$

A term like

$$\left(v_{\parallel} - U\right) \frac{e}{T} \frac{B_\theta}{B_T} \frac{\partial \phi_1}{r \partial \theta} f_0 \sim V_{\parallel} e \frac{1}{qR} \frac{\partial \phi_1}{\partial \theta} \times \frac{\partial f_0}{\partial \epsilon} \sim V_{\parallel} \, e \widetilde{E}_{\parallel} \times \frac{\partial f_0}{\partial \epsilon}$$
$$= \text{work done by parallel flow against } \widetilde{E}_{\parallel}$$

is energetic. It is added in the LHS to the poloidal advection of the order-1 function f_1 .

Wong Burrell consider that the first term in the RHS

$$\varepsilon \frac{B_\theta}{B_T} \frac{1}{r} \left[\frac{v_\perp^2}{(2T/m)} \left(U + \frac{1}{B_\theta} \frac{d\phi_0}{dr} \right) + 2 \frac{v_{\parallel} \left(v_{\parallel} - U\right)}{(2T/m)} \frac{1}{B_\theta} \frac{d\phi_0}{dr} \right] \sin\theta$$

is related with the parallel velocity and with $E \times B$ velocity, in the inhomogeneous magnetic field.

We note

$$\varepsilon \frac{B_\theta}{B_T} \frac{1}{r} \sin\theta = \frac{B_\theta}{B_T} \frac{1}{R} \sin\theta$$

and we have

$$U + \frac{1}{B_\theta} \frac{d\phi_0}{dr} \sim \text{parallel velocity incl. } E_r \times B_\theta$$

$$2 \frac{v_{\parallel} \left(v_{\parallel} - U\right)}{(2T/m)} \frac{1}{B_\theta} \frac{d\phi_0}{dr} \sim \text{parallel velocity incl. } E_r \times B_\theta$$

These terms act to produce poloidal advection of the perturbed distribution function f_1.

They are called *magnetic pumping* terms.

The other terms in the RHS are due to the drift: curvature and gradient, and the self-consistent perturbation of the potential on the surface ϕ_1,

$$\frac{T}{eB_\theta} \left[\frac{v_{\parallel}^2 + \frac{v_\perp^2}{2}}{(T/m)} \sin\theta + \frac{1}{\varepsilon} \frac{e}{T} \frac{\partial \phi_1}{\partial \theta} \right] \sim \text{drift} + \frac{1}{B_\theta} \frac{\partial \phi_1}{r \partial \theta}$$
$$\sim \text{drift} + \frac{E_\theta^{(1)}}{B_\theta}$$

See below the comparison.

6.3 First order distribution function and electrostatic potential

The terms due to the parallel velocity in the RHS (*magnetic pumping*) are larger by a factor

$$\frac{L_n}{\rho_\theta}$$

than the drift + ϕ_1 terms in the RHS.

Then the drift $+\phi_1$ terms are neglected.

The first order distribution function for the species α is *assumed* to have a harmonic variation along the poloidal coordinate, i.e. $\sim \cos\theta$. The choice $(\cos\theta)$ is suggested by the Pfirsch Schluter current (which is the toroidal current in response to the non-vanishing divergence of the diamagnetic current, caused by the toroidal geometry).

$$\begin{aligned} f_{\alpha 1} &= h_{\alpha 1} \cos\theta \\ \frac{e\phi_1}{T} &= \varepsilon W \cos\theta \end{aligned}$$

$$W \equiv \frac{\sum_\alpha Z_\alpha n_\alpha m_\alpha U^2}{\sum_\alpha Z_\alpha^2 n_\alpha T}$$

or

$$W \equiv \frac{\sum_\alpha [Z_\alpha n_\alpha m_\alpha] U^2}{\sum_\alpha Z_\alpha \; [Z_\alpha n_\alpha m_\alpha] (T/m_\alpha)}$$

is a ratio of the parallel energy U^2 weighted by the charge Z_α to the thermal energy (T/m_α) weighted by the charge.

This W should be small if U is much smaller than the thermal speed of the species-α particles.

Where

$$\begin{aligned} h_{\alpha 1} &= \varepsilon \left(\frac{v_\| U}{T/m_\alpha} - Z_\alpha \; W \right) f_{\alpha 0} \\ \alpha &\equiv \text{species, electrons, ions,...} \end{aligned}$$

Interesting.

The first-order f_1 correction to the zeroth distribution function (a $U-$shifted Maxwellian) is

- variable on θ like $\cos\theta$;
- is of the order $\sim \varepsilon$ (toroidality) $\times$ (parallel flow).

Note
the velocity U is here *imposed* from outside (for ex. NBI).

The first order f_1 would be zero if we can assume that the imposed velocity U is zero.

But this is not possible, U cannot be zero.

Constraint in this order

$$\frac{1}{B_\theta}\frac{d\phi_0}{dr} + U = 0$$

well known resonance condition of *parallel* flow.

The electrostatic potential ϕ_0 results from ambipolarity condition.

The presence of an imposed parallel velocity U means that there is an electric potential ϕ_0.

See **Kagan Catto enhanced bootstrap** where it is stated that normal situation is that the poloidal projection of the parallel velocity is equal to the ion diamagnetic velocity. This is the *rho_ effective*.

End.

From Wang Burrell 1982
"The magnetic pumping terms are typically larger than the magnetic drift terms by a factor

$$\frac{L_n}{\rho_\theta}$$

for each species"

The *magnetic pumping* is the variation of the magnitude of the magnetic field along the line, due to the toroidality.

The *magnetic drift* is

$$\mathbf{v}_D^{mag} = \frac{1}{\Omega}\widehat{\mathbf{n}} \times (\mu \boldsymbol{\nabla} B)$$

with neglect of the curvature drift.

6.3.1 The magnetic pumping terms

The group of terms in the equation for the correction to the distribution function

$$\sin\theta \times \varepsilon \frac{B_\theta}{B_T}\frac{1}{r}\left[\frac{v_\perp^2}{v_{th}^2}\left(U + \frac{1}{B_\theta}\frac{d\phi_0}{dr}\right) + \frac{mv_\parallel\left(v_\parallel - U\right)}{T}\frac{1}{B_\theta}\frac{d\phi_0}{dr}\right] f_0$$

arises from *parallel* and $E \times B$ *flow* and are called magnetic pumping terms.

The second term in the square paranthesis

$$\begin{aligned} & \sin\theta \times \varepsilon \frac{B_\theta}{B_T}\frac{1}{r} \times \frac{mv_\parallel\left(v_\parallel - U\right)}{T}\frac{1}{B_\theta}\frac{d\phi_0}{dr} f_0 \\ \rightarrow \quad & \sin\theta \times \varepsilon \frac{B_\theta}{B_T}\frac{1}{r}\frac{1}{B_\theta}\frac{d\phi_0}{dr} \times v_\parallel \times \frac{\partial}{\partial v_\parallel}\exp\left[-\frac{\left(v_\parallel - U\right)^2}{(2T/m)}\right] \\ \rightarrow \quad & \sin\theta \frac{1}{R}\left(\frac{1}{B_T}\frac{d\phi_0}{dr}\right) \times v_\parallel \frac{\partial f_0}{\partial v_\parallel} \end{aligned}$$

we recall that $d\phi_0/dr$ is the radial electric field, and with B_T gives a poloidal velocity, $\sim v_E$.

Now, from the equation of motion

$$\frac{dv_\parallel}{dt} = -\left(\frac{v_\perp^2}{2}\right)\frac{B_\theta}{B_T}\frac{\sin\theta}{R} + v_\parallel \frac{1}{B_0}\left(\frac{d\phi_0}{dr}\right)\frac{\sin\theta}{R} - \frac{e}{m}\frac{B_\theta}{B_T}\frac{\partial\phi_1}{r\partial\theta}$$

we retain the second term

$$\left(\frac{dv_\parallel}{dt}\right)^{electric} = v_\parallel \frac{1}{B_0}\left(\frac{d\phi_0}{dr}\right)\frac{\sin\theta}{R}$$

and the two expressions (*i.e.* second term in the square paranthesis above) are coincident since they represent the *energetic* effect along the parallel motion

$$\left(\frac{dv_\parallel}{dt}\right)^{electric}\frac{\partial f_0}{\partial v_\parallel}$$

where the particle parallel motion $v_\parallel$ is against the *parallel* electric field (originated from ϕ_0 and B_θ).

It is the energetic *magnetic mirror* effect.

6.3.2 The drift terms

The second group of terms is

$$\varepsilon\frac{B_\theta}{B_T}\frac{1}{r}\frac{T}{eB_\theta}\left[\frac{m}{T}\left(v_\parallel^2+\frac{v_\perp^2}{2}\right)\sin\theta+\frac{1}{\varepsilon}\frac{e}{T}\frac{\partial\phi_1}{\partial\theta}\right]$$
$$\times\left[\frac{d\ln n}{dr}+\left(\frac{m\left(v_\parallel-U\right)^2}{2T}+\frac{mv_\perp^2}{2T}-\frac{3}{2}\right)\frac{d\ln T}{dr}+\frac{m\left(v_\parallel-U\right)}{T}\frac{dU}{dr}\right]$$

This comes from *curvature* and *gradient drifts*. It contains ϕ_1 which is the variation of the electrostatic potential in the magnetic surface.

$$\begin{aligned}
&\varepsilon\frac{B_\theta}{B_T}\frac{1}{r}\frac{T}{eB_\theta}\left[\frac{m}{T}\left(v_\parallel^2+\frac{v_\perp^2}{2}\right)\sin\theta+\frac{1}{\varepsilon}\frac{e}{T}\frac{\partial\phi_1}{\partial\theta}\right]\\
=\;&\frac{1}{R}\frac{1}{B_T}\frac{T}{e}\times\frac{m}{T}R\left[\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{R}\sin\theta\times\frac{\Omega}{\Omega}+\frac{e}{m}\frac{\partial\phi_1}{r\partial\theta}\right]\\
=\;&\frac{1}{B_T}\frac{m}{e}\left[\frac{eB}{m}\,v_r^{drift}+\frac{e}{m}\frac{\partial\phi_1}{r\partial\theta}\right]\approx v_r^{drift}+\frac{1}{B}\frac{\partial\phi_1}{r\partial\theta}\\
=\;&v_r^{drift}+v_r^{\widetilde{E}}\qquad\text{(radial velocity)}
\end{aligned}$$

This factor is the radial advection (by drift and poloidal electric field) of the other factor, which consists of radial gradients of n, of T and of the flow velocity U

$$\left(v_r^{drift}+v_r^{\widetilde{E}}\right)\frac{\partial f_0}{\partial r}$$

6.3.3 Comparison *mirror* to *drift* terms

Now we want to make a comparison between the two terms.

For this we simplify $U=0$, and take $\phi_1\equiv 0$, and write

$$\begin{aligned}
&\sin\theta\times\left[\frac{v_\perp^2}{v_{th}^2}\frac{1}{B_\theta}\frac{d\phi_0}{dr}+\frac{mv_\parallel^2}{T}\frac{1}{B_\theta}\frac{d\phi_0}{dr}\right]\\
=\;&\sin\theta\times\frac{1}{B_\theta}\frac{d\phi_0}{dr}\frac{m}{T}\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)
\end{aligned}$$

and the second group of terms

$$\sin\theta\times\frac{T}{eB_\theta}\frac{m}{T}\left(v_\parallel^2+\frac{v_\perp^2}{2}\right)\frac{1}{L_n}\left[1+\eta\left(\frac{v^2}{v_{th}^2}-\frac{3}{2}\right)\right]$$

and we recall that

$$\frac{T}{eB_\theta}\frac{1}{L_n}\left[1+\eta\left(\frac{v^2}{v_{th}^2}-\frac{3}{2}\right)\right]=\frac{B}{B_\theta}v_T^{dia}$$

then

$$\sin\theta\times\frac{m}{T}\left(v_\parallel^2+\frac{v_\perp^2}{2}\right)\frac{B}{B_\theta}v_T^{dia}$$

Now we have to compare

$$\frac{\text{(mirror, energetic)}}{\text{(drift, convection)}}\sim\frac{\sin\theta\times\frac{1}{B_\theta}\frac{d\phi_0}{dr}\frac{m}{T}\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)}{\sin\theta\times\frac{m}{T}\left(v_\parallel^2+\frac{v_\perp^2}{2}\right)\frac{B}{B_\theta}v_T^{dia}}$$

$$=\frac{\frac{1}{B_\theta}\frac{d\phi_0}{dr}}{\frac{B}{B_\theta}v_T^{dia}}=\frac{\frac{1}{B}\frac{d\phi_0}{dr}}{v_T^{dia}}$$

then we have to compare the poloidal electric $E\times B$ velocity due to radial variation of the electric potential, with the diamagnetic velocity.

Note For us the rotation velocity V_E is greater than the diamagnetic velocity

$$V_E\gg v_{dia}$$

(see *rho-effective*) but the diamagnetic velocity is steadly increasing $L_n\to\rho_\theta$.

$$\frac{\frac{1}{B}\frac{d\phi_0}{dr}}{v_T^{dia}}\sim\frac{V_E}{v_{dia}}\sim\frac{L_n}{\rho_\theta}$$

When

$$\frac{1}{B}\frac{d\phi_0}{dr}\sim v_T^{dia}$$

$$\begin{aligned}\frac{\rho_\theta}{L_n}&\to 1\\ \text{then } 1-\frac{v_{dia}}{V_E}&\to 0\ \text{ and }\ \frac{1}{\rho_{eff}^2}\to 0\end{aligned}$$

Reaching comparative magnitudes $L_n\searrow\rho_\theta$.

For the distribution function

a the time variation due to energetic effect of mirror (magnetic modulation)

is equal to

b the time variation due to the convection by drift velocity

End

Wang Burrell assert that the first group of terms ("pumping", energetic, mirror, $\left(\frac{dv_{\parallel}}{dt}\right)^{electric} \frac{\partial f_0}{\partial v_{\parallel}}$) is larger than the second group of terms ("drift", advection of equilibrium gradients, $\left(v_r^{drift} + v_r^{\widetilde{E}}\right) \frac{\partial f_0}{\partial r}$) as

$$\frac{(\text{mirror})}{(\text{drift})} \sim \frac{\left(\frac{dv_{\parallel}}{dt}\right)^{electric} \frac{\partial f_0}{\partial v_{\parallel}}}{\left(v_r^{drift} + v_r^{\widetilde{E}}\right) \frac{\partial f_0}{\partial r}} \sim \frac{L_n}{\rho_\theta}$$

It is difficult to understand why.

For trapped part of the velocity space is OK. The modulations due to the magnetic mirror is substantial, it turns back $v_{\parallel}$.

NOTE
This is however *favorable* for the idea of *rho-effective*.
Here seems to involve the equality between L_n and ρ_θ.
END

6.4 The two constraints: neutrality and ambipolarity

The neutrality

$$\sum e \int d^3v \; f_1 = 0$$

This equation will determine ϕ_1, the variation of the electrostatic potential on the surface.

The ambipolarity.
We have for the radial flux

$$\Gamma = \int \frac{d\theta}{2\pi} \; h \int d^3v \; \left(\mathbf{v}_D + \mathbf{v}_E\right) \cdot \widehat{\mathbf{e}}_r \; f$$

The condition of ambipolarity

$$\sum Z_\alpha e \Gamma_\alpha = 0$$

is an equation for the electrostatic potential ϕ.

"To maintain neutrality the radial electric field will acquire time dependence

$$\frac{\partial E_r}{\partial t}$$

producing a *polarization* current which will completely cancel the radial current $\sum e\Gamma$.

The polarization current is

$$\begin{aligned} & \text{polarization current} \\ = \quad & -\sum_{\alpha} m_{\alpha} n_{\alpha} \frac{1}{B_0^2} \frac{\partial}{\partial t} \frac{d\phi}{dr} \end{aligned}$$

Assume

$$\sum e\Gamma \sim n e v_{drift}\ \varepsilon$$

taking equality it results

$$\frac{\partial}{\partial t} \sim \omega^{ion-transit}$$

This time scale cannot be present in a *transport* calculation."

This implies that the ambipolarity is not automatic and must be imposed and the equation will provide a value for $\phi_0 (r)$.

The existence of a potential on surfaces $\phi_0 (r)$ implicitely makes U nonzero.

Then both $h_{\alpha 1}$ and ϕ_1 cannot be zero since U and W cannot be zero.

6.5 Extend the first order

When the expressions like

$$\begin{aligned} f_{\alpha 1} &= h_{\alpha 1} \cos\theta \\ & \ldots \end{aligned}$$

have been assumed we actually neglected the effect of the drift $+\phi_1$ terms (the second set of terms in RHS).

Now we advance beyond this level of approximation and add corrections

$$f_{\alpha 1} = h_{\alpha 1} \cos\theta + g_{\alpha 1}$$

$$\frac{e\phi_1}{T} = e\ W\ \cos\theta + \widetilde{\phi}_1$$

$$\frac{1}{B_\theta}\frac{d\phi_0}{dr}+U=u$$

Note that u is a small corection to the basic equilibrium which consists of cancelling the parallel flow (the *resonance* $\frac{1}{B_\theta}\frac{d\phi_0}{dr}+U=0$), **Kagan Catto**.

Note that $\widetilde{\phi}_1$, the supplement, actually is normalized to e/T.

the next step is the change to the *moving referential*

$$\begin{aligned} v_\parallel &\rightarrow v'_\parallel = v_\parallel - U \\ w' &= \frac{1}{2}v'^2_\parallel+\frac{1}{2}v_\perp^2 \quad \text{(new energy)} \end{aligned}$$

The equation becomes

$$\begin{aligned} &\left(v'_\parallel+u\right)\frac{B_\theta}{B_\varphi}\frac{\partial g_{\alpha 1}}{r\partial\theta} \\ &+\left[v'_\parallel-\frac{T/m_\alpha}{\left(\frac{Z_\alpha e B_\theta}{m_\alpha}\right)}\left(\frac{d}{dr}\ln n_\alpha+\overline{\frac{w'}{T/m_\alpha}-\frac{3}{2}}\frac{d}{dr}\ln T+\frac{v'_\parallel}{T/m_\alpha}\frac{dU}{dr}\right)\right]\frac{B_\theta}{B_\varphi}Z_\alpha\frac{\partial\widetilde{\phi}_1}{r\partial\theta}f_{\alpha 0} \\ &-C^{lin}\left[g_{\alpha 1}\right] \\ =\ &\varepsilon\frac{1}{r}\frac{B_\theta}{B_\varphi}H_\alpha\left[u+\frac{T/m_\alpha}{\left(\frac{Z_\alpha e B_\theta}{m_\alpha}\right)}\left(\frac{d}{dr}\ln n_\alpha+\overline{\frac{w'}{T/m_\alpha}-\frac{3}{2}}\frac{d}{dr}\ln T+\frac{v'_\parallel}{T/m_\alpha}\frac{dU}{dr}\right)\right]f_{\alpha 0}\ \sin\theta \end{aligned}$$

with the notation

$$H_\alpha\equiv\frac{1}{T/m_\alpha}\left(v'^2_\parallel+\frac{v_\perp^2}{2}\right)+\frac{U^2}{T/m_\alpha}-Z\ W+\frac{v'_\parallel}{T/m_\alpha}\frac{dU}{dr}$$

The velocity $v'_\parallel$ is measured in the referential moving with the flow velocity U. Here the first term $\frac{1}{T/m_\alpha}\left(v'^2_\parallel+\frac{v_\perp^2}{2}\right)$ in H_α compares the energy in the moving frame with the thermal energy.

The next term $\frac{U^2}{T/m_\alpha}$ in H_α is a ratio between the energy of the flow to the thermal energy.

If we will have this H_α in combination with other factors, we would see

$$H_\alpha\rightarrow\frac{1}{v_{th}^2}R\Omega\times\frac{1}{\Omega}\frac{v'^2_\parallel+\frac{v_\perp^2}{2}}{R}$$

which is

$$\frac{R}{\rho_s}\frac{1}{v_{th}}\times\left(v_{drift}\right)$$

Let us see

$$\left[v_\parallel' - \frac{T/m_\alpha}{\left(\frac{Z_\alpha e B_\theta}{m_\alpha}\right)}\left(\frac{d}{dr}\ln n_\alpha + \overline{\frac{w'}{T/m_\alpha} - \frac{3}{2}}\,\frac{d}{dr}\ln T + \frac{v_\parallel'}{T/m_\alpha}\frac{dU}{dr}\right)\right]\frac{B_\theta}{B_\varphi}Z_\alpha\frac{\partial\widetilde{\phi}_1}{r\partial\theta}f_{\alpha 0}$$

The gradients are

$$\left(\frac{d}{dr}\ln n_\alpha + \overline{\frac{w'}{T/m_\alpha} - \frac{3}{2}}\,\frac{d}{dr}\ln T + \frac{v_\parallel'}{T/m_\alpha}\frac{dU}{dr}\right) \sim \frac{\partial}{\partial r}\ln f_0$$

if we ignore the shear of U. Also

$$\frac{T/m_\alpha}{\left(\frac{Z_\alpha e B_\theta}{m_\alpha}\right)} = \frac{v_{th}^2}{\Omega_{Z,\theta}} \sim v_{th}\ \rho_\theta$$

$$\begin{aligned} & v_\parallel' - v_{th}\left[\rho_\theta\frac{\partial}{\partial r}\ln f_0\right] \\ \sim\ & V_\parallel \end{aligned}$$

then

$$\begin{aligned} & V_\parallel \times \frac{B_\theta}{B_\varphi} \times Z_\alpha\frac{\partial\widetilde{\phi}_1}{r\partial\theta} \\ =\ & V_\theta\ \widetilde{E}_\theta \\ \rightarrow\ & V_\theta\ \widetilde{E}_\theta \times f_0 \end{aligned}$$

We should have $\frac{df_0}{d\epsilon}$. This is an energetic term.

6.6 Approximations in the equation for $g_{\alpha 1}$

It is made the substitution

$$g_{\alpha 1}' = g_{\alpha 1} + Z_\alpha\widetilde{\phi}_1\ f_{\alpha 0}$$

(remember $\widetilde{\phi}_1$ contains e/T). After this, the neutrality condition is

$$\sum Z_\alpha^2 n_\alpha\widetilde{\phi}_1 = \sum Z_\alpha\int d^3v\ g_{\alpha 1}'$$

and the ambipolarity condition

$$\sum Z_\alpha\Gamma_\alpha = 0$$

which determine the velocity u, *i.e.* the perturbation that prevents, in higher order, the resonance $\frac{1}{B_\theta}\frac{d\phi_0}{dr} + U = 0$.

The explanation of the definition of the new $g'_{\alpha 1}$ results from an approximation that can be done on the equation for $g_{\alpha 1}$.

6.6.1 First term

First term is

$$\left(v'_\parallel + u\right)\frac{B_\theta}{B_\varphi}\frac{\partial g_{\alpha 1}}{r\partial\theta}$$

and in it the velocity u is neglected. Then we have

$$v'_\parallel \frac{B_\theta}{B_\varphi}\frac{\partial g_{\alpha 1}}{r\partial\theta}$$

6.6.2 Second term

In the square paranthesis of the second term

$$v'_\parallel - \frac{T/m_\alpha}{\left(\frac{Z_\alpha e B_\theta}{m_\alpha}\right)}\left(\frac{d}{dr}\ln n_\alpha + \overline{\frac{w'}{T/m_\alpha} - \frac{3}{2}}\frac{d}{dr}\ln T + \frac{v'_\parallel}{T/m_\alpha}\frac{dU}{dr}\right)$$

the second part contains at the denominator $\Omega_{\alpha\theta}$ which means that it will contain ρ_θ. It is neglected.

The second term will be

$$v'_\parallel \times \frac{B_\theta}{B_\varphi} Z_\alpha \frac{\partial\widetilde{\phi}_1}{r\partial\theta} f_{\alpha 0}$$

6.7 Collision term

Now adopt a strong simplification of the collision term.

At the end, one takes

$$\nu \to 0+$$

6.8 After approximations

We have for the LHS

$$v'_{\parallel}\frac{B_\theta}{B_\varphi}\frac{\partial g_{\alpha 1}}{r\partial\theta}+v'_{\parallel}\times\frac{B_\theta}{B_\varphi}Z_\alpha\frac{\partial\widetilde{\phi}_1}{r\partial\theta}f_{\alpha 0}-C$$

or

$$\begin{aligned}&v'_{\parallel}\frac{B_\theta}{B_\varphi}\frac{\partial}{r\partial\theta}\left(g_{\alpha 1}+Z_\alpha\widetilde{\phi}_1\ f_{\alpha 0}\right)\\ \rightarrow\quad &v_{\parallel}\frac{B_\theta}{B_\varphi}\frac{\partial}{r\partial\theta}g'_{\alpha 1}\end{aligned}$$

The drift kinetic equation becomes

$$\begin{aligned}&v'_{\parallel}\frac{B_\theta}{B_\varphi}\frac{\partial}{r\partial\theta}g'_{\alpha 1}-C^{lin}\left[g'_{\alpha 1}\right]\\ =\quad &\varepsilon\frac{B_\theta}{B_\varphi}\frac{1}{r}H_\alpha\left[u+\frac{T/m_\alpha}{\left(\frac{Z_\alpha eB_\theta}{m_\alpha}\right)}\left(\frac{d}{dr}\ln n_\alpha+\overline{\frac{w'}{T/m_\alpha}-\frac{3}{2}}\frac{d}{dr}\ln T+\frac{v'_{\parallel}}{T/m_\alpha}\frac{dU}{dr}\right)\right]f_{\alpha 0}\ \sin\theta\end{aligned}$$

To see what is this

$$\begin{aligned}&\varepsilon\frac{B_\theta}{B_\varphi}\frac{1}{r}\times\frac{R}{\rho_s}\frac{1}{v_{th}}\times(v_{drift})\times\left[u+v_{th}\left(\rho_\theta\frac{d}{dr}\ln f_0\right)\right]\sin\theta\\ \rightarrow\quad &\frac{B_\theta}{B_\varphi}\frac{1}{\rho_s v_{th}}\times\left[v_{drift}\sin\theta\right]\times\left[V_\parallel\right]\\ \rightarrow\quad &\frac{1}{\rho_s v_{th}}\times\left[v_{drift}^{radial}\right]\times\left[V_\theta\right]f_0\end{aligned}$$

and

$$\begin{aligned}&v'_{\parallel}\frac{B_\theta}{B_\varphi}\frac{\partial}{r\partial\theta}g'_{\alpha 1}\\ \rightarrow\quad &v'_{\parallel}\nabla_\parallel g'_{\alpha 1}\end{aligned}$$

then

$$\begin{aligned}v'_{\parallel}\nabla_\parallel g'_{\alpha 1}-C^{lin}\quad &\sim\quad\frac{1}{\rho_s v_{th}}\times\left[v_{drift}^{radial}\right]\times\left[V_\theta\right]f_0\\ &\sim\quad(\text{velocity})\times\frac{f_0}{\rho_s}\end{aligned}$$

The order of magnitude

$$\frac{g'_{\alpha 1}}{f_{\alpha 0}} \sim \varepsilon \frac{\rho_\theta}{L_n}$$

The transport fluxes

$$\Gamma = \int \frac{d\theta}{2\pi} \, h \, \int d^3v \, \left(\mathbf{v}_D + \mathbf{v}_E\right) \cdot \widehat{\mathbf{e}}_r \, f$$

$$\Gamma_\alpha = -\frac{T/m_\alpha}{\Omega_{0\alpha}} \frac{1}{R} \int \frac{d\theta}{2\pi} \sin\theta \int d^3v \, \left(H_\alpha \, g'_{\alpha 1} + \frac{Z_\alpha}{\varepsilon \sin\theta} \frac{\partial \widetilde{\phi}_1}{\partial \theta} \, g_{\alpha 1} \right)$$

Here will be replaced the solution $g'_{\alpha 1}$ obtained with P and δ below.

6.9 Comments on the contributions

The density gradients enter in the equations through the combination

$$u + \frac{T}{Z_\alpha e B_\theta} \frac{d}{dr} \ln n_\alpha$$

The velocity-deffect u will be determined from the *condition of ambipolarity.*

Then in the expression of the flux there will be a combination like

$$\frac{1}{Z_\alpha} \frac{d}{dr} \ln n_\alpha - \frac{1}{Z_\beta} \frac{d}{dr} \ln n_\beta$$

In other treatments this combination occurs due to conservation of momentum in the Coulomb collisions.

6.10 Solution

Solution of the equation

$$v'_\parallel \frac{B_\theta}{B_T} \frac{\partial g'_{\alpha 1}}{r \partial \theta} + \nu g'_{\alpha 1} = A \left(v'_\parallel, v_\perp^2\right) f_0 \sin\theta$$

is

$$g'_{\alpha 1} = \frac{r B_\varphi}{B_\theta} A \, f_0 \left[-P \left(\frac{1}{v'_\parallel} \right) \cos\theta + \pi \delta \left(v'_\parallel\right) \sin\theta \right]$$

The principal part to be taken at the integration over the variable $v_\parallel$.

Similar to **Galeev** and with **Rozhansky Tendler**.
Note that this is *drift kinetic* and is NOT an instability kinetic equation.
See **Rutherford1970**.

What shows the formal solution.
It contains a $\delta\left(v'_{\|}\right)$.

Therefore the transport comes exclusively from the particles that correspond to the resonance condition

$$v'_{\|} = v_{\|} - U = 0$$

which is selected by the δ function.

Using this expression for the distribution function $g'_{\alpha 1}$ one can calculate the fluxes of transport.

First for every species one introduces

$$z_\alpha \equiv \frac{U^2}{2T/m_\alpha} - \frac{1}{2} Z_\alpha \, W$$

This variable is a measure of the magnitude of the energy in the parallel flow U compared with the *thermal* energy.

It is the two middle terms in H_α (the last in H_α is the shear of the parallel flow, and the first is the relative magnitude of the energy in the moving frame to the thermal energy).

Then, with

$$\begin{aligned} a_\alpha &= 1 + 2z_\alpha + 2z_\alpha^2 \\ b_\alpha &= \frac{3}{2} + z_\alpha - z_\alpha^2 \\ c_\alpha &= 3 + 4z_\alpha + 2z_\alpha^2 \end{aligned}$$

and

$$v_{th,\alpha} = \sqrt{\frac{2T}{m_\alpha}}$$

The transport fluxes are

$$\Gamma_\alpha = -n_\alpha \frac{T/m_\alpha}{\Omega_{0\alpha}} \frac{1}{R} \varepsilon \frac{1}{v_{th,\alpha}} \sqrt{\pi} \left[a_\alpha \left(u + \frac{T/m_\alpha}{\Omega_{\theta\alpha}} \frac{d}{dr} \ln n_\alpha \right) + b_\alpha \frac{T/m_\alpha}{\Omega_{\theta\alpha}} \frac{d}{dr} \ln T \right]$$

$$\frac{\widetilde{Q}}{T} =$$

$$\Pi = \sum_\alpha m_\alpha \Gamma_\alpha U$$

We note that in these expressions occurs u which is the departure from the resonance and which is NOT yet determined. Only ambipolarity will determine it.

Particular case, only ions Z_i , m_i.

$$\Gamma_i = 0 \quad \text{the ambipolarity}$$

(neglect of electron fluxes).

From Γ_i

$$\left[a_\alpha \left(u + \frac{T/m_\alpha}{\Omega_{\theta\alpha}} \frac{d}{dr} \ln n_\alpha\right) + b_\alpha \frac{T/m_\alpha}{\Omega_{\theta\alpha}} \frac{d}{dr} \ln T\right] = 0$$

$$U + \frac{1}{B_\theta} \frac{d\phi}{dr} + \frac{T/m_i}{\Omega_{\theta i}} \left(\frac{d}{dr} \ln n_i + K \frac{d}{dr} \ln T\right) = 0$$

where

$$K \equiv \frac{b_i}{a_i}$$

For no parallel flow $U = 0$

$$K \rightarrow \frac{3}{2}$$

then the ambipolarity which is simply $\Gamma_{ion} = 0$ gives for the toroidal flow velocity of the ions

$$U_{ion} = -\frac{1}{B_\theta} \frac{d\phi}{dr} - \frac{T/m_i}{\Omega_{\theta i}} \left(\frac{d}{dr} \ln n_i + \frac{3}{2} \frac{d}{dr} \ln T\right)$$

When however U is known (and it fulfills the ambipolarity constraint) it can be taken as input in the previous formula to calculate the electrostatic potential ϕ_0.

Then with $\frac{d\phi_0}{dr}$ replaced in the expression of $\Gamma_{electrons}$ one has

$$\begin{aligned}
&\Gamma^{ambipolar} \\
&= -\frac{\sqrt{\pi}}{4} \left(1 + \frac{1}{Z_i}\right) q^2 \omega_{transit}^{elect} \rho_e^2 \\
&\quad \times \left[\left(1 + \frac{2 z_{ion}}{Z_i}\right) \frac{d}{dr} \ln n_e + \left(\frac{3}{2} + \frac{3 - Z_i}{1 + Z_i} \frac{z_{ion}}{Z_i}\right) n_e \frac{d}{dr} \ln T\right]
\end{aligned}$$

$$\begin{aligned}
z_{ion} &= \frac{1}{1 + Z_i} \frac{V^2}{2T/m_i} \\
\omega_{transit}^{elect} &= \frac{B_\theta}{B_T} \frac{\overline{v}_e}{r}
\end{aligned}$$

It is also obtained the perturbation of the potential, with harmonic components $\sin\theta$ and $\cos\theta$,

$$\begin{aligned}\widetilde{\phi}_1 &= \frac{\pi\varepsilon}{1+Z_i}\frac{T/m_i}{\Omega_{\theta i}}\left[\frac{1}{\sqrt{\pi}}\left(\frac{1}{2}-z_i-(1+2z_i)K\right)\frac{d}{dr}\ln T\,\frac{1}{v_{th,i}}\times\sin\theta\right.\\ &\qquad\left.+\frac{U}{T/m_i}\left(3\frac{d}{dr}\ln T-2\frac{d}{dr}\ln U\right)\times\cos\theta\right]\end{aligned}$$

The $\sin\theta$ part is most important.

Then $\widetilde{\phi}_1 \sim \sin\theta$.

Important case.

The presence of a single impurity species (I) beside the background ions and electrons.

Negligible terms

$$\frac{U^2}{2T/m_e} \ll 1$$

$$\frac{U^2}{2T/m_i} \ll 1$$

Define

$$\gamma \equiv \sqrt{\frac{m_I}{m_i}}\frac{n_I}{n_i}$$

$$\alpha_I \equiv \frac{Z_I^2 n_I}{n_e + Z_i^2 n_i}$$

$$y \equiv \frac{U^2}{2T/m_I}$$

The variables become

$$z_i = -\frac{Z_i}{Z_I}\frac{\alpha_I}{1+\alpha_I}y$$

$$z_I = \frac{1}{1+\alpha_I}y$$

The ambipolar condition

$$Z_i\Gamma_i + Z_I\Gamma_I = 0$$

becomes

$$\begin{aligned} & U + \frac{1}{B_\theta}\frac{d\phi}{dr} \qquad \text{(should be 0 at resonance)}\\ & +\frac{T/m_i}{\Omega_{\theta i}}\,\frac{1}{a_i+\gamma a_I}\left[\frac{a_i}{Z_i}\frac{d}{dr}\ln n_i + \gamma\frac{a_I}{Z_I}\frac{d}{dr}\ln n_I + \left(\frac{b_i}{Z_i}+\gamma\frac{b_I}{Z_I}\right)\frac{d}{dr}\ln T\right]\\ = & \; 0\end{aligned}$$

This constraint allows to determine the velocity u.

The flux is then

$$\begin{aligned} Z_I \Gamma_I &= -\frac{\sqrt{\pi}}{4} Z_i^2 n_i q^2 \ \omega_{ti} \ \rho_i^2 \\ &\times \left[F_1 \left(\frac{1}{Z_I} \frac{1}{n_I} \frac{dn_i}{dr} - \frac{1}{Z_i} \frac{d}{dr} \ln n_i \right) + F_2 \frac{d}{dr} \ln T \right] \end{aligned}$$

$$F_1 = \gamma \frac{a_I a_i}{a_i + \gamma a_I}$$

$$\begin{aligned} F_2 &= \gamma \frac{a_I a_i}{a_i + \gamma a_I} \\ &\times \left(\frac{1}{Z_I} \frac{b_I}{a_I} - \frac{1}{Z_i} \frac{b_i}{a_I} \right) \end{aligned}$$

(to be checked).

F_1 is positive, F_2 is negative.

Assume that the impurities are very rare, just a trace.

Then $\frac{n_I}{n_i} \ll 1$ and is neglected.

The radial flux of the impurity atoms is

$$\begin{aligned} \Gamma_I &= -\frac{\sqrt{\pi}}{4} q^2 \omega_{tI} \ \rho_I^2 \ \times a_I \times \left[\left(\frac{d}{dr} \ln n_I - \frac{Z_I n_I}{Z_i n_i} \frac{d}{dr} \ln n_i \right) \right. \\ &\qquad \left. - \left(\frac{3}{2} \frac{Z_I}{Z_i} - \frac{b_I}{a_I} \right) n_I \frac{d}{dr} \ln T \right] \end{aligned}$$

Here one observes the <u>*inward*</u> *convection of impurities due to the gradient of temperature.*

$$\Gamma_I \sim \frac{\sqrt{\pi}}{4} q^2 \omega_{tI} \ \rho_I^2 \ \times a_I \times \left[\left(\frac{3}{2} \frac{Z_I}{Z_i} - \frac{b_I}{a_I} \right) n_I \frac{d}{dr} \ln T \right]$$

Both the main ion density n_i gradient and the temperature gradient $\frac{d}{dr} \ln T$ correspond to *inward* convection

$$\begin{aligned} F_1 &> 0 \\ F_2 &< 0 \end{aligned}$$

Possibly this is what is called *temperature screening* for impurities.

We need a physical picture of how the ambipolarity (determination of u) leads to the final signs in Γ_I.

7 Drift kinetic equation in a rotating plasma (Hinton Wong)

This is partly in *plasma general, rotation*.

See **Wong Burrell**.
The approach has been adopted by **Fulop Helander**.
in the *rotating frame* there is a new velocity and a new set of spatial variables.

$$\mathbf{v}' = \mathbf{v} - \mathbf{u}_0 (\mathbf{x}, t)$$

and the kinetic equation is

$$\begin{aligned} & \frac{\partial f}{\partial t} + (\mathbf{v}' + \mathbf{u}_0) \cdot \boldsymbol{\nabla}' f + \frac{e}{m} [\mathbf{E} + (\mathbf{v}' + \mathbf{u}_0) \times \mathbf{B}] \cdot \frac{\partial f}{\partial \mathbf{v}'} \\ & - \left\{ \frac{\partial \mathbf{u}_0}{\partial t} + [(\mathbf{v}' + \mathbf{u}_0) \cdot \boldsymbol{\nabla}] \, \mathbf{u}_0 \right\} \frac{\partial f}{\partial \mathbf{v}'} \\ = \; & C + S \end{aligned}$$

Here

$$\mathbf{u}_0 = \omega R \widehat{\mathbf{e}}_\varphi + F\mathbf{B}$$

the rotation ω is toroidal and includes a term which is the rotation of a surface as a rigid object, $F\mathbf{B}$.

There is an expansion on multiple time scales

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} + ...$$

where the terms have order

$$\frac{v_{th,i}}{L} \;, \; \varepsilon \frac{v_{th,i}}{L} \;, \; \varepsilon^2 \frac{v_{th,i}}{L} \;, ...$$

About the electric field.
It is expanded

$$\mathbf{E} = \mathbf{E}_{-1} + \mathbf{E}_0 + E_1 + ...$$

where the order -1 occurs because it is assumed that an electric field may be of the order of the *thermal speed*. It then is outside the neoclassical expansion.

$$\omega = -\frac{\partial \Phi_{-1}}{\partial \psi}$$

The potentials are

$$\begin{aligned}
\mathbf{E}_{-1} &= -\boldsymbol{\nabla}\Phi_{-1} \\
\mathbf{E}_{0} &= -\boldsymbol{\nabla}\Phi_{0} \\
&\ldots
\end{aligned}$$

This is the frequency of rotation, $\omega \sim v/R$ where $v \sim E/B_\theta$ (since v is toroidal, the magnetic field is the poloidal component) and $E \equiv$ radial, $E \sim -\partial\Phi/\partial r = (-\partial\Phi/\partial\psi) \times (\partial\psi/\partial r)$. We have

$$|\boldsymbol{\nabla}\psi| = RB_\theta$$

then

$$\begin{aligned}
E &\sim -\frac{\partial\Phi}{\partial\psi}|\boldsymbol{\nabla}\psi| = -\frac{\partial\Phi}{\partial\psi}RB_\theta \\
v &\sim \frac{E}{B_\theta} = -\frac{\partial\Phi}{\partial\psi}R \;\; \text{(toroidal)}
\end{aligned}$$

from where

$$\omega \sim \frac{v}{R} = -\frac{\partial\Phi}{\partial\psi}$$

We **note** that the rotation is considered supported by an electrostatic potential Φ which occurs in order -1. This electric potential is constant on surfaces. It leads to a *radial* electric field. Higher orders correspond to variations on surfaces.

$$n\mathbf{u}_0 = \int d^3v\ \mathbf{v}f$$

and

$$\mathbf{u}_0 \cdot \boldsymbol{\nabla}\psi = 0 \quad \text{(in surface)}$$

$$\begin{aligned}
\mathbf{v}' &= \widehat{\mathbf{n}}v_\parallel + \mathbf{v}_\perp \\
&= \widehat{\mathbf{n}}v_\parallel + v_\perp\left(\widehat{\mathbf{e}}_1\cos\zeta + \widehat{\mathbf{e}}_2\sin\zeta\right) \\
\zeta &\equiv \text{gyroangle}
\end{aligned}$$

which implies the transformations

$$\frac{\partial f}{\partial \mathbf{v}'} = \widehat{\mathbf{n}}\frac{\partial f}{\partial v_\parallel} + \frac{\mathbf{v}_\perp}{v_\perp}\frac{\partial f}{\partial v_\perp} + \frac{\widehat{\mathbf{n}}\times\mathbf{v}_\perp}{v_\perp^2}\frac{\partial f}{\partial\zeta}$$

and

$$\begin{aligned} \boldsymbol{\nabla}' f &= \boldsymbol{\nabla} f + (\boldsymbol{\nabla} \cdot \mathbf{b}) \cdot \mathbf{v}_\perp \left(\frac{\partial f}{\partial v_\parallel} - \frac{v_\parallel}{v_\perp} \frac{\partial f}{\partial v_\perp} \right) \\ & \qquad + [(\boldsymbol{\nabla} \cdot \widehat{\mathbf{e}}_1) \cos\zeta + (\boldsymbol{\nabla} \cdot \widehat{\mathbf{e}}_2) \sin\zeta] \cdot \frac{\mathbf{v}'}{v_\perp} \frac{\partial f}{\partial \zeta} \end{aligned}$$

The invariants, redefined in the rotating frame

$$\begin{aligned} \mu &= \frac{v_\perp^2}{2B} \\ \epsilon &= \frac{1}{2}\left(v_\parallel^2 + v_\perp^2\right) + \frac{e\widetilde{\Phi}}{m} - \frac{\omega^2 R^2}{2} \end{aligned}$$

The part of the distribution function that depends on the *gyroangle* ζ is

$$\begin{aligned} \widetilde{f}_1 &= \frac{\mathbf{v}' \times \widehat{\mathbf{n}}}{\Omega} \cdot \boldsymbol{\nabla}\psi \left[\frac{1}{N}\frac{\partial N}{\partial \psi} + \frac{e}{T}\frac{\partial}{\partial \psi} \langle \Phi_0 \rangle \right. \\ & \left. + \left(\frac{m\varepsilon}{T} - \frac{3}{2} \right) \frac{1}{T}\frac{\partial T}{\partial \psi} + \frac{m}{T}\left(\frac{I v_\parallel}{B} + \omega R^2 \right) \frac{\partial \omega}{\partial \psi} \right] \\ & + \frac{\mathbf{v}'\ \mathbf{v}' : (\widehat{\mathbf{e}}_1\widehat{\mathbf{e}}_1 - \widehat{\mathbf{e}}_2\widehat{\mathbf{e}}_2) \left|\boldsymbol{\nabla}\psi\right|^2}{2\Omega B v_{th,i}^2} \frac{\partial \omega}{\partial \psi} f_0 \end{aligned}$$

With this "separation" gyroaverage + gyroangle dependent, we must return to **Frieman, Rewoldt Tang**, the first days derivation of a drift kinetic equation.

(See *instabilities*).

The detailed form

$$(\mathbf{v}' \times \widehat{\mathbf{n}}) \cdot \boldsymbol{\nabla}\psi = v_\parallel \sin\zeta \ \ |\boldsymbol{\nabla}\psi|$$

$$\mathbf{v}'\ \mathbf{v}' : (\widehat{\mathbf{e}}_1\widehat{\mathbf{e}}_1 - \widehat{\mathbf{e}}_2\widehat{\mathbf{e}}_2) = v_\perp^2 \cos(2\zeta)$$

The drift velocity is

$$\begin{aligned} \mathbf{v}_D &= \frac{1}{\Omega}\widehat{\mathbf{n}} \times \Big(\mu \boldsymbol{\nabla} B + v_\parallel^2 (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} + \frac{e}{m}\boldsymbol{\nabla}\Phi_0 \\ & \qquad -\omega^2 \mathbf{R} + 2\omega \widehat{\mathbf{e}}_z \times \widehat{\mathbf{n}} v_\parallel \Big) \end{aligned}$$

(remember $\omega = -\frac{\partial \Phi_{-1}}{\partial \psi} \sim \frac{\frac{B}{B_\theta} V_E}{R}$)

The new terms

- the centrifugal force (including the mass factor m)

$$m\omega^2\mathbf{R}$$

The drift resulting from this term is

$$\frac{1}{\Omega}\widehat{\mathbf{n}}\times\left(-\omega^2\right)\mathbf{R} = \text{vertical}$$

because this is a force and the drift Braginskii is $\mathbf{F}\times\mathbf{B}$. Then, it is perpendicular.

- the Coriolis term (including the mass factor m)

$$2mv_{\|}\widehat{\mathbf{n}}\times\omega\widehat{\mathbf{e}}_z$$

The drift resulting from this term is

$$\begin{aligned}
\frac{1}{\Omega}\widehat{\mathbf{n}}\times\left(2v_{\|}\omega\right)\left(\widehat{\mathbf{n}}\times\widehat{\mathbf{e}}_z\right) &= \frac{2v_{\|}\omega}{\Omega}\left[\widehat{\mathbf{n}}\left(\widehat{\mathbf{n}}\cdot\widehat{\mathbf{e}}_z\right)-\widehat{\mathbf{e}}_z\right] \\
&\approx -\frac{2v_{\|}\omega}{\Omega}\widehat{\mathbf{e}}_z \\
&= \text{vertical}
\end{aligned}$$

The physical representation of the Coriolis effect in this context can be obtained from planetary physics: descend of a point from the North Pole toward the equator. But, remember, to see the Coriolis effect we must be in a system of reference that rotates with the planet. Similarly, to see the Coriolis effect in the equations of a rotating plasma, one must choose a referential that moves with the plasma.

Hence the formula above are in the *rotating reference system.*
Comparing the two new terms

$$\frac{\text{centrifugal}}{\text{Coriolis}}\sim\frac{\omega^2 R}{2v_{\|}\omega}\sim\frac{1}{2}\frac{\omega}{v_{\|}/R}$$

where $v_{\|}$ = velocity of the particle. The ratio can be put in connection with the definition of a Mach number which compares the speed of rotation with the speed of sound.

the formulas

$$\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\psi = I\widehat{\mathbf{n}}-BR\widehat{\mathbf{e}}_{\varphi}$$

$$\begin{aligned}\widehat{\mathbf{e}}_{\varphi}R\cdot\ (\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla})\,\widehat{\mathbf{n}} &= I\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\left(\frac{1}{B}\right)\\ &\sim (\text{toroidal, } R)\cdot\text{curvature}\sim I\nabla_{\|}\left(\frac{1}{B}\right)\end{aligned}$$

In circular surfaces it is

$$I\left(-\frac{1}{B^2}\right)\nabla_{\|}B = -B_{\varphi}R\ \frac{1}{B}\nabla_{\|}\ln B$$

then

$$\begin{aligned}\boldsymbol{\kappa}_{\varphi} &\equiv \boldsymbol{\kappa}\cdot\widehat{\mathbf{e}}_{\varphi}\\ &= -\left(\frac{B_{\varphi}}{B}\right)\nabla_{\|}\ln B\end{aligned}$$

Since the curvature is directed mostly toward the major symmetry axis $\sim -\widehat{\mathbf{e}}_R$, this quantity $\boldsymbol{\kappa}\cdot\widehat{\mathbf{e}}_{\varphi}$ should be very small.

$$\begin{aligned}\widehat{\mathbf{e}}_{\varphi}R\cdot(\text{Coriolis}) &=\\ \widehat{\mathbf{e}}_{\varphi}R\cdot\omega\widehat{\mathbf{e}}_z\times\widehat{\mathbf{n}} &= \omega\widehat{\mathbf{n}}\cdot\mathbf{R}\end{aligned}$$

lead to the *radial* projection of the drift velocity

$$\mathbf{v}_D\cdot\boldsymbol{\nabla}\psi = \frac{m}{e}v_{\|}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\left(\frac{Iv_{\|}}{B}+\omega R^2\right)$$

where

$$\boldsymbol{\nabla}\equiv\boldsymbol{\nabla}_{\epsilon=ct,\mu=ct}$$

and

$$v_{\|} = \left\{2\left[\epsilon-\mu B-\frac{e}{m}\widetilde{\Phi}_0+\frac{\omega^2R^2}{2}\right]\right\}^{1/2}$$

Identities that involve the versors

$$-\widehat{\mathbf{e}}_2\widehat{\mathbf{e}}_2:(\boldsymbol{\nabla}\ \widehat{\mathbf{n}}) = \widehat{\mathbf{e}}_1\widehat{\mathbf{e}}_1:(\boldsymbol{\nabla}\ \widehat{\mathbf{n}})+\frac{\nabla_{\|}B}{B}$$

which must be rewritten for clarity

$$\begin{aligned}\widehat{\mathbf{e}}_2\ \cdot[(\widehat{\mathbf{e}}_2\cdot\boldsymbol{\nabla})\,\widehat{\mathbf{n}}]+\widehat{\mathbf{e}}_1\ \cdot[(\widehat{\mathbf{e}}_1\cdot\boldsymbol{\nabla})\,\widehat{\mathbf{n}}] &= -\widehat{\mathbf{n}}\cdot[(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla})\,\widehat{\mathbf{n}}]+\boldsymbol{\nabla}\cdot\widehat{\mathbf{n}}\\ &= 0 \qquad\qquad\qquad\quad -\nabla_{\|}\ln B\end{aligned}$$

since the first term is $\widehat{\mathbf{n}}\cdot\boldsymbol{\kappa}=0$.

And

$$\begin{aligned}
\widehat{\mathbf{e}}_1 \cdot (\boldsymbol{\nabla}\ \widehat{\mathbf{n}}) \cdot \widehat{\mathbf{e}}_1 &= \widehat{\mathbf{e}}_2 \cdot (\boldsymbol{\nabla}\times\widehat{\mathbf{e}}_1) \\
&= (\widehat{\mathbf{e}}_2 \cdot \widehat{\mathbf{e}}_\varphi)\, \widehat{\mathbf{e}}_\varphi \cdot (\boldsymbol{\nabla}\times\widehat{\mathbf{e}}_1)
\end{aligned}$$

written as

$$[(\widehat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}] \cdot \widehat{\mathbf{e}}_1 = \widehat{\mathbf{e}}_2 \cdot (\boldsymbol{\nabla}\times\widehat{\mathbf{e}}_1)$$

Other formulas

$$\widehat{\mathbf{e}}_\varphi \cdot (\boldsymbol{\nabla}\times\widehat{\mathbf{e}}_1) = \frac{R\mathbf{B}\cdot\boldsymbol{\nabla}\,|\boldsymbol{\nabla}\psi|}{|\boldsymbol{\nabla}\psi|^2}$$

$$\widehat{\mathbf{e}}_2 \cdot \widehat{\mathbf{e}}_\varphi = -\frac{|\boldsymbol{\nabla}\psi|}{BR}$$

In circular surfaces

$$\begin{aligned}
|\boldsymbol{\nabla}\psi| &= RB_\theta \\
\widehat{\mathbf{e}}_2 \cdot \widehat{\mathbf{e}}_\varphi &= -\frac{RB_\theta}{RB} = -\frac{B_\theta}{B} \ll 1
\end{aligned}$$

And

$$\begin{aligned}
(\widehat{\mathbf{e}}_1\widehat{\mathbf{e}}_1 - \widehat{\mathbf{e}}_2\widehat{\mathbf{e}}_2) &: (\boldsymbol{\nabla}\widehat{\mathbf{n}}) \\
&= -\frac{B}{|\boldsymbol{\nabla}\psi|^2}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\left(\frac{|\boldsymbol{\nabla}\psi|^2}{B}\right)
\end{aligned}$$

The operation of averaging over the gyrophase ζ.

The linearized drift kinetic equation for $\overline{f}_1$ the averaged function of distribution in order 1, dependent on

$$(\mathbf{x}, t, \epsilon, \mu, \sigma = \pm 1)$$

$$\begin{aligned}
& v_\parallel \nabla_\parallel \overline{f}_1 - C^{lin}\overline{f}_1 \\
= & -\frac{e}{T} v_\parallel \nabla_\parallel \Phi_1\ f_0 \\
& -v_\parallel \left(\nabla_\parallel \alpha_1\right) \left[\frac{1}{N}\frac{\partial N}{\partial \psi} + \frac{e}{T}\frac{\partial \langle \Phi_0 \rangle}{\partial \psi} + \frac{1}{T}\frac{\partial T}{\partial \psi}\right] f_0 \\
& -v_\parallel \left(\nabla_\parallel \alpha_2\right) \left[\frac{1}{T}\frac{\partial T}{\partial \psi}\right] f_0 \\
& -v_\parallel \left(\nabla_\parallel \alpha_3\right) \left[\frac{1}{\omega}\frac{\partial \omega}{\partial \psi}\right] f_0
\end{aligned}$$

where

$$\alpha_1 = \frac{m}{e}\left(\frac{Iv_\parallel}{B} + \omega R^2\right)$$

$$\alpha_2 = \left(\frac{m\epsilon}{T} - \frac{5}{2}\right)\alpha_1$$

$$\alpha_3 = \frac{m\omega}{ev_{th,i}^2}\left[\left(\frac{Iv_\parallel}{B} + \omega R^2\right)^2 + \mu\frac{|\boldsymbol{\nabla}\psi|^2}{B}\right]$$

with

$$I = RB_\varphi$$

It is introduced the set of *thermodynamical forces*

$$\begin{aligned}
A_1 &= \frac{1}{N}\frac{\partial N}{\partial \psi} + \frac{e}{T}\frac{\partial\langle\Phi_0\rangle}{\partial\psi} + \frac{1}{T}\frac{\partial T}{\partial \psi} \\
A_2 &= \frac{1}{T}\frac{\partial T}{\partial \psi} \\
A_3 &= \frac{1}{\omega}\frac{\partial \omega}{\partial \psi}
\end{aligned}$$

we note that the operator

$$v_\parallel \nabla_\parallel$$

occurs in both sides of the drift kinetic equation and this suggests a substitution

$$\begin{aligned}
\overline{f}_1 &= f - \frac{e\Phi_1}{T}f_0 \\
&\quad - \left[\left(\frac{Iv_\parallel}{B} + \omega R^2\right)A_1\right. \\
&\quad + \omega R^2\left(\frac{m\epsilon}{T} - \frac{5}{2}\right)A_2 \\
&\quad \left. + \frac{\omega}{v_{th,i}^2}\left(\frac{2Iv_\parallel}{B}\omega R^2 + \omega^2 R^4\right)A_3\right]f_0
\end{aligned}$$

The term depending on the electric potential Φ_1 is the *adiabatic* response.

Hinton Wong make the observation that the linearized ion-ion collision operator anihilates terms like

$$\left(a + bv_\parallel + cv^2\right)f_0$$

which means

$$C^{lin}\left[\left(a+bv_{\|}+cv^2\right)f_0\right]=0$$

(due to the conservation in collisions of - respectively, number, momentum and energy).

The drift kinetic equation becomes

$$\begin{aligned}
& v_{\|}\nabla_{\|}f-C^{lin}f \\
= & -v_{\|}\left(\nabla_{\|}\beta_2\right)A_2\ f_0 \\
& -v_{\|}\left(\nabla_{\|}\beta_3\right)A_3\ f_0
\end{aligned}$$

where

$$\begin{aligned}
\beta_2 &= \frac{Iv_{\|}}{\Omega_i}\frac{m_i\epsilon}{T}-\frac{5}{2} \\
\beta_3 &= \frac{\omega}{\Omega_i v_{th,i}^2}\left(\frac{I^2v_{\|}^2}{B}+\mu\left|\boldsymbol{\nabla}\psi\right|^2\right)
\end{aligned}$$

8 Drift-Boltzmann equation in Stix

The **Drift-Boltzmann equation** in **Stix** is

$$\begin{aligned}
& \left\{\frac{\partial}{\partial t}+\right. \\
& +\left[\widehat{\mathbf{n}}v_{\|}+\frac{1}{\Omega}\widehat{\mathbf{n}}\times\left(\frac{1}{m}\mu\boldsymbol{\nabla}B+v_{\|}^2\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}+\frac{q}{m}\boldsymbol{\nabla}\phi\right)\right]\times \\
& \qquad\times\left.\left[\boldsymbol{\nabla}+\frac{\mu\boldsymbol{\nabla}B}{m}\frac{\partial}{\partial\left(v_{\perp}^2/2\right)}-\left(\frac{\mu\boldsymbol{\nabla}B+q\boldsymbol{\nabla}\phi}{m}\right)\frac{\partial}{\partial\left(v_{\|}^2/2\right)}\right]\right\}f \\
= & \ -\nu\left(f-f_0\right)
\end{aligned}$$

This is due to the change of variables, as follows.

In the equation, the partial derivatives are performed *holding five of the six variables* $(\epsilon,\mu,\mathbf{r},t)$ constant. The velocities are

$$\mathbf{v}_D=\widehat{\mathbf{n}}v_{\|}+\frac{1}{\Omega}\widehat{\mathbf{n}}\times\left[\frac{1}{m}\mu\boldsymbol{\nabla}B+v_{\|}^2\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}+\frac{q}{m}\boldsymbol{\nabla}\phi\right]$$

where

$$v_{\|}=\left[\frac{2}{m}\left(\epsilon-\mu B-q\phi\right)\right]^{1/2}$$

Changing to the system of coordinates

$$(\epsilon, \mu, \mathbf{r}, t) \rightarrow \left(\frac{v_\perp^2}{2}, v_\parallel, \mathbf{r}, t \right)$$

the operators becomes

$$\left. \frac{\partial}{\partial t} \right|_{\epsilon,\mu,\mathbf{r}} = \left. \frac{\partial}{\partial t} \right|_{v_\perp, v_\parallel, \mathbf{r}} - \frac{q}{m} \frac{\partial \phi}{\partial t} \left. \frac{\partial}{\partial \left(v_\parallel^2 / 2 \right)} \right|_{v_\perp, v_\parallel, \mathbf{r}}$$

and

$$\begin{aligned} \left. \boldsymbol{\nabla} \right|_{\epsilon,\mu,t} &= \left. \boldsymbol{\nabla} \right|_{v_\perp, v_\parallel, t} + \\ & + \frac{\mu \boldsymbol{\nabla} B}{m} \left. \frac{\partial}{\partial \left(v_\perp^2 / 2 \right)} \right|_{v_\perp, v_\parallel, t} - \left(\frac{\mu \boldsymbol{\nabla} B + q \boldsymbol{\nabla} \phi}{m} \right) \left. \frac{\partial}{\partial \left(v_\parallel^2 / 2 \right)} \right|_{v_\perp, v_\parallel, t} \end{aligned}$$

and

$$q \frac{\partial \phi}{\partial t} \left. \frac{\partial}{\partial \epsilon} \right|_{\mu, \mathbf{r}, t} = \frac{q}{m} \frac{\partial \phi}{\partial t} \left. \frac{\partial}{\partial \left(v_\parallel^2 / 2 \right)} \right|_{v_\perp, v_\parallel, t}$$

NOTE
The change of variables

$$(\mathbf{x}, t; \epsilon, \mu) \rightarrow \left(\mathbf{x}, t; \frac{v_\perp^2}{2}, v_\parallel, \sigma \right)$$

has a version

$$(\mathbf{x}, t; \epsilon, \mu) \rightarrow (\mathbf{x}, t; v, \xi)$$

where

$$\xi = \frac{v_\parallel}{v}$$

In **Sugama Nishimura** it is explicitely shown the difference.

In **Galeev Sagdeev Liu Novakovskii** the drift kinetic equation keeps $\left(\frac{dv_\parallel}{dt} \right) \frac{\partial f}{\partial v_\parallel}$.

In **Novakovskii Liu Sagdeev Rosenbluth** the two energy terms are retained, variations due to the changes in parallel velocity and in squared velocity (*rotation*).

8.1 Change of variables in preparation for the drift-kinetic equation Catto Tsang

8.1.1 Time dependence of the parameters of a particle's orbit

In the paper **linearized gyro-kinetic equation with collisions, Catto Tsang**.

The distribution function is composed of two parts:

- Maxwellian F_M,
- a part due to the diamagnetic flow, F_{dia}.

Then

$$F = F_M + F_{dia}$$

where

$$\begin{aligned} F_M &= F_M(r,\epsilon) \\ &= N(r)\frac{1}{[2\pi T(r)/m]^{3/2}}\exp\left(-\frac{\epsilon}{T/m}\right) \end{aligned}$$

$$\begin{aligned} F_{dia} &= F_{dia}(r,\theta,\mu,\epsilon,\zeta) \\ &= -\frac{1}{\Omega}\widehat{\mathbf{n}}\times\mathbf{v}\cdot\boldsymbol{\nabla}F_M \end{aligned}$$

the linearized equation is

$$\begin{aligned} & \frac{\partial f}{\partial t}+\left(\mathbf{v}\cdot\nabla-\Omega\frac{\partial}{\partial\zeta}\right)\left(f+\frac{Z|e|}{T}F_M\phi\right) \\ & -Z|e|v_{\perp}\phi\ \cos\zeta\ \frac{\partial}{\partial r}\left(\frac{F_M}{T}\right) \\ & +\frac{Z|e|(-\boldsymbol{\nabla}\phi)}{T}\cdot\boldsymbol{\nabla}_{\mathbf{v}}F_{dia} \\ & +\dot{\mu}\frac{\partial f}{\partial\mu}+\dot{\zeta}\frac{\partial f}{\partial\zeta} \\ = \ & C(f) \end{aligned}$$

where

$$\frac{d\mu}{dt}=\frac{\partial\mu}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})\mu$$

$$
\begin{aligned}
\dot{\mu} &= -\mu \mathbf{v} \cdot \boldsymbol{\nabla} \ln B \\
&\quad -\frac{v_\perp v_\parallel}{B} \left\{ v_\parallel \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} \cdot \mathbf{v}_\perp \right. \\
&\qquad\qquad -v_\perp \left[\left(\widehat{\mathbf{e}}_r \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{e}}_r \ \cdot \widehat{\mathbf{n}} \ \cos^2 \zeta + \right. \\
&\qquad\qquad\quad + \left(\widehat{\mathbf{e}}_\theta \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{e}}_\theta \cdot \widehat{\mathbf{n}} \ \sin^2 \zeta + \\
&\qquad\qquad\quad \left. \left. + \left(\left(\widehat{\mathbf{e}}_\theta \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{e}}_r \cdot \widehat{\mathbf{n}} + \left(\widehat{\mathbf{e}}_r \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{e}}_\theta \cdot \widehat{\mathbf{n}} \right) \ \sin \zeta \cos \zeta \right] \right\}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\zeta} &= v_\parallel \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{e}}_r \cdot \widehat{\mathbf{e}}_r \\
&\quad -\frac{v_\parallel^2}{v_\perp} \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} \cdot \widehat{\mathbf{e}}_\zeta \\
&\quad +v_\parallel \left[\left(\widehat{\mathbf{e}}_r \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{e}}_\theta \cdot \widehat{\mathbf{n}} \ \cos^2 \zeta \right. \\
&\qquad - \left(\widehat{\mathbf{e}}_\theta \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{e}}_r \cdot \widehat{\mathbf{n}} \ \sin^2 \zeta \\
&\qquad \left. + \left(\left(\widehat{\mathbf{e}}_\theta \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{e}}_\theta \cdot \widehat{\mathbf{n}} - \left(\widehat{\mathbf{e}}_r \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{e}}_r \cdot \widehat{\mathbf{n}} \right) \ \sin \zeta \cos \zeta \right] \\
&\quad -v_\perp \left(\left(\widehat{\mathbf{e}}_r \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{e}}_r \cdot \widehat{\mathbf{e}}_\theta \ \cos \zeta - \left(\widehat{\mathbf{e}}_\theta \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{e}}_\theta \cdot \widehat{\mathbf{e}}_r \ \sin \zeta \right)
\end{aligned}
$$

See above for the change of variables.

8.1.2 The distribution function

The paper **Catto Tsang** introduces the *integration factor* in the developments of the Fokker-Planck equation.

First, as in other cases, one separates in the form of the distribution function a

- fast variation in the poloidal direction; this is retained in a Fourier expansion along the coordinate θ,

$$
\begin{aligned}
f &= \widehat{f} \exp\left(-i\omega t + im\theta - il\varphi\right) \\
\phi &= \widehat{\phi} \exp\left(-i\omega t + im\theta - il\varphi\right)
\end{aligned}
$$

- slow variation of the eigenmode in the poloidal angle θ, which is represented by the dependence of the Fourier coefficients on θ, on a *slower space scale*

$$
\begin{aligned}
\widehat{f} &= \widehat{f}\left(r, \theta, \mathbf{v}\right) \\
\widehat{\phi} &= \widehat{\phi}\left(r, \theta\right)
\end{aligned}
$$

It is assumed that

$$m \gg 1$$

The operator of spatial gradient will also be splitted

$$\boldsymbol{\nabla} = \boldsymbol{\nabla}_\perp + \boldsymbol{\nabla}_s$$

The part that is perpendicular will combine the variations along the

$$\begin{array}{ccc} \text{radius } r & & \widehat{\mathbf{e}}_r \frac{\partial}{\partial r} \\ \text{poloidal, fast} & k_\theta = \frac{m}{r} & \widehat{\mathbf{e}}_\theta \left(ik_\theta \right) \end{array}$$

which is

$$\boldsymbol{\nabla}_\perp = \widehat{\mathbf{e}}_r \frac{\partial}{\partial r} + \widehat{\mathbf{e}}_\theta \left(ik_\theta \right)$$

(usually we note $k = k_\theta = \frac{m}{r}$).

The part that combines the variations along

$$\begin{array}{ccc} \text{poloidal, slow} & & \widehat{\mathbf{e}}_\theta \frac{\partial}{\partial \theta} \\ \text{parallel} & ik_\parallel = i\frac{m-lq}{qR_0} & \widehat{\mathbf{n}} \left(ik_\parallel \right) \end{array}$$

Visibly, this operator acts in the magnetic surface and is connected with the *magnetic shear*.

Then

$$\boldsymbol{\nabla}_s = \widehat{\mathbf{e}}_\theta \frac{\partial}{\partial \theta} + \widehat{\mathbf{n}} \left(ik_\parallel \right)$$

With these we return to the linearized Fokker-Planck equation and note that the part of the second term

$$\left(\mathbf{v} \cdot \nabla - \Omega \frac{\partial}{\partial \zeta} \right) \left(\widehat{f} + \frac{Z\left|e\right|}{T} F_M \widehat{\phi} \right)$$

where the gradient operator is restricted to the perpendicular direction $\boldsymbol{\nabla}_\perp$ is dominant

$$\left(\mathbf{v} \cdot \nabla_\perp - \Omega \frac{\partial}{\partial \zeta} \right) \left(\widehat{f} + \frac{Z\left|e\right|}{T} F_M \widehat{\phi} \right)$$

and this is expanded by specifying the velocity, as

$$\begin{aligned} & \left[\left(\overline{v}_\perp \widehat{\mathbf{e}}_r \ \cos\zeta + v_\perp \widehat{\mathbf{e}}_\theta \ \sin\zeta \right) \cdot \boldsymbol{\nabla}_\perp - \Omega \frac{\partial}{\partial \theta} + i\overline{v}_\perp \ \cos\zeta \frac{\partial L}{\partial r} \right] \\ & \times \left(\widehat{f} + \frac{Z\left|e\right|}{T} F_M \widehat{\phi} \right) \\ = & \ 0 \end{aligned}$$

where

$$L \equiv \frac{k_\theta v_\perp}{\Omega} \ \cos\zeta$$

$$\overline{v}_\perp = \frac{v_\perp}{1 + \sin\zeta \frac{\partial}{\partial r}\left(\frac{v_\perp}{\Omega}\right)}$$

NOTE

We note the combination

$$\overline{v}_\perp \widehat{\mathbf{e}}_r \ \cos\zeta \cdot \boldsymbol{\nabla}_\perp + i\overline{v}_\perp \ \cos\zeta \frac{\partial L}{\partial r}$$

where

$$\widehat{\mathbf{e}}_r \cdot \boldsymbol{\nabla}_\perp = \frac{\partial}{\partial r}$$

and

$$\begin{aligned}
& \frac{v_\perp}{1 + \sin\zeta \frac{\partial}{\partial r}\left(\frac{v_\perp}{\Omega}\right)} \cos\zeta \frac{\partial}{\partial r} + i \frac{v_\perp}{1 + \sin\zeta \frac{\partial}{\partial r}\left(\frac{v_\perp}{\Omega}\right)} \cos\zeta \frac{\partial}{\partial r}\left(\frac{k_\theta v_\perp}{\Omega} \ \cos\zeta\right) \\
= \ & \frac{v_\perp}{1 + \sin\zeta \frac{\partial}{\partial r}\left(\frac{v_\perp}{\Omega}\right)} \cos\zeta \frac{\partial}{\partial r} \\
& + i \frac{v_\perp}{1 + \sin\zeta \frac{\partial}{\partial r}\left(\frac{v_\perp}{\Omega}\right)} k_\theta \cos^2\zeta \frac{\partial}{\partial r}\left(\frac{v_\perp}{\Omega}\right)
\end{aligned}$$

END

The change of variables

$$\begin{array}{ccc}
r & \rightarrow & r' = r + \frac{v_\perp}{\Omega} \sin\zeta \\
\theta & \rightarrow & \theta' = \theta \\
\zeta & \rightarrow & \zeta' = \zeta
\end{array}$$

$$\begin{array}{ccc}
\mu & \rightarrow & \mu' = \mu \\
\epsilon & \rightarrow & \epsilon' = \epsilon
\end{array}$$

We "recognize" the variable

$$x + \frac{v_y}{\Omega}$$

which is one of the invariants usually taken as arguments of the distribution function.

The Fokker Planck equation in the *primmed* variables becomes

$$\frac{\partial}{\partial\zeta'}\left\{\left[\widehat{f} + \frac{Z\,|e|}{T} F_M \widehat{\phi}\right] \exp\left(iL\right)\right\} = 0$$

with the solution

$$\widehat{f} + \frac{Z\left|e\right|}{T} F_M \widehat{\phi} = \widehat{g}' \exp\left(-iL\right)$$

where

$$\begin{aligned} g' &= g\left(r + \frac{v_\perp}{\Omega}\sin\zeta, \theta, \mu, \epsilon\right) \\ & \quad \text{independent of } \zeta' \end{aligned}$$

The expansion of $\widehat{f}$ and $\widehat{\phi}$ in series of terms ordered in powers neoclassical small parameter (ratio of the spatial drift step and the length of the gradients) the Fokker Planck equation is linearized

$$\begin{aligned} & \left[-i\omega + \left(v_\parallel + \frac{1}{\Omega}\mu B \ \widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\times\widehat{\mathbf{n}}\right)\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}_\perp \right. \\ & \left. + \frac{1}{\Omega}\widehat{\mathbf{n}}\times\left(\mu\boldsymbol{\nabla}B + v_\parallel^2\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right)\cdot\boldsymbol{\nabla}_s\right] g' \\ & - \left\langle \exp\left(iL\right) \ C\left\{g'\exp\left(iL\right)\right\}\right\rangle \\ = \ & i\frac{Z\left|e\right|}{T}F_M\left(\omega - \omega_*^T\right)\left\langle\widehat{\phi}\exp\left(iL\right)\right\rangle \end{aligned}$$

(see **Frieman Rewoldt Tang 1978, 1980**)

where

$$\omega_*^T = k_\theta \frac{T/m}{\Omega}\frac{1}{F_M}\frac{\partial F_M}{\partial r}$$

$$\begin{aligned} & \left\langle\widehat{\phi}\exp\left(iL\right)\right\rangle \\ = \ & J_0\left(\frac{k_\theta v_\perp}{\Omega}\right)\left[\widehat{\phi}\left(r\right) + \frac{v_\perp}{\Omega}\sin\zeta\frac{\partial\widehat{\phi}}{\partial r} + \frac{1}{2}\frac{v_\perp^2}{\Omega^2}\sin^2\zeta\frac{\partial^2\widehat{\phi}}{\partial r^2}\right] \\ & + J_1\left(\frac{k_\theta v_\perp}{\Omega}\right) \ \frac{v_\perp}{2k_\theta\Omega}\frac{\partial^2\widehat{\phi}}{\partial r^2} \end{aligned}$$

Note that in **Frieman Rewoldt Tang 1980** this gyrophase averageleads to an argument of the Bessel function which consists of the square root of a sum of two squared coefficients of trigonometric functions. And also involves

$$\frac{\partial}{\partial\psi}S\left(\psi\right) - \frac{dq}{d\psi}\left(\theta - \theta_0\right)$$

9 Linearized gyrokinetics Catto Tsang 1977 and 1978

Take as reference for comparisons **Hazeltine 1973**.

The equations

$$\mathbf{R} = \mathbf{r} + \frac{1}{\Omega}\mathbf{v}\times\widehat{\mathbf{n}}$$

$$\begin{aligned}
\mathbf{v} &= v_{\parallel}\widehat{\mathbf{n}} + v_{\perp}\widehat{\mathbf{e}}_{\perp} \\
&= v_{\parallel}\widehat{\mathbf{n}} + v_{\perp}\left(\widehat{\mathbf{e}}_1 \cos\zeta + \widehat{\mathbf{e}}_2 \sin\zeta\right)
\end{aligned}$$

$$\begin{aligned}
\widehat{\mathbf{e}}_{\perp}^{V} &= \widehat{\mathbf{e}}_1 \cos\zeta + \widehat{\mathbf{e}}_2 \sin\zeta \\
& \quad \text{versor perpendicular on magnetic line}
\end{aligned}$$

and

$$\widehat{\mathbf{e}}_2 = \widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_1$$

$$\begin{aligned}
\left(\widehat{\mathbf{n}}, \widehat{\mathbf{e}}_{\psi}, \widehat{\mathbf{e}}_{\theta}\right) &\equiv \text{orthogonal spatial system} \\
\widehat{\mathbf{e}}_{\theta} &= \widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_{\psi}
\end{aligned}$$

Note these are the old notations

$$\begin{aligned}
\widehat{\mathbf{e}}_1 &\equiv \widehat{\mathbf{e}}_x \\
\widehat{\mathbf{e}}_2 &\equiv \widehat{\mathbf{e}}_y
\end{aligned}$$

with $\widehat{\mathbf{n}}$ they are the trieder. **End.**

Variables and notations

$$\begin{aligned}
\epsilon &= \frac{1}{2}v^2 \\
\mu &= \frac{v_{\perp}^2}{2B} \\
v_{\parallel} &= 2\left(\epsilon - \mu B\right) \\
v_{\perp}^2 &= \left[\left(\mathbf{I} - \widehat{\mathbf{n}}\,\widehat{\mathbf{n}}\right)\cdot\mathbf{v}\right]^2
\end{aligned}$$

Change to the guiding center variables

$$\begin{aligned}
\frac{\partial}{\partial t} &\rightarrow \frac{\partial}{\partial t} \\
\boldsymbol{\nabla} &\rightarrow \boldsymbol{\nabla}_R - \left[\boldsymbol{\nabla}\left(\frac{1}{\Omega}\widehat{\mathbf{n}}\right)\times\mathbf{v}\right]\cdot\boldsymbol{\nabla}_R + \boldsymbol{\nabla}\zeta\frac{\partial}{\partial\zeta} + \boldsymbol{\nabla}\mu\frac{\partial}{\partial\mu} \\
\boldsymbol{\nabla}_{\mathbf{v}} &\rightarrow \boldsymbol{\nabla}_V + \frac{1}{\Omega}\mathbf{I}\times\widehat{\mathbf{n}}\ \cdot\boldsymbol{\nabla}_R
\end{aligned}$$

where

$$\boldsymbol{\nabla}_V = \mathbf{v}\frac{\partial}{\partial \epsilon} + \mathbf{v}_\perp \frac{1}{B}\frac{\partial}{\partial \mu} + \widehat{\mathbf{e}}_\zeta \frac{1}{v_\perp}\frac{\partial}{\partial \zeta}$$

with the definition

$$\widehat{\mathbf{e}}_\zeta = \widehat{\mathbf{n}} \times \widehat{\mathbf{v}}_\perp$$

and

$$\boldsymbol{\nabla}\zeta = \frac{v_\parallel}{v_\perp}\left[-\left(\widehat{\mathbf{e}}_\theta \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\ \cos\zeta + \left(\widehat{\mathbf{e}}_\psi \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\ \sin\zeta\right] + \left(\widehat{\mathbf{e}}_\psi \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{e}}_\theta$$

$$\boldsymbol{\nabla}\mu = -\frac{\mu}{B}\boldsymbol{\nabla}B - \frac{v_\parallel v_\perp}{B}\left[\left(\widehat{\mathbf{e}}_\theta \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\ \cos\zeta + \left(\widehat{\mathbf{e}}_\psi \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\ \sin\zeta\right]$$

The derivation starts from the projections

$$\begin{aligned}
\mathbf{v}\cdot\widehat{\mathbf{e}}_1 &= v_\perp \cos\zeta \\
\mathbf{v}\cdot\widehat{\mathbf{e}}_2 &= v_\perp \sin\zeta
\end{aligned}$$

on which one applies gradients in real space and velocity space.

The distribution function is a sum of

Maxwellian F_M
diamagnetic correction F_D

The second is a correction, *diamagnetic*

$$\begin{aligned}
F_D &= F_D\left(r,\theta,\mu,\epsilon,\zeta\right) \\
&= -\left(\frac{1}{\Omega_c}\widehat{\mathbf{n}} \times \mathbf{v}\right)\cdot \boldsymbol{\nabla}F_M
\end{aligned}$$

The linearized Fokker Planck

$$\begin{aligned}
&\frac{\partial f}{\partial t} + \left(\mathbf{v}\cdot\boldsymbol{\nabla} - \Omega_c \frac{\partial}{\partial \zeta}\right)\left(f + \frac{Ze_a\Phi}{T}F_M\right) \\
&-Ze_a\Phi\ v_\perp \cos\zeta \frac{\partial}{\partial r}\left(\frac{F_M}{T}\right) \\
&-\frac{Ze_a}{m_a}\boldsymbol{\nabla}\Phi \cdot \frac{\partial F_D}{\partial \mathbf{v}} \\
&+\dot{\mu}\frac{\partial f}{\partial \mu} + \dot{\zeta}\frac{\partial f}{\partial \zeta} \\
=\ & C\left(f\right)
\end{aligned}$$

where

$$
\begin{aligned}
& B\dot{\mu} \\
= \; & -\mu \mathbf{v}\cdot\boldsymbol{\nabla}B \\
& -v_\perp v_\parallel \left\{ v_\parallel \left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\widehat{\mathbf{n}}\cdot\widehat{\mathbf{e}}_\perp^V\right) \right. \\
& \qquad -v_\perp \left[\widehat{\mathbf{e}}_1\cdot\boldsymbol{\nabla}\widehat{\mathbf{e}}_1\cdot\widehat{\mathbf{n}}\ \cos^2\zeta \right. \\
& \qquad\qquad +\widehat{\mathbf{e}}_2\cdot\boldsymbol{\nabla}\widehat{\mathbf{e}}_2\cdot\widehat{\mathbf{n}}\ \sin^2\zeta \\
& \qquad\qquad \left.\left. +\left(\widehat{\mathbf{e}}_2\cdot\boldsymbol{\nabla}\widehat{\mathbf{e}}_1\cdot\widehat{\mathbf{n}}+\widehat{\mathbf{e}}_1\cdot\boldsymbol{\nabla}\widehat{\mathbf{e}}_2\cdot\widehat{\mathbf{n}}\right)\sin\zeta\cos\zeta\right]\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{\zeta} \;=\; & v_\parallel\ \widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\widehat{\mathbf{e}}_2\cdot\widehat{\mathbf{e}}_1 \;-\; \frac{v_\parallel^2}{v_\perp}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\widehat{\mathbf{n}}\cdot\widehat{\mathbf{e}}_\zeta \\
& +v_\parallel\left[\widehat{\mathbf{e}}_1\cdot\boldsymbol{\nabla}\widehat{\mathbf{e}}_2\cdot\widehat{\mathbf{n}}\ \cos^2\zeta\right. \\
& \qquad -\widehat{\mathbf{e}}_2\cdot\boldsymbol{\nabla}\widehat{\mathbf{e}}_1\cdot\widehat{\mathbf{n}}\ \sin^2\zeta \\
& \left. +\left(\widehat{\mathbf{e}}_2\cdot\boldsymbol{\nabla}\widehat{\mathbf{e}}_2\cdot\widehat{\mathbf{n}}-\widehat{\mathbf{e}}_1\cdot\boldsymbol{\nabla}\widehat{\mathbf{e}}_1\cdot\widehat{\mathbf{n}}\right)\sin\zeta\cos\zeta\right] \\
& -v_\perp\left(\widehat{\mathbf{e}}_1\cdot\boldsymbol{\nabla}\widehat{\mathbf{e}}_1\cdot\widehat{\mathbf{e}}_2\ \cos\zeta-\widehat{\mathbf{e}}_2\cdot\boldsymbol{\nabla}\widehat{\mathbf{e}}_2\cdot\widehat{\mathbf{e}}_1\ \sin\zeta\right)
\end{aligned}
$$

The Fourier expansion

$$
\begin{aligned}
f &= \widehat{f}\left(r,\theta,\mathbf{v}\right)\exp\left(-i\omega t+im\theta-il\varphi\right) \\
\Phi &= ...
\end{aligned}
$$

This is eqquivalent to extract as factor the *fast* θ variation, in the exponential,

$$m \gg 1$$

and keep the *slow* θ variation in the coefficient.

$$
\begin{aligned}
& \boldsymbol{\nabla}\left[\widehat{Q}\left(r,\theta\right)\ \exp\left(im\theta-il\varphi\right)\right] \\
\approx\; & \exp\left(im\theta-il\varphi\right)\left[\boldsymbol{\nabla}_{slow}+\boldsymbol{\nabla}_{fast}\right]\ \widehat{Q}
\end{aligned}
$$

where

$$
\begin{aligned}
\boldsymbol{\nabla}_{fast} &= \widehat{\mathbf{e}}_r\frac{\partial}{\partial r}+\widehat{\mathbf{e}}_\theta ik \quad \text{where} \quad k=\frac{m}{r} \\
\boldsymbol{\nabla}_{slow} &= \widehat{\mathbf{e}}_\theta\frac{\partial}{r\partial\theta}+\widehat{\mathbf{n}}ik_\parallel \quad \text{where} \quad k_\parallel=\frac{m-lq}{qR_0}
\end{aligned}
$$

The ordering in the drift kinetic equation

Instead of

$$\Omega_c \frac{\partial}{\partial \zeta} \left(\widehat{f} + \frac{Ze_a \widehat{\Phi}}{T} F_M \right) = 0$$

it is considered as dominant the combination

$$\left[\left(\widetilde{v}_\perp \widehat{\mathbf{e}}_1 \cos\zeta + v_\perp \widehat{\mathbf{e}}_2 \sin\zeta \right) \cdot \boldsymbol{\nabla}_{fast} - \Omega_c \frac{\partial}{\partial \zeta} + i \widetilde{v}_\perp \cos\zeta \frac{\partial L}{\partial r} \right] \left(\widehat{f} + \frac{Ze_a \widehat{\Phi}}{T} F_M \right) = 0$$

It has been introduced the function

$$L \equiv \frac{k v_\perp}{\Omega_c} \cos\zeta$$

$$\widetilde{v}_\perp = \frac{v_\perp}{1 + \sin\zeta \frac{\partial}{\partial r} \left(\frac{v_\perp}{\Omega_c} \right)}$$

Catto Tsang use this form by adopting a change of variables

$$\begin{array}{lll} r & \rightarrow & r' = r + \frac{v_\perp}{\Omega_c} \sin\zeta \\ \theta' & \rightarrow & \theta' = \theta \\ \zeta & \rightarrow & \zeta' = \zeta \\ \mu & \rightarrow & \mu' = \mu \\ \epsilon & \rightarrow & \epsilon' = \epsilon \end{array}$$

after which the equation becomes

$$\frac{\partial}{\partial \zeta'} \left[\left(\widehat{f} + \frac{Ze_a \widehat{\Phi}}{T} F_M \right) \exp(iL) \right] = 0$$

or

$$\widehat{f} + \frac{Ze_a \widehat{\Phi}}{T} F_M = \widehat{g}' \exp(-iL)$$

where

$$g' = g \left(r + \frac{v_\perp}{\Omega_c} \sin\zeta \; , \; \theta, \mu, \epsilon \right)$$

The following part is from **1977.**
Now one transforms the Vlasov equation to guiding center variables.
The following relations are noted

$$\frac{Z|e|}{m} \mathbf{v} \times \mathbf{B} \cdot \boldsymbol{\nabla}_{\mathbf{v}} = -\Omega \frac{\partial}{\partial \zeta}$$

$$\Omega \mathbf{v}\times\widehat{\mathbf{n}} \ \cdot \ \left(\frac{1}{\Omega}\mathbf{I}\times\widehat{\mathbf{n}} \ \cdot \ \boldsymbol{\nabla}_R\right) = -\mathbf{v}_\perp \cdot \boldsymbol{\nabla}_R$$

and the Vlasov equation becomes

$$\begin{aligned} & \frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} + \frac{Z\left|e\right|}{m}\mathbf{v}\times\mathbf{B}\cdot\boldsymbol{\nabla}_{\mathbf{v}} \\ \rightarrow \quad & \frac{\partial}{\partial t} - \Omega\frac{\partial}{\partial\zeta} + v_\parallel\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}_R + \mathbf{v}\cdot\left[\boldsymbol{\nabla}\zeta\frac{\partial}{\partial\zeta} + \boldsymbol{\nabla}\mu\frac{\partial}{\partial\mu} - \boldsymbol{\nabla}\left(\frac{1}{\Omega}\widehat{\mathbf{n}}\right)\times\mathbf{v} \ \cdot \ \boldsymbol{\nabla}_R\right] \end{aligned}$$

The gyro-averaging is

$$\frac{1}{2\pi}\oint d\zeta \ \mathbf{v} \ \mathbf{v} = \frac{v_\perp^2}{2}\left(\mathbf{I} - \widehat{\mathbf{n}} \ \widehat{\mathbf{n}}\right) + v_\parallel^2 \ \widehat{\mathbf{n}} \ \widehat{\mathbf{n}}$$

$$\begin{aligned} \mathbf{v}_{drift} &= -\frac{1}{2\pi}\oint d\zeta\left\{\mathbf{v}\cdot\left[\boldsymbol{\nabla}\left(\frac{1}{\Omega}\widehat{\mathbf{n}}\right)\times\mathbf{v}\right] : \left(\mathbf{I} - \widehat{\mathbf{n}} \ \widehat{\mathbf{n}}\right)\right\} \quad \text{(tensorial)} \\ &= \frac{1}{\Omega}\widehat{\mathbf{n}}\times\left[\frac{v_\perp^2}{2B}\boldsymbol{\nabla}B + v_\parallel^2\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right] \end{aligned}$$

$$\begin{aligned} u_\parallel &= -\frac{1}{2\pi}\oint d\zeta\left\{\mathbf{v}\cdot\left[\boldsymbol{\nabla}\left(\frac{1}{\Omega}\widehat{\mathbf{n}}\right)\times\mathbf{v}\right]\cdot\widehat{\mathbf{n}}\right\} \quad \text{(tensorial)} \\ &= -\frac{v_\perp^2}{2\Omega}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\times\widehat{\mathbf{n}} \end{aligned}$$

$$\frac{1}{2\pi}\oint d\zeta \ \mathbf{v}\cdot\boldsymbol{\nabla}\mu \approx 0$$

The result is the gyro-averaged form of the Vlasov operator

$$\begin{aligned} & \frac{1}{2\pi}\oint d\zeta\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} + \frac{Z\left|e\right|}{m}\mathbf{v}\times\mathbf{B}\cdot\boldsymbol{\nabla}_{\mathbf{v}}\right] \\ \rightarrow \quad & \frac{\partial}{\partial t} + \left(v_\parallel\widehat{\mathbf{n}} + \mathbf{v}_{drift}\right)\cdot\boldsymbol{\nabla}_R \end{aligned}$$

For calculations of the responses of plasma to perturbations (instabilities) one uses the unperturbed distribution function

$$\begin{aligned} F_0 &\equiv \text{function of } v^2 \\ & \quad\ \text{and of the canonical angular momentum} \\ P_\varphi &\equiv -Z\left|e\right|\left(\psi - \frac{m}{Z\left|e\right|}R\,\widehat{\mathbf{e}}_\varphi\cdot\mathbf{v}\right) \end{aligned}$$

with $\widehat{\mathbf{e}}_\varphi \equiv$ the toroidal versor.

We introduce the notation

$$\psi_0 \equiv -\frac{P_\varphi}{Z\,|e|} = \psi - \frac{m}{Z\,|e|} R\,\widehat{\mathbf{e}}_\varphi \cdot \mathbf{v}$$

$$\begin{aligned} F_0 &= F_0\left(P_\varphi, v_\perp^2\right) \\ &= N\left(P_\varphi\right)\left[\frac{1}{\pi}\frac{1}{2T\left(P_\varphi\right)/m}\right]^{3/2} \exp\left[-\frac{v^2}{2T\left(P_\varphi\right)/m}\right] \\ &\approx F_M\left\{1 - \widehat{\mathbf{e}}_\varphi \cdot \mathbf{v}\frac{m}{Z\,|e|}R\frac{1}{N}\frac{\partial N}{\partial \psi}\left[1 + \eta\left(\frac{v^2}{2T/m} - \frac{3}{2}\right)\right]\right\} \end{aligned}$$

where

$$\eta \equiv \frac{d\ln T}{d\ln N} = \frac{L_N}{L_T}$$

The study of instabilities imposes to work with a distribution function from which one extracts the adiabatic part

$$f = -\frac{Z\,|e|}{T}F_0\phi + g$$

with

$$\begin{aligned} &\frac{\partial g}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} g + \frac{Z\,|e|}{m}\mathbf{v}\times\mathbf{B}\ \cdot\boldsymbol{\nabla}_{\mathbf{v}} g \\ =\ &\frac{Z\,|e|}{T}F_0\frac{\partial \phi}{\partial t} - \frac{\partial F_0}{\partial \psi_0}\frac{\partial \phi}{\partial \zeta} \end{aligned}$$

To zeroth order the function g does not depend on the gyrophase

$$\frac{\partial g}{\partial \zeta} = 0$$

Then

$$\begin{aligned} &\frac{\partial g}{\partial t} + \left(v_\parallel \widehat{\mathbf{n}} + \mathbf{v}_{drift}\right)\cdot\boldsymbol{\nabla}_R g \\ =\ &\Omega\frac{1}{2\pi}\oint d\zeta\left[\frac{Z\,|e|}{T}F_0\frac{\partial \phi\left(\mathbf{r},t\right)}{\partial t} - \frac{\partial F_0}{\partial \psi_0}\frac{\partial \phi\left(\mathbf{r},t\right)}{\partial \varphi}\right] \end{aligned}$$

The spatial coordinates are

$$\left(\psi, \theta, \varphi\right)$$

and the potential is expanded

$$\phi(\mathbf{r},t) = \frac{1}{(2\pi)^2} \sum_{l,m} \int_{-\infty}^{\infty} d\kappa \int_{Landau} d\omega \ \phi_{lm} \exp\left[-i\omega t + i\kappa\psi + im\theta - il\varphi\right]$$

The Landau contour goes in the upper half plane of the complex ω variable, above all singularities.

The transformation of spatial variables

$$\mathbf{R} \rightarrow (\psi', \theta', \varphi')$$

$$\psi' = \psi + \frac{1}{\Omega}\mathbf{v}\times\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\psi$$

$$\theta' = \theta + \frac{1}{\Omega}\mathbf{v}\times\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta$$

$$\varphi' = \varphi + \frac{1}{\Omega}\mathbf{v}\times\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\varphi$$

Then the spatial part of the Fourier representation ca be re-expressed

$$\begin{aligned} & \exp\left(i\kappa\psi + im\theta - il\varphi\right) \\ = & \exp\left[i\left(\kappa\psi' + im\theta' - il\varphi'\right) - i\frac{1}{\Omega}\mathbf{v}\times\widehat{\mathbf{n}} \cdot \mathbf{k}\right] \end{aligned}$$

where a new term has been separated

$$\mathbf{k} = \boldsymbol{\kappa}\boldsymbol{\nabla}\psi + m\boldsymbol{\nabla}\theta - l\boldsymbol{\nabla}\varphi$$

The *normalized* (to $-Z|e|$) toroidal invariant becomes

$$\psi_0 = \psi' + \frac{m}{Z|e|}R\widehat{\mathbf{e}}_{\varphi} \cdot \mathbf{v} - \frac{1}{\Omega}\mathbf{v}\times\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\psi$$

In the Right Hand Side of the Vlasov equation for the non-adiabatic part g it occur

$$F_0 \ \text{ and } \ \frac{\partial F_0}{\partial \psi_0}$$

These two functions are dependent of

$$\left(v^2, \psi_0\right)$$

These are the invariants, one is the total energy (no static electric field) and the toroidal invariant ψ_0 resulting from P_{φ}.

The functions are expanded around

$$\psi_0 \approx \psi'$$

for

$$\rho_i \ll L \ \text{(lengths of equilibrium)}$$

The part that has been separated in the exponent can be expanded in Bessel function.

$$\begin{aligned} & \frac{1}{2\pi}\oint d\zeta \ \exp\left[-i\frac{1}{\Omega}\mathbf{v}\times\widehat{\mathbf{n}}\cdot\mathbf{k}\right] \\ \approx & \ J_0\left(\frac{k_\perp v_\perp}{\Omega}\right) \end{aligned}$$

then the average over the gyromotion of the potential ϕ is

$$\begin{aligned} \widetilde{\phi} &= \frac{1}{2\pi}\oint d\zeta \ \phi(\mathbf{r},t) \\ &= \frac{1}{(2\pi)^2}\sum_{l,m}\int_{-\infty}^{\infty}d\kappa\int_{Landau}d\omega \ J_0\left(\frac{k_\perp v_\perp}{\Omega}\right)\exp\left(-i\omega t+i\kappa\psi'+im\theta'-il\varphi\right) \end{aligned}$$

where the projection of the wavenumber is

$$k_\perp = \left|\widehat{\mathbf{n}}\times(\mathbf{k}\times\widehat{\mathbf{n}})\right|$$

The result after this expansion

$$\begin{aligned} & \frac{\partial g}{\partial t}+\left(v_\parallel\widehat{\mathbf{n}}+\mathbf{v}_{drift}\right)\cdot\boldsymbol{\nabla}_R g \\ = & \ \frac{Z\left|e\right|}{T}F_M\left\{\frac{\partial\widetilde{\phi}}{\partial t}-\frac{T}{Z\left|e\right|}\frac{1}{N}\frac{\partial N}{\partial\psi'}\left[1+\eta\left(\frac{v^2}{2T/m}-\frac{3}{2}\right)\right]\frac{\partial\widetilde{\phi}}{\partial\varphi'}\right\} \end{aligned}$$

This equation can be solved by integrating along the orbits, the inverse of the operator acting on g in the Left Hand Side.

But there is an obstacle: the particle velocity

$$v_\parallel\widehat{\mathbf{n}}+\mathbf{v}_{drift}$$

is also dependent on the point of the orbit.

An approximative result can be obtained considering that this velocity is *independent* on the space variables $(\psi', \theta', \varphi')$.

Then it is effectively a constant for the integration along the orbits.

It is then possible to expand the function g,

$$g = \frac{1}{(2\pi)^2} \sum_{l,m} \int_{-\infty}^{\infty} d\kappa \int_L d\omega \; g'_{lm}(\kappa,\omega) \exp\left(-i\omega t + i\kappa\psi' + im\theta' - il\varphi'\right)$$

which the allows

$$g'_{lm}(\kappa,\omega) = \frac{Z|e|}{T} F_M \left(\omega - \omega_*^T\right) J_0\left(\frac{k_\perp v_\perp}{\Omega}\right) \frac{1}{\omega - \mathbf{k}\cdot\left(v_\parallel \widehat{\mathbf{n}} + \mathbf{v}_{drift}\right)} \phi_{lm}(\kappa,\omega)$$

with the notation

$$\omega_*^T = l \frac{T}{Z|e|} \frac{1}{N} \frac{\partial N}{\partial \psi} \left[1 + \eta\left(\frac{v^2}{2T/m} - \frac{3}{2}\right)\right]$$

The function g has a similar Fourier expansion in which the Fourier eigenfunctions (harmonics) are expressed in terms of the un-primed variables

$$(\psi,\theta,\varphi) \quad \text{instead of} \quad (\psi',\theta',\varphi')$$

The expansion is

$$g = \frac{1}{(2\pi)^2} \sum_{lm} \int_{-\infty}^{\infty} d\kappa \int_{Landau} d\omega \; g_{lm}(\kappa,\omega) \exp\left(-i\omega t + i\kappa\psi + im\theta - il\varphi\right)$$

the relationship between the two Fourier trasformed functions

$$g_{lm} \quad \text{and} \quad g'_{lm}$$

comes from the transformation of the expression in the exponential

$$g_{lm}(\kappa,\omega) = g'_{lm}(\kappa,\omega) \exp\left(i\frac{1}{\Omega}\mathbf{v}\times\widehat{\mathbf{n}}\cdot\mathbf{k}\right)$$

Then

- the Fourier transform g_{lm} that results from the Fourier expansion that uses the variables (ψ,θ,φ) is in terms of the variables of the *particle*;
- the Fourier transform g_{lm} that results from the Fourier expansion that uses the variables (ψ',θ',φ') is in terms of the *guiding center*.

Further the Fourier expansion is extended to the distribution function solution of the Vlasov equation.

$$f_{lm} = -\frac{Z\left|e\right|}{T} F_M \phi_{lm} + g_{lm}$$

Now we can calculate vthe perturbation of the density

$$n_{lm} = \int d^3 v \ f_{lm}$$

The integration in the velocity space is made in terms of the variables

$$\left(v_{\perp}, v_{\|}, \zeta\right)$$

with the result

$$\begin{aligned} n_{lm} &= -\frac{Z\left|e\right|}{T}\phi_{lm} N \\ &\quad -\frac{Z\left|e\right|}{T}\phi_{lm}\frac{\Gamma_0}{\left|k_{\|}\right| v_{th}}\left(\left\{\omega - \omega_* \left[1 - \eta\left(\frac{1}{2} + b - b\frac{\Gamma_1}{\Gamma_0} - \xi^2\right)\right]\right\} Z\left(\xi\right) - \eta\omega_*\xi\right) \end{aligned}$$

where

$$\omega_* = l\frac{T}{Z\left|e\right|}\frac{1}{N}\frac{\partial N}{\partial \psi} \quad \text{diamagnetic frequency}$$

$$v_{th}^2 = \frac{2T}{m}$$

$$\Gamma_{\nu} = I_{\nu}\left(b\right)\exp\left(-b\right)$$

$$b = \frac{k_{\perp}^2\left(T/m\right)}{\Omega_c^2} = \frac{1}{2}\frac{k_{\perp}^2 v^2}{\Omega_c^2}$$

$$\xi \equiv \frac{\omega - \mathbf{k}\cdot\mathbf{v}_{drift}}{\left|k_{\|}\right| v_{th}}$$

$$Z\left(\xi\right) \equiv \text{plasma dispersion function}$$

10 Change of variables adequate for the boundary trapped/circulating

in **turbulence driven bootstrap McdeWitt** the "pitch angle variable"

$$\kappa = \sqrt{\frac{1-\lambda(1-\varepsilon)}{2\varepsilon\lambda}}$$

where

$$\begin{aligned}\lambda &= \frac{2\mu B_0}{v^2} = \frac{v_\perp^2}{v^2}\frac{B_0}{B} \\ &= \frac{v_\perp^2}{v^2}h\end{aligned}$$

the change of variables

$$\left(\mathbf{x}, v^2, \mu\right) \rightarrow (\mathbf{x}, v, \kappa)$$

takes place with

$$\left.\frac{\partial}{\partial \mathbf{x}}\right|_{\mu, v^2} = \left.\frac{\partial}{\partial \mathbf{x}}\right|_{v,\kappa} + \frac{\frac{1}{2}-\kappa^2}{2\kappa}\frac{\partial \ln\varepsilon}{\partial \mathbf{x}}\left.\frac{\partial}{\partial\kappa}\right|_{\mathbf{x},v}$$

$$\left.\frac{\partial}{\partial v^2}\right|_{\mathbf{x},\mu} = \frac{1}{2v}\left.\frac{\partial}{\partial v}\right|_{\mathbf{x},\kappa} + \frac{1-\varepsilon+2\varepsilon\kappa^2}{4\varepsilon\kappa v^2}\left.\frac{\partial}{\partial\kappa}\right|_{\mathbf{x},v}$$

The Jacobian

$$\mathcal{I} = 2\pi\varepsilon\kappa\left(\frac{v}{v_{th,e}}\right)^3 \lambda^2 \left(\frac{B}{B_0}\right)\frac{v_{th,e}}{v_\parallel}$$

11 Equation for the gyrokinetic turbulent heating Hinton Waltz 2006

There is turbulence mixed on neoclassics.

There is only electric field, since the turbulence is considered electrostatic.

The equation

$$\frac{\partial F}{\partial t} + \Lambda F + \frac{e}{m}\mathbf{v}\times\mathbf{B}\cdot\frac{\partial F}{\partial \mathbf{v}} = D + C$$

$$\Lambda F = \mathbf{v}\cdot\boldsymbol{\nabla}F + \frac{e}{m}\mathbf{E}\cdot\frac{\partial F}{\partial \mathbf{v}}$$
$$\Lambda F = \frac{d\mathbf{z}}{dt}\cdot\frac{\partial F}{\partial \mathbf{z}}$$

$$\begin{aligned}\mathbf{z} &= (\mathbf{x},\mathbf{v})\\ \frac{d\mathbf{z}}{dt} &= \left(\mathbf{v},\frac{e}{m}\mathbf{E}\right)\end{aligned}$$

The definition of $\Lambda F = \frac{d\mathbf{z}}{dt}\cdot\frac{\partial F}{\partial \mathbf{z}}$ can be expressed ina conservative form

$$\Lambda F = \frac{\partial}{\partial \mathbf{z}}\left(\frac{d\mathbf{z}}{dt}F\right)$$

since

$$\begin{aligned}\frac{\partial}{\partial \mathbf{z}}\cdot\frac{d\mathbf{z}}{dt} &= \boldsymbol{\nabla}\cdot\mathbf{v}+\frac{e}{m}\frac{\partial}{\partial \mathbf{v}}\cdot\mathbf{E}\\ &= 0\end{aligned}$$

the change of variables

$$(\mathbf{x},\mathbf{v}) \rightarrow \left(\mathbf{x}', v_{\|}, \mu, \zeta\right)$$

In the new coordinates

$$\left(v_{\|}, \mu, \zeta\right)$$
$$\Lambda = \mathbf{v}\cdot\boldsymbol{\nabla}+\frac{dv_{\|}}{dt}\frac{\partial}{\partial v_{\|}}+\frac{d\mu}{dt}\frac{\partial}{\partial \mu}+\frac{d\zeta}{dt}\frac{\partial}{\partial \zeta}$$
$$\frac{dv_{\|}}{dt} = \mathbf{v}\cdot\boldsymbol{\nabla}v_{\|} + \frac{e}{m}\mathbf{E}\cdot\frac{\partial v_{\|}}{\partial \mathbf{v}}$$
$$\frac{d\mu}{dt} = \mathbf{v}\cdot\boldsymbol{\nabla}\mu + \frac{e}{m}\mathbf{E}\cdot\frac{\partial \mu}{\partial \mathbf{v}}$$
$$\frac{d\zeta}{dt} = \mathbf{v}\cdot\boldsymbol{\nabla}\zeta + \frac{e}{m}\mathbf{E}\cdot\frac{\partial \zeta}{\partial \mathbf{v}}$$

The explicit form derived from these equations

$$\frac{dv_{\|}}{dt} = (\mathbf{v}\cdot\boldsymbol{\nabla})\ \widehat{\mathbf{n}}\cdot\mathbf{v}_{\perp} + \frac{e}{m}E_{\|}$$

$$B\frac{d\mu}{dt} = -\mathbf{v}\cdot\ \left[\mu\boldsymbol{\nabla}B + v_{\|}\left[(\ \boldsymbol{\nabla})\ \widehat{\mathbf{n}}\right]\cdot\mathbf{v}_{\perp}\right] + \frac{e}{m}\mathbf{E}_{\perp}\cdot\mathbf{v}_{\perp}$$

$$\frac{d\zeta}{dt} = -\mathbf{v}\cdot \left([(\ \boldsymbol{\nabla}) \ \widehat{\mathbf{e}}_x] \cdot \widehat{\mathbf{e}}_y + \frac{v_\parallel}{v_\perp^2} [(\ \boldsymbol{\nabla}) \ \widehat{\mathbf{n}}] \times \widehat{\mathbf{n}} \ \cdot \mathbf{v}_\perp \right) + \frac{e}{m} \frac{\mathbf{E}_\perp \cdot \ \widehat{\mathbf{n}} \times \mathbf{v}_\perp}{v_\perp^2}$$

Note

Taking into account the use of a particular writting of the successive scalar vector products, we have to read the last formula as

$$\begin{aligned} \frac{d\zeta}{dt} &= -\left[(\mathbf{v} \cdot \boldsymbol{\nabla}) \, \widehat{\mathbf{e}}_x \right] \cdot \widehat{\mathbf{e}}_y \\ &\quad -\frac{v_\parallel}{v_\perp^2} \left\{ \left[(\mathbf{v} \cdot \boldsymbol{\nabla}) \, \widehat{\mathbf{n}} \right] \times \widehat{\mathbf{n}} \right\} \cdot \mathbf{v}_\perp \\ &\quad + \frac{e}{m} \frac{\mathbf{E}_\perp \cdot \ \widehat{\mathbf{n}} \times \mathbf{v}_\perp}{v_\perp^2} \end{aligned}$$

End

Note in **Catto Tsang** (CT) *contains a typo*

$$\frac{d\mu}{dt} = -\frac{\mu}{B} \frac{dB}{dt} - v_\parallel \widehat{\mathbf{n}} \cdot \frac{d\widehat{\mathbf{n}}}{dt} + \frac{e}{mB} \mathbf{v}_\perp \cdot \mathbf{E}$$

This formula is derived by **Hazeltine 1973** but there are two differences

- there is a factor $1/B$ in the middle term, and
- also for the middle term, instead of $-v_\parallel \widehat{\mathbf{n}} \cdot \frac{d\widehat{\mathbf{n}}}{dt}$ in Hazeltine1973 it is $-v_\parallel \mathbf{v}_\perp \cdot \frac{d\widehat{\mathbf{n}}}{dt}$;

Then the Hazeltine1973 formula is

$$\frac{d\mu}{dt} = -\frac{\mu}{B} \frac{dB}{dt} - \frac{v_\parallel}{B} \mathbf{v}_\perp \cdot \frac{d\widehat{\mathbf{n}}}{dt} + \frac{e}{mB} \mathbf{v}_\perp \cdot \mathbf{E}$$

(Catto Tsang Hazeltine)

Note

On the other hand we have an alternative expression in **CattoTsang**, [we preserve the *notations* of CT, and later adopt another writting]

$$\boldsymbol{\nabla}\mu = -\frac{\mu}{B} \boldsymbol{\nabla} B - \frac{v_\parallel v_\perp}{B} \left[(\boldsymbol{\nabla} \widehat{\mathbf{n}}) \cdot \widehat{\mathbf{e}}_\theta \ \cos\zeta + (\boldsymbol{\nabla} \widehat{\mathbf{n}}) \cdot \widehat{\mathbf{e}}_\psi \ \sin\zeta \right]$$

and it looks that, in the full time derivative

$$\frac{d\mu}{dt} = (\mathbf{v} \cdot \boldsymbol{\nabla}) \, \mu = \mathbf{v} \cdot (\boldsymbol{\nabla} \mu)$$

we have to identify the second term

$$\mathbf{v}\cdot\left\{-\frac{v_{\|}v_{\perp}}{B}\left[(\boldsymbol{\nabla}\widehat{\mathbf{n}})\cdot\widehat{\mathbf{e}}_{\theta}\ \cos\zeta+(\boldsymbol{\nabla}\widehat{\mathbf{n}})\cdot\widehat{\mathbf{e}}_{\psi}\ \sin\zeta\right]\right\}$$

with

$$-\frac{v_{\|}}{B}\mathbf{v}_{\perp}\cdot\frac{d\widehat{\mathbf{n}}}{dt}$$

We exclude the factors that appear in both expressions, $v_{\|}$, $1/B$, $-$, ignore the explicit time derivatives $\partial/\partial t$ and write more explicitely the scalar product, to exhibit the convective derivations

$$v_{\perp}\left\{\left[(\mathbf{v}\cdot\boldsymbol{\nabla})\,\widehat{\mathbf{n}}\right]\cdot\widehat{\mathbf{e}}_{\theta}\ \cos\zeta+\left[(\mathbf{v}\cdot\boldsymbol{\nabla})\,\widehat{\mathbf{n}}\right]\cdot\widehat{\mathbf{e}}_{\psi}\ \sin\zeta\right\}$$

with

$$\mathbf{v}_{\perp}\cdot\left[(\mathbf{v}\cdot\boldsymbol{\nabla})\,\widehat{\mathbf{n}}\right]$$

these two expressions are identical if we define

$$v_{\perp}\left(\widehat{\mathbf{e}}_{\theta}\ \cos\zeta+\widehat{\mathbf{e}}_{\psi}\ \sin\zeta\right)=\mathbf{v}_{\perp}$$

End

Now return to the formula of **Catto Tsang** corected after **Hazeltine**

$$\frac{d\mu}{dt} = -\frac{\mu}{B}\frac{dB}{dt}-\frac{v_{\|}}{B}\mathbf{v}_{\perp}\cdot\frac{d\widehat{\mathbf{n}}}{dt}+\frac{e}{mB}\mathbf{v}_{\perp}\cdot\mathbf{E}$$

(Catto Tsang Hazeltine)

Now this must be compared with

$$B\frac{d\mu}{dt} = -\mathbf{v}\cdot\ \left[\mu\boldsymbol{\nabla}B+v_{\|}\left(\boldsymbol{\nabla}\ \widehat{\mathbf{n}}\right)\cdot\mathbf{v}_{\perp}\right]+\frac{e}{m}\mathbf{E}_{\perp}\cdot\mathbf{v}_{\perp}$$

(Hinton Waltz)

The first term.

Comparing with the above **Hinton Waltz** (HW) equation (after multiplying with B) we have

$$-\mathbf{v}\cdot\mu\boldsymbol{\nabla}B\ \text{HW}\ \rightarrow-\mu\left(\frac{\partial B}{\partial t}+\mathbf{v}\cdot\boldsymbol{\nabla}B\right)\ \text{CTH}$$

No explicit time variation of B is assumed but convective, then for the first term HW=CTH.

Last term

$$\frac{e}{m}\mathbf{E}_\perp \cdot \mathbf{v}_\perp$$

HW=CTH

The middle term after $B \times$ (CT) after correction like in Hazeltine1973

$$-v_\parallel \mathbf{v}_\perp \cdot \frac{d\widehat{\mathbf{n}}}{dt} \text{(CTH)} \;\rightarrow\; -\mathbf{v}\cdot \left[v_\parallel \left(\boldsymbol{\nabla}\ \widehat{\mathbf{n}}\right) \cdot \mathbf{v}_\perp\right] \text{ (HW)}$$

After excluding the explicit time derivative $\partial\widehat{\mathbf{n}}/\partial t$ we must compare

$$\begin{array}{l} -\mathbf{v}_\perp \cdot \left[\left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right] \text{ with} \\ -\left[\left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\cdot\mathbf{v}_\perp \end{array}$$

This is OK

Conclusion

$$\text{(Catto Tsang Hazeltine)} = \text{(Hinton Waltz)}$$

$$\text{(CTH)} = \text{(HW)}$$

End.

NOTE

In the paper **Belli Candy** the following formula is used

$$\frac{1}{\mu}\frac{d\mu}{dt} = \frac{v_\parallel B}{\Omega_{ca}}\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\left(\frac{1}{B}v_\parallel \widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\times\widehat{\mathbf{n}}\right)$$

and

$$\begin{aligned}\frac{1}{\mu}\frac{d\mu}{dt} &= \frac{1}{\Omega_{ca}}\frac{v_\parallel^2+\mu B}{B}\frac{I}{\mathcal{J}_r}\frac{1}{B}\frac{\partial B}{\partial\theta}\left(\frac{d}{dr}\ln I + \frac{4\pi}{B^2}\frac{dp}{dr}\right) \\ &\quad +2\frac{v_\parallel^2}{\Omega_{ca}}\frac{1}{B}\frac{I}{\mathcal{J}_r}\frac{1}{B}\frac{\partial B}{\partial\theta}\frac{4\pi}{B^2}\frac{dp}{dr}\end{aligned}$$

where

$$\mathcal{J}_\psi = \frac{1}{\left(\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\theta\right)\cdot\boldsymbol{\nabla}\varphi}$$

$$\mathcal{J}_r = \frac{\partial\psi}{\partial r}\mathcal{J}_\psi$$

$$\nabla_\parallel = \frac{1}{\mathcal{J}_\psi}\frac{1}{B}\frac{\partial}{\partial\theta}$$

where we know that

$$\nabla_{\|} = \frac{1}{qR}\frac{\partial}{\partial\theta}$$

then

$$\mathcal{J}_{\psi} = \frac{qR}{B}$$

and

$$\begin{aligned}
\mathcal{J}_r &= |\nabla\psi|\,\mathcal{J}_{\psi} \\
&= RB_{\theta}\frac{qR}{B} = RB_{\theta}\frac{r}{R}\frac{B_{\varphi}}{B_{\theta}}R\frac{1}{B} = rR\frac{B_{\varphi}}{B}
\end{aligned}$$

In connection with the formula of **Belli Candy**

$$\frac{1}{\mu}\frac{d\mu}{dt} = \frac{v_{\|}B}{\Omega_{ca}}\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\left(\frac{1}{B}v_{\|}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\times\widehat{\mathbf{n}}\right)$$

we find in **Hinton Waltz**

$$\boldsymbol{\nabla}\times\widehat{\mathbf{n}} = \widehat{\mathbf{n}}\left[\widehat{\mathbf{n}}\cdot\left(\boldsymbol{\nabla}\times\widehat{\mathbf{n}}\right)\right] + \widehat{\mathbf{n}}\times\boldsymbol{\kappa}$$

which does not help.

Another formula

$$-\left(\widehat{\mathbf{e}}_1\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{e}}_2 + \left(\widehat{\mathbf{e}}_2\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{e}}_1 = \boldsymbol{\nabla}\times\widehat{\mathbf{n}}$$

END

Note the particular notation adopted for the tensorial contraction, like

$$\mathbf{v}\cdot\left(\boldsymbol{\nabla}\ \widehat{\mathbf{n}}\right)\cdot\mathbf{v}_{\perp} = v_{\mu}\ \left(\partial_{\mu}\ \widehat{n}_{\beta}\right)v_{\perp\beta}$$

because the convective derivative is

$$\left[\left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\cdot\mathbf{v}_{\perp}$$

End

One has to return to the change of variables

$$\begin{aligned}
\mathbf{z} \ &\equiv\ \left(\mathbf{x},\mathbf{v}\right)\rightarrow\left(\mathbf{x}',v_{\|},\mu,\zeta\right) \\
&\quad\ \text{Jacobian } B
\end{aligned}$$

Then

$$\Lambda F = \frac{\partial}{\partial \mathbf{z}} \left(\frac{d\mathbf{z}}{dt} F \right)$$

becomes

$$\begin{aligned}
\Lambda F \quad = \quad & \frac{1}{B} \left\{ \boldsymbol{\nabla}' \cdot \left[B \left(\frac{\partial \mathbf{x}'}{\partial \mathbf{z}} \right) \cdot \frac{d\mathbf{z}}{dt} F \right] \right. \\
& + \frac{\partial}{\partial v_\parallel} \left[B \left(\frac{\partial v_\parallel}{\partial \mathbf{z}} \right) \cdot \frac{d\mathbf{z}}{dt} F \right] \\
& + \frac{\partial}{\partial \mu} \left[B \left(\frac{\partial \mu}{\partial \mathbf{z}} \right) \cdot \frac{d\mathbf{z}}{dt} F \right] \\
& \left. + \frac{\partial}{\partial \zeta} \left[B \left(\frac{\partial \zeta}{\partial \mathbf{z}} \right) \cdot \frac{d\mathbf{z}}{dt} F \right] \right\}
\end{aligned}$$

Each factor has its explicit form

$$\left(\frac{\partial \mathbf{x}'}{\partial \mathbf{z}} \right) \cdot \frac{d\mathbf{z}}{dt} = \mathbf{v}$$

$$\left(\frac{\partial v_\parallel}{\partial \mathbf{z}} \right) \cdot \frac{d\mathbf{z}}{dt} = \frac{\partial v_\parallel}{\partial \mathbf{x}} \cdot \mathbf{v} + \frac{\partial v_\parallel}{\partial \mathbf{v}} \cdot \left(\frac{e}{m} \mathbf{E} \right) = \frac{dv_\parallel}{dt}$$

$$\left(\frac{\partial \mu}{\partial \mathbf{z}} \right) \cdot \frac{d\mathbf{z}}{dt} = \frac{\partial \mu}{\partial \mathbf{x}} \cdot \mathbf{v} + \frac{\partial \mu}{\partial \mathbf{v}} \cdot \left(\frac{e}{m} \mathbf{E} \right) = \frac{d\mu}{dt}$$

$$\left(\frac{\partial \zeta}{\partial \mathbf{z}} \right) \cdot \frac{d\mathbf{z}}{dt} = \frac{\partial \zeta}{\partial \mathbf{x}} \cdot \mathbf{v} + \frac{\partial \zeta}{\partial \mathbf{v}} \cdot \left(\frac{e}{m} \mathbf{E} \right) = \frac{d\zeta}{dt}$$

whose expressions are known and written above.

Finally, we recall the set of velocity-space variables

$$\left(v_\parallel, \mu, \zeta \right)$$

and then

$$\Lambda F = \frac{1}{B} \boldsymbol{\nabla}' \cdot \left(B \mathbf{v} F \right) + \frac{\partial}{\partial v_\parallel} \left(\frac{dv_\parallel}{dt} F \right) + \frac{\partial}{\partial \mu} \left(\frac{d\mu}{dt} F \right) + \frac{\partial}{\partial \zeta} \left(\frac{d\zeta}{dt} F \right)$$

Later it will be necessary to consider the gyro-averaged part

$$\begin{aligned}
\overline{()} \quad & \equiv \quad \text{gyroaveraged} \\
\widetilde{()} \quad & \equiv \quad \text{dependent on gyroangle } \zeta
\end{aligned}$$

The gyro-averaged parallel acceleration has two contributions

$$\overline{\frac{dv_\parallel}{dt}} = -\mu\nabla_\parallel B + \frac{e}{m}E_\parallel$$

Here we have

$$\begin{aligned}
\overline{\frac{dv_\parallel}{dt}} &= \overline{\left[(\mathbf{v}\cdot\boldsymbol{\nabla})\ \widehat{\mathbf{n}}\right]\cdot\mathbf{v}_\perp + \frac{e}{m}E_\parallel} \\
&= \int_0^{2\pi}\frac{d\zeta}{2\pi}\left\{\left[\left(\left(v_\parallel\widehat{\mathbf{n}}+\mathbf{v}_\perp\ \right)\cdot\boldsymbol{\nabla}\right)\ \widehat{\mathbf{n}}\right]\cdot\mathbf{v}_\perp\right\} + \frac{e}{m}E_\parallel
\end{aligned}$$

We take into account that

$$\begin{aligned}
\int_0^{2\pi}\frac{d\zeta}{2\pi}\mathbf{v}_\perp &= v_\perp\int_0^{2\pi}\frac{d\zeta}{2\pi}\left(\widehat{\mathbf{e}}_x\cos\zeta+\widehat{\mathbf{e}}_y\sin\zeta\right) \\
&= 0
\end{aligned}$$

Then the first term

$$\int_0^{2\pi}\frac{d\zeta}{2\pi}\ v_\parallel\left[(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla})\ \widehat{\mathbf{n}}\right]\cdot\mathbf{v}_\perp = 0$$

and the next one

$$\begin{aligned}
&\int_0^{2\pi}\frac{d\zeta}{2\pi}\left\{\left[(\mathbf{v}_\perp\cdot\boldsymbol{\nabla})\ \widehat{\mathbf{n}}\right]\cdot\mathbf{v}_\perp\right\} \\
= &\int_0^{2\pi}\frac{d\zeta}{2\pi}\left\{\left[\left(\left(\widehat{\mathbf{e}}_x\cos\zeta+\widehat{\mathbf{e}}_y\sin\zeta\right)\cdot\boldsymbol{\nabla}\right)\ \widehat{\mathbf{n}}\right]\cdot\left(\widehat{\mathbf{e}}_x\cos\zeta+\widehat{\mathbf{e}}_y\sin\zeta\right)\right\} \\
= &\ v_\perp^2\int_0^{2\pi}\frac{d\zeta}{2\pi}\Big\{\left[(\widehat{\mathbf{e}}_x\cdot\boldsymbol{\nabla})\ \widehat{\mathbf{n}}\right]\cdot\widehat{\mathbf{e}}_x\ \cos^2\zeta+ \\
&\quad+\left[(\widehat{\mathbf{e}}_x\cdot\boldsymbol{\nabla})\ \widehat{\mathbf{n}}\right]\cdot\widehat{\mathbf{e}}_y\ \cos\zeta\sin\zeta \\
&\quad+\left[(\widehat{\mathbf{e}}_y\cdot\boldsymbol{\nabla})\ \widehat{\mathbf{n}}\right]\cdot\widehat{\mathbf{e}}_x\ \sin\zeta\cos\zeta \\
&\quad+\left[(\widehat{\mathbf{e}}_y\cdot\boldsymbol{\nabla})\ \widehat{\mathbf{n}}\right]\cdot\widehat{\mathbf{e}}_y\ \sin^2\zeta\Big\}
\end{aligned}$$

or

$$\begin{aligned}
&\int_0^{2\pi}\frac{d\zeta}{2\pi}\left\{\left[(\mathbf{v}_\perp\cdot\boldsymbol{\nabla})\ \widehat{\mathbf{n}}\right]\cdot\mathbf{v}_\perp\right\} \\
= &\ \frac{v_\perp^2}{2}\left\{\left[(\widehat{\mathbf{e}}_x\cdot\boldsymbol{\nabla})\ \widehat{\mathbf{n}}\right]\cdot\widehat{\mathbf{e}}_x+\left[(\widehat{\mathbf{e}}_y\cdot\boldsymbol{\nabla})\ \widehat{\mathbf{n}}\right]\cdot\widehat{\mathbf{e}}_y\right\}
\end{aligned}$$

On the other hand we have the *tensorial* product

$$\left[(\widehat{\mathbf{e}}_x\cdot\boldsymbol{\nabla})\ \widehat{\mathbf{n}}\right]\cdot\widehat{\mathbf{e}}_x+\left[(\widehat{\mathbf{e}}_y\cdot\boldsymbol{\nabla})\ \widehat{\mathbf{n}}\right]\cdot\widehat{\mathbf{e}}_y = -\left[(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla})\ \widehat{\mathbf{n}}\right]\cdot\widehat{\mathbf{n}}+\boldsymbol{\nabla}\cdot\widehat{\mathbf{n}}$$

In the RHS it is the parallel projection of the *curvature*

$$\boldsymbol{\kappa} = (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \, \widehat{\mathbf{n}} \quad \text{(def. curvature)}$$

and

$$[(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \ \widehat{\mathbf{n}}] \cdot \widehat{\mathbf{n}} = \boldsymbol{\kappa} \cdot \widehat{\mathbf{n}}$$

this is zero, since $\boldsymbol{\kappa}$ and $\widehat{\mathbf{n}}$ are perpendicular

$$\boldsymbol{\kappa} \cdot \widehat{\mathbf{n}} = 0$$

This could also have been seen from

$$\begin{aligned} & [(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \ \widehat{\mathbf{n}}] \cdot \widehat{\mathbf{n}} \\ = \ & n_i \left[(n_k \partial_k) \, n_i \right] = (n_k \partial_k) \left(\frac{1}{2} n_i^2 \right) = (n_k \partial_k) \left(\frac{1}{2} \right) \\ = \ & 0 \end{aligned}$$

We have

the divergence of the versor of the magnetic field is the parallel gradient of the logarithm of the magnetic field

$$\boldsymbol{\nabla} \cdot \widehat{\mathbf{n}} = -\nabla_{\|} \ln B$$

where

$$\nabla_{\|} \ln B = \frac{\varepsilon}{q} \frac{\sin\theta}{R}$$

Note

In the previous comparison

We have

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\ \boldsymbol{\nabla} \cdot (B \widehat{\mathbf{n}}) &= (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \, B + B \, (\boldsymbol{\nabla} \cdot \widehat{\mathbf{n}}) = 0 \\ \nabla_{\|} B &= -B \, (\boldsymbol{\nabla} \cdot \widehat{\mathbf{n}}) \\ \nabla_{\|} \ln B &= -\boldsymbol{\nabla} \cdot \widehat{\mathbf{n}} \end{aligned}$$

$$\nabla_{\|} \ln B = -\boldsymbol{\nabla} \cdot \widehat{\mathbf{n}}$$

then

$$-\boldsymbol{\nabla} \cdot \widehat{\mathbf{n}} = \frac{\varepsilon}{q} \frac{\sin\theta}{R}$$

We can use

$$\widehat{\mathbf{n}} = \frac{\mathbf{B}_T + \mathbf{B}_\theta}{B}$$

$$\mathbf{B}_\theta = \frac{b(r)}{h}\widehat{\mathbf{e}}_\theta$$

End

Then the gyroaverage of the second term of the time derivative of the parallel velocity is

$$\begin{aligned}
& \int_0^{2\pi} \frac{d\zeta}{2\pi} \left\{ \left[\left(\mathbf{v}_\perp \ \cdot \boldsymbol{\nabla} \right) \ \widehat{\mathbf{n}} \right] \cdot \mathbf{v}_\perp \right\} \\
= & \frac{v_\perp^2}{2} \left\{ \boldsymbol{\nabla} \cdot \widehat{\mathbf{n}} \right\} = -\frac{v_\perp^2}{2} \nabla_\parallel \ln B \\
= & -\mu \nabla_\parallel B
\end{aligned}$$

and finally

$$\overline{\frac{dv_\parallel}{dt}} = -\mu \nabla_\parallel B + \frac{e}{m} E_\parallel$$

Note this is the (gyrophase-averaged) acceleration during the parallel flow, caused by the magnetic mirror, the modulation of B along the line. **End.**

Further

$$\begin{aligned}
& \overline{\frac{d\mu}{dt}} \\
= & \int_0^{2\pi} \frac{d\zeta}{2\pi} \frac{1}{B} \left\{ -\mathbf{v} \cdot \ \left[\mu \boldsymbol{\nabla} B + v_\parallel \left(\boldsymbol{\nabla} \ \widehat{\mathbf{n}} \right) \cdot \mathbf{v}_\perp \right] + \frac{e}{m} \mathbf{E}_\perp \cdot \mathbf{v}_\perp \right\} \\
& \text{written differently} \\
= & \int_0^{2\pi} \frac{d\zeta}{2\pi} \frac{1}{B} \left\{ -\mu \left(\mathbf{v} \cdot \boldsymbol{\nabla} \right) B + v_\parallel \left[\left(\mathbf{v} \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} \right] \cdot \mathbf{v}_\perp + \frac{e}{m} \mathbf{E}_\perp \cdot \mathbf{v}_\perp \right\}
\end{aligned}$$

The last term is zero.

The first term is

$$\begin{aligned}
& \frac{1}{B} \int_0^{2\pi} \frac{d\zeta}{2\pi} \left\{ -\mathbf{v} \cdot \mu \boldsymbol{\nabla} B \right\} \\
= & \frac{1}{B} \mu \int_0^{2\pi} \frac{d\zeta}{2\pi} \left(v_\parallel \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} B + \mathbf{v}_\perp \cdot \boldsymbol{\nabla} B \right) \\
= & \mu v_\parallel \nabla_\parallel \ln B
\end{aligned}$$

The second term is

$$\begin{aligned}
& \frac{1}{B} \int_0^{2\pi} \frac{d\zeta}{2\pi} \left[-\mathbf{v} \cdot v_\parallel \left(\boldsymbol{\nabla} \ \widehat{\mathbf{n}} \right) \cdot \mathbf{v}_\perp \right] \\
= & \frac{1}{B} v_\parallel \int_0^{2\pi} \frac{d\zeta}{2\pi} \left\{ - \left[\left(\mathbf{v} \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} \right] \cdot \mathbf{v}_\perp \right\}
\end{aligned}$$

For this we have the result obtained above

$$
\begin{aligned}
& \frac{1}{B} v_{\|} \int_0^{2\pi} \frac{d\zeta}{2\pi} \left\{ - \left[\left(\left(v_{\|} \widehat{\mathbf{n}} + \mathbf{v}_{\perp} \right) \cdot \boldsymbol{\nabla} \right) \ \widehat{\mathbf{n}} \right] \cdot \mathbf{v}_{\perp} \right\} \\
= & \ \frac{1}{B} v_{\|} \int_0^{2\pi} \frac{d\zeta}{2\pi} \left\{ - v_{\|} \left[\left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \ \right) \widehat{\mathbf{n}} \right] \cdot \mathbf{v}_{\perp} \right\} + \frac{1}{B} v_{\|} \int_0^{2\pi} \frac{d\zeta}{2\pi} \left\{ \left[\left(\mathbf{v}_{\perp} \cdot \boldsymbol{\nabla} \right) \ \widehat{\mathbf{n}} \right] \cdot \mathbf{v}_{\perp} \right\} \\
= & \ 0 + \frac{v_{\perp}^2}{2B} v_{\|} \left\{ - \left[\left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \right) \ \widehat{\mathbf{n}} \right] \cdot \widehat{\mathbf{n}} \right\} = - \frac{v_{\perp}^2}{2B} v_{\|} \nabla_{\|} \ln B
\end{aligned}
$$

In the last operations we have used

$$
\frac{1}{B} v_{\|} \int_0^{2\pi} \frac{d\zeta}{2\pi} \left\{ \left[\left(\mathbf{v}_{\perp} \ \cdot \boldsymbol{\nabla} \right) \ \widehat{\mathbf{n}} \right] \cdot \mathbf{v}_{\perp} \right\} = - \frac{v_{\perp}^2}{2} v_{\|} \nabla_{\|} \ln B
$$

Now we sum the two contributions

$$
\begin{aligned}
& \overline{\frac{d\mu}{dt}} \\
= & \ \mu v_{\|} \nabla_{\|} \ln B + \left(- \frac{v_{\perp}^2}{2B} v_{\|} \nabla_{\|} \ln B \right) = 0
\end{aligned}
$$

$$
\overline{\frac{d\mu}{dt}} = 0
$$

which means that, it may ONLY exist a gyrophase-dependent part of the time-variation (acceleration) of the magnetic momentum.

It has been excluded an explicit time dependence $\partial\mu/\partial t = 0$. But in **Hazeltine 1973** it is present.

Note
In the work of **Candy Belii** there is a non-zero derivative of μ to time
End

The part that depends on gyroangle
The gyrophase-dependent part of the parallel acceleration

$$
\widetilde{\frac{dv_{\|}}{dt}} = v_{\|} \boldsymbol{\kappa} \cdot \mathbf{v}_{\perp} + \mathbf{w}_{\perp\perp} : \left(\boldsymbol{\nabla} \ \widehat{\mathbf{n}} \right)
$$

and the gyrophase-dependent part of the magnetic moment ($\sim v_{\perp}^2/2$)

$$
B \widetilde{\frac{d\mu}{dt}} = - \mu \mathbf{v}_{\perp} \cdot \boldsymbol{\nabla} B - v_{\|}^2 \ \boldsymbol{\kappa} \cdot \mathbf{v}_{\perp} + \mathbf{v}_{\perp} \cdot \frac{e}{m} \mathbf{E}_{\perp} - v_{\|} \ \mathbf{w}_{\perp\perp} : \left(\boldsymbol{\nabla} \ \widehat{\mathbf{n}} \right)
$$

where

$$\boldsymbol{\kappa} = (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}$$

curvature

$$\mathbf{v}_\perp = v_\perp \left(\widehat{\mathbf{e}}_x \cos\zeta + \widehat{\mathbf{e}}_y \sin\zeta \right)$$

$$\begin{aligned} \mathbf{w}_{\perp\perp} &= \mathbf{v}_\perp\ \mathbf{v}_\perp - \langle \mathbf{v}_\perp\ \mathbf{v}_\perp \rangle \\ &= \mu B \left[\boldsymbol{\sigma}_{\perp\perp} \cos(2\zeta) + \boldsymbol{\tau}_{\perp\perp}\ \sin(2\zeta) \right] \end{aligned}$$

$$\boldsymbol{\sigma}_{\perp\perp} = \widehat{\mathbf{e}}_x\ \widehat{\mathbf{e}}_x - \widehat{\mathbf{e}}_y\ \widehat{\mathbf{e}}_y$$

$$\boldsymbol{\tau}_{\perp\perp} = \widehat{\mathbf{e}}_x\ \widehat{\mathbf{e}}_y + \widehat{\mathbf{e}}_y\ \widehat{\mathbf{e}}_x$$

Note
In **Hazeltine 1973** we have

$$\widehat{\mathbf{e}}_{\mathbf{v}_\perp} = \widehat{\mathbf{e}}_x \cos\zeta + \widehat{\mathbf{e}}_y \sin\zeta$$

$$\langle \widehat{\mathbf{e}}_{\mathbf{v}_\perp}\ \widehat{\mathbf{e}}_{\mathbf{v}_\perp} \rangle = \frac{1}{2} \left(\mathbf{I} - \widehat{\mathbf{n}}\ \widehat{\mathbf{n}} \right)$$

End

Formulas for the detailed drift equation.
Perpendicular drift

$$\mathbf{v}_D = \frac{1}{\Omega_c} \widehat{\mathbf{n}} \times \left(\mu \boldsymbol{\nabla}_\perp B + v_\parallel^2\ \boldsymbol{\kappa} - \frac{e}{m} \mathbf{E}_\perp \right)$$

perpendicular drift

where

$$\boldsymbol{\kappa} = (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}} \quad \text{curvature}$$

Parallel drift

$$\mathbf{u}_D = \frac{m}{e}\ \mu\ \widehat{\mathbf{n}}\ \left[\widehat{\mathbf{n}} \cdot (\boldsymbol{\nabla} \times \widehat{\mathbf{n}}) \right]$$

parallel drift

Formulas

$$\boldsymbol{\nabla} \times \widehat{\mathbf{n}} = \widehat{\mathbf{n}}\ \left[\widehat{\mathbf{n}} \cdot (\boldsymbol{\nabla} \times \widehat{\mathbf{n}}) \right] + \widehat{\mathbf{n}} \times \boldsymbol{\kappa}$$

This is simply a formula showing the components of the vector $\boldsymbol{\nabla} \times \widehat{\mathbf{n}}$ in terms of *parallel* and *whatever* (actually perpendicular $\widehat{\mathbf{n}} \times$) , components.

The first term is a simple projection, nothing new.

The second uses $\boldsymbol{\kappa} = (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}$ the curvature, and is $\widehat{\mathbf{n}} \times [(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}]$, perpendicular on the curvature vector. Since (approximately) the curvature is directed toward the major axis of symmetry of the torus, $\widehat{\mathbf{n}} \times \boldsymbol{\kappa}$ is directed vertically. But a vector like $\boldsymbol{\nabla}\times\widehat{\mathbf{n}}$ cannot simply be a geometrical quantity. It depends on the current density.

Even further

$$\begin{aligned} \widehat{\mathbf{n}} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\kappa}) &= (\boldsymbol{\nabla}\cdot\widehat{\mathbf{n}})\ \widehat{\mathbf{n}} \cdot (\boldsymbol{\nabla}\times\widehat{\mathbf{n}}) \\ &\quad + \widehat{\mathbf{n}} \cdot [(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla})\, (\boldsymbol{\nabla}\times\widehat{\mathbf{n}})] \end{aligned}$$

and

$$\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \times \boldsymbol{\kappa} = \mathbf{B} \cdot \boldsymbol{\nabla} \left(\frac{\widehat{\mathbf{n}} \cdot (\boldsymbol{\nabla}\times\widehat{\mathbf{n}})}{B} \right)$$

for velocities, the average on tensorial products

$$\langle \mathbf{v}_\perp\ \mathbf{v}_\perp \rangle = \mu B \,(\mathbf{I} - \widehat{\mathbf{n}}\ \widehat{\mathbf{n}})$$

$$\left\langle \mathbf{v}_\perp\ \widetilde{\frac{dv_\parallel}{dt}} \right\rangle = \mu B\ v_\parallel\ \boldsymbol{\kappa}$$

$$\left\langle \mathbf{v}_\perp \widetilde{\frac{d\mu}{dt}} \right\rangle = -\mu \left[\mu \boldsymbol{\nabla}_\perp B + v_\parallel^2\ \boldsymbol{\kappa} - \frac{e}{m}\mathbf{E}_\perp \right]$$

$$\begin{aligned} \left\langle \mathbf{w}_{\perp\perp}\ \widetilde{\frac{dv_\parallel}{dt}} \right\rangle &= \langle \mathbf{w}_{\perp\perp}\ \mathbf{w}_{\perp\perp} \rangle : (\boldsymbol{\nabla}\ \widehat{\mathbf{n}}) \\ &= \frac{\mu^2 B^2}{2} \left(\sigma_{\perp\perp}\ \sigma_{\perp\perp} + \tau_{\perp\perp}\tau_{\perp\perp} \right) : (\boldsymbol{\nabla}\ \widehat{\mathbf{n}}) \end{aligned}$$

$$\left\langle \mathbf{w}_{\perp\perp}\ \widetilde{\frac{d\mu}{dt}} \right\rangle = -\frac{v_\parallel}{B} \left\langle \mathbf{w}_{\perp\perp}\ \widetilde{\frac{dv_\parallel}{dt}} \right\rangle$$

The equation of convective part

$$\begin{aligned} \Lambda F &= \mathbf{v} \cdot \boldsymbol{\nabla} F + \frac{e}{m}\mathbf{E}\cdot\frac{\partial F}{\partial \mathbf{v}} \\ &= \frac{1}{B}\boldsymbol{\nabla}' (B\mathbf{v}F) + \frac{\partial}{\partial v_\parallel}\left(\frac{dv_\parallel}{dt} F \right) + \frac{\partial}{\partial \mu}\left(\frac{d\mu}{dt} F \right) + \frac{\partial}{\partial \zeta}\left(\frac{d\zeta}{dt} F \right) \end{aligned}$$

The two parts after averaging

$$\frac{\partial \overline{F}}{\partial t} + \left\langle \Lambda \overline{F} \right\rangle + \left\langle \Lambda \widetilde{F} \right\rangle$$

The two parts

$$\begin{aligned}
\left\langle \Lambda \overline{F} \right\rangle &= v_\parallel \nabla_\parallel \overline{F} + \overline{\frac{dv_\parallel}{dt}} \frac{\partial \overline{F}}{\partial v_\parallel} \\
&= \frac{1}{B} \boldsymbol{\nabla} \cdot \left(B \widehat{\mathbf{n}}\, v_\parallel \overline{F} \right) + \frac{\partial}{\partial v_\parallel} \left(\overline{\frac{dv_\parallel}{dt}}\, \overline{F} \right)
\end{aligned}$$

Further

$$\begin{aligned}
\left\langle \Lambda \widetilde{F} \right\rangle &= \frac{1}{B} \boldsymbol{\nabla} \cdot \left(B \left\langle \mathbf{v}_\perp \widetilde{F} \right\rangle \right) \\
&\quad + \frac{\partial}{\partial v_\parallel} \left\langle \widetilde{\frac{dv_\parallel}{dt}}\, \widetilde{F} \right\rangle \\
&\quad + \frac{\partial}{\partial \mu} \left\langle \widetilde{\frac{d\mu}{dt}}\, \widetilde{F} \right\rangle
\end{aligned}$$

The formulas that are involved here are

$$\left\langle \mathbf{v}_\perp\, \widetilde{F} \right\rangle = \widehat{\mathbf{n}} \times \left\langle \mathbf{v}_\perp\, \frac{\partial \widetilde{F}}{\partial \zeta} \right\rangle$$

This is obtained writting the average as an integration over the angle ζ and performing an integration by parts.

$$\left\langle \mathbf{w}_{\perp\perp}\, \widetilde{F} \right\rangle = \frac{1}{2} \widehat{\mathbf{n}} \times \left\langle \mathbf{w}_{\perp\perp}\, \frac{\partial \widetilde{F}}{\partial \zeta} \right\rangle$$

where

$$\Omega \frac{\partial \widetilde{F}}{\partial \zeta} = \Lambda \overline{F} - \left\langle \Lambda\, \overline{F} \right\rangle - \left(D - \overline{D} \right)$$

The last terms come from the fluctuations

$$D = -\frac{e}{m} \left\{ \left\{ \left(\delta \mathbf{E} + \mathbf{v} \times \delta \mathbf{B} \right) \cdot \frac{\partial \delta f}{\partial \mathbf{v}} \right\} \right\}_{statistical-mean}$$

The equation that is finally obtained

$$\begin{aligned}
& \frac{\partial \overline{F}}{\partial t} + \left(v_\parallel \widehat{\mathbf{n}} + \mathbf{v}_D + \mathbf{v}_\parallel^D \right) \cdot \boldsymbol{\nabla} \overline{F} \\
& + a_{v_\parallel} \frac{\partial \overline{F}}{\partial v_\parallel} + a_\mu \frac{\partial \overline{F}}{\partial \mu} \\
= \; & \overline{C}
\end{aligned}$$

The parallel drift is

$$\mathbf{v}_\parallel^D = \frac{m}{e} \mu \, \widehat{\mathbf{n}} \; \left\{ \widehat{\mathbf{n}} \cdot \left(\boldsymbol{\nabla} \times \widehat{\mathbf{n}} \right) \right\}$$

and

$$\begin{aligned}
a_{v_\parallel} \; = \; & -\mu \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} B + \frac{e}{m} E_\parallel \\
& + v_\parallel \mathbf{v}_D \cdot \boldsymbol{\kappa} \\
& - \mu \frac{B v_\parallel}{\Omega} \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \times \boldsymbol{\kappa}
\end{aligned}$$

This is the "acceleration" $\left(dv_\parallel / dt \right)$ in the energetic effect of the changes of $v_\parallel$

$$a_\mu = \frac{\mu}{\Omega} \left[v_\parallel^2 \; \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \times \boldsymbol{\kappa} - \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \times \widehat{\mathbf{n}} \right) \; \widehat{\mathbf{n}} \cdot \left(\mu \boldsymbol{\nabla} B - \frac{e}{m} \mathbf{E} \right) \right]$$

This is "acceleration" $(d\mu/dt)$ in the energetic effect of the changes of μ

Returning to the formulas

$$\boldsymbol{\nabla} \times \widehat{\mathbf{n}} = \widehat{\mathbf{n}} \; \left[\widehat{\mathbf{n}} \cdot \left(\boldsymbol{\nabla} \times \widehat{\mathbf{n}} \right) \right] + \widehat{\mathbf{n}} \times \boldsymbol{\kappa}$$

$$\begin{aligned}
\widehat{\mathbf{n}} \cdot \left(\boldsymbol{\nabla} \times \boldsymbol{\kappa} \right) \; = \; & \left(\boldsymbol{\nabla} \cdot \widehat{\mathbf{n}} \right) \; \widehat{\mathbf{n}} \cdot \left(\boldsymbol{\nabla} \times \widehat{\mathbf{n}} \right) \\
& + \widehat{\mathbf{n}} \cdot \left[\left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \right) \left(\boldsymbol{\nabla} \times \widehat{\mathbf{n}} \right) \right]
\end{aligned}$$

and

$$\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \times \boldsymbol{\kappa} = \mathbf{B} \cdot \boldsymbol{\nabla} \left(\frac{\widehat{\mathbf{n}} \cdot \left(\boldsymbol{\nabla} \times \widehat{\mathbf{n}} \right)}{B} \right)$$

The terms in the drift kinetic equation in the *conservative* form

$$\begin{aligned}
& \boldsymbol{\nabla} \cdot \left[B \left(\widehat{\mathbf{n}} v_\parallel + \mathbf{v}_\parallel^D + \mathbf{v}_D \right) \right] \\
= \; & \frac{m}{e} \left[\left(\boldsymbol{\nabla} \times \widehat{\mathbf{n}} \right) \cdot \left(\mu \boldsymbol{\nabla} B - \frac{e}{m} \mathbf{E} \right) \right. \\
& \left. - v_\parallel^2 \; \mathbf{B} \cdot \boldsymbol{\nabla} \left(\frac{\widehat{\mathbf{n}} \cdot \left(\boldsymbol{\nabla} \times \widehat{\mathbf{n}} \right)}{B} \right) + \mu \mathbf{B} \cdot \boldsymbol{\nabla} \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \times \widehat{\mathbf{n}} \right) \right] \\
& + \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \times \mathbf{E}
\end{aligned}$$

and for the *accelerations*

$$\begin{aligned}\frac{\partial}{\partial v_\parallel}\left(B a_{v_\parallel}\right) &= \frac{m}{e}\widehat{\mathbf{n}}\times\left(\mu\boldsymbol{\nabla}B-\frac{e}{m}\mathbf{E}\right)\cdot\boldsymbol{\kappa} \\ &\quad -\frac{m}{e}\mu B\ \mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\times\widehat{\mathbf{n}}}{B}\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial\mu}\left(B a_\mu\right) &= \frac{m}{e}\left[v_\parallel^2\ \mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\times\widehat{\mathbf{n}}}{B}\right)-2\mu\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}B\right)\ \widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\times\widehat{\mathbf{n}}\right] \\ &\quad +\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\times\widehat{\mathbf{n}}\right)\ E_\parallel\end{aligned}$$

The drifts are separated
perpendicular drift

$$\mathbf{v}_D=\frac{1}{\Omega}\widehat{\mathbf{n}}\times\left(\mu\boldsymbol{\nabla}_\perp B+v_\parallel^2\boldsymbol{\kappa}-\frac{e}{m}\mathbf{E}\right)$$

and the parallel drift

$$\mathbf{u}_D=\frac{m}{e}\mu\ \widehat{\mathbf{n}}\ \left(\widehat{\mathbf{n}}\ \cdot\boldsymbol{\nabla}\times\widehat{\mathbf{n}}\right)$$

12 General notes on the Drift-kinetic equation

In **Hazeltine 1974**

$$\frac{\partial f}{\partial t}+\left(v_\parallel\widehat{\mathbf{n}}+\mathbf{v}_D\right)\cdot\boldsymbol{\nabla}f+\frac{e}{m}\frac{\partial\phi}{\partial t}\frac{\partial f}{\partial\epsilon}=C\left(f\right)$$

where

$$\mathbf{v}_D=-v_\parallel\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\left(\frac{v_\parallel}{\Omega}\right)$$

$$\begin{aligned}\epsilon &\equiv \frac{v^2}{2}+\frac{e\phi}{m} \\ \mu &\equiv \frac{v_\perp^2}{2B}\end{aligned}$$

$$\boldsymbol{\nabla}v_\parallel=-\frac{\mu\boldsymbol{\nabla}B+\frac{e}{m}\boldsymbol{\nabla}\phi}{v_\parallel}$$

13 Axisymmetric torus (Rutherford 1970)

The calculation is made for the drift-kinetic equation, *i.e.* after averaging over the gyroangle.

The **particle diffusion in axisymmetric torus Rutherford 1970**.

Main steps.

- conservation of the longitudinal invariant, $\rightarrow$ shift of the basic variable $\psi' = \psi + \frac{m}{e} R v_\varphi$;
- separation of the distribution function
  - part f that is non-collisional and reflects the drift (or the *shift* of ψ)
  - part g that is the result of collisions
- expansion of f (in $\delta = \rho_\theta / L$) and expansion of g in ν_{ei}/ω_b;
- both functions f and g have guiding center and gyrophase - dependent parts;
- the Fokker-Planck equation with Landau collision operator
- the *continuity* equation
- the *zero-radial total current* through a magnetic surface.

The geometry

$$\mathbf{B}_T = B_T \widehat{\mathbf{e}}_\varphi$$

and

$$R B_T = \text{const}$$

$$\mathbf{B}_\perp = \boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\varphi = \boldsymbol{\nabla}\chi$$

The same geometric variable χ, a "potential" for the poloidal magnetic field, is used by **Hirshman Sigmar**

We have

$$|\boldsymbol{\nabla}\psi| = R B_\theta$$

$$\mathbf{B}_\perp = R B_\theta \frac{1}{R} \widehat{\mathbf{e}}_r \times \widehat{\mathbf{e}}_\varphi = B_\theta \, \widehat{\mathbf{e}}_\theta$$

we see that the coordinate χ contains a factor B_θ,

$$\boldsymbol{\nabla}\chi = B_\theta \widehat{\mathbf{e}}_\theta$$

and

$$d\chi = \left|\boldsymbol{\nabla}\chi\right| dl_\theta = B_\theta \ r d\theta$$

Later it is given in detail

$$d\chi = B_\perp \ r d\theta$$

Later we will need the annihilator

$$\begin{aligned} \oint \frac{dl}{B} ... &= \oint \frac{d\chi}{B_\perp^2} ... \\ &= \oint \frac{r d\theta}{B_\theta} \end{aligned}$$

The kinetic equation

$$\frac{df}{dt} = C\left(f, f\right)$$

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} f + \frac{e}{m}\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)\cdot\frac{\partial f}{\partial \mathbf{v}} = C\left(f, f\right)$$

$$C\left(f, f\right) = \sum_k C_{jk}\left(f, f\right)$$

The **Fokker-Planck** collision operator in the **Landau** form

$$\begin{aligned} C_{jk}\left(f, f\right) &= c_{jk}\frac{\partial}{\partial \mathbf{v}} \cdot \int d^3v' \frac{\partial^2 u}{\partial \mathbf{v}\partial \mathbf{v}} \\ &\times \left(\frac{\partial f_j\left(\mathbf{v}\right)}{\partial \mathbf{v}} f_k\left(\mathbf{v}'\right) - \frac{m_j}{m_k} f_j\left(\mathbf{v}\right)\frac{\partial f_k\left(\mathbf{v}'\right)}{\partial \mathbf{v}'}\right) \end{aligned}$$

$$c_{jk} = 2\pi \frac{e_j^2 e_k^2}{m_j^2} \ln \Lambda$$

where the relative velocity is

$$\mathbf{u} \equiv \mathbf{v} - \mathbf{v}'$$

$$\frac{\partial^2 u}{\partial \mathbf{v}\partial \mathbf{v}} = \frac{1}{u^3}\left(u^2 \mathbf{I} - \mathbf{u}\mathbf{u}\right)$$

Axisymmetry, or conservation of the canonical angular momentum in toroidal geometry

$$\frac{d}{dt}\left[R\left(mv_\varphi + eA_\varphi\right)\right] = 0$$

or

$$Rmv_\varphi + eRA_\varphi = \text{const}$$

Now we have

$$\begin{aligned}
\psi &= \iint\limits_{\Sigma} d\mathbf{s}\cdot\mathbf{B} \\
&= \iint\limits_{\Sigma} d\mathbf{s}\cdot B_\theta \widehat{\mathbf{e}}_\theta = \iint\limits_{\Sigma} d\mathbf{s}\cdot\left(\boldsymbol{\nabla}\times A_\varphi \widehat{\mathbf{e}}_\varphi\right) \\
&= \oint\limits_{\Gamma} d\mathbf{l}_\varphi \cdot A_\varphi \widehat{\mathbf{e}}_\varphi = \oint\limits_{\Gamma} \left(R_0 + r\cos\theta\right) d\varphi \ A_\varphi \\
&= 2\pi\left(R_0 + r\cos\theta\right) A_\varphi \\
&= 2\pi R A_\varphi
\end{aligned}$$

The factor 2π is kept also by **Yushmanov**.

Leaving the factor 2π for a normalization of the surface integral in the definition of ψ we note that

$$\psi = RA_\varphi$$

and this allows us to express the longitudinal invariant as

$$\psi + \frac{Rmv_\varphi}{e} = \text{const}$$

Note the conservation of the logitudinal invariant leads to a shift of the position of the particle relative the magnetic surface, $\psi \rightarrow \psi'$. This is the *drift surface* used by **Kagan Catto**.

For the Maxwellian, this is a perturbation, resulted from the neoclassical drift of the particle. **End.**

The other invariant, energy per unit mass

$$\epsilon = \frac{v^2}{2} + \frac{e\phi}{m}$$

The distribution function solution of the *drift-kinetic* equation can be expressed in terms of the two invariants

$$f = f\left(\epsilon, \psi + \frac{Rmv_\varphi}{e}\right)$$

This is a distribution function that solves the drift-kinetic equation *without collisions*

$$\frac{df}{dt} = 0$$

Since there is a shift relative to the magnetic surface ψ, we can introduce a distribution function that is expressed only in terms of the basic invariants

$$\epsilon, \psi$$

and a correction to this distribution function f, expressed as a series expansion of the departure

$$\psi + \frac{Rmv_\varphi}{e} \quad \text{relative to} \quad \psi$$

(*i.e.* the usual small neoclassical parameter $\delta = \rho / L$). Then the distribution function is a series

$$f = f_0 + f_1 + f_2 + ...$$

and the successive orders are given as powers of

$$\frac{Rmv_\varphi}{e}$$

which is a small departure from the value ψ of the poloidal flux function for the magnetic surface.

The terms of the series f_0, f_1 , f_2, ... represent corrections that are necessary to the Maxwellian because there is an invariant, the longitudinal invariant. A particle cannot stay on only one magnetic surface since the magnetic gauge potential present in the invariant imposes a variation which can only be compensated by departures from the surface ψ.

This is the situation *if there are NO collisions.*

The parameter of the expansion of f until now is

$$\delta = \frac{\rho_\theta}{L}$$

and is the measure of the departure of the particle in the neoclassical drift, relative to the magnetic surface, ψ.

For collisions, the small parameter results from comparing the transit frequency with the collision frequency.

When we take into account the collisions, we have to consider the a correction to f. This correction is denoted

$$g$$

and the new distribution function is written

$$f \equiv f\left(\epsilon, \psi + \frac{Rmv_\varphi}{e}\right) + g$$

and it is a solution of the drift-kinetic equation which now contains the collisions

$$\frac{df}{dt} = C(f, f)$$

To solve the drift kinetic equation taking into account the small departure from (ϵ, ψ) and the *collisions*, we write series expansions

$$\begin{aligned} f &= f_0 + f_1 + f_2 + \dots \\ g &= \qquad\; g_1 + g_2 + \dots \end{aligned}$$

The zeroth order f_0 is Maxwellian

$$f_0 = \frac{N(\psi)}{(2\pi)^{3/2} v_{th}^3} \exp\left(-\frac{\epsilon}{v_{th}^2}\right)$$

$$v_{th}^2 = \frac{T}{m}$$

It is obvious that the first order in f is obtained by Taylor expansion of the shifted variable $\psi' = \psi + \frac{Rmv_\varphi}{e}$:

$$f_1 = \frac{mRv_\varphi}{e} \frac{\partial f_0(\epsilon, \psi)}{\partial \psi}$$

Note This first order neoclassical correction, purely geometrical, determined by the drift of the particles in toroidal system has an gyration-averaged part

$$\overline{f}_1 = RB_T \frac{1}{\Omega} v_\parallel \frac{\partial f_M}{\partial \psi}$$

and a gyration-dependent part

$$\begin{aligned} \widetilde{f}_1 &= RB_T \frac{1}{\Omega} v_\perp \sin\zeta \frac{\partial f_M}{\partial \psi} \\ &= \frac{1}{\Omega} (\mathbf{v} \times \widehat{\mathbf{n}}) \cdot \boldsymbol{\nabla}\psi \; \frac{\partial f_M}{\partial \psi} \end{aligned}$$

End.

COMMENT

We see that taking

$$\begin{aligned}\psi &= R\int_0^r B_\theta(r)\,dr \\ \frac{\partial}{\partial\psi} &= \frac{1}{RB_\theta}\frac{\partial}{\partial r}\end{aligned}$$

then the first correction (exclusively coming from neoclassic orbit, and NOT from collisions) is

$$\begin{aligned}f_1 &= RB_T\frac{1}{\Omega}v_\parallel\frac{\partial f_M}{\partial\psi} \\ &= \frac{v_\varphi}{eB_\theta/m}\frac{\partial f_0}{\partial r} = \frac{v_\varphi}{\Omega_\theta}\frac{\partial f_0}{\partial r} \\ &\sim \rho_\theta\frac{\partial f_0}{\partial r}\end{aligned}$$

is the change of the equilibrium distribution function over a distance v_φ/Ω_θ from the magnetic surface, which is a Larmor radius for the poloidal field.

END

NOTE

In **Hirshman Sigmar Clarke** the first part of the correction

$$f_{a1} = -I\frac{v_\parallel}{\Omega_a}\frac{\partial f_{a0}}{\partial\psi} + g_a(\psi,\epsilon,\mu)$$

is called *diamagnetic response* of the species a.

The second part g_a is the *collisional response* of the species a.

This is zero for trapped particles $g_a^{trapped} = 0$.

The equation of the collisional correction g_a is obtained by considering the solubility conditions for the next higher order equation in the "collisionality parameter" expansion, ν_{*a}.

END

Rutherford proves that the correction g is zero in the trapped region.

With $\epsilon = \frac{v^2}{2} + \frac{e\phi}{m}$ the density is Boltzmann, as obtained from the integration over the space of velocities of the zero order function f_0

$$n(\psi) = N\exp\left(-\frac{e\phi}{\kappa T}\right)$$

The order of magnitude

$$\begin{array}{ccc} \frac{e}{m}\mathbf{v}\times\mathbf{B}\cdot\frac{\partial}{\partial\mathbf{v}} & \rightarrow & \begin{array}{c}\text{order } 0\\ \text{gyration } \Omega\end{array} \\ \left.\begin{array}{c}\frac{e}{m}\boldsymbol{\nabla}\phi\\ \mathbf{v}\cdot\boldsymbol{\nabla}\end{array}\right\} & \rightarrow \text{ order } 1 & \begin{array}{c}\text{energy, electric field}\\ \text{convection, drift}\end{array} \\ \dfrac{\partial}{\partial t} \text{ is of order } 3 & & \end{array}$$

With f_0 and f_1 calculated, we can go to the g functions, which represent the effect of collisions.

Then

$$\begin{aligned} \frac{e}{m}\mathbf{v}\times\mathbf{B}\cdot\frac{\partial}{\partial\mathbf{v}}g_1 &= C(f_0,f_0) \\ &= 0 \end{aligned}$$

This refers to gyration. It should detect the variation of g_1 with the *angle of gyration.*

$$\begin{aligned} & \mathbf{v}\cdot\boldsymbol{\nabla}g_1+\frac{e}{m}(-\boldsymbol{\nabla}\phi)\cdot\frac{\partial}{\partial\mathbf{v}}g_1 \\ & \qquad +\frac{e}{m}\mathbf{v}\times\mathbf{B}\cdot\frac{\partial}{\partial\mathbf{v}}g_2 \\ = \; & C(f_0,f_1)+C(f_0,g_1) \end{aligned}$$

This is: convection of g_1 and energy effect of an electric field on g_1 is compared with *gyration* of g_2. The balance is ensured by collisions.

In velocity space we have *acceleration* in the electric field $-\boldsymbol{\nabla}\phi$ and effect of gyration.

The equation of continuity.

In order three (the time variation)

$$\frac{\partial n}{\partial t}+\boldsymbol{\nabla}\cdot\int d^3v\;\mathbf{v}g=0$$

The collision operator

$$C_j(f_0,f_1)=\sum_k C_{jk}(f_0,f_1)$$

is *linearized*

$$
\begin{aligned}
& C_{jk}\left(f_0, f_1\right) \\
= \quad & c_{jk}\frac{\partial}{\partial \mathbf{v}} \cdot \int d^3v' \; \frac{\partial^2 u}{\partial \mathbf{v}\;\partial \mathbf{v}} \cdot \left[f_{0k}\left(\frac{\mathbf{v} f_{1j}}{v_{th,j}^2} + \frac{\partial f_{1j}}{\partial \mathbf{v}} \right) - \frac{m_j}{m_k} f_{0j} \left(\frac{\mathbf{v}' f_{1k}}{v_{th,k}^2} + \frac{\partial f_{1k}}{\partial \mathbf{v}'} \right) \right]
\end{aligned}
$$

using

$$\mathbf{u}\cdot\frac{\partial^2 u}{\partial \mathbf{v}\;\partial \mathbf{v}} = 0$$

$$c_{jk} = 2\pi \frac{e_j^2 e_k^2}{m_j^2} \ln \Lambda$$

NOTE

In **Gaffey** one finds

$$
\begin{aligned}
\boldsymbol{\omega} &= \frac{\partial^2 u}{\partial \mathbf{v}_i \partial \mathbf{v}_i} \\
&= \frac{1}{u}\,\mathbf{I} - \frac{\mathbf{u}\;\mathbf{u}}{u^3}
\end{aligned}
$$

and, after definition

$$x_{ij} \equiv \frac{v_i}{v_{th,j}}$$

also

$$\frac{\partial^2 x_{ij}}{\partial \mathbf{v}_i\;\partial \mathbf{v}_i} = \frac{1}{v_{th,j}} \frac{1}{v_i^3}\left(v_i^2\;\mathbf{I} - \mathbf{v}_i\;\mathbf{v}_i\right)$$

$$
\begin{aligned}
\frac{\partial x_{ij}}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 x_{ij}}{\partial \mathbf{v}_i\;\partial \mathbf{v}_i} &= 0 \\
\frac{1}{v_{th,j}} \frac{\mathbf{v}_i}{v_i} \cdot \frac{1}{v_{th,j}} \frac{1}{v_i^3}\left(v_i^2\;\mathbf{I} - \mathbf{v}_i\;\mathbf{v}_i\right) &= 0
\end{aligned}
$$

END

The charge neutrality

$$\sum e \int d^3v \; \left(f_1 + g_1 - \frac{e\phi}{\kappa T} f_0 \right) = 0$$

This equation has a particularity.

The functions f_1 and g_1 will be found to have a variation with the poloidal angle θ since they result from *magnetic modulation (mirror) effect.*

For the neutrality to be verified in every point (in particular for every θ) it is necessary to admit existence of a electrostatic potential with θ variation.

This electrostatic potential must occur in the energetic distribution of f_0, like Boltzmann distribution.

The derivation of the drift kinetic equation proceeds through the change of variables that emphasizes the gyroangle variable

$$\begin{aligned}
\mathbf{v} &= v_{\parallel}\widehat{\mathbf{n}} + \mathbf{v}_{\perp} \\
\mathbf{v}_{\perp} &= v_{\perp}\left(\widehat{\mathbf{e}}_1 \cos\zeta + \widehat{\mathbf{e}}_2 \sin\zeta\right)
\end{aligned}$$

for the system

$$\begin{aligned}
& \left(\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2, \widehat{\mathbf{n}}\right) \quad \text{or} \\
\widehat{\mathbf{e}}_1 &= \widehat{\mathbf{e}}_{\psi} \quad \text{(radial)} \\
\widehat{\mathbf{e}}_2 &= \widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_{\psi} \quad \text{(perpendicular)}
\end{aligned}$$

The new variables instead of (v_x, v_y, v_z) are

$$\epsilon = \frac{v^2}{2} + \frac{e\phi}{m} \quad \text{and} \quad \mu = \frac{v_{\perp}^2}{2B} \quad \text{and} \quad \zeta$$

$$v_{\parallel} = \sigma\sqrt{2\left(\epsilon - \mu B - \frac{e\phi}{m}\right)}$$

$$d^3v = \frac{B}{\left|v_{\parallel}\right|}\, d\epsilon d\mu d\zeta$$

For the Jacobian $B/\left|v_{\parallel}\right|$ see the change of variables calculated in this text.

The new form of the equation for g_1

$$\frac{e}{m}\mathbf{v} \times \mathbf{B} \cdot \frac{\partial}{\partial \mathbf{v}} g_1 = 0$$

is

$$\begin{aligned}
-\Omega_c \frac{\partial g_1}{\partial \zeta} &= 0 \\
&\rightarrow g_1\left(\mathbf{x},\ \epsilon, \mu\right) \\
&\quad \text{does NOT depend on gyrophase } \zeta
\end{aligned}$$

The next order

$$\begin{aligned}
& \mathbf{v}\cdot\boldsymbol{\nabla} g_1 \quad \text{spatial advection} \\
& +\frac{e}{m}\left(-\boldsymbol{\nabla}\phi\right)\cdot\frac{\partial}{\partial\mathbf{v}}g_1 \quad \text{electric field, acceleration} \\
& \qquad +\frac{e}{m}\mathbf{v}\times\mathbf{B}\cdot\frac{\partial}{\partial\mathbf{v}}g_2 \quad \text{gyration, at order 2} \\
= \; & C\left(f_0,f_1\right)+C\left(f_0,g_1\right)
\end{aligned}$$

is modified after adoption of the new variables and this suggests the separation in g_2 as follows

$$\begin{aligned}
g_2 \; = \; & -\frac{1}{\Omega_c}\left(\widehat{\mathbf{n}}\times\mathbf{v}\right)\cdot\boldsymbol{\nabla} g_1 \quad \text{perpendicular advection } \sim\rho \\
& -\left[\mathbf{v}\cdot\mathbf{v}_D+\frac{v_\parallel}{\Omega_c}\int d\zeta\left(\mathbf{v}\;\mathbf{v}_\perp:\boldsymbol{\nabla}\;\mathbf{n}-\frac{v_\perp^2}{2}\boldsymbol{\nabla}\cdot\mathbf{n}\right)\right]\frac{\partial g_1}{B\partial\mu} \quad \text{(transversal energy)} \\
& +g_2'
\end{aligned}$$

We note that the second line consists of variation in velocity space related to the changes in μ. Since μ is an invariant for a single particle, the change of μ comes only by collisions.

NOTE

In a previous Section it is derived the result

$$\begin{aligned}
\widetilde{f}_1 \; = \; & -\boldsymbol{\rho}\cdot\boldsymbol{\nabla}\overline{f}+\boldsymbol{\rho}\cdot\left(\mu\boldsymbol{\nabla} B+v_\parallel^2\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}+\frac{e}{m}\boldsymbol{\nabla}\phi\right)\frac{\partial\overline{f}}{B\partial\mu} \\
& +\frac{1}{\Omega}v_\parallel\mu\frac{\partial\overline{f}}{\partial\mu}\left[\widehat{\boldsymbol{\rho}}\widehat{\mathbf{v}}_\perp:\boldsymbol{\nabla}\widehat{\mathbf{n}}-\frac{1}{2}\widehat{\mathbf{n}}\cdot\left(\boldsymbol{\nabla}\times\widehat{\mathbf{n}}\right)\right]
\end{aligned}$$

This is also in **Hazeltine 1972**, see *instabilities*.

END

NOTE

In **Hsu Shaing Gromley poloidal damping** the equation for the distribution function that will be used for calculation of the averaged parallel viscosity $\langle\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}\rangle$ is

$$\begin{aligned}
& \frac{\partial\overline{f}}{\partial t}+\left(v_\parallel\widehat{\mathbf{n}}+\mathbf{V}\right)\cdot\boldsymbol{\nabla}\overline{f} \\
& -\left[\mathbf{V}\cdot\boldsymbol{\nabla}\left(\mu B\right)+\mu B\left(\boldsymbol{\nabla}\cdot\mathbf{V}-\widehat{\mathbf{n}}\cdot\left[\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\;\mathbf{V}\right]\right)\right]\frac{\partial\overline{f}}{\partial\mu} \\
& +\left[v_\parallel\frac{1}{nm}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\cdot\mathbf{P}-\mu B\left(\boldsymbol{\nabla}\cdot\mathbf{V}\right)-\left(v_\parallel^2-\mu B\right)\widehat{\mathbf{n}}\cdot\left[\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\;\mathbf{V}\right]\right]\frac{\partial\overline{f}}{\partial w} \\
= \; & C\left(\overline{f}\right)
\end{aligned}$$

We notice

$$\frac{d\mu}{dt} = (\mathbf{V} \cdot \boldsymbol{\nabla})(\mu B) + \mu B \left(\boldsymbol{\nabla} \cdot \mathbf{V} - \widehat{\mathbf{n}} \cdot [(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla})\ \mathbf{V}]\right)$$

which is the coefficient of $\frac{\partial \overline{f}}{\partial \mu}$ in the second line. The full term was

$$\frac{d\mu}{dt}\frac{\partial \overline{f}}{\partial \mu}$$

and is in *energy (velocity) space.*

END

Here

$$\begin{aligned}
\mathbf{v}_D &= \frac{1}{\Omega_c}\widehat{\mathbf{n}} \times \left[\left(\mu + \frac{v_\parallel^2}{B}\right)\boldsymbol{\nabla} B + \frac{e}{m}\boldsymbol{\nabla}\phi\right] \\
&= -v_\parallel \widehat{\mathbf{n}} \times \boldsymbol{\nabla}\left(\frac{v_\parallel}{\Omega_c}\right)
\end{aligned}$$

we **NOTE** the expression

$$\begin{aligned}
\frac{1}{\Omega_c}\left(\mu + \frac{v_\parallel^2}{B}\right)\boldsymbol{\nabla} B &= \frac{1}{\Omega_c}\left(\frac{v_\perp^2}{2} + v_\parallel^2\right)\boldsymbol{\nabla} \ln B \\
&\rightarrow \frac{1}{\Omega_c}\frac{\frac{v_\perp^2}{2} + v_\parallel^2}{R}\widehat{\mathbf{e}}_{vertical}
\end{aligned}$$

Then the equation becomes

$$v_\parallel \nabla_\parallel g_1 - \Omega_c \frac{\partial g_2'}{\partial \zeta} = C(f_0, f_1) + C(f_0, g_1)$$

(see also **Novakovskii**).

The functions will be separated in a part that is gyroaveraged and a part that retains the dependence on ζ,

$$g = \overline{g} + \widetilde{g}$$

with

$$\overline{g} = \int_0^{2\pi} \frac{d\zeta}{2\pi}\ g$$

which is integration over gyroangle.

After this separation

$$\begin{aligned}\widetilde{g}_1 &= 0 \\ & g_1 \text{ does not depend on the gyroangle } \zeta\end{aligned}$$

The equation $v_\parallel \nabla_\parallel g_1 - \Omega_c \frac{\partial g_2'}{\partial \zeta} = C(f_0, f_1) + C(f_0, g_1)$ is also separated and reduces to

$$v_\parallel \nabla_\parallel g_1 = \overline{C}(f_0, f_1) + \overline{C}(f_0, g_1)$$

and the other part

$$-\Omega_c \frac{\partial \widetilde{g}_2}{\partial \zeta} = \widetilde{C}(f_0, f_1) + \widetilde{C}(f_0, g_1)$$

This equation is strictly determined by gyration and collisions that affect the gyration.

The relations

$$\begin{aligned}\overline{C}\left(f_0, \widetilde{f}_1\right) &= 0 \\ \widetilde{C}(f_0, g_1) &= 0\end{aligned}$$

can be derived from the definition of the collision operator.

They lead to

$$v_\parallel \nabla_\parallel g_1 = C\left(f_0, \overline{f}_1\right) + C(f_0, g_1)$$

since $\overline{g}_1 = g_1$ since $\widetilde{g}_1 = 0$. In this equation there is no gyroangle-dependence left. It is the equation for the neoclassical part of the distribution function, g_1. This will produce the neoclassical radial transport flux.

For the next order, the part that depends on the gyroangle ζ is

$$-\Omega_c \frac{\partial g_2'}{\partial \zeta} = C\left(f_0, \widetilde{f}_1\right)$$

The two components (one independent on gyration and the other dependent on ζ) of the first order correction f_1 are

$$\begin{aligned}f_1 &= \frac{mR}{e} v_\varphi \frac{\partial f_0}{\partial \psi} \\ &= RB_T \frac{1}{eB_T/m} v_\varphi \frac{\partial f_0}{\partial \psi}\end{aligned}$$

The velocity along the toroidal direction, v_φ has two contributions: one $v_\varphi^{(1)}$ from the *parallel* velocity $v_\parallel$ and the other $v_\varphi^{(2)}$ from the poloidal velocity. The parallel velocity is independent on ζ. The poloidal component is a projection of $\mathbf{v}_\perp$ along the poloidal direction. The velocity $\mathbf{v}_\perp$ is NOT the guiding center velocity, it is the non-averaged-on-ζ velocity of a particle, therefore it depends on ζ.

The poloidal direction is

$$\sim \widehat{\mathbf{e}}_\psi \times \widehat{\mathbf{n}} = -\widehat{\mathbf{e}}_2$$

and this $\widehat{\mathbf{e}}_2$ component in $\mathbf{v}_\perp$ is

$$v_\perp \sin\zeta$$

Then

$$v_\varphi^{(1)} = v_\parallel \frac{B_T}{B}$$

$$\begin{aligned} \overline{f}_1 &= \frac{mR}{e} v_\varphi \frac{\partial f_0}{\partial \psi} = R \frac{1}{e/m} \frac{B_T}{B} v_\parallel \frac{\partial f_0}{\partial \psi} = RB_T \frac{1}{\Omega_c} v_\parallel \frac{\partial f_0}{\partial \psi} \\ &= RB_T \frac{1}{eB/m} v_\parallel \frac{1}{RB_\theta} \frac{\partial f_0}{\partial r} \approx \rho_\theta \frac{\partial f_0}{\partial r} \end{aligned}$$

and

$$v_\varphi^{(2)} = \mathbf{v}_\perp \cdot \widehat{\mathbf{e}}_\varphi$$

The contribution of the velocity that is oriented transversal on $\mathbf{B}$, $\mathbf{v}_\perp$, projected on the φ (toroidal direction) is

$$\mathbf{v}_\perp \cdot \widehat{\mathbf{e}}_\varphi = \left[v_\perp \left(\widehat{\mathbf{e}}_1 \cos\zeta + \widehat{\mathbf{e}}_2 \sin\zeta \right) \right] \cdot \widehat{\mathbf{e}}_\varphi$$

Since $\widehat{\mathbf{e}}_1 \equiv \widehat{\mathbf{e}}_\psi$ is perpendicular on ψ it does not contribute to this scalar product,

$$\begin{aligned} v_\varphi^{(2)} &= \mathbf{v}_\perp \cdot \widehat{\mathbf{e}}_\varphi = v_\perp \sin\zeta \ \left(\widehat{\mathbf{e}}_2 \cdot \widehat{\mathbf{e}}_\varphi \right) \\ &= v_\perp \sin\zeta \ \frac{B_\theta}{B} \end{aligned}$$

$$\begin{aligned} \widetilde{f}_1 &= \frac{mR}{e} v_\varphi^{(2)} \frac{\partial f_0}{\partial \psi} \\ &= \frac{R}{e/m} \frac{B_\theta}{B} \ v_\perp \sin\zeta \frac{\partial f_0}{\partial \psi} \end{aligned}$$

To express the perpendicular velocity from a vector operation, let us start from

$$\widehat{\mathbf{e}}_2 = \widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_\psi$$

and take

$$\begin{aligned}
\widehat{\mathbf{e}}_2 \times \widehat{\mathbf{n}} &= -\widehat{\mathbf{n}} \times (\widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_\psi) = -\widehat{\mathbf{n}} (\widehat{\mathbf{n}} \cdot \widehat{\mathbf{e}}_\psi) + \widehat{\mathbf{e}}_\psi (\widehat{\mathbf{n}} \cdot \widehat{\mathbf{n}}) \\
&= \widehat{\mathbf{e}}_\psi
\end{aligned}$$

this suggests to use

$$\begin{aligned}
& \mathbf{v}_\perp \times \widehat{\mathbf{n}} \\
= \ & v_\perp (\widehat{\mathbf{e}}_1 \cos\zeta + \widehat{\mathbf{e}}_2 \sin\zeta) \times \widehat{\mathbf{n}} = v_\perp (\widehat{\mathbf{e}}_1 \times \widehat{\mathbf{n}}) \cos\zeta + v_\perp (\widehat{\mathbf{e}}_2 \times \widehat{\mathbf{n}}) \sin\zeta \\
= \ & -v_\perp \widehat{\mathbf{e}}_2 \cos\zeta + v_\perp \widehat{\mathbf{e}}_\psi \sin\zeta
\end{aligned}$$

so if we take the scalar vector product with a vector that has direction $\widehat{\mathbf{e}}_\psi$, like $\boldsymbol{\nabla}\psi$ we select precisely the second term

$$[v_\perp (\widehat{\mathbf{e}}_1 \cos\zeta + \widehat{\mathbf{e}}_2 \sin\zeta) \times \widehat{\mathbf{n}}] \cdot \widehat{\mathbf{e}}_\psi = v_\perp \sin\zeta$$

Then we construct the expression

$$\begin{aligned}
[v_\perp (\widehat{\mathbf{e}}_1 \cos\zeta + \widehat{\mathbf{e}}_2 \sin\zeta) \times \widehat{\mathbf{n}}] \cdot \widehat{\mathbf{e}}_\psi &= v_\perp \sin\zeta \\
(\mathbf{v}_\perp \times \widehat{\mathbf{n}}) \cdot \widehat{\mathbf{e}}_\psi &= v_\perp \sin\zeta
\end{aligned}$$

and return to the part of f_1 that depends on ζ,

$$\begin{aligned}
\widetilde{f}_1 &= \frac{R}{e/m} \frac{B_\theta}{B} v_\perp \sin\zeta \frac{\partial f_0}{\partial \psi} \\
&= \frac{R}{\Omega_c} B_\theta (\mathbf{v}_\perp \times \widehat{\mathbf{n}}) \cdot \widehat{\mathbf{e}}_\psi \frac{\partial f_0}{\partial \psi}
\end{aligned}$$

here we can simply replace $\mathbf{v}_\perp$ with $\mathbf{v}$.

We take into account that

$$R B_\theta \widehat{\mathbf{e}}_\psi = \boldsymbol{\nabla}\psi$$

and obtain

$$\widetilde{f}_1 = \frac{1}{\Omega_c} (\mathbf{v}_\perp \times \widehat{\mathbf{n}}) \cdot \boldsymbol{\nabla}\psi \ \frac{\partial f_0}{\partial \psi}$$

The correction f_1 and its two sub-components $\overline{f}_1$ and $\widetilde{f}_1$ are purely geometrical, they are NOT determined by the collisions. They are generated by the shift in the spatial position of the current particle $\psi \rightarrow \psi'$ caused by the need to preserve the longitudinal invariant J.

The equation of continuity becomes

$$\frac{\partial n}{\partial t}+\boldsymbol{\nabla}\cdot\int d^3v\ \left(\mathbf{v}_\perp\ \widetilde{g}_2+v_\parallel\widehat{\mathbf{n}}\ \overline{g}_2\right)=0$$

We would like to eliminate the function $\overline{g}_2$ at this stage of calculations.
It is possible by applying the annihilator

$$\oint\frac{dl}{B}...$$

using

$$\boldsymbol{\nabla}\cdot\mathbf{B}=0$$

or

$$\oint\frac{rd\theta}{B_\theta}...$$

The result is

$$\frac{\partial n}{\partial t}+\frac{1}{\left(\oint\frac{rd\theta}{B_\theta}\right)}\frac{\partial\Gamma}{\partial\psi}=0$$

where the flux of particles averaged over the surface is

$$\Gamma=\oint\frac{rd\theta}{B_\theta}\int d^3v\ \left(\mathbf{v}_\perp\cdot\boldsymbol{\nabla}\psi\right)\ \widetilde{g}_2$$

Note
In the work by **Rutherford1970** it is

$$\Gamma=\oint\frac{d\chi}{B_\perp^2}\int d^3v\ \left(\mathbf{v}_\perp\cdot\boldsymbol{\nabla}\psi\right)\ \widetilde{g}_2$$

End.

Regarding this radial flux of transport, one now has to use an expression for $\widetilde{g}_2$.
The source is the equation for the second order collisional correction, g_2.
This equation includes a part expressed in terms of g_1 and a part that is g_2'.

The function g_1 does NOT depend on the gyrangle ζ.
It is the solution of the equation

$$v_\parallel\nabla_\parallel g_1=C\left[f_0,\overline{f}_1\right]+C\left[f_0,g_1\right]$$

The function g_2' is the solution of

$$-\Omega_c \frac{\partial g_2'}{\partial \zeta} = C\left[f_0, \widetilde{f}_1\right]$$

After solving these two equations, obtaining g_1 and g_2', we calculate $\widetilde{g}_2$ from the *tilde* version $\widetilde{()}$ of the equation for g_2

$$\begin{aligned}
\widetilde{g}_2 &= -\frac{1}{\Omega_c}\widehat{\mathbf{n}} \times \mathbf{v} \cdot \boldsymbol{\nabla} g_1 \\
&\quad -\left[\mathbf{v} \cdot \mathbf{v}_D + \frac{v_\parallel}{\Omega_c}\int d\zeta \left(\mathbf{v}\ \mathbf{v}_\perp : \boldsymbol{\nabla}\ \mathbf{n} - \frac{v_\perp^2}{2}\boldsymbol{\nabla}\cdot\mathbf{n}\right)\right]\frac{\partial g_1}{B\partial\mu} \\
&\quad +\widetilde{g}_2'
\end{aligned}$$

Here we have $\widetilde{g}_2' = g_2'$ since this part of g_2 is function of ζ . The first term in g_2 is a function of ζ due to the presence of $\mathbf{v}\times\widehat{\mathbf{n}}$ and the second term (with $\frac{\partial g_1}{B\partial\mu}$) has variation with ζ because of $\mathbf{v}$.

We have determined above the part of f_1 which depends on gyroangle ζ,

$$\widetilde{f}_1 = -\frac{1}{\Omega_c}\left(\widehat{\mathbf{n}} \times \mathbf{v}\right)\cdot\boldsymbol{\nabla}\psi\frac{\partial f_0}{\partial\psi} = RB_\theta\frac{1}{\Omega_c}v_\perp \sin\zeta\frac{\partial f_0}{\partial\psi}$$

and this is inserted in the collision operator in the equation $-\Omega_c\frac{\partial g_2'}{\partial\zeta} = C\left[f_0, \widetilde{f}_1\right]$. This equation is multiplied with

$$-\frac{1}{\Omega_c}\left(\widehat{\mathbf{n}} \times \mathbf{v}\right)\cdot\boldsymbol{\nabla}\psi$$

and the equation is integrated over velocity space. By this procedure the function g_1 is excluded. It will be calculated later.

The flux is of classical collisional transport flux, integrated over the volume inside the magnetic surface

$$\Gamma_j = -\oint\frac{d\theta}{B_\theta^2}\sum_k \frac{2}{3}\frac{1}{\Omega_j^2}c_{jk}\left|\boldsymbol{\nabla}\psi\right|^2\iint d^3v\ d^3v'\frac{1}{u}\left(f_{0k}\frac{\partial f_{0j}}{\partial\psi} - \frac{e_j}{e_k}f_{0j}\frac{\partial f_{0k}}{\partial\psi}\right)$$

for

$$u = \left|\mathbf{v} - \mathbf{v}'\right|$$

using

$$\left|\boldsymbol{\nabla}\psi\right| = RB_\theta$$

$$\begin{aligned}\Gamma_{classical} &= -\oint d\theta \frac{R^2}{B^2}\ \eta_\perp\ p\frac{\partial n}{\partial \psi} \\ \text{where } p &= 2n\kappa T \\ \eta_\perp &= \frac{4\sqrt{2\pi}}{3}\sqrt{m_e}e^2 \ln\Lambda\ \ (\kappa T)^{3/2}\end{aligned}$$

To calculate the effect of g_1 (which has been eliminated by the preceding operation) one has to insert the function $\widetilde{g}_2$, obtained taking the *tilde* of the expression of g_2, in the formula for the flux

$$\Gamma = \oint \frac{rd\theta}{B_\theta}\int d^3v\ \mathbf{v}_\perp \cdot \boldsymbol{\nabla}\psi \left(-\frac{1}{\Omega_c}\left(\widehat{\mathbf{n}}\times\mathbf{v}\right)\cdot\boldsymbol{\nabla} g_1 - \mathbf{v}\cdot\mathbf{v}_D\frac{\partial g_1}{B\partial\mu}\right)$$

We **note** the suppression of the term from the paranthesis $\frac{v_\parallel}{\Omega_c}\int d\zeta\left(\mathbf{v}\ \mathbf{v}_\perp : \boldsymbol{\nabla}\ \mathbf{n} - \frac{v_\perp^2}{2}\boldsymbol{\nabla}\cdot\mathbf{n}\right)$.

This can be seen after the adoption of the system

$$\epsilon,\mu,\zeta \to d^3v = 2\pi\frac{B}{|v_\parallel|}d\epsilon d\mu$$

and integrating by parts in χ (equivalent θ) and μ.

End.

Then

$$\begin{aligned}\Gamma &= -\oint\frac{rd\theta}{B_\theta}\int d^3v\ \mu B\left(\frac{1}{\Omega}\left(\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\psi\right)\cdot\boldsymbol{\nabla} g_1 + \mathbf{v}_D\cdot\boldsymbol{\nabla}\psi\frac{\partial g_1}{B\partial\mu}\right) \\ &= -\oint\frac{rd\theta}{B_\theta}\int d^3v\mu B\ \frac{1}{\Omega_c}\left(\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\psi\right)\ \left[\boldsymbol{\nabla} g_1 + v_\parallel\boldsymbol{\nabla}\left(\frac{v_\parallel}{B}\right)\frac{\partial g_1}{\partial\mu}\right] \\ &= \oint rd\theta\ B_\theta\int d^3v\ \frac{\mu R B_T}{\Omega_c}\left[\frac{\partial g_1}{\partial\chi} + v_\parallel\frac{\partial}{\partial\chi}\left(\frac{v_\parallel}{B}\right)\frac{\partial g_1}{\partial\mu}\right]\end{aligned}$$

We recall that

$$\frac{\partial}{\partial\chi} = \frac{1}{B_\theta}\frac{\partial}{r\partial\theta}$$

taking

$$d^3v = 2\pi\frac{B}{|v_\parallel|}d\epsilon d\mu$$

It is made an integration by parts on θ and in μ.

$$\begin{aligned}\Gamma &= RB_T\oint d\chi\int 2\pi d\epsilon d\mu\ \frac{|v_\parallel|}{\Omega_c}\frac{\partial g_1}{\partial\chi} \\ &= -RB_T\oint d\chi\int 2\pi d\epsilon d\mu\ g_1\frac{\partial}{\partial\chi}\left(\frac{|v_\parallel|}{\Omega_c}\right)\end{aligned}$$

In qualitative terms, the flux comes from the poloidal θ variation of ρ_θ.

Now we must return to calculate the function g_1.
The equation

$$v_{\|}\frac{B_\theta^2}{B}\frac{\partial g_1}{\partial \chi} = C\left(f_0,\overline{f}_1\right) + C\left(f_0, g_1\right)$$

This equation looks like $0 = -\nabla_{\|}p + R_{\|}^{\nu}$. The parallel convection is balanced by collisions.

[this is

$$v_{\|}\frac{B_\theta}{B}\frac{\partial g_1}{r\partial \theta} = v_{\|}\frac{1}{qR}\frac{\partial g_1}{\partial \theta} = v_{\|}\nabla_{\|}g_1$$

]

However its content is more complicated.

There is a variation in space of g_1, function of θ. However the spatial convection is balanced by a change in the *velocity space* since the collision consists of

- pitch angle scattering, affecting trapped / circulating
- friction, involving the neoclassical addition to the maxwellian due to drift $\psi \rightarrow \psi'$.

Here

$$\overline{f}_1 = RB_T\frac{1}{\Omega_c}v_{\|}\frac{\partial f_0}{\partial \psi}$$

is the shift of the Maxwellian with the distance ρ_θ relative to the magnetic surface

And must be replaced in

$$C\left[f_0\ ,\ \overline{f}_1\right] = \sum_k C_{jk}\left[f_0, \overline{f}_1\right]$$

$$\begin{aligned} C_{jk}\left[f_0\ ,\ \overline{f}_1\right] &= c_{jk}\ RB_T\frac{v_{\|}}{\Omega_i}\left(\frac{\partial}{\partial \epsilon}\frac{\mathbf{v}}{v_{\|}} + \frac{\partial}{B\partial \mu}\frac{\mathbf{v}_\perp}{v_{\|}}\right) \\ &\times \int d^3v'\ \frac{\partial^2 u}{\partial \mathbf{v}\ \partial \mathbf{v}} \cdot \widehat{\mathbf{n}}\left(f_{0k}\frac{\partial f_{0j}}{\partial \psi} - \frac{e_j}{e_k}f_{0j}\frac{\partial f_{0k}}{\partial \psi}\right) \end{aligned}$$

This formula is adapted for collisions electron-ions and the function g_1,

$$C_{ei}\left[f_0\ ,\ g_1\right] = \nu_{ei}\left(v\right)\frac{v_\parallel}{B}\frac{\partial}{\partial\mu}\left(\mu v_\parallel\frac{\partial g_{1e}}{\partial\mu}\right)$$

where

$$\begin{aligned}
\nu_{ei}\left(v\right) &= 2c_{ei}\,\frac{n}{v^3} \\
\text{with } c_{jk} &= 2\pi\frac{e_j^2 e_k^2}{m_j^2}\ln\Lambda \\
&\quad \text{Lorentz gas approximation}
\end{aligned}$$

The collisions electron-ions where the second function is neoclassic correction

$$C_{ei}\left[f_0,\overline{f}_1\right] = 2\nu_{ei}\left(v\right)\ RB_T\,\frac{v_\parallel}{B}\,\frac{m_e}{n}f_{0e}\frac{\partial n}{\partial\psi}$$

Let us comment

$$\begin{aligned}
& 2\nu_{ei}\left(v\right)\ RB_T\frac{1}{B}\frac{1}{RB_\theta}v_\parallel m_e f_{0e}\frac{d}{dr}\ln n \\
\sim\ & \nu_{ei}e\frac{v_\parallel}{\Omega_{e,\theta}}\frac{1}{L_n}f_{0e} = \nu_{ei}\ e\ \frac{v_\parallel}{\Omega_e}\frac{B}{B_\theta}\frac{1}{L_n}f_{0e}
\end{aligned}$$

and we have

$$v_{dia} \sim \frac{T}{eB}\frac{1}{L_n} = \frac{v_{th}^2}{\Omega_e}\frac{1}{L_n}$$

$$2\nu_{ei}\ e\frac{v_{th}^2}{v_{th}^2}\ \frac{v_\parallel}{\Omega_e}\frac{B}{B_\theta}\frac{1}{L_n}f_{0e} = 2\nu_{ei}\ e\frac{v_\parallel}{v_{th,e}^2}v_{dia}\frac{B}{B_\theta}f_{0e}$$

The diamagnetic velocity is perpendicular on magnetic field.

The equation for g_{1e} is

$$\begin{aligned}
B_\theta^2\frac{v_\parallel}{B}\frac{\partial g_{1e}}{\partial\chi} &= \nu_{ei}\left(v\right)\frac{v_\parallel}{B}\left[\frac{\partial}{\partial\mu}\left(\mu v_\parallel\frac{\partial g_{1e}}{\partial\mu}\right) + 2RB_T\frac{m_e}{e}f_{0e}\frac{1}{n}\frac{\partial n}{\partial\psi}\right] \\
&\quad + C_{ee}\left(f_0, g_{1e}\right)
\end{aligned}$$

The solution is obtained by expansion

$$g_{1e} = g_{1e}^{(0)} + g_{1e}^{(1)} + \ldots$$

for small collision frequency, ν_*/ω_b,

$$\frac{\partial g_{1e}^{(0)}}{\partial\chi}=0$$

equivalent to NO dependence on poloidal angle θ.

$$\begin{aligned}\frac{\partial g_{1e}^{(1)}}{\partial\chi} &= \nu_{ei}(v)\frac{1}{B_\theta^2}\left[\frac{\partial}{\partial\mu}\left(\mu v_\parallel\frac{\partial g_{1e}^{(1)}}{\partial\mu}\right)+2RB_\theta\frac{m_e}{e}f_{0e}\frac{1}{n}\frac{\partial n}{\partial\psi}\right]\\ &+C_{ee}\end{aligned}$$

The calculation proceeds through the exploitation of the *periodicity*. This is suggested by the LHS, $\partial/\partial\chi$ which is $\sim\theta$.

We integrate over the poloidal circle $\sim\theta$ or $\sim\chi$.
For untrapped particles the integration is over $[0,2\pi]$.
The LHS is zero due to periodicity and the RHS is

$$\frac{\partial}{\partial\mu}\left(\mu\frac{\partial g_{1e}^{(0)}}{\partial\mu}\oint\frac{v_\parallel d\chi}{B_\theta^2}\right)+2RB_T\frac{m_e}{e}f_{0e}\frac{1}{n}\frac{\partial n}{\partial\psi}\oint\frac{d\chi}{B_\theta^2}=0$$

The pitch angle (velocity space) part must be balanced by friction on the shifted distribution function.

This is integrated in μ, the second term gets a μ which is simplified with the one in the LHS,

$$\frac{\partial g_{1e}^{(0)}}{\partial\mu}=-2RB_T\frac{m_e}{e}f_{0e}\frac{1}{n}\frac{\partial n}{\partial\psi}\;\frac{\oint\frac{rd\theta}{B_\theta}}{\oint\frac{rd\theta}{B_\theta}v_\parallel}$$

The procedure for trapped particles takes into account that the function $g_{1e}^{(0)}$ does NOT change sign at the change

$$v_\parallel\rightarrow -v_\parallel$$

The equation

$$\frac{\partial}{\partial\mu}\left(\mu\frac{\partial g_{1e}^{(0)}}{\partial\mu}\oint\frac{\left|v_\parallel\right|d\chi}{B_\theta^2}\right)=0$$

is integrated between the two turning points $\pm\theta_0$,

$$\begin{aligned}\frac{\partial g_{1e}^{(0)}}{\partial\mu} &= \frac{\text{const}}{\mu\oint\frac{rd\theta}{B_\theta}\left|v_\parallel\right|} \\ \text{below it is shown that const} &= 0\end{aligned}$$

The equation $\frac{\partial}{\partial\mu}...$ for $g_{1e}^{(0)}$ is integrated

- over θ
- over μ
- over a thin layer across the boundary between trapped and circulating

Then

$$\left(\frac{\partial g_{1e}^{(0)}}{\partial\mu}\right)_{+v_\parallel}^{untrapped} + \left(\frac{\partial g_{1e}^{(0)}}{\partial\mu}\right)_{-v_\parallel}^{untrapped} = 2\left(\frac{\partial g_{1e}^{(0)}}{\partial\mu}\right)_{trapped}$$

The LHS must be zero, since the derivatives to μ at points (velocity-space) $\pm v_\parallel$ must be equal and opposite.

Then

$$\left(\frac{\partial g_{1e}^{(0)}}{\partial\mu}\right)_{trapped} = 0$$

The radial (transport) flux is

$$\Gamma = -c_{ei}\,2RB_T\frac{m_e}{e}n\oint\frac{rd\theta}{B_\theta}\int\frac{2\pi d\epsilon d\mu}{v^3}\frac{\left|v_\parallel\right|}{B}\left[\frac{\partial}{\partial\mu}\left(\mu v_\parallel\frac{\partial g_{1e}}{\partial\mu}\right)+2RB_T\frac{m_e}{e}f_{0e}\frac{1}{n}\frac{\partial n}{\partial\psi}\right]$$

The first term in square brackets can be integrated once by parts and $\frac{\partial g_{1e}^{(0)}}{\partial\mu}$ can be replaced by its expression derived above, $\frac{\partial g_{1e}^{(0)}}{\partial\mu} = -2RB_T\frac{m_e}{e}f_{0e}\frac{1}{n}\frac{\partial n}{\partial\psi}\frac{\oint\frac{rd\theta}{B_\theta}}{\oint\frac{rd\theta}{B_\theta}v_\parallel}$ on only the UNTRAPPED region in $\mathbf{v}$ space,

$$\Gamma = -c_{ei}\,4R^2B_T^2\frac{m_e^2}{e^2}\frac{\partial n}{\partial\psi}\oint\frac{rd\theta}{B_\theta}\int\frac{2\pi d\epsilon d\mu}{v^3}\mu\ f_{0e}\left(\frac{1}{\left|v_\parallel\right|}-\mathrm{H}\frac{\oint\frac{rd\theta}{B_\theta}}{\oint\frac{rd\theta}{B_\theta}\left|v_\parallel\right|}\right)$$

where H is nonzero only for circulating particles.

Substitution

$$\mu = \lambda \frac{v^2}{2}$$

Comparison with the classical diffusion

$$\begin{aligned}
\frac{D}{D_{classical}} &= \frac{\Gamma_{total}}{\Gamma_{classical}} \\
&= 1 + \frac{1}{\left(\oint \frac{rd\theta}{B_\theta}\frac{1}{B^2}\right)} \left\{ \oint \frac{rd\theta}{B_\theta}\frac{1}{B^2} \right. \\
&\qquad \left. - \frac{3}{4}\left(\oint \frac{rd\theta}{B_\theta}\right)^2 \int_0^{1/B_{\max}} \frac{\lambda d\lambda}{\oint \frac{rd\theta}{B_\theta}\sqrt{1-\lambda B}} \right\}
\end{aligned}$$

In small aspect ratio

$$B = \frac{B_0}{h}$$

$$\begin{aligned}
\frac{D}{D_{classical}} &= 1 + \frac{3\sqrt{2}}{2}\frac{B^2}{B_\theta^2}\sqrt{\frac{r}{R}}\left[1 - \int_0^1 \frac{dk}{k^2}\left(\frac{\pi}{2\mathbf{E}(k)} - 1\right)\right] \\
&\approx 1 + 1.6\frac{B^2}{B_\theta^2}\sqrt{\frac{r}{R}}
\end{aligned}$$

It is

$$1 + 1.6\left(\frac{rB}{RB_\theta}\right)^2 \frac{R^2}{r^2}\sqrt{\frac{r}{R}} = 1 + 1.6q^2\varepsilon^{-3/2}$$

For large collision frequency

$$\frac{D}{D_{classical}} = 1 + \frac{2\eta_\parallel}{\eta_\perp}\left(\frac{rB}{RB_\theta}\right)^2$$

Pfirsch Schluter enhancement factor

In **Hassam Drake** the *effective inertia* of the plasma at poloidal rotation is calculated and is similar, but it is a different physical effect.

14 Drift-kinetic equation with rotation Ware Wiley

This is also in *plasma general rotation*.

In the neoclassical theory (see **Ware Wiley)** it is found the velocity of plasma poloidal rotation which gives the *equilibrium*. An estimate of this mass velocity is

$$V = \frac{\rho_{i\theta}}{L} v_{thi}$$

where L is a typical scale length, like L_n. To obtain this it was supposed that $V \ll v_{thi}$.

To calculate the mass velocity of plasma at equilibrium V with better precision or for **higher values of** V, it is necessary to solve the drift-kinetic equation to second order in the small parameter ρ_θ / L. In [?], it is developed a model of drift-kinetic equation taking:

- first order in ρ_θ / L
- keeping V as a zeroth-order quantity (*i.e.* not very small compared to the thermal velocity)
- taking the zeroth-order expression of the distribution function as a **shifted Maxwellian**.

The computation is performed in the *frame moving with the velocity* V.

$$\text{absolute particle velocity} = \mathbf{V} + \mathbf{u} + \mathbf{s}$$

where $u\widehat{\mathbf{n}}$ is the **parallel velocity relative to the moving frame** and $\mathbf{s}$ is the perpendicular velocity in the moving frame

$$\mathbf{s} = s\widehat{\mathbf{e}}_n \cos\zeta - s\widehat{\mathbf{e}}_\perp \sin\zeta$$

where the versors

$$(\widehat{\mathbf{n}}, \widehat{\mathbf{e}}_n, \widehat{\mathbf{e}}_\perp)$$

correspond to the directions: parallel to the magnetic field, perpendicular (normal) to the magnetic surface and perpendicular to the vectorial product of these two ones.

The particle **equation of motion**

$$\frac{d}{dt}(\mathbf{V} + \mathbf{u} + \mathbf{s}) = \frac{e\mathbf{E}}{m} + \Omega(\mathbf{V} + \mathbf{s}) \times \widehat{\mathbf{n}}$$

$$\frac{\partial}{\partial t}(\mathbf{V}+\mathbf{u}+\mathbf{s})+(\mathbf{V}+\mathbf{u}+\mathbf{s})\cdot\boldsymbol{\nabla}(\mathbf{V}+\mathbf{u}+\mathbf{s})=\frac{e\mathbf{E}}{m}+\frac{e}{m}\mathbf{V}\times\mathbf{B}+\Omega\mathbf{s}\times\widehat{\mathbf{n}}$$

or

$$\frac{\partial}{\partial t}(\mathbf{u}+\mathbf{s})+(\mathbf{V}+\mathbf{u}+\mathbf{s})\cdot\boldsymbol{\nabla}(\mathbf{u}+\mathbf{s})=\mathbf{F}-(\mathbf{u}+\mathbf{s})\cdot\boldsymbol{\nabla}\mathbf{V}+\Omega\mathbf{s}\times\widehat{\mathbf{n}}$$

where the force on the unit mass is

$$\mathbf{F}=\frac{e}{m}(\mathbf{E}+\mathbf{V}\times\mathbf{B})-\frac{\partial\mathbf{V}}{\partial t}-\mathbf{V}\cdot\boldsymbol{\nabla}\mathbf{V}$$

Nota. It is not clear if the change to the moving frame system should be reflected in a change of the electric field. Instead of $\mathbf{E}$ we should put the transformed electric field.

Conclusion: there is no change of the coordinate system. Only some convenient quantities are used in the velocity space. The space coordinates remain unchanged.

NOTA on the simultaneous appearence in the equation of motion of the

- particle velocity $\mathbf{u}$ (**parallel)** and $\mathbf{s}$ (**perpendicular on $\mathbf{B}$**) and
- plasma rotation velocity $\mathbf{V}$.

This presence of two velocities of different nature generates particular effects. In the expression of the particle drift velocity (which normally should have to consist of only: gradB, curvature and electric ExB drifts) now appears the *force which is exerted on the plasma mass* and which is related with the plasma rotation velocity $\mathbf{V}$ by the equation of plasma momentum conservation. Certain effects of the **force** appearing in the plasma momentum equation will be transfered in the formula for the particle drift velocity, $\mathbf{v}_D$. In particular the *diamagnetic velocity* will be present in $\mathbf{v}_D$, which is rather unusual.

This change, not only in the formula of $\mathbf{v}_D$ but also in the nature of its comosition is obviously related to the change of coordinates in the velocity space:

$\mathbf{v}$ is the ion velocity in the REST frame

END OF THE NOTE

NOTE on the energetic term. When the term of convection of $\boldsymbol{\nabla} f$ is dominated by the velocity parallel arising from a plasma poloidal rotation:

$$v_{\|}\widehat{\mathbf{n}}+\mathbf{V} \text{ where}$$

$$\mathbf{V} = \frac{K(\psi)}{n}\mathbf{B}+R\left(-\frac{\partial\phi}{\partial\psi}\right)\widehat{\mathbf{e}}_{\varphi}$$

the energetic term is

$$v_{\parallel}\left(\frac{\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\cdot\mathbf{P}}{mn}-\nabla_{\parallel}\frac{v_{\parallel}KB}{n}\right)\frac{\partial\overline{f}}{\partial w}$$

This part also arises from the consideration of the force exerted on the particle by the fluid in motion. The forces arising in the fluid in motion are affecting the particle. This appears automatically since the equation drift-kinetic is written in the **REST FRAME** of the rotation of the plasma.

This is why we obtain the **presence of the (Pressure) viscous force $\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\cdot\mathbf{P}$ appears in the drift-kinetic equation, which is not a fluid equation for balance of forces.**

GENERAL CONCLUSION: when the drift-kinetic equation is written in the rest frame of the rotating plasma,

the fluid motion = plasma rotation
and the particle motion = drift motion and energy change are mixed

This is reflected by the drift-kinetic equation.

END of the NOTE.

The velocity space coordinates are

$$w=\frac{v^2}{2}\text{ (total energy, but relative to the moving frame)}$$

$$\mu=\frac{s^2}{2B}\text{ (magnetic moment, relative to the moving frame)}$$

ζ the gyroangle

The calculation is similar to the gyroaveraging performed in neoclassical theory to obtain the drift-kinetic equation.

$$\frac{dw}{dt}=\mathbf{F}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{V}$$

$$B\frac{d\mu}{dt}=\mu\frac{dB}{dt}-u\mathbf{s}\cdot\frac{d\widehat{\mathbf{n}}}{dt}+\mathbf{F}\cdot\mathbf{s}-\mathbf{s}\cdot\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{V}$$

$$\frac{d\zeta}{dt}=\Omega+\widehat{\mathbf{e}}_{\perp}\cdot\frac{d\widehat{\mathbf{e}}_n}{dt}+\frac{\widehat{\boldsymbol{\rho}}}{s}\cdot\left(u\frac{d\widehat{\mathbf{e}}_{\perp}}{dt}-\mathbf{F}+\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{V}\right)$$

where

$$\widehat{\boldsymbol{\rho}}=\widehat{\mathbf{e}}_n\sin\zeta+\widehat{\mathbf{e}}_{\perp}\cos\zeta$$

Now the gyroaveraging is performed

$$\langle g \rangle = \oint \frac{d\zeta}{2\pi} g$$

and the notation

$$g = \langle g \rangle + \widetilde{g}$$

The results:

$$\begin{aligned}
\mathbf{v}_d &= \frac{\mathbf{F}\times\widehat{\mathbf{n}}}{\Omega} + \\
&+\widehat{\mathbf{n}}\frac{\mu B}{\Omega}\left(\frac{j_{\|}}{B}\right) \\
&+\frac{1}{\Omega}\widehat{\mathbf{n}} \times \left(\mu \boldsymbol{\nabla} B + u^2 \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right) \widehat{\mathbf{n}}\right. \\
&\qquad \left. + \, \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{V} + \mathbf{V} \cdot \boldsymbol{\nabla} \mathbf{u}\right)
\end{aligned}$$

where

$$j_{\|} = \widehat{\mathbf{n}} \cdot \frac{1}{\mu_0} \left(\boldsymbol{\nabla} \times \mathbf{B}\right)$$

The change in the particle energy

$$\begin{aligned}
\dot{w} &= \mathbf{F}\cdot \left(\mathbf{u} + \mathbf{v}_d\right) \\
&-\frac{\mu B}{\Omega}\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \times \mathbf{F} \\
&-\mu B \boldsymbol{\nabla} \cdot \mathbf{V} \\
&-\left(u^2 - \mu B\right) \widehat{\mathbf{n}} \cdot \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \mathbf{V} \\
&-\mu B u \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{\pi}}{p}\right) \\
&+\frac{2u}{\Omega} \left(\widehat{\mathbf{e}}_n e_{n\|} + \widehat{\mathbf{e}}_{\perp} e_{n\perp}\right) \cdot \{-\mathbf{F}\times\widehat{\mathbf{n}} \\
&\qquad \left(3\mu B - u^2\right) \left(\widehat{\mathbf{n}} \times \boldsymbol{\nabla} \widehat{\mathbf{n}}\right) \\
&\qquad \left. \frac{\left(\mu B - u^2\right)}{u} \frac{\widehat{\mathbf{n}} \times \left(\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{V} + \mathbf{V} \cdot \boldsymbol{\nabla} \mathbf{u}\right)}{u} \right\}
\end{aligned}$$

where $\boldsymbol{\pi}$ is the **magnetic viscosity part of the pressure tensor**

$$\pi_{nn} = -\pi_{\perp\perp} = -\frac{p}{\Omega} e_{n\perp}$$

$$\pi_{\|\|} = 0$$

$$\pi_{n\perp} = \pi_{\perp n} = \frac{p}{2\Omega} \left(e_{nn} - e_{\perp\perp}\right)$$

where $\mathbf{e}$ is **the velocity stress tensor**

$$(\mathbf{e})_{\alpha\beta} = \frac{1}{2}\left(\widehat{\mathbf{e}}_\alpha \cdot \widehat{\mathbf{e}}_\beta \cdot \boldsymbol{\nabla}\mathbf{V} + \widehat{\mathbf{e}}_\beta \cdot \widehat{\mathbf{e}}_\alpha \cdot \boldsymbol{\nabla}\mathbf{V} - \frac{2}{3}\delta_{\alpha\beta}\boldsymbol{\nabla}\cdot\mathbf{V}\right)$$

The drift-kinetic equation, in its most general form:

$$\begin{aligned} \frac{\partial \overline{f}}{\partial t} + & \\ & + (\mathbf{u} + \mathbf{v}_d + \mathbf{V}) \cdot \boldsymbol{\nabla}\overline{f} \\ & + \left\langle \frac{d\mu}{dt} \right\rangle \frac{\partial \overline{f}}{\partial \mu} \\ & + \dot{w}\frac{\partial \overline{f}}{\partial w} \\ = \ & 0 \end{aligned}$$

The ion drift-kinetic equation

$$\begin{aligned} & \frac{\partial \overline{f}}{\partial t} + \left(v_\parallel \widehat{\mathbf{n}} + \mathbf{V}\right) \cdot \boldsymbol{\nabla}\overline{f} - \\ & - \left[\mathbf{V} \cdot \boldsymbol{\nabla}\mu B + \mu B \left(\boldsymbol{\nabla}\cdot\mathbf{V} - \widehat{\mathbf{n}}\widehat{\mathbf{n}} \colon \boldsymbol{\nabla}\mathbf{V}\right)\right] \frac{\partial \overline{f}}{\partial \mu} + \\ & + \left(v_\parallel \frac{\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\cdot\mathbf{P}}{n m_i} - \mu B \boldsymbol{\nabla}\cdot\mathbf{V} - \left(v_\parallel^2 - \mu B\right)\widehat{\mathbf{n}}\widehat{\mathbf{n}} \colon \boldsymbol{\nabla}\mathbf{V}\right) \frac{\partial \overline{f}}{\partial w} \\ = \ & C\left(\overline{f}\right) \end{aligned}$$

where the **ion STRESS tensor** is

$$\begin{aligned} \mathbf{P} & \equiv \\ & = nT\,\mathbf{I} + \boldsymbol{\Pi} \\ & \simeq nT\mathbf{I} + \frac{3}{2}\pi_\parallel\left(\widehat{\mathbf{n}}\widehat{\mathbf{n}} - \frac{1}{3}\mathbf{I}\right) \end{aligned}$$

where $\boldsymbol{\Pi}$ is the **ion VISCOSITY tensor**.

15 Drift kinetic equation in neoclassical theory

Take as example the drift kinetic equation from **Sugama Nishimura**.

This is also in *viscosity*.

It is prepared for non-axisymmetric systems, like LHD, stellarator and also for tokamak

$$V_{\parallel}\left[f_a^{(1)}\right]+\mathbf{v}_D\cdot\boldsymbol{\nabla}f_{Ma}-v_{\parallel}e_aB\frac{\langle BE_{\parallel}\rangle}{\langle B^2\rangle}\frac{1}{T_a}f_{Ma}=C_a^{lin}\left[f_a^{(1)}\right]$$

The drift velocity is

$$\mathbf{v}_D=\frac{1}{\Omega_a}\widehat{\mathbf{n}}\times\left(\mu\boldsymbol{\nabla}B+v_{\parallel}^2\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}+\frac{e_a}{m_a}\phi\right)$$

Here the parallel velocity is replaced by an operator.

If the system of variables is

$$\begin{aligned}&(\mathbf{x},\epsilon,\mu)\\ \epsilon &= \frac{1}{2}m_av^2+e_a\phi\end{aligned}$$

then the operator

$$V_{\parallel}\left[f_a^{(1)}\right]=v_{\parallel}\nabla_{\parallel}$$

If the system of variables is

$$(\mathbf{x},v,\xi)$$

then

$$V_{\parallel}\left[.\right]=v\xi\ \nabla_{\parallel}-\frac{1}{2}\left(1-\xi^2\right)\nabla_{\parallel}\ln B\ \frac{\partial}{\partial\xi}$$

This form has the structure

$$v_{\parallel}\nabla_{\parallel}+\frac{dv_{\parallel}}{dt}\frac{\partial}{\partial v_{\parallel}}$$

the last term is the energy echanges between the parallel and perpendicular components along the orbit.

Here

$$\begin{aligned}\frac{1}{2}\left(1-\xi^2\right)\nabla_{\parallel}\ln B &= \frac{1}{2}\frac{v^2-v_{\parallel}^2}{v^2}\nabla_{\parallel}\ln B=\frac{1}{v^2}\frac{v_{\perp}^2}{2B}\nabla_{\parallel}B=\frac{1}{v^2}\mu\nabla_{\parallel}B\\ &= \frac{1}{v^2}\times(\text{mirror force})\end{aligned}$$

This form seems more adequate to make explicit the existance of trapped particles.

However the other system of variables can show trapped particles when the collision operator introduces λ.

The last term in the drift-kinetic equation is the work done by the particle with parallel velocity $v_{\|}$ against the parallel electric field, if it exists

$$-v_{\|} e_a B \frac{\langle B E_{\|} \rangle}{\langle B^2 \rangle} \frac{1}{T_a} f_{Ma}$$

where $v_{\|} e E_{\|} \sim$ (velocity) $\times$ (force) $\sim$ (energy / time) and $-\frac{1}{T_e} f_{Ma} \sim \frac{\partial f_{Ma}}{\partial \epsilon}$.

We **note** that $E_{\|}$ *must exist* according to Pfirsch Schluter current and finite resistivity. The explanation given by **Stringer**.

Most of the treatments in neoclassical problems start from

$$v_{\|} \nabla_{\|} f_a^{(1)} + \mathbf{v}_D \cdot \boldsymbol{\nabla} f_{Ma} = \text{collisions}$$

But a choice of the system of variables $(\mathbf{x}, \epsilon, \mu)$ or $(\mathbf{x}, v, \xi)$ must be made when we adopt the form of the collision operator. The pitch angle scattering is usually expressed in terms of $\lambda = (v_\perp^2 / v^2)\, h$ which can be changed to ξ.

The trapped particles are made visible when one makes an integration by parts in λ to avoid the second derivative of the perturbed function g_a to λ, since this is singular. Then the limits in λ are adapted to trapped or circulating.

Two apparently distinct things.

- the term in the drift kinetic equation

  $$v_{\|} \nabla_{\|} f^{(1)}$$

  comes from the convective derivative

  $$\left(v_{\|} \widehat{\mathbf{n}} \right) \cdot \boldsymbol{\nabla} f$$

  and retains the first order correction to the distribution function since this can have a variation along the magnetic field line, or, equivalently, has variation on θ. This is related with the trapped particles, a purely neoclassical effect.

- the term

  $$\mathbf{v}_D \cdot \boldsymbol{\nabla} f_M$$

is the effect of the neoclassical drifts $\mathbf{v}_D$ on the equilibrium (Maxwellian) distribution function. This gives a correction to the distribution function that results from the convection of the equilibrium function f_M away from the magnetic surface where it is constant $[f_M(\psi)]$. This correction is

$$\sim \rho_\theta \frac{\partial f_M}{\partial r}$$

and also hides trapped (banana) particles.

Note In tokamak, due to the axisymmetry, the *neoclassical toroidal viscosity VANISHES*. This will be seen as

$$\begin{aligned}
\left\langle R^2 \boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi} \right\rangle &= \boldsymbol{\pi}^{(1)} + \boldsymbol{\pi}^{(2)} \sim \frac{\rho_i^2}{a^2} \ll 1 \\
\boldsymbol{\pi}^{(1)} &\approx 0 \\
\boldsymbol{\pi}^{(2)} &\sim \text{very small, the only way} \\
& \quad\ \text{to transport toroidal momentum} \\
& \quad\ \text{(Honda)}
\end{aligned}$$

(see *viscosity.tex*).

16 Drift-kinetic equation *Lebedev Diamond et al.*

The drift kinetic equation

$$\frac{\partial F}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} F + \frac{dU}{dt}\frac{\partial F}{\partial U} = St\{F\} + \frac{\delta F}{\delta t}$$

In the first order of the

$$\frac{\rho}{L}$$

expansion, the trajectories are

$$\frac{d\mathbf{X}}{dt} \equiv \mathbf{V} = \frac{1}{B}(\mathbf{B}U + \mathbf{V}_E + \mathbf{V}_d)$$

where U is the parallel velocity of a particle, with time variation determined by the variation of the magnetic field along the magnetic line (mirror effect) and by the presence of an electric field E,

$$\frac{dU}{dt} = -\frac{\mu}{m}\nabla_{\|} B + \frac{U}{B}\mathbf{V}_E \cdot \boldsymbol{\nabla} B$$

where

$$\mathbf{V}_d = \frac{1}{\Omega_c}\widehat{\mathbf{n}} \times \frac{\mu B + mU^2}{m}\boldsymbol{\nabla} \ln B$$

$$\mu = \frac{mv_\perp^2}{2B}$$

Now it is introduced a reference velocity directed along the toroidal angle

$$\mathbf{V}_\Phi = -\frac{1}{B_T}\left(2\pi R B_T \frac{\partial \overline{\Phi}}{\partial \psi}\right)(R\boldsymbol{\nabla}\varphi)$$

where

$$\varphi \equiv \text{toroidal angle}$$

The system of reference is changed to this rotating frame as if the plasma would rotate as a rigid body

$$S \to S_{\mathbf{V}_\Phi}$$

where the average radial electric field on the surface is *zero*.

$$F = f_M + f$$

where

$$f \sim \frac{\rho}{L}$$

The equation is

$$\begin{aligned}
&\frac{\partial f}{\partial t} + U\nabla_\| f - \left\{\frac{v_\perp^2}{2}\nabla_\| \ln B - \frac{1}{B}\frac{\partial}{\partial t}\left[\frac{1}{B}\left(2\pi R B_T \frac{d\overline{\Phi}}{d\psi}\right)\right]\right\}\frac{\partial f}{\partial U} - C^* f \\
= \;& -U\nabla_\| \left(\frac{e\widetilde{\Phi}}{T}\right) f_M \\
& + \frac{U}{T/m}\frac{\partial}{\partial t}\left[\frac{1}{B}\left(2\pi R B_T \frac{d\overline{\Phi}}{dt}\right)\right] f_M \\
& + \frac{1}{\Omega_c}\left(U^2 + \frac{v_\perp^2}{2}\right)\frac{1}{T}\left[\left(2\pi R B_T \frac{d\,(P/n)}{d\psi}\right) + \right. \\
& \qquad \left. + \left(\frac{U^2}{2T/m} + \frac{v_\perp^2}{2T/m} - \frac{5}{2}\right)\left(2\pi R B_T \frac{dT}{d\psi}\right)\right] \nabla_\| \ln B \; f_M + \frac{\delta F}{\delta t}
\end{aligned}$$

where has variation on the magnetic surface, *i.e.* it contains harmonic components (as suggested by Pfirsch Schluter current, see *variation on surface* text)

$$\widetilde{\Phi} \sim \sin\theta \;,\; \cos\theta$$

The parallel derivation is done with the operator

$$\nabla_{\|} \equiv \frac{B_T}{qRB} \frac{\partial}{\partial \theta}$$

which actually is

$$\nabla_{\|} = \frac{B_\theta}{B} \frac{\partial}{r \partial \theta}$$

The most important term in the equation is

$$U \nabla_{\|} f$$

and this *parallel free streaming* term results in a solution which is resonant at

$$U = 0$$

NOTE that this is clarified by the formal solution used by **Galeev** and by **Tendler Rozhansky** which consists of

$$\frac{1}{x} = P \left(\frac{1}{x} \right) + i \pi \delta (x)$$

where in this case $x = U$.

END.

The *resonance is broadened* by the term of the Left Hand Side

$$\frac{\partial}{\partial t} \left[\frac{1}{B} \left(2\pi R B_T \frac{d\overline{\Phi}}{d\psi} \right) \right] \frac{\partial f}{\partial U}$$

which represents the time derivative of the radial electric field which combined with the magnetic field B gives the poloidal velocity. Then the time derivative of the poloidal velocity (projected on poloidal direction) is multiplied by the variation of the distribution function with the parallel velocity U.

This term $\frac{1}{B} \frac{\partial}{\partial t} \left[\frac{1}{B} \left(2\pi R B_T \frac{d\overline{\Phi}}{d\psi} \right) \right]$ is actually connected with polarization, since the *radial* electric field $\frac{d\overline{\Phi}}{d\psi}$ is in zero order.

17 Drift kinetic equation Rosenbluth, Hazeltine and Hinton 1972

See below for the *solution of the drift kinetic equation by these authors.*

Remark:

The Drift Kinetic equation (from where many applications are developed) is

$$\frac{\partial f}{\partial t} + (\mathbf{v} + \mathbf{v}_D) \cdot \boldsymbol{\nabla} f + \frac{eE_\parallel}{m} v_\parallel \frac{\partial f}{\partial \epsilon} = C$$

One starts from the Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} f + \frac{e}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} = C(f)$$

for

$$f(\mathbf{x}, \mathbf{v}) = f(r, \theta, \epsilon, \mu)$$

The adiabatic invariance of μ gives

$$\frac{df}{dt} = \dot{r} \frac{\partial f}{\partial r} + \dot{\theta} \frac{\partial f}{\partial \theta} + \frac{e}{m} E_\parallel \frac{\partial f}{\partial \epsilon}$$

and introduce here the equations of motion

$$\begin{aligned} \dot{r} &= v_{D,r} \quad \text{(drift radial)} \\ r\dot{\theta} &= \left(v_\parallel\right)_\theta + v_{D,\theta} \end{aligned}$$

Then

$$\mathbf{v}_D \cdot \boldsymbol{\nabla} f + \left(v_\parallel \frac{B_\theta}{B}\right) \frac{\partial f}{\partial \theta} + \frac{e}{m} E_\parallel \frac{\partial f}{\partial \epsilon} = C(f)$$

where

$$\mathbf{v}_D = \text{drift velocity of a particle}$$

When the distribution function is written as

$$f = f_0 \left(1 + \widehat{f}\right)$$

it is obtained

$$f_M \left[v_{D,r} \left[A_1 + A_2 (\epsilon - e\Phi)\right] + v_\parallel \frac{B_\theta}{B} \frac{\partial \widehat{f}}{r \partial \theta} - v_\parallel \frac{eE_\parallel}{T} \right] = C(f)$$

where

$$\begin{aligned} A_{1a} &= \frac{n'}{n} - \frac{3}{2} \frac{T'}{T} + \frac{e_a \Phi'}{T} \\ A_2 &= \frac{T'}{T^2} \end{aligned}$$

$$()' = \frac{\partial}{\partial r}()$$

and the parameters did not have been labeled by type of particles $a = e, i$ because it was assumed that the temperatures of electrons and ions are equal.

The distribution function is written as

$$f = f_0 \left(1 + \widehat{f}\right)$$

and $\widehat{f}$ has variation in the surface (poloidal variation).

The equation

$$v_{\parallel} \frac{B_\theta}{B} f_0 \frac{\partial \widehat{f}}{r \partial \theta} + v_{D,r} \frac{\partial f_0}{\partial r} + e E_{\parallel} v_{\parallel} \frac{\partial f_0}{\partial \epsilon} = C(f)$$

We **Note**

$$\begin{aligned} \left(e E_{\parallel}\right) v_{\parallel} &= \text{force } \times \text{ velocity } = \text{ power } = \text{ energy/time} \\ -e \mathbf{v} \cdot \boldsymbol{\nabla} \phi &= -e \left(\frac{d}{dt} - \frac{\partial}{\partial t}\right) \phi \end{aligned}$$

The second term gives $i\omega$ after Fourier transform.

The first, after integration on time (which removes the time derivative) gives ω_* (diamagnetic). **End.**

We take the Maxwellian

$$f_0 = \frac{n}{\left(\pi \frac{2T}{m}\right)^{3/2}} \exp\left[-\frac{\epsilon - e\Phi}{T}\right]$$

where

$$\begin{aligned} \epsilon &= \frac{m v^2}{2} + e\Phi \\ \mu &= \frac{m v_\perp^2}{2B} \end{aligned}$$

The equation must be solved for the function $\widehat{f}$. A variational procedure will be used.

The term with $E_{\parallel}$ is treated invoking first the *Spitzer and Harm* solution of the problem

$$-v_{\parallel} e E_{\parallel} \frac{1}{T} f_M = C\left(v_{\parallel} E_{\parallel} f_s\right)$$

Spitzer and Harm, using a Fokker-Planck collisional operator have solved the problem and f_s is tabulated.

Then, since the collision operator is linearized, we can extract from the right hand side $C(f)$ the term $-v_\parallel \frac{eE_\parallel}{T} f_M$ since we can express it as $C\left(v_\parallel E_\parallel f_s\right)$. Then

$$\begin{aligned} f_M \left[v_{D,r} \left[A_1 + A_2 (\epsilon - e\Phi) \right] + v_\parallel \frac{B_\theta}{B} \frac{\partial \widehat{f}}{r\partial\theta} - v_\parallel \frac{eE_\parallel}{T} \right] &= C(f) \\ f_M \left[v_{D,r} \left[A_1 + A_2 (\epsilon - e\Phi) \right] + v_\parallel \frac{B_\theta}{B} \frac{\partial \widehat{f}}{r\partial\theta} \right] &= -\left(-v_\parallel \frac{eE_\parallel}{T} \right) + C(f) \\ &= -C\left(v_\parallel E_\parallel f_s\right) + C(f) \\ &= C\left(f - v_\parallel E_\parallel f_s\right) \text{ linearized} \end{aligned}$$

Until this point we used the first small parameter of the neoclassical drift kinetic theory

$$\delta \equiv \frac{\rho}{l} \ll 1$$

Now it is time to introduce the second small parameter

$$\begin{aligned} \Delta &\equiv \frac{\nu_{eff}}{\omega_{bounce}} \\ &= \frac{\text{freq.of collisions}}{\text{freq. of bounce on banana}} \\ &\ll 1 \end{aligned}$$

This means that the bounce on banana is very rapid compared with collisions and the average over the bounce is justified.

We expand the correction $f_0 \widehat{f}$ (which is order 1 in the spatial-like small parameter δ) as a series in powers of $\Delta \equiv \nu_{eff}/\omega_{bounce}$;

$$\widehat{f} = \widehat{f}^{(0)} + \widehat{f}^{(1)} + ...$$

We take the zeroth order in $\nu_{eff}/\omega_{bounce}$ in the drift kinetic equation

$$f_M \left[v_{D,r} \left[A_1 + A_2 (\epsilon - e\Phi) \right] + v_\parallel \frac{B_\theta}{B} \frac{\partial \widehat{f}}{r\partial\theta} \right] = C\left(f - v_\parallel E_\parallel f_s\right)$$

We replace the radial drift velocity $v_{D,r}$ by

$$v_{D,r} = \frac{1}{\frac{eB_0}{m}} v_\parallel \frac{\partial}{r\partial\theta} \left(v_\parallel h\right)$$

and using

$$B = \frac{B_0}{h}$$

we have

$$v_{D,r} = \frac{1}{\frac{eBh}{m}} v_{\parallel} \frac{\partial}{r\partial\theta}\left(v_{\parallel} h\right)$$

and obtain

$$\frac{1}{\frac{eBh}{m}} v_{\parallel} \frac{\partial}{r\partial\theta}\left(v_{\parallel} h\right)\left[A_1 + A_2\left(\epsilon - e\Phi\right)\right] + v_{\parallel}\frac{B_\theta}{B}\frac{\partial \widehat{f}}{r\partial\theta} = 0$$

$$\frac{\partial \widehat{f}}{r\partial\theta} = -\frac{1}{\frac{eB_\theta h}{m}} \frac{\partial}{r\partial\theta}\left(v_{\parallel} h\right)\left[A_1 + A_2\left(\epsilon - e\Phi\right)\right]$$

But

$$B_\theta h = b\left(r\right)$$

does not depend on θ and can be introduced inside the derivation

$$\begin{aligned}
\frac{1}{\frac{eB_\theta h}{m}} \frac{\partial}{r\partial\theta}\left(v_{\parallel} h\right) &= \frac{\partial}{r\partial\theta}\left[\left(\frac{1}{\frac{eB_\theta h}{m}}\right) v_{\parallel} h\right] \\
&= \frac{\partial}{r\partial\theta}\left[v_{\parallel}\frac{1}{\frac{eB_\theta}{m}}\right] \\
&= \frac{\partial}{r\partial\theta}\left[\frac{v_{\parallel}}{\Omega_\theta}\right] \sim \frac{\partial}{r\partial\theta}\rho_\theta
\end{aligned}$$

This allows us to proceed

$$\begin{aligned}
\frac{\partial \widehat{f}^{(0)}}{r\partial\theta} &= -\frac{1}{\frac{eB_\theta h}{m}} \frac{\partial}{r\partial\theta}\left(v_{\parallel} h\right)\left[A_1 + A_2\left(\epsilon - e\Phi\right)\right] \\
&= -\frac{\partial}{r\partial\theta}\left[v_{\parallel}\frac{1}{\frac{eB_\theta}{m}}\right]\left[A_1 + A_2\left(\epsilon - e\Phi\right)\right] \\
&= -\frac{\partial}{r\partial\theta}\left\{v_{\parallel}\frac{1}{\frac{eB_\theta}{m}}\left[A_1 + A_2\left(\epsilon - e\Phi\right)\right]\right\}
\end{aligned}$$

and this can be integrated

$$\begin{aligned}
\widehat{f}^{(0)}\left(r,\theta,\epsilon,\mu\right) &= -v_{\parallel}\frac{1}{\left(\frac{eB_\theta}{m}\right)}\left[A_1 + A_2\left(\epsilon - e\Phi\right)\right] \\
&\quad + g\left(\mu,\epsilon,\sigma\right)
\end{aligned}$$

We see that the zeroth order in the time-like small parameter the distribution function has a variation on the magnetic surface, *i.e.* in the angle θ. This is given as

$$\sim -\frac{v_\parallel}{\Omega_\theta}\nabla f_M \sim \rho_\theta \nabla f_M$$

18 Drift Kinetic Equation: Hazeltine Hinton Rosenbluth 1973

Arbitrary aspect ratio.

the *flux coordinates*

$$(\psi, \chi, \varphi)$$

The coordinate χ labels magnetic field lines. Usually it is taken θ, but it may contain B_θ as in **Rutherford1970**.

The field

$$\mathbf{B} = I(\psi)\,\boldsymbol{\nabla}\varphi + \boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi$$

The equations

$$p = p(\psi)$$

$$R^2 \boldsymbol{\nabla}\cdot \left(\frac{1}{R^2}\boldsymbol{\nabla}\psi\right) = II' - \mu_0 R^2 p'$$

This is **Shafranov** equation.

The parallel current

$$j_\parallel = -\frac{I}{B}p' - \frac{1}{\mu_0}I'B$$

$$\frac{\partial \psi}{\partial t} = RE_\varphi$$

the magnetic surfaces have slow time variation, generated by the electric field in the toroidal direction E_φ (induction).

The DKEq

$$\frac{\partial f}{\partial t} + (\mathbf{v} + \mathbf{v}_D)\cdot \boldsymbol{\nabla} f + \frac{e}{m}\frac{\partial f}{\partial \epsilon}\left(E_\parallel v_\parallel + \frac{\partial \Phi}{\partial t}\right) = C$$

with variables

$$\begin{aligned} \mu &= \frac{v_\perp^2}{2B} \\ \epsilon &= \frac{1}{2}v^2 + e\Phi \end{aligned}$$

$$f_0 = f_M = \frac{n}{\left(\pi v_{th}^2\right)^{3/2}} \exp\left[-\frac{2\left(\epsilon - e\Phi\right)}{v_{th}^2}\right]$$

$$v_{th} = \sqrt{\frac{2T}{m}}$$

$$\mathbf{v}_D = -v_\parallel \widehat{\mathbf{n}} \times \boldsymbol{\nabla}\left(\frac{v_\parallel}{\Omega_c}\right)$$

In flux coordinates

$$\begin{aligned}
& \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \psi}\frac{\partial \psi}{\partial t} + \frac{\partial f}{\partial \chi}\frac{\partial \chi}{\partial t} \\
& + v_\parallel \frac{\boldsymbol{\nabla}\chi \cdot \mathbf{B}}{B}\left\{\frac{\partial f}{\partial \chi} + I\left[\frac{\partial}{\partial \chi}\left(\frac{v_\parallel}{\Omega_c}\right)\frac{\partial f}{\partial \psi} - \frac{\partial}{\partial \psi}\left(\frac{v_\parallel}{B}\right)\frac{\partial f}{\partial \chi}\right]\right\} \\
& + \frac{e}{m}\frac{\partial f}{\partial \epsilon}\left(E_\parallel v_\parallel + \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial \psi}\frac{\partial \psi}{\partial t} + \frac{\partial \Phi}{\partial \chi}\frac{\partial \chi}{\partial t}\right) \\
= {} & C
\end{aligned}$$

The curly paranthesis comes from

$$\mathbf{v}_D \cdot \boldsymbol{\nabla} f \rightarrow \left[-v_\parallel \widehat{\mathbf{n}} \times \boldsymbol{\nabla}\left(\frac{v_\parallel}{\Omega_c}\right)\right] \cdot \boldsymbol{\nabla} f$$

where

$$\boldsymbol{\nabla} \sim (\psi, \chi)$$

The expansion is introduced

$$f = f_0\left(1 + \widehat{f}\right)$$

and the equation is linearized

$$\begin{aligned}
& v_\parallel \frac{\boldsymbol{\nabla}\chi \cdot \mathbf{B}}{B}\frac{\partial \widehat{f}}{\partial \chi} - \frac{C\left(\widehat{f}\right)}{f_0} \\
= {} & \frac{e}{T}E_\parallel v_\parallel - v_\parallel \frac{\boldsymbol{\nabla}\chi \cdot \mathbf{B}}{B} I\left(\frac{\partial}{\partial \chi}\frac{v_\parallel}{\Omega_c}\right)\frac{\partial}{\partial \psi}\ln f_0
\end{aligned}$$

The boundary conditions

$$\begin{aligned}
\widehat{f}_\sigma\left(\chi = -\pi\right) &= \widehat{f}_\sigma\left(\chi = +\pi\right) \\
& \quad \text{for passing particles} \\
0 &< \lambda < 1 - \varepsilon \quad \left(\text{small } v_\perp^2\right)
\end{aligned}$$

$$\begin{aligned} \widehat{f}_{+}(\pm\chi_c) &= \widehat{f}_{-}(\pm\chi_c) \\ &\quad \text{for trapped particles} \\ 1-\varepsilon &< \lambda < 1+\varepsilon \end{aligned}$$

where

$$\chi_c \equiv \text{bounce angle}$$

The flux-force relation ($J \sim$ fluxes)

$$\begin{aligned} J_1^{\psi} &= \Gamma^{\psi} \\ J_2^{\psi} &= Q_e^{\psi} \\ J_3^{\psi} &= \frac{1}{T_e}\left\langle \frac{j_{\|} - j_{\| s}}{R} \right\rangle \end{aligned}$$

where

$$\begin{aligned} j_{\| s} &= \sigma_s E_{\|} \\ \sigma_s &\equiv \text{Spitzer conductivity} \end{aligned}$$

$$\begin{aligned} A_{1\psi} &= \frac{p'(\psi)}{nT_e} - \frac{5}{2}\frac{T_e'(\psi)}{T_e} + \left(y - \frac{5}{2}\right)\frac{T_i'(\psi)}{T_e} \\ A_{2\psi} &= \frac{T_e'(\psi)}{T_e^2} \\ A_{3\psi} &= I\left\langle E_{\|} B \right\rangle \end{aligned}$$

The introduction of the parameter y is intended to make the currents J dependent on T_i' only through the first Force $A_{1\psi}$.

Then

$$J_k^{\psi} = -\sum L_{kj}^{\psi} A_{j\psi}$$

The variable

$$\lambda \equiv \frac{\mu}{w} = \frac{v_\perp^2}{v^2 B}$$

Small λ means more *parallel* energy than the *perpendicular* one and this means circulating particles. The region *small*-λ means circulating. Beyond the threshold λ_c, there are trapped particles.

The first order distribution function has a standard structure *neoclassical* $\sim \rho_\theta$ plus the function g which exists only for passing particles

$$\widehat{f}_{\sigma} = -\frac{I}{\Omega_c} v_{\|} \frac{d}{d\psi} \ln f_0 + g_{\sigma}(\psi, \chi, \lambda)$$

We return to the equation for $\widehat{f}$,

$$
\begin{aligned}
& v_{\|} \frac{\boldsymbol{\nabla}\chi \cdot \mathbf{B}}{B} \frac{\partial \widehat{f}}{\partial \chi} - \frac{C\left(\widehat{f}\right)}{f_0} \\
= \; & \frac{e}{T} E_{\|} v_{\|} - v_{\|} \frac{\boldsymbol{\nabla}\chi \cdot \mathbf{B}}{B} I \left(\frac{\partial}{\partial \chi} \frac{v_{\|}}{\Omega_c} \right) \frac{\partial}{\partial \psi} \ln f_0
\end{aligned}
$$

or

$$
\begin{aligned}
& v_{\|} \frac{\boldsymbol{\nabla}\chi \cdot \mathbf{B}}{B} \frac{\partial \widehat{f}}{\partial \chi} + v_{\|} \frac{\boldsymbol{\nabla}\chi \cdot \mathbf{B}}{B} I \left(\frac{\partial}{\partial \chi} \frac{v_{\|}}{\Omega_c} \right) \frac{\partial}{\partial \psi} \ln f_0 \\
= \; & \frac{C\left(\widehat{f}\right)}{f_0} + \frac{e}{T} E_{\|} v_{\|}
\end{aligned}
$$

In the LHS we have

$$
\begin{aligned}
& \text{factor } \; \frac{v_{\|}}{B} \\
& \text{a common derivation operator } \quad \left(\boldsymbol{\nabla}\chi \cdot \mathbf{B} \right) \frac{\partial}{\partial \chi}
\end{aligned}
$$

the operator is derivation over a *periodic* domain, poloidal. Then we use *periodicity*,

The condition

$$
\oint \frac{d\chi}{\boldsymbol{\nabla}\chi \cdot \mathbf{B}} \frac{B}{v_{\|}} \left[C\left(\widehat{f}\right) + \frac{e}{T} E_{\|} v_{\|} \; f_0 \right] = 0
$$

$$
g_{\sigma} = 0 \quad \text{for trapped particles} \quad \lambda > \lambda_c
$$

The change of variables

$$
\int d^3 v = \int dw d\lambda \frac{wB}{\left| v_{\|} \right|}
$$

19 Comment on the derivation of the drift kinetic equation

The direct procedure consists of replacing in the Fokker Plank equation the analytical form of the particle trajectory $\left(\frac{d\mathbf{x}}{dt} \text{ and } \frac{d\mathbf{v}}{dt}\right)$. For a more convenient formulation this is preceded by a change of variables adapted to the geometry of the trajectories and to the energy changes. No approximation is done at this level.

However the disparity of scales (essentially the gyration) suggests to split the task: the exact equation is averaged over the fast motion and equations for the two parts are separated: one for the distribution independent of the gyration and the other for the part dependent on it.

This technical operation has a very useful result: it exhibits the velocity of the guiding centre (the *drift velocity* $\mathbf{v}_D$) expressed in terms of the geometry of the field, i.e. gradient of B and curvature of lines. The convection that is due to this drift shifts the Maxwellian distribution across the magnetic lines and the small perturbation part of the distribution function must compensate by its convection, carried by the parallel velocity.

Inevitably a perturbative treatment must be adopted.

The basic distribution function is Maxwellian. Averaging over gyration reveals a correction of the order of the small Larmor radius, ρ_L. But the drift velocity $\mathbf{v}_D$ modifies this (classical) result since the convection induced by it $(\mathbf{v}_D \cdot \boldsymbol{\nabla})$ is of the order ρ_θ which is much larger than ρ_L.

The drift kinetic equation expresses the competition between the drift of the Maxwellian across the magnetic lines, convection by the parallel velocity of the first order perturbation to the distribution function, energy effect due to electric field and collisions.

20 The Pfirsch Schluter and the bootstrap currents

See *bootstrap.tex*.

First one defines the parallel velocity of each component

$$\begin{aligned} U_{\| a} &= -I \frac{v_{th,a}^2}{\Omega_{ca}} \left(\frac{p_a'}{p_a} + \frac{e_a \Phi'}{T_a} \right) \text{ (parallel projection of poloidal rotations } dia,\, E_r\text{)} \\ &\quad + \frac{B K_a(\psi)}{n_a} \text{ (specific parallel, not coming from projection of poloidal)} \end{aligned}$$

The definition of K_a is a *surface function* $\sim \psi$.

$$K_a(\psi) = \frac{1}{B} \int d^3v \; v_\| \; f_{0a} g_a$$

We see that K_a corresponds to that part of the parallel flow that comes directly from the correction (only existing for *circulating* particles) to the flow velocity.

$K_a(\psi)$ comes from g_a.

For trapped particles $g_a \equiv 0$ and $K_a = 0$. [this is good: the bananas cannot move toroidally, except for the precession, which is not here]

With the expressions for $U_{\|a}$ we can create the parallel current expression.

[also in *bootstrap*]

It results

$$j_{\|} = -\frac{I}{B}\frac{dp}{d\psi} + eB(ZK_i - K_e)$$

NOTE

The factors in the first term are

$$\begin{aligned} -\frac{I}{B}\frac{dp}{d\psi} &\sim -\frac{RB_T}{B}\frac{dp}{dr}\frac{dr}{|\nabla\psi|} \approx -R\frac{1}{B_\theta R}\frac{dp}{dr} \\ &= -\frac{1}{B_\theta}\frac{dp}{dr} \end{aligned}$$

which is the BOOTSTRAP current.

We can retain the approx

$$\frac{I}{B}\frac{d}{d\psi} \to \frac{1}{B_\theta}\frac{d}{dr}$$

END

The parallel current $j_{\|} = -\frac{I}{B}\frac{dp}{d\psi} + eB(ZK_i - K_e)$ is divided into two parts:

$$j_{\|} = j_{PS} + j_{NC}$$

This is done by adding and subtracting the term

$$I\frac{dp}{d\psi}\frac{\mathbf{B}}{B_0^2}$$

The Pfirsch Schluter expression is obtained by taking the first term in j, i.e. $-\frac{I}{B}\frac{dp}{d\psi}$ and one term that has been added/subtracted, $+I\frac{dp}{d\psi}\frac{\mathbf{B}}{B_0^2}$,

$$\mathbf{j}_{PS} = I\frac{dp}{d\psi}\mathbf{B}\left(\frac{1}{B_0^2} - \frac{1}{B^2}\right)$$

Pfirsch Schluter is a parallel current

We note the structure of this PS expression

$$\begin{aligned} \mathbf{j}_{PS} = & -I\frac{1}{B}\frac{dp}{d\psi}\widehat{\mathbf{e}}_{\|} \\ & +I\frac{B}{B_0^2}\frac{dp}{d\psi}\widehat{\mathbf{e}}_{\|} \end{aligned}$$

The reason to add and subtract a term

$$I\frac{1}{B_0^2}\frac{dp}{d\psi}\ \mathbf{B}$$

in $j_{\|}$ and group it with $\mathbf{B}/B^2$ is the necessity to exhibit the fact that the PS current changes sign on the poloidal section.

See **Hirshman neoclassical current** (in *bootstrap diamagnetic*) where

$$j_{PS} = -F\frac{1}{B}\frac{dp}{d\psi}\left(1 - \frac{B^2}{\langle B^2\rangle}\right)$$

The rest in the expression of $j_{\|}$, is

$$\begin{aligned} & eB\left(ZK_i - K_e\right)\widehat{\mathbf{e}}_{\|} - I\frac{B}{B_0^2}\frac{dp}{d\psi}\widehat{\mathbf{e}}_{\|} \\ \equiv\ & \mathbf{j}_{NC} \end{aligned}$$

or

$$\mathbf{j}_{NC} = \mathbf{B}\ \left[-I\frac{1}{B_0^2}\frac{dp}{d\psi} + e\left(ZK_i - K_e\right)\right]$$

NOTE

Just for reminding the notations of **Hirschman neoclassical current impurities**

$$\mathbf{j} = \mathbf{j}_{\perp} + j_{\|}\widehat{\mathbf{n}}$$

$$\mathbf{j}_{\perp} = \frac{(-\boldsymbol{\nabla}p)\times\mathbf{B}}{B^2}$$

(this is compatible with the drift produced by a force $\mathbf{G}$ (gravity, electric field, gradient of pressure) $\mathbf{u}_D = \frac{1}{q}\frac{\mathbf{G}\times\mathbf{B}}{B^2}$, where q is the charge, take $\mathbf{G} = \frac{(-\boldsymbol{\nabla}p)}{n}$, resulting

$$qn\mathbf{u}_D = \frac{(-\boldsymbol{\nabla}p)\times\mathbf{B}}{B^2}\quad \text{current due to the drift in the force field}$$

Hirschman adopts

$$\mathbf{B} = \mathbf{B}_T + \mathbf{B}_p$$

$$\begin{aligned} \mathbf{B}_T &= F\left(\psi\right)\ \boldsymbol{\nabla}\varphi \\ \varphi &\equiv \text{toroidal angle} \end{aligned}$$

Note this is $\mathbf{B}_T = -I\frac{1}{R}\widehat{\mathbf{e}}_\varphi \approx -B_T\widehat{\mathbf{e}}_\varphi$ for axisymmetric systems where $I = RB_T$ and $F = -I$. **End.**

$$\mathbf{B}_p = \boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi$$

The poloidal flux is $2\pi\psi$.

$$j_\| = F\frac{1}{B}\frac{dp}{d\psi} + KB$$

where

$$K = \frac{\mathbf{j}\cdot\mathbf{B}_p}{B_p^2}$$

proportional with the poloidal current which has a projection on parallel direction

Then, to compare, the formula for the parallel current

$$j_\| = -\frac{I}{B}\frac{dp}{d\psi} + eB\left(ZK_i - K_e\right)$$

is at Hirschman

$$j_\| = F\frac{1}{B}\frac{dp}{d\psi} + KB \quad \text{(Hirschman)}$$

and we identify

$$-\frac{I}{B}\frac{dp}{d\psi} \to F\frac{1}{B}\frac{dp}{d\psi}$$

or

$$F = -I$$

functions of surface ψ in an axisymmetric system.

The poloidal component of the Ampere's law

$$K\left(\psi\right) = -\frac{1}{4\pi}\frac{dF}{d\psi}$$

END

Here, the neoclassical part

$$\mathbf{j}_{NC} = \mathbf{B}\left[e\left(ZK_i - K_e\right) - \frac{I}{B_0^2}\frac{dp}{d\psi}\right]$$

where

$$B_0 = \text{field on the magnetic axis}$$

One can see that

$$\nabla \cdot \mathbf{j}_{PS} = -\nabla \cdot \left(\frac{1}{B} \widehat{\mathbf{n}} \times \nabla p \right)$$

which means that the PS current has a divergence that compensates the divergence of the diamagnetic flux.

The other current has *zero divergence* and is

$$\mathbf{j}_{NC} = \mathbf{B} \left[-\frac{I}{B_0^2} \frac{dp}{d\psi} + e\left(ZK_i - K_e \right) \right]$$

$$\nabla \cdot \mathbf{J}_{NC} = 0$$

See also **Hirshman neoclassical current,** $j_{\|} = -F \frac{1}{B} \frac{dp}{d\psi} + KB$ (in *bootstrap diamagnetic*). In **Rosenbluth Hazeltine Hinton 1972** we have a more complex representation of the *poloidal* part (as distinct from the Pfirsch Schluter part which is parallel)

$$KB \rightarrow \mathbf{B} e\left(ZK_i - K_e \right)$$

Continuation with the model of **Hirschman current**.

Consider the momentum balance for electrons (the electrons carry the current)

$$\begin{aligned} 0 &= -\nabla p - \nabla \cdot \boldsymbol{\pi} \\ &\quad -en\ \mathbf{E} \\ &\quad +\frac{\mathbf{j}}{\sigma_s} \end{aligned}$$

1. We multiply with $\mathbf{B}$ (which will suppress $-\nabla p$ since it is perpendicular)
2. take the surface average

Then

$$0 = -\left\langle \mathbf{B} \cdot \nabla \cdot \boldsymbol{\pi} \right\rangle - |e|\, n \left\langle \mathbf{B} \cdot \mathbf{E} \right\rangle + |e|\, n \frac{1}{\sigma_s} \left\langle \mathbf{B} \cdot \mathbf{j} \right\rangle$$

The classical Ohm's law

$$\begin{aligned} \mathbf{B} \cdot \int d^3 v\ m_e \mathbf{v}\ C\left(f_e \right)\ \ \text{(collisional} \quad & \| \quad \text{transfer of momentum)} \\ &= \quad |e|\, n_e\ \frac{1}{\sigma_s} \mathbf{J} \cdot \mathbf{B} \end{aligned}$$

where

$$\sigma_s \equiv \text{Spitzer conductivity}$$

The surface-averged parallel current is composed of
- the Spitzer current
- the neoclassical current

Then

$$\langle \mathbf{j} \cdot \mathbf{B} \rangle = \langle j_s B \rangle + \langle j_{NC} B \rangle$$

where it has been introduced the classical Spitzer current

$$j_s = \sigma_s \frac{\langle \mathbf{E} \cdot \mathbf{B} \rangle}{\langle B^2 \rangle} B$$

First, return to the equation of momentum (surface averaged after scalar multiplied with $\mathbf{B}$) and replace $\langle \mathbf{j} \cdot \mathbf{B} \rangle = \langle j_s B \rangle + \langle j_{NC} B \rangle$,

$$\begin{aligned}
0 &= -\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi} \rangle - |e| \, n \langle \mathbf{B} \cdot \mathbf{E} \rangle + |e| \, n \frac{1}{\sigma_s} \langle \mathbf{B} \cdot \mathbf{j} \rangle \\
0 &= -\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi} \rangle - |e| \, n \langle \mathbf{B} \cdot \mathbf{E} \rangle + |e| \, n \frac{1}{\sigma_s} \left[\langle j_s B \rangle + \langle j_{NC} B \rangle \right] \\
0 &= -\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi} \rangle - |e| \, n \langle \mathbf{B} \cdot \mathbf{E} \rangle + |e| \, n \frac{1}{\sigma_s} \left\langle \sigma_s \frac{\langle \mathbf{E} \cdot \mathbf{B} \rangle}{\langle B^2 \rangle} B \; B \right\rangle + |e| \, n \frac{1}{\sigma_s} \langle j_{NC} B \rangle
\end{aligned}$$

Separately, the term

$$|e| \, n \frac{1}{\sigma_s} \left\langle \sigma_s \frac{\langle \mathbf{E} \cdot \mathbf{B} \rangle}{\langle B^2 \rangle} B \; B \right\rangle = |e| \, n \; \langle \mathbf{E} \cdot \mathbf{B} \rangle$$

and it will cancel the term $-|e| \, n \langle \mathbf{B} \cdot \mathbf{E} \rangle$ that precedes it

$$0 = -\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi} \rangle + |e| \, n \frac{1}{\sigma_s} \langle j_{NC} B \rangle$$

Multiply and divide the last term with B,

$$j_{NC} = \sigma_s \frac{\langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi} \rangle}{|e| \, n} \frac{B}{\langle B^2 \rangle}$$

21 Approximations to the neoclassical drift-kinetic equation

21.1 Plateau

Approximate drift-kinetic equation in the *plateau* regime can use the following **collision operator**

$$C_e\left(f_{e1}\right)=\nu_{ei}\epsilon\frac{\partial^2 f_{e1}}{\partial v_{\parallel}^2}$$

where ϵ is the kinetic energy per unit mass, $\epsilon=v^2/2$.

This is a kind of diffusion in the parallel velocity variable, $v_{\parallel}$.

This form of the collision operator comes from the *pitch angle scattering* operator, expressed in terms of μ.

NOTE

This choice is also made in **Shaing Hsu Domiguez resonance** in the calculation of the function g from the kinetic equation. The purpose is the *viscosity.*

See *viscosity.tex* in *general.*

More explanation are in **orbit squeezing shaing hazeltine**, where

$$C\left(f\right)=\nu_D\frac{v^2}{2R^2q^2}\frac{\partial^2 f}{\partial\omega^2}$$

and a first order derivative

$$\frac{\partial f}{\partial\omega}\text{ is neglected}$$

END

In **Novakovskii Galeev Sagdeev Liu** it is used the following collision operator for *ion-ion* collisions in *plateau*

$$St\left(\widetilde{f}\right)=-\nu_{eff}\widetilde{f}$$

where

$$\nu_{eff}\approx\nu_{ii}\frac{v_{i,Th}^2}{v_{\parallel}^2}$$

and $\widetilde{f}$ is extracted from the full distribution function for ions in plateau, by the formula

$$\begin{aligned} f &= \left(1 - \frac{mv_{\|}U_0}{T}\right) f_M \\ &+\varepsilon \left(\frac{mv_{\|}U_0}{T}\right) \cos\theta \ f_M \\ &+\widetilde{f} \end{aligned}$$

21.2 Pfirsch-Schluter

The equation is

$$\frac{v_{\|}}{qR}\frac{\partial f_{e1}}{\partial \theta} + \mathbf{v}_D \cdot \boldsymbol{\nabla} f_{e0} = C_{ei}\left(f_{e1}\right) + C_{ee}\left(f_{e1}\right)$$

which can be described in literary terms: the electron distribution function has a small (order 1) variation in the poloidal direction: $f_{e1}(\theta)$, in the magnetic surface, generated by the advection of the Maxwellian (order 0) distribution function caused by the particle *drift* motion. The drift motion v_D *depends on the poloidal angle* θ and it is a *neoclassical effect.*

$$v_{\|}\frac{B_\theta}{B_T} = v_\theta$$

The poloidal velocity is just the parallel velocity $v_{\|}$ projected along θ.

$$\begin{aligned} \frac{v_{\|}}{qR} &= \frac{v_{\|}}{\frac{rB_T}{RB_\theta}R} = v_{\|}\frac{B_\theta}{B_T}\frac{1}{r} \\ &= v_\theta \frac{1}{r} \end{aligned}$$

Then

$$\frac{v_{\|}}{qR}\frac{\partial}{\partial\theta} = v_\theta \frac{\partial}{r\partial\theta}$$

Then the operator is

$$\mathbf{v}\cdot\boldsymbol{\nabla} = \mathbf{v}_{Dr}\cdot\boldsymbol{\nabla} + v_\theta\frac{\partial}{r\partial\theta}$$

The Drift velocity $\mathbf{v}_D$ is directed approximately vertical.

NOTE this separation of the operator $\mathbf{v}\cdot\boldsymbol{\nabla}$ is very important.

The first part

$$v_{Dr}\frac{\partial}{\partial r}$$

is applied on the equilibrium distribution function $f_0 = f_M$.

The second part

$$v_\theta \frac{\partial}{r\partial\theta}$$

will obtain the the variation of the *first correction* to the distribution function, f_1 . This actually reflects the variation of the distribution function in the magnetic surface, which means variation with θ.

END

NOTE the absence of an energetic term. There is no $\partial f/\partial\epsilon$ term yet.
END.

21.3 Expressions for the neoclassical distribution function

In the banana regime the distortion of the electron distribution function due to

- eqilibrium gradients, and
- trapped electrons

is:

$$f_{e1} = f_{e1}^{(1)} + f_{e1}^{(2)}$$

$$\begin{aligned} f_{e1} &= \frac{1}{\Omega_\theta}\left\{ v_\parallel \left[\frac{d}{dr}\ln n_e + \left(\frac{m\epsilon}{T_e} - \frac{3}{2}\right)\frac{d}{dr}\ln T_e\right] + \right. \\ &\quad \left. + \mathrm{H}\left(\lambda_c - \lambda\right)\left[\frac{d}{dr}\ln n_e + \left(\frac{m\epsilon}{T_e} - \frac{3}{2}\right)\frac{d}{dr}\ln T_e\right]\epsilon \int_{\lambda_c}^{\lambda} \frac{d\lambda}{\langle v_\parallel \rangle}\right\} f_{e0} \end{aligned}$$

where ϵ is the **kinetic energy per unit mass**

$$\epsilon = \frac{v^2}{2}$$

$$\lambda = \frac{v_\perp^2}{v^2}\frac{1}{B}$$

λ_c is the value of the pitch angle variable which separates the **trapped** from the **untrapped** regions; H is the Heaviside function.

It can be seen that the first term is the distortion introduced by the **drift**

$$\begin{aligned} f_{e1}^{(1)} &= \rho_\theta \nabla_r f_{e0} \\ &\sim \text{radial variation of the Maxwellian} \\ &\quad \text{over a distance of a poloidal gyration radius, } \rho_\theta \end{aligned}$$

and the second is due to the *circulating*, valid only for

$$0 < \lambda < \lambda_c = 1 - \varepsilon$$

NOTE that in **Rosenbluth Hazeltine Hinton it is defined** λ as

$$\lambda \equiv \frac{\mu B_0}{\epsilon - e\Phi} = \frac{\frac{mv_\perp^2}{2B} B_0}{\left(\frac{mv^2}{2} + e\Phi\right) - e\Phi} = \frac{v_\perp^2}{v^2}\frac{B_0}{B}$$

we have

$$\begin{aligned} \frac{B_0}{B} &= 1 + \varepsilon \cos\theta \\ &= h \end{aligned}$$

Then

$$\begin{aligned} \lambda &= h\frac{v^2 - v_\parallel^2}{v^2} = h\left(1 - \frac{v_\parallel^2}{v^2}\right) \\ &= h\left(1 - \xi^2\right) \end{aligned}$$

where

$$\xi \equiv \frac{v_\parallel}{v}$$

and introducing the *pitch angle* ζ

$$\frac{v_\perp^2}{v^2} \equiv (\sin\zeta)^2$$

Then

$$\begin{aligned} \lambda &= h\frac{v_\perp^2}{v^2} \\ &= h\sin^2\zeta \end{aligned}$$

The parallel velocity is

$$v_\parallel = |v|\sqrt{1 - \frac{\lambda}{h}}$$

and the *limits* of the trapped particle region in the variable λ are the external region of the *small*-λ interval

$$0 < \lambda < 1 - \frac{r}{R}$$

circulating, $v_{\perp}$ is small, $v_{\parallel}$ is larger

END.

Here

$$\begin{aligned} \lambda &\equiv \frac{\sqrt{1-\xi^2}}{B} = \frac{\sqrt{1-\frac{v_{\parallel}^2}{v^2}}}{B} \\ &= \frac{v_{\perp}}{v}\frac{1}{B} \end{aligned}$$

$$f_{e1}^{(2)} = \epsilon \int_{\lambda_c}^{\lambda} \frac{d\lambda}{\langle v_{\parallel} \rangle} \nabla_r f_{e0}$$

The *particle flux* which results from this distortion of the distribution function in the banana regime:

$$\Gamma = -0.73 \frac{\rho_{e\theta}^2}{\tau_e} n_e \sqrt{\varepsilon} \left(\frac{d}{dr} \ln n_e - \frac{1}{2} \frac{d}{dr} \ln T_e \right)$$

The particle flux is calculated in general from the formula

$$\Gamma = B_{\varphi} \int \frac{d\theta}{2\pi} (1 + \varepsilon \cos\theta) \int dv \frac{v_{\parallel}}{\Omega_{\theta}} (1 + \varepsilon \cos\theta)\, C_{ei} (f_{e1})$$

A general expression of the flux (**Hazeltine and Hinton, RMP 1976**)

$$\Gamma = -\frac{\rho_{e\theta}^2}{\tau_e} n_e \sqrt{\varepsilon} \left[K_{11} \left(\frac{d}{dr} \ln n_e - \frac{3}{2} \frac{d}{dr} \ln T_e \right) + K_{12} \left(\frac{d}{dr} \ln T_e \right) \right]$$

where

$$\begin{aligned} K_{11} &= 0.73... \\ K_{12} &= 0.73... \end{aligned}$$

The **perturbation in the magnetic surface of the density and temperature**, which are associated to the neoclassical distortion of the distribution function resulting from the drift-kinetic equation, is:

$$\frac{\widetilde{T}_e}{T_e} = \frac{3\pi}{32} \frac{q^2 R^2}{\tau_e} \frac{\sin\theta}{R\,|\Omega_e|} \left(\frac{3}{4} \frac{d}{dr} \ln n_e - \frac{1}{2} \frac{d}{dr} \ln T_e \right)$$

$$\frac{\widetilde{n}_e}{n_e} = -\frac{3\pi}{32}\frac{q^2R^2}{\tau_e}\frac{\sin\theta}{R\left|\Omega_e\right|}\left(\frac{25}{8}\frac{d}{dr}\ln n_e + \frac{5}{4}\frac{d}{dr}\ln T_e\right)$$

In **Stacey** the neoclassical perturbation in the magnetic surface of $n(r,\theta)$ and of $\phi(r,\theta)$ are calculated.

A kinetic calculation of the variation in the surface is made by **Stringer**.

NOTE

that this gives some sense to the notations introduced by **shaing hsu dominguez resonance** (and **shaing hazeltine squeezing**) where the RHS (which contains f_M) of the equation for the function g is replaced as

$$\begin{aligned} & 2\frac{v^2}{v_{th}^2}\left(\frac{1}{2}-\frac{3}{2}\frac{u^2}{v^2}\right)f_M\ \mathcal{D} - \frac{8}{15}f_M L_2^{(1/2)}\left[-\frac{5}{2}\frac{1}{NeB}\left(\widehat{\mathbf{n}}\times\boldsymbol{\nabla}T\right)\cdot\boldsymbol{\nabla}N + \frac{1}{p}\boldsymbol{\nabla}\cdot\mathbf{q}\right] \\ = & \ \mathcal{N} f_M\ \sin\theta \end{aligned}$$

and later, the substitution

$$B\left(\nabla_{\|}\ln B - \frac{2}{3}\nabla_{\|}\ln N\right) = \mathcal{M}\sin\theta$$

In the comment for **Hinton Waltz** above, we have

$$\nabla_{\|}\ln B = \frac{1}{B}\nabla_{\|}B = \frac{1}{B}\frac{1}{qR}\frac{\partial}{\partial\theta}B$$

and

$$\begin{aligned} B &= \frac{B_0}{h} \approx B_0\left(1-\varepsilon\cos\theta\right) \\ \frac{\partial B}{\partial\theta} &= B_0\varepsilon\sin\theta \end{aligned}$$

$$\begin{aligned} \nabla_{\|}\ln B &= \frac{1}{B}\frac{1}{qR}B_0\varepsilon\sin\theta \\ &\approx \frac{\varepsilon}{q}\frac{\sin\theta}{R} \end{aligned}$$

then it is normal to take

$$\nabla_{\|}\ln B \sim \sin\theta$$

The same is for $N(r,\theta)$

$$\nabla_{\|}\ln N \sim \sin\theta$$

END

22 Drift kinetic multiple species Hirshman Sigmar 1977

The intention is to place in a unique formalism all collisional regimes: Pfirsch Schluter, plateau, banana.

Some species can be in one regime and others in a different regime.

Variables

$$(v, v_{\parallel})$$

There is no parallel flow that is assumed from the start.

If there is a parallel flow, it results from the gradients of density and temperature.

The equation

$$\begin{aligned}
& v_{\parallel}\nabla_{\parallel} f_a^{(1)} \text{ parallel advection} \\
& -\frac{v^2 - v_{\parallel}^2}{2}\nabla_{\parallel}\ln B \frac{\partial f_a^{(1)}}{\partial v_{\parallel}} \quad \text{mirror} - B \text{ force} \\
& -v_{\parallel}\frac{e_a E_{\parallel}}{T_a} f_a^{(0)} \quad \text{electric parallel acceleration} \\
= & \sum_b C_{ab} \\
& +\nabla_{\parallel}\ln B \frac{1}{\Omega_a}\frac{v^2 + v_{\parallel}^2}{2} I(\psi) \frac{df_a^{(0)}}{d\psi} \quad \text{drift } v_{D,r} \text{ advection of } f_0
\end{aligned}$$

$$\frac{df_a^{(0)}}{d\psi} = \left[A_{1a} + \frac{e_a}{T_a}\frac{d\Phi}{d\psi} + x_a^2\ A_{2a}\right] f_a^{(0)}$$

$$A_{1a} = \frac{d}{d\psi}\ln n - \frac{3}{2}\frac{d}{d\psi}\ln T_a$$

$$A_{2a} = \frac{d}{d\psi}\ln T_a$$

$$x_a = \frac{v}{v_{th,a}}$$

$$E_{\parallel} = E_{\parallel}^{(A)} = -\widehat{\mathbf{n}}\cdot\frac{\partial \mathbf{A}}{\partial t}$$

The term

$$-\frac{v^2 - v_{\parallel}^2}{2}\nabla_{\parallel}\ln B \frac{\partial f_a^{(1)}}{\partial v_{\parallel}}$$

mirror force modulation along $\mathbf{B}$

introduces the modulation of the parallel velocity of slow resonant particles. It is $\sim \mu \nabla_{\parallel} B \equiv$acceleration$\sim \frac{\partial v_{\parallel}}{\partial t}$ along the travel on the line.

This modulation of $v_{\parallel}$, due to the *magnetic field mirror* effect is essential in the existence of the viscosity

$$\sim \left(p_{\parallel} - p_{\perp}\right)$$

NOTE

Same in Galeev reference.

In **Hirshman Sigmar 1977** the equation is

$$\begin{aligned}
& v_{\parallel} \nabla_{\parallel} f_{a1} \quad \text{(advection parallel)} \\
& -\frac{1}{2}\left(v^2 - v_{\parallel}^2\right) \nabla_{\parallel} \ln B \ \frac{\partial f_{a1}}{\partial v_{\parallel}} \quad \text{(mirror force)} \\
& -v_{\parallel} \ e_a E_{\parallel} \ \frac{1}{T_a} f_{a0} \quad \text{(electric)} \\
= & \sum_b C_{ab} \quad \text{(collisions)} \\
& +\nabla_{\parallel} \ln B \ \frac{1}{\Omega_a} \frac{v^2 + v_{\parallel}^2}{2} I\left(\psi\right) \ \frac{df_{a0}}{d\psi} \quad \text{(drift } \mathbf{v}_D \text{ advection of } f_0\text{)}
\end{aligned}$$

Let us compare with **Galeev Sagdeev Liu Novakovskii JETP**.

The first term is the parallel advection plus the electric term

$$\begin{aligned}
& v_{\parallel} \nabla_{\parallel} f_{a1} - v_{\parallel} \ e_a E_{\parallel} \ \frac{1}{T_a} f_{a0} \quad \text{(Hirshman Sigmar 1977)} \\
& \left(v_{\parallel} \frac{B_{\theta}}{B_{\varphi}} + V_E\right) \frac{\partial f^{(1)}}{r \partial \theta} \quad \text{(Galeev Sagdeev Liu Novakovskii)}
\end{aligned}$$

There is a visible difference. The first terms are the same

$$v_{\parallel} \nabla_{\parallel} f_{a1} = v_{\parallel} \frac{\partial f_{a1}}{\partial l_{\parallel}} = v_{\parallel} \frac{1}{qR} \frac{\partial f_{a1}}{\partial \theta} = v_{\parallel} \frac{B_{\theta}}{B_{\varphi}} \frac{\partial f_{a1}}{r \partial \theta} \quad \text{(HS)}$$

but the electric terms are diferent. In HS1977

$$-v_{\parallel} \ e_a E_{\parallel} \ \frac{1}{T_a} f_{a0} = v_{\parallel} F_{\parallel}^{electric} \frac{\partial}{\partial \epsilon} f_{a0}$$

where

$$f_{a0} \ \text{(Maxwellian)} \sim \exp\left(-\frac{\epsilon}{T}\right)$$

is the change of the equilibrium (Maxwellian) distribution function due to the acceleration of charged particles in the parallel electric field. All equilibrium distribution (Maxwellian) participates. It is a term of the kind

$$\frac{d\epsilon}{dt}\frac{\partial f_{Ma}}{\partial \epsilon}$$

In GSLN

$$V_E \frac{\partial f^{(1)}}{r\partial\theta}$$

is the advection of the perturbed distribution function, with poloidal variation, by the poloidal rotation velocity V_E, produced by the gradient of the electrostatic potential across surfaces, $V_E(r,t) = \frac{\partial\phi(r,t)}{\partial r}\frac{1}{B}$. This velocity of poloidal rotation is the main interest in GSLN. The damping rate is calculated. The term is of the type

$$\mathbf{v}\cdot\boldsymbol{\nabla} f^{(1)}$$

and quantitatively is important, it almost cancels the parallel advection term $v_{\|}\nabla_{\|} f^{(1)}$ (projected on poloidal direction) in GSLN, leaving only the part of v that depends on θ

$$2\sigma\sqrt{\frac{\mu B}{m}\varepsilon S}\,\sqrt{\kappa^2 - \sin^2\left(\frac{\theta}{2}\right)}$$

This is actually the part of the poloidal velocity that *advects* poloidally the perturbed, $\sim\theta$, part of the distribution function, $f^{(1)}$.

The second term is the *mirror force*

$$-\frac{1}{2}\left(v^2 - v_{\|}^2\right)\nabla_{\|}\ln B\,\frac{\partial f_{a1}}{\partial v_{\|}} \quad \text{(Hirshman Sigmar 1977)}$$

$$-\frac{\mu}{m}\nabla_{\|}B\,\frac{\partial f^{(1)}}{\partial v_{\|}} \quad \text{(Galeev Sagdeev Liu Novakovskii)}$$

which is the same in HS and GSLN since

$$-\frac{v_{\perp}^2}{2}\nabla_{\|}\ln B = \frac{dv_{\|}}{dt}$$

for

$$\mu = \frac{mv_{\perp}^2}{2B}$$

There is in **HS1977** a term that is absent in GLSN

$$\nabla_{\parallel} \ln B \ \frac{1}{\Omega_a} \frac{v^2 + v_{\parallel}^2}{2} \ I(\psi) \ \frac{\partial f_{a0}}{\partial \psi}$$

which is the advection (i.e. $\mathbf{v} \cdot \boldsymbol{\nabla}$) of the equilibrium (Maxwellian) distribution by the *neoclassical drift velocity*.

In **GSLN** the focus is on the rate of damping of the poloidal rotation,

$$\frac{\partial V_E}{\partial t}$$

and the Maxwellian distribution function does not play a role. The time evolution of $V_E(r,t)$ results from the competition of terms: advection of the perturbed distribution function, - which is done by $V_E(r,t)$ - is balanced by collisions. The problem is treated as "initial value".

In HS1977 the objective is the viscosity.
END

The way the drift kinetic equation is formulated naturally introduces the thermodynamical "forces"

$$\frac{\partial f_{a0}}{\partial \psi} = \left(A_{1a} + \frac{e}{T_a} \frac{\partial \Phi}{\partial \psi} + x_a^2 \ A_{2a} \right) f_{a0}$$

$$\begin{aligned} A_{1a} &= \frac{\partial}{\partial \psi} \ln n_a - \frac{3}{2} \frac{\partial}{\partial \psi} \ln T_a \\ A_{2a} &= \frac{\partial}{\partial \psi} \ln T_a \end{aligned}$$

where

$$\begin{aligned} x_a &= \frac{v}{v_{th,a}} \\ v_{th,a} &= \left(\frac{2T_a}{m_a} \right)^{1/2} \end{aligned}$$

The statement in **HS1977**: the mirror force is weak and the modulation of the parallel velocity along the magnetic line is weak.

The modulation of $v_\parallel\left(l_\parallel\right)$ due to magnetic mirror *removes the degeneracy of the particle velocities along the line* and this is essential in the determination of the friction forces that arise from the relative parallel particle streaming.

The term in the RHS that is the advection by neoclassical drift velocity of the Maxwellian distribution function is the *drive* for the transport processes.

The collision operator

$$C_{ab}\left(f_{a1}, f_{b0}\right) + C_{ab}\left(f_{a0,f_{b1}}\right) \equiv C_{ab}$$

this is

$$\begin{aligned} C_{ab} &= \nu_{ab}^{deflection}\ \mathcal{L}\left[f_{a1}\right] \\ &\quad + \left(\nu_{ab}^{deflection} - \nu_{ab}^{slowing}\right) \frac{v_\parallel u_{a1}\left(v\right)}{v^2} f_{a0} \\ &\quad + \nu_{ab}^{slowing}\left(v\right)\ \frac{2v_\parallel}{v_{th,a}^2} r_{ba}\ f_{a0} \end{aligned}$$

where

$$\mathcal{L} = \frac{v^2}{2}\frac{\partial}{\partial v_\parallel}\left[1-\left(\frac{v_\parallel}{v}\right)^2\right]\frac{\partial}{\partial v_\parallel}$$

pitch angle scattering operator

$$u_{a1}\left(v\right) = \frac{1}{f_{a0}}\ \frac{3}{4\pi}\int d\Omega\ v_\parallel f_{a1}$$

with the solid angle infinitesimal

$$d\Omega = \frac{2\pi}{v} dv_\parallel$$

$$r_{ba} = \frac{\int d^3v\ v_{ba}^{slowing}\ \ m_b v_\parallel f_{b1}}{m_a n_a \left\{\nu_{ab}^{slowing}\right\}}$$

momentum restoring coefficient

A new operator

$$\left\{W_{ab}\left(v\right)\right\} \equiv 2\int d^3v\ \left(\frac{v_\parallel}{v_{th,a}}\right)^2 W_{ab}\left(v\right)\ \left(\frac{f_{a0}}{n_a}\right)$$

To obtain an explicit form for u_{a1} and r_{ba} one must solve the drift kinetic equation.

But the solution of the drift kinetic equation already can be separated in terms that can be explicitly written and a part that requires to effectively solve the equation.

$$\begin{aligned} u_{a1} &= -I\frac{v^2}{\Omega_a}\frac{1}{f_{a0}}\frac{\partial f_{a0}}{\partial \psi} \\ &\quad +\widehat{u}_{a1}\left(\psi, v\right)\ B \end{aligned}$$

The second term is obtained from the drift kinetic equation for f_{a1} after integration over the solid angle in velocity space.

The momentum restoring coefficient r_{ba} is obtained from the drift kinetic equation for f_{a1} after integration over the velocity magnitude

$$\begin{aligned} r_{ab} &= -I\frac{v_{th,a}^2}{2\Omega_a}\left[A_{a1}+\frac{\left\{x_a^2\nu_{ab}^{slowing}\right\}}{\left\{\nu_{ab}^{slowing}\right\}}\ A_{2a}\right] \\ &\quad +\widehat{r}_{ab}\left(\psi\right)\ B \end{aligned}$$

The two "supplementary" terms $\widehat{u}_{a1}\left(\psi, v\right)$ and $\widehat{r}_{ab}\left(\psi\right)$ are functions only of surface coordinate, $\psi \sim r$.

The first terms in u_{a1} and in r_{ab} are dependent of θ through B.

To solve the drift kinetic equation one has to adopt a procedure similar to that used for **Rutherford 1970**.

One uses the periodicity on θ since the advection terms along the magnetic field line are converted into derivations on poloidal angle θ. Taking "surface average" one obtains a constraint on the sum of the collision operator and the first-neoclassical correction which is due to drift. Then the difference defines a new variable

$$\begin{aligned}
f_{a1} &= -I\frac{v_\parallel}{\Omega_a}\frac{\partial f_{a0}}{\partial\psi}+I\frac{v_\parallel}{\overline{\Omega}_a}\frac{\partial f_{a0}}{\partial\psi}\frac{B}{\langle B^2\rangle^{1/2}} \\
&\quad +\frac{v_\parallel\ \overline{u}_{a1}}{v^2}\frac{\sum_b\left(\nu_{ab}^{deflection}-\nu_{ab}^{slowing}\right)}{\sum_b\nu_{ab}^{deflection}}\ f_{a0}\ \frac{B}{\langle B^2\rangle^{1/2}} \\
&\quad +\sum_b\frac{2v_\parallel\ \overline{r}_{ba}}{v_{th,a}^2}\ \frac{\nu_{ab}^{slowing}}{\sum_b\nu_{ab}^{deflection}}\ f_{a0}\ \frac{B}{\langle B^2\rangle^{1/2}} \\
&\quad +\frac{1}{\sum_b\nu_{ab}^{deflection}}\frac{e_a v_\parallel\overline{E}_\parallel}{T_a}\ f_{a0}\ \frac{B}{\langle B^2\rangle^{1/2}} \\
&\quad +h_{a1}
\end{aligned}$$

The first line

$$-I\frac{v_\parallel}{\overline{\Omega}_a}\frac{\partial f_{a0}}{\partial\psi}\left(1-\frac{B}{\langle B^2\rangle^{1/2}}\right)$$

The first factor

$$\begin{aligned}
-I\frac{v_\parallel}{\overline{\Omega}_a}\frac{\partial f_{a0}}{\partial\psi} &\approx -RB_T\frac{v_\parallel}{e_aB/m_a}\frac{1}{|\boldsymbol{\nabla}\psi|}\frac{\partial f_{a0}}{\partial r} \\
&\approx -RB_T\frac{v_\parallel}{e_aB/m_a}\frac{1}{RB_\theta}\frac{\partial f_{a0}}{\partial r} \\
&\approx -\frac{v_\parallel}{\Omega_{\theta a}}\frac{\partial f_{a0}}{\partial r}
\end{aligned}$$

This is the neoclassical perturbation, in the first order, to the Maxwellian distribution function, due to the drift of particles. It results from the advective term, $\mathbf{v}_D\cdot\boldsymbol{\nabla}f_{aM}$ and is proportional with the poloidal Larmor radius, ρ_θ.

23 Drift kinetic Hirshman Sigmar Clarke 1976 multiple ions

The eq

$$\begin{aligned}
& \frac{\partial f_a}{\partial t} + \left(v_\parallel \widehat{\mathbf{n}} + \mathbf{v}_{Da}\right) \cdot \boldsymbol{\nabla} f_a \\
& + \left[\mu \frac{\partial B}{\partial t} + v_\parallel \frac{e_a E_\parallel^{(A)}}{m_a} + \frac{1}{m_a} \frac{\partial \left(e_a \Phi\right)}{\partial t}\right] \frac{\partial f_a}{\partial \epsilon} \\
= & \sum_b C_{ab}\left(f_a, f_b\right)
\end{aligned}$$

where the *drift velocity* of species a is

$$\begin{aligned}
\mathbf{v}_{Da} = & -v_\parallel \widehat{\mathbf{n}} \times \boldsymbol{\nabla}\left(\frac{v_\parallel}{\Omega_a}\right) \\
& + \frac{v_\parallel^2}{\Omega_a} \frac{\left(\boldsymbol{\nabla} \times \mathbf{B}\right)_\perp}{B}
\end{aligned}$$

$$v_\parallel = \sigma \sqrt{2\left(\epsilon - \mu B - \frac{e_a}{m_a}\Phi\right)}$$

The expansion is made in series of powers of Larmor radius

$$f_a = f_{a0} + f_{a1} + \ldots$$

$$f_{a0} = \frac{N_a(\psi)}{\left(\pi \frac{2T_a(\psi)}{m_a}\right)^{3/2}} \exp\left[-\frac{\epsilon}{\frac{T_a(\psi)}{m_a}}\right]$$

$$N_a(\psi) = n_{a0} \exp\left[\frac{e_a \Phi}{T_{a0}}\right]$$

The first order

$$\begin{aligned}
& v_\parallel \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} f_{a1} \\
& + v_\parallel \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\left(I(\psi) \frac{v_\parallel}{\Omega_a} \frac{\partial f_{a0}}{\partial \psi}\right) \text{ the radial shift of } f_0 \\
& - e_a E_\parallel^{(A)} \, v_\parallel \, \frac{1}{T_{a0}} f_{a0} \text{ parallel electric field acceleration} \\
= & \sum_b C_{ab}\left(f_{a1}, f_{b1}\right)
\end{aligned}$$

NOTE

We used to write this equation with a slight rearrangement of terms

$$\begin{aligned} & v_{\parallel}\nabla_{\parallel}f_{a1}-\sum_{b}C_{ab}\left(f_{a1},f_{b1}\right) \\ = & -v_{\parallel}\nabla_{\parallel}\left(I\left(\psi\right)\frac{v_{\parallel}}{\Omega_{a}}\frac{\partial f_{a0}}{\partial\psi}\right)+e_{a}E_{\parallel}^{(A)}\ v_{\parallel}\ \frac{1}{T_{a0}}f_{a0} \end{aligned}$$

to separate (in the LHS) the terms of perturbation of the distribution function, keeping the source in the RHS. Then we see that the perturbation f_{a1} exists for it to compensate the effect of the modulation of B along the line (plus the work done against a parallel electric field by a parallel velocity which is modulated).

END

The solution

$$f_{a1}=-I\frac{v_{\parallel}}{\Omega_{a}}\frac{\partial f_{a0}}{\partial\psi}+g_{a}\left(\psi;\epsilon,\mu\right)$$

The equation for g_a is derived from the exploitation of periodicity and is, in the domain of circulating particles

$$\left\langle\frac{B}{v_{\parallel}}\left[e_{a}E_{\parallel}^{(A)}\ v_{\parallel}\ \frac{1}{T_{a0}}f_{a0}+\sum_{b}C\left(f_{a1},f_{b1}\right)\right]\right\rangle=0$$

and

$$g_{a}=0$$

in the domain of trapped particles.

Why *time variation* of the potential $\partial\Phi/\partial t$? Is-it polarization ? Or, damping of the poloidal rotation ?

24 Drift Kinetic eq. and numerical solution Santarius and Hinton

The neoclassical drift convection of the drift wave instability.

In *drift kinetic solutions* comment on **Santarius Hinton** numerical solution of drift wave equation.

The term

$$
\begin{aligned}
& \exp\left[il\left(\varphi-q\theta\right)\right]\left(\mathbf{v}+\mathbf{v}_{D}\right)\cdot\boldsymbol{\nabla}f_{m} \\
= & \exp\left[il\left(\varphi-q\theta\right)\right]\left(\frac{d\theta}{dt}\frac{\partial f_{m}}{\partial\theta}+\frac{d\varphi}{dt}\frac{\partial f_{m}}{\partial\varphi}+\frac{dr}{dt}\frac{\partial f_{m}}{\partial r}\right) \\
= & \left[\frac{\xi v}{qR}+\frac{1}{\Omega_{0}}\frac{1}{R}\frac{v^{2}\left(1+\xi^{2}\right)}{2}\cos\theta\right]\frac{\partial\widetilde{f}_{m}}{r\partial\theta} \\
& +\frac{1}{\Omega_{0}}\frac{1}{R}\frac{v^{2}\left(1+\xi^{2}\right)}{2}\frac{il}{r}\left(q\cos\theta+\frac{\varepsilon^{2}}{q}\cos\theta+\frac{dq}{dr}r\ \theta\sin\theta\right)\widetilde{f}_{m} \\
& +\frac{1}{\Omega_{0}}\frac{1}{R}\frac{v^{2}\left(1+\xi^{2}\right)}{2}\sin\theta\frac{\partial\widetilde{f}_{m}}{\partial r}
\end{aligned}
$$

This comes after an expansion in toroidal l harmonic components

$$
\widetilde{f}_{m}\left(r,\theta\right)=f_{m}\left(r,\theta,\varphi\right)\exp\left[il\left(\varphi-q\theta\right)\right]
$$

of the *poloidal* harmonic component m of the first order distribution function.

This is the part that we are interested in, a line in the formula above

$$
+\frac{1}{\Omega_{0}}\frac{1}{R}\frac{v^{2}\left(1+\xi^{2}\right)}{2}\frac{il}{r}\left(q\cos\theta+\frac{dq}{dr}r\ \theta\sin\theta\right)\widetilde{f}_{m}
$$

The first factors are

$$
\frac{1}{\Omega_{0}}\frac{1}{R}\frac{v^{2}\left(1+\xi^{2}\right)}{2}=\frac{1}{\Omega_{0}}\frac{\frac{v_{\perp}^{2}}{2}+v_{\parallel}^{2}}{R}=v_{D}
$$

and the wavenumbers

$$
q=\frac{m}{l}\rightarrow l\sim\frac{m}{q}
$$

and

$$
k_{\theta}\sim\frac{m}{r}
$$

which leads to

$$
\frac{l}{r}=\frac{m}{rq}=k_{\theta}\frac{1}{q}
$$

and

$$
\begin{aligned}
& \frac{1}{\Omega_{0}}\frac{1}{R}\frac{v^{2}\left(1+\xi^{2}\right)}{2}\frac{il}{r}\left(q\cos\theta+\frac{dq}{dr}r\ \theta\sin\theta\right) \\
= & \ i\ v_{D}k_{\theta}\ \left(\cos\theta+\widehat{s}\theta\sin\theta\right)
\end{aligned}
$$

where

$$\widehat{s} = \frac{rq'}{q}$$

and we note

$$v_D k_\theta = \omega_D$$

The term can now be written

$$\omega_D \left(\cos\theta + \widehat{s}\theta \sin\theta \right)$$

which will occur in the treatment of drift kinetic equation for ions (like ITG).

The equation is Fokker Planck

$$\frac{\partial f_e}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} f_e + \frac{\partial f_e}{\partial \mu}\frac{d\mu}{dt} + \frac{\partial f_e}{\partial \epsilon}\frac{d\epsilon}{dt} + \frac{\partial f_e}{\partial \zeta}\frac{d\zeta}{dt} = C\left(f_e\right)$$

and after gyro-averaging one gets the *drift-kinetic* equation

$$\frac{\partial \overline{f}_e}{\partial t} + \left(\mathbf{v}_\parallel + \mathbf{v}_D + \mathbf{v}_E \right) \cdot \boldsymbol{\nabla} \overline{f}_e - \frac{|e|}{m_e}\frac{\partial \phi}{\partial t}\frac{\partial \overline{f}_e}{\partial \epsilon} = C\left(f_e\right)$$

One notices that the energy term is actually adequate for the *wave perturbations*, which means for instabilities, not for pure neoclassics. Except if we are interested in time variation of poloidal rotation by the change of the electrostatic potential associated to surfaces ψ due to polarization.

One would expect

$$|e| \, E_\parallel v_\parallel \frac{\partial \overline{f}_e}{\partial \epsilon}$$

which allows to represent the effect of an electric field in tokamak.

Accordingly, the term of the type

$$\mathbf{v}_D \cdot \boldsymbol{\nabla} f_M = v_{Dr} \frac{\partial f_M}{\partial r}$$

is *absent.*

Instead we have

$$\begin{aligned} \mathbf{v}_E \cdot \boldsymbol{\nabla} f_M &= \frac{-\boldsymbol{\nabla}\phi \times \widehat{\mathbf{n}}}{B} \cdot \boldsymbol{\nabla} f_M \\ &= \frac{1}{B}\frac{\partial \phi}{r\partial \theta}\frac{\partial f_M}{\partial r} \end{aligned}$$

which is again of the type of *wave perturbation* since we expect that the fluctuations of the potential along the poloidal direction will produce the fluctuating advection of the density-temperature gradients from the equilibrium profiles, along the radial direction.

Therefore we are not surprized to see

$$\begin{aligned}
& -i\omega f_e^{(1)} + \left(\mathbf{v}_\parallel + \mathbf{v}_D\right) \cdot \boldsymbol{\nabla} f_e^{(1)} - C\left(f_e^{(1)}\right) \\
= \; & \frac{|e|}{T_e} f_{Me} \left\{ i\omega\phi + \frac{T_e}{|e|\,B} \frac{\partial\phi}{r\partial\theta} \frac{d\ln n}{dr} \left[1 + \eta_e \left(\frac{v^2}{v_{th,e}^2} - \frac{3}{2}\right)\right]\right\}
\end{aligned}$$

We notice the wave -like time variations, harmonic as $i\omega$, for the fluctuating potential ϕ and for the fluctuating distribution function $f_e^{(1)}$.

Naturally, it follows an expansion in the helical geometry

$$\begin{array}{ll}
\text{poloidal} & \theta \\
\text{toroidal} & \exp\left(-il\varphi\right)
\end{array}$$

See further the text *solutions drift kinetic.*

25 Reason for the order -1 in the expansion in ω/ν

There is an explanation in **Frieman 1970 collisional diffusion**.

This is given in **Hazeline Hinton review** around 4.33.

In the colision-dominated regime the pressure is isotropic and

$$\begin{aligned}
\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} p &= \nabla_\parallel p \\
0 &= -\nabla_\parallel p + F_\parallel + enE_\parallel
\end{aligned}$$

The expansion parameter is

$$\Delta \equiv \frac{\omega}{\nu}$$

which compares the frequency of bounce with the frequency of collisions.

It is the second small parameter, after

$$\delta \equiv \frac{\rho}{L}$$

Here $\omega/\nu \ll 1.$ and

$$f_1 = f_1^{(0)} + f_1^{(1)} + ...$$

The fact that the parallel force (friction) is proportional with ν makes that

$$\nabla_{\|}p \sim O\left(\delta\Delta^{-1}\right)$$

and since the variation of the pressure in space is obtained through integration of the distribution function over the velocity space, this term is proportional with the distribution function and this means that this one must also contain a term of order Δ^{-1}

$$f_1 = f_1^{(-1)} + f_1^{(0)} + f_1^{(1)} + ...$$

26 Hazeltine Hinton review

26.1 Fluxes

The expressions after the definition of the first small parameter

$$\delta \equiv \frac{\rho}{L}$$

are, in the first order δ^1,

$$(n\mathbf{u}_{\perp})_1 = \frac{1}{m\Omega}\widehat{\mathbf{n}} \times (\boldsymbol{\nabla}p + en\boldsymbol{\nabla}\phi)$$

This is the fluid derivation of the *diamagnetic* flow.

$$\mathbf{q}_{\perp,1} = \frac{5}{2}\frac{1}{m\Omega}p\ \widehat{\mathbf{n}} \times \boldsymbol{\nabla}T$$

there is the constraint that the total particle and heat fluxes are divergence-free

$$\begin{aligned}\boldsymbol{\nabla}\cdot(n\mathbf{u})_1 &= 0 \\ \boldsymbol{\nabla}\cdot\mathbf{q}_1 &= 0\end{aligned}$$

and this implies the existence of *return flows.*

The general extressions of the total flows are written as sums of components that are (1) along $\mathbf{B}$ and (2) toroidal. The difference between these two directions is small.

$$\begin{aligned}n\mathbf{u} &= \widehat{K}\mathbf{B} & \text{(parallel)} \\ &+\widetilde{K}\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta & \text{(toroidal)}\end{aligned}$$

The functions $\widetilde{K}$ and $\widehat{K}$ must be determined.

A relation between $\widetilde{K}$ and $\widehat{K}$ can be obtained from the zero-divergence condition

$$\frac{1}{\sqrt{g}}\frac{\partial \widetilde{K}}{\partial \varphi} = -\mathbf{B}\cdot\boldsymbol{\nabla}\widehat{K}$$

or

$$\frac{\partial \widetilde{K}}{\partial \varphi} = \left(\sqrt{g}B\right)\ \nabla_{\parallel}\widehat{K}$$

Now we compare two equations. The equation for the perpendicular flux

$$\left(n\mathbf{u}_{\perp}\right)_1 = \frac{1}{m\Omega}\widehat{\mathbf{n}}\times\left(\boldsymbol{\nabla}p + en\boldsymbol{\nabla}\phi\right)$$

and the *formal* equation for the total flux, which simply defines the two unknown functions $\widehat{K}$ and $\widetilde{K}$.

$$n\mathbf{u} = \widehat{K}\mathbf{B} + \widetilde{K}\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\theta$$

where we note that the perpendicular component of the total flux, $(n\mathbf{u})_{\perp}$ can only come from the term with $\widetilde{K}\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\theta$.

We start from

$$\left|\boldsymbol{\nabla}\varphi\cdot\left(\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\theta\right)\right| = \frac{1}{\sqrt{g}}$$

where

$$\frac{\partial g}{\partial \varphi} = 0$$

We write

$$\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\theta \sim W\boldsymbol{\nabla}\varphi \quad \text{(toroidal)}$$

and we multiply

$$\begin{aligned}
\boldsymbol{\nabla}\varphi\cdot\left(\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\theta\right) &\sim W\left|\boldsymbol{\nabla}\varphi\right|^2 \\
\frac{1}{\sqrt{g}} &= W\frac{1}{R^2} \\
W &= \frac{R^2}{\sqrt{g}}
\end{aligned}$$

so that the relationship is

$$\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\theta = \frac{R^2}{\sqrt{g}}\boldsymbol{\nabla}\varphi$$

Now we want to take the *parallel* component of this product $\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta$, so that

$$\begin{aligned} & (\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta) \cdot \mathbf{B} \\ = & \frac{R^2}{\sqrt{g}} \boldsymbol{\nabla}\varphi \cdot \mathbf{B} \\ = & \frac{R^2}{\sqrt{g}} \frac{1}{R} \widehat{\mathbf{e}}_\varphi \cdot \mathbf{B} \\ = & B \frac{R}{\sqrt{g}} \left(\widehat{\mathbf{e}}_\varphi \cdot \widehat{\mathbf{e}}_\parallel \right) \end{aligned}$$

We take

$$\widehat{\mathbf{e}}_\varphi \cdot \widehat{\mathbf{e}}_\parallel = \frac{B_T}{B}$$

$$(\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta) \cdot \mathbf{B} = B \frac{R}{\sqrt{g}} \frac{B_T}{B}$$

$$R B_T = \sqrt{g} \left(\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta \right) \cdot \mathbf{B}$$

The quantity

$$I \equiv \sqrt{g} \left(\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta \right) \cdot \mathbf{B}$$

only depends on ψ, the magnetic surface

$$\begin{aligned} I &= I(\psi) \\ &= R B_T \quad \text{(for circular surfaces)} \end{aligned}$$

To extract the *perpendicular* component from $\widetilde{K} (\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta)$ we multiply with $\widehat{\mathbf{e}}_\perp$, where

$$\widehat{\mathbf{e}}_\perp = \widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_\psi$$

and obtain

$$\left[\widetilde{K} \left(\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta \right) \right] \cdot \left(\widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_\psi \right)$$

where

$$\widehat{\mathbf{e}}_\psi = \frac{\boldsymbol{\nabla}\psi}{|\boldsymbol{\nabla}\psi|}$$

To extract the *parallel* component from $\widetilde{K} (\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta)$ we multiply with the versor $\widehat{\mathbf{n}}$, and take into account that $I \equiv \sqrt{g} (\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta) \cdot \mathbf{B}$,

$$\widetilde{K} \left(\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta \right) \cdot \mathbf{B} \frac{1}{B} = \widetilde{K} \quad I g^{-1/2} \frac{1}{B}$$

or, $\approx \frac{RB_T}{\sqrt{g}}\frac{1}{B}$. This gives

$$
\begin{aligned}
(n\mathbf{u})_{\parallel} &= \left\{\left[\widehat{K}\mathbf{B}+\widetilde{K}\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\theta\right]\cdot\widehat{\mathbf{n}}\right\}\widehat{\mathbf{n}} \\
&= \left\{\widehat{K}B+\left(\widetilde{K}\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\theta\right)\cdot\mathbf{B}/B\right\}\widehat{\mathbf{n}} \\
&= \left\{\widehat{K}B+\widetilde{K}\;\; Ig^{-1/2}\frac{1}{B}\right\}\widehat{\mathbf{n}}
\end{aligned}
$$

To obtain the *perpendicular* component we must extract from the full expression of the flow

$$
\begin{aligned}
n\mathbf{u} &= (n\mathbf{u})_{\parallel}\widehat{\mathbf{n}}+(n\mathbf{u})_{\perp}\widehat{\mathbf{e}}_{\perp} \\
\widehat{K}\mathbf{B}+\widetilde{K}\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\theta &= \left\{\widehat{K}B+\widetilde{K}\;\; Ig^{-1/2}\frac{1}{B}\right\}\widehat{\mathbf{n}} \\
&\quad +(n\mathbf{u})_{\perp}\widehat{\mathbf{e}}_{\perp}
\end{aligned}
$$

$$
\begin{aligned}
(n\mathbf{u})_{\perp}\widehat{\mathbf{e}}_{\perp} &= \widetilde{K}\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\theta \\
&\quad -\widetilde{K}\;\; Ig^{-1/2}\frac{1}{B}\widehat{\mathbf{n}}
\end{aligned}
$$

Then

$$
(n\mathbf{u})_{\perp}=\widetilde{K}\;\;\widehat{\mathbf{e}}_{\perp}\cdot\left(\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\theta-Ig^{-1/2}\frac{1}{B}\widehat{\mathbf{n}}\right)
$$

The last term is suppressed, $\widehat{\mathbf{e}}_{\perp}\cdot\widehat{\mathbf{n}}=0$. The first term is

$$
\widehat{\mathbf{e}}_{\perp}\cdot(\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\theta)=(\boldsymbol{\nabla}\theta\times\widehat{\mathbf{e}}_{\perp})\cdot\boldsymbol{\nabla}\psi
$$

Separately

$$
\widehat{\mathbf{e}}_{\perp}=\widehat{\mathbf{n}}\times\widehat{\mathbf{e}}_{\psi}
$$

and

$$
\begin{aligned}
\boldsymbol{\nabla}\theta\times\widehat{\mathbf{e}}_{\perp} &= \boldsymbol{\nabla}\theta\times(\widehat{\mathbf{n}}\times\widehat{\mathbf{e}}_{\psi}) \\
&= \widehat{\mathbf{n}}\,(\boldsymbol{\nabla}\theta\cdot\widehat{\mathbf{e}}_{\psi})-\widehat{\mathbf{e}}_{\psi}\,(\boldsymbol{\nabla}\theta\cdot\widehat{\mathbf{n}}) \\
&= -\widehat{\mathbf{e}}_{\psi}\,(\boldsymbol{\nabla}\theta\cdot\widehat{\mathbf{n}})
\end{aligned}
$$

Then

$$
(n\mathbf{u})_{\perp}=[n\mathbf{u}\cdot\widehat{\mathbf{e}}_{\perp}]\;\;\widehat{\mathbf{e}}_{\perp}
$$

$$
\begin{aligned}
(n\mathbf{u})_{\perp}\cdot\widehat{\mathbf{e}}_{\perp} &= \widetilde{K}\;\widehat{\mathbf{e}}_{\perp}\cdot(\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\theta)=\widetilde{K}\;\;(\boldsymbol{\nabla}\theta\times\widehat{\mathbf{e}}_{\perp})\cdot\boldsymbol{\nabla}\psi \\
&= \widetilde{K}\;\;[-\,|\boldsymbol{\nabla}\psi|\,(\boldsymbol{\nabla}\theta\cdot\widehat{\mathbf{n}})]
\end{aligned}
$$

$$\begin{aligned}\left[-\widetilde{K}\ \ |\nabla\psi|\,(\nabla\theta\cdot\widehat{\mathbf{n}})\right]\widehat{\mathbf{e}}_{\perp} &= \frac{1}{m\Omega}\widehat{\mathbf{n}}\times(\nabla p+en\nabla\phi)\\ &= \frac{1}{m\Omega}\widehat{\mathbf{n}}\times\nabla\psi\ \left(\frac{dp}{d\psi}+en\frac{d\phi}{d\psi}\right)\end{aligned}$$

we recall that

$$\widehat{\mathbf{e}}_{\perp}=\widehat{\mathbf{n}}\times\widehat{\mathbf{e}}_{\psi}$$

and it results

$$\begin{aligned}&\left[-\widetilde{K}\ \ |\nabla\psi|\,(\nabla\theta\cdot\widehat{\mathbf{n}})\right](\widehat{\mathbf{n}}\times\widehat{\mathbf{e}}_{\psi})\\ =\ \ &(\widehat{\mathbf{n}}\times\widehat{\mathbf{e}}_{\psi})\,|\nabla\psi|\,\frac{1}{m\Omega}\left(\frac{dp}{d\psi}+en\frac{d\phi}{d\psi}\right)\\ &-\widetilde{K}\ \ (\nabla\theta\cdot\widehat{\mathbf{n}})=\frac{1}{m\Omega}\left(\frac{dp}{d\psi}+en\frac{d\phi}{d\psi}\right)\end{aligned}$$

We find

$$\widetilde{K}=-\frac{1}{\nabla\theta\cdot\widehat{\mathbf{n}}}\frac{1}{eB}\left(\frac{dp}{d\psi}+en\frac{d\phi}{d\psi}\right)$$

It appears that

$$\frac{1}{\nabla\theta\cdot\widehat{\mathbf{n}}}\frac{1}{B}=\sqrt{g}$$

where

$$\sqrt{g}=\frac{1}{2\pi}\frac{qR}{B_T}$$

or

$$\nabla\theta\cdot\mathbf{B}=\nabla\theta\cdot(\nabla\varphi\times\nabla\psi)$$

This leads to

$$\begin{aligned}\widetilde{K} &= -\frac{1}{\nabla\theta\cdot\widehat{\mathbf{n}}}\frac{1}{eB}\left(\frac{dp}{d\psi}+en\frac{d\phi}{d\psi}\right)\\ &= -\sqrt{g}\frac{1}{e}\left(\frac{dp}{d\psi}+en\frac{d\phi}{d\psi}\right)\end{aligned}$$

It still is necessary to determine $\widehat{K}$ from the expression of the *total* particle flux $(n\mathbf{u})^{(1)}$ in order δ^1.

The formal expression for the parallel flux can be obtained after we have determined $\widetilde{K}$. We multiply $(n\mathbf{u})^{(1)}$ by $\widehat{\mathbf{n}}$.

$$\begin{aligned}nu_{\parallel} &= (n\mathbf{u})^{(1)}\cdot\widehat{\mathbf{n}}\\ &= \widehat{\mathbf{n}}\cdot\left(\widehat{K}\mathbf{B}+\widetilde{K}\nabla\psi\times\nabla\theta\right)\\ &= \widehat{K}\ B\\ &\quad +\widetilde{K}\ \widehat{\mathbf{n}}\cdot(\nabla\psi\times\nabla\theta)\end{aligned}$$

We use

$$I \equiv \sqrt{g}\left(\nabla\psi \times \nabla\theta\right) \cdot \mathbf{B}$$

which gives

$$\left(\nabla\psi \times \nabla\theta\right) \cdot \widehat{\mathbf{n}} = \frac{I}{B\sqrt{g}}$$

In addition we can now replace

$$\widetilde{K} = -\sqrt{g}\frac{1}{e}\left(\frac{dp}{d\psi} + en\frac{d\phi}{d\psi}\right)$$

to obtain

$$\begin{aligned} nu_{\parallel} &= \widehat{K}\ B \\ &\quad -\sqrt{g}\frac{1}{e}\left(\frac{dp}{d\psi} + en\frac{d\phi}{d\psi}\right)\frac{I}{B\sqrt{g}} \end{aligned}$$

$$\left(nu_{\parallel}\right)^{(1)} = -\frac{1}{m\Omega_c} I\left(\frac{dp}{d\psi} + en\frac{d\phi}{d\psi}\right) + \widehat{K}\ B$$

(**Note** that $\frac{1}{m\Omega_c}\widehat{\mathbf{n}} \times \nabla p$ is nu^{dia} and that the factor I and the derivation to ψ provides projection on the parallel direction).

The similar calculation for the flow of heat starts from

$$\mathbf{q}_{\perp,1} = \frac{5}{2}\frac{1}{m\Omega}p\ \widehat{\mathbf{n}} \times \nabla T$$

and writes the expression in terms of *parallel* and *toroidal* components

$$\left(q_{\parallel}\right)^{(1)} = -\frac{5}{2}\frac{1}{m\Omega_c} I\ p\frac{dT}{d\psi} + \widehat{L}\left(\psi\right) B$$

For the *current* the expression uses the electron + ion $\left(nu_{\parallel}\right)^{(1)}$ formulas for the parallel flows, multiplied by e_a and added. The potential $d\phi/d\psi$ disappears.

$$j_{\parallel} = -I\frac{1}{B}\frac{dp}{d\psi} + K\left(\psi\right) B$$

where

$$K \equiv \sum_{a=e,i} e_a \widehat{K}_a$$

NOTE

In **Hirshman neoclassical current 1978** (see *bootstrap.tex*) we find

$$j_{\|} = -F\frac{p'}{B} + KB$$

where

$$p' = \frac{dp}{d\psi}$$

and finally

$$K(\psi) = -\frac{1}{2\pi}\frac{dF}{d\psi}$$

We should identify

$$F \rightarrow I$$

but there are problems with θ dependence for noncircular surfaces.

END

The *current* contains the *poloidal* component, projected along the magnetic line, $\|$.

We conclude that at this moment we have to determine two functions

$$\widehat{K}_a \text{ and } \widehat{L}$$

The distribution function in the first order must depend on the constants of motion

$$\begin{aligned} &\psi - \frac{Iv_{\|}}{\Omega} \\ &\epsilon \\ &\mu \end{aligned}$$

then

$$\begin{aligned} f(\mathbf{x}, \epsilon, \mu) &= F\left(\psi - \frac{Iv_{\|}}{\Omega}, \epsilon, \mu\right) \\ &= F(\psi, \epsilon, \mu) - \frac{Iv_{\|}}{\Omega}\frac{\partial F}{\partial \psi} \\ &\quad +O\left(\delta^2\right) \end{aligned}$$

It results

$$\begin{aligned} f &= f_M(\psi, \epsilon) \\ &\quad -\frac{Iv_{\|}}{\Omega}\frac{\partial f_M}{\partial \psi} \\ &\quad +g(\psi, \epsilon, \mu) \end{aligned}$$

then

$$\widehat{K}(\psi) = 2\pi \sum_{\sigma} \sigma \int d\mu d\epsilon \ g$$

$$\widehat{L}(\psi) = 2\pi \sum_{\sigma} \int d\mu d\epsilon \ m \left(\epsilon - \frac{e\phi}{m} - \frac{5}{2}\frac{T}{m} \right) \ g$$

With these expressions, the formulas for the fluid and heat flows are complete, if we are able to determine the solution g of the drift kinetic equation.

See the application of these formulas in the Pfirsch Schluter transport enhancement (*drift kinetic derivation text*).

NOTE
The structure of the function of distribution in the first approximation

$$f = f_M - \frac{I v_{\|}}{\Omega} \frac{\partial f_M}{\partial \psi} + g$$

rises a problem. For trapped particles we must have a non-uniformity of the distribution function (for trapped) on the poloidal variable θ. This is because the trapped particles are staying longer at the low field side and only the tip of their banana reaches regions of high magnetic field, *i.e.* inside the torus. The only way a space variation can be visible in the expression of f is the function

$$v_{\|} \equiv v_{\|}(\theta)$$

This is because g will finally be proved to be zero in the trapped region.

See expressions in **Galeev Sagdeev JETP**.
END

26.2 Drift-kinetic equation

There are three expansions that are made for deriving the drift-kinetic equation

- separation of the gyroangle averaged part from the part that depends on the gyration angle

$$f = \overline{f} + \widetilde{f}$$

- expansion in the small spatial parameter: Larmor versus equilibrium length

$$\delta = \frac{\rho}{L}$$

- expansion in small temporal parameter: bounce versus collision

$$\Delta = \frac{\omega_b}{\nu}$$

With the variables

$$\begin{aligned} w &\equiv \epsilon - \frac{e}{m}\phi \\ &= \frac{v^2}{2} \end{aligned}$$

The kinetic equation 4.33

$$\begin{aligned} &\frac{\partial f}{\partial t} + \left(v_\parallel \widehat{\mathbf{n}} + \mathbf{v}_D\right) \cdot \boldsymbol{\nabla} f \\ &+\frac{e}{m} E_\parallel v_\parallel \frac{\partial f}{\partial w} \\ = \;& C\left(f, f\right) \end{aligned}$$

After separation of the equilibrium (Maxwellian) distribution function and the correction of order δ^1,

$$\begin{aligned} &v_\parallel \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} f_1 - C\left(f_1\right) \\ = \;& -\mathbf{v}_D \cdot \boldsymbol{\nabla} f_M + v_\parallel e E_\parallel \frac{1}{T} f_M \end{aligned}$$

Note that in **Trapped electron mode Rewoldt Tang Friemann** there is a derivation

$$\frac{\partial}{\partial \epsilon}$$

done for a banana, from which two terms arise:

$$\frac{m v_\parallel}{T_e} - \frac{1}{m v_\parallel}$$

of which only the second survives for cuasi-trapped particles, that have small $v_\parallel$.

End.

There is a flow of the ions

$$n_i u_{i\parallel} \widehat{\mathbf{n}} = \int d^3v \; \mathbf{v} \; f_i$$

This flow is parallel with the lines.

The *linearized* collision operator

$$\begin{aligned} C_{ei}^{l} &= \nu_{ei}(v)\ L[f_{e1}] \qquad \text{(pitch angle)} \\ &+\nu_{ei}(v)\, 2\frac{v_{\parallel}u_{i\parallel}}{v_{th,i}^{2}} f_{eM} \qquad \text{(collisional coupling with a parallel flow)} \end{aligned}$$

where

$$\nu_{ei}(v) \equiv \frac{3\sqrt{\pi}}{4\tau_e}\left(\frac{v_{th,e}}{v}\right)^{3}$$

Now the expression for the collision operator L is given in terms of the *cosine of the pitch angle*

$$\xi \equiv \frac{v_{\parallel}}{v}$$

(**note** the difference relative to

$$\lambda = \frac{v_{\perp}^{2}}{v^{2}}\frac{1}{B(\mathbf{x})}$$

, or, other definition, $\lambda' = \frac{v_{\perp}^{2}}{v^{2}}h$.) PITCH ANGLE operator, takes care of trapped/untrapped region.

$$L = \frac{1}{2}\frac{\partial}{\partial \xi}\left(1-\xi^{2}\right)\frac{\partial}{\partial \xi}$$

Due to the structure of the expression of the collision operator, it is extracted from f_{1e} the second term, which contains the ion parallel flow velocity $\mathbf{u}_{\parallel i}$.

$$\begin{aligned} f_{1e} &= 2\frac{v_{\parallel}u_{i\parallel}}{v_{th,i}^{2}} f_{eM} \\ &\quad +f_{e}^{(u)} \end{aligned}$$

NOTE

See **Rewoldt Tang Frieman integral formulation two-dim trapped**, eq. (10), for circulating electrons.

END

The equation for f_{1e} is

$$\begin{aligned} & v_{\parallel}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla} f_{e}^{(u)} - C_{0e}^{l}\left(f_{e}^{(u)}\right) \\ = \; & -ev_{\parallel}E_{\parallel}\frac{f_{Me}}{T_e} - \mathbf{v}_D\cdot\boldsymbol{\nabla} f_{Me} \\ & -f_{Me}v_{\parallel}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\left(2\frac{v_{\parallel}u_{i\parallel}}{v_{th,i}^{2}}\right) \end{aligned}$$

It follows the expansion of this function

$$f_1^{(u)} = f_1^{(u)(-1)} + f_1^{(u)(0)} + f_1^{(u)(1)} + ...$$

and the following term is Δ^{-2},

$$C^l \left(f_{1e}^{(u)(-1)} \right) = 0$$

which means that

$$\begin{aligned} f_{1e}^{(u)(-1)} &= \text{perturbed Maxwellian} \\ &= f_{Me} \left[\frac{p'}{p} + \frac{T'}{T} \left(\frac{v^2}{v_{th}^2} - \frac{5}{2} \right) + 2 \frac{v_\parallel u_{i\parallel}}{v_{th,i}^2} \right] \end{aligned}$$

The parallel flow is of the order

$$u_{i\parallel} \sim u_{e\parallel} \sim O\left(\delta \Delta^0\right)$$

and will be omitted.

$$\begin{aligned} C_{0e}^l \left(f_{1e}^{(u)(0)} \right) &= \frac{1}{T_e} f_{Me} v_\parallel E_\parallel \\ &\quad + f_{Me} v_\parallel \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} f_{1e}^{(u)(-1)} \end{aligned}$$

Here we can introduce the function $f_{1e}^{(u)(-1)}$ which is the Maxwellian, after the neglect of the parallel flow $u_{\parallel e}$.

We obtain

$$\begin{aligned} & C_{0e}^l \left(f_{1e}^{(u)(0)} \right) \\ &= f_{Me} v_\parallel \left[A_1 + \left(\frac{v^2}{v_{th,e}^2} - \frac{5}{2} \right) A_2 \right] \end{aligned}$$

We remark that the forces actually exist only due to the variation of the plasma parameters in the magnetic surface, or, equivalently, along the magnetic field

$$A_1 = \nabla_\parallel \ln p + \frac{eE_\parallel}{T_e}$$

$$A_2 = \nabla_\parallel \ln T_e$$

these are very small variations, usually.

The transport coefficients are the entries of the matix connecting the fluxes (J) with the forces (A)

$$
\begin{aligned}
\frac{J_{\|}}{e} &= K_{11}A_1 + K_{12}A_2 \\
\frac{q_{\|e}}{T_e} &= -K_{12}A_1 - K_{22}A_2 \\
\frac{q_{\|i}}{T_i} &= -K_iA_i
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 &= \nabla_{\|}p + \frac{eE_{\|}}{T_e} \\
A_2 &= \nabla_{\|}\ln T_e \\
A_i &= \nabla_{\|}\ln T_i
\end{aligned}
$$

Substitution to extract as factor the collision τ,

$$K_{mn} = \left(\frac{\tau_e n T_e}{m_e}\right)\kappa_{mn}$$

The next step consists of finding the connection between the parallel gradients and the perpendicular gradients of the main parameters.

Compare two expressions for the parallel flux of heat. The formal one, in terms of the formal function $K_i = K_i(\psi)$,

$$q_{\|i} = -T_iK_iA_i$$

with

$$q_{\|i} = -\frac{5}{2}\frac{1}{ZeB}\ Ip_i\frac{dT_i}{d\psi} + \widehat{L}(\psi)\,B$$

and one derives an expression for the force A_i.

$$A_i = \frac{5}{2}\frac{1}{ZeB}\ \frac{1}{K_i}\ n_i\frac{1}{T_i}\frac{dT_i}{d\psi} + \widehat{L}(\psi)\,\frac{B}{K_iT_i}$$

Since we know that

$$A_i = \nabla_{\|}\ln T_i$$

we can use the annihilator equivalent to multiplication by B followed by the surface averaging

$$\langle A_iB\rangle = 0$$

Then

$$A_i = \frac{5}{2}\frac{1}{ZeB}\,\frac{1}{K_i}\,n_i\frac{dT_i}{d\psi}\left(I - \frac{B^2\langle I\rangle}{\langle B^2\rangle}\right)$$

The heat flux on the direction perpendicular on the surface, averaged over the surface

$$\begin{aligned} & \left\langle \mathbf{q}_i^{neocl}\cdot\boldsymbol{\nabla}\psi\right\rangle \\ = & -1.6\left(\frac{1}{eZ}\right)^2\frac{m_i}{\tau_i}p_i\frac{dT_i}{d\psi} \\ & \times\left(\left\langle\frac{I^2}{B^2}\right\rangle - \frac{\langle I^2\rangle}{\langle B^2\rangle}\right)\end{aligned}$$

Question: how the collision time τ_e has appeared in this expression?

We must note that we actually have a connection between a parallel and a perpendicular flux.

Let us consider the fluid result

$$(n\mathbf{u}_\perp)_1 = \frac{1}{m\Omega}\widehat{\mathbf{n}}\times(\boldsymbol{\nabla}p + en\boldsymbol{\nabla}\phi)$$

We multiply vectorially with the magnetic field $\mathbf{B}$,

$$\begin{aligned} n\mathbf{u}_\perp\times\mathbf{B} &= \frac{1}{m\Omega}\left[\widehat{\mathbf{n}}\times(\boldsymbol{\nabla}p + en\boldsymbol{\nabla}\phi)\right]\times\mathbf{B} \\ &= \frac{1}{m\Omega}\left[\widehat{\mathbf{n}}\left(\mathbf{B}\cdot\boldsymbol{\nabla}p + \mathbf{B}\cdot en\boldsymbol{\nabla}\phi\right) - (\boldsymbol{\nabla}p + en\boldsymbol{\nabla}\phi)\left(\widehat{\mathbf{n}}\cdot\mathbf{B}\right)\right] \\ &= \frac{1}{e}\left[\widehat{\mathbf{n}}\left(\nabla_\parallel p + en\nabla_\parallel\phi\right) - (\boldsymbol{\nabla}p + en\boldsymbol{\nabla}\phi)\right]\end{aligned}$$

The term in the left hand side is

$$n\mathbf{u}_\perp\times\mathbf{B} = n\mathbf{u}\times\mathbf{B}$$

Now we use the equation

$$\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\theta = \frac{1}{q}\mathbf{B}_T$$

and multiply the two terms of the equation above by them and by $\sqrt{g}$

Left hand side

$$\begin{aligned} & \sqrt{g}\,(\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\theta)\cdot(n\mathbf{u}\times\mathbf{B}) \\ = & \;n\mathbf{u}\cdot\boldsymbol{\nabla}\psi\end{aligned}$$

We note that actually this is what we look for: the flux perpendicular on the magnetic surfaces.

The right hand side, where we use the other expression

$$\sqrt{g}\frac{1}{q}\mathbf{B}_T\left\{\frac{1}{e}\left[\widehat{\mathbf{n}}\left(\nabla_{\|}p+en\nabla_{\|}\phi\right)-(\boldsymbol{\nabla}p+en\boldsymbol{\nabla}\phi)\right]\right\}$$

However when we want to prepare the averaging over the magnetic surface we take separately the last term and return to the first expression, which means to multiply by $\sqrt{g}\left(\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\theta\right)\cdot$,

$$\begin{aligned}
& \sqrt{g}\left(\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\theta\right)\cdot\left(\boldsymbol{\nabla}p+en\boldsymbol{\nabla}\phi\right) \\
= \; & \sqrt{g}\left(\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\theta\right)\cdot\boldsymbol{\nabla}\varphi\frac{\partial p}{\partial\varphi}+en\sqrt{g}\left(\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\theta\right)\cdot\boldsymbol{\nabla}\varphi\frac{\partial\phi}{\partial\varphi} \\
= \; & \frac{\partial p}{\partial\varphi}+en\frac{\partial\phi}{\partial\varphi}
\end{aligned}$$

The flux surface average is zero.

Then from the two terms above, we just remain with the first, parallel

$$\sqrt{g}\frac{1}{q}\left(\mathbf{B}_T\cdot\mathbf{B}\right)\frac{1}{eB}\left(\nabla_{\|}p+en\nabla_{\|}\phi\right)$$

and for surface average we multiply with B

$$\left\langle\sqrt{g}\frac{1}{q}\left(\mathbf{B}_T\cdot\mathbf{B}\right)\frac{1}{e}\left(\nabla_{\|}p+en\nabla_{\|}\phi\right)\right\rangle$$

Now we recognize the expression

$$I=\frac{1}{q}\sqrt{g}\left(\mathbf{B}_T\cdot\mathbf{B}\right)$$

27 Review of Hirshman Sigmar

The definitions.

Momentum collisional

$$\mathbf{F}_{a1}=\int d\mathbf{v}\; m_a\mathbf{v}\; C_a\left(f_a\right)$$

Energy collisional

$$Q_a=\int d\mathbf{v}\;\frac{1}{2}m_a\left|\mathbf{v}-\mathbf{u}_a\right|^2\; C_a\left(f_a\right)$$

Energy flux

$$\mathbf{Q}_a = \int d\mathbf{v} \left(\frac{1}{2} m_a \mathbf{v}^2\right) \mathbf{v} \; f_a$$

it is

$$\mathbf{Q}_a = \mathbf{q}_a + \frac{5}{2} p_a \mathbf{u}_a + \left(n_a \frac{1}{2} m_a u_a^2\right) \mathbf{u}_a + \mathbf{u}_a \cdot \mathbf{\Pi}_a$$

Energy-weighted stress tensor

$$\mathbf{R}_a \equiv \int d\mathbf{v} \left(\frac{1}{2} m_a v^2\right) \mathbf{v} \; \mathbf{v} \; f_a$$

Collisional rate of heat flux generation (heat friction)

$$\begin{aligned}
\mathbf{G}_a &\equiv \int d\mathbf{v} \left(\frac{1}{2} m_a v^2\right) \mathbf{v} \; C_a(f_a) \\
&= \frac{T_a}{m_a} \left(\frac{5}{2} \mathbf{F}_{a1} + \mathbf{F}_{a2}\right)
\end{aligned}$$

and

$$\mathbf{F}_{a2} = \int d\mathbf{v} \; m_a \mathbf{v} \left(\frac{v^2}{v_{th,a}^2} - \frac{5}{2}\right) C_a$$

The TOTAL *radial* $\sim \psi$ fluxes are

$$\Gamma_a^{\psi} = -\frac{2\pi}{\chi'} \frac{1}{e_a} \left\langle R^2 \boldsymbol{\nabla}\varphi \cdot (\mathbf{F}_{a1} + e_a n_a \mathbf{E}) \right\rangle$$

$$\frac{q_a^{\psi}}{T_a} = -\frac{2\pi}{\chi'} \frac{1}{e_a} \left\langle R^2 \boldsymbol{\nabla}\varphi \cdot \mathbf{F}_{a2} \right\rangle$$

and after substracting the *classical* fluxes

$$\overline{\Gamma}_a^{\psi} = -\frac{2\pi}{\chi'} \left\langle \frac{I}{m_a \Omega_a} \left(F_{a1}^{\parallel} + e_a n_a E_{\parallel}^{(A)}\right) \right\rangle$$

$$\frac{\overline{q}_a^{\psi}}{T_a} = -\frac{2\pi}{\chi'} \left\langle \frac{I}{m_a \Omega_a} F_{a2}^{\parallel} \right\rangle$$

In this way we express the perpendicular fluxes, along ψ, in terms of parallel friction forces, $\parallel$.

NOTE

Actually the two *friction* forces $\mathbf{F}_{a1}$ and $\mathbf{F}_{a2}$ are involved in the calculation of the *classical* fluxes,

$$
\begin{aligned}
\widetilde{\Gamma}_{\psi} &\equiv \left\langle -\frac{1}{m_a\Omega_a}\left(\widehat{\mathbf{n}}\times\mathbf{F}_{a1}\right)\cdot\boldsymbol{\nabla}\psi\right\rangle \\
\frac{\widetilde{q}_{\psi}}{T_a} &\equiv \left\langle -\frac{1}{m_a\Omega_a}\left(\widehat{\mathbf{n}}\times\mathbf{F}_{a2}\right)\right\rangle
\end{aligned}
$$

END

the gyroaveraged fluxes are

$$
\begin{aligned}
\overline{\Gamma}_a^{\psi} &= \left\langle -\frac{1}{m_a\Omega_a}\left[\widehat{\mathbf{n}}\times\left(-\boldsymbol{\nabla}p-\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_a+e_a n_a\boldsymbol{\nabla}\phi\right)\right]\cdot\boldsymbol{\nabla}\psi\right\rangle \quad \text{(fluid)} \\
&= \left\langle \int d\mathbf{v}\ \left(\mathbf{v}_{Da}\cdot\boldsymbol{\nabla}\psi\right)\ \overline{f}_a\right\rangle \qquad \text{(kinetic)}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\overline{q}_a}{T_a} &= \left\langle -\frac{1}{m_a\Omega_a}\left[\widehat{\mathbf{n}}\times\left(-\boldsymbol{\nabla}\cdot\boldsymbol{\Theta}_a-\frac{5}{2}n_a\boldsymbol{\nabla}T_a\right)\cdot\boldsymbol{\nabla}\psi\right]\right\rangle \\
&= \left\langle \int d\mathbf{v}\ \left(\mathbf{v}_{Da}\cdot\boldsymbol{\nabla}\psi\right)\left(\frac{v^2}{v_{th,a}^2}-\frac{5}{2}\right)\ \overline{f}_a\right\rangle
\end{aligned}
$$

Geometric equations

$$
\begin{aligned}
\mathbf{B}\cdot\boldsymbol{\nabla} &\equiv B\nabla_{\|} \\
&= \frac{\chi'}{2\pi}\frac{1}{\sqrt{g}}\frac{\partial}{\partial\theta}
\end{aligned}
$$

where

$$
\chi' = \frac{\partial\chi}{\partial\psi}
$$

and χ is the poloidal flux function.

$$
\frac{dV}{d\psi} = 2\pi\int_0^{2\pi}\sqrt{g}\ d\theta
$$

The averages

$$
\langle A\rangle = \frac{\int_0^{2\pi}\sqrt{g}\ A\ d\theta}{\int_0^{2\pi}\sqrt{g}\ d\theta}
$$

and

$$\langle \mathbf{B} \cdot \boldsymbol{\nabla} A \rangle = 0$$

$$\langle \boldsymbol{\nabla} \cdot \mathbf{A} \rangle = \frac{1}{\frac{dV}{d\psi}} \frac{\partial}{\partial \psi} \left(\frac{dV}{d\psi} \langle \mathbf{A} \cdot \boldsymbol{\nabla} \psi \rangle \right)$$

The fluxes can be derived from the basic fluid equation.

$$\begin{aligned} m_a n_a \frac{d\mathbf{u}_a}{dt} &= -\boldsymbol{\nabla} p_a - \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a \\ &\quad + e_a n_a \left(\mathbf{E} + \mathbf{u}_a \times \mathbf{B} \right) \\ &\quad + \mathbf{F}_{a1} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \mathbf{Q}_a}{\partial t} &= \frac{e_a}{m_a} \left[\mathbf{E} \cdot \left(\frac{5}{2} p_a \ \mathbf{I} + \boldsymbol{\pi}_a + m_a n_a \ \mathbf{u}_a \mathbf{u}_a \right) + \mathbf{Q}_a \times \mathbf{B} \right] \\ &\quad - \boldsymbol{\nabla} \cdot \mathbf{r}_a \\ &\quad + \mathbf{G}_a \end{aligned}$$

where

$$\begin{aligned} \mathbf{Q}_a &\equiv \int d\mathbf{v} \ \left(\frac{1}{2} m_a \mathbf{v}^2 \right) \ \mathbf{v} \ f_a \\ &= \mathbf{q}_a + \frac{5}{2} p_a \mathbf{u}_a + n_a \left(\frac{1}{2} m_a \mathbf{u}_a^2 \right) \mathbf{u}_a + \mathbf{u}_a \cdot \boldsymbol{\pi}_a \\ &= \text{total energy flux} \end{aligned}$$

$$\begin{aligned} \mathbf{r}_a &\equiv \int d\mathbf{v} \ \left(\frac{1}{2} m_a v^2 \ \mathbf{v} \ \mathbf{v} \right) f_a \\ &= \text{energy-weighted stress tensor} \end{aligned}$$

$$\begin{aligned} \mathbf{G}_a &\equiv \int d\mathbf{v} \ \left(\frac{1}{2} m_a v^2 \ \mathbf{v} \right) C_a \\ &= \frac{T_a}{m_a} \left(\frac{5}{2} \mathbf{F}_{a1} + \mathbf{F}_{a2} \right) \\ &= \text{colisional rate of heat flux generation} \end{aligned}$$

In order to obtain the relation between perpendicular fluxes (for transport) and parallel fluxes, one multiplies the momentum equation by $\mathbf{B}$ and takes the average over the magnetic surface. Assuming stationarity

$$\begin{aligned} 0 &= \langle -\mathbf{B} \cdot \boldsymbol{\nabla} p_a - \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a \\ &\quad + e_a n_a \mathbf{B} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{F}_{a1} \rangle \end{aligned}$$

We take into account that

$$\langle \mathbf{B} \cdot \boldsymbol{\nabla} p \rangle = 0$$

The definition

$$\begin{aligned} \boldsymbol{\pi}_a &= \underset{\text{pressure anisotropy}}{\left(p_{\|a} - p_{\perp a}\right)} \left(\widehat{\mathbf{n}}\widehat{\mathbf{n}} - \frac{1}{3}\mathbf{I}\right) \end{aligned}$$

and we have to calculate

$$\begin{aligned} & \langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a \rangle \\ =& \left\langle \left(p_{\perp a} - p_{\|a}\right) \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} B \right\rangle \end{aligned}$$

Then from the full equation $\mathbf{B}\cdot$ (momentum conservation) it results

$$\begin{aligned} & \left\langle \left(p_{\perp a} - p_{\|a}\right) \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} B \right\rangle \\ =& \left\langle \left(e_a n_a E^{(A)\|} + F_{a1}^{\|}\right) B \right\rangle \\ & - \langle e_a \widetilde{n}_a \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \phi \rangle \end{aligned}$$

This formula connects the surface average of the parallel friction forces to the stress anisotropy $\left(p_{\perp a} - p_{\|a}\right)$.

To go further one multiplies the *stationary form* of the momentum equation by the toroidal vector

$$R^2 \boldsymbol{\nabla} \varphi \cdot (\text{momentum conservation})$$

and takes the average over the magnetic surface

$$\begin{aligned} 0 =& \left\langle R^2 \boldsymbol{\nabla} \varphi \cdot \left(-\boldsymbol{\nabla} p_a - \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a\right) \right\rangle \\ & + \left\langle R^2 \boldsymbol{\nabla} \varphi \cdot \left[e_a n_a \left(\mathbf{E} + \mathbf{u}_a \times \mathbf{B}\right)\right] \right\rangle \\ & + \left\langle R^2 \boldsymbol{\nabla} \varphi \cdot \mathbf{F}_{a1} \right\rangle \end{aligned}$$

where

$$R^2 \boldsymbol{\nabla} \varphi = \frac{I}{B} \widehat{\mathbf{n}} - \frac{\chi'}{2\pi} \frac{\widehat{\mathbf{n}} \times \boldsymbol{\nabla} \psi}{B}$$

and we have

$$-\left\langle R^2 \boldsymbol{\nabla} \varphi \cdot \boldsymbol{\nabla} p_a \right\rangle = 0$$

$$\begin{aligned} & \left\langle R^2 \boldsymbol{\nabla} \varphi \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_a \right\rangle \\ =& \left\langle \boldsymbol{\nabla} \cdot \frac{\chi'}{2\pi} \frac{p_{\|a} - p_{\perp a}}{B} \widehat{\mathbf{n}} \times \boldsymbol{\nabla} \psi \right\rangle \end{aligned}$$

The general formula

$$\left\langle \boldsymbol{\nabla}\cdot\mathbf{A}\right\rangle = \frac{1}{\frac{dV}{d\psi}}\frac{\partial}{\partial\psi}\left(\frac{dV}{d\psi}\left\langle \mathbf{A}\cdot\boldsymbol{\nabla}\psi\right\rangle\right)$$

is applied

$$\begin{aligned} & \left\langle R^2\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_a\right\rangle \\ = & \frac{1}{\frac{dV}{d\psi}}\frac{\partial}{\partial\psi}\left\{\frac{dV}{d\psi}\left\langle\left(\frac{\chi'}{2\pi}\frac{p_{\parallel a}-p_{\perp a}}{B}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\psi\right)\cdot\boldsymbol{\nabla}\psi\right\rangle\right\} \\ = & \ 0 \end{aligned}$$

the next terms

$$\left\langle R^2\boldsymbol{\nabla}\varphi\cdot\left(e_a n_a \mathbf{u}_a\times\mathbf{B}\right)\right\rangle$$

applying the formula above

$$\begin{aligned} & \left\langle R^2\boldsymbol{\nabla}\varphi\cdot\left(e_a n_a \mathbf{u}_a\times\mathbf{B}\right)\right\rangle \\ = & \left\langle\left(\frac{I}{B}\widehat{\mathbf{n}}-\frac{\chi'}{2\pi}\frac{\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\psi}{B}\right)\cdot\left(e_a n_a \mathbf{u}_a\times\mathbf{B}\right)\right\rangle \\ = & \left\langle -\frac{\chi'}{2\pi}\frac{\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\psi}{B}\cdot\left(e_a n_a \mathbf{u}_a\times\mathbf{B}\right)\right\rangle \\ = & -\frac{\chi'}{2\pi}\left\langle e_a n_a\left(\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\psi\right)\cdot\left(\mathbf{u}_a\times\widehat{\mathbf{n}}\right)\right\rangle \end{aligned}$$

We expand the scalar product of the two vector products

$$\left(\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\psi\right)\cdot\left(\mathbf{u}_a\times\widehat{\mathbf{n}}\right)$$

by introducing the notation

$$\mathbf{k}\equiv\mathbf{u}_a\times\widehat{\mathbf{n}}$$

$$-\mathbf{k}\times\left(\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\psi\right)=\left(\mathbf{k}\times\widehat{\mathbf{n}}\right)\cdot\boldsymbol{\nabla}\psi$$

and

$$\begin{aligned} \mathbf{k}\times\widehat{\mathbf{n}} &= -\widehat{\mathbf{n}}\times\left(\mathbf{u}_a\times\widehat{\mathbf{n}}\right) = -\mathbf{u}_a\left(\widehat{\mathbf{n}}\cdot\widehat{\mathbf{n}}\right)+\widehat{\mathbf{n}}\left(\mathbf{u}_a\cdot\widehat{\mathbf{n}}\right) \\ &= -\mathbf{u}_a+\widehat{\mathbf{n}}\left(\mathbf{u}_a\cdot\widehat{\mathbf{n}}\right) \end{aligned}$$

Then

$$\begin{aligned} & \left(\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\psi\right)\cdot\left(\mathbf{u}_a\times\widehat{\mathbf{n}}\right) \\ = & \left[-\mathbf{u}_a+\widehat{\mathbf{n}}\left(\mathbf{u}_a\cdot\widehat{\mathbf{n}}\right)\right]\cdot\boldsymbol{\nabla}\psi \\ = & -\mathbf{u}_a\cdot\boldsymbol{\nabla}\psi \end{aligned}$$

and the term is

$$\begin{aligned}
&\left\langle R^2 \boldsymbol{\nabla}\varphi \cdot (e_a n_a \mathbf{u}_a \times \mathbf{B})\right\rangle \\
= & \frac{\chi'}{2\pi} \left\langle e_a n_a \mathbf{u}_a \cdot \boldsymbol{\nabla}\psi \right\rangle \\
= & e_a \frac{\chi'}{2\pi} \Gamma^{\psi}
\end{aligned}$$

The other two terms in the equation: $R^2 \boldsymbol{\nabla}\varphi \cdot$ (momentum conservation) are

$$\left\langle R^2 \boldsymbol{\nabla}\varphi \cdot [e_a n_a \mathbf{E} + \mathbf{F}_{a1}]\right\rangle$$

Finally we collect the results

$$0 = e_a \frac{\chi'}{2\pi} \Gamma^{\psi} + \left\langle R^2 \boldsymbol{\nabla}\varphi \cdot [e_a n_a \mathbf{E} + \mathbf{F}_{a1}]\right\rangle$$

or

$$\Gamma^{\psi} = -\frac{2\pi}{\chi'} \frac{1}{e_a} \left\langle R^2 \boldsymbol{\nabla}\varphi \cdot [e_a n_a \mathbf{E} + \mathbf{F}_{a1}]\right\rangle$$

Since, we remind

$$R^2 \boldsymbol{\nabla}\varphi = \frac{I}{B}\widehat{\mathbf{n}} - \frac{\chi'}{2\pi} \frac{\widehat{\mathbf{n}} \times \boldsymbol{\nabla}\psi}{B}$$

we can rewrite

$$\begin{aligned}
\Gamma_a^{\psi} = & -\frac{2\pi}{\chi'} \frac{1}{e_a} \\
& \times \left\langle \left(\frac{I}{B}\widehat{\mathbf{n}} - \frac{\chi'}{2\pi} \frac{\widehat{\mathbf{n}} \times \boldsymbol{\nabla}\psi}{B} \right) \cdot [e_a n_a \mathbf{E} + \mathbf{F}_{a1}] \right\rangle
\end{aligned}$$

We will find

$$\begin{aligned}
\left\langle \frac{\widehat{\mathbf{n}} \times \boldsymbol{\nabla}\psi}{m_a \Omega_a} \cdot \mathbf{F}_{a1} \right\rangle & = \widetilde{\Gamma}_a^{\psi} \\
& = \text{classical flux}
\end{aligned}$$

It can be expressed

$$\widetilde{\Gamma}_a^{\psi} = -\left\langle \frac{\widehat{\mathbf{n}} \times \mathbf{F}_{a1}}{m_a \Omega_a} \cdot \boldsymbol{\nabla}\psi \right\rangle$$

from where it results that the perpendicular forces drive the classical fluxes.

It remains, after excluding the classical flux

$$\begin{aligned}
\overline{\Gamma}^{\psi} &= -\frac{2\pi}{\chi'}\frac{1}{e_a}\left\langle \frac{I}{B}\widehat{\mathbf{n}}\cdot\left[e_a n_a \mathbf{E}+\mathbf{F}_{a1}\right]\right\rangle \\
&= -\frac{2\pi}{\chi'}\left\langle I\frac{e_a n_a E^{\parallel}+F_{a1}^{\parallel}}{m_a\Omega_a}\right\rangle
\end{aligned}$$

Returning to a previous calculation

$$\begin{aligned}
& \left\langle \left(p_{\perp a}-p_{\parallel a}\right)\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}B\right\rangle \\
= & \left\langle \left(e_a n_a E^{(A)\parallel}+F_{a1}^{\parallel}\right)B\right\rangle \\
& -\left\langle e_a\widetilde{n}_a\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\phi\right\rangle
\end{aligned}$$

This is a relation between the viscous stress and the friction.

The expression for the *neoclassical* flux of particles across the magnetic surfaces is separated into three pieces, by addind and substracting some terms

$$\begin{aligned}
\overline{\Gamma}_{\alpha}^{\psi} &= -\frac{2\pi}{\chi'}\frac{1}{e_a}\left\langle \frac{I}{B}\widehat{\mathbf{n}}\cdot\left[e_a n_a \mathbf{E}+\mathbf{F}_{a1}\right]\right\rangle \\
&= \Gamma_a^{BP}+\Gamma_a^{PS}+\Gamma_a^{E}
\end{aligned}$$

where

$$\Gamma_a^{BP} = -\frac{2\pi}{\chi'}\left\langle I\right\rangle\frac{\left\langle\left(e_a n_a E^{(A)\parallel}+F_{a1}^{\parallel}\right)B\right\rangle}{e_a\left\langle B^2\right\rangle}$$

$$\Gamma_a^{PS} = -\frac{2\pi}{\chi'}\left\langle\frac{F_{a1}}{m_a\Omega_a}\left(I-\left\langle I\right\rangle\frac{B^2}{\left\langle B^2\right\rangle}\right)\right\rangle$$

$$\Gamma_a^{E} = -\frac{2\pi}{\chi'}\left\langle I\right\rangle\frac{\left\langle n_a\nabla_{\parallel}\phi\right\rangle}{\left\langle B^2\right\rangle}$$

Question. What means the surface average of I ? It is function of ψ.

28 Pfirsch Schluter regime

It is a problem of *equilibrium flows* in tokamak.

Many aspects are discussed in **Stringer**.tex

The paper **ionthermchanghinton** is about full consideration of finite r/R in the *ion thermal conductivity*.

The formulas

$$R_0' = \frac{dR}{dr} = \text{Shafranov shift}$$

$$\varepsilon = \frac{r}{R}$$

Then

$$\left\langle \frac{B_0^2}{B^2} \right\rangle = \frac{1 + \frac{3}{2}\left(\varepsilon^2 + \varepsilon R_0'\right) + \frac{3}{8}\varepsilon^3 R_0'}{1 + \frac{3}{2}\varepsilon R_0'}$$

Neglecting the Shafranov shift

$$\left\langle \frac{B_0^2}{B^2} \right\rangle \approx 1 + \frac{3}{2}\varepsilon^2$$

And

$$\frac{1}{\left\langle \frac{B^2}{B_0^2} \right\rangle} = \frac{\sqrt{1-\varepsilon^2}\left(1 + \frac{1}{2}\varepsilon R_0'\right)}{1 + \frac{R_0'}{\varepsilon}\left(\sqrt{1-\varepsilon^2} - 1\right)}$$

or

$$\begin{aligned} \frac{1}{\left\langle \frac{B^2}{B_0^2} \right\rangle} &\approx \sqrt{1-\varepsilon^2} \\ &\approx 1 - \frac{1}{2}\varepsilon^2 \end{aligned}$$

An approximation used by **Hazeltine Hinton Rosenbluth 1973** was

$$\left\langle \frac{B_0^2}{B^2} \right\rangle \approx 1$$

Definition of the **Pfirsch Schluter** factor

$$F = \frac{1}{2\sqrt{\varepsilon}} \left[\left\langle \frac{B_0^2}{B^2} \right\rangle - \frac{1}{\left\langle \frac{B^2}{B_0^2} \right\rangle} \right]$$

or

$$\begin{aligned} F &= \frac{1}{2\sqrt{\varepsilon}} \left[1 + \frac{3}{2}\varepsilon^2 - \left(1 - \frac{1}{2}\varepsilon^2\right) \right] \\ &= \frac{1}{2\sqrt{\varepsilon}} \left(2\varepsilon^2\right) = \varepsilon^{3/2} \end{aligned}$$

NOTE
A derivation can be found in **Hirshman neoclassical current**.

In *boostrap and diamagnetic.*
END

NOTE a similar factor is **Rutherford**

$$\begin{aligned}\frac{D}{D_{classical}} &= 1+\frac{3\sqrt{2}}{2}\frac{B^2}{B_\theta^2}\sqrt{\frac{r}{R}}\left[1-\int_0^1\frac{dk}{k^2}\left(\frac{\pi}{2\mathbf{E}\left(k\right)}-1\right)\right] \\ &\approx 1+1.6\frac{B^2}{B_\theta^2}\sqrt{\frac{r}{R}}=1+1.6q^2\varepsilon^{-3/2}\end{aligned}$$

for the transport enhancement coming from Pfirsch Schluter effect.
END

In the paper **Bolton Ware** in which there is a numerical implementation of a collision operator, it is also said

the PS heat flux Q^{PS} is driven by the energy-weighted friction produced by the parallel heat flow

$$\widetilde{q}_{\parallel}=-\frac{5}{2}\frac{p_i}{eB_\theta}\frac{dT_i}{dr}\ 2\varepsilon\cos\theta$$

and, in turn, this parallel heat flow ("harmonic on the section" as described somehwere) arises from the non-zero divergence of the heat flux

$$\mathbf{Q}=-\frac{5}{2}\frac{p_i}{eB}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}T_i$$

(they call it magnetic heat conduction).

It is the diamagnetic heat flow.

The problem that is discussed is the collisionality affecting the heat conductibility, between regimes (PS, plateau and banana)

NOTE we must understand how is possible to have an effective flow (diamagnetic) in the absence of collisions, such level of absence of collisions where the *pressure* itself cannot be defined. **END**