Equations of charged particle motion in toroidal (tokamak) magnetic field

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Abstract

A large number of versions of derivation and formulation of the equations of motion of particles is reviewed. This rather strange approach (instead of a simple and concise presentation) is useful when there are applications with particular requests of representation of the particle motion or, to simplify the lecture of the large number of papers.

This is one of the Lectures in the Work Session of Plasma Theory. The text is never final.

1 Particle motion in tokamak, introduction

The knowledge of the distribution function $f(\mathbf{x}, \mathbf{v}, t)$, with its time-dependence on the phase-space variables: real-space $\mathbf{x}$ and velocity space $\mathbf{v}$) allows to calculate flows, currents, etc. and to analyse stability to perturbations of the states of the plasma. The theory of instabilities, the transport and the evaluation of fluxes of particles and energy are based on the distribution function. The equations for particle motion in tokamak are necessary for the determination of the kinetic distribution function.

The motion of particles in the strong magnetic field in tokamak has some particularities. The Larmor gyration is by far the most rapid motion with a space and time scale that are widely separated relative to other motions. The averaging over this rapid rotation exhibits the guiding center drift motion. The gyroangle is removed from the arguments of the (gyro-averaged) distribution function. The Fokker Planck equation becomes the drift-kinetic equation and the convective term

$$\left(\mathbf{v}_{\|} + \mathbf{v}_D\right) \cdot f$$

now includes explicitly the drift of the guiding center $\mathbf{v}_D$. It is the cause of the departure of the particle trajectory from the simple helical (magnetic) line and explains the neoclassical transport.

Similar to the gyration but at much larger space and time scales there is a periodic motion of the guiding centers, along orbits defined by the magnetic mirrors. These orbits (called "banana" orbits) are for trapped particles between the reflection points where $v_\parallel = 0$. Averaging over this periodic motion (bounce along the "banana" orbit) exhibits a drift of the "center of banana orbit", a precession along the toroidal direction.

We would be tempted to underline the similarity between
- Larmor gyration and
- periodic motion (bounce) on "banana" orbits
and respectively between
- gyro-averaging, leading to guiding center motion
- bounce-averaging on "banana" orbits, which identifies the toroidal precession

However there are essential differences between the two cases.

Note first that, in the case of Larmor gyration, all charged particles perform this motion, with no exception.

In the case of the large scale motion of the guiding centers, there are two distinct type of trajectories
- between magnetic mirror points, i.e. trapped particles on banana orbits
- helical motion, with no trapping, i.e. those particles go beyond any magnetic mirror

These two types of large scale trajectories define two populations: (1) trapped particles and (2) circulating (passing) particles. The displacement of trapped particles relative to the magnetic surface where lie the two mirror points (limits of the banana orbit) is a considerable enhancement of transport in plasma. Especially the collisions between the trapped/passing populations lead to important transport processes in tokamak.

The text will present derivation of the equations of motion of particles, which means five equations for three spatial variables and two velocity variables. The invariants of the motion in the magnetic geometry of tokamak explains why in the velocity space two variables are sufficient.

Next, we will discuss the derivation of the equations for the kinetic distribution function. The trajectories, as mentioned, are an essential input in this derivation. In addition, the collision operator allows to represent the friction and respectively the transitions between the trapped and passing populations. The solutions of the drift kinetic equations are only indicated (a separate text exists for this subject).

1.1 Invariants and variables

The energy

$$\epsilon = \frac{1}{2} m v_{\|}^2 + \mu B + e\phi$$

The magnetic moment

$$\begin{aligned} \text{action } S &= \oint p dq = \text{ const} \\ p &\equiv \text{generalized momentum} \\ q &\equiv \text{generalized coordinate} \end{aligned}$$

In the case of Larmor rotation

$$\begin{aligned} dq &= \rho_L d\theta \;\; \text{element of distance along the circumference} \\ p &= p_\theta = m v_\theta \;\; \text{conjugate momentum, along the circumference} \end{aligned}$$

Since $v_\theta = v_\perp = $ const on the circle of gyration,

$$\begin{aligned} S &= \oint m v_\theta \; \rho_L d\theta = m v_\perp \rho_L \int_0^{2\pi} d\theta = 2\pi m v_\perp \rho_L \\ &= 2\pi m v_\perp \frac{v_\perp}{\Omega_c} = 2\pi m v_\perp^2 \frac{1}{eB/m} = \frac{4\pi m}{e} \times \frac{m v_\perp^2}{2B} \end{aligned}$$

$$\mu = \frac{m v_\perp^2}{2B}$$

Here the *adiabatic invariant* is the amount of magnetic flux that traverses the surface of the Larmor circle of gyration.

The longitudinal invariant. Due to axisymmetry, a consequence of the Noether theorem (a symmetry $\rightarrow$ an invariant)

$$J = m v_{\|} \left(1 + \varepsilon \cos\theta\right) - e \int_0^r B_\theta h dr$$

For this invariant: the *generalized* momentum

$$\mathbf{p}^{gen} = \mathbf{p} - e\mathbf{A}$$

multiplied by the Radius R

$$\mathbf{R} \times \mathbf{p}^{gen} = \text{angular momentum}$$

projected on the direction φ of axisymmetry.

$$R^2 \boldsymbol{\nabla}\varphi \cdot (m\mathbf{v} - e\mathbf{A}) = \text{const}$$

1.1.1 Variables

These are

$$(\mathbf{x}, \mathbf{v}) \rightarrow (\mathbf{x}, \epsilon, \mu, \zeta)$$

The gyration

$$\zeta : \text{ gyration angle}$$

Volume in velocity space

$$\int d^3v = \sum_\sigma \int d\mu d\epsilon d\zeta \, \frac{B}{\left|v_\parallel\right|}$$

The Jacobian $B/\left|v_\parallel\right|$ is obtained in the section on change of variables.

The domain of integration

$$\begin{aligned} \frac{e\phi}{m} &< \epsilon < \infty \\ 0 &< \mu < \frac{\epsilon - e\phi/m}{B} \end{aligned}$$

NOTE that ϵ contains the electrostatic potential, while w does not. **End.**

For untrapped particles

$$\int^{pass} d^3v = \sum_\sigma \int d\lambda \; w dw \; d\zeta \, \frac{B}{\left|v_\parallel\right|}$$

domain of integration

$$\begin{aligned} 0 &< w < \infty \\ 0 &< \lambda < \lambda_c \quad \text{CIRCULATING} \\ & \quad \text{because small } v_\perp \end{aligned}$$

where

$$w = \frac{v^2}{2} \quad \text{(energy, no } \phi\text{)}$$

$$\begin{aligned}
\lambda &= \frac{v_\perp^2}{v^2}\frac{1}{B} = \frac{v_\perp^2}{v^2}\frac{1}{B_0}h(\theta) \\
&= \frac{\mu}{w}
\end{aligned}$$

NOTE.
In **Rewoldt Tang Frieman**

$$\int d^3v = \frac{\pi}{2}\left(\frac{2}{m_j}\right)^{3/2}\sum_{\sigma_\parallel}\int_0^\infty d\epsilon\ \epsilon^{1/2}\int_0^{h(\theta)} d\lambda \frac{1}{h(\theta)}\frac{1}{\sqrt{1-\frac{\lambda}{h(\theta)}}}$$

Here the definition of λ does NOT contain the magnetic field but only $\lambda = \frac{v_\perp^2}{v^2}h(\theta)$. Then

$$\sqrt{1-\frac{\lambda}{h(\theta)}} = \sqrt{1-\frac{v_\perp^2}{v^2}} = \frac{|v_\parallel|}{v}$$

This would be $\frac{|v_\parallel|}{v} = \sigma\xi$ where $\xi = v_\parallel/v$. And

$$\begin{aligned}
&\int_0^{h(\theta)} d\lambda \frac{1}{h(\theta)}\frac{v}{|v_\parallel|}(...) \\
&= \int_0^{h(\theta)} d\left(\frac{v_\perp^2}{v^2}\right)\frac{v}{|v_\parallel|}(...) = \int_0^{h(\theta)} d\left(\frac{v_\perp^2}{2B}\frac{1}{v^2}\right) v\frac{B}{|v_\parallel|}(...) \\
&= \frac{1}{v}\int_0^{h(\theta)} d\mu\ \frac{B}{|v_\parallel|}(...)
\end{aligned}$$

an integration over ϵ is still necessary.

Alternatively:

$$\left(\mathbf{x}, \frac{v^2}{2}, \frac{v_\parallel}{v}\right)$$

Usual notation

$$\begin{aligned}
w &\equiv \frac{v^2}{2} \\
\xi &\equiv \frac{v_\parallel}{v}
\end{aligned}$$

w distinct of ϵ by absence of ϕ.

1.1.2 Invariants

The generalized toroidal momentum of the particle

$$J = mv_{\parallel}(1+\varepsilon\cos\theta) - e\int_0^r B_\theta h dr$$

where

$$\begin{aligned} B_\theta &= \frac{b(r)}{h} \\ \text{with } h &= 1+\varepsilon\cos\theta \end{aligned}$$

NOTE
Verifying the units

$$\begin{aligned} e\int_0^r B_\theta h dr &\rightarrow C\ T\ m \\ mv_{\parallel}(1+\varepsilon\cos\theta) &\rightarrow kg\frac{m}{s} \end{aligned}$$

in the file *Notes-sci, General,* we find

$$\frac{kg}{s} = C\ T$$

which confirms the units. OK.
END

Adopting the approximation

$$v_\varphi \approx v_\parallel$$

$$\frac{d}{dt}\left(mv_\varphi h - e\int^r b(r)\, dr\right) = 0$$

in **Rosenbluth Hazeltine Hinton**. This is the way they find v_{Dr}.

From **Hazeltine Hinton review**
First some notation from this review.
(φ is the toroidal angle) the definition of the poloidal flux function

$$\psi(\mathbf{x}) = \frac{1}{2\pi^2}\int d^3x\ \boldsymbol{\nabla}\theta\cdot\mathbf{B}$$

poloidal flux

$$\Phi(\mathbf{x}) = \frac{1}{2\pi}\int d^3x \; \boldsymbol{\nabla}\varphi \cdot \mathbf{B}$$

toroidal flux under a surface ψ

The magnetic field

$$\mathbf{B} = \mathbf{B}_\theta + \mathbf{B}_T$$

$$\begin{aligned} \mathbf{B}_T &= \frac{1}{2\pi}\boldsymbol{\nabla}\Phi \times \boldsymbol{\nabla}\theta \\ &= q(\psi)\,\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta \end{aligned}$$

$$\mathbf{B}_\theta = \boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi$$

and

$$q(\psi) = \frac{1}{2\pi}\frac{d\varphi}{d\psi}$$

The function I,

$$I(\psi,\theta,\varphi) = \sqrt{g}\;\left(\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta\right)\cdot \mathbf{B}$$

This function will appear with notation F in **Hirschman** in the expression of the density of current $\mathbf{j}$, with poloidal part and parallel part.

From here, the invariant

$$\begin{aligned} R^2\boldsymbol{\nabla}\varphi \cdot (m\mathbf{v} - e\mathbf{A}) &= \text{const} \\ Rmv_\varphi - eRA_\varphi &= \text{const} \end{aligned}$$

But

$$\begin{aligned} R^2\boldsymbol{\nabla}\varphi \cdot e\mathbf{A} &= eRA_\varphi \\ &= e\psi \text{ poloidal flux function} \end{aligned}$$

or

$$\begin{aligned} \psi &= -RA_T \\ A_T &= R\boldsymbol{\nabla}\varphi \cdot \mathbf{A} \quad \text{toroidal component of magnetic potential} \end{aligned}$$

and RA_T measures the poloidal flux *outside* a magnetic surface.

Then

$$Rmv_\varphi - e\psi = \text{const}$$

$$\psi - \frac{m v_\varphi R}{e} = \text{const}$$

The last form is equivalent to a "distance" shift relative to the magnetic surface ψ.

This equation seems like a redefinition of the poloidal magnetic potential ψ.

Because

$$\psi = -R A_\varphi$$

in **Morozov Soloviev** page 229 (and in **Hazeltine Hinton** at page 254)

$$\mathbf{A}^* = \mathbf{A} + \frac{m v_\parallel}{e} \widehat{\mathbf{n}}$$

and the magnetic field

$$\mathbf{B}^* = \boldsymbol{\nabla} \times \mathbf{A}^*$$

Then

$$\mathbf{v}_\parallel + \mathbf{v}_D = \frac{\mathbf{B}^*}{B} v_\parallel$$

One can define a function $\psi^* (\mathbf{x}, \epsilon, \mu)$ that allows to write

$$\mathbf{B}^* \cdot \boldsymbol{\nabla} \psi^* = 0$$

(which is similar to $\mathbf{B} \cdot \boldsymbol{\nabla} \psi = 0$). The surfaces

$$\begin{aligned} \mu &= \text{const} \\ \epsilon &= \text{const} \\ \psi^* &= \text{const} \end{aligned}$$

are *drift surfaces.*

See **Kogan Catto** pedestal.

Adiabatic invariant: the formula for trapped particles

$$\begin{aligned} J &\approx \oint v_\parallel d\varphi = \sqrt{\frac{2}{m}} \oint d\varphi \sqrt{\epsilon - \mu B - e\phi} \\ \varphi &\equiv \text{toroidal angle} \end{aligned}$$

The magnetic part $-R A_\varphi = \psi$ is not included here. It is implicitely assumed that the orbit does not depart significantly from the magnetic surface, or that this departure can be neglected for the particular problem.

Boozer Kuo-Petravic 1981.

When secondary magnetic wells along the field line are present, the longitudinal invariant

$$\oint m v_{\parallel} dl$$

suffers jumps when the particles are trapped or detrapped, without collisions, in/from the secondary wells.

1.2 The variables of the Clebsch representation

In **Jenko2008** the definition is adopted

$$q\left(r\right)=\int_{0}^{2\pi}\frac{d\theta}{2\pi}\,\frac{\mathbf{B}\cdot\boldsymbol{\nabla}\varphi}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}$$

The integration on the circle $[0,2\pi]$ on θ is intended to ensure the constant q on surfaces r.

The new variable

$$\beta=q\left(r\right)\ \chi-\varphi$$

and

$$\mathbf{B}=\boldsymbol{\nabla}\beta\times\boldsymbol{\nabla}\psi$$

or

$$\mathbf{B}=\boldsymbol{\nabla}\left[q\left(r\right)\ \chi-\varphi\right]=\frac{dq}{dr}\boldsymbol{\nabla}r\ \chi+q\left(r\right)\boldsymbol{\nabla}\chi-\boldsymbol{\nabla}\varphi$$

$$q=\frac{\mathbf{B}\cdot\boldsymbol{\nabla}\varphi}{\mathbf{B}\cdot\boldsymbol{\nabla}\chi}$$

and

$$\mathbf{B}\cdot\boldsymbol{\nabla}\chi=\mathbf{B}\cdot\boldsymbol{\nabla}\theta\ \frac{\partial\chi}{\partial\theta}$$

Then

$$\mathbf{B}\cdot\boldsymbol{\nabla}\theta\ \frac{\partial\chi}{\partial\theta}=\frac{\mathbf{B}\cdot\boldsymbol{\nabla}\varphi}{q}$$

$$\begin{aligned}
\frac{\partial\chi}{\partial\theta} &= \frac{1}{q\left(r\right)}\frac{\mathbf{B}\cdot\boldsymbol{\nabla}\varphi}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}=\frac{1}{q\left(r\right)}\frac{\frac{B_0R_0}{R}\widehat{\mathbf{e}}_{\varphi}\cdot\frac{1}{R}\widehat{\mathbf{e}}_{\varphi}}{\frac{B_0R_0}{R}\frac{1}{q(r)}\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\widehat{\mathbf{e}}_{\theta}\cdot\frac{1}{r}\widehat{\mathbf{e}}_{\theta}}\\
&= \frac{r}{R}\frac{\sqrt{1-\varepsilon^2}}{\varepsilon}=\frac{r}{R_0\left(1+\varepsilon\cos\theta\right)}\frac{\sqrt{1-\left(\frac{r}{R_0}\right)^2}}{\frac{r}{R_0}}\\
&= \sqrt{1-\varepsilon^2}\frac{1}{1+\varepsilon\cos\theta}
\end{aligned}$$

It results

$$\chi = \sqrt{1-\varepsilon^2} \int_0^\theta \frac{d\theta}{1+\varepsilon\cos\theta}$$

The Formula **2.553-3 Gradshtein Ryzhik** is

$$\begin{aligned} & \int \frac{dx}{a+b\cos x} \\ = & \frac{2}{\sqrt{a^2-b^2}} \arctan\left[\frac{a-b}{\sqrt{a^2-b^2}} \tan\left(\frac{x}{2}\right)\right] \end{aligned}$$

here

$$\begin{aligned} a &= 1 \\ b &= \varepsilon \end{aligned}$$

then

$$\chi = \sqrt{1-\varepsilon^2}\frac{2}{\sqrt{1-\varepsilon^2}} \arctan\left[\frac{1-\varepsilon}{\sqrt{1-\varepsilon^2}} \tan\left(\frac{\theta}{2}\right)\right]$$

or

$$\chi = \arctan\left[\sqrt{\frac{1-\varepsilon}{1+\varepsilon}} \tan\left(\frac{\theta}{2}\right)\right]$$

2 Pitch angle variable κ Beer thesis

2.1 Drift velocity Beer Thesis

Here is the drift velocity derived in **Beer thesis**.

The $\mathbf{\nabla}B$ and curvature drifts are included

$$\begin{aligned} \mathbf{v}_D &= \frac{1}{\Omega} v_\parallel^2 \, \widehat{\mathbf{n}} \times \left[\left(\widehat{\mathbf{n}} \cdot \mathbf{\nabla}\right) \widehat{\mathbf{n}}\right] \\ & + \frac{1}{\Omega} \widehat{\mathbf{n}} \times \mu \mathbf{\nabla} B \end{aligned}$$

Now the following formula are used to transform the curvature term

$$0 = -\mathbf{\nabla} p + \mathbf{j} \times \mathbf{B}$$

$$\mathbf{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j}$$

$$\left(\widehat{\mathbf{n}} \cdot \mathbf{\nabla}\right) \widehat{\mathbf{n}} = \left(\mathbf{\nabla} \times \widehat{\mathbf{n}}\right) \times \widehat{\mathbf{n}}$$

We consider

$$\boldsymbol{\nabla} \times \mathbf{B} = \boldsymbol{\nabla}\times (B\widehat{\mathbf{n}}) = B\,(\boldsymbol{\nabla}\times\widehat{\mathbf{n}}) + \boldsymbol{\nabla}B \times \widehat{\mathbf{n}}$$

from where

$$\begin{aligned}
\boldsymbol{\nabla}\times\widehat{\mathbf{n}} &= \frac{1}{B}\,(\boldsymbol{\nabla}\times\mathbf{B} - \boldsymbol{\nabla}B\times\widehat{\mathbf{n}}) \\
&= \frac{1}{B}\,(\mu_0\mathbf{j} - \boldsymbol{\nabla}B\times\widehat{\mathbf{n}})
\end{aligned}$$

Returning to curvature

$$\begin{aligned}
(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla})\,\widehat{\mathbf{n}} &= (\boldsymbol{\nabla}\times\widehat{\mathbf{n}})\times\widehat{\mathbf{n}} \\
&= \frac{1}{B}\,(\mu_0\mathbf{j} - \boldsymbol{\nabla}B\times\widehat{\mathbf{n}})\times\widehat{\mathbf{n}} \\
&= \frac{1}{B^2}\mu_0\mathbf{j}\times\mathbf{B} \\
&\quad -\frac{1}{B}\,(\boldsymbol{\nabla}B\times\widehat{\mathbf{n}})\times\widehat{\mathbf{n}}
\end{aligned}$$

The first term can use the gradient of the pressure and the second will expand the double product

$$\begin{aligned}
(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla})\,\widehat{\mathbf{n}} &= -\frac{1}{B^2}\mu_0\;\boldsymbol{\nabla}p \\
&\quad +\frac{1}{B}\left[\boldsymbol{\nabla}B\;\;(\widehat{\mathbf{n}}\cdot\widehat{\mathbf{n}}) - \widehat{\mathbf{n}}\;\;(\boldsymbol{\nabla}B\cdot\widehat{\mathbf{n}})\right] \\
&= -\frac{1}{B^2}\mu_0\boldsymbol{\nabla}p + \frac{1}{B}\boldsymbol{\nabla}B - \frac{1}{B}\widehat{\mathbf{n}}\;\;(\boldsymbol{\nabla}B\cdot\widehat{\mathbf{n}})
\end{aligned}$$

Note that the last term will be cancelled by the vectro product with $\widehat{\mathbf{n}}$, in $\mathbf{v}_D$.

This will be replaced in the expression of the drift velocity

$$\begin{aligned}
\mathbf{v}_D &= \frac{1}{\Omega}v_\parallel^2\;\widehat{\mathbf{n}}\times[(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla})\,\widehat{\mathbf{n}}] + \frac{1}{\Omega}\widehat{\mathbf{n}}\times\mu\boldsymbol{\nabla}B \\
&= \frac{1}{\Omega}v_\parallel^2\;\widehat{\mathbf{n}}\times\left\{-\frac{1}{B^2}\mu_0\boldsymbol{\nabla}p + \frac{1}{B}\boldsymbol{\nabla}B\right\} + \frac{1}{\Omega}\widehat{\mathbf{n}}\times\mu\boldsymbol{\nabla}B
\end{aligned}$$

The second term in the paranthesis and the last term are combined

$$\frac{1}{\Omega}\frac{v_\parallel^2 + \frac{v_\perp^2}{2}}{B}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}B$$

and the final form is

$$
\begin{aligned}
\mathbf{v}_D &= \frac{1}{\Omega}\frac{v_\parallel^2 + \frac{v_\perp^2}{2}}{B}\widehat{\mathbf{n}} \times \nabla B \\
&\quad - \frac{1}{\Omega}\frac{v_\parallel^2}{B^2}\mu_0 \widehat{\mathbf{n}} \times \nabla p
\end{aligned}
$$

When $\beta \ll 1$ the term with gradient of pressure can be neglected. In this case, since we have the expression

$$
(\widehat{\mathbf{n}} \cdot \nabla)\,\widehat{\mathbf{n}} = -\frac{1}{B^2}\mu_0 \nabla p + \frac{1}{B}\nabla B - \frac{1}{B}\widehat{\mathbf{n}}\ \left(\nabla B \cdot \widehat{\mathbf{n}}\right)
$$

this gives an approximation

$$
(\widehat{\mathbf{n}} \cdot \nabla)\,\widehat{\mathbf{n}} \approx \nabla \ln B
$$

these formulas are used by **Jenko Hauf**.

The denominator in the expression of the particle guiding center velocity , $\mathbf{B}^*$, is derived according to Littlejohn in a previous Section.

2.1.1 Variables in velocity space (v,κ) Beers

The variables in the velocity space are

$$
\begin{aligned}
& (v,\kappa) \\
\kappa &\equiv \text{pitch angle}
\end{aligned}
$$

$$
E = \frac{1}{2}v^2
$$

In the absence of an electrostatic potential with variation on the trajectory, the velocity v is invariant for the particle [actually, the energy $v^2/2$]

In the following

$$
\varepsilon = \frac{r}{R_0}
$$

and

$$
\kappa^2 = \frac{1 - \frac{\mu B_{\min}}{E}}{2\varepsilon_B}
$$

where

$$
\varepsilon_B = \frac{B_{\max} - B_{\min}}{2B_{\max}}
$$

and

$$
\begin{aligned}
B &= \frac{B_0}{1+\varepsilon\cos\theta} \\
B_{\min} &= \frac{B_0}{1+\varepsilon} \quad \text{at} \quad \theta = 0 \quad \text{(outermost midplane point)} \\
B_{\max} &= \frac{B_0}{1-\varepsilon} \quad \text{at} \quad \theta = \pi \quad \text{(innermost midplane point)}
\end{aligned}
$$

and

$$
\begin{aligned}
\varepsilon_B &= \frac{B_{\max}-B_{\min}}{2B_{\max}} = \frac{\frac{B_0}{1-\varepsilon}-\frac{B_0}{1+\varepsilon}}{2\frac{B_0}{1-\varepsilon}} = \frac{1+\varepsilon-1+\varepsilon}{2\,(1+\varepsilon)} \\
&= \frac{\varepsilon}{1+\varepsilon}
\end{aligned}
$$

NOTE

$B_{\min}$ and $B_{\max}$ are NOT related to a particular orbit but are geometrical characteristics of the magnetic field.

END

Here κ^2 is the pitch angle at the outer midplane, normalized to 1 at the boundary trapped/passing in velocity space. At the trapped/passing boundary

$$
\begin{aligned}
E &= \mu B_{\max} \\
\kappa^2 &= 1
\end{aligned}
$$

Then

$$
\begin{aligned}
& \quad \text{trapped particles} \\
\kappa &< 1
\end{aligned}
$$

Consider the point of turning of the banana.

Let $B_t \equiv$ the magnetic field at the turning point $v_\parallel = 0$.

There the relation between the magnetic field B_t at the turning point and the energy (which is conserved) is written

$$E = \mu B_t$$

The magnetic field is

$$B_t = \frac{B_0}{1+\varepsilon\cos\theta_t}$$

Let us calculate the poloidal angle θ_t where the turning of banana takes place.

Start from

$$\begin{aligned}
\kappa^2 &= \frac{1-\frac{\mu B_{\min}}{E}}{2\varepsilon_B} = \frac{1-\frac{v_\perp^2}{2B}\frac{B_0}{1+\varepsilon}\frac{1}{v^2/2}}{\frac{2\varepsilon}{1+\varepsilon}} = \frac{1-\frac{v_\perp^2}{v^2}\frac{1+\varepsilon\cos\theta}{1+\varepsilon}}{\frac{2\varepsilon}{1+\varepsilon}} \\
&= \frac{1}{2\varepsilon}\left[1+\varepsilon-\lambda\right] \\
&\quad \text{Beers}
\end{aligned}$$

where the new variable is defined as

$$\lambda \equiv \frac{v_\perp^2}{v^2}h$$

where

$$h = 1+\varepsilon\cos\theta$$

or $h = \frac{B_0}{B}$.

This means

$$\kappa^2 = \frac{1}{2\varepsilon}\left[1+\varepsilon-\frac{v_\perp^2}{v^2}\left(1+\varepsilon\cos\theta\right)\right]$$

NOTE on Approximations

One can continue to transform

$$\begin{aligned}
\kappa^2 &= \frac{1}{2\varepsilon}\frac{1}{v^2}\left[v^2+\varepsilon v^2-v_\perp^2-\varepsilon v_\perp^2\cos\theta\right] \\
&= \frac{1}{2\varepsilon}\frac{1}{v^2}\left[v_\parallel^2+\varepsilon v_\parallel^2+\varepsilon v_\perp^2-\varepsilon v_\perp^2\cos\theta\right] \\
&= \frac{1}{2\varepsilon}\frac{1}{v^2}\left[v_\parallel^2\left(1+\varepsilon\right)+\varepsilon v_\perp^2 2\sin^2\left(\frac{\theta}{2}\right)\right]
\end{aligned}$$

This equation suggests some approximations.

(A) For deeply trapped particles, $v_\parallel \approx 0$,

$$\begin{aligned}
k^2 &\approx \frac{1}{2\varepsilon}\frac{1}{v^2}\varepsilon v_\perp^2 2\sin^2\left(\frac{\theta}{2}\right) \\
&= \frac{v_\perp^2}{v^2}\sin^2\left(\frac{\theta}{2}\right)
\end{aligned}$$

(B) Or, from the full formula, keeping $v_\parallel$,

$$\begin{aligned}
\kappa^2 &= \frac{1}{2\varepsilon}\frac{1}{v^2}\left[v_\parallel^2(1+\varepsilon)+\varepsilon v_\perp^2 2\sin^2\left(\frac{\theta}{2}\right)\right] \\
\kappa^2 &= v_\parallel^2\frac{1}{v^2}\frac{1+\varepsilon}{2\varepsilon} \\
&\quad +\frac{v_\perp^2}{v^2}\sin^2\left(\frac{\theta}{2}\right)
\end{aligned}$$

For deeply trapped particles one can take

$$v_\perp^2 \approx v^2$$

which leads to

$$\kappa^2-\sin^2\left(\frac{\theta}{2}\right)\approx\frac{v_\parallel^2}{v^2}\frac{1+\varepsilon}{2\varepsilon}$$

which allows to extract $v_\parallel$,

$$\frac{v_\parallel}{v}\approx\sigma\sqrt{\frac{2\varepsilon}{1+\varepsilon}}\sqrt{\kappa^2-\sin^2\left(\frac{\theta}{2}\right)}$$

($\sigma=\pm1$) and this formula may be used when in the integrals along the bounce-orbits (banana) the time variable of integration is replaced with the angle variable

$$dt=\frac{dl}{v_\parallel}\rightarrow d\theta$$

For small ε the neglect of ε^2 gives

$$\frac{v_\parallel}{v}\approx\sigma\sqrt{2\varepsilon}\sqrt{\kappa^2-\sin^2\left(\frac{\theta}{2}\right)}$$

Note that one should include here the parallel projection of a poloidal rotation - of electrc origin, $v_E\times\frac{B_\theta}{B_T}$. **End**

The first to have adopted κ as velocity-space variable are **Galeev Sagdeev**. Another definition, taking only *initial* values, in **Fong Hahm**

$$\kappa^2\equiv\frac{1}{2}\frac{1}{\varepsilon}\frac{v_{\parallel0}^2}{v_{\perp0}^2}$$

END

Now we apply this formula for κ^2 at the turning point where

$$\begin{aligned} v_{\parallel} &= 0 \rightarrow v_{\perp}^2 = v^2 \\ \lambda_t &= h \end{aligned}$$

and

$$\begin{aligned} \kappa_t^2 &= \frac{1}{2\varepsilon}\left[1+\varepsilon-\lambda_t\right] = \frac{1+\varepsilon-1-\varepsilon\cos\theta_t}{2\varepsilon} = \frac{1}{2}\left(1-\cos\theta_t\right) = \sin^2\left(\frac{\theta_t}{2}\right) \\ &\quad \text{at the turning points} \quad \text{[Beers]} \end{aligned}$$

We obtain

$$\kappa_t = \sin\left(\frac{\theta_t}{2}\right)$$

Near the boundary trapped/passing, the bananas have their tips very close to the equatorial plane in the inner part of the torus (toward the main symmetry axis). This is a state where the trapped particles can easily become passing i.e. the tips touch and the motion become helical. Then

$$\begin{aligned} \theta_t &\rightarrow \pi \\ & \quad \text{and} \\ \kappa_t &\rightarrow 1 \end{aligned}$$

The choice

$$\kappa \approx 1$$

will be made below to find limit analytical expressions of particle fluxes, assuming they are dominated by the particles in the region of the velocity space near the boundary trapped/passing.

2.1.2 The parallel velocity $v_{\parallel}$ in terms of κ^2 (Beers)

The formula for κ^2 is expanded for small inverse aspect ratio $\varepsilon \ll 1$ as

$$\begin{aligned} \frac{1-\frac{v_{\perp}^2}{v^2}\frac{1+\varepsilon\cos\theta}{1+\varepsilon}}{\frac{2\varepsilon}{1+\varepsilon}} &= \frac{1-\frac{v_{\perp}^2}{2B}B_0\frac{1}{1+\varepsilon}\frac{1}{v^2/2}}{\frac{2\varepsilon}{1+\varepsilon}} \\ &= \frac{1}{v^2/2}\frac{\frac{v^2}{2}-\mu B_0\frac{1}{1+\varepsilon}}{\frac{2\varepsilon}{1+\varepsilon}} \end{aligned}$$

$$\kappa^2 \approx \frac{v^2/2 - \mu B_0 (1-\varepsilon)}{2\varepsilon \ (v^2/2)}$$

Beers

For trapped particles the parallel velocity, from

$$\kappa^2 = \frac{1 - \frac{\mu B_{\min}}{E}}{2\varepsilon_B}$$

is

$$\begin{aligned} 1 - 2\varepsilon_B \kappa^2 &= \frac{v_\perp^2}{2} \frac{1}{\frac{v^2}{2}} \frac{B_{\min}}{B} \rightarrow \\ v_\perp^2 &= v^2 \frac{B}{B_{\min}} \left(1 - 2\varepsilon_B \kappa^2\right) \end{aligned}$$

$$\left|v_\parallel\right| = v\sqrt{1 - \left(\frac{B}{B_{\min}}\right)(1 - 2\varepsilon_B \ \kappa^2)}$$

Beers

We can find another expression for $\left|v_\parallel\right|$ using explicit variables

$$\varepsilon_B = \frac{B_{\max} - B_{\min}}{2B_{\max}} = \frac{\varepsilon}{1+\varepsilon}$$

and

$$B_{\min} = \frac{B_0}{1+\varepsilon}$$

$$B = \frac{B_0}{1 + \varepsilon\cos\theta}$$

and

$$\begin{aligned} \frac{B}{B_{\min}} &= \frac{B_0}{1+\varepsilon\cos\theta} \frac{1+\varepsilon}{B_0} \\ &= \frac{1+\varepsilon}{1+\varepsilon\cos\theta} \end{aligned}$$

Then

$$\begin{aligned}
\frac{v_\parallel}{v} &= \sqrt{1-\left(\frac{B}{B_{\min}}\right)(1-2\varepsilon_B\ \kappa^2)} \\
&= \sqrt{1-\frac{1+\varepsilon}{1+\varepsilon\cos\theta}\left(1-2\frac{\varepsilon}{1+\varepsilon}\kappa^2\right)} \\
&= \sqrt{1-\frac{1+\varepsilon}{1+\varepsilon\cos\theta}+\frac{2\varepsilon\kappa^2}{1+\varepsilon\cos\theta}} \\
&= \sqrt{\frac{1+\varepsilon\cos\theta-1-\varepsilon+2\varepsilon\kappa^2}{1+\varepsilon\cos\theta}}
\end{aligned}$$

Further

$$\begin{aligned}
&\sqrt{\frac{1+\varepsilon\cos\theta-1-\varepsilon-2\varepsilon\kappa^2}{1+\varepsilon\cos\theta}} \\
=&\sqrt{\left[\frac{1}{2}(\cos\theta-1)-\kappa^2\right]\frac{2\varepsilon}{1+\varepsilon\cos\theta}}
\end{aligned}$$

For small ε the fast fraction is

$$\begin{aligned}
\frac{2\varepsilon}{1+\varepsilon\cos\theta} &\approx 2\varepsilon\,(1-\varepsilon\cos\theta) \\
&\approx 2\varepsilon
\end{aligned}$$

neglecting ε^2.

$$\begin{aligned}
\frac{|v_\parallel|}{v} &= \sqrt{2\varepsilon}\sqrt{\frac{1}{2}(\cos\theta-1)+\kappa^2} \\
&= \sqrt{2\varepsilon}\sqrt{\kappa^2-\sin^2\left(\frac{\theta}{2}\right)}
\end{aligned}$$

$$\frac{v_\parallel}{v} = \sigma\sqrt{1-\left(\frac{B}{B_{\min}}\right)(1-2\varepsilon_B\ \kappa^2)} \approx \sigma\sqrt{2\varepsilon}\ \sqrt{\kappa^2-\sin^2\left(\frac{\theta}{2}\right)}$$

2.1.3 The bounce time

The bounce time is obtained integrating over the closed orbit along the time variable

$$dt = \frac{dl}{|v_\parallel|}$$

$$\tau_{bounce}(\kappa) = \oint \frac{dl}{|v_{\parallel}|}$$

$$\tau_{bounce} = 2 \times \frac{1}{\frac{v}{qR}} \int_{-\theta_t}^{\theta_t} \frac{d\theta}{\sqrt{1 - \left(\frac{B}{B_{\min}}\right)(1 - 2\varepsilon_B\ \kappa^2)}}$$

The factor $2\times$ comes from the two sides of the banana orbit (say right and then left parts) that compose the closed loop.

Comment

It results that the transformation of the variable of integration on the loop

- from $dt = \frac{dl}{|v_{\parallel}|}$,

- to $d\theta$ is done as

$$\begin{aligned}
|v_{\parallel}| &= v\sqrt{1 - \left(\frac{B}{B_{\min}}\right)(1 - 2\varepsilon_B\ \kappa^2)} = v\sqrt{2\varepsilon}\sqrt{\kappa^2 - \sin^2\left(\frac{\theta}{2}\right)} \\
dt &= \frac{dl}{|v_{\parallel}|} = \frac{qR\ d\theta}{|v_{\parallel}|} = \frac{1}{\left(\frac{v}{qR}\right)} \frac{1}{\sqrt{2\varepsilon}} \frac{d\theta}{\sqrt{\kappa^2 - \sin^2\left(\frac{\theta}{2}\right)}}
\end{aligned}$$

SubNote

the coefficient is

$$\frac{1}{\left(\frac{v}{qR}\right)} = \frac{rB_T}{RB_\theta} R \frac{1}{v} = \frac{r}{v\frac{B_\theta}{B_T}}$$

In the paper **GS1968** the formula is

$$v_{\parallel} = v_E \frac{B_\theta}{B_T} + \sigma\ 2\sqrt{\varepsilon}\sqrt{\frac{\mu B_0}{m}}\sqrt{\kappa^2 - \sin^2\left(\frac{\theta}{2}\right)}$$

We remark

- the factor $\sqrt{\frac{\mu B_0}{m}} \approx \sqrt{\frac{v_\perp^2}{2}} \approx v\frac{1}{\sqrt{2}}$ if we admit that the particles are deep trapped. Then

$$\begin{aligned}
\sigma\ 2\sqrt{\varepsilon}\sqrt{\frac{\mu B_0}{m}}\sqrt{\kappa^2 - \sin^2\left(\frac{\theta}{2}\right)} &= \sigma\ \sqrt{\varepsilon} 2v \frac{1}{\sqrt{2}}\sqrt{\kappa^2 - \sin^2\left(\frac{\theta}{2}\right)} \\
&= \sigma\ v\sqrt{2\varepsilon}\sqrt{\kappa^2 - \sin^2\left(\frac{\theta}{2}\right)}
\end{aligned}$$

which is the answer above.

- one must always include $v_E \frac{B_\theta}{B_T}$ in $v_\parallel$; this is the reason that in **GS1968** there is the factor $\sqrt{v^2 + v_E^2 \left(\frac{B_T}{B_\theta}\right)^2}$.

END comment.

3 Variables in velocity space $\left(v, \kappa^2\right)$ Galeev Sagdeev

From **Galeev Sagdeev**.

Return to the expression for τ_{bounce},

$$\tau_{bounce} = 2 \times \frac{1}{\frac{v}{qR}} \int_{-\theta_t}^{\theta_t} \frac{d\theta}{\sqrt{1 - \left(\frac{B}{B_{\min}}\right)(1 - 2\varepsilon_B\ \kappa^2)}}$$

As above, the factor $2\times$ is necessary to extend the integration over the two parts of the banana orbit, between the mirror points.

Here we replace two factors that can be rewritten

$$\frac{1}{\left(\frac{v}{qR}\right)} = \frac{r}{v\frac{B_\theta}{B_T}}$$

and

$$\sqrt{1 - \left(\frac{B}{B_{\min}}\right)(1 - 2\varepsilon_B\ \kappa^2)} = \sqrt{2\varepsilon}\ \sqrt{\kappa^2 - \sin^2\left(\frac{\theta}{2}\right)}$$

i.e.

$$\begin{aligned}
\tau_{bounce} &= 2 \times \frac{1}{\frac{v}{qR}} \int_{-\theta_t}^{\theta_t} \frac{d\theta}{\sqrt{1 - \left(\frac{B}{B_{\min}}\right)(1 - 2\varepsilon_B\ \kappa^2)}} \\
&= 2 \times \frac{r}{v\frac{B_\theta}{B_T}} \int_{-\theta_t}^{\theta_t} \frac{d\theta}{\sqrt{2\varepsilon}\ \sqrt{\kappa^2 - \sin^2\left(\frac{\theta}{2}\right)}} \\
&= 2 \times \frac{r}{v\frac{B_\theta}{B_T}} \frac{1}{\sqrt{2\varepsilon}} \int_{-\theta_t}^{\theta_t} \frac{d\theta}{\sqrt{\kappa^2 - \sin^2\left(\frac{\theta}{2}\right)}}
\end{aligned}$$

Now the integration between the mirror points is $2\times$ the integration between 0 and θ_t.

The result for τ_{bounce} is

$$\begin{aligned}\tau_{bounce} &= 2\times 2\times \frac{1}{v\frac{B_\theta}{B_T}}\frac{1}{\sqrt{2\varepsilon}}r \\ &\times \int_0^{\vartheta_0}\frac{d\theta}{\sqrt{\kappa^2-\sin^2\left(\frac{\theta}{2}\right)}}\end{aligned}$$

Below it is calculated (Gradshtein Ryzhik)

$$\int_0^{\pi}\frac{dx}{\sqrt{\kappa^2-\sin^2\left(\frac{x}{2}\right)}} = 2\frac{1}{\kappa}\mathbf{K}\left(\frac{1}{\kappa}\right)$$

then we obtain

$$\tau_{bounce} = 2\times 2\times \frac{1}{v\frac{B_\theta}{B_T}}\frac{1}{\sqrt{2\varepsilon}}r2\frac{1}{\kappa}\mathbf{K}\left(\frac{1}{\kappa}\right) \approx 4\sqrt{2}\frac{1}{v\frac{B_\theta}{B_T}}\frac{1}{\sqrt{\varepsilon}}r\ \frac{1}{\kappa}\mathbf{K}\left(\kappa\right)$$

NOTE of caution

To find relation with the result of **GS** we must admit an approximation.

Since the upper limit π is for the turning point (angle θ_t) of a particle that is at the boundary trapped/passing, we have

$$\kappa_t^2 = \frac{1}{2\varepsilon}\left[1+\varepsilon-\lambda_t\right] = \frac{1+\varepsilon-1-\varepsilon\cos\theta_t}{2\varepsilon} = \frac{1}{2}\left(1-\cos\theta_t\right) = \sin^2\left(\frac{\theta_t}{2}\right)$$

and if we chose

$$\begin{aligned}\theta_t &\equiv \text{angle of mirror} \\ &\approx \pi\end{aligned}$$

then

$$\sin^2\left(\frac{\theta_t}{2}\right) = \sin^2\left(\frac{\pi}{2}\right) = 1$$

and this means

$$\kappa \approx \kappa_t = 1$$

THIS IS JUST AN APPROXIMATION

It means that the factor κ can sometimes be approximated 1

$$\begin{aligned}\int_0^{\pi}\frac{dx}{\sqrt{\kappa^2-\sin^2\left(\frac{x}{2}\right)}} &= 2\frac{1}{\kappa}\mathbf{K}\left(\frac{1}{\kappa}\right) \\ &\approx 2\mathbf{K}\left(\kappa\right)\end{aligned}$$

for particles close to the boundary trapped/passing. The expression in **GS**

$$\tau^{GS} = 4\sqrt{2}\frac{1}{v\frac{B_\theta}{B_T}}\frac{1}{\sqrt{\varepsilon}}r\ \frac{1}{\kappa}\mathbf{K}\left(\kappa\right)$$

i.e. only the factor $1/\kappa$ is missing.

However we will keep this factor.

It will be used in other averages, below.

In the paper **GS 1968** the formula is

$$\begin{aligned}\tau &= \frac{4r}{\sqrt{\varepsilon\ \left[\left(\frac{B_\theta}{B_T}\right)^2 v^2 + v_E^2\right]}}\int_0^{\vartheta_0}\frac{d\theta}{\sqrt{2\left[\kappa^2 - \sin^2\left(\frac{\theta}{2}\right)\right]}} \\ &= 4\sqrt{2}\frac{r}{(B_\theta/B_T)}\frac{1}{\sqrt{\varepsilon\left[v^2 + \left(v_E\frac{B_T}{B_\theta}\right)^2\right]}}K\left(\kappa\right)\end{aligned}$$

There is a major difference,
- the coefficient $1/\kappa$, is absent (explained above) and
- there is the factor in the denominator

$$\sqrt{\varepsilon\left[v^2 + \left(v_E\frac{B_T}{B_\theta}\right)^2\right]} \approx \sqrt{\varepsilon}v$$

then, **GS**

$$\tau_{bounce} \approx 4\sqrt{2}\frac{r}{(B_\theta/B_T)}\frac{1}{v}\frac{1}{\sqrt{\varepsilon}}K\left(\kappa\right)$$

since $\kappa \approx 1$ for the particles that are close to the boundary trapped/passing.

END note of caution

With these physical variables one can calculate the average of the electrostatic potential on the "bounce" orbit of the particle. The average is made with *time* as variable along the orbit, $dt = \frac{dz}{|v_{\parallel}|}$,

$$\begin{aligned}
& \langle \Phi \rangle_{bounce} (x, y; \kappa) \\
= & \frac{\oint \frac{dz}{|v_{\parallel}|} \Phi (x, y, z)}{\oint \frac{dz}{|v_{\parallel}|}} \\
= & \frac{1}{\tau_{bounce}} \frac{1}{\left(\frac{v}{qR}\right)} \int_{-\theta_t}^{\theta_t} \frac{d\theta}{\sqrt{1 - \left(\frac{B}{B_{\min}}\right) (1 - 2\varepsilon_B \ \kappa^2)}} \Phi (x, y, \theta)
\end{aligned}$$

In the second expression it has been made a change of variable of integration

- first it was the time $dt = \frac{dz}{|v_{\parallel}|}$, and the domain of integration was the time duration of a bounce

- the other form is integration over the angle θ, the variable of periodicity on the closed orbit $d\theta = \frac{dz}{qR}$.

Then

$$\begin{aligned}
& \frac{1}{\tau_{bounce}} \frac{1}{\left(\frac{v}{qR}\right)} \int_{-\theta_t}^{\theta_t} \frac{d\theta}{\sqrt{1 - \left(\frac{B}{B_{\min}}\right) (1 - 2\varepsilon_B \ \kappa^2)}} \Phi (x, y, \theta) \\
= & \frac{1}{\tau_{bounce}} \frac{r}{v \frac{B_\theta}{B_T}} \int_{-\theta_t}^{\theta_t} \frac{d\theta}{\sqrt{2\varepsilon} \ \sqrt{\kappa^2 - \sin^2 \left(\frac{\theta}{2}\right)}} \Phi (x, y, \theta)
\end{aligned}$$

Now we replace the τ_{bounce}

$$\tau_{bounce} = 4\sqrt{2} \frac{1}{v \frac{B_\theta}{B_T}} \frac{1}{\sqrt{\varepsilon}} r \ \frac{1}{\kappa} \mathbf{K} (\kappa)$$

to obtain

$$\begin{aligned}
& \langle \Phi \rangle_{bounce} (x, y; \kappa) \ \text{(two times integration over one branch of banana)} \\
= & \frac{1}{\tau_{bounce}} \times 2 \times \frac{r}{v \frac{B_\theta}{B_T}} \int_{-\theta_t}^{\theta_t} \frac{d\theta}{\sqrt{2\varepsilon} \ \sqrt{\kappa^2 - \sin^2 \left(\frac{\theta}{2}\right)}} \Phi (x, y, \theta) \\
= & \frac{1}{4\sqrt{2} \frac{1}{v \frac{B_\theta}{B_T}} \frac{1}{\sqrt{\varepsilon}} r \ \frac{1}{\kappa} \mathbf{K} (\kappa)} \times 2 \times \frac{r}{v \frac{B_\theta}{B_T}} \frac{1}{\sqrt{2\varepsilon}} \int_{-\theta_t}^{\theta_t} \frac{d\theta}{\sqrt{\kappa^2 - \sin^2 \left(\frac{\theta}{2}\right)}} \Phi (x, y, \theta) \\
= & \frac{\kappa}{4\mathbf{K} (\kappa)} \int_{-\theta_t}^{\theta_t} \frac{d\theta}{\sqrt{\kappa^2 - \sin^2 \left(\frac{\theta}{2}\right)}} \Phi (x, y, \theta)
\end{aligned}$$

Approximations for small ε. For

$$B = \frac{B_0}{1 + \varepsilon \cos\theta}$$

then

$$\langle \Phi \rangle_{bounce} = \frac{\kappa}{4\mathbf{K}(\kappa)} \int_{-\theta_t}^{\theta_t} \frac{d\theta}{\sqrt{\kappa^2 - \sin^2\left(\frac{\theta}{2}\right)}} \Phi(x, y, \theta)$$

Now, $\langle \Phi \rangle_{bounce}$ is a function of κ. This variable κ selects orbit of the particle.

NOTE

In **Galeev Sagdeev** it is made the transformation from the time variable along the periodic motion of banana to the geometrical variable, the angle θ, and averages are on time of bounce

$$\begin{aligned} \langle W(t) \rangle &= \frac{1}{\tau} \int_0^{\tau} dt\ W(t) \\ &= \frac{1}{\tau} \int_0^{\tau} dt\ W[\theta(t)] \end{aligned}$$

Here we must replace the explicit form of the *bounce* time

$$\tau_{bounce} = 4\sqrt{2} \frac{1}{v \frac{B_\theta}{B_T}} \frac{1}{\sqrt{\varepsilon}} r\ \frac{1}{\kappa} \mathbf{K}(\kappa)$$

and the transformation from the variable of integration "time" on the loop motion to angle θ of the points on the loop

$$dt = \frac{dl}{|v_\parallel|} = \frac{qR\ d\theta}{|v_\parallel|}$$

The parallel velocity is

$$|v_\parallel| = v\sqrt{2\varepsilon} \sqrt{\kappa^2 - \sin^2\left(\frac{\theta}{2}\right)}$$

These two formulas lead to

$$dt = \frac{1}{\left(\frac{v}{qR}\right)} \frac{1}{\sqrt{2\varepsilon}} \frac{d\theta}{\sqrt{\kappa^2 - \sin^2\left(\frac{\theta}{2}\right)}}$$

The symmetry of a banana is reflected in

- a factor $2\times$ the integration along one of the parts (left side, right side) of the banana

- a factor $2\times$ the integration over *half* the angle, i.e. between the equatorial plane to the point of mirror

$$\langle W(t)\rangle = \frac{1}{4\sqrt{2}\frac{1}{v\frac{B_\theta}{B_T}}\frac{1}{\sqrt{\varepsilon}}r\ \frac{1}{\kappa}\mathbf{K}(\kappa)} \times 2 \times \frac{r}{v\frac{B_\theta}{B_T}}\int_{-\theta_t}^{\theta_t}\frac{d\theta}{\sqrt{2\varepsilon}\ \sqrt{\kappa^2-\sin^2\left(\frac{\theta}{2}\right)}}W(\theta)$$

and after splitting the integral on $(-\theta_t, +\theta_t)$ in $2\times$(integral on $(0,\theta_t)$) the factor 4 at the denominator disappears

$$\langle W(t)\rangle = \frac{\kappa}{K(\kappa)}\int_0^{\theta_0/2}\frac{1}{2}d\theta\ \frac{W(\theta)}{\sqrt{\kappa^2-\sin^2(\theta/2)}}$$

END

3.1 Equations of particle motion (Galeev Sagdeev 1968)

In **Galeev Sagdeev JETP 26 (1968) 233**,

The longitudinal invariant

$$J = \Omega_c\int_0^r \frac{r}{qR}dr + v_\parallel(1+\varepsilon\cos\theta)$$

(note that $\frac{eB}{m}\int^r \frac{B_\theta}{B_T}dr \approx \frac{e}{m}\int^r B_\theta dr$)

The generalized momentum of the particle

$$J = mv_\parallel h - eA_\varphi$$

$$J = mv_\parallel(1+\varepsilon\cos\theta) - e\int_0^r B_\theta dr$$

in the limit of small Larmor radius, can be expanded to second order in the deviation from the point

$$\begin{aligned} r &= r_0 \\ \theta &= 0 \end{aligned}$$

(this is on the equatorial plane. In **Fong Hahm** it is the *center* of the banana, which means the radius of the magnetic surface where the center of the banana is located).

There are *always* parallel flow $v_\parallel$ and poloidal flow due to radial electric field v_E. We will see that the effective poloidal rotation is inhibited (by a strong damping) and the parallel+poloidal flow must combine to eliminate the poloidal plasma flow: this, in *lowest order* of toroidality.

See **Hassam Kulsrud**. Also **Peeters**.

Notations

$$\begin{aligned} q &= \frac{rB_T}{RB_\theta} \quad , \quad \varepsilon = \frac{r}{R} \\ \frac{\varepsilon}{q} &\equiv \Theta = \frac{B_\theta}{B_T} \ll 1 \\ &= \text{factor to project parallel to poloidal} \end{aligned}$$

and

$$\Omega_c = \frac{eB}{m}$$

The equation for the particle motion is derived after a second order expansion of the *longitudinal invariant* J. The longitudinal invariant is

$$J = mv_\parallel (1 + \varepsilon \cos\theta) - e\int_0^r B_\theta dr$$

and its expansion in small $(r - r_0)$ is

$$\frac{1}{2}\Omega_c \left(\frac{\varepsilon}{q}\right)^2 (r - r_0)^2 + \delta v_\parallel \left(\frac{\varepsilon}{q}\right)(r - r_0) - v_g r_0 (\cos\theta - 1) = 0 \qquad (1)$$

where

$$\delta v (r_0, 0) = v_\parallel (r_0, 0) - \frac{v_E (r_0)}{r/(qR)}$$

we note that the difference in RHS should be zero, if the only rotation would be electric, due to $\Phi_0 (r)$: the parallel flow $v_\parallel$ projected on the poloidal direction (by multiplication with $\frac{r}{qR} \ll 1$) must cancel the electric poloidal flow, $v_E (r_0)$.

$$\delta v_\parallel \equiv v_\parallel (r_0, 0) - \frac{v_E}{\varepsilon/q}$$

and

$$\begin{aligned} v_g &= \frac{\mu B_0 + mv_\parallel^2}{m\Omega_c R} = \frac{1}{\Omega_c}\frac{\frac{v_\perp^2}{2} + v_\parallel^2}{R} \\ &\equiv v_D \quad \text{(the drift velocity)} \end{aligned}$$

We have noted Ω_c the cyclotron frequency.

The factor

$$\frac{1}{\varepsilon/q}=\frac{R}{r}\frac{rB_{\varphi}}{RB_{\theta}}=\frac{B_{\varphi}}{B_{\theta}}\gg 1$$

is the projection (of v_E which is poloidal) on the *parallel* direction. In this way $\delta v_{\|}$ is the parallel velocity (with electric rotation subtracted) at the outermost point, initially.

The solution of the equation of motion Eq.(1) is

$$r-r_0=\frac{1}{\Omega_c\left(\varepsilon/q\right)}\left[-\left(\delta v_{\|}\right)\pm\sqrt{\left(\delta v_{\|}\right)^2+2\Omega_c v_g r_0\left(\cos\theta-1\right)}\right]$$

The equations of motion in **GS1968**

$$\frac{dr}{dt}=-\frac{1}{\Omega_{ci}}\frac{v_{\perp}^2/2+v_{\|}^2}{R}\sin\theta$$

$$\frac{rd\theta}{dt}=-v_{\|}\frac{B_{\theta}}{B_T}-\frac{1}{\Omega_{ci}}\frac{v_{\perp}^2/2+v_{\|}^2}{R}\cos\theta+\frac{1}{B_0}\frac{\partial\phi}{\partial r}$$

NOTE

- In **GS1968** the first term in $rd\theta/dt$ is $-v_{\|}\frac{B_{\theta}}{B_T}$.
- in **GS1968** the electric velocity term is $\frac{1}{B_0}\frac{\partial\phi}{\partial r}=v_E$; it occurs with sign $+$ in the equation

in contrast

- in **GS review** the first term in $rd\theta/dt$ is $v_{\|}\frac{B_{\theta}}{B_T}$ (opposite sign relative to **GS1968** but the same as in **Fong Hahm**)
- in **GS review** the electric velocity term is $\frac{1}{B_0}\frac{\partial\phi}{\partial r}=v_E$; it occurs with sign $+$ in the equation

We adopt **GS review**.

The set of equations is

$$\frac{dr}{dt}=-\frac{1}{\Omega_{ci}}\frac{v_{\perp}^2/2+v_{\|}^2}{R}\sin\theta$$

$$\frac{rd\theta}{dt}=v_{\|}\frac{B_{\theta}}{B_T}-\frac{1}{\Omega_{ci}}\frac{v_{\perp}^2/2+v_{\|}^2}{R}\cos\theta+v_E$$

END

NOTE In **Fong Hahm** the equations of motion are

$$\begin{aligned}
\frac{dr}{dt} &\approx -\frac{1}{\Omega_c}\frac{v_\parallel^2+\frac{v_\perp^2}{2}}{R_0}\sin\theta \\
\frac{rd\theta}{dt} &\approx v_\parallel\frac{r}{qR}-\frac{1}{\Omega_c}\frac{v_\parallel^2+\frac{v_\perp^2}{2}}{R_0}\cos\theta \\
\frac{d\varphi}{dt} &\approx v_\parallel\frac{1}{R_0}
\end{aligned}$$

and in **Mikhailovskii** the sign in the drift tarm is opposite. It should be kept consistent if we want coincidence of results.

END

and from these,

$$r\left(\theta\right)-r_0=\pm\frac{v}{\Omega_c}\;q_0\varepsilon^{-1/2}\;\sqrt{\frac{1}{\varepsilon}\frac{v_{\parallel 0}^2}{v_{\perp 0}^2}-2\sin^2\left(\frac{\theta}{2}\right)}$$

where

$$\begin{aligned}
\varepsilon &= \frac{r}{R_0} \\
r_0 &\equiv \text{radius of the bounce limit points} \\
&\sim \text{bounce "center"}
\end{aligned}$$

$$v_{\parallel 0}\;,\;v_{\perp 0}\;\equiv\text{velocities at the outer mid-plane}$$

$$\begin{aligned}
\Lambda_B &\equiv \frac{v}{\Omega_c}q_0\varepsilon^{-1/2} \\
&= \frac{1}{\sqrt{\varepsilon}}\frac{rB_T}{RB_\theta}\frac{v}{eB/m}=\sqrt{\varepsilon}\frac{v}{\Omega_\theta}=\sqrt{\varepsilon}\;\rho_\theta \\
&= \text{"radius" of the banana}
\end{aligned}$$

The **GS** constant for a single orbit is introduced (the same definition in Fong Hahm)

$$\kappa^2\equiv\frac{1}{2}\frac{1}{\varepsilon}\frac{v_{\parallel 0}^2}{v_{\perp 0}^2}$$

SubNOTE

The formula previously

$$\kappa^2 = \frac{1 - \frac{\mu B_{\min}}{E}}{2\varepsilon_B}$$

Beers

where

$$\varepsilon_B = \frac{B_{\max} - B_{\min}}{2B_{\max}}$$

and in detail

$$\begin{aligned}
B &= \frac{B_0}{1+\varepsilon\cos\theta} \\
B_{\min} &= \frac{B_0}{1+\varepsilon} \quad \text{at} \quad \theta = 0 \quad \text{(outermost midplane point)} \\
B_{\max} &= \frac{B_0}{1-\varepsilon} \quad \text{at} \quad \theta = \pi \quad \text{(innermost midplane point)}
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon_B &= \frac{B_{\max} - B_{\min}}{2B_{\max}} = \frac{\frac{B_0}{1-\varepsilon} - \frac{B_0}{1+\varepsilon}}{2\frac{B_0}{1-\varepsilon}} = \frac{1+\varepsilon-1+\varepsilon}{2(1+\varepsilon)} \\
&= \frac{\varepsilon}{1+\varepsilon}
\end{aligned}$$

From where we get

$$\kappa^2 = \frac{1}{2\varepsilon}\left[1+\varepsilon - \frac{v_\perp^2}{v^2}(1+\varepsilon\cos\theta)\right]$$

Beers

ENDsub

SubNote compare with

$$\lambda = \frac{v_\perp^2}{v^2} h$$

and for a single, specified orbit, in κ^2 only *initial* values $\left(v_{\parallel 0}\ ,\ v_{\perp 0}^2\right)$ are used (**GS** and **Fong Hahm**).

The domain of trapped is

$$\begin{aligned}
\lambda &\sim \text{large, i.e. comparable to 1, } v_\perp^2 \approx v^2 \\
1-\varepsilon &\leq \lambda \leq 1+\varepsilon
\end{aligned}$$

End Sub.

Here

$$\begin{aligned} \kappa^2 \leq & \; 1 \;\; \text{trapped} \;\; \left(v_{\|0}^2 \ll v_{\perp 0}^2\right) \\ & \text{(almost all energy is perpendicular)} \\ & \text{(actually this is deep trapped)} \end{aligned}$$

It is introduced the angle ϑ defined

$$\vartheta(\theta) = \arcsin\left(\frac{\sin(\theta/2)}{\kappa}\right)$$

which will become the argument of the elliptic functions.

This is introduce because the combination is

$$\sqrt{\kappa^2 - \sin^2\left(\frac{\theta}{2}\right)}$$

and after dividing by κ^2,

$$\begin{aligned} & \pm\kappa\sqrt{1 - \frac{\sin^2\left(\frac{\theta}{2}\right)}{\kappa^2}} \\ = & \pm\kappa\sqrt{\cos^2\vartheta} \\ = & \pm\kappa \;\cos\vartheta \end{aligned}$$

where

$$\begin{aligned} \cos^2\vartheta &= 1 - \frac{\sin^2\left(\frac{\theta}{2}\right)}{\kappa^2} \\ \sin^2\vartheta &= \frac{\sin^2\left(\frac{\theta}{2}\right)}{\kappa^2} \\ \sin\vartheta &= \frac{\sin\left(\frac{\theta}{2}\right)}{\kappa} \end{aligned}$$

or

$$\vartheta(\theta) = \arcsin\left(\frac{\sin(\theta/2)}{\kappa}\right)$$

END

When $\delta v_{\|}$ is <u>large</u> enough we can factorize $\delta v_{\|}$ under the square root and then expand. The solution can be approximated

$$r - r_0 = \left[\frac{\mu B_0 + m v_{\|}^2}{m\Omega_c R}\frac{1}{\delta v_{\|}}\right] r_0 (\cos\theta - 1) \tag{2}$$

Thus, positively charged particles (IONS, $\Omega_c = e_i B / m_i > 0$) will traverse the magnetic surface in the counterclockwise direction and will be deflected

$$\begin{aligned} \text{inward from it, if } \delta v_\parallel &> 0 \\ \text{outward, if } \delta v_\parallel &< 0 \end{aligned}$$

It follows that the particles for which the condition

$$\left(\delta v_\parallel\right)^2 < 4\Omega_c v_g r_0$$

is satisfied, will be *trapped*.

In the following the average $\langle\rangle$ is over <u>time</u>, on a bounce period τ_{bounce} (note that $v_E \equiv v_0$ in GS).

In the set of equations of motion of a particles **Galeev Sagdeev** the second is

$$\frac{r d\theta}{dt} = v_\parallel \frac{B_\theta}{B_T} - \frac{1}{\Omega_{ci}} \frac{v_\perp^2/2 + v_\parallel^2}{R} \cos\theta + \frac{1}{B_0} \frac{\partial \phi}{\partial r}$$

The last term is

$$v_E = \frac{1}{B_0} \frac{\partial \phi}{\partial r}$$

The term that comes from the poloidal projection of the drift velocity $\mathbf{v}_D \cdot \widehat{\mathbf{e}}_\theta$

$$-\frac{1}{\Omega_{ci}} \frac{v_\perp^2/2 + v_\parallel^2}{R} \cos\theta$$

is approximated for *deep trapped* by taking $v_\parallel \approx 0$

$$\approx -\frac{1}{\Omega_{ci}} \frac{v_\perp^2/2}{R} \cos\theta = -\frac{1}{\Omega_c} \frac{\frac{\mu B}{m}}{R} \cos\theta$$

The full equation is divided by r,

$$\frac{d\theta}{dt} = v_\parallel \frac{B_\theta}{r B_T} - \frac{1}{\Omega_c} \frac{\frac{\mu B}{m}}{R} \frac{\cos\theta}{r} + \frac{v_E}{r}$$

This equation is averaged over the bounce motion (an operation that is similar to gyroaveraging).

When we calculate the average over the bounce of the trapped particle on its closed orbit, we must take into account the width. This occurs in the first term

$$v_\parallel \frac{B_\theta}{r B_T}$$

as space dependence of the magnetic field, multiplied with the variable that takes the largest values, $v_{\parallel}$. It is made an expansion to first order in the departure $r-r_0$ of the orbit from the surface of radius r_0, where both mirror points are located

$$\begin{aligned} v_{\parallel}\frac{B_{\theta}}{rB_T} &= v_{\parallel}\frac{1}{qR} \\ &= v_{\parallel}\left(\frac{1}{qR}\right)_0 + v_{\parallel}\left(r-r_0\right)\frac{d}{dr}\left(\frac{1}{qR}\right) \end{aligned}$$

And in the term

$$\frac{v_E}{r} = \left(\frac{v_E}{r}\right)_0 + \left(r-r_0\right)\frac{d}{dr}\left(\frac{v_E}{r}\right)$$

Digression

It is considered that R is too big to be affected by average. Its variation along the closed banana orbit is not significant.

However, just to see.

If we make in detail the derivative

$$\begin{aligned} \frac{d}{dr}\left(\frac{1}{qR}\right) &= \frac{1}{R}\frac{d}{dr}\left(\frac{1}{q}\right) + \frac{1}{q}\frac{d}{dr}\left(\frac{1}{R}\right) \\ &= \frac{1}{R}\left(-\frac{1}{q^2}\frac{dq}{dr}\right) + \frac{1}{q}\left(-\frac{1}{R^2}\cos\theta\right) \end{aligned}$$

NOTE

The average of $1/R$ on the bounce motion is neglected, $1/R$ is treated as constant. This is because R is large compared with the width of a banana orbit and the variationof R on the two sides of the banana is very small.

END

The expression to be averaged is

$$\begin{aligned} & \left\langle v_{\parallel}\left(r-r_0\right)\frac{d}{dr}\left(\frac{1}{qR}\right)\right\rangle \\ &= \left\langle v_{\parallel}\left(r-r_0\right)\left[\frac{1}{R}\left(-\frac{1}{q^2}\frac{dq}{dr}\right)\right]\right\rangle \\ &= \left\langle v_{\parallel}\left(r-r_0\right)\right\rangle\frac{1}{R}\left(-\frac{1}{q^2}\frac{dq}{dr}\right) \end{aligned}$$

Later below it will be found that

$$v_{\parallel}\left(r-r_0\right) = \left(\frac{2\mu B_0}{m}\right)\frac{1}{\Omega_c}q\left(r_0\right)\left(2\kappa^2-1+\cos\theta\right)$$

End Digression

Using this, one can calculate the average over the time of bounce

$$\begin{aligned}\left\langle \frac{d\theta}{dt} \right\rangle_{time} &= \left\langle v_{\parallel} \frac{B_\theta}{r B_T} \right. \\ &\quad - \frac{1}{\Omega_c} \frac{\frac{\mu B_0}{m}}{R} \frac{\cos\theta}{r} \\ &\quad \left. + \frac{v_E}{r} \right\rangle_{time}\end{aligned}$$

After introducing the expansions to first order in the width of the orbit $(r - r_0)$ we have

$$\begin{aligned}\left\langle \frac{d\theta}{dt} \right\rangle_{time} &= \frac{1}{qR} \left\langle v_{\parallel} \right\rangle + \left\langle v_{\parallel} (r - r_0) \right\rangle \frac{d}{dr} \left(\frac{1}{q(r) R} \right) \\ &\quad - \frac{1}{\Omega_c} \frac{\frac{\mu B_0}{m}}{R} \left\langle \frac{\cos\theta}{r_0} \right\rangle \\ &\quad + \frac{v_E(r_0)}{r_0} + \frac{d}{dr} \left(\frac{v_E}{r} \right) \langle (r - r_0) \rangle\end{aligned}$$

The projection of a banana orbit on the poloidal section remains the same, even if there is toroidal drift. Then θ remains a periodic variable for the bouncing motion, and LHS is zero

$$\left\langle \frac{d\theta}{dt} \right\rangle_{time} = 0$$

The variable $r - r_0$ is also bounce-periodic and its average is zero

$$\langle (r - r_0) \rangle = 0$$

We must calculate

$$\left\langle v_{\parallel} (r - r_0) \right\rangle$$

Both factors have strong variation along the orbit.

After the expansion of the longitudinal invariant to second order in the displacement $r - r_0$, one can use the approximate result for $r - r_0$

$$r(\theta) - r_0 = \pm \frac{v}{\Omega_c} q_0 \varepsilon^{-1/2} \sqrt{\frac{1}{\varepsilon} \frac{v_{\parallel 0}^2}{v_{\perp 0}^2} - 2 \sin^2 \left(\frac{\theta}{2} \right)}$$

and the (see below) approximation for $v_{\parallel}$,

$$\frac{v_{\parallel}}{v} \approx \sigma\sqrt{2\varepsilon}\ \sqrt{\kappa^2 - \sin^2\left(\frac{\theta}{2}\right)}$$

to calculate (as in **GS**)

$$\left\langle v_{\parallel}\left(r - r_0\right)\right\rangle_{time-bounce} = -\left\langle \delta v_{\parallel}\left(r,\theta\right) \frac{\delta v_{\parallel}\left(r_0, 0\right) - \delta v_{\parallel}\left(r,\theta\right)}{\Omega_c\ \left(\frac{B_\theta}{B_T}\right)} \right\rangle$$

Taking into account the GS notation

$$\frac{1}{\varepsilon}\frac{v_{\parallel 0}^2}{v_{\perp 0}^2} = 2\kappa^2$$

we will use the expressions of the two factors $v_{\parallel}$ and $(r - r_0)$,

$$\begin{aligned}
&\left\langle v_{\parallel}\left(r - r_0\right)\right\rangle_{time-bounce} \\
= &\left\langle \left(\sigma v\sqrt{2\varepsilon}\ \sqrt{\kappa^2 - \sin^2\left(\frac{\theta}{2}\right)}\right) \left(\pm \frac{v}{\Omega_c}\ q_0 \varepsilon^{-1/2}\ \sqrt{\frac{1}{\varepsilon}\frac{v_{\parallel 0}^2}{v_{\perp 0}^2} - 2\sin^2\left(\frac{\theta}{2}\right)}\right)\right\rangle \\
= &\frac{1}{\Omega_c} v^2 q_0 \sqrt{2}\sqrt{2}\left\langle \left[\kappa^2 - \sin^2\left(\frac{\theta}{2}\right)\right]\right\rangle
\end{aligned}$$

For deep trapped almost all energy is in the gyration, the parallel velocity is small

$$v^2 \approx v_\perp^2 \approx \frac{2\mu B_0}{m}$$

$$\begin{aligned}
&\left\langle v_{\parallel}\left(r - r_0\right)\right\rangle_{time-bounce} \\
= &\left(\frac{2\mu B_0}{m}\right)\frac{1}{\Omega_c} q\left(r_0\right)\left\langle 2\kappa^2 - 2\sin^2\left(\frac{\theta}{2}\right)\right\rangle \\
= &\left(\frac{2\mu B_0}{m}\right)\frac{1}{\Omega_c} q\left(r_0\right)\left\langle 2\kappa^2 - 1 + \cos\theta\right\rangle
\end{aligned}$$

or

$$\begin{aligned}
&\left\langle v_{\parallel}\left(r - r_0\right)\right\rangle_{time-bounce} \\
= &\left(\frac{2\mu B_0}{m}\right)\frac{1}{\Omega_c} q\left(r\right)\left[\left\langle\cos\theta\right\rangle - 1 + 2\kappa^2\right]
\end{aligned}$$

Which is the result of **GS review**.

Note that here the average over the bounce is not yet done, $\langle\cos\theta\rangle_{bounce}$. **End**.

We return to the equation

$$\left\langle \frac{d\theta}{dt} \right\rangle_{time} = 0$$

to extract the quantity that is interesting for us, the bounce-average parallel velocity. This is equivalent to a toroidal drift of the whole banana orbit.

$$\begin{aligned}
\frac{1}{qR}\left\langle v_{\parallel}\right\rangle &= -\left\langle v_{\parallel}\left(r-r_0\right)\right\rangle \frac{d}{dr}\left(\frac{1}{q\left(r\right)R}\right) \\
&\quad +\frac{1}{\Omega_c}\frac{\frac{\mu B_0}{m}}{R}\left\langle \frac{\cos\theta}{r_0}\right\rangle \\
&\quad -\frac{v_E\left(r_0\right)}{r_0}
\end{aligned}$$

or

$$\begin{aligned}
\left\langle v_{\parallel}\right\rangle &= -v_E\frac{qR}{r_0}+qR\frac{1}{\Omega_c}\frac{\frac{\mu B_0}{m}}{R}\left\langle \frac{\cos\theta}{r_0}\right\rangle \\
&\quad -qR\left(\frac{2\mu B_0}{m}\right)\frac{1}{\Omega_c}q\left(r\right)\left[\left\langle\cos\theta\right\rangle-1+2\kappa^2\right]\frac{d}{dr}\left(\frac{1}{q\left(r\right)R}\right)
\end{aligned}$$

we organize this expression by collecting the factors of $\langle\cos\theta\rangle$

$$\begin{aligned}
\left\langle v_{\parallel}\right\rangle &= -v_E\frac{qR}{r_0} \\
&\quad +\left\langle\cos\theta\right\rangle\left[qR\frac{1}{\Omega_c}\frac{\frac{\mu B_0}{m}}{R}\frac{1}{r_0}-qR\left(\frac{\mu B_0}{m}\right)\frac{1}{\Omega_c}2q\left(r\right)\frac{d}{dr}\left(\frac{1}{q\left(r\right)R}\right)\right] \\
&\quad -qR\left(\frac{\mu B_0}{m}\right)\frac{1}{\Omega_c}2q\left(r\right)\left(-1+2\kappa^2\right)\frac{1}{R_0}\left(-\frac{1}{q^2}\frac{dq}{dr}\right)
\end{aligned}$$

$$\begin{aligned}
\left\langle v_{\parallel}\right\rangle &= -v_E\frac{qR}{r_0} \\
&\quad +\left\langle\cos\theta\right\rangle\frac{1}{\Omega_c}qR\frac{\mu B_0}{m}\left[\frac{1}{r_0 R}-2q\frac{1}{R}\left(-\frac{1}{q^2}\frac{dq}{dr}\right)\right] \\
&\quad +q\frac{1}{\Omega_c}\left(\frac{\mu B_0}{m}\right)2\left(-1+2\kappa^2\right)\frac{d}{dr}\ln q
\end{aligned}$$

The square paranthesis is

$$\begin{aligned}
&\left[\frac{1}{r_0 R} - 2q\frac{1}{R}\left(-\frac{1}{q^2}\frac{dq}{dr}\right)\right] \\
= &\ \frac{1}{R}\left(\frac{1}{r_0} + 2\frac{d}{dr}\ln q\right)
\end{aligned}$$

Now we must calculate $\langle\cos\theta\rangle$ and for this we will use the formula

$$\tau_{bounce} = 4\sqrt{2}\frac{1}{v\frac{B_\theta}{B_T}}\frac{1}{\sqrt{\varepsilon}}r\ \frac{1}{\kappa}\mathbf{K}(\kappa) = 4\sqrt{\frac{2}{\varepsilon}}qR\frac{1}{\kappa}\mathbf{K}(\kappa)$$

$$\langle W(t)\rangle = \frac{\kappa}{K(\kappa)}\int_0^{\theta_0/2}\ \frac{1}{2}d\theta\ \frac{W(\theta)}{\sqrt{\kappa^2 - \sin^2(\theta/2)}}$$

$$\langle\cos\theta\rangle = \frac{\kappa}{K(\kappa)}\int_0^{\theta_0/2}\ \frac{1}{2}d\theta\ \frac{\cos\theta}{\sqrt{\kappa^2 - \sin^2(\theta/2)}}$$

In order to calculate this integral, we have two formulas Gradshtein Ryzhik 3.671

$$\int_0^\pi \frac{dx}{\sqrt{a \pm b\cos x}} = \frac{2}{\sqrt{a+b}}\mathbf{K}\left(\sqrt{\frac{2b}{a+b}}\right)$$

$$\int_0^\pi\ dx\ \sqrt{a \pm b\cos x} = 2\sqrt{a+b}\mathbf{E}\left(\sqrt{\frac{2b}{a+b}}\right)$$

for

$$\begin{aligned}
a &> 0 \\
b &> 0
\end{aligned}$$

We must transform these formulas

$$\begin{aligned}
&\int_0^\pi dx\frac{1}{\sqrt{a+b\cos x - b + b}} \\
= &\int_0^\pi dx\frac{1}{\sqrt{a+b+b(\cos x - 1)}} \\
= &\int_0^\pi dx\frac{1}{\sqrt{a+b-b2\sin^2\left(\frac{x}{2}\right)}} \\
= &\ \frac{1}{\sqrt{2b}}\int_0^\pi dx\frac{1}{\sqrt{\left(\frac{a+b}{2b}\right) - \sin^2\left(\frac{x}{2}\right)}}
\end{aligned}$$

we introduce the notation

$$\kappa^2 = \frac{a+b}{2b}$$

then we have obtained the form that is interesting for us

$$\sqrt{2b}\int_0^{\pi} \frac{dx}{\sqrt{a+b\cos x}} = \int_0^{\pi} \frac{dx}{\sqrt{\kappa^2 - \sin^2\left(\frac{x}{2}\right)}}$$

and we can use the known formula for the integral in the LHS

$$\sqrt{2b}\frac{2}{\sqrt{a+b}}\mathbf{K}\left(\sqrt{\frac{2b}{a+b}}\right) = \int_0^{\pi} \frac{dx}{\sqrt{\kappa^2 - \sin^2\left(\frac{x}{2}\right)}}$$

or

$$\begin{aligned}
\int_0^{\pi} \frac{dx}{\sqrt{\kappa^2 - \sin^2\left(\frac{x}{2}\right)}} &= 2\sqrt{\frac{2b}{a+b}}\mathbf{K}\left(\sqrt{\frac{2b}{a+b}}\right) \\
&= 2\frac{1}{\kappa}\mathbf{K}\left(\frac{1}{\kappa}\right)
\end{aligned}$$

Now let us calculate

$$\int_0^{\theta_0/2} \frac{1}{2}d\theta \, \frac{\cos\theta}{\sqrt{\kappa^2 - \sin^2(\theta/2)}}$$

We return to the form with $\cos\theta$ in the radical

$$\sqrt{\kappa^2 - \sin^2(\theta/2)} = \sqrt{\kappa^2 - \frac{1-\cos\theta}{2}} = \frac{1}{\sqrt{2}}\sqrt{2\kappa^2 - 1 + \cos\theta}$$

$$\begin{aligned}
\frac{\cos\theta}{\sqrt{\kappa^2 - \sin^2(\theta/2)}} &= \sqrt{2}\frac{\cos\theta}{\sqrt{2\kappa^2 - 1 + \cos\theta}} \\
&= \sqrt{2}\frac{2\kappa^2 - 1 - 2\kappa^2 + 1 + \cos\theta}{\sqrt{2\kappa^2 - 1 + \cos\theta}} \\
&= \sqrt{2}\left(\sqrt{2\kappa^2 - 1 + \cos\theta} + \frac{-2\kappa^2 + 1}{\sqrt{2\kappa^2 - 1 + \cos\theta}}\right) \\
&= \sqrt{2}\sqrt{\alpha + \cos\theta} \\
&\quad + \left(1 - 2\kappa^2\right)\sqrt{2}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{\kappa^2 - \sin^2\left(\frac{\theta}{2}\right)}}
\end{aligned}$$

where

$$\alpha \equiv 2\kappa^2 - 1$$

Using

$$\int_0^\pi dx\ \sqrt{a + b\cos x} = 2\sqrt{a+b}\mathbf{E}\left(\sqrt{\frac{2b}{a+b}}\right)$$

for $a = \alpha = 2\kappa^2 - 1$ and $b = 1$,

$$\begin{aligned}
\int_0^\pi d\theta\ \sqrt{\alpha + \cos\theta} &= 2\sqrt{2\kappa^2 - 1 + 1}\mathbf{E}\left(\sqrt{\frac{2}{2\kappa^2 - 1 + 1}}\right) \\
&= 2\sqrt{2}\ \kappa\ \mathbf{E}\left(\frac{1}{\kappa}\right)
\end{aligned}$$

Then

$$\begin{aligned}
&\int_0^\pi \frac{1}{2}d\theta\frac{\cos\theta}{\sqrt{\kappa^2 - \sin^2(\theta/2)}} \\
&= \frac{1}{2}\sqrt{2}2\sqrt{2}\ \kappa\ \mathbf{E}\left(\frac{1}{\kappa}\right) \\
&\quad +\frac{1}{2}\left(1 - 2\kappa^2\right)\sqrt{2}\frac{1}{\sqrt{2}}2\frac{1}{\kappa}\mathbf{K}\left(\frac{1}{\kappa}\right) \\
&= 2\kappa\mathbf{E}\left(\frac{1}{\kappa}\right) + \frac{1-2\kappa^2}{\kappa}\mathbf{K}\left(\frac{1}{\kappa}\right)
\end{aligned}$$

This permits to write

$$\begin{aligned}
\langle\cos\theta\rangle &= \frac{\kappa}{\mathbf{K}(\kappa)}\int_0^{\theta_0/2}\frac{1}{2}d\theta\ \frac{\cos\theta}{\sqrt{\kappa^2 - \sin^2(\theta/2)}} \\
&\approx \frac{\kappa}{\mathbf{K}(\kappa)}\left[2\kappa\mathbf{E}\left(\frac{1}{\kappa}\right) + \frac{1-2\kappa^2}{\kappa}\mathbf{K}\left(\frac{1}{\kappa}\right)\right]
\end{aligned}$$

In short it is

$$\langle\cos\theta\rangle = 2\kappa^2\frac{\mathbf{E}}{\mathbf{K}} + 1 - 2\kappa^2$$

And we can go further with

$$\begin{aligned}
\left\langle v_\parallel\right\rangle &= -v_E\frac{qR}{r_0} \\
&\quad + \langle\cos\theta\rangle\frac{1}{\Omega_c}qR\frac{\mu B_0}{m}\frac{1}{R}\left(\frac{1}{r_0} + 2\frac{d}{dr}\ln q\right) \\
&\quad +q\frac{1}{\Omega_c}\left(\frac{\mu B_0}{m}\right)2\left(-1 + 2\kappa^2\right)\frac{d}{dr}\ln q
\end{aligned}$$

separately the last two terms

$$
\begin{aligned}
& q\frac{1}{\Omega_c}\left(\frac{\mu B_0}{m}\right) \\
& \times\left[\langle\cos\theta\rangle\left(\frac{1}{r_0}+2\frac{d}{dr}\ln q\right)+2\left(-1+2\kappa^2\right)\frac{d}{dr}\ln q\right]
\end{aligned}
$$

where

$$
\langle\cos\theta\rangle = 2\kappa^2\frac{\mathbf{E}}{\mathbf{K}}+1-2\kappa^2
$$

$$
\begin{aligned}
& \left(2\kappa^2\frac{\mathbf{E}}{\mathbf{K}}+1-2\kappa^2\right)\left(\frac{1}{r_0}+2\frac{d}{dr}\ln q\right)+2\left(-1+2\kappa^2\right)\frac{d}{dr}\ln q \\
= \quad & \frac{2\kappa^2}{r_0}\frac{\mathbf{E}}{\mathbf{K}}+4\kappa^2\frac{\mathbf{E}}{\mathbf{K}}\frac{d}{dr}\ln q+\frac{1-2\kappa^2}{r_0}+2\left(1-2\kappa^2\right)\frac{d}{dr}\ln q \\
& \qquad -2\left(1-2\kappa^2\right)\frac{d}{dr}\ln q
\end{aligned}
$$

This is further

$$
\begin{aligned}
& \frac{1}{r_0}2\kappa^2\left(\frac{\mathbf{E}}{\mathbf{K}}+\frac{1-2\kappa^2}{2\kappa^2}\right) \\
& +2\kappa^2\frac{d}{dr}\ln q\left(2\frac{\mathbf{E}}{\mathbf{K}}\right)
\end{aligned}
$$

TO CHECK

Comment

If κ is close to 1 (particles are at the boundary trapped/passing) then the result is

$$
\begin{aligned}
& \frac{1}{r_0}2\kappa^2\left(\frac{\mathbf{E}}{\mathbf{K}}+\frac{1-2\kappa^2}{2\kappa^2}\right)+2\kappa^2\frac{d}{dr}\ln q\left(2\frac{\mathbf{E}}{\mathbf{K}}\right) \\
\approx \quad & 2\kappa^2\left[\frac{1}{r_0}\left(\frac{\mathbf{E}}{\mathbf{K}}-\frac{1}{2}\right)+2\frac{\mathbf{E}}{\mathbf{K}}\frac{d}{dr}\ln q\right]
\end{aligned}
$$

with the final form

$$
\begin{aligned}
\left\langle v_{\parallel}\right\rangle^{our} \quad = \quad & -v_E\frac{B_T}{B_\theta} \\
& +2q\frac{\frac{\mu B_0}{m}}{\Omega_c}\kappa^2\left[\frac{1}{r_0}\left(\frac{\mathbf{E}}{\mathbf{K}}-\frac{1}{2}\right)+2\left(\frac{d}{dr}\ln q\right)\frac{\mathbf{E}}{\mathbf{K}}\right]
\end{aligned}
$$

which is very close to the formula of **GS review**

$$
\begin{aligned}
\left\langle v_{\parallel}\right\rangle^{GS} &= -v_E\frac{B_T}{B_\theta} \\
&\quad +2\frac{\frac{\mu B_0}{m}}{\Omega_c}\frac{1}{R}\frac{B_T}{B_\theta}\left[\left(\frac{\mathbf{E}}{\mathbf{K}}-\frac{1}{2}\right)+2\left(\frac{d}{dr}\ln q\right)\left(\frac{\mathbf{E}}{\mathbf{K}}-1+\kappa^2\right)\right]
\end{aligned}
$$

We NOTE the following. In **GS review** formula

$$
\frac{1}{R}\frac{B_T}{B_\theta}=q\frac{1}{r}
$$

and the coefficients in front of the square paranthesis become similar

$$
2q\frac{\frac{\mu B_0}{m}}{\Omega_c}\kappa^2 \quad \text{versus} \quad 2q\frac{\frac{\mu B_0}{m}}{\Omega_c}\frac{1}{r}
$$

if we still assume that $\kappa \approx 1$.

But: in **GS review** there is a factor $\frac{1}{r}$ in front of the square paranthesis.

This rises a problem of dimension, because the term $\left(\frac{d}{dr}\ln q\right)$ inside has $(length)^{-1}$ and no other similar dimension is found there.

In our formula, $\frac{1}{r_0}$ in front of the first paranthesis is compatible, being $(length)^{-1}$, like $\left(\frac{d}{dr}\ln q\right)$.

End Comment

Remark on physics

the toroidal drift of the banana orbits $\left\langle v_{\parallel}\right\rangle$ arises exclusively from

- the fact that the banana have a finite width

- the magnetic field has spatial variation which is explored by the partice on a (wide) banana

NOTE

This is the place where

$$
\frac{1}{\Omega_c}\sim \text{dependence on charge}
$$

enters our formulas. Later we will have

- $v_{\parallel}$ independent of charge but
- $\left\langle v_{\parallel}\right\rangle$ i.e. the *precession*, depending on charge.

In this formula the dependence on charge enters through Ω_c and is introduced by the radial displacement $r-r_0$, produced by the neoclassical drift $\mathbf{v}_D$. The drift is different for electrons and ions.

End.

Returning to **GS review**

$$
\begin{aligned}
& \left\langle v_{\|}\right\rangle \\
= \;& -v_E \frac{B_T}{B_\theta} \\
& +\frac{1}{\Omega_c} \frac{\frac{\mu B_0}{m}}{R} \frac{B_T}{B_\theta}\left[\left(\frac{\mathbf{E}(\kappa)}{\mathbf{K}(\kappa)}-\frac{1}{2}\right)+2 \frac{d}{d r} \ln q(r)\left(\frac{\mathbf{E}}{\mathbf{K}}-1+\kappa^2\right)\right]
\end{aligned}
$$

Remember

$$
\begin{aligned}
v_0 & \equiv v_E \text{ poloidal, electric, in the first term GS} \\
v_0 \frac{B_T}{B_\theta} & \equiv \text{parallel projection of } v_E
\end{aligned}
$$

and **GS** take $\mu=\frac{m v_{\perp}^2}{2 B}$.

The part

$$
\begin{aligned}
\left\langle v_{\|}\right\rangle+v_E \frac{B_T}{B_\theta} & \rightarrow v_E+\frac{B_\theta}{B_T}\left\langle v_{\|}\right\rangle \\
& = \text{poloidal velocity} \\
& \approx 0
\end{aligned}
$$

The rest of the formula is the residual part, the *precession*.

The conclusion of **Galeev Sagdeev**
- the radial electric field E_r
- the centrifugal force
- the diamagnetic force
lead to a precession of the banana orbits

NOTE
- there is proportionality with m;
- the derivation is purely cinematic, no pressure gradient
- there is proportionality with $\frac{1}{\Omega_c}$, which introduces the difference due to charge $\pm e$. The supplementary term is positive for ions and is negative for electrons. This is a current, part of the elusive "bootstrap" current.

END

NOTE

The equation for the toroidal motion, in the article by **Fong Hahm** is for

$$\begin{aligned} & \varphi(\theta) \quad \text{(toroidal angle coordinate of the particle)} \\ \equiv\ & \text{motion equation} \end{aligned}$$

END

An approximation of the parallel velocity. This formula comes from the definition of κ^2,

$$\kappa^2 = \frac{1}{2\varepsilon}\left[1+\varepsilon-\frac{v_\perp^2}{v^2}(1+\varepsilon\cos\theta)\right]$$
$$\text{Beers}$$

from where

$$\frac{v_\parallel}{v} \approx \sigma\sqrt{2\varepsilon}\ \sqrt{\kappa^2-\sin^2\left(\frac{\theta}{2}\right)}$$

$$v_\parallel = -v_0\frac{B_T}{B_\theta}+\sigma\sqrt{2x_e\varepsilon}\ v_{th,e}\ \sqrt{\kappa^2-\sin^2(\theta/2)}$$

where

$$x_e = \frac{v^2}{v_{th,e}^2}$$

This is NOT the precession, is the velocity along the orbit. But, strangely, it is not sensitive to charge sign although its bounce average $\langle v_\parallel \rangle$ is proved to be dependent (on $\frac{1}{\Omega_c}$). This is because the average over the banana orbit introduces the radial displacement $r-r_0$ which depends on $\mathbf{v}_D$ i.e. on $1/\Omega_c$.

From **Fong Hahm**
"Unlike $r(\theta)$, the toroidal position does not return to its original value after a complete orbit. The difference in the poloidal angular velocity between the first and second halves of the orbit, due to the effects of the magnetic field strength inhomogeneity and curvature along with the magnetic shear, gives rise to a net precessional motion in the toroidal direction."

No poloidal flow is the basic state. An element of plasma moving along the magnetic line $v_\parallel$ would also advance in poloidal direction $\sim\theta$. But the plasma tube moves, due to the electric flow, in opposite poloidal direction such that it compensates the first poloidal displacement and so the only displacement is in toroidal direction $\sim \varphi$.

See **Hassam Kulsrud**.

See **Peeters bootstrap review**.

Everywhere it is approximated

$$v_{\parallel} - \frac{v_E}{\Theta} \approx 0$$

and in particular this is treated as $1/x = P(1/x) + i\pi\delta(x)$ in **Galeev** (kinetic equation) and in **Rozhansky Tendler**. Also it is physically explained in **Hassam Drake** using the motion of *tubes of plasma* and of plasma along the tubes.

$$v_E = \frac{1}{B_T} \frac{d\Phi}{dr}$$

$$v_g = \frac{1}{\Omega_c} \frac{\frac{\mu B_0}{m} + \frac{v_E^2}{\Theta^2}}{R} \equiv v_D$$

We **Note**: the parameter of the problem Δv is a difference relative to 0 of the competing velocities: one is poloidal velocity generated by the radial electric field, - projected on the parallel direction. And the other is the genuine parallel velocity. Normally the two contributions almost cancels each other: the poloidal velocity is cancelled by the poloidal projection of the parallel velocity

$$v_E - \Theta v_{\parallel} \sim 0$$

When there is poloidal rotation there is also parallel flow such that the amount of advancing on poloidal direction is precisely cancelled by the displacement of the "point" along a magnetic line, such that after projection, the effective θ-motion is zero. The two contributions compensate and the point is at the same θ.

Where from comes this constraint? Mention Magnetic Damping mechanism.

Condtion for trapping

$$(\delta v)^2 < 4 r_0 \Omega_c v_D$$

The radial displacement of the barely trapped particles is

$$\begin{aligned} & \Delta r_t (\theta = 0) \\ = \quad & \frac{B_T}{B_\theta} \frac{4}{\Omega_c} \sqrt{\frac{\mu B_0}{m} + \frac{v_E^2}{\Theta^2}} \end{aligned}$$

The equation of motion is

$$\frac{rd\theta}{dt} = -\sigma\frac{B_\theta}{B_T}\sqrt{\varepsilon\left(v^2+v_E^2\left(\frac{B_T}{B_\theta}\right)^2\right)}$$
$$\times\sqrt{2\kappa^2-1+\cos\theta}$$

where

$$2\kappa^2 = \frac{\left[\delta v\left(r_0,0\right)\right]^2}{\varepsilon\left(v^2+\frac{v_E^2}{\Theta^2}\right)}$$
$$v^2 = 2\frac{\epsilon - e\Phi}{m}$$
$$\sigma \equiv sign\left[\delta v\left(r,\theta\right)\right]$$

The result is purely kinematic, motion of a particle in the magnetic field geometry.

Individual parallel velocity does not depend on the charge sign

$$v_\parallel = -v_0\frac{B_T}{B_\theta} + \sigma\sqrt{2x_e\varepsilon}\ v_{th,e}\ \sqrt{\kappa^2-\sin^2\left(\theta/2\right)}$$

THE RESIDUAL PARALLEL DRIFT $\langle v_\parallel\rangle$ *DEPENDS on the SIGN OF CHARGE* via Ω_c

$$\langle v_\parallel\rangle = -v_E\frac{B_T}{B_\theta} + \frac{1}{\Omega_c}\frac{\frac{\mu B_0}{m}}{R}\frac{B_T}{B_\theta}\ \Sigma$$

where $\Sigma \equiv \left(\frac{E(\kappa)}{K(\kappa)}-\frac{1}{2}\right) + 2\frac{d}{dr}\ln q\left(r\right)\left(\frac{E}{K}-1+\kappa^2\right)$ (see **Mikhailovskii** for corrected result).

This is A CURRENT

This current does NOT come from the gradient of pressure.

This current does NOT come from the collisions affecting the populations of trapped and passing particles.

3.1.1 For comparison: Mikhailovskii book

This is discussed below.

The intersting aspect is the formation of an angular variable (that follows a helix on the torus) and the toroidal drift of trapped orbits is calculated by bounce averaging the equation for this variable.

In the following there are some differences of notations and signs.

$$\begin{aligned}
v_{D,r} &= \frac{1}{\Omega}\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{R}\ \sin\theta \\
v_{D,\theta} &= \frac{1}{\Omega}\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{R}\ \frac{\cos\theta}{r} \\
v_{D,\varphi} &= \frac{1}{\Omega}\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{R}\ \left(-\frac{r}{qR^2}\cos\theta\right)
\end{aligned}$$

(the signs are opposed to **Berk Galeev**, **Galeev Sagdeev, Fong Hahm**, see below, in *trapped particles*)

Define

$$\varphi_D = \varphi - q\theta$$

Its equation is

$$\frac{d\varphi_D}{dt} = \frac{1}{\Omega}\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{R}\left(-\frac{q}{r}\right)\left(\cos\theta + \frac{rq'}{q}\theta\sin\theta\right)$$

Note the factor that we will find in ITG analytical theories,

$$\cos\theta + \widehat{s}\ \theta\ \ \sin\theta$$

End.

Define the *average of* $\frac{d\varphi_D}{dt}$ over a bounce

$$\overline{\left(\frac{d\varphi_D}{dt}\right)} = \frac{1}{\tau_{bounce}}\oint\frac{dl_\parallel}{v_\parallel}\ \left(\frac{d\varphi_D}{dt}\right)$$

It is

$$\overline{\left(\frac{d\varphi_D}{dt}\right)} = \frac{2\varepsilon\ \epsilon q}{\Omega r^2}\left[\frac{\mathbf{E}}{\mathbf{K}} - \frac{1}{2} + 2\frac{rq'}{q}\left(\frac{\mathbf{E}}{\mathbf{K}} - 1 + \kappa^2\right)\right]$$

(note $\widehat{s} = rq'/q$) or

$$\overline{\left(\frac{d\varphi_D}{dt}\right)} = \frac{1}{\tau_{bounce}}\frac{q}{\Omega r}\ \frac{dJ_\parallel}{dr}$$

This is the frequency of precession of a banana in the toroidal direction.

3.2 Bootstrap current derived by Galeev Sagdeev

In the text it is adopted a collision operator.

The assupmtions
- the ei collisions are more important than the ee collisions

Collisions

$$C^{ei}\left[f_e\right]=\frac{\partial}{\partial v_\alpha}\left[\frac{1}{2}\frac{\mu_0 e^4 Z^2}{m_e^2}n\ln\Lambda\frac{1}{v}\left(\delta_{a\beta}-\frac{v_\alpha v_\beta}{v^2}\right)\left(\frac{\partial f_e}{\partial v_\beta}-\frac{v_\beta}{T_e}f_e\right)\right]$$

The ion distribution is taken Maxwellian.

Note
At this stage we cannot distinguich between
- slowing down (friction)
- pitch angle transfer across the boundary trapped/passing

End

The electron drift kinetic equation

$$\begin{aligned}&\frac{\partial f_e}{\partial t}+\left(\frac{B_\theta}{B_T}v_\parallel\right)\frac{\partial f_e}{r\partial\theta}+\frac{d}{dr}\left(v_{Dr}f_e\right)\\&+\nabla\ v_\parallel\left(-\frac{\mu\nabla_\parallel B_\varphi}{m_e}f_e\right)\\=\ &C^{ei}\left[f_e\right]\end{aligned}$$

NOTE
We have

$$-\frac{\mu\nabla_\parallel B_\varphi}{m_e}=\text{force per unit mass that a particle must confront moving along the line}$$

and

$$\begin{aligned}&v_\parallel\left(-\frac{\mu\nabla_\parallel B_\varphi}{m_e}\right)\\=\ &\text{force}\times\text{velocity}=\frac{\text{work}}{\text{time}}\ (=\text{power})\end{aligned}$$

The divergence of the flow

$$\left[v_\parallel \left(-\frac{\mu \nabla_\parallel B_\varphi}{m_e} \right) f \right]$$

This is an energetic term.

END

The two distribution functions are

$$f_j^{trap} = f_j^{trap,(0)} \left(\mu, E, J \left(r, \theta \right) \right)$$
$$f_j^{pass} = f_j^{pass} \left(\mu, E, J; \sigma \right)$$

Expansion

$$f_j = f_j^{(0)} + f_j^{(1)}$$

We must use for the first order neoclassical correction, the expression of the parallel velocity

$$v_\parallel = -v_0 \frac{B_T}{B_\theta} + \sigma \sqrt{2 x_e \varepsilon}\ v_{th,e} \ \sqrt{\kappa^2 - \sin^2 \left(\theta / 2 \right)}$$

The additional term is a parallel velocity. Since the neoclassical perturbation is obtained from the variation of the distribution function on θ (poloidal) we must project this parallel velocity, multiplying it with $\frac{B_\theta}{B_T}$, i.e.

$$\frac{B_\theta}{B_T} \left[\sigma \sqrt{2 x_e \varepsilon}\ v_{th,e} \ \sqrt{\kappa^2 - \sin^2 \left(\theta / 2 \right)} \right] = \widetilde{v}_\theta$$

and

$$\widetilde{v}_\theta \frac{\partial f_e^{(1)}}{r \partial \theta} = C \left(f \right)$$

Then the equation for the first order is

$$\sigma v_{th,e} \frac{B_\theta}{B_T} \sqrt{2 x_e \varepsilon} \sqrt{\kappa^2 - \sin^2 \left(\frac{\theta}{2} \right)}\ \frac{\partial f_e^{(1)}}{r \partial \theta} = C \left[f \right]$$

where

$$\begin{aligned} C \left[f \right] &= \frac{3\sqrt{\pi}}{4} \nu_{ei} \frac{1}{\varepsilon} \frac{1}{x_e^{3/2}}\ \sigma \sqrt{\kappa^2 - \sin^2 \left(\frac{\theta}{2} \right)} \\ &\times \frac{\partial}{\partial \kappa^2} \left[\sigma \sqrt{\kappa^2 - \sin^2 \left(\frac{\theta}{2} \right)} \left(\frac{\partial}{\partial \kappa^2} + 2 x_e \varepsilon \right) - \frac{v_0}{\left(\frac{B_\theta}{B_T} \right) v_{th,e}} \sqrt{2 x_e \varepsilon} \right] f_e \end{aligned}$$

No spatial gradient of f_e.

3.3 The distribution functions

There is no spatial derivative of the zero-order (equilibrium) distribution function in the collision operator, everything is related to the velocity space.

But the first order neoclassical correction of the distribution function, f_1, is due to the neoclassical drift $(\mathbf{v}_D \cdot \boldsymbol{\nabla}) f_0$, and this introduces the gradient, e.g.

$$\frac{d}{dr} \ln n_0 (r)$$

or of pressure. This is where the "pattern" of unequal fluxes counted on an infinitesimal transversal surface (on the parallel direction) arises. This corresponds to the usual expansion of f,

$$f = f_0 + f_1$$

where f_1 consists of the neoclassical part, $\sim \rho_\theta / L_n$ plus a function g that results from collisions (**Rutherford**, **Connor, Hellander**, etc.).

The function g only exists for *passing* particles.

The equation is

$$\left\langle \frac{B}{v_\parallel} C (f) \right\rangle = 0$$

and the unknown is

$$\frac{\partial g}{\partial \lambda}$$

This is what **Galeev Sagdeev** obtain.

3.3.1 The distribution function for the trapped particles

This is

$$f_e^{trap,(0)} = \frac{n_e^{(0)}}{\left(\sqrt{\pi} v_{th,e}\right)^3} \exp\left(-\frac{\mathcal{E}}{T_e}\right) \left[1 + \sigma \sqrt{2 x_e \varepsilon} \frac{B_T}{B_\theta} \frac{d}{dr} \ln n_e^{(0)} \sqrt{\kappa^2 - \sin^2\left(\frac{\theta}{2}\right)}\right]$$

a factor is missing ρ_e.

The space derivative of the density arises because the density is expanded for small space interval

$$n^{(0)} + \Delta r \ \frac{dn^{(0)}}{dr}$$

The presence of the gradient of density is essential for the creation of the "pattern" of unequal fluxes that is propagated through collisions to the passing particles and generates the "bootstrap" current.

3.3.2 The distribution function of the passing particles

This is

$$
\begin{aligned}
f_e^{pass,(0)} &= \frac{n_e^{(0)}}{\left(\sqrt{\pi}v_{th,e}\right)^3}\exp\left(-\frac{\mathcal{E}}{T_e}\right) \\
&\quad\times\left\{1-\sigma\sqrt{2x_e\varepsilon}\frac{B_T}{B_\theta}\rho_e\frac{d}{dr}\ln n_e^{(0)}\right. \\
&\quad\left.\times\left(\sqrt{\kappa^2-\sin^2\left(\frac{\theta}{2}\right)}-\frac{\pi}{4}\int_1^{\kappa^2}\frac{dt}{\sqrt{t}\,E\left(\frac{1}{\sqrt{t}}\right)}\right)\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
f_e^{pass,(0)} &= \frac{n_e^{(0)}}{\left(\sqrt{\pi}v_{th,e}\right)^3}\exp\left(-\frac{\mathcal{E}}{T_e}\right) \\
&\quad\times\left\{1-\sigma\frac{v\sqrt{2\varepsilon}}{\Omega_{ce}}\frac{B_T}{B_\theta}\rho_e\frac{d}{dr}\ln n_e^{(0)}\right. \\
&\qquad\left.\times\left(\frac{\sqrt{\kappa^2-\sin^2\left(\frac{\theta}{2}\right)}}{\sqrt{1+2\kappa^2\varepsilon}}-\frac{\pi}{4}\int_1^{\kappa^2}\frac{dt}{\sqrt{t}\left(1+2\varepsilon t\right)^{3/2}E\left(\frac{1}{\sqrt{t}}\right)}\right)\right\}
\end{aligned}
$$

The current

$$
j = 2\alpha_{ei}\sqrt{\varepsilon}\frac{1}{B_\theta}\frac{dp_e}{dr}
$$

We recognize here the bootstrap current. The collisions have been essential to this result.

NOTE
We must identify in this derivation the basic idea: there is in the population of trapped particles a reservoir of directed momentum and this is available to passing particles via collisions.

Here **GS** the trapped particles are ions and the passing particles are electrons. They carry the current.
END

NOTE
Regarding the structure of these two distribution functions.
In **Rutherford** etc. there is
- the neoclassical shift $\sim \rho_\theta/L_n$, and

- the function g that arises from collisions and is (a) finite for passing (b) zero for trapped.

END

NOTE

In alternative treatment for $f_{t,u}$ the condition is based on periodicity

$$\left\langle \frac{B}{v_{\|}} C(f) \right\rangle = 0$$

where f is expanded in the neoclassical correction plus the function g, which intervenes only as $\frac{\partial g}{\partial \lambda}$, and only on passing region of velocity space.

END

3.4 Other applications

A trapped particle has in every point a drift velocity, essentially the magnetic drift and the curvature drift.

It is possible to derive, as above, the bounce average of the drift *frequency*. For electrons

$$\begin{aligned} \langle \omega_D \rangle_{bounce} &= \frac{T_e}{eB} \frac{k_y}{R} \frac{v^2}{2\, v_{th,e}^2} G(\widehat{s}; \kappa) \\ &\equiv \text{frequency of toroidal precession of the banana} \end{aligned}$$

Here

$$G(\widehat{s}, \kappa) = \left(2 \frac{\mathbf{E}(\kappa)}{\mathbf{K}(\kappa)} - 1 \right) + 4\widehat{s} \left(\frac{\mathbf{E}(\kappa)}{\mathbf{K}(\kappa)} - 1 + \kappa^2 \right)$$

It is found also by **Mikhailovskii (book)**. Compare

$$\overline{\left(\frac{d\varphi_D}{dt} \right)} = \frac{2\varepsilon\ \epsilon q}{\Omega r^2} \left[\frac{\mathbf{E}}{\mathbf{K}} - \frac{1}{2} + 2 \frac{r q'}{q} \left(\frac{\mathbf{E}}{\mathbf{K}} - 1 + \kappa^2 \right) \right]$$

The new variables in velocity space and the operator of bounce averaging allow to calculate momenta, like integral of the bounce-averaged distribution function $\langle f_e \rangle_{bounce}$ over the *trapped* particle region of the velocity space

$$v \text{ has variation between } 0 \text{ and } \infty$$

$$\kappa \text{ has variation between } \sin\left(\frac{\theta}{2}\right) \text{ and } 1$$

(this is because there is $\sqrt{\kappa^2 - \sin^2\left(\frac{\theta}{2}\right)}$).

The following quantity is the "density of trapped particles" since it is the integral of the distribution function over the velocity space that corresponds to the trapped orbits.

$$\begin{aligned}&\int_{trapped} d^3v \ \langle f_e\rangle_{bounce} \\ = &\int_0^\infty 4\pi v^2 dv \int_{\sin(\theta/2)}^1 \kappa \ d\kappa \ \frac{B}{B_{\min}} \frac{2\varepsilon_B}{\sqrt{1-\left(\frac{B}{B_{\min}}\right)(1-2\varepsilon_B \ \kappa^2)}} \langle f_e\rangle_{bounce}\end{aligned}$$

Here $\varepsilon_B = \frac{B_{\max}-B_{\min}}{2B_{\max}} = \frac{\varepsilon}{1+\varepsilon}$ and $B_{\min} = \frac{B_0}{1+\varepsilon}$. $B = \frac{B_0}{1+\varepsilon\cos\theta}$. And $\frac{v_\parallel}{v} = \sqrt{1-\left(\frac{B}{B_{\min}}\right)(1-2\varepsilon_B \ \kappa^2)} \approx \sqrt{\frac{\varepsilon}{1+\varepsilon}}\sqrt{\kappa^2-\sin^2\left(\frac{\theta}{2}\right)}$.

$$\begin{aligned}\frac{B}{B_{\min}} 2\varepsilon_B &= \frac{B_0}{1+\varepsilon\cos\theta}\frac{1+\varepsilon}{B_0} 2\frac{\varepsilon}{1+\varepsilon} = 2\varepsilon\left(1-\varepsilon\cos\theta\right) \\ &\rightarrow \ \approx 2\varepsilon\end{aligned}$$

Expanding for small ε,

$$\begin{aligned}&\int_{trapped} d^3v \ \langle f_e\rangle_{bounce} \\ = &\sqrt{2\varepsilon}\int_0^\infty 4\pi v^2 dv \int_{\sin(\theta/2)}^1 \kappa \ d\kappa \frac{1}{\sqrt{\kappa^2-\sin^2\left(\frac{\theta}{2}\right)}} \langle f_e\rangle_{bounce}\end{aligned}$$

4 Equations in space (Berk Galeev 1966)

Note that the paper **Berk Galeev Velocity space instabilities 1966** gives a different differential equation for the particle displacement in the poloidal direction, by taking into account the projection of the vertical drift of the particle on the poloidal tangent

$$\frac{dr}{dt} = -\frac{1}{\Omega_{ci}}\frac{v_\perp^2/2+v_\parallel^2}{R}\sin\theta$$

$$\frac{rd\theta}{dt} = -v_\parallel \frac{B_\theta}{B_T} - \frac{1}{\Omega_{ci}}\frac{v_\perp^2/2+v_\parallel^2}{R}\cos\theta + \frac{1}{B_0}\frac{\partial\phi}{\partial r}$$

The "$-$ " sign comes from the reference system adopted by the definition of **Berk Galeev**.

4.0.1 Inverse connection: from equations to invariance of longitudinal momentum J

From this same paper we see the derivation of the constant of motion which is the *longitudinal invariant.* The two equations of motion are combined.

$$\frac{1}{\Omega_{ci}}\frac{v_\perp^2/2+v_\parallel^2}{R}\frac{d}{dr}\left(r\cos\theta\right)=-v_\parallel\frac{B_\theta}{B_T}+\frac{1}{B_0}\frac{\partial\phi}{\partial r}$$

This is a point-like equation.

We are going to verify this equation.

Consider the dynamical version, with θ a function of time

$$\begin{aligned}
& \frac{1}{\Omega_{ci}}\frac{v_\perp^2/2+v_\parallel^2}{R}\frac{d}{dr}\left\{r\cos\left[\theta\left(t\right)\right]\right\} \\
= & \frac{1}{\Omega_{ci}}\frac{v_\perp^2/2+v_\parallel^2}{R}\cos\theta-\frac{1}{\Omega_{ci}}\frac{v_\perp^2/2+v_\parallel^2}{R}\sin\theta\frac{r\partial\theta}{\partial r} \\
= & \frac{1}{\Omega_{ci}}\frac{v_\perp^2/2+v_\parallel^2}{R}\cos\theta-\frac{1}{\Omega_{ci}}\frac{v_\perp^2/2+v_\parallel^2}{R}\sin\theta\frac{rd\theta}{dt}\frac{dt}{dr}
\end{aligned}$$

we replace

$$\frac{dt}{dr}=\frac{1}{-\frac{1}{\Omega_{ci}}\frac{v_\perp^2/2+v_\parallel^2}{R}\sin\theta}$$

and obtain

$$\begin{aligned}
& \frac{1}{\Omega_{ci}}\frac{v_\perp^2/2+v_\parallel^2}{R}\frac{d}{dr}\left(r\cos\theta\right) \\
= & \frac{1}{\Omega_{ci}}\frac{v_\perp^2/2+v_\parallel^2}{R}\cos\theta-\frac{1}{\Omega_{ci}}\frac{v_\perp^2/2+v_\parallel^2}{R}\sin\theta\frac{rd\theta}{dt}\frac{1}{-\frac{1}{\Omega_{ci}}\frac{v_\perp^2/2+v_\parallel^2}{R}\sin\theta} \\
= & \frac{1}{\Omega_{ci}}\frac{v_\perp^2/2+v_\parallel^2}{R}\cos\theta+\frac{rd\theta}{dt}
\end{aligned}$$

Now replace the last term $\frac{rd\theta}{dt}$ and obtain

$$\begin{aligned}
& \frac{1}{\Omega_{ci}}\frac{v_\perp^2/2+v_\parallel^2}{R}\frac{d}{dr}\left(r\cos\theta\right) \\
= & \frac{1}{\Omega_{ci}}\frac{v_\perp^2/2+v_\parallel^2}{R}\cos\theta-v_\parallel\frac{B_\theta}{B_T}-\frac{1}{\Omega_{ci}}\frac{v_\perp^2/2+v_\parallel^2}{R}\cos\theta+\frac{1}{B_0}\frac{\partial\phi}{\partial r} \\
= & -v_\parallel\frac{B_\theta}{B_T}+\frac{1}{B_0}\frac{\partial\phi}{\partial r}
\end{aligned}$$

which confirms the equation

$$\frac{1}{\Omega_{ci}}\frac{v_\perp^2/2+v_\parallel^2}{R}\frac{d}{dr}\left(r\cos\theta\right)=-v_\parallel\frac{B_\theta}{B_T}+\frac{1}{B_0}\frac{\partial\phi}{\partial r}$$

In order to proceed further we need the derivative of the velocities along the orbit.

The invariant

$$\epsilon=\frac{v_\perp^2}{2}+\frac{v_\parallel^2}{2}+\frac{e}{m}\phi=\mu B+\frac{v_\parallel^2}{2}+\frac{e}{m}\phi$$

and calculate the derivation to r,

$$\frac{d\epsilon}{dr}=0$$

which means a constraint on the radial variation of the electrostatic potential $\phi\left(r\right)$

$$\frac{e}{m}\frac{\partial\phi}{\partial r}=-\mu\frac{dB}{dr}-v_\parallel\frac{dv_\parallel}{dr}$$

Since

$$B=\frac{B_0}{1+\frac{r}{R}\cos\theta}$$

the variation of the *magnitude of the magnetic field* B with the minor radius coordinate is

$$\begin{aligned}\frac{dB}{dr}&=-B_0\frac{1}{\left(1+\frac{r}{R}\cos\theta\right)^2}\frac{d}{dr}\left(\frac{r\cos\theta}{R}\right)\\&=-\frac{B_0}{1+\frac{r}{R}\cos\theta}\frac{1}{1+\frac{r}{R}\cos\theta}\frac{d}{dr}\left(\frac{r\cos\theta}{R}\right)\\&=-B\frac{1}{1+\frac{r}{R}\cos\theta}\frac{1}{R}\frac{d}{dr}\left(r\cos\theta\right)\\&\approx-\frac{B}{R}\frac{d}{dr}\left(r\cos\theta\right)\end{aligned}$$

Then

$$\begin{aligned}\frac{e}{m}\frac{\partial\phi}{\partial r}&=-\mu\frac{dB}{dr}-v_\parallel\frac{dv_\parallel}{dr}\\&=\mu\frac{B}{R}\frac{d}{dr}\left(r\cos\theta\right)-v_\parallel\frac{dv_\parallel}{dr}\end{aligned}$$

Now we replace this expression in the previous equation

$$\begin{aligned}
\frac{1}{\Omega_{ci}}\frac{v_{\perp}^{2}/2+v_{\parallel}^{2}}{R}\frac{d}{dr}(r\cos\theta) &= -v_{\parallel}\frac{B_{\theta}}{B_{T}}+\frac{1}{B_{0}}\frac{\partial\phi}{\partial r} \\
&= -v_{\parallel}\frac{B_{\theta}}{B_{T}}+\frac{1}{eB_{0}/m}\left[\mu\frac{B}{R}\frac{d}{dr}(r\cos\theta)-v_{\parallel}\frac{dv_{\parallel}}{dr}\right] \\
&= v_{\parallel}\frac{B_{\theta}}{B_{T}}+\frac{1}{\Omega_{ci}}\frac{v_{\perp}^{2}}{2}\frac{1}{R}\frac{d}{dr}(r\cos\theta)-\frac{v_{\parallel}}{\Omega_{ci}}\frac{dv_{\parallel}}{dr}
\end{aligned}$$

The last line is

$$\begin{aligned}
&\frac{1}{\Omega_{ci}}\frac{1}{R}\frac{v_{\perp}^{2}}{2}\frac{d}{dr}(r\cos\theta)+\frac{v_{\parallel}^{2}}{\Omega_{ci}}\frac{1}{R}\frac{d}{dr}(r\cos\theta) \\
=\ &\frac{1}{\Omega_{ci}}\frac{v_{\perp}^{2}}{2}\frac{1}{R}\frac{d}{dr}(r\cos\theta)+v_{\parallel}\frac{B_{\theta}}{B_{T}}-\frac{v_{\parallel}}{\Omega_{ci}}\frac{dv_{\parallel}}{dr}
\end{aligned}$$

The first terms in the left and right sides cancels each other and we divide by $v_{\parallel}$ to obtain

$$\frac{v_{\parallel}}{\Omega_{ci}}\frac{1}{R}\frac{d}{dr}(r\cos\theta)=\frac{B_{\theta}}{B_{T}}-\frac{1}{\Omega_{ci}}\frac{dv_{\parallel}}{dr}$$

The sign of the first term in the right hand side differs from that of **Berk Galeev** since they define

$$\frac{1}{\iota}=-q$$

We consider that $v_{\parallel}$ in the left side has not too fast radial variation. Then it is possible to integrate and obtain

$$\begin{aligned}
J &= -\frac{\Omega_{ci}}{R}\int_{0}^{r}\frac{r}{q}dr+v_{\parallel}\left(1+\frac{r}{R}\cos\theta\right) \\
&= \text{const}
\end{aligned}$$

which is the invariant.

As **noted** by **Berk Galeev** this is *an invariant.*

We have actually made a reversed construction.

Normally the logitudinal invariant (toroidal generalized angular momentum) is invoked to derive the equations of motion for the particle.

In the calculation we have presented from **Berk Galeev** the equations of motion are used to re-construct the invariant and show that it is constant. It is just a verification.

5 The drift velocity derived from the longitudinal invariant

This is **Rosenbluth Hazeltine Hinton**.

The field

$$\begin{aligned} B_r &= 0 \\ B_\theta &= \frac{b(r)}{h} \\ B_\varphi &= \frac{B_0}{h} \end{aligned}$$

$$|B| \approx \frac{B_0}{h}$$

The longitudinal invariant

$$\frac{d}{dt}\left(hmv_\varphi - e\int^r dr\ b(r)\right) = 0$$

where

$$\int^r dr\ b(r) = A_\varphi$$

The derivative of the first term: it is purely convective

$$\begin{aligned} \frac{d}{dt}(hmv_\varphi) &= \left[\frac{\partial}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\right](hmv_\varphi) \\ &= \mathbf{v}.\boldsymbol{\nabla}(hmv_\varphi) \quad \text{(no explicit } \partial/\partial t \text{ only convective term)} \\ &\approx \mathbf{v}\cdot\boldsymbol{\nabla}\left(hmv_\parallel\right) \\ &\approx m\left(v_\parallel \frac{B_\theta}{B}\right)\frac{\partial}{r\partial\theta}\left(hv_\parallel\right) \quad \text{(the variation is poloidal } \theta\text{)} \end{aligned}$$

The energy is

$$\epsilon = \frac{1}{2}mv^2 + e\Phi$$

and the magnetic momentum

$$\mu = \frac{mv_\perp^2}{2B}$$

The derivative of the second term: we keep the time variation and its derivative

$$\begin{aligned} \frac{d}{dt}\left(e\int^r dr\ b(r)\right) &= eR\frac{\partial A_\varphi}{\partial t} + e\frac{dr}{dt}b(r) \\ &= eh\mathcal{E}_\varphi + ev_{D,r}\ hB_\theta \\ &\ \mathcal{E}_\varphi \text{ is the inductive electric field} \end{aligned}$$

Since the electric field is small

$$m\left(v_{\|}\frac{B_\theta}{B}\right)\frac{\partial}{r\partial\theta}\left(hv_{\|}\right)-ev_{D,r}\ hB_\theta\approx 0$$

Then

$$\begin{aligned} v_{D,r} &= \frac{m}{eB_0}v_{\|}\frac{\partial}{r\partial\theta}\left(v_{\|}h\right) \\ &= \frac{v_{\|}}{\Omega_c}\frac{\partial}{r\partial\theta}\left(v_{\|}h\right) \end{aligned}$$

This is another expression for the drift velocity of the particle

$$\mathbf{v}_D=\frac{1}{\Omega_c}\widehat{\mathbf{n}}\times\left(\mu\boldsymbol{\nabla}B+v_{\|}^2\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}+\frac{e}{m}\boldsymbol{\nabla}\phi\right)$$

$$\begin{aligned} \mathbf{v}_D &= \frac{1}{\Omega}\frac{\frac{v_\perp^2}{2}+v_{\|}^2}{R}\widehat{\mathbf{n}}\times\widehat{\mathbf{e}}_R \\ &\sim \text{vertical} \end{aligned}$$

and

$$v_{D,r}=\frac{v_{\|}}{\Omega_c}\frac{\partial}{r\partial\theta}\left(v_{\|}h\right)$$

6 The equations in space and velocity (Wang Burrell 1982)

Cited by **Novakovski, et al.**

The Invariants are

$$\epsilon \ \text{ and } \ \mu$$

where

$$\begin{aligned} \epsilon &\equiv \text{total energy} \\ &= \frac{mv_{\|}^2}{2}+\frac{mv_\perp^2}{2}+e\phi \end{aligned}$$

and

$$\begin{aligned} \mu &\equiv \text{magnetic moment} \\ &= \frac{v_\perp^2}{2B} \end{aligned}$$

From here we will calculate the variation of the velocity along the orbit. We need to consider

$$\frac{d}{dt} \text{ as } \frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla}$$

since there is space variation of the magnetic field and of the potential along the orbits.

First we take as velocity variables

$$v_{\parallel} \text{ and } \frac{v_{\perp}^2}{2}$$

and find the expression of each in terms of the invariants (ϵ, μ) First define the energy per unit mass

$$\begin{aligned}
\epsilon \quad \rightarrow \quad \frac{\epsilon}{m} &= \frac{1}{2}v^2 + \frac{e}{m}\phi \\
&= \frac{1}{2}v_{\parallel}^2 + \frac{v_{\perp}^2}{2} + \frac{e}{m}\phi
\end{aligned}$$

$$\begin{aligned}
v_{\parallel} &= \sqrt{2\epsilon - v_{\perp}^2 - 2\frac{e}{m}\phi} \\
&= \left(2\epsilon - 2\mu B - 2\frac{e}{m}\phi\right)^{1/2} \\
&= \sqrt{2}\left(\epsilon - \mu B - \frac{e}{m}\phi\right)^{1/2}
\end{aligned}$$

$$\frac{v_{\perp}^2}{2} = \mu B$$

Now we take the time derivatives along the orbits

$$\frac{dv_{\parallel}}{dt} = \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla}\right) v_{\parallel}$$

where for the ions (almost the motion of the plasma)

$$\mathbf{v} = \widehat{\mathbf{n}}v_{\parallel} + \mathbf{v}_E + \mathbf{v}_D$$

where

$$\mathbf{v}_E = \frac{-\boldsymbol{\nabla}\phi \times \widehat{\mathbf{n}}}{B}$$

$$\mathbf{v}_D = \frac{1}{\Omega_{ci}}\widehat{\mathbf{n}} \times \left[\mu \boldsymbol{\nabla} B + v_{\parallel}^2 \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right) \widehat{\mathbf{n}}\right]$$

Then

$$(\mathbf{v}\cdot\boldsymbol{\nabla})\, v_{\|} = \left[\left(\widehat{\mathbf{n}} v_{\|} + \mathbf{v}_E + \mathbf{v}_D\right)\cdot\boldsymbol{\nabla}\right] v_{\|}$$

We must take into account the fact that the derivations are taken at fixed invariants.

In order to expand this equation we will use the following formulas.

6.0.2 Useful Formulas

Divergence of the flow of particles due to the electric drift

$$\begin{aligned}
& \boldsymbol{\nabla}\cdot(n\mathbf{V}_E) \\
= \; & \boldsymbol{\nabla}\cdot\left(n\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\right) = n\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\times\left(\frac{\widehat{\mathbf{n}}}{B}\right) \\
= \; & n\boldsymbol{\nabla}\phi\cdot\left[\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\ln B + \widehat{\mathbf{n}}\times(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla})\,\widehat{\mathbf{n}}\right]
\end{aligned}$$

where it has been used the formula

$$\boldsymbol{\nabla}\times\left(\frac{\widehat{\mathbf{n}}}{B}\right) = \widehat{\mathbf{n}}\times\boldsymbol{\nabla}\ln B + \widehat{\mathbf{n}}\times(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla})\,\widehat{\mathbf{n}}$$

$$\begin{aligned}
\mathbf{v}_{De} &= -\rho_s c_s\,\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\left(\frac{\widehat{\mathbf{n}}}{B}\right) \\
&\approx 2\rho_s c_s\frac{1}{R_0}\left(\widehat{\mathbf{e}}_\theta\;\cos\theta + \widehat{\mathbf{e}}_r\;\sin\theta\right)
\end{aligned}$$

is the drift velocity of the electrons.

The curvature of the magnetic field line

$$\boldsymbol{\kappa} \equiv (\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla})\,\widehat{\mathbf{n}} \approx -\frac{1}{R}\widehat{\mathbf{e}}_R$$

is directed toward the main axis.

Let us consider

$$B = R_0\frac{B_0}{R} \approx B_0\,(1-\varepsilon\cos\theta)$$

(also adopted by **Wong Burrell 1982**). Then the gradient of the scalar function B is

$$\begin{aligned}
\boldsymbol{\nabla} B &= \widehat{\mathbf{e}}_r \frac{\partial}{\partial r} B + \widehat{\mathbf{e}}_\theta \frac{\partial}{r\partial\theta} B + \widehat{\mathbf{e}}_Z \frac{\partial}{\partial Z} B \qquad (Z \text{ is along the toroidal direction}) \\
&= B_0 \left(-\frac{1}{R} \cos\theta \right) \widehat{\mathbf{e}}_r + B_0 \frac{1}{r} \varepsilon \sin\theta \, \widehat{\mathbf{e}}_\theta \\
&= B_0 \frac{1}{r} \varepsilon \left[-\cos\theta \, \widehat{\mathbf{e}}_r + \sin\theta \, \widehat{\mathbf{e}}_\theta \right] = B_0 \frac{1}{R} \left(-\widehat{\mathbf{e}}_R \right) = \frac{B_0}{R_0} \frac{R_0}{R} \left(-\widehat{\mathbf{e}}_R \right) \\
&\approx B \frac{1}{R} \left(-\widehat{\mathbf{e}}_R \right)
\end{aligned}$$

or

$$\boldsymbol{\nabla} \ln B \approx \frac{1}{R} \left(-\widehat{\mathbf{e}}_R \right) = \boldsymbol{\kappa}$$

Since we can use the zero-divergence of the magnetic vector field, we find

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\
\boldsymbol{\nabla} \cdot \left(\widehat{\mathbf{n}} B \right) &= B \left(\boldsymbol{\nabla} \cdot \widehat{\mathbf{n}} \right) + \widehat{\mathbf{n}} \cdot \left(\boldsymbol{\nabla} B \right) = 0 \\
\frac{\boldsymbol{\nabla} B}{B} &= -\boldsymbol{\nabla} \cdot \widehat{\mathbf{n}}
\end{aligned}$$

or

$$\boldsymbol{\nabla} \cdot \widehat{\mathbf{n}} = -\boldsymbol{\nabla} \ln B$$

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \widehat{\mathbf{n}} &= -\boldsymbol{\nabla} \ln B = \frac{1}{R} \widehat{\mathbf{e}}_R = -\left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} \\
&\equiv -\boldsymbol{\kappa}
\end{aligned}$$

or

$$\boldsymbol{\nabla} \ln B = \boldsymbol{\kappa}$$

Now we can use these equations to evaluate the terms.

6.0.3 The terms

We also have to take into account that

$$\frac{\partial \mu}{\partial x} = -\frac{\mu}{B} \frac{\partial B}{\partial x} - \frac{v_\parallel}{B} \mathbf{v} \cdot \frac{\partial \widehat{\mathbf{n}}}{\partial x}$$

as is derived in *drif kinetic derivation.tex* by the change of variables.

This is important only if μ is taken as a variable.

If we focus on a single particle, then ϵ and μ are invariants, they do not depend on $(\mathbf{x}, t)$.

(1) First term is

$$v_\parallel \left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right) v_\parallel = \nabla_\parallel \left(\frac{1}{2}v_\parallel^2\right)$$

$$\begin{aligned}
& \left(\widehat{\mathbf{n}}v_\parallel\cdot\boldsymbol{\nabla}\right) v_\parallel \quad \text{taken at fixed } \epsilon \text{ and } \mu \quad \text{(i.e. for a particle)} \\
= \; & \widehat{\mathbf{n}}v_\parallel\cdot\boldsymbol{\nabla}\sqrt{2}\left(\epsilon-\mu B-\frac{e}{m}\phi\right)^{1/2} \\
= \; & v_\parallel\sqrt{2}\frac{1}{2}\frac{1}{\left(\epsilon-\mu B-\frac{e}{m}\phi\right)^{1/2}}\nabla_\parallel\left(-\mu B-\frac{e}{m}\phi\right) \\
= \; & -\mu\nabla_\parallel B-\frac{e}{m}\nabla_\parallel\phi
\end{aligned}$$

must be corrected ? no

(2) Second term

$$\begin{aligned}
\left(\mathbf{v}_E\cdot\boldsymbol{\nabla}\right) v_\parallel &= \left(\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}\right)\sqrt{2}\left(\epsilon-\mu B-\frac{e}{m}\phi\right)^{1/2} \\
&= \frac{1}{v_\parallel}\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}\left(-\mu B-\frac{e}{m}\phi\right)
\end{aligned}$$

or

$$\begin{aligned}
&= -\frac{\mu}{v_\parallel}\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}B \\
&\quad -\frac{e}{m}\frac{1}{v_\parallel}\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}\phi \quad \text{(identically zero)} \\
&\approx -\frac{\mu}{v_\parallel}\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}B
\end{aligned}$$

This is also written

$$\begin{aligned}
\left(\mathbf{v}_E\cdot\boldsymbol{\nabla}\right) v_\parallel &= -\frac{\mu}{v_\parallel}\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}B = -\frac{\mu B}{v_\parallel}\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}\ln B \\
&= -\frac{\mu B}{v_\parallel}\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\kappa}
\end{aligned}$$

must be corrected ? no

Note that

$$\left(\mathbf{v}_E\cdot\boldsymbol{\nabla}\right) v_\parallel \approx 0 \qquad\qquad \text{(in this order)}$$

is an important part of the analytical form written by **Galeev Sagdeev 1968** for the term

$$\frac{dv_\parallel}{dt}\frac{\partial f_j}{\partial v_\parallel}$$

in their drift-kinetic equation with which they calculate the *trapped* and *circulating* distribution functions.

End.

Note

In **Hirshman Sigmar p1109**

$$\nabla_\parallel \left(\frac{v_\parallel^2}{2}\right)\Bigg|_{\epsilon,\mu} = -\mu \nabla_\parallel B$$

The density of energy in the parallel motion is $v_\parallel^2/2$; the gradient of the density of energy is a *force.*

This is $-\mu\nabla_\parallel B$, the force acting along the magnetic field line, the *mirror force.*

End

(3) The third term

$$\begin{aligned}
& (\mathbf{v}_D \cdot \boldsymbol{\nabla}) v_\parallel \\
= \quad & \left\{\frac{1}{\Omega_{ci}}\widehat{\mathbf{n}} \times \left[\mu \boldsymbol{\nabla} B + v_\parallel^2 (\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla})\widehat{\mathbf{n}}\right]\cdot\boldsymbol{\nabla}\right\}\sqrt{2}\left(\epsilon - \mu B - \frac{e}{m}\phi\right)^{1/2} \\
= \quad & \frac{1}{\Omega_{ci}}\widehat{\mathbf{n}} \times \left[\mu \boldsymbol{\nabla} B + v_\parallel^2 (\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla})\widehat{\mathbf{n}}\right]\sqrt{2}\frac{1}{2}\frac{1}{\left(\epsilon - \mu B - \frac{e}{m}\phi\right)^{1/2}}\cdot\boldsymbol{\nabla}\left(-\mu B - \frac{e}{m}\phi\right) \\
= \quad & \frac{1}{\Omega_{ci}}\widehat{\mathbf{n}} \times \left[\mu \boldsymbol{\nabla} B + v_\parallel^2 (\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla})\widehat{\mathbf{n}}\right]\frac{1}{v_\parallel}\cdot\boldsymbol{\nabla}\left(-\mu B - \frac{e}{m}\phi\right)
\end{aligned}$$

This consists of $2\times 2 = 4$ terms.

$$\frac{1}{\Omega_{ci}}\widehat{\mathbf{n}} \times [\mu \boldsymbol{\nabla} B]\frac{1}{v_\parallel}\cdot\boldsymbol{\nabla}(-\mu B) \tag{3.1}$$

$$+\frac{1}{\Omega_{ci}}\widehat{\mathbf{n}} \times [\mu \boldsymbol{\nabla} B]\frac{1}{v_\parallel}\cdot\boldsymbol{\nabla}\left(-\frac{e}{m}\phi\right) \tag{3.2}$$

$$+\frac{1}{\Omega_{ci}}\widehat{\mathbf{n}} \times \left[v_\parallel^2 (\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla})\widehat{\mathbf{n}}\right]\frac{1}{v_\parallel}\cdot\boldsymbol{\nabla}(-\mu B) \tag{3.3}$$

$$+\frac{1}{\Omega_{ci}}\widehat{\mathbf{n}} \times \left[v_\parallel^2 (\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla})\widehat{\mathbf{n}}\right]\frac{1}{v_\parallel}\cdot\boldsymbol{\nabla}\left(-\frac{e}{m}\phi\right) \tag{3.4}$$

Now we take separately the contributions to the third term.

(3.1) the first line in the formula

$$\frac{1}{\Omega_{ci}}\widehat{\mathbf{n}}\times\mu\boldsymbol{\nabla}B\cdot\frac{1}{v_{\parallel}}\left(-\mu\right)\boldsymbol{\nabla}B=0$$

(mixed product is zero).

(3.2) The second contribution to the third term

$$\begin{aligned}
&\frac{1}{\Omega_{ci}}\widehat{\mathbf{n}}\times\left[\mu\boldsymbol{\nabla}B\right]\frac{1}{v_{\parallel}}\cdot\boldsymbol{\nabla}\left(-\frac{e}{m}\phi\right)\\
=&\left(-\frac{e}{m}\right)\frac{1}{\Omega_{ci}}\frac{1}{v_{\parallel}}\mu\left(\widehat{\mathbf{n}}\times\boldsymbol{\nabla}B\right)\cdot\boldsymbol{\nabla}\phi\\
=&-\frac{\mu}{v_{\parallel}}\frac{\widehat{\mathbf{n}}\times\boldsymbol{\nabla}B}{B}\cdot\boldsymbol{\nabla}\phi
\end{aligned}$$

This is identical to a previous contribution in $\left(\mathbf{v}_E\cdot\boldsymbol{\nabla}\right)v_{\parallel}$. It can also be written

$$\frac{\mu}{v_{\parallel}}\left(-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}\right)\cdot\boldsymbol{\nabla}\ln B$$

which emphasize the factor

$$\boldsymbol{\nabla}\ln B\approx\frac{1}{R}\left(-\widehat{\mathbf{e}}_R\right)=\boldsymbol{\kappa}$$

From this, the contribution can be written

$$\frac{\mu B}{v_{\parallel}}\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\kappa}$$

and is the projection of the *electric* drift velocity $\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}$ along the curvature vector $\boldsymbol{\kappa}$.

Note
this is

$$\begin{aligned}
&\frac{1}{\Omega_{ci}}\widehat{\mathbf{n}}\times\mu\boldsymbol{\nabla}B\cdot\frac{1}{v_{\parallel}}\boldsymbol{\nabla}\left(-\frac{e}{m}\phi\right)\\
=&\mu\frac{1}{v_{\parallel}}\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}B=\mu B\frac{1}{v_{\parallel}}\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}\ln B
\end{aligned}$$

End

(3.3) The third contribution to the third term

$$\begin{aligned}
& \frac{1}{\Omega_{ci}}\widehat{\mathbf{n}}\times\left[v_\parallel^2\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\frac{1}{v_\parallel}\cdot\boldsymbol{\nabla}\left(-\mu B\right) \\
= & \frac{1}{\Omega_{ci}}v_\parallel\left(-\mu\right)\left\{\widehat{\mathbf{n}}\times\left[\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\right\}\cdot\boldsymbol{\nabla}B
\end{aligned}$$

We note that $\boldsymbol{\kappa}=\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\approx-\frac{1}{R}\widehat{\mathbf{e}}_R$ and $\widehat{\mathbf{n}}\times\left[\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\sim\widehat{\mathbf{n}}\times\widehat{\mathbf{e}}_R\sim\widehat{\mathbf{e}}_z$; the other factor is $\boldsymbol{\nabla}B\approx R_0\boldsymbol{\nabla}\left(\frac{B_0}{R}\right)=B_0R_0\left(-\frac{1}{R^2}\right)\boldsymbol{\nabla}R$. Here $\boldsymbol{\nabla}R\sim\widehat{\mathbf{e}}_R$, then $\left\{\widehat{\mathbf{n}}\times\left[\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\right\}\cdot\boldsymbol{\nabla}B\sim\left(\widehat{\mathbf{n}}\times\widehat{\mathbf{e}}_R\right)\cdot\widehat{\mathbf{e}}_R=0$ contains an identical vector factor in the mixed product and it is zero.

Note
this is

$$\frac{1}{\Omega_{ci}}\widehat{\mathbf{n}}\times\left[v_\parallel^2\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\frac{1}{v_\parallel}\cdot\left(-\mu\right)\boldsymbol{\nabla}B$$

and we see that

$$\begin{aligned}
\widehat{\mathbf{n}}\times\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}} & \approx \widehat{\mathbf{n}}\times\frac{-\widehat{\mathbf{e}}_R}{R} \\
& = \frac{1}{R}\widehat{\mathbf{e}}_z
\end{aligned}$$

from where we find

$$\frac{v_\parallel^2}{\Omega_{ci}}\frac{1}{v_\parallel}\frac{1}{R}\widehat{\mathbf{e}}_z\left(-\mu\right)\boldsymbol{\nabla}B\sim\frac{v_\parallel}{\Omega_{ci}}\frac{1}{R}\left(-\mu\right)\left(\widehat{\mathbf{e}}_z\cdot\widehat{\mathbf{e}}_R\right)\frac{1}{R}\approx 0$$

and this term does not have a contribution.
End

(3.4) The fourth contribution to the third term

$$\begin{aligned}
& \frac{1}{\Omega_{ci}}\left\{\widehat{\mathbf{n}}\times\left[v_\parallel^2\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\frac{1}{v_\parallel}\cdot\boldsymbol{\nabla}\right\}\left(-\frac{e}{m}\phi\right) \\
= & \frac{1}{\frac{eB}{m_i}}v_\parallel\left(-\frac{e}{m}\right)\left\{\widehat{\mathbf{n}}\times\left[\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\right\}\cdot\boldsymbol{\nabla}\phi \\
= & -\frac{1}{B}v_\parallel\left\{\widehat{\mathbf{n}}\times\left[\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\right\}\cdot\boldsymbol{\nabla}\phi
\end{aligned}$$

or

$$
\begin{aligned}
& -\frac{1}{B}v_{\|}\left\{\widehat{\mathbf{n}}\times\left[\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\right\}\cdot\boldsymbol{\nabla}\phi \\
= & \frac{v_{\|}}{B}\left(-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}\right)\cdot\left[\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right] \\
= & v_{\|}\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\kappa}
\end{aligned}
$$

The electric velocity $v_E = \frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}$ is projected along the curvature vector $\boldsymbol{\kappa}$.

Note
This is

$$
\begin{aligned}
& \frac{1}{\Omega_{ci}}\widehat{\mathbf{n}}\times v_{\|}^2\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\frac{1}{v_{\|}}\cdot\boldsymbol{\nabla}\left(-\frac{e}{m}\phi\right) \\
= & v_{\|}\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\left[\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right] \\
= & v_{\|}\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\kappa}
\end{aligned}
$$

End
Note
Then the term $-\frac{1}{B}v_{\|}\left\{\widehat{\mathbf{n}}\times\left[\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\right\}\cdot\boldsymbol{\nabla}\phi$ (the fourth in 3, *i.e.* 3.4) we want to find is

$$
\begin{aligned}
(3.4) \quad = & -\frac{1}{B}v_{\|}\left\{\widehat{\mathbf{n}}\times\left[\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\right\}\cdot\boldsymbol{\nabla}\phi \\
= & -\frac{1}{B}v_{\|}\left[\widehat{\mathbf{n}}\times\frac{\boldsymbol{\nabla}B}{B}\right]\cdot\boldsymbol{\nabla}\phi
\end{aligned}
$$

which can be arranged as follows:

$$
v_{\|}\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\frac{1}{B}\cdot\boldsymbol{\nabla}B
$$

End

Now we collect the results from detailed expressions of the term $\mathbf{v}_D\cdot\boldsymbol{\nabla}v_{\|}$. It is

$$
\begin{aligned}
& (3) \\
= & (3.1) + (3.2) + (3.3) + (3.4)
\end{aligned}
$$

or

$$
\begin{aligned}
&0 && (3.1)\\
&+\frac{\mu B}{v_\parallel}\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\kappa} && (3.2)\\
&+0 && (3.3)\\
&+v_\parallel\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\kappa} && (3.4)
\end{aligned}
$$

Note that and we rewrite this as

$$\mathbf{v}_D\cdot\boldsymbol{\nabla}v_\parallel=\left(v_\parallel+\frac{\mu B}{v_\parallel}\right)\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\kappa}$$

End

And now we put together the terms

$$\mathbf{v}\cdot\boldsymbol{\nabla}v_\parallel=(1)+(2)+(3)$$

Finally we have

$$
\begin{aligned}
\mathbf{v}\cdot\boldsymbol{\nabla}v_\parallel &= \left(\widehat{\mathbf{n}}v_\parallel+\mathbf{v}_E+\mathbf{v}_D\right)\cdot\boldsymbol{\nabla}v_\parallel\\
&= \widehat{\mathbf{n}}v_\parallel\cdot\boldsymbol{\nabla}v_\parallel \quad \text{this is} \quad -\mu\nabla_\parallel B-\frac{e}{m}\nabla_\parallel\phi\\
&\quad +\mathbf{v}_E\cdot\boldsymbol{\nabla}v_\parallel \quad \text{this is} \quad -\frac{\mu B}{v_\parallel}\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\kappa}\\
&\quad +\mathbf{v}_D\cdot\boldsymbol{\nabla}v_\parallel \quad \text{this is} \quad \left(v_\parallel+\frac{\mu B}{v_\parallel}\right)\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\kappa}
\end{aligned}
$$

or

$$\mathbf{v}\cdot\boldsymbol{\nabla}v_\parallel=-\mu\nabla_\parallel B-\frac{e}{m}\nabla_\parallel\phi+v_\parallel\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\kappa}$$

Now we can build the convective derivative of $v_\parallel$,

$$
\begin{aligned}
\frac{dv_\parallel}{dt} &= \left[\frac{\partial}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})\right]v_\parallel\\
&= \mathbf{v}\cdot\boldsymbol{\nabla}v_\parallel\\
&= -\mu\nabla_\parallel B-\frac{e}{m}\nabla_\parallel\phi+v_\parallel\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\frac{1}{B}\cdot\boldsymbol{\nabla}B
\end{aligned}
$$

This will be expressed in terms of the two variables $\left(v_{\perp}^{2}/2, v_{\|}\right)$ as follows

$$\frac{dv_{\|}}{dt} = -\left(\frac{v_{\perp}^{2}}{2}\right)\nabla_{\|}\ln B + v_{\|}\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}\ln B - \frac{e}{m}\nabla_{\|}\phi$$

This is one of the equations for the variation of the velocity variables with the particle motion.

The other equation regards $v_{\perp}^{2}/2$.

$$\begin{aligned}
\frac{d}{dt}\left(\frac{v_{\perp}^{2}}{2}\right) &= \left[\frac{\partial}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\right]\left(\frac{v_{\perp}^{2}}{2}\right) \\
&= \left(\widehat{\mathbf{n}}v_{\|} + \mathbf{v}_{E} + \mathbf{v}_{D}\right)\cdot\boldsymbol{\nabla}(\mu B) \\
&= v_{\|}\left(\frac{v_{\perp}^{2}}{2}\right)\frac{1}{B}\nabla_{\|}B + \\
&\quad + \left(\frac{v_{\perp}^{2}}{2}\right)\frac{1}{B}\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}B
\end{aligned}$$

$$\frac{d}{dt}\left(\frac{v_{\perp}^{2}}{2}\right) = v_{\|}\left(\frac{v_{\perp}^{2}}{2}\right)\frac{1}{B}\nabla_{\|}B + \left(\frac{v_{\perp}^{2}}{2}\right)\frac{1}{B}\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}B$$

and using the notation of **Wong Burrell 1982**

$$\lambda \equiv \frac{v_{\perp}^{2}}{2} \quad \text{(Wong Burrell)}$$

we write

$$\frac{d\lambda}{dt} = \lambda v_{\|}\boldsymbol{\nabla}_{\|}\ln B + \lambda\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}\ln B$$

Finally the two equations are

$$\begin{aligned}
\frac{dv_{\|}}{dt} &= -\frac{v_{\perp}^{2}}{2}\nabla_{\|}\ln B + v_{\|}\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}\ln B - \frac{e}{m}\nabla_{\|}\phi \\
\frac{d}{dt}\left(\frac{v_{\perp}^{2}}{2}\right) &= \frac{v_{\perp}^{2}}{2}v_{\|}\boldsymbol{\nabla}_{\|}\ln B + \frac{v_{\perp}^{2}}{2}\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}\ln B
\end{aligned}$$

NOTE regarding the derivation of the two equations for the variation of the velocity parameters along the orbits: $v_{\|}$ and $\lambda \equiv v_{\perp}^{2}/2$.

It was based on the hypothesis that there is **no change in the energy of the particle**. No acceleration of the particles in the toroidal direction.

No explicit variation with time of the velocities due to the explicit variation in time of the potential ϕ.

This variation $\phi(t)$ should now be considered.

End.

6.0.4 Include the explicit time derivative

We return to the equations of variation of the parallel and perpendicular velocities.

$$
\begin{aligned}
\frac{dv_{\|}}{dt} &= \left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\boldsymbol{\nabla}\right)v_{\|} \\
&= \frac{\partial v_{\|}}{\partial t} \\
&\quad +(\mathbf{v}\cdot\boldsymbol{\nabla})\,v_{\|} \\
&= \frac{\partial}{\partial t}\sqrt{2}\left(\epsilon-\mu B-\frac{e}{m}\phi\right)^{1/2} \\
&\quad -\left(\frac{v_{\perp}^2}{2}\right)\nabla_{\|}B+v_{\|}\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}\ln B-\frac{e}{m}\nabla_{\|}\phi
\end{aligned}
$$

The explicit time derivative acts only on ϕ.

$$
\begin{aligned}
\frac{\partial}{\partial t}\sqrt{2}\left(\epsilon-\mu B-\frac{e}{m}\phi\right)^{1/2} &= \sqrt{2}\frac{1}{2}\frac{1}{\left(\epsilon-\mu B-\frac{e}{m}\phi\right)^{1/2}}\frac{\partial}{\partial t}\left(\epsilon-\mu B-\frac{e}{m}\phi\right) \\
&= \frac{1}{v_{\|}}\frac{\partial}{\partial t}\left(-\frac{e}{m}\phi\right)=\left(-\frac{e}{m}\right)\frac{1}{v_{\|}}\frac{\partial\phi}{\partial t}
\end{aligned}
$$

Then the final form of the equation for the parallel component of the velocity is

$$
\begin{aligned}
\frac{dv_{\|}}{dt} &= \left(-\frac{e}{m}\right)\frac{1}{v_{\|}}\frac{\partial\phi}{\partial t} \\
&\quad -\left(\frac{v_{\perp}^2}{2}\right)\nabla_{\|}B+v_{\|}\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}\ln B-\frac{e}{m}\nabla_{\|}\phi
\end{aligned}
$$

The other equation,

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{v_{\perp}^2}{2}\right) &= \frac{d}{dt}(\mu B) \\
&= \left[\frac{\partial}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})\right](\mu B)
\end{aligned}
$$

is not modified by considering the explicit time variation of the electric potential, since there is no explicit time variation of the magnetic field.

For a circular geometry, when the potential can be separated into a main part (constant on magnetic surface) and a part that depends on (r,θ),

$$
\phi=\phi_0(r)+\phi_1(r,\theta)
$$

$$
\begin{aligned}
\frac{dx_\theta}{dt} &= v_\parallel \frac{B_\theta}{B_T} + \frac{1}{B_0}\frac{d\phi_0}{dr} \\
\frac{dx_r}{dt} &= -\frac{1}{B_0}\frac{\partial\phi_1}{r\partial\theta} - \frac{1}{\Omega_{ci}}\frac{v_\perp^2/2 + v_\parallel^2}{R}\sin\theta \\
\frac{d}{dt}\left(\frac{v_\perp^2}{2}\right) &= \left(\frac{v_\perp^2}{2}\right) v_\parallel \frac{B_\theta}{B_T}\frac{\sin\theta}{R} + \left(\frac{v_\perp^2}{2}\right)\frac{1}{B_0}\left(\frac{d\phi_0}{dr}\right)\frac{\sin\theta}{R} \\
\frac{dv_\parallel}{dt} &= -\left(\frac{v_\perp^2}{2}\right)\frac{B_\theta}{B_T}\frac{\sin\theta}{R} + v_\parallel \frac{1}{B_0}\left(\frac{d\phi_0}{dr}\right)\frac{\sin\theta}{R} - \frac{e}{m}\frac{B_\theta}{B_T}\frac{\partial\phi_1}{r\partial\theta}
\end{aligned}
$$

This is **Wong Burrell 1982**.

The drift-kinetic equation is written as

$$
\frac{\partial f}{\partial t} + \boldsymbol{\nabla}\cdot\left(\frac{d\mathbf{x}}{dt}f\right) + \frac{\partial}{\partial v_\parallel}\left(\frac{dv_\parallel}{dt}f\right) + \frac{\partial}{\partial\left(v_\perp^2/2\right)}\left(\frac{d\left(v_\perp^2/2\right)}{dt}f\right) = C\left(f,f\right)
$$

It is remarked that the volume in phase space is NOT preserved by the equations of motion

$$
\begin{aligned}
&\boldsymbol{\nabla}\cdot\frac{d\mathbf{x}}{dt} + \frac{\partial}{\partial v_\parallel}\left(\frac{dv_\parallel}{dt}\right) + \frac{\partial}{\partial\left(v_\perp^2/2\right)}\left(\frac{d\left(v_\perp^2/2\right)}{dt}\right) \\
= \;& \frac{1}{B^2}\left(\boldsymbol{\nabla}\times\mathbf{B}\right)\cdot\boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\cdot\mathbf{v}_D
\end{aligned}
$$

This is not so large and later it will be neglected.

From **Wong Burrell** we have

$$
\mathbf{v}_D = \frac{1}{\Omega}\left(v_\parallel^2 + \frac{v_\perp^2}{2}\right)\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\ln B
$$

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{v}_D &= \frac{m}{e}\left(v_\parallel^2 + \frac{v_\perp^2}{2}\right)\frac{\mathbf{J}\cdot\boldsymbol{\nabla}B}{B^3} \\
&\approx \frac{\varepsilon^2 v_D}{r}
\end{aligned}
$$

and

$$
\frac{J_\theta}{B} \sim \varepsilon^2\frac{1}{r}
$$

This divergence of the drift velocity is neglected.

Then the drift kinetic equation is

$$
\frac{d\mathbf{x}}{dt}\cdot\boldsymbol{\nabla}f + \frac{dv_\parallel}{dt}\frac{\partial f}{\partial v_\parallel} + \left[\frac{d}{dt}\left(\frac{v_\perp^2}{2}\right)\right]\frac{\partial f}{\partial\left(v_\perp^2/2\right)} = C\left(f,f\right)
$$

and an expansion is made

$$\begin{aligned} f &= f_0 + f_1 + ... \\ &\quad \text{(multiple time-scale expansion)} \end{aligned}$$

$$\frac{dx_\theta}{dt}\frac{\partial f_0}{r\partial\theta} = C\left(f_0, f_0\right)$$

or

$$\frac{dx_\theta}{dt} = \frac{d\left(r\theta\right)}{dt}$$

$$f_0 = \frac{1}{\left[\pi\left(\frac{2T}{m}\right)\right]^{3/2}} n \ \exp\left[-\frac{\left(v_\parallel - U\right)^2}{(2T/m)} - \frac{v_\perp^2}{(2T/m)}\right]$$

where $U \equiv$parallel flow velocity

$$\begin{aligned} & \left(v_\parallel + \frac{1}{B_\theta}\frac{d\phi_0}{dr}\right)\frac{B_\theta}{B_T}\frac{\partial f_1}{r\partial\theta} \\ & + \left(v_\parallel - U\right)\frac{e}{T}\frac{B_\theta}{B_T}\frac{\partial\phi_1}{r\partial\theta} f_0 - C^{lin}\left(f_1\right) \\ = \ & \varepsilon\frac{B_\theta}{B_T}\frac{1}{r}\left\{\left[\frac{v_\perp^2}{(2T/m)}\left(U + \frac{1}{B_\theta}\frac{d\phi_0}{dr}\right) + 2\frac{v_\parallel\left(v_\parallel - U\right)}{(2T/m)}\frac{1}{B_\theta}\frac{d\phi_0}{dr}\right]\sin\theta\right. \\ & + \frac{T}{eB_\theta}\left[\frac{v_\parallel^2 + \frac{v_\perp^2}{2}}{(T/m)}\sin\theta + \frac{1}{\varepsilon}\frac{e}{T}\frac{\partial\phi_1}{\partial\theta}\right] \times \\ & \qquad \left.\times\left[\frac{d}{dr}\ln n + \left(\frac{\left(v_\parallel - U\right)^2}{(2T/m)} + \frac{v_\perp^2}{(2T/m)} - \frac{3}{2}\right)\frac{d}{dr}\ln T + \frac{\left(v_\parallel - U\right)\frac{dU}{dr}}{(T/m)}\right]\right\} \end{aligned}$$

Wong Burrell consider that the first term in the RHS is related with the parallel velocity and with $E \times B$ velocity, in the inhomogeneous magnetic field.

They are called *magnetic pumping* terms.

The other terms in the RHS are due to the drift: curvature and gradient.

The first order distribution function for the species α is

$$\begin{aligned} f_{\alpha 1} &= h_{\alpha 1}\cos\theta \\ \frac{e\phi_1}{T} &= \varepsilon W\cos\theta \end{aligned}$$

$$W \equiv \frac{\sum_\alpha Z_\alpha n_\alpha m_\alpha U^2}{\sum_\alpha Z_\alpha^2 n_\alpha T}$$

or

$$W \equiv \frac{\sum_{\alpha} [Z_\alpha n_\alpha m_\alpha] U^2}{\sum_{\alpha} Z_\alpha \ [Z_\alpha n_\alpha m_\alpha] (T/m_\alpha)}$$

is a ratio of the parallel energy U^2 weighted by the charge Z_α to the thermal energy (T/m_α) wighted by the charge.

Where

$$h_{\alpha 1} = \varepsilon \left(\frac{v_\parallel U}{T/m_\alpha} - Z_\alpha \ W \right) f_{\alpha 0}$$

Interesting.

The correction to the zeroth distribution function (a $U-$shifted Maxwellian) is $\sim \varepsilon$ (toroidality) $\times$ (parallel flow).

Constraint in this order

$$\frac{1}{B_\theta} \frac{d\phi_0}{dr} + U = 0$$

well known resonance condition of *parallel* flow.

The electrostatic potential ϕ_0 results from ambipolarity condition.

From Wang Burrell

"The magnetic pumping terms are typically larger than the magnetic drift terms by a factor

$$\frac{L_n}{\rho_\theta}$$

for each species"

The *magnetic pumping* is the variation of the magnitude of the magnetic field along the line, due to the toroidality.

The *magnetic drift* is

$$\mathbf{v}_D^{mag} = \frac{1}{\Omega} \widehat{\mathbf{n}} \times (\mu \boldsymbol{\nabla} B)$$

with neglect of the curvature drift.

The group of terms in the equation for the correction to the distribution function

$$\sin\theta \times \epsilon \frac{B_\theta}{B_T} \frac{1}{r} \left[\frac{v_\perp^2}{v_{th}^2} \left(U + \frac{1}{B_\theta} \frac{d\phi_0}{dr} \right) + \frac{m v_\parallel \left(v_\parallel - U \right)}{T} \frac{1}{B_\theta} \frac{d\phi_0}{dr} \right]$$

arises from *parallel* and $E \times B$ *flow* and can be called magnetic pumping terms.

The second group of terms is

$$\begin{aligned}
&\epsilon \frac{B_\theta}{B_T} \frac{1}{r} \frac{T}{eB_\theta} \left[\frac{m}{T} \left(v_\parallel^2 + \frac{v_\perp^2}{2} \right) \sin\theta + \frac{1}{\epsilon} \frac{e}{T} \frac{\partial \phi_1}{\partial \theta} \right] \\
&\times \left[\frac{d \ln n}{dr} + \left(\frac{m \left(v_\parallel - U \right)^2}{2T} + \frac{m v_\perp^2}{2T} - \frac{3}{2} \right) \frac{d \ln T}{dr} + \frac{m \left(v_\parallel - U \right)}{T} \frac{dU}{dr} \right]
\end{aligned}$$

This comes from *curvature* and *gradient drifts*.

The second contains ϕ_1 which is the variation of the electrostatic potential in the magnetic surface, $\phi_1(\theta)$.

Now we want to make a comparison between the two terms.

For this we simplify $U = 0$, and take $\phi_1 \equiv 0$, and write

$$\begin{aligned}
& \sin\theta \times \left[\frac{v_\perp^2}{v_{th}^2} \frac{1}{B_\theta} \frac{d\phi_0}{dr} + \frac{m v_\parallel^2}{T} \frac{1}{B_\theta} \frac{d\phi_0}{dr} \right] \\
= \quad & \sin\theta \times \frac{1}{B_\theta} \frac{d\phi_0}{dr} \frac{m}{T} \left(\frac{v_\perp^2}{2} + v_\parallel^2 \right)
\end{aligned}$$

and the second group of terms

$$\sin\theta \times \frac{T}{eB_\theta} \frac{m}{T} \left(v_\parallel^2 + \frac{v_\perp^2}{2} \right) \frac{1}{L_n} \left[1 + \eta \left(\frac{v^2}{v_{th}^2} - \frac{3}{2} \right) \right]$$

and we recall that

$$\frac{T}{eB_\theta} \frac{1}{L_n} \left[1 + \eta \left(\frac{v^2}{v_{th}^2} - \frac{3}{2} \right) \right] = \frac{B}{B_\theta} v_T^{dia}$$

then

$$\sin\theta \times \frac{m}{T} \left(v_\parallel^2 + \frac{v_\perp^2}{2} \right) \frac{B}{B_\theta} v_T^{dia}$$

Now we have to compare

$$\begin{aligned}
& \frac{\sin\theta \times \frac{1}{B_\theta} \frac{d\phi_0}{dr} \frac{m}{T} \left(\frac{v_\perp^2}{2} + v_\parallel^2 \right)}{\sin\theta \times \frac{m}{T} \left(v_\parallel^2 + \frac{v_\perp^2}{2} \right) \frac{B}{B_\theta} v_T^{dia}} \\
= \quad & \frac{\frac{1}{B_\theta} \frac{d\phi_0}{dr}}{\frac{B}{B_\theta} v_T^{dia}} = \frac{\frac{1}{B} \frac{d\phi_0}{dr}}{v_T^{dia}}
\end{aligned}$$

then we have to compare the electric $E \times B$ velocity due to radial variation of the electric potential, with the diamagnetic velocity.

$$\frac{1}{B}\frac{d\phi_0}{dr} \sim v_T^{dia}$$

Or, for us the rotation velocity is greater than the diamagnetic velocity.

Wang Burrell asserts that the first group of terms is larger than the second group of terms as

$$\frac{L_n}{\rho_\theta}$$

It is difficult to understand why.

Stix and **Rosenbluth (Vienna)** show damping $\sim 1/\tau_{ii}$, highly efficient.

Now we advance beyond this level of approximation

$$f_{\alpha 1} = h_{\alpha 1}\cos\theta + g_{\alpha 1}$$

$$\frac{e\phi_1}{T} = e\ W\ \cos\theta + \widetilde{\phi}_1$$

$$\frac{1}{B_\theta}\frac{d\phi_0}{dr} + U = u$$

[**note** this kind of explicit assumption on the trigonometrical variation of f and ϕ on surface (i.e. with θ) is also in **Stacey**]

From now on the parallel velocity of plasma U and the parallel velocity from which the $E_r \times B_\theta$ (toroidal) has been removed, u, will occur in the formulas.

the next step is the change to the *moving referential*

$$\begin{aligned} v_\parallel &\rightarrow v'_\parallel = v_\parallel - U \\ w' &= \frac{1}{2}v'^2_\parallel + \frac{1}{2}v_\perp^2 \end{aligned}$$

The equation becomes

$$\begin{aligned} &\left(v'_\parallel + u\right)\frac{B_\theta}{B_\varphi}\frac{\partial g_{\alpha 1}}{r\partial\theta} \\ &+\left[v'_\parallel - \frac{T/m_\alpha}{\left(\frac{Z_\alpha e B_\theta}{m_\alpha}\right)}\left(\frac{d}{dr}\ln n_\alpha + \overline{\frac{w'}{T/m_\alpha} - \frac{3}{2}}\,\frac{d}{dr}\ln T + \frac{v'_\parallel}{T/m_\alpha}\frac{dU}{dr}\right)\right]\frac{B_\theta}{B_\varphi}Z_\alpha\frac{\partial\widetilde{\phi}_1}{r\partial\theta}f_{\alpha 0} \\ &-C^{lin}\left[g_{\alpha 1}\right] \\ = \ &\varepsilon\frac{1}{r}\frac{B_\theta}{B_\varphi}H_\alpha\left[u + \frac{T/m_\alpha}{\left(\frac{Z_\alpha e B_\theta}{m_\alpha}\right)}\left(\frac{d}{dr}\ln n_\alpha + \overline{\frac{w'}{T/m_\alpha} - \frac{3}{2}}\,\frac{d}{dr}\ln T + \frac{v'_\parallel}{T/m_\alpha}\frac{dU}{dr}\right)\right]f_{\alpha 0}\ \sin\theta \end{aligned}$$

with the notation

$$H_\alpha \equiv \frac{1}{T/m_\alpha}\left(v_\parallel'^2 + \frac{v_\perp^2}{2}\right) + \frac{U^2}{T/m_\alpha} - Z\ W + \frac{v_\parallel'}{T/m_\alpha}\frac{dU}{dr}$$

It is made the substitution

$$g_{\alpha 1}' = g_{\alpha 1} + Z_\alpha \widetilde{\phi}_1\ f_{\alpha 0}$$

After this, the neutrality condition is

$$\sum Z_\alpha^2 n_\alpha \widetilde{\phi}_1 = \sum Z_\alpha \int d^3v\ g_{\alpha 1}'$$

and the ambipolarity condition

$$\sum Z_\alpha \Gamma_\alpha = 0$$

which determine the velocity u, which is the perturbation that inhibits, in higher order, the resonance $\frac{1}{B_\theta}\frac{d\phi_0}{dr} + U = 0$.

The drift kinetic equation becomes

$$\begin{aligned} & v_\parallel' \frac{B_\theta}{B_\varphi}\frac{\partial}{r\partial\theta} g_{\alpha 1}' - C^{lin}\left[g_{\alpha 1}'\right] \\ = \ & \varepsilon \frac{B_\theta}{B_\varphi}\frac{1}{r} H_\alpha \left[u + \frac{T/m_\alpha}{\left(\frac{Z_\alpha e B_\theta}{m_\alpha}\right)}\left(\frac{d}{dr}\ln n_\alpha + \overline{\frac{w'}{T/m_\alpha} - \frac{3}{2}}\frac{d}{dr}\ln T + \frac{v_\parallel'}{T/m_\alpha}\frac{dU}{dr}\right)\right] f_{\alpha 0}\ \sin\theta \end{aligned}$$

The order of magnitude

$$\frac{g_{\alpha 1}'}{f_{\alpha 0}} \sim \varepsilon \frac{\rho_\theta}{L_n}$$

Solution of the equation

$$v_\parallel \frac{B_\theta}{B_T}\frac{\partial g_{\alpha 1}'}{r\partial\theta} + \nu g_{\alpha 1}' = A\left(v_\parallel', v_\perp^2\right) f_0 \sin\theta$$

is

$$g_{\alpha 1}' = \frac{r B_\varphi}{B_\theta} A\ f_0 \left[-P\left(\frac{1}{v_\parallel'}\right)\cos\theta + \pi\delta\left(v_\parallel'\right)\sin\theta\right]$$

The principal part to be take at the integration over the variable $v_\parallel$.

Similar to **Galeev** and to **Rozhansky Tendler**.

Note that this is *drift kinetic* and is NOT an instability kinetic equation. See **Rutherford 1970**.

What shows the formal solution.
It contains a $\delta\left(v_{\|}'\right)$.
Therefore the transport comes exclusively from the particles that correspond to the resonance condition

$$v_{\|}' = v_{\|} - U = 0$$

Using this expression for the distribution function $g_{\alpha 1}'$ one can calculate the fluxes of transport.
First for every species one introduces

$$z_\alpha \equiv \frac{U^2}{2T/m_\alpha} - \frac{1}{2} Z_\alpha \, W$$

then

$$a_\alpha = 1 + 2z_\alpha + 2z_\alpha^2$$
$$b_\alpha = \frac{3}{2} + z_\alpha - z_\alpha^2$$
$$c_\alpha = 3 + 4z_\alpha + 2z_\alpha^2$$

and

$$v_{th,\alpha} = \sqrt{\frac{2T}{m_\alpha}}$$

The transport fluxes are

$$\Gamma_\alpha = -n_\alpha \frac{T/m_\alpha}{\Omega_{0\alpha}} \frac{1}{R} \varepsilon \frac{1}{v_{th,\alpha}} \sqrt{\pi} \left[a_\alpha \left(u + \frac{T/m_\alpha}{\Omega_{\theta\alpha}} \frac{d}{dr} \ln n_\alpha \right) + b_\alpha \frac{T/m_\alpha}{\Omega_{\theta\alpha}} \frac{d}{dr} \ln T \right]$$

$$\frac{\widetilde{Q}}{T} =$$

$$\Pi = \sum_\alpha m_\alpha \Gamma_\alpha U$$

Particular case, only ions Z_i , m_i.

$$\Gamma_i = 0 \quad \text{the ambipolarity}$$

(neglect of electron fluxes)

$$U + \frac{1}{B_\theta} \frac{d\phi}{dr} + \frac{T/m_i}{\Omega_{\theta i}} \left(\frac{d}{dr} \ln n_i + K \frac{d}{dr} \ln T \right) = 0$$

where

$$K \equiv \frac{b_i}{a_i}$$

For no parallel flow $U = 0$

$$K \to \frac{3}{2}$$

It is also obtained the pertrubation of the potential

$$\begin{aligned}
\widetilde{\phi}_1 &= \frac{\pi\varepsilon}{1+Z_i}\frac{T/m_i}{\Omega_{\theta i}}\left[\frac{1}{\sqrt{\pi}}\left(\frac{1}{2} - z_i - (1+2z_i)K\right)\frac{d}{dr}\ln T\,\frac{1}{v_{th,i}} \times \sin\theta \right. \\
&\quad \left. + \frac{U}{T/m_i}\left(3\frac{d}{dr}\ln T - 2\frac{d}{dr}\ln U\right) \times \cos\theta\right]
\end{aligned}$$

The $\sin\theta$ part is most important.

Important case.

The presence of a single impurity species (I) beside the background ions and electrons.

The radial flux is

$$\begin{aligned}
\Gamma_I &= -\frac{\sqrt{\pi}}{4}q^2\omega_{tI}\,\rho_I^2 \times a_I \times \left[\left(\frac{d}{dr}\ln n_I - \frac{Z_I n_I}{Z_i n_i}\frac{d}{dr}\ln n_i\right)\right. \\
&\quad \left. - \left(\frac{3}{2}\frac{Z_I}{Z_i} - \frac{b_i}{a_i}\right) n_I \frac{d}{dr}\ln T\right]
\end{aligned}$$

Here one observes the *inward convection of impurities due to the gradient of temperature.*

The factor that contains the gradient of temperature

$$\begin{aligned}
-\left(\frac{3}{2}\frac{Z_I}{Z_i} - \frac{b_i}{a_i}\right) &\sim -\left(\frac{3}{2}\frac{Z_I}{Z_i} - \frac{3}{2}\right) \\
&\sim \text{negative, since } Z_I \gg Z_i
\end{aligned}$$

then

$$-\left(\frac{3}{2}\frac{Z_I}{Z_i} - \frac{b_i}{a_i}\right) n_I \frac{d}{dr}\ln T$$

is positive for normal T radial profile

Taking into account the sign $-$ of the overall Γ, it results

$$\Gamma < 0$$

toward the center of plasma

What is the physical reason for this? Follow the derivation of this last term $-\left(\frac{3}{2}\frac{Z_I}{Z_i}-\frac{b_i}{a_i}\right)n_I\frac{d}{dr}\ln T$.

See also *drift kinetic equation derivation.tex.*

Comment

From **Diamond instability ionization**.

The divergence of the *electric velocity* in the ion continuity equation produces poloidal mode coupling

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\left(n\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\right) &= n\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\times\left(\frac{\widehat{\mathbf{n}}}{B}\right) \\
&= n\boldsymbol{\nabla}\phi\cdot\left[\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\ln B+\widehat{\mathbf{n}}\times\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]
\end{aligned}$$

The two terms in the paranthesis: variation of the magnetic field and curvature. They produce neoclassical drifts of the ions.

End Comment

7 Rutherford 1970: electric field

The discussion on this work is in *drift-kinetic derivation.*

Axisymmetry, or conservation of the canonical angular momentum in toroidal geometry

$$\frac{d}{dt}\left[R\left(mv_{\|}+eA_{\varphi}\right)\right]=0$$

or

$$Rmv_{\varphi}+eRA_{\varphi}=\mathrm{const}$$

The derivation to time consists of only the convective part $(\partial/\partial t=0)$

$$m\left(v_{\|}\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\left[Rv_{\|}\right]+ev_{drift}\frac{\partial}{\partial r}\left[RA_{\varphi}\right]=0$$

For the magnetic field

$$\begin{aligned}
B_{\theta} &= \frac{B_{\theta 0}}{h} \\
B_{\varphi} &= \frac{B_{\varphi 0}}{h} \\
h &= 1+\varepsilon\cos\theta
\end{aligned}$$

For the magnetic flux part

$$RA_{\varphi} = -\int RB_{\theta} dr$$

from these formulas one obtains the drift velocity

$$\begin{aligned} v_{drift} &= \frac{m}{e} v_{\parallel} \left(\widehat{\mathbf{n}} \cdot \mathbf{\nabla}\right) \left(\frac{v_{\parallel}}{B_{\theta}}\right) \\ &= \frac{m}{e} v_{\parallel} \nabla_{\parallel} \left(\frac{v_{\parallel}}{B_{\theta}}\right) \end{aligned}$$

One must note that *the spatial derivatives are taken at constant* ϵ *and* μ.

When there is an electric field E_{φ} the energy ϵ has spatial variations.

For trapped particles

$$\begin{aligned} \epsilon_{trapped} &= \epsilon - \frac{e}{m} \int^{l} E dl \\ &= \text{const} \end{aligned}$$

where

$$dl = \frac{B}{B_{\theta}} r d\theta$$

It is defined an averaged drift velocity through the operation

$$\begin{aligned} \overline{v}_{D,r} \int \frac{dl}{v_{\parallel}} &= \int \frac{dl}{v_{\parallel}} v_{D,r} \\ &= \frac{m}{e} \int dl \left(\widehat{\mathbf{n}} \cdot \mathbf{\nabla}\right) \left(\frac{v_{\parallel}}{B_{\theta}}\right) \\ &= \frac{m}{e} \int dl \left[\left(\widehat{\mathbf{n}} \cdot \mathbf{\nabla}\right) \left(\frac{v_{\parallel}}{B_{\theta}}\right)_{\epsilon_T} - \frac{eE}{m} \frac{\partial}{\partial \epsilon_T} \left(\frac{v_{\parallel}}{B_{\theta}}\right) \right] \\ &= -\frac{E}{B_{\theta}} \int \frac{dl}{v_{\parallel}} \end{aligned}$$

This is the Ware drift of trapped banana orbits when a toroidal electric field E is applied.

For the collisional regime.

The *electron* distribution function $f_e \equiv f$ is expanded in series of powers of gyroradius.

$$v_{D,r} \frac{\partial f_0}{\partial r} + v_{\parallel} \left(\widehat{\mathbf{n}} \cdot \mathbf{\nabla}\right) f_1 + \frac{eE}{m} \frac{\partial f_0}{\partial v_{\parallel}} = C\left(f\right)$$

NOTE

The expansion of f is different of that adopted in **Rutherford1970 PF13**, where there are two series:

- one for neoclassic $\delta = \rho_\theta / L$, $f_0 + f_1 + f_2 + ...$
- one for collisions, ν_{ei}/ω_b , $g_1 + g_2 + ...$

END

Here f_1 includes neoclassic (shift due to the drift) and collisions (pitch angle and friction).

The surface average of the radial particle flux

$$\begin{aligned}
\Gamma_r &= \left\langle \int d^3v \; v_{d,r} f_1 \right\rangle \\
\text{with the definition } \langle A \rangle &= \int \frac{d\theta}{2\pi} h \; A \\
\text{and } d^3v &= 2\pi \frac{B}{v_\parallel} d\epsilon d\mu
\end{aligned}$$

Then

$$\begin{aligned}
\Gamma_r &= \left\langle \frac{m}{e} \int d^3v \; v_\parallel \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right) \left(\frac{v_\parallel}{B_\theta}\right) \right\rangle \\
&= -\left\langle \frac{m}{eB_\theta} \int d^3v \; v_\parallel^2 \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right) f_1 \right\rangle \\
&= \left\langle \frac{m}{eB_\theta} \int d^3v \; v_\parallel \left[\frac{eE}{m} \frac{\partial f_0}{\partial v_\parallel} - C\left(f_1\right)\right] \right\rangle
\end{aligned}$$

(after an integration by parts on θ from surface average). The term

$$\int d^3v \; v_{D,r} \frac{\partial f_0}{\partial r} = 0$$

due to the integration over $v_\parallel$ with the term being odd in $v_\parallel$.

Let us take

$$C\left(f_1\right) = \nu\left(v\right) v_\parallel \frac{\partial}{\partial \mu} \left(\mu \frac{v_\parallel}{B} \frac{\partial f_1}{\partial \mu}\right)$$

The hypothesis that *there is ZERO mean ion velocity* imposes the existence of an electrostatic potential. This potential will create a drift velocity equal and opposite to the diamagnetic velocity

$$ne \frac{\partial \Phi}{\partial r} = -T_i \frac{\partial n}{\partial r}$$

In the drift kinetic equation the derivatives are at constant ϵ.

$$\frac{\partial f_0}{\partial r} = f_0 \left(1 + \frac{T_i}{T_e}\right) \frac{1}{n} \frac{\partial n}{\partial r}$$

the solution of the drift kinetic equation is obtained in the lowest order by neglecting E and collisions.

$$\begin{aligned} f_1 &= -\frac{m}{eB_\theta} v_\parallel \frac{\partial f_0}{\partial r} + g \\ &= -\frac{v_\parallel}{\Omega_\theta} \frac{\partial f_0}{\partial r} + g \end{aligned}$$

where g has NO poloidal variation

$$\frac{\partial g^{(0)}}{\partial \theta} = 0$$

In the next order $g^{(1)}(\theta)$, it is made the averaging over θ, (on surface).

It is used the *periodicity* on θ.

This is the moment where the surface averages appear in the equation, $\langle v_\parallel \rangle$.

one obtains the equation that constraints the function $g \equiv g^{(0)}$:

$$-\frac{eE}{T_e} f_0 = \nu \frac{\partial}{\partial \mu} \mu \left(\langle v_\parallel \rangle \frac{\partial g}{B_0 \partial \mu} + \frac{m}{eB_{\theta 0}} \frac{\partial f_0}{\partial r} \right)$$

In the RHS we take the first derivation

$$\begin{aligned} \frac{1}{v} \left(-\frac{eE}{T_e} f_0 \right) &= \frac{\langle v_\parallel \rangle}{B_0} \frac{\partial g}{\partial \mu} + \frac{m}{eB_{\theta 0}} \frac{\partial f_0}{\partial r} \\ &\quad + \mu \frac{\partial}{\partial \mu} \left(\frac{\langle v_\parallel \rangle}{B_0} \frac{\partial g}{\partial \mu} + \frac{m}{eB_{\theta 0}} \frac{\partial f_0}{\partial r} \right) \end{aligned}$$

Divide the equation by $\frac{\langle v_\parallel \rangle}{B_0}$ to isolate $\frac{\partial g}{\partial \mu}$ and take the integral on μ between μ and the maximum value for circulating particles.

The solution which is well-behaved at $\mu = 0$ (which is $v_\perp = 0$, *i.e.* fully circulating),

$$g^{(0)} = f_0 \left[\frac{eE}{T_e} \frac{1}{\nu} + \frac{m}{eB_{\theta 0}} \frac{1}{n} \frac{\partial n}{\partial r} \left(1 + \frac{T_i}{T_e}\right) \right] \int_\mu^{\epsilon / B_{\max}} \frac{B_0 d\mu}{\langle v_\parallel \rangle}$$

for circulating particles

and

$$g = 0$$
$$\text{for trapped particles}$$

(The demonstration of the fact that $g_{1e}^{(0)}$ is *zero* in the trapped region is given by **Rutherford 1970 PF13**).

With this expression for g one constitutes the first order distribution function f_1, after adding the neoclassical $\sim \rho_\theta/L$ part.
With f_1 one returns to the expression of the radial particle flux, Γ_r.

Here, however, there will be an integration in velocity space over the collision operator which already contains a derivation to μ. Since f_1 also contains a derivation to μ, it is better to first perform an integration by part over μ, to avoid the occurence of second order derivative to μ. This would be singular. *Explanation.* In **Galeev Sagdeev** and **Berk Galeev** it is shown that there is *discontinuity* of the first derivative of the distribution function at the boundary between trapped/circulating. Then the double derivative would be singular.

$$\Gamma_r = -\int d^3v \left[\frac{nE}{B_\theta} + \frac{m}{e^2 B_\theta^2}\left(T_e + T_i\right)\nu\frac{\partial n}{\partial r}\right]\frac{\mu B_0}{n v_{th,e}^2} f_0 \left(1 - \frac{v_\parallel}{\langle v_\parallel\rangle}\right)$$

where

$$v_{th,e}^2 = \frac{T_e}{m_e}$$

The term $\frac{v_\parallel}{\langle v_\parallel\rangle}$ is taken into account only over CIRCULATING particles.
NOTE
See also **Connor1973** for this integral (containing the factor $\left(\frac{1}{v_\parallel} - \frac{1}{\langle v_\parallel\rangle}\right)$), attributed to **Hinton Oberman** and with a coefficient

$$-1.46\,\sqrt{\varepsilon} + ...$$

and in Hirshman bootstrap.
see below, related to the fraction of trapped
END.

(See also **Helander ECRH**).
The function

$$\langle v_\parallel\rangle$$

can be expressed in terms of elliptic integrals, as shown above.

The collisionality

$$\nu = \frac{1}{\tau_e} \times 3\sqrt{\frac{\pi}{2}} \left(\frac{v_{th,e}}{v}\right)^3$$

$$\frac{1}{\tau_e} = \frac{4\sqrt{2\pi}e^4}{3\sqrt{m_e}} \ln \Lambda \frac{n}{T_e^{3/2}}$$

After integrations over the velocity space

$$\begin{aligned} \Gamma_r &= -1.6\sqrt{\frac{r}{R}} \left[\frac{eE}{B_\theta} + \frac{\rho_\theta^2}{\tau_e}\frac{\partial n}{\partial r}\right] \\ &= \text{Ware flux } + \text{ banana diffusion flux} \end{aligned}$$

where

$$\rho_\theta^2 = \frac{m\left(T_e + T_i\right)}{e^2 B_\theta^2}$$

and the diffusion coefficient of the bananas is approximated as usual

$$D \sim \sqrt{\varepsilon}\frac{\rho_\theta^2}{\tau_e} \text{ (for bananas)}$$

NOTE

The factor

$$-1.46\sqrt{\varepsilon}$$

is mentioned by **Connor1973** and attributed to **Hinton Oberman** for the integral

$$\frac{1}{n_k}\sum_{\sigma=\pm 1}\iint 2\pi B \; d\mu \; dw \; \frac{\mu B}{T_k} \; m_k F_{M,k}\left(\frac{1}{\overline{v}_\parallel} - \frac{1}{v_\parallel}\right) = -1.46\;\sqrt{\varepsilon} \\ +O\left(\varepsilon\right)$$

$$\left(2\pi d\mu dw\right)\frac{v_\perp^2}{2T_k/m_k}B\left(\frac{1}{\langle v_\parallel\rangle} - \frac{1}{v_\parallel}\right)$$

The factor 1.46 is calculated in the fraction of trapped particles, f_t, in this text.

END

For the electric field one has to take

$$\begin{aligned} E &= \frac{E_0}{h} \\ \text{such that } \boldsymbol{\nabla}\times\mathbf{E} &= 0 \end{aligned}$$

to exclude time variation of the magnetic field, *i.e.* of the magnetic surfaces. In **Hinton Hazeltine** they are order 3.

8 Equations in imposed electric field (KH) Horton Tajima kamimura

The equations

$$\begin{aligned}
\frac{d\mathbf{R}_j}{dt} &= \frac{\mathbf{E}\times\mathbf{B}}{B^2} + v_{\parallel}\ \widehat{\mathbf{n}} + \frac{1}{\Omega_j B}\frac{d\mathbf{E}_{\perp}}{dt} \\
\frac{dv_{\parallel}}{dt} &= \frac{e_j}{m_j}\mathbf{E}\cdot\widehat{\mathbf{n}}
\end{aligned}$$

9 Equations for the particle's motion and for the variation of the velocities along the orbits

9.1 Notes

After derivation of the equations of motion for the guiding center (by averaging over the gyromotion) we should stop making confusions between the *guiding center* and the *charged particle*.

For example if we have an electric field and a GC particle we should not expect the GC to move in the sense of the electric field.

All intuitions about the *charged particle* are NOT valid for GC.

In the paper on **intrinsic rotation DIIID.**

Creation of a charged particle with only perpendicular velocity will lead to an average parallel flow.

The particle is created with only $\mathbf{v}\perp\mathbf{B}$. This means that there will be *gyration*. It is sufficient that the particle has gyration, without any other initial displacement for the particle to feel the space variation of the magnetic field magnitude $\boldsymbol{\nabla}B$. Then the drift exists

$$\widehat{\mathbf{n}}\times\mu\boldsymbol{\nabla}B$$

and this is perpendicular to the magnetic surface. It exists, for the *guiding center* a velocity with radial component $\mathbf{v}_r$. From this it results

$$\mathbf{v}_r\times\mathbf{B}_\theta \rightarrow \text{motion in toroidal direction}$$

The invariant is

$$Rmv_{\parallel} + eRA_{\varphi} = \text{const}$$

and means that the fact that the particle traverses the magnetic surfaces (equivalently, it changes A_φ) induces parallel velocity $v_\parallel$.

The invariants refer to the motion of the particle itself, not only the guiding center.

9.2 Expressions for the velocities

9.2.1 Neoclassical drift of the guiding centers

A formula for the drift of particles

$$\begin{aligned}\mathbf{v} &= \left(\frac{v_\parallel}{B}\right)\left[\mathbf{B}+\boldsymbol{\nabla}\times\left(\rho_\parallel\mathbf{B}\right)\right] \\ &= v_\parallel\widehat{\mathbf{n}}+\frac{v_\parallel}{B}\boldsymbol{\nabla}\times\left(\rho_\parallel\mathbf{B}\right)\end{aligned}$$

where

$$\rho_\parallel = \frac{v_\parallel}{\left(\frac{eB}{m}\right)}=\frac{v_\parallel}{\Omega_c}$$

a kind of Larmor radius of gyration

We have

$$\mathbf{v}=v_\parallel\widehat{\mathbf{n}}+\mathbf{v}_D$$

$$\begin{aligned}\mathbf{v}_D &= \frac{e}{m}\rho_\parallel\boldsymbol{\nabla}\times\left(\rho_\parallel\mathbf{B}\right) \\ &= \frac{e}{m}\rho_\parallel\left[\rho_\parallel\boldsymbol{\nabla}\times\mathbf{B}+\boldsymbol{\nabla}\rho_\parallel\times\mathbf{B}\right] \\ &= \frac{e}{m}\rho_\parallel^2\left(\mu_0\mathbf{j}\right) \\ &\quad -\frac{eB}{m}\rho_\parallel\left(\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\rho_\parallel\right)\end{aligned}$$

Ignoring for the moment the term depending on the current density and only retaining the second term, we have

$$\mathbf{v}_D=-v_\parallel\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\left(\frac{v_\parallel}{\Omega}\right)$$

which is one of the expression used frequently (**Rosenbluth Hinton** alphas).

9.2.2 Projection of drift velocity

In **Taguchi 1992 poloidal**

$$V_r = \frac{v_\parallel}{(e_a B_0/m_a)} \frac{\partial}{r\partial\theta}\left(h v_\parallel\right)$$

note this is for the kinetic equation

$$\begin{aligned} & v_\parallel \nabla_\parallel f_a^{(1)} - C_a\left[f_a^{(1)}\right] \\ = & -V_r \frac{\partial f_a^{(0)}}{\partial r} - v_\parallel \nabla_\parallel \left(\frac{e_a \Phi^{(1)}}{T_a}\right) f_a^{(0)} \end{aligned}$$

after which the substitution is made

$$\begin{aligned} f_a^{(1)} &= -\left(\frac{e_a \Phi^{(1)}}{T_a}\right) f_a^{(0)} \\ & +g_a \end{aligned}$$

The first part in the content of $f_a^{(1)}$ is determined by the electrostatic potential with poloidal variation, $\Phi^{(1)} \sim \theta$.

Here

$$\begin{aligned} & v_\parallel \nabla_\parallel g_a - C_a\left[g_a\right] \\ = & -V_r \frac{\partial f_a^{(0)}}{\partial r} \end{aligned}$$

When the radial flux is calculated averaged over the magnetic surface, one notice a cancelation of the terms that contain $\Phi^{(1)}$.

$$\begin{aligned} \Gamma_a &= \left\langle n_{a0} \left(\frac{-\boldsymbol{\nabla}\Phi^{(1)} \times \widehat{\mathbf{n}}}{B}\right)_{radial}\right\rangle + \left\langle \int d^3v \; V_r \; f_a^{(1)} \right\rangle \\ &= \left\langle \int d^3v \; V_r \; g_a \right\rangle \end{aligned}$$

end.

The radial part in the neoclassical drift

$$\mathbf{v}_D \cdot \boldsymbol{\nabla}\psi = v_\parallel \frac{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}{B} \frac{\partial}{\partial\theta}\left(\rho_\parallel \frac{\mathbf{B}\cdot\left(\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta\right)}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}\right)$$

This calculation can be further developed. We take into account that in the case of circular surface

$$\begin{aligned}
\frac{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}{B}\frac{\partial}{\partial\theta} &= \frac{B_\theta}{B_T}\frac{1}{r}\frac{\partial}{\partial\theta} \\
&= \frac{1}{qR}\frac{\partial}{\partial\theta} \\
&\simeq \frac{\partial}{\partial z}
\end{aligned}$$

which makes these factors to combine in

$$v_\parallel\frac{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}{B}\frac{\partial}{\partial\theta} = v_\parallel\frac{1}{qR}\frac{\partial}{\partial\theta}$$

from here we retain

$$\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta = \frac{1}{qR}$$

The other factor of the full expression requires the calculation of

$$\begin{aligned}
\frac{\mathbf{B}\cdot(\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\theta)}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta} &= \frac{\widehat{\mathbf{n}}\cdot(RB_\varphi)(-\widehat{\mathbf{e}}_r)\times(1/r)\widehat{\mathbf{e}}_\theta}{\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta} \\
&= \frac{1}{1/(qR)}RB_\varphi\frac{1}{r}\widehat{\mathbf{n}}\cdot\widehat{\mathbf{e}}_\varphi \\
&= (RB_\varphi)(qR)\frac{1}{r}\left(\frac{B_\theta}{B_\varphi}\right) \\
&= RB_\varphi
\end{aligned}$$

(check however the signs) and

$$\rho_\parallel\frac{\mathbf{B}\cdot(\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\theta)}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta} = \rho_\parallel(RB_\varphi)$$

Now we return to the initial expression of the projection of the drift velocity on the radial direction and we obtain

$$\begin{aligned}
\mathbf{v}_D\cdot\boldsymbol{\nabla}\psi &= v_\parallel\frac{1}{qR}\frac{\partial}{\partial\theta} \\
&\quad\times\left(\rho_\parallel RB_\varphi\right)
\end{aligned}$$

since (this will be denoted I)

$$\begin{aligned}
RB_\varphi &= R_0B_{0\varphi} = const \\
&\equiv I
\end{aligned}$$

it will get out of the derivation to θ,

$$\mathbf{v}_D \cdot \boldsymbol{\nabla}\psi = v_\parallel \frac{1}{qR} R B_\varphi \frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{\Omega}\right)$$

Introduce the expressions

$$I \equiv R B_\varphi$$

$$\frac{1}{qR}\frac{\partial}{\partial\theta} = \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta \frac{\partial}{\partial\theta}$$

We **note** that this formula consists of the usual projection from the parallel to the poloidal direction

$$\begin{aligned}
\frac{1}{qR}\frac{\partial}{\partial\theta} &= \frac{RB_\theta}{rB_T}\frac{1}{R}\frac{\partial}{\partial\theta} = \frac{B_\theta}{B_T}\frac{\partial}{r\partial\theta} \\
&= \nabla_\parallel
\end{aligned}$$

The expression can be re-written

$$\mathbf{v}_D \cdot \boldsymbol{\nabla}\psi = I v_\parallel \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta \frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{\Omega}\right)$$

and this is the expression used by **Rosenbluth Hinton** for alphas.

It is useful because allows to average explicitely the terms in the expansion of the distribution function over bounce.

We take into account that in the case of circular surface

$$\begin{aligned}
\frac{\mathbf{B} \cdot \boldsymbol{\nabla}\theta}{B}\frac{\partial}{\partial\theta} &= \frac{B_\theta}{B_T}\frac{1}{r}\frac{\partial}{\partial\theta} \\
&= \frac{1}{qR}\frac{\partial}{\partial\theta} \\
&\simeq \frac{\partial}{\partial z}
\end{aligned}$$

we obtain

$$\begin{aligned}
v_{D,r} &= \frac{v_\parallel}{Rq}\frac{\partial}{\partial\theta}\left(\rho_\parallel R B_\varphi\right) \\
&= I v_\parallel \frac{1}{qR}\frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{\Omega_{ci}}\right)
\end{aligned}$$

Separately

$$\frac{I}{qR} = \frac{RB_\varphi}{\frac{rB_\varphi}{RB_\theta}R} = RB_\theta = |\boldsymbol{\nabla}\psi|$$

then

$$|\boldsymbol{\nabla}\psi| = \frac{I}{qR}$$

We can take as variables ϵ = the total particle energy

$$\epsilon = \frac{mv_\perp^2}{2} + \frac{mv_\parallel^2}{2} + e\Phi$$

μ = the magnetic moment

$$\mu = \frac{mv_\perp^2}{2B}$$

and then, **for IONS**

$$v_\parallel = \left[\frac{2}{m}(\epsilon - \mu B - \Phi\right]^{1/2}$$

$$\begin{aligned}
\frac{\partial}{\partial\theta}v_\parallel &= \frac{1}{mv_\parallel}\left(-\mu\frac{\partial B}{\partial\theta} - |e|\,\frac{\partial\Phi}{\partial\theta}\right) \\
&= \frac{1}{mv_\parallel}\left(-\mu\frac{\varepsilon B_0}{(1+\varepsilon\cos\theta)^2}\sin\theta - |e|\,\frac{\partial\Phi}{\partial\theta}\right) \\
&= -\frac{1}{v_\parallel}\frac{v_\perp^2}{2}\varepsilon\sin\theta
\end{aligned}$$

The variation of the *parallel velocity* $v_\parallel$ is due to the variation of the modulus of the magnetic field along the magnetic field line as it turns from exterior to the interior of the torus.

The derivative of the magnetic field is

$$\begin{aligned}
\frac{\partial B}{\partial\theta} &= \frac{\partial}{\partial\theta}\left(\frac{B_0}{1+\varepsilon\cos\theta}\right) \\
&= \frac{\varepsilon B_0\sin\theta}{(1+\varepsilon\cos\theta)^2}
\end{aligned}$$

It results the known formula:

$$v_{D,r}^{ions} = \frac{1}{\Omega_i}\frac{v_\perp^2/2 + v_\parallel^2}{R}\sin\theta$$

The component of the neoclassical drift projected on the toroidal angle φ direction

$$\mathbf{v}_D\cdot\boldsymbol{\nabla}\varphi = -v_\parallel\frac{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}{B}\frac{\partial}{\partial\theta}\left(\rho_\parallel\frac{\mathbf{B}\cdot(\boldsymbol{\nabla}\theta\times\boldsymbol{\nabla}\varphi)}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}\right)$$

or, the *toroidal drift*,

$$\mathbf{v}_D \cdot \boldsymbol{\nabla}\varphi = -v_\parallel \frac{1}{qR}\frac{\partial}{\partial\theta}\left(\frac{\widehat{\mathbf{n}} \cdot (\boldsymbol{\nabla}\theta \times \boldsymbol{\nabla}\varphi)}{\frac{1}{qR}}\right)$$

If the vectorial product $\boldsymbol{\nabla}\theta \times \boldsymbol{\nabla}\varphi$ is $\frac{1}{rR}\widehat{\mathbf{e}}_r$ then the scalar product is zero and there are no component of the drift velocity on the direction of the gradient $\boldsymbol{\nabla}\varphi$. We must conclude that this vector-gradient is not transported along the line up to the current point.

This toroidal component of the drift velocity should represent the toroidal precession of the particles.

See **Mikhailovskii**.

$$\mathbf{v}_D = \frac{1}{\Omega}\mathbf{n}\times\left(\mu\boldsymbol{\nabla}B + v_\parallel^2\left(\mathbf{n}\cdot\boldsymbol{\nabla}\right)\mathbf{n} + \frac{e}{m}\phi\right)$$

(slow temporal variation of the radial electric field). And

$$\mathbf{v}_D = -v_\parallel \widehat{\mathbf{n}} \times \boldsymbol{\nabla}\left(\frac{v_\parallel}{\Omega}\right)$$

A formula written in general terms

$$\frac{d\xi_{1,2}}{dt} = \pm\frac{1}{\Omega}\sqrt{\frac{g_{33}}{g}}\frac{\left(\frac{\partial J}{\partial\xi_{2,1}}\right)}{\left(\oint\frac{d\zeta}{v_\parallel}\right)}$$

where $\xi_{1,2}$ are the coordinates defined by the magnetic line, g_{33} is the longitudinal component of the metric tensor, whose determinant is g. As coordinates, one can choose (r, φ_0).

A simple expression

$$\begin{aligned} v_{D\perp} &= \frac{1}{eB_0}\left(\frac{mv_\parallel^2}{R_0} + \frac{mv_\perp^2}{2R_0}\right) \\ &= \frac{1}{\Omega}\frac{v_\perp^2/2 + v_\parallel^2}{R_0} \quad \text{it is almost vertical} \end{aligned}$$

where the formula is used in [**?**]. This formula combines the *grad-B* drift and the *curvature* drift from the general expression of the drift velocity. The direction is approximately *vertical* which means parallel with the torus main axis of symmetry.

The drift of particles in the RADIAL (r) direction, averaged on the bounce is ZERO.

For *trapped particles* one can use

$$v_{\parallel} \ll v_{\perp}$$

which simplifies the expression for the curvature drift

$$v_{D\perp}^{trapped} = -\frac{1}{\Omega}\frac{v_{\perp}^{2}/2}{R_0}$$

with $e < 0$ for electrons. This velocity is directed vertically, almost parallel with the torus main symmetry axis. The projection on the radial direction, for the electrons

$$\left. v_{D,r}^{trapped} \right|_{electrons} = -\frac{v_{\perp}^{2}}{2R_0 \left|\Omega_e\right|} \sin\theta$$

This is often used as an equation of motion in the radial direction, for the trapped electrons

$$\dot{r} = v_{D,r}^{trapped} = -\frac{v_{\perp}^{2}}{2R_0 \left|\Omega_e\right|} \sin\theta \tag{3}$$

For exemple, for the ions at the temperature T_i, $v_{\perp}^{2} = \frac{2T_i}{m_i}$.

9.2.3 Parallel drift motion

The equation (valid for passing and trapped particles) of balance of forces acting upon the particle

$$m\frac{d\mathbf{v}}{dt} = e\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right) - \mu\boldsymbol{\nabla}B$$

then the parallel component is

$$m\frac{dv_{\parallel}}{dt} = eE_{\parallel} - \mu\frac{\partial B}{\partial\zeta}$$

(where $\zeta \equiv l_{\parallel}$ is a coordinate *along* the magnetic field, obtained from the *toroidal* angle) and, for

$$B = B_0\left(1 - \varepsilon\cos\left(k_0\zeta\right)\right)$$

with $k_0 = 1/qR_0$, and taking $E_{\parallel} = 0$ we have

$$\dot{v}_{\parallel} = -\frac{v_{\perp}^{2}}{2}\varepsilon k_0 \sin\theta \tag{4}$$

(**Note** that this suggests

$$k_0\zeta = \theta \quad \text{(poloidal angle)}$$

$$\begin{aligned}
\theta &= \frac{l_{\|}}{qR_0} \equiv \frac{\zeta}{qR_0} \quad \text{or} \\
r\theta &= \frac{B_\theta}{B_T} l_{\|} \\
\frac{r\theta}{l_{\|}} &= \frac{B_\theta}{B_T} \\
\frac{dl_\theta}{dl_{\|}} &= \frac{B_\theta}{B_T}
\end{aligned}$$

which is approximately correct).

Comparing (3) and (4) it results

$$r = r_0 + \frac{v_{\|}}{\Omega_\theta}$$

or

$$r = r_0 + \rho_\theta$$

as the radial displacement of the particles, due to the drift. The parallel velocity $v_{\|}$ in this formula is oscillatory for both passing and trapped particles. But for trapped it changes the sign.

In **Mikhailovskii** (see *trapped particles*) the toroidal drift is calculated, $v_{D,\varphi}$.

10 Trapped particles

The fraction is (**bootstrap ECRH Helander Hastie Connor**)

$$f_t = \sqrt{\varepsilon\,(1+\cos\theta)}$$

with

$$\langle f_t \rangle = \frac{2\sqrt{2\varepsilon}}{\pi}$$

In **Hirshman Sigmar Clarke 1976**, the fraction of trapped particles

$$\begin{aligned}
f_t &\sim \sqrt{\frac{\delta B}{B}} \\
\delta B &\equiv \text{the variation of the magnetic field along the line}
\end{aligned}$$

this can seen as the strength of the magnetic mirror.

In **Tang review instabilities**

$$\begin{aligned} \text{fraction of trapped at } \theta &= 0 \\ \sqrt{\frac{B_{\max}}{B_{\min}} - 1} &\approx \sqrt{2\varepsilon} \end{aligned}$$

and averaged over the magnetic surface is

$$f_T \approx \sqrt{\varepsilon}$$

The mean parallel velocity of the trapped particles

$$\sqrt{\varepsilon}\; v_{th,a}$$

Average bounce frequency

$$\omega_{b,a} \approx \frac{\sqrt{\varepsilon} v_{th}}{qR}$$

The drift velocity

$$v_{d,a} \approx v_{th,a} \frac{\rho_j}{R}$$

Half-width of a banana

$$\begin{aligned} \delta r &\approx \frac{v_{d,a}}{\omega_{b,a}} \\ &\sim \rho_a \frac{q}{\sqrt{\varepsilon}} \end{aligned}$$

Effective collision frequency

$$\nu_{eff,a} \sim \frac{\nu_a}{\varepsilon}$$

where $\nu_a \equiv$Spitzer frequency.

Remember the figure from **Peeters** on the fluxes of particles in velocity space. It is a collisional flux.

10.1 Banana regime

Condition for the banana regime

$$\Delta = \frac{\nu_{ii}}{\omega_b} \ll 1$$

where ν_{ii} is the ion-ion collision and ω_b is the bounce frequency. This is **the small parameter** of the banana regime.

The bounce of the trapped particle is much faster than the collisions.

i.e. *the banana orbits are visible*

Note in **Novakovskii** (*rotation.tex*) it is said that the trapped particles are much less collisional than the circulating ones. This is the reason:

1. the trapped particles are few.
2. the trapped particles have small parallel velocity

End.

Collisional diffusion **Frieman 1970**.
Rosenbluth Frieman Hazeltine.
Shaing trapping detrapping.

Region in the space of parameters

$$w = \frac{v^2}{2} = \epsilon - \frac{e\phi}{m}$$

$$\lambda = \frac{\mu}{w}$$

or

$$\lambda = \frac{v_\perp^2}{v^2} \frac{1}{B(\mathbf{x})}$$

We have

$$B(\mathbf{x}) = \frac{B_0}{h}$$

and, extracting a factor B_0,

$$\lambda \to \lambda' = \frac{v_\perp^2}{v^2} h$$

then, since here $\lambda = \frac{v_\perp^2}{v^2} \frac{1}{B(\mathbf{x})}$ we have

$$v_\| = \sigma \sqrt{2w(1 - \lambda B)}$$

We define

$$\begin{aligned}
\lambda_m &= \frac{1}{B(\mathbf{x})} \\
&= \text{the largest } \lambda \text{ for which the function} \\
& \quad\; f(\mathbf{x},\lambda,w) \text{ is defined} \\
\text{corresponds to } v_\perp^2 &= v^2 \text{ (no parallel velocity, deep trapped)}
\end{aligned}$$

and

$$\begin{aligned}
\lambda_c &= \frac{1}{B_{\max}} \\
& \quad \text{where} \\
B_{\max} &= \text{the maximum of } |\mathbf{B}| \text{ along a field line}
\end{aligned}$$

or

$$B_{\max} = \frac{B_0}{h_{\min}} = \frac{B_0}{1-(r/R_0)}$$

then

$$\lambda_c = \frac{1-(r/R_0)}{B_0}$$

and λ_c is the critical λ for trapping.

The *trapped region* (high $v_\perp^2$) is

$$\begin{aligned}
\lambda_c &< \lambda < \lambda_m \\
& \quad \text{trapped}
\end{aligned}$$

the *untrapped* (passing, circulating) region, small $v_\perp$,

$$\begin{aligned}
0 &< \lambda < \lambda_c \\
& \quad \text{passing}
\end{aligned}$$

All *circulating* particles have the perpendicular velocity sufficiently small (*i.e.* λ small) such that at the given energy the parallel velocity to be high enough for the particle to overcome the magnetic barrier along the line.

Then λ must be *small* for the particle to be *circulating.*

10.2 Motion of trapped particles as function of time

The equation of motion (**Galeev Sagdeev Rev. Plasma Phys. vol.7**)

$$\frac{rd\theta}{dt} = \sigma\frac{\varepsilon}{q}v\sqrt{\varepsilon}\sqrt{2\varkappa^2 - 1 + \cos\theta}$$

where

$$
\begin{aligned}
v^2 &= \frac{2}{m}\left[\epsilon - e\phi_0\left(r_0\right)\right] \\
2\varkappa^2 &= \frac{\left|\Delta v_{\|}\left(r_0, 0\right)\right|^2}{v^2\varepsilon} \\
\sigma &= \pm 1
\end{aligned}
$$

NOTE
The formula for κ^2 in **Fong Hahm** is based on *initial* values

$$
\kappa^2 = \frac{v_{\|0}^2}{2\varepsilon_a v_{\perp 0}^2}
$$

END

This equation is suitable only for trapped particles and slowly transiting particles, since use has been made of an expansion in terms of the **small deviation of the particle velocity from the value**

$$
\begin{aligned}
-\frac{v_\theta}{\varepsilon/q} &= -\frac{v_\theta}{\frac{r}{R}\frac{RB_\theta}{rB_T}} = -v_\theta\frac{B_T}{B_\theta} = -\frac{v_\theta}{\Theta} \\
&= \text{parallel projection of the poloidal velocity } v_0 \equiv v_\theta
\end{aligned}
$$

which requires

$$
\Delta v_{\|} \ll v_{th}
$$

Then the motion of trapped particles can be described in terms of elliptic functions with modulus

$$
\varkappa^2 < 1
$$

The **period of oscillation along the closed trajectory**

$$
\begin{aligned}
\tau &= \frac{4r}{\left(\varepsilon/q\right)\sqrt{2\varepsilon}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\varkappa^2 - \sin^2\theta/2}} \\
&= \frac{4\sqrt{2}r}{\left(\varepsilon/q\right)v\sqrt{\varepsilon}}\mathbf{K}\left(\varkappa\right) = 4\sqrt{\frac{2}{\varepsilon}}qR\,\mathbf{K}\left(\varkappa\right)
\end{aligned}
$$

where $K\left(\varkappa\right)$ is the complete elliptic integral of the first kind and θ_0 is the angle θ where the expression under the radical is zero

$$
\varkappa^2 - \sin^2\left(\frac{\theta}{2}\right) = 0
$$

It results that

$$
\varkappa < 1
$$

for a particle to be trapped. As explained several times, it is missing $1/\kappa$.

10.3 Trapped particle modes (Galeev Sagdeev Wong)

The paper **PF 10 (1967) 1535 Galeev Sagdeev Wong** shows that a radial electric field (plasma rotation) can stabilize the mode found by Kadomtsev Pogutse for the trapped particles.

The idea is that the electric field produces rotation and some trapped particles get higher parallel velocity and they become circulating. The mode which is based on trapped particles is then less sustained, possibly stabilized.

NOTE The argument is similar to the one advocated by **Yankov Nycander** for the suppression of the ITG.
END.

NOTE
In the zero order approximation the equations for the orbits are

$$\begin{aligned}
\frac{dr}{dt} &= 0 \\
\frac{rd\theta}{dt} &= -\frac{1}{q}\frac{r}{R_0}v_{\|} + \frac{1}{B_0}\frac{\partial \Phi^{(0)}}{\partial r} \\
\frac{R_0 d\varphi}{dt} &= v_{\|}
\end{aligned}$$

Note the set of equations in **Fong Hahm** is

$$\begin{aligned}
\frac{dr}{dt} &\approx -\frac{1}{\Omega_c}\frac{\frac{v_\perp^2}{2}+v_{\|}^2}{R_0}\sin\theta \\
\frac{d\theta}{dt} &\approx \frac{1}{qR_0}v_{\|} - \frac{1}{\Omega_c}\frac{\frac{v_\perp^2}{2}+v_{\|}^2}{R_0}\cos\theta \\
\frac{d\zeta}{dt} &\approx \frac{1}{R_0}v_{\|}
\end{aligned}$$

No radial electric field. **End**

The second equation can be written

$$\frac{rd\theta}{dt} = \frac{R}{R_0}\frac{B_\theta}{B_T}v_{\|} + \frac{1}{B_0}\frac{\partial \Phi^{(0)}}{\partial r}$$

and we have

$$\begin{aligned}
\frac{R}{R_0}\frac{B_\theta}{B_T} &= \frac{R_0 h}{R_0}\frac{b(r)}{h}\frac{h}{B_0} = \frac{b(r)}{B_0/h} \\
&\approx \frac{B_\theta}{B_T} = \Theta
\end{aligned}$$

$$\frac{rd\theta}{dt} = -\Theta v_{\parallel} + \frac{1}{B_0}\frac{\partial \Phi^{(0)}}{\partial r}$$

Note that normally the poloidal projection of the parallel velocity, $\Theta v_{\parallel}$ should *cancel* the poloidal velocity induced by the radial electric field. Then $rd\theta/dt \rightarrow 0$. If not, it remains a poloidal rotation. **End.**

We use this form of the equation to obtain formally the parallel velocity

$$v_{\parallel} = -\frac{1}{\Theta}\left[\frac{rd\theta}{dt} - \frac{1}{B_0}\frac{\partial \Phi^{(0)}}{\partial r}\right]$$

Since we know $v_{\parallel}$ the equation for φ can now be integrated

$$\begin{aligned}
\frac{R_0 d\varphi}{dt} &= -\frac{1}{\Theta}\left[\frac{rd\theta}{dt} - \frac{1}{B_0}\frac{\partial \Phi^{(0)}}{\partial r}\right] \\
\frac{d\varphi}{dt} &= -\frac{B_T}{B_\theta}\frac{r}{R_0}\left(\frac{d\theta}{dt} - \frac{1}{r}\frac{1}{B_0}\frac{\partial \Phi^{(0)}}{\partial r}\right)
\end{aligned}$$

with the result

$$\varphi = -q\left(\theta - \frac{1}{r}\frac{1}{B_0}\frac{\partial \Phi^{(0)}}{\partial r}\ t\right)$$

We **NOTE** again that the absence of poloidal displacement

$$\frac{d\theta}{dt} = 0$$

can be expressed as a relationship between the poloidal velocities

$$\begin{aligned}
-\Theta v_{\parallel} + \frac{1}{B_0}\frac{\partial \Phi^{(0)}}{\partial r} &= 0 \\
v_{\parallel} - \frac{1}{\Theta} v_E &= 0
\end{aligned}$$

END.

In the absence of a radial electric field $\Phi^{(0)} = 0$ the equation is

$$\varphi(\theta) = -q\theta$$

We **note** that in **Hahm Fong**

$$\begin{aligned}
\zeta(\theta) &= q_0\theta \\
&\quad + \sqrt{2}\Lambda_B\left[\left(2\frac{dq_0}{dr} + \frac{q_0}{r}\right)C_1(\varphi(\theta), \kappa) - \left(2\frac{dq_0}{dr}\left(1-\kappa^2\right) + \frac{q_0}{2r_0}\right)C_2(\varphi(\theta), \kappa)\right]
\end{aligned}$$

see below.

End.
A new variable is defined

$$\varphi' = \varphi + q\left(\theta - \frac{1}{r}\frac{1}{B_0}\frac{\partial \Phi^{(0)}}{\partial r}\ t\right)$$

In the equations are included now the *drifts* of the particles

$$\begin{aligned}
\frac{dr}{dt} &= -v_D \sin\theta \\
\frac{rd\theta}{dt} &= -\Theta v_{\|} + v_E - v_D\cos\theta \\
\frac{d\varphi'}{dt} &= -\frac{v_D}{r}\left(q\cos\theta + \frac{dq}{dr} r\ \theta\ \sin\theta\right)
\end{aligned}$$

where

$$v_D = \frac{1}{\Omega_c}\frac{\frac{v_\perp^2}{2} + v_\|^2}{R}$$

$$v_E = \frac{1}{B_0}\frac{\partial \phi^{(0)}}{\partial r}$$

with $\phi^{(0)}$ a static electric potential.

The angle in the toroidal direction is

$$\varphi' = \varphi + \frac{q}{r}\left(r\theta - \frac{1}{B_0}\frac{\partial \varphi^{(0)}}{\partial r} t\right)$$

or

$$R\varphi' = R\varphi + \frac{B_T}{B_\theta}\left(r\theta - \frac{1}{B_0}\frac{\partial \varphi^{(0)}}{\partial r} t\right)$$

This angle is the usual geometrical toroidal angle φ (or the toroidal distance $R\varphi$) corrected to include the toroidal component (or projection) caused by a poloidal motion:

- the factor B_T/B_θ produces the projection along the toroidal direction of a distance on the poloidal direction $r\theta$.
- the distance travelled along the poloidal direction is

$$r\theta - \frac{1}{B}\frac{\partial \varphi^{(0)}}{\partial r} \times t$$

  with the electric poloidal speed constant, then multiplied by the time.

This variable is the analogue of the variable

$$v_{\|} - \frac{v_E}{\Theta}$$

whose importance is stressed.

NOTE that this combination is actually determined by the sum of *poloidal* velocities: V_E and $\Theta V_{\|}$. This sum must be very close to zero, as adopted by **Rozhansky Tendler** with

$$\begin{aligned} \frac{1}{x} &= \mathbf{PV}\left(\frac{1}{x}\right) + i\pi\delta\left(x\right) \\ \text{where } x &= V_E + \Theta V_{\|} \end{aligned}$$

END

Condition of trapping for the particles

$$\left[v_{\|} - \frac{v_E}{\Theta}\right]^2 < 4v_D r\left[\Omega_c + v_{\|}\frac{d}{dr}\left(\frac{1}{\Theta}\right) + \frac{1}{\Theta^2}\frac{dv_E}{dr}\right]$$

It is said (**Galeev Sagdeev Wong** and **Rozhanski Tendler**) that the trapped particles have

$$\begin{aligned} v_{\|} - \frac{v_E}{\Theta} &\approx 0 \\ &\text{actually very small} \end{aligned}$$

For

- large v_E, or
- small $\Theta \equiv \frac{B_\theta}{B_T} << 1$

the trapped particles can be found in the *tail* of the distribution function.

In the absence of the electric field,

$$v_{\|} < \sqrt{\varepsilon} v$$

and the trapped particles are in the main part of the distribution function.

The equation for the poloidal displacement

$$\frac{rd\theta}{dt} = -\Theta v_{\|} + v_E - v_D \cos\theta$$

is approximated since

$$|v_D \cos\theta| \sim \frac{\rho_L}{r} q << 1$$

can be ignored.

$$\begin{aligned}
& \frac{rd\theta}{dt} \\
= & -\Theta v_\parallel + v_E \quad \text{(the lowest order)} \\
= & \pm \sqrt{\left(\Theta v_\parallel - v_E\right)^2_{r=r_0} + \frac{1}{\Omega_c} \frac{\frac{v_\perp^2}{2} + v_\parallel^2}{R} \left(\Theta^2 \Omega_c - v_\parallel \frac{d\Theta}{dr} + \frac{dv_E}{dr}\right)(\cos\theta - 1)}
\end{aligned}$$

$$\begin{aligned}
&= \pm\sqrt{\left(\Theta^2 v^2 + v_E^2\right)\varepsilon} \times \sqrt{2\kappa^2 - 1 + \cos\theta} \\
&= \pm\sqrt{2\left(\mu B_0 \Theta^2 + v_E^2\right)} \times \sqrt{2\kappa^2 - 1 + \cos\theta}
\end{aligned}$$

where

$$\varepsilon = \frac{r}{R_0 \left[1 + \frac{d}{dr}\left(\frac{v_E}{\Theta\Omega_c}\right)\Theta\right]}$$

The derivatives of v_E and of Θ will be neglected

$$\frac{dv_E}{dr} = 0 \quad , \quad \frac{d\Theta}{dr} = 0$$

$$\mu = \frac{v_\perp^2}{2B}$$

$$2\kappa^2 = \frac{\left(\Theta v_\parallel - v_E\right)^2}{2\left(\mu B_0 \Theta^2 + v_E^2\right)\varepsilon} \frac{1}{\varepsilon}$$

terms with

$$\frac{d\Theta}{dr} \quad \text{and} \quad \frac{dv_E}{dr} \quad \text{are neglected}$$

The limit angle for a banana is

$$\cos\theta_0 = 1 - 2\kappa^2$$

The time of bounce

$$\begin{aligned}
\tau &= 4\int_0^{\theta_0} \frac{rd\theta'}{v_E - \Theta v_\parallel} \\
&= 4\frac{r}{\sqrt{2\left(\mu B_0 \Theta^2 + v_E^2\right)\varepsilon}} \int_0^{\theta_0} \frac{d\theta'}{\sqrt{2\kappa^2 - 1 + \cos\theta'}} \\
&= 4\frac{r}{\sqrt{2\left(\mu B_0 \Theta^2 + v_E^2\right)\varepsilon}} \sqrt{2} K\left(\sin\frac{\theta_0}{2}\right)
\end{aligned}$$

where K is the complete elliptic function.

The distribution function of trapped particles

$$\begin{aligned} f_j^{(0)} &= \frac{n_j(r)}{\left(\sqrt{\pi} v_{th}\right)^3} \exp\left(-\frac{2E}{v_{th}^2}\right) \\ &\times \left[1 - \frac{v_\parallel}{\Theta\Omega_{cj}} \frac{1}{n_j(r)} \frac{dn_j(r)}{dr}\right] \end{aligned}$$

In this form there is a dependence on θ given by the parallel velocity. Then an approximation is adopted, where $\overline{v}_\parallel$

$$\begin{aligned} \overline{v}_\parallel \equiv &\ \text{velocity along the magnetic line} \\ &\ \text{with neglect of the } \theta\text{-dependence} \end{aligned}$$

NOTE that this is the usual neoclassical correction to the equilibrium distribution function

$$\begin{aligned} f_0 &= f_M - \rho_\theta \frac{\partial f_M}{\partial r} \\ &= f_M \left(1 - \rho_\theta \frac{1}{f_M} \frac{\partial f_M}{\partial r}\right) \end{aligned}$$

since

$$\begin{aligned} & \frac{\overline{v}_\parallel}{\Theta\Omega_{cj}} \frac{1}{n_j(r)} \frac{dn_j(r)}{dr} \\ = & \frac{B}{B_\theta} \frac{1}{e_j B / m_j} \overline{v}_\parallel \frac{1}{n_j(r)} \frac{dn_j(r)}{dr} \\ = & \rho_\theta \frac{1}{f_M} \frac{\partial f_M}{\partial r} \\ = & \frac{\rho_\theta}{L_n} \end{aligned}$$

END
The linearized equation

$$\begin{aligned} & \frac{\partial f_j^{(1)}}{\partial t} + \mathbf{v}_D \cdot \boldsymbol{\nabla} f_j^{(1)} \\ = & \frac{1}{B_0} \frac{\partial \phi}{r \partial \theta} \frac{\partial f_j^{(0)}}{\partial r} \\ & + \frac{e_j}{m_j} \boldsymbol{\nabla}_\parallel \phi \cdot \frac{\partial f_j^{(0)}}{\partial \mathbf{v}_\parallel} \end{aligned}$$

where here

$$\phi \equiv \text{electric potential of the wave perturbation (of the instability)}$$

NOTE

we remark the difference between this treatment, adapted probably to the wave, and the treatment for the first order neoclassical correction.

The latter is based on the balance between

$$\text{most important term at first order,} \quad v_{\|}\nabla_{\|}f_1$$

which will be calculated from the parallel projection of the *poloidal variation* of the first order function $f_1(\theta)$. And

$$\text{small term of change of the zero-order } f_0, \quad v_{Dr}\frac{\partial f_0}{\partial r}$$

coming from advection by the drift velocity $\mathbf{v}_D$ along the radial direction (the only possible variation of f_0).

This balance can be integrated since the drift velocity contains the same derivative $\frac{\partial}{r\partial\theta}$.

END

10.4 Trapped particles, Mikhailovskii

This is the **book Mikhailovskii p260.**

Some specific choices in this book.

The longitudinal invariant of a trapped particle is the integral over the loop

$$J_{\|} = \oint v_{\|}\ dl$$

We **note** that this is the *circulation* also introduced fluid theory. This is used also by **Hassam Kulsrud** in the work about the poloidal rotation. We have

$$J_{\|} = \sqrt{2\epsilon}\ I_2(\lambda)$$

where

$$I_2(\lambda) = \oint dl_{\|}\ \sigma\sqrt{1-\lambda B}$$

The presence of σ $(=\pm 1)$ reminds us that we must integrate forward/backward on the two parts of the loop (banana).

Consider simpler form of *invariants*, without electrostatic potential and divided to the mass

$$\epsilon = \frac{1}{2}\left(v_{\perp}^{2} + v_{\parallel}^{2}\right)$$

$$\mu = \frac{v_{\perp}^{2}}{2B}$$

Then

$$v_{\perp} = 2\mu B$$

$$v_{\parallel} = \pm\sqrt{2\left(\epsilon - \mu B\right)}$$

Define

$$\lambda \equiv \frac{\mu}{\epsilon}$$

It results

$$\begin{aligned} v_{\parallel} &= \pm\sqrt{2\epsilon - 2\mu B} = \pm\sqrt{2\epsilon - 2\lambda\epsilon B} \\ &= \pm\sqrt{2\epsilon}\,\sqrt{1 - \lambda B} \end{aligned}$$

The *trapped* particles

$$\frac{1}{B_{\max}} < \lambda < \frac{1}{B} \quad \text{trapped}$$

where $B_{\max}$ is the maximum of the magnetic field on a line.

The *circulating* particles

$$0 < \lambda < \frac{1}{B_{\max}} \quad \text{passing}$$

The *bounce period* for a trapped particle

$$\tau_{bounce} = \oint \frac{dl_{\parallel}}{v_{\parallel}}$$

$$\tau_{bounce} = \frac{1}{\sqrt{2\epsilon}} I_{1}\left(\lambda\right)$$

where

$$I_{1}\left(\lambda\right) = \oint dl_{\parallel}\, \frac{\sigma}{\sqrt{1 - \lambda B}}$$

NOTE

Remember **Galeev Sagdeev review**,

$$\begin{aligned}
\tau_{bounce}^{GS} &= 2\times 2\times \frac{1}{v\frac{B_\theta}{B_T}}\frac{1}{\sqrt{2\varepsilon}}r\int_0^{\vartheta_0}\frac{d\theta}{\sqrt{\kappa^2-\sin^2\left(\frac{\theta}{2}\right)}} \\
\tau_{bounce}^{GS} &= 4\sqrt{2}\frac{1}{v\frac{B_\theta}{B_T}}\frac{1}{\sqrt{\varepsilon}}r\ \frac{1}{\kappa}\mathbf{K}\left(\kappa\right) \quad \text{or} \quad 4\sqrt{\frac{2}{\varepsilon}}\frac{qR}{v}\frac{1}{\kappa}\mathbf{K}\left(\kappa\right)
\end{aligned}$$

If we use this formula, we have

$$\begin{aligned}
\tau_{bounce} &= \frac{1}{\sqrt{2\epsilon}}I_1\left(\lambda\right) \quad \text{with } \tau_{bounce}^{GS} = 4\sqrt{\frac{2}{\varepsilon}}\frac{qR}{v}\frac{1}{\kappa}\mathbf{K}\left(\kappa\right) \\
&= 4\sqrt{2}\frac{1}{v\frac{B_\theta}{B_T}}\frac{1}{\sqrt{\varepsilon}}r\ \frac{1}{\kappa}\mathbf{K}\left(\kappa\right) = \frac{1}{\sqrt{2\varepsilon}}\left(8\frac{1}{v\frac{B_\theta}{B_T}}r\ \frac{1}{\kappa}\mathbf{K}\left(\kappa\right)\right)
\end{aligned}$$

then

$$\begin{aligned}
I_1^{GS}\left(\lambda\right) &= \sqrt{2\epsilon}\ \tau_{bounce}^{GS} = v\ 4\sqrt{\frac{2}{\varepsilon}}\frac{qR}{v}\frac{1}{\kappa}\mathbf{K}\left(\kappa\right) \\
&= 4\sqrt{\frac{2}{\varepsilon}}qR\frac{1}{\kappa}\mathbf{K}\left(\kappa\right)
\end{aligned}$$

In **Mikhailovskii** the result is

$$I_1^{Mikha} = 4\sqrt{\frac{2}{\varepsilon}}qR\ \mathbf{K}\left(\kappa\right)$$

They coincide if $\kappa \approx 1$.

$$\begin{aligned}
\kappa^2 &\equiv \frac{1}{2}\frac{1}{\varepsilon}\frac{v_{\|0}^2}{v_{\perp 0}^2} \quad \text{in GS} \\
\kappa &= \frac{1}{\sqrt{2\varepsilon}}\frac{v_{\|0}}{v_{\perp 0}}
\end{aligned}$$

END

Mikhailovskii defines

$$B_s = \text{toroidal magnetic field averaged over surface}$$

(in the book the magnetic field B is denoted B_0 which can create some confusion, since usually B_0 is the magnetic field on the axis, i.e. at $\theta = \frac{\pi}{2}$). Then

$$B = B_s\left(1+\varepsilon\cos\theta\right)$$

The more familiar notation is

$$
\begin{aligned}
B &= \frac{B_0}{h} \\
& (B_0 \equiv \text{ here the magnetic field on the tok. axis})
\end{aligned}
$$

and this means

$$
\begin{aligned}
B_s h &= \frac{B_0}{h} \\
B_s &\approx \frac{B_0}{1 + 2\varepsilon \cos\theta}
\end{aligned}
$$

Using B in terms of B_s (the magnetic field averaged over the surface) we have for the parallel velocity

$$
v_\parallel = \pm\sqrt{2\epsilon}\ \sqrt{1 - \lambda B_s \left(1 + \varepsilon \cos\theta\right)}
$$

where $\epsilon = v^2/2$. This is

$$
\begin{aligned}
v_\parallel &= \pm\sqrt{2\epsilon}\ \sqrt{1 - \lambda B_0 \frac{\left(1 + \varepsilon \cos\theta\right)}{\left(1 + \varepsilon \cos\theta\right)^2}} \\
&= \pm\sqrt{2\epsilon}\sqrt{1 - \lambda \frac{B_0}{1 + \varepsilon \cos\theta}} \\
&= \sigma\sqrt{2\epsilon}\sqrt{1 - \lambda B} \quad \text{as GS}
\end{aligned}
$$

NOTE
Remember the result of **GS review**,

$$
\frac{v_\parallel}{v} = \sigma\sqrt{1 - \left(\frac{B}{B_{\min}}\right)\left(1 - 2\varepsilon_B\ \kappa^2\right)}
$$

$$
\frac{v_\parallel}{v} \approx \sigma\sqrt{2\varepsilon}\ \sqrt{\kappa^2 - \sin^2\left(\frac{\theta}{2}\right)}
$$

(no v_E here).
END

The element of length along a magnetic field *line* is

$$
\begin{aligned}
dl_\parallel &\approx \sqrt{g_{33}}\, d\varphi \\
&\approx R\ d\varphi
\end{aligned}
$$

The poloidal angle variable is expressed in terms of the variable φ, the toroidal angle

$$\theta = \theta_0 + \frac{\varphi}{q}$$

The meaning is clear, advancing in the toroidal direction with angle φ the current point on the helical line will have a new poloidal coordinate θ, given by the characteristic q of the line.

And

$$B = B_s \left[1 + \varepsilon \cos\left(\theta_0 + \frac{\varphi}{q}\right)\right]$$

The two integrations are

$$\begin{aligned}
I_1(\lambda) &= \oint dl_{\parallel} \, \frac{\sigma}{\sqrt{1-\lambda B}} \\
&= \oint R d\varphi \, \frac{\sigma}{\sqrt{1 - \lambda B_s \left[1 + \varepsilon \cos\left(\theta_0 + \frac{\varphi}{q}\right)\right]}} \\
&= 4\sqrt{\frac{2}{\varepsilon}} qR \, \mathbf{K}(\kappa) \qquad \text{(M)}
\end{aligned}$$

and

$$\begin{aligned}
I_2(\lambda) &= \oint dl_{\parallel} \, \sigma \sqrt{1-\lambda B} \\
&= \oint R d\varphi \, \sigma \sqrt{1 - \lambda B_s \left[1 + \varepsilon \cos\left(\theta_0 + \frac{\varphi}{q}\right)\right]} \\
&= 8\sqrt{2\varepsilon} \, qR \, \left[\mathbf{E}(\kappa) - \left(1-\kappa^2\right) \mathbf{K}(\kappa)\right]
\end{aligned}$$

where

$$\kappa^2 = \frac{1}{2}\left[1 + \frac{1}{\varepsilon}(1 - \lambda B_s)\right]$$

NOTE various forms

$\kappa^2 = \frac{1}{2\varepsilon}[1+\varepsilon-\lambda]$	GS review
$\kappa^2 \equiv \frac{1}{2}\frac{1}{\varepsilon}\frac{v_{\parallel 0}^2}{v_{\perp 0}^2}$	GS and Fong Hahm
$\kappa^2 = \frac{1-\frac{\mu B_{\min}}{E}}{2\varepsilon_B}$	Beers
$\kappa^2 = \frac{1}{2}\left[1 + \frac{1}{\varepsilon}(1-\lambda B_s)\right]$	Mikhailovskii
$\kappa = \sqrt{\frac{1-\lambda(1-\varepsilon)}{2\varepsilon\lambda}}$	deWitt

END.

The equations of motion derived in **Mikhailovskii**

$$\frac{dv_\perp}{dt} = -\frac{1}{2} v_\perp v_\parallel \ (\boldsymbol{\nabla}\cdot\widehat{\mathbf{n}})$$

$$\frac{dv_\parallel}{dt} = \frac{v_\perp^2}{2} \ (\boldsymbol{\nabla}\cdot\widehat{\mathbf{n}})$$

where the divergence of the versor of the direction of the magnetic field is

$$\boldsymbol{\nabla}\cdot\widehat{\mathbf{n}} = \mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{1}{B}\right)$$

The equations are detailed further in **Mikhailovskii**

$$\begin{aligned} \frac{dv_\perp}{dt} &= -\frac{1}{2} v_\perp v_\parallel \ (\boldsymbol{\nabla}\cdot\widehat{\mathbf{n}}) \\ &\quad + \frac{v_\perp}{2B}\left(\frac{\partial B}{\partial t} + \mathbf{v}_E\cdot\boldsymbol{\nabla} B\right) \end{aligned}$$

and

$$\begin{aligned} \frac{dv_\parallel}{dt} &= \frac{v_\perp^2}{2} (\boldsymbol{\nabla}\cdot\widehat{\mathbf{n}}) \\ &\quad + \frac{eE_\parallel}{m} + v_\parallel \ \mathbf{v}_E\cdot\left(\nabla_\parallel \widehat{\mathbf{n}}\right) \end{aligned}$$

The drift of particles is

$$\begin{aligned} \mathbf{v}_D &= \frac{1}{\Omega_c}\widehat{\mathbf{n}}\times\left(\frac{v_\perp^2}{2}\boldsymbol{\nabla}\ln B + v_\parallel^2 \ (\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla})\,\widehat{\mathbf{n}}\right) \\ &= \frac{1}{\Omega_c}\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{R}\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\ln B_0 \end{aligned}$$

$$\text{curvature} \quad \boldsymbol{\kappa} = (\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla})\,\widehat{\mathbf{n}} = \boldsymbol{\nabla}\ln B$$

Then, for Mikhailovskii

$$\begin{aligned} v_{D,r} &= \frac{1}{\Omega}\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{R} \ \sin\theta \qquad \text{M} \\ v_{D,\theta} &= \frac{1}{\Omega}\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{R} \ \frac{\cos\theta}{r} \qquad \text{M} \\ v_{D,\varphi} &= \frac{1}{\Omega}\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{R} \ \left(-\frac{r}{qR^2}\cos\theta\right) \qquad \text{M} \end{aligned}$$

Explanation

The drift $\mathbf{v}_D$ is perpendicular on the magnetic field line, due to

$$\widehat{\mathbf{n}} \times \boldsymbol{\nabla} \ln B$$

The result can then to be projected on the pure toroidal direction

$$\frac{1}{B}\left[\widehat{\mathbf{n}} \times \boldsymbol{\nabla} B\right] \cdot \widehat{\mathbf{e}}_\varphi$$

We have

$$B = \frac{B_0}{1+\varepsilon\cos\theta}$$

Now we will attempt to simplify our task and start the operations *after* expanding for small ε,

$$\begin{aligned}
\boldsymbol{\nabla} B &= B_0 \ \boldsymbol{\nabla}\left(1 - \frac{r}{R}\cos\theta\right) \\
&\approx -B_0\frac{1}{R}\boldsymbol{\nabla} r \ \cos\theta - B_0\frac{1}{R}r \ \left(-\sin\theta\right)\boldsymbol{\nabla}\theta \\
&= -\frac{B_0}{R}\cos\theta \ \widehat{\mathbf{e}}_r + \frac{B_0\sin\theta}{R}r\frac{1}{r}\widehat{\mathbf{e}}_\theta
\end{aligned}$$

and using this,

$$\begin{aligned}
\frac{1}{B}\left[\widehat{\mathbf{n}} \times \boldsymbol{\nabla} B\right] &= \frac{1}{B}\left[-\frac{B}{R_0}\cos\theta\right]\widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_r + \frac{1}{B}\left[\frac{B}{R_0}\sin\theta\right]\widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_\theta \\
&= \frac{1}{R_0}\left[-\cos\theta\right]\widehat{\mathbf{e}}_\perp + \frac{1}{R_0}\left[\sin\theta\right]\widehat{\mathbf{e}}_r \sin\left(\widehat{\mathbf{n}}, \widehat{\mathbf{e}}_\theta\right)
\end{aligned}$$

NOTE of caution

Here intevenes a particular choice of the directions of the three versors along (r, θ, φ). The scalar product $\widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_r$ is

$$\widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_r = \begin{cases} \widehat{\mathbf{e}}_\perp & \text{GS} \\ -\widehat{\mathbf{e}}_\perp & \text{M} \end{cases}$$

END NOTE

We continue with **GS** $\widehat{\mathbf{n}} \times \widehat{\mathbf{e}}_r = \widehat{\mathbf{e}}_\perp$.

The angle between $\widehat{\mathbf{n}}$ and $\widehat{\mathbf{e}}_\theta$ is very close to $\pi/2$, and

$$\sin\left(\widehat{\mathbf{n}}, \widehat{\mathbf{e}}_\theta\right) = \cos\left(\widehat{\mathbf{n}}, \widehat{\mathbf{e}}_\varphi\right) = \frac{B_T}{B}$$

We can now project onto $\widehat{\mathbf{e}}_\varphi$,

$$\begin{aligned}
& \frac{1}{B}\left[\widehat{\mathbf{n}}\times\boldsymbol{\nabla}B\right]\cdot\widehat{\mathbf{e}}_\varphi \\
= & \left(\frac{1}{R_0}\left[-\cos\theta\right]\widehat{\mathbf{e}}_\perp+\frac{1}{R_0}\left[\sin\theta\right]\widehat{\mathbf{e}}_r\frac{B_T}{B}\right)\cdot\widehat{\mathbf{e}}_\varphi \\
= & \frac{1}{R_0}\left[-\cos\theta\right]\left(\widehat{\mathbf{e}}_\perp\cdot\widehat{\mathbf{e}}_\varphi\right)
\end{aligned}$$

The angle between $\widehat{\mathbf{e}}_\perp$ and $\widehat{\mathbf{e}}_\varphi$ is $\frac{\pi}{2}$+angle between $\widehat{\mathbf{e}}_\perp$ and $\widehat{\mathbf{e}}_\theta$, let us note this η. The angle η is the same as the one between $\widehat{\mathbf{e}}_\varphi$ and $\widehat{\mathbf{n}}$. Then

$$\left(\widehat{\mathbf{e}}_\perp\cdot\widehat{\mathbf{e}}_\varphi\right)=\cos\left(\frac{\pi}{2}+\eta\right)=-\sin\eta=-\frac{B_\theta}{B}$$

and

$$\frac{1}{B}\left[\widehat{\mathbf{n}}\times\boldsymbol{\nabla}B\right]\cdot\widehat{\mathbf{e}}_\varphi=\frac{1}{R_0}\left[-\cos\theta\right]\left(-\frac{B_\theta}{B}\right)$$

Now we can return to the drift, with its toroidal component

$$\begin{aligned}
v_{D,\varphi} &= \frac{1}{\Omega}\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{R}\frac{1}{R_0}\cos\theta\left(\frac{B_\theta}{B}\right) \\
&\approx \frac{1}{\Omega}\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{R}\frac{r}{R^2}\frac{1}{q}\cos\theta \qquad \text{(according to GS)}
\end{aligned}$$

If, we had adopted

$$\widehat{\mathbf{n}}\times\widehat{\mathbf{e}}_r=-\widehat{\mathbf{e}}_\perp \quad \text{(according to M)}$$

then

$$v_{D,\varphi}=\frac{1}{\Omega}\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{R}\left(-\frac{r}{qR^2}\cos\theta\right) \qquad \text{(M)}$$

This is $\frac{Rd\varphi}{dt}$.

IMPORTANT note of caution

The signs in these **Mikhailovskii** equations are opposite to those of other papers. This comes from the choice of the system of versors on r,θ,φ.

In **Berk Galeev**

$$\frac{dr}{dt}=-\frac{1}{\Omega_c}\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{R}\sin\theta \qquad \text{BG}$$

$$\frac{rd\theta}{dt} = v_{\parallel}\frac{B_\theta}{B_T} - \frac{1}{\Omega_c}\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{R}\cos\theta + \frac{1}{B}\frac{\partial\phi}{\partial r} \quad \text{BG}$$

End note

Define

$$\varphi_D = \varphi - q\theta$$

This function is the position (θ,φ) of the current point on the helix laying on the surface $q(\psi)$ =const.

Its time variation includes the possibility that the particle evolve on other surfaces i.e. $q(r)$ will change too.

Comment on this combination of angle variables.

In **Wong Sagdeev Galeev** (above)

$$\frac{rd\theta}{dt} = -\Theta v_\parallel + \frac{1}{B_0}\frac{\partial\Phi^{(0)}}{\partial r}$$

and it is extracted $v_\parallel = -\frac{1}{\Theta}\left[\frac{rd\theta}{dt} - \frac{1}{B_0}\frac{\partial\Phi^{(0)}}{\partial r}\right]$. Since we know $v_\parallel$ the equation for φ can now be integrated

$$\frac{d\varphi}{dt} = -\frac{B_T}{B_\theta}\frac{r}{R_0}\left(\frac{d\theta}{dt} - \frac{1}{r}\frac{1}{B_0}\frac{\partial\Phi^{(0)}}{\partial r}\right)$$

with the result

$$\varphi = -q\left(\theta - \frac{1}{r}\frac{1}{B_0}\frac{\partial\Phi^{(0)}}{\partial r}\, t\right)$$

It is clear that a simple combination of the equation places emphasis on the angle variable

$$\varphi - q\theta$$

which is φ_D.

End.

The equation is obtained by replacing $d\varphi/dt$ and $d\theta/dt$, and dr/dt, from drift equations

$$\frac{d\varphi_D}{dt} = \frac{1}{\Omega}\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{R}\left(-\frac{q}{r}\right)\left(\cos\theta + \frac{rq'}{q}\theta\sin\theta\right)$$

Define the *average of* $\frac{d\varphi_D}{dt}$ over a bounce

$$\overline{\left(\frac{d\varphi_D}{dt}\right)} = \frac{1}{\tau_{bounce}}\oint\frac{dl_\parallel}{v_\parallel}\left(\frac{d\varphi_D}{dt}\right)$$

It is

$$\overline{\left(\frac{d\varphi_D}{dt}\right)} = \frac{2\varepsilon\ \epsilon q}{\Omega r^2}\left[\frac{\mathbf{E}}{\mathbf{K}} - \frac{1}{2} + 2\frac{rq'}{q}\left(\frac{\mathbf{E}}{\mathbf{K}} - 1 + \kappa^2\right)\right]$$

NOTE this is the precession $\langle v_\parallel \rangle$ calculated by **GS review**, but corrected. **END.**

or

$$\overline{\left(\frac{d\varphi_D}{dt}\right)} = \frac{1}{\tau_{bounce}}\frac{q}{\Omega r}\frac{dJ_\parallel}{dr}$$

This is the frequency of precession of a banana in the toroidal direction.

The distribution function

$$f_0^{(0)} = F\left(\epsilon, \mu, \sigma, \mathbf{r}_\perp\right)$$

Momenta

$$\begin{aligned} n_0 &= \int d^3v\ f_0^{(0)} \\ &= \int 2\pi v_\perp\ dv_\perp\ dv_\parallel\ f_0^{(0)} \end{aligned}$$

the element of volume is converted to (ϵ, μ)

$$\int (...)\, v_\perp dv_\perp dv_\parallel = \sum_\sigma \int (...)\, \frac{B}{\left|v_\parallel\right|}\ d\epsilon\ d\mu$$

The longitudinal pressure

$$\begin{aligned} p_\parallel &= \int d^3v\ mv_\parallel^2\ F \\ &= \sum_\sigma 2\pi \int mv_\parallel^2\ \frac{B}{\left|v_\parallel\right|} d\epsilon d\mu \\ &= 2\pi \sum_\sigma \int m\left|v_\parallel\right| B\ d\epsilon\ d\mu \end{aligned}$$

The perpendicular pressure

$$\begin{aligned} p_\perp &= \int d^3v\ \frac{mv_\perp^2}{2}\ F \\ &= 2\pi \sum_\sigma \int \frac{mv_\perp^2}{2}\ F \frac{B}{\left|v_\parallel\right|}\ d\epsilon\ d\mu \\ &= 2\pi m \sum_\sigma \int F \frac{B^2\mu}{\left|v_\parallel\right|}\ d\epsilon\ d\mu \end{aligned}$$

or, after an integration by parts

$$p_{\perp} = -2\pi m \sum_{\sigma} \int B^2 \mu \left| v_{\parallel} \right| \frac{\partial F}{\partial \epsilon} \, d\epsilon \, d\mu$$

Calculate the variation of the pressure components along a magnetic field line.

It is acted upon with the operator

$$\nabla_{\parallel}$$

One has to take into account

$$\nabla_{\parallel} v_{\parallel} = -\frac{\mu}{v_{\parallel}} \nabla_{\parallel} B$$

(which means $\nabla_{\parallel} \left(\frac{v_{\parallel}^2}{2} \right) = -\mu \nabla_{\parallel} B$, the parallel gradient of the density of energy is the *mirror* force)

Then

$$\nabla_{\parallel} p_{\parallel} = \pi_{\parallel} \, \nabla_{\parallel} \left(\frac{B^2}{2\mu_0} \right)$$

parallel mirror force

where

$$\pi_{\parallel} = \frac{p_{\parallel} - p_{\perp}}{B^2/\mu_0}$$

And

$$\nabla_{\parallel} p_{\perp} = \pi_{\perp} \nabla_{\parallel} \left(\frac{B^2}{2\mu_0} \right)$$

where

$$\pi_{\perp} = \frac{2p_{\perp} + \widehat{c}}{B^2 \mu_0}$$

$$\widehat{c} = 2\pi m \sum_{\sigma} \int \frac{B}{v_{\parallel}} \frac{\partial F}{\partial \epsilon} \, (\mu B)^2 \, d\epsilon \, d\mu$$

A geometrical relation

$$\begin{aligned} \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{B}) &= \boldsymbol{\nabla} \left(\frac{B^2}{2} \right) - (\mathbf{B} \cdot \boldsymbol{\nabla}) \mathbf{B} \\ &= \nabla_{\perp} \left(\frac{B^2}{2} \right) - B^2 \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \right) \widehat{\mathbf{n}} \end{aligned}$$

from where

$$
\begin{aligned}
(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} &\equiv \text{curvature} \\
&= \frac{1}{B^2} \nabla_\perp \left(\frac{B^2}{2} \right) - \frac{1}{B^2} \mathbf{B} \times \mu_0 \mathbf{j}
\end{aligned}
$$

(See **Mikhailovskii** and **Beer thesis**)

10.5 Drift of bananas in the toroidal (angle φ) direction. (Toroidal precession drift of the banana orbit)

GS

$$v_{D,\varphi} = \frac{v^2}{R\Omega_\theta}$$

(**note** the similarity with the approximate expression used by **Rosenbluth Hazeltine Hinton** for the *transversal* drift velocity

$$v_D \sim \frac{v_{th}^2}{\Omega_c R}$$

which is due to

$$v_D \approx \frac{1}{\Omega_c} \frac{1}{R} \left(\frac{v_\perp^2}{2} + v_\parallel^2 \right)$$

).

It results that the drift of bananas in the toroidal direction is obtained from the same expression as the transversal drift velocity by replacing the cyclotron frequency Ω_c in the total magnetic field with the cyclotron frequency in the poloidal magnetic field B_θ.

A more detailed expression can be obtained starting with the equation of motion in the form (7) and averaging over the period τ of the periodic motion. The velocity of the drift in the toroidal direction can be found from this averaged equation and from the fact that the trajectory as a whole remains fixed in the (r, θ) (poloidal) plane; **GS review** see above

$$
\begin{aligned}
\langle \theta \rangle &= -\frac{\mu B_0}{m \Omega_c R} \left\langle \frac{\cos \theta_0}{r_0} \right\rangle + \frac{v_0 (r_0)}{r_0} + \frac{d}{dr} \left(\frac{v_0}{r} \right) \langle (r - r_0) \rangle \\
&\quad + \frac{1}{qR} \langle v_\parallel \rangle + \langle v_\parallel (r - r_0) \rangle \frac{d}{dr} \frac{1}{q(r) R} \\
&= 0
\end{aligned}
$$

The variables $(r - r_0)$ and θ are periodic and their averages are zero. Using the equation (2) the following average can be calculated

$$\begin{aligned}\left\langle v_{\parallel}(r-r_0)\right\rangle &= -\left\langle \Delta v_{\parallel}(r,\theta)\frac{\Delta v_{\parallel}(r_0,0)-\Delta v_{\parallel}(r,\theta)}{\Omega_c\theta}\right\rangle \\ &= \frac{2\mu B_0}{m\Omega_c}q(r)\left[\langle\cos\theta\rangle - 1 + 2\varkappa^2\right]\end{aligned}$$

This was a *time average*. It will be replaced now by a *poloidal angle average*, between limits of bounce, as in the general formula

$$\langle F(\theta)\rangle = \frac{1}{\tau}\int_0^{\tau} F[\theta(t)]\,dt = \frac{1}{K(\varkappa)}\int_0^{\theta_0/2}\frac{F(\theta)}{\sqrt{\varkappa^2-\sin^2\theta/2}}\frac{d\theta}{2}$$

Then the average velocity of drift of the banana along the toroidal direction is given by the average of $v_{\parallel}$:

$$\underbrace{\left\langle v_{\parallel}\right\rangle}_{\text{velocity in toroidal direction}} = -\frac{v_0}{\varepsilon/q}+\frac{2\mu B_0}{m\Omega_c}\frac{1}{R(\varepsilon/q)}\left[\left(\frac{E(\varkappa)}{K(\varkappa)}-\frac{1}{2}\right)+2\frac{d\ln q}{dr}\left(\frac{E(\varkappa)}{K(\varkappa)}-1+\varkappa^2\right)\right]$$

where $E(\varkappa)$ is the complete elliptic integral of the second kind.

See **Mikhailovskii** for a corrected expression, here, as noted above, it is $\widehat{s}$ instead of the $\frac{d}{dr}\ln q$.

In conclusion:

- the *radial electric field*; this is v_0 since in their notation $v_0 = \phi'/B$ is the poloidal velocity due to the radial electric field.
- the *centrifugal force;* this is the curvature drift $\sim v_{\parallel}^2(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla})\widehat{\mathbf{n}}$
- the *diamagnetic force* in the r direction; [**note** that the *diamagnetic force* in the Russian denominations is actually the grad-B force since they prove that the gradient of the magnitude of the magnetic field, which gives $\mu\boldsymbol{\nabla}B$ is equal with $grad{-}p$, from the momentum equation].

lead to a **precessional motion of the trapped particles along the torus**.

Comment about the toroidal precession of the bananas.

In a cylinder there is *no* trapping, no banana. The poloidal rotation is independent of the axial motion of the plasma.

In a torus, with a direct dependence on the ratio $\varepsilon \equiv a/R$, there is trapping and banana trajectories. And there is a toroidal precession. The radial electric field induces a toroidal precession of bananas on geometrical basis.

In a torus the poloidal and the toroidal rotations are coupled.

10.6 The bounce-averaged kinetic equation (Hahm Fong)

The objective is to derive a kinetic equation based on bounce-averaged motion of trapped particles.

Note the *bounce-averaging* operation is similar in its intention to the averaging over the gyromotion.

The latter has led us to the drift $\mathbf{v}_D$. Further, in the presence of a gradient of pressure, this averaging has emphasized the *diamagnetic* flow and current.

Now we expect a similar result, the *drift* of banana orbits. Further, it is possible to find a toroidal flow and current when a gradient of pressure is present. This would be the basic component of the *bootstrap* current.

End.

Three expansion parameters

$$\varepsilon_B \sim \frac{\Lambda_B}{L_B} = \frac{\text{width of banana}}{B/\left|\boldsymbol{\nabla} B\right| \text{ (length of variation of magnetic field)}} \ll 1$$

$$\varepsilon_\phi = \frac{e\phi}{T_i} \sim \frac{1}{k_\perp L_p} \ll 1$$

$$\begin{aligned} \varepsilon_k &\sim \frac{\omega}{\omega_{bounce}} = \frac{\text{freq. of fluctuations}}{\text{freq. of bounce}} \\ &= k_\perp \Lambda_B \\ &\ll 1 \end{aligned}$$

The velocity of the drift of the particle

$$\begin{aligned} \mathbf{v} &= v_\parallel \widehat{\mathbf{n}} \\ &\quad + \frac{1}{eB/m}\left(\frac{v_\perp^2}{2} + v_\parallel^2\right)\frac{1}{B}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} B \\ &\quad \left(\text{this is simply } \frac{1}{\Omega}\widehat{\mathbf{n}} \times \left[\mu \boldsymbol{\nabla} B + v_\parallel^2 \left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right]\right) \end{aligned}$$

Actually the first term is

$$\begin{aligned}
\mu \boldsymbol{\nabla} B &= \frac{v_\perp^2}{2}\frac{1}{B}\boldsymbol{\nabla} B \\
&= \frac{v_\perp^2}{2}\boldsymbol{\nabla}\ln B
\end{aligned}$$

The magnetic field is slightly different in definition

$$\mathbf{B} = \frac{\varepsilon_a B_0}{q\left(1+\varepsilon_a\cos\theta\right)}\widehat{\mathbf{e}}_\theta + \frac{B_0}{1+\varepsilon_a\cos\theta}\widehat{\mathbf{e}}_\varphi$$

where

$$\begin{aligned}
\varepsilon_a &= \frac{r}{R_0} = \text{local inverse aspect ratio} \\
\text{defined with the } R_0 &\equiv \text{center of bounce} \\
& \quad \left(\text{no connection with } R^{mag-axis} \equiv R_0^{center} \text{ or } R\right)
\end{aligned}$$

then

$$\begin{aligned}
\frac{B_0}{\frac{rB_T}{RB_\theta}\left(1+\varepsilon_a\cos\theta\right)}\frac{r}{R_0} &= B_\theta\frac{1}{B_T}\frac{B_0}{1+\varepsilon\cos\theta}\frac{1+\varepsilon\cos\theta}{1+\varepsilon_a\cos\theta}\frac{R}{R_0} \\
&= B_\theta\frac{1+\varepsilon\cos\theta}{1+\varepsilon_a\cos\theta}\frac{R^{mag-axis}}{1+\varepsilon\cos\theta}\frac{1}{\frac{R^{mag-axis}}{1+\varepsilon_a\cos\theta}} \\
&= B_\theta
\end{aligned}$$

The equations of motion (**Berk Galeev**)

$$\frac{dr}{dt} \approx -\frac{1}{\Omega}\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)\frac{\sin\theta}{R_0}$$

$$\frac{d\theta}{dt} \approx \frac{v_\parallel}{qR_0} - \frac{1}{r}\frac{1}{\Omega}\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{R_0}\cos\theta$$

$$\frac{d\varphi}{dt} \approx \frac{v_\parallel}{R_0}$$

NOTE If we want to integrate these equations we need equations also for the *velocities* $v_\parallel$ and $v_\perp$. See **Berk Galeev**, etc. **END.**

NOTE that the first equation is the projection of the drift velocity on the radial direction

$$\begin{aligned}
\frac{dr}{dt} &= -\frac{1}{\Omega_c}\frac{1}{R_0}\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)\sin\theta = \left(\mathbf{v}_D\right)_r \\
&= \left[\frac{1}{\Omega_c}\widehat{\mathbf{n}}\times\left(\mu\boldsymbol{\nabla}B + v_\parallel^2\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right)\right]_r
\end{aligned}$$

Comment
This is because $\mu\boldsymbol{\nabla} B = \frac{v_\perp^2}{2}\boldsymbol{\nabla}\ln B$, and the vector product

$$\widehat{\mathbf{n}} \times \mu\boldsymbol{\nabla} B = \widehat{\mathbf{n}} \times \frac{v_\perp^2}{2}\boldsymbol{\nabla}\ln B = \frac{v_\perp^2}{2}\frac{1}{B}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} B$$

involves

$$\begin{aligned}
\widehat{\mathbf{n}} \times \boldsymbol{\nabla} B &= \widehat{\mathbf{n}} \times \boldsymbol{\nabla}\left(\frac{B_0}{R}\right) = -B_0\frac{1}{R^2}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} R \\
&= -\frac{B_0}{R^2}\widehat{\mathbf{n}} \times (-\widehat{\mathbf{e}}_R) = \frac{B_0}{R^2}\left(-\widehat{\mathbf{e}}_{vertical}^{up}\right)
\end{aligned}$$

returning

$$\widehat{\mathbf{n}} \times \mu\boldsymbol{\nabla} B = \frac{v_\perp^2}{2}\frac{1}{B}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} B = \frac{v_\perp^2}{2}\frac{1}{B}\frac{B_0}{R^2}\left(-\widehat{\mathbf{e}}_{vertical}^{up}\right)$$

and, after replacing $B = \frac{B_0}{R}$,

$$\begin{aligned}
\left(\widehat{\mathbf{n}} \times \mu\boldsymbol{\nabla} B\right)_r &= \left(\widehat{\mathbf{n}} \times \mu\boldsymbol{\nabla} B\right) \cdot \widehat{\mathbf{e}}_r \\
&= -\frac{v_\perp^2}{2}\frac{1}{R}\left(\widehat{\mathbf{e}}_{vertical}^{up} \cdot \widehat{\mathbf{e}}_r\right) = \\
&= -\frac{v_\perp^2}{2}\frac{1}{R}\sin\theta
\end{aligned}$$

Similar result for the other term, that is also a factor of $\sin\theta$.
End Comment.

For the second equation we recognize the first term as the poloidal projection of the parallel velocity

$$\begin{aligned}
\left(\frac{rd\theta}{dt}\right)^{(1)} &= \left(v_\parallel\right)_\theta = v_\parallel\frac{B_\theta}{B} \approx v_\parallel\frac{r}{R}\left(\frac{RB_\theta}{rB_T}\right) \approx r\frac{v_\parallel}{qR_0} \\
\text{or} \quad \left(\frac{d\theta}{dt}\right)^{(1)} &= \frac{v_\parallel}{qR_0} \quad \text{the poloidal projection of the parallel velocity}
\end{aligned}$$

The second term in the θ motion equation is due to the poloidal projection of the *drift velocity*

$$\begin{aligned}
\left(\mathbf{v}_D\right)_\theta &= \left[\frac{1}{\Omega_c}\widehat{\mathbf{n}} \times \left(\mu\boldsymbol{\nabla} B + v_\parallel^2\left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\right)\right]_\theta \\
&\approx \frac{1}{\Omega_c}\frac{1}{R_0}\left(\frac{v_\perp^2}{2} + v_\parallel^2\right)\left(\widehat{\mathbf{e}}_{vertical}\right)_\theta \\
&= \frac{1}{\Omega_c}\frac{1}{R_0}\left(\frac{v_\perp^2}{2} + v_\parallel^2\right)\cos\theta
\end{aligned}$$

END NOTE.

NOTE

that for trapped particles the θ function of time is periodic.

We conclude that what is supplementary in this equation will produce a systematic toroidal drift.

END

The equations of motion, **Berk Galeev**, mentioned by **Fong Hahm** by quoting **Kadomtsev Pogutse**

$$\frac{dr}{dt} \approx -\frac{1}{\Omega}\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)\frac{\sin\theta}{R_0}$$

$$\frac{d\theta}{dt} \approx \frac{v_\parallel}{qR_0} - \frac{1}{r}\frac{1}{\Omega}\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)\frac{\cos\theta}{R_0}$$

$$\frac{d\varphi}{dt} \approx \frac{v_\parallel}{R_0}$$

These equations are integrated as functions of θ.

See **Galeev Sagdeev Wong**. In this text.

$$r(\theta) - r_0 = \pm\frac{vq_0}{\Omega\sqrt{\varepsilon}}\left[\frac{v_{\parallel 0}^2}{\varepsilon v_{\perp 0}^2} - 2\sin^2\left(\frac{\theta}{2}\right)\right]^{1/2}$$

where

$$\begin{aligned} r_0 &\equiv \text{radius of the surface to which the bounce points belong} \\ q_0 &= q(r_0) \\ v_{\parallel 0} &\equiv \text{velocity at the outer mid-plane} \\ v_{\perp 0} &\equiv \text{velocity at the outer mid-plane} \end{aligned}$$

Characteristic parameters of the motion

$$\Lambda_b = \frac{vq_0}{\Omega\sqrt{\varepsilon}} = \frac{v}{\Omega}q_0\varepsilon^{-1/2} \equiv \text{banana radius}$$

We note that this width of the banana trajectory is obtained from the *coefficient* of the expression $r(\theta) - r_0$ written above and has the following meaning:

- the departure of the particle from the magnetic surface where its center is located is
$$+\frac{v}{\Omega}\frac{q_0}{\sqrt{\varepsilon}}$$
when the particle advances in the positive toroidal direction: this is the *first half of the banana*.

- the departure of the particle trajectory from the magnetic surface is

$$-\frac{v}{\Omega}\frac{q_0}{\sqrt{\varepsilon}}$$

  when the particle moves back: this is the *second half of the banana.*

The other parameter is

$$\kappa^2 \equiv \frac{1}{2\varepsilon}\frac{v_{\parallel 0}^2}{v_{\perp 0}^2} \equiv \text{pitch angle parameter}$$

Note in **turbulence driven bootstrap McdeWitt** the "pitch angle variable"

$$\kappa = \sqrt{\frac{1-\lambda\left(1-\varepsilon\right)}{2\varepsilon\lambda}}$$

End.

We can understand how this parameter arises and how we are led to introduce it. The expression under the radical in the equation of the trajectory $r\left(\theta\right)-r_0$ must be transformed such as to make possible the integration

$$\begin{aligned}
&\left[\frac{v_{\parallel 0}^2}{v_{\perp 0}^2}\frac{1}{\varepsilon}-2\sin^2\left(\frac{\theta}{2}\right)\right]^{1/2} \\
= &\sqrt{2}\left[\frac{v_{\parallel 0}^2}{v_{\perp 0}^2}\frac{1}{2\varepsilon}-\sin^2\left(\frac{\theta}{2}\right)\right]^{1/2} \\
= &\sqrt{2}\left[\kappa^2-\sin^2\left(\frac{\theta}{2}\right)\right]^{1/2} \\
= &\pm\sqrt{2}\kappa\left[1-\frac{\sin^2\left(\frac{\theta}{2}\right)}{\kappa^2}\right]^{1/2}
\end{aligned}$$

and a substitution of the variable of integration is made

$$\sin^2\left[\varphi\left(\theta\right)\right] \equiv \frac{\sin^2\left(\frac{\theta}{2}\right)}{\kappa^2}$$

which defines the function $\varphi\left(\theta\right)$ by the equation

$$\varphi\left(\theta\right) = \arcsin\left[\frac{\sin\left(\frac{\theta}{2}\right)}{\kappa}\right]$$

We can separate the motion, taking the picture of the meridional plane

$$r_0 \equiv \text{radius of the surface of the "center" of the banana}$$

$$r(t) = r_0 + \delta r(\theta)$$

The structure of the trajectory (**Fong Hahm**):

1. bounce center on the surface $r = r_0$,
2. oscillatory motion $r = r_0 \pm \delta r$
   (a) $+\delta r$ when the motion is in positive sense along the line with length $l_{\|}$; this is the *first* half of the banana.
   (b) $-\delta r$ when the motion is in negative sense along the line with length $l_{\|}$; this is the *second* half of the banana.

These equations are integrated as functions of θ

$$r(\theta) - r_0 = \pm \frac{v q_0}{\Omega \sqrt{\varepsilon}} \left[\frac{v_{\|0}^2}{\varepsilon v_{\perp 0}^2} - 2 \sin^2 \left(\frac{\theta}{2} \right) \right]^{1/2}$$

where

$$\begin{aligned} r_0 &\equiv \text{radius of the surface to which the bounce points belong} \\ q_0 &= q(r_0) \\ v_{\|0} &\equiv \text{velocity at the outer mid-plane} \\ v_{\perp 0} &\equiv \text{velocity at the outer mid-plane} \end{aligned}$$

Characteristic parameters of the motion

$$\Lambda_b = \frac{v q_0}{\Omega \sqrt{\varepsilon}} \equiv \text{banana radius}$$

$$\begin{aligned} \kappa^2 &= \frac{1}{2\varepsilon} \frac{v_{\|0}^2}{v_{\perp 0}^2} \equiv \sin \text{ of pitch angle} \\ &< 1 \quad \text{(for trapped, to ensure } \exists \text{ of real solution } \theta) \end{aligned}$$

We can separate the motion, taking the picture of the meridional plane

$$r_0 \equiv \text{radius of the surface of the "center" of the banana}$$

$$r(t) = r_0 + \delta r(\theta)$$

The structure of the trajectory (**Fong Hahm**):

1. bounce center on the surface $r = r_0$,
2. oscillatory motion $r = r_0 \pm \delta r$
   (a) $+\delta r$ when the motion is in positive sense along the line with length $l_{\|}$; this is the *first* half of the banana.
   (b) $-\delta r$ when the motion is in negative sense along the line with length $l_{\|}$; this is the *second* half of the banana.

The motion in toroidal direction (**Fong Hahm**)

$$\begin{aligned}\varphi(\theta) &= q_0\theta \\ &+\sqrt{2}\Lambda_b\left[\left(2q_0' + \frac{q_0}{r_0}\right)C_1(\xi(\theta),\kappa)\right. \\ &\left.-\left(2q_0'\left(1-\kappa^2\right)+\frac{q_0}{2r_0}\right)C_2(\xi(\theta),\kappa)\right]\end{aligned}$$

where

$$C_1(\xi(\theta),\kappa) = \begin{cases} E(\xi,\kappa) + \mathbf{E}(\kappa) & \text{first half of orbit} \\ 3\mathbf{E}(\kappa) - E(\xi,\kappa) & \text{second half of orbit}\end{cases}$$

$$C_2(\xi(\theta),\kappa) = \begin{cases} F(\xi,\kappa) + \mathbf{K}(\kappa) & \text{first half of orbit} \\ 3\mathbf{K}(\kappa) - F(\xi,\kappa) & \text{second half of orbit}\end{cases}$$

$$\xi(\theta) \equiv \arcsin\left(\frac{\sin(\theta/2)}{\kappa}\right)$$

NOTE
the toroidal drift of bananas may be a part of the *bootstrap* current.
We must find here the proportionality with

$$\frac{1}{B_\theta}$$

as it is the general formula for the bootstrap current.

But the analysis made in parallel to the analysis of gyromotion, diamagnetic flow, suggests that this toroidal flow of banana orbits must arise as a result of the average over the periodic motion on the banana orbits [see **Galeev Sagdeev**]. This means that this toroidal motion is a *drift* and this drift is a consequence of the deformation of the purely periodic orbit (banana) caused by the presence of a *force* = gradient of pressure.

A purely toroidal precession, without the participation of the radial gradient of pressure (force) and of B_θ does not support the similarity with the diamagentic case.

Then we note that this *precession* is purely geometrical, it does not need gradient of pressure (force) so it is a motion that has nothing to do with the $(-\boldsymbol{\nabla} p) \times \mathbf{B}_\theta$. The two halves of the banana traverse regions where the magnetic configuration $[q(r)]$ is different. This is pure geometry.

This is not the current we are looking for.

We still need the $(-\boldsymbol{\nabla} p)$ to deform the bananas such that they get a *drift* in the toroidal direction and this should be a current.

END

The motion of particle is along the magnetic line with the deviations

$$\begin{aligned} \beta &\equiv \delta\psi_b(r) \quad \text{deviation in the radial direction} \\ \alpha &\equiv \varphi - q(r)\theta \quad \text{deviation non-radial perpendicular} \end{aligned}$$

We know that a magnetic line is given by a fixed value of

$$\varphi - q(r)\theta = \alpha_0$$

and a change of this variable means to quit one line for another line.

In the definition of $\alpha(\theta, t)$ we have φ (the toroidal angle) and θ which can increase then decrease in the case of trapped particles. In the lowest order, in which we neglect the differences between the two halfs of the bananas (inhomogeneous magnetic field, shear) the trapped particle must return precisely a the same position, with no advancement in ζ. Then if there is a difference

$$\alpha(\theta_{fin}) \neq \alpha(\theta_{ini})$$

this is because the two halves of the banana are different. At the end of a time interval

$$\tau_{bounce}$$

the difference between $\alpha's$ is the *precession* in φ.

The average *precessional* motion can be characterized by the change in α, (which is $\varphi - q(r)\theta = \alpha$), given by a precessional frequency

$$\omega_{pr} = \frac{\alpha(\theta_f) - \alpha(\theta_i)}{\tau_{bounce}}$$

The center of bounce has a coordinate α which varies in time according to

$$\alpha_{pr}(t) = \omega_{pr} t(\theta)$$

(this means a linear increase of the coordinate that labels the successive magnetic field lines that the center of the banana changes when it moves along its *precessional* path) where the time has been expressed as function of the poloidal angle by the integrating

$$\frac{dt}{d\theta} \rightarrow t(\theta)$$

This linear increase is extracted from the full function $\alpha(\theta)$. What remains is the the deviation from the average

$$\delta\alpha(\theta) \equiv \alpha(\theta) - \omega_{pr} t(\theta)$$

$$\begin{aligned} \delta\alpha(\theta) &= \mp\sqrt{2}\Lambda_b \left[\left(2q_0' + \frac{q_0}{r_0} \right) \left(\frac{\mathbf{E}(\kappa)}{\mathbf{K}(\kappa)} F(\xi,\kappa) - E(\xi,\kappa) \right) \right. \\ & \left. + q_0' \theta \sqrt{\kappa^2 - \sin^2(\theta/2)} \right] \end{aligned}$$

The motion of the bounce center is

$$\begin{aligned} & r_0 \\ & \alpha_{pr}(\theta) \end{aligned}$$

and the motion of the particle is

$$\begin{aligned} r(\theta) &= r_0 + \delta r(\theta) \\ \alpha(\theta) &= \alpha_{pr}(\theta) + \delta\alpha(\theta) \end{aligned}$$

The particularities of the motion of the trapped particles in the toroidal direction

the orbit does not return to the same initial position

The reason is the difference that exists between the travel in the first half of the banana relative to the travel in the second half of the banana

- the intensity of the magnetic field (B_T, B_θ) is different
- the curvature of the magnetic lines $\sim \frac{-\widehat{\mathbf{e}}_R}{R}$ is different

- the magnetic shear $q(r)$ is different

on the two halves.
The result is that the orbit does not close and it is generated a *precession.*

The motion is essentially along the magnetic field line, with deviations

- in the radial direction
$$\beta \equiv \psi_p(r)$$
- nonradial direction
$$\begin{aligned} \alpha &= \varphi - q(r)\theta \\ &\quad \text{nonradial perpendicular} \end{aligned}$$
- It is defined an average precessional frequency
$$\omega_{pr} = \frac{\alpha(\theta_{fin}) - \alpha(\theta_{ini})}{\tau_{bounce}}$$

The average position of the bounce motion in α cooerdinate

$$\alpha_{pr}(\theta) = \omega_{pr} \times t(\theta)$$

where $t(\theta)$ is obtained from the equations, but solving for

$$\frac{dt}{d\theta}$$

The deviation is

$$\begin{aligned} \Delta\alpha(\theta) &= \alpha(\theta) - \omega_{pr}\, t(\theta) \\ &= \mp\sqrt{2}\Lambda_B \left[\left(2\frac{dq_0}{dr} + \frac{q_0}{r_0}\right) \left(\frac{\mathbf{E}(\kappa)}{\mathbf{K}(\kappa)} F(\xi,\kappa) - E(\xi,\kappa)\right) \right. \\ &\quad \left. + \frac{dq_0}{dr}\,\theta\,\sqrt{\kappa^2 - \sin^2\left(\frac{\theta}{2}\right)} \right] \end{aligned}$$

These new variables have been introduced to separate the two motions

- the motion of the center of bounce
$$(r_0, \alpha_{pr}(\theta))$$

- the motion of the guiding center on banana

$$
\begin{aligned}
r(\theta) &= r_0 + \Delta r(\theta) \\
\alpha(\theta) &= \alpha_{pr}(\theta) + \Delta\alpha(\theta)
\end{aligned}
$$

In view of calculation of the density of ions $n_i(\mathbf{x})$ taking into account the trapping and the precession, one defines the averaging operator

$$
\begin{aligned}
\langle g(\theta)\rangle_\theta &= \frac{\oint d\theta \; \frac{dt}{d\theta} \; g(\theta)}{\oint d\theta \; \frac{dt}{d\theta}} \\
&= \frac{\oint d\theta \; \frac{qR}{v_\parallel} \; g(\theta)}{\tau_{bounce}}
\end{aligned}
$$

The operator will be used to calculate the *bounce average* of the density of ions.

First the deviations

$$\Delta r(\theta) \quad \text{and} \quad \Delta\alpha(\theta)$$

The first average

$$
\begin{aligned}
\langle \Delta r(\theta)\rangle_\theta &= 0 \\
\langle \Delta\alpha(\theta)\rangle_\theta &= 0
\end{aligned}
$$

Now the second moment

$$\left\langle [\Delta r(\theta)]^2 \right\rangle_\theta = \Lambda_B^2 \; \kappa^2$$

$$\left\langle [\Delta\alpha(\theta)]^2 \right\rangle = \frac{1}{16}\Lambda_B^2 \; \left(2\frac{dq_0}{dr} - \frac{q_0}{r_0}\right)^2 \; \kappa^4$$

$$\langle \Delta r \; \Delta\alpha\rangle_\theta = 0$$

The ion density is expanded about the position of the *bounce-center* in terms of deviations

$$
\begin{aligned}
n_i(\mathbf{X} + \mathbf{\Lambda}) &= n_i(\mathbf{X}) \\
&\quad + \Delta r(\theta) \left.\frac{\partial n_i}{\partial r}\right|_{\mathbf{X}} + \Delta\alpha(\theta) \left.\frac{\partial n_i}{\partial \alpha}\right|_{\mathbf{X}} \\
&\quad + \frac{1}{2}(\Delta r)^2 \left.\frac{\partial^2 n_i}{\partial r^2}\right|_{\mathbf{X}} + \frac{1}{2}(\Delta\alpha)^2 \left.\frac{\partial^2 n_i}{\partial \alpha^2}\right|_{\theta} \\
&\quad + (\Delta r \; \Delta\alpha) \left.\frac{\partial^2 n_i}{\partial r \partial \alpha}\right|_{\mathbf{X}}
\end{aligned}
$$

This expression is bounce-averaged. Then integrated over the space of velocities for trapped particles

$$\begin{aligned}
& n_i(\mathbf{X}) - \left\langle \left\langle n_i(\mathbf{X}+\mathbf{\Lambda}) \right\rangle_\theta \right\rangle_v \\
\approx\ & \frac{1}{4}\Lambda_{B,th}^2 \ \sqrt{2\varepsilon_a}\ n_{i0}\frac{e}{T_i}\frac{\partial^2\phi}{\partial r^2} \\
& +\frac{3}{320}\Lambda_{B,th}^2 \ \left(2\frac{dq_0}{dr}-\frac{q_0}{r_0}\right)^2 \sqrt{2\varepsilon_a} n_{i0}\frac{e\phi}{T_i}\ \frac{\partial^2\phi}{\partial\alpha^2}
\end{aligned}$$

This difference in density is the *polarization*.

The difference in density is expressed in terms of B_θ, as expected for a *neoclassical polarization*

$$\begin{aligned}
& n_i(\mathbf{X}) - \left\langle \left\langle n_i(\mathbf{X}+\mathbf{\Lambda}) \right\rangle_\theta \right\rangle_v \\
=\ & \frac{1}{4}\frac{m_i n_{i0} c^2}{eB_\theta^2}\left(2\varepsilon_a\right)^{3/2}\ n_{i0}\frac{e}{T_i}\frac{\partial^2\phi}{\partial r^2} \\
& +\frac{3}{320}\frac{m_i n_{i0} c^2}{eB_\theta^2}\ \left(2\frac{dq_0}{dr}-\frac{q_0}{r_0}\right)^2 \left(2\varepsilon_a\right)^{3/2} n_{i0}\frac{e\phi}{T_i}\ \frac{\partial^2\phi}{\partial\alpha^2}
\end{aligned}$$

The density of trapped ions has been approximated

$$\delta n_{i,trapped} \sim \sqrt{2\varepsilon_a} n_{i0}\frac{e\phi}{T_i}$$

In the formula above the first term is due to the radial excursions of the trapped ion.

The second term is due to the toroidal precession.

The results of previous calculations

$$\mathbf{j}_{pol} \approx \frac{16}{3\pi\sqrt{2}}\ \varepsilon_a^{3/2}\ \frac{n_i m_i c^2}{B_\theta^2}\ \frac{\partial \mathbf{E}_r}{\partial t}$$

The factor is

$$\sim \frac{c^2}{v_A^2}$$

which is the neoclassical polarization.

10.7 Radial drift of bananas

About the radial drift of banana it is relevant the concept of *turbulent equipartition*, in particular **Nycander Yankov** and the comparison with the *atmosphere.*

It is mentioned sometimes the invariant that consists of the integral of the poloidal magnetic field through the surface of the banana. This flux may be an invariant and plays a role when the banana are moving across the magnetic surfaces.

See also **Robertson Hinton,** and **Novakovskii Galeev Liu Sagdeev Hassam**.

The combination of a **toroidal electric field** E_φ and the **poloidal magnetic field** B_θ gives a *radial drift of the bananas*:

$$v_r = -\frac{E_\varphi}{B_\theta}$$

See **Robertson Hinton** neoclassical polarization.

10.8 Time of flight along the banana orbit

This is the time of bounce

$$\tau_{bounce} = \frac{1}{\sqrt{\varepsilon}} \frac{qR}{v_{th,i}}$$

However at **Rosenbluth Hazeltine Hinton** the bounce time on a banana orbit is

$$\tau_B \sim \frac{r}{v_\parallel \frac{B_\theta}{B}}$$

where we recognize $v_\parallel \frac{B_\theta}{B} = v_\theta$ the poloidal velocity. Then $\tau_B \sim (2\pi r)/v_\theta$.

$$\tau_B \sim \frac{1}{v_\parallel} \frac{1}{\frac{RB_\theta}{rB}\frac{1}{R}} = \frac{qR}{v_\parallel}$$

It looks like an error: the factor $\varepsilon^{-1/2}$ is missing. However the two expressions are identical because here $v_\parallel$ is the velocity of a trapped particle and this is $\sqrt{\varepsilon}$ fraction of v_{th}

$$v_\parallel^{trapped} = \sqrt{\varepsilon} v_{th}$$

The parallel velocity of a trapped particle is a fraction $\sqrt{\varepsilon}$ of the thermal velocity

$$v_{\parallel} \sim \sqrt{\varepsilon} v_{th}$$

and it is normal since the the parallel velocity must be very small, for trapped particles.

10.9 Special points of the trajectory

Rome Peng.

There are two points with special role in the description of the particle trajectories:

- **Stagnation point**; fast-ions injected with a certain pitch angle (angle between the ion velocity vector and the magnetic field vector) less than $\pi/4$ for the conditions described below ($x/a = 0.3$, energy $= 10\ keV$) will have quasi-circular orbits centered on the *co-injection stagnation point*; This is located at a larger major radius than the magnetic axis. Fast-ions injected in the same space-energy conditions but with pitch angle greater than $3\pi/4$ will follow nearly identical trajectories which are allmost circles centered at the *counter-injection stagnation point.*
- **Pinch point**. Where the vertical component of the velocity due to the **toroidal drift** $\mathbf{v}_D$ (curvature and gradB) is equal and opposite to the vertical component of the velocity along the magnetic line. At the *pinch point* the particle stangates in its vertical motion.

10.10 Squeezing of banana orbits in electric fields

This is described by **Shaing**.

And **poloidal rotation Hmode Hinton**.

It is used to explain the *dynamic* viscosity.

In the H mode layer there is a strong electric field.

10.11 Fraction of trapped particles

The fraction of trapped particles from **Hirschman 1988**

$$
\begin{aligned}
f_c &= \frac{3}{4}\left\langle B^2\right\rangle \int_0^{B_{\max}^{-1}} \frac{\lambda d\lambda}{\left\langle (1-\lambda B)^{1/2}\right\rangle} \quad \text{(fraction of circulating)} \\
f_t &= 1 - f_c
\end{aligned}
$$

10.12 The upper and lower bounds for the fraction of trapped particles Liu Liu

The general formula

$$
f_t = 1 - \frac{3}{4}\left\langle h^2\right\rangle \int_0^1 \frac{\lambda \, d\lambda}{\left\langle \sqrt{1-h\lambda}\right\rangle}
$$

where

$$
h = \frac{B}{B_{\max}}
$$

$$
B_{\max} \equiv \text{ max of } B \text{ on a magnetic surface}
$$

$$
\begin{aligned}
B_{\max} &= \frac{B_0}{1-\varepsilon} \\
B &= \frac{B_0}{1+\varepsilon\cos\theta}
\end{aligned}
$$

$$
h = \frac{B}{B_{\max}} = \frac{B_0/\left(1+\varepsilon\cos\theta\right)}{B_0/\left(1-\varepsilon\right)} = \frac{1-\varepsilon}{1+\varepsilon\cos\theta}
$$

and

$$
\langle h\rangle = \int_0^{2\pi} \frac{d\theta}{2\pi}\frac{1-\varepsilon}{1+\varepsilon\cos\theta}
$$

Using the symmetry

$$
\int_0^{\pi} \frac{d\theta}{1+\varepsilon\cos\theta} + \int_{\pi}^{2\pi} \frac{d\theta}{1+\varepsilon\cos\theta}
$$

substitution

$$
\begin{aligned}
\theta &= \varphi + \pi \\
d\theta &= d\varphi \\
\theta|_{\pi}^{2\pi} &= \varphi|_0^{\pi}
\end{aligned}
$$

and

$$\begin{aligned}\int_{\pi}^{2\pi}\frac{d\theta}{1+\varepsilon\cos\theta} &= \int_0^{\pi}\frac{d\varphi}{1+\varepsilon\cos(\pi+\varphi)}=\int_0^{\pi}\frac{d\varphi}{1+\varepsilon\cos\pi\cos\varphi}\\ &= \int_0^{\pi}\frac{d\varphi}{1-\varepsilon\cos\varphi}\end{aligned}$$

In **Gradshtein Ryzhik 3.613**

$$\begin{aligned}\int_0^{\pi}\frac{\cos nx}{1+a\cos x}dx &= \frac{\pi}{\sqrt{1-a^2}}\left(\frac{\sqrt{1-a^2}-1}{a}\right)^n\\ \text{for}\quad a^2 &< 1 \quad \text{and} \quad n\geq 0\end{aligned}$$

$$\int_0^{\pi}\frac{d\theta}{2\pi}\frac{1}{1+\varepsilon\cos\theta}=\frac{1}{2\pi}\frac{\pi}{\sqrt{1-\varepsilon^2}}$$

and

$$\frac{1}{2\pi}\int_0^{\pi}\frac{d\varphi}{1-\varepsilon\cos\varphi}=\frac{1}{2\pi}\frac{\pi}{\sqrt{1-\varepsilon^2}}$$

After summation

$$\begin{aligned}\langle h\rangle &= \int_0^{2\pi}\frac{d\theta}{2\pi}\frac{1-\varepsilon}{1+\varepsilon\cos\theta}\\ &= (1-\varepsilon)\left[\frac{1}{2}\frac{1}{\sqrt{1-\varepsilon^2}}+\frac{1}{2}\frac{1}{\sqrt{1-\varepsilon^2}}\right]\\ &= \frac{1-\varepsilon}{\sqrt{1-\varepsilon^2}}\end{aligned}$$

A good approximation is

$$\langle h\rangle = 1-\varepsilon$$

It will be necessary

$$\begin{aligned}\left\langle h^2\right\rangle &= \int_0^{2\pi}\frac{d\theta}{2\pi}\left(\frac{1-\varepsilon}{1+\varepsilon\cos\theta}\right)^2\\ &= \frac{(1-\varepsilon)^2}{2\pi}\int_0^{2\pi}d\theta\frac{1}{(1+\varepsilon\cos\theta)^2}\end{aligned}$$

In **Gradshtein Ryzhik 3.661** we find

$$\begin{aligned}
&\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^{n+1}} \\
= \quad & 2\times \frac{\pi}{(a^2-b^2)^{\frac{n+1}{2}}} P_n\left(\frac{a}{\sqrt{a^2-b^2}}\right) \\
= \quad & 2\times \frac{\pi}{2^n (a+b)^n \sqrt{a^2-b^2}} \sum_{k=0}^{n} \frac{(2n-2k-1)!!\,(2k-1)!!}{(n-k)!k!} \left(\frac{a+b}{a-b}\right)^k
\end{aligned}$$

This is valid for

$$a > |b|$$

Take

$$a = 1 \quad , \quad b = \varepsilon$$

We use the first formula, for $n = 1$

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{(1+\varepsilon\cos\theta)^2} &= \frac{2\pi}{(1-\varepsilon^2)^{\frac{2}{2}}} P_1\left(\frac{1}{\sqrt{1-\varepsilon^2}}\right) \\
&= \frac{2\pi}{1-\varepsilon^2} P_1\left(\frac{1}{\sqrt{1-\varepsilon^2}}\right)
\end{aligned}$$

whose approximation is, neglecting ε^2 and taking

$$P_1(x) = x \text{ i.e. } P_1\left(\frac{1}{\sqrt{1-\varepsilon^2}}\right) = \frac{1}{\sqrt{1-\varepsilon^2}}$$

we have

$$\int_0^{2\pi} \frac{d\theta}{(1+\varepsilon\cos\theta)^2} = \frac{2\pi}{1-\varepsilon^2}\frac{1}{\sqrt{1-\varepsilon^2}}$$

Then

$$\begin{aligned}
\langle h^2 \rangle &= \frac{(1-\varepsilon)^2}{2\pi}\frac{2\pi}{1-\varepsilon^2}\frac{1}{\sqrt{1-\varepsilon^2}} \\
&= \frac{(1-\varepsilon)^{3/2}}{\sqrt{1+\varepsilon}}
\end{aligned}$$

The fraction f_t is necessary for the bootstrap current.

The *upper* bound for the effective trapped fraction

$$f_t^{upper} = 1 - \frac{3}{4}\langle h^2 \rangle \int_0^1 \frac{\lambda\, d\lambda}{\sqrt{1-\lambda\langle h \rangle}}$$

or

$$f_t^{upper} = 1 - \frac{\langle h^2 \rangle}{\langle h \rangle^2} \left[1 - \sqrt{1 - \langle h \rangle} \ \left(1 + \frac{1}{2} \langle h \rangle \right) \right]$$

It is derived

$$f_t^{lower} = 1 - \langle h^2 \rangle \left\langle \frac{1}{h^2} \left(1 - \sqrt{1 - \langle h \rangle} \ \left(1 + \frac{1}{2} \langle h \rangle \right) \right) \right\rangle$$

Application

$$\langle h \rangle = 1 - \varepsilon$$

$$\left\langle h^2 \right\rangle = \frac{(1-\varepsilon)^{3/2}}{\sqrt{1+\varepsilon}}$$

In the limit

$$\varepsilon \to 0$$

we have

$$f_t = \gamma \sqrt{\varepsilon} + ...$$

where

$$\begin{aligned} \gamma &= \frac{3\sqrt{2}}{2} \left[1 + \int_0^1 \frac{dk}{k^2} \left(1 - \frac{\pi}{2E(k)} \right) \right] \\ &\approx 1.46 \end{aligned}$$

To verify this result, we calculate

$$\begin{aligned} f_t &= 1 - \frac{3}{4} \left\langle h^2 \right\rangle \int_0^1 \frac{\lambda \, d\lambda}{\left\langle \sqrt{1 - h\lambda} \right\rangle} \\ &= 1 - \frac{3}{4} \frac{(1-\varepsilon)^{3/2}}{\sqrt{1+\varepsilon}} \int_0^1 \frac{\lambda d\lambda}{\left\langle \sqrt{1 - \lambda \frac{1-\varepsilon}{1+\varepsilon\cos\theta}} \right\rangle} \end{aligned}$$

Separately the average at the denominator

$$\begin{aligned} & \left\langle \sqrt{1 - \lambda \frac{1-\varepsilon}{1+\varepsilon\cos\theta}} \right\rangle \\ &= \int_0^{2\pi} \frac{d\theta}{2\pi} \sqrt{1 - \lambda \frac{1-\varepsilon}{1+\varepsilon\cos\theta}} \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\theta \sqrt{\frac{1+\varepsilon\cos\theta - \lambda + \lambda\varepsilon}{1+\varepsilon\cos\theta}} \end{aligned}$$

Digression
Consider $\varepsilon \ll 1$, neglect ε^2,

$$\begin{aligned}
\left\langle \sqrt{1-\lambda\frac{1-\varepsilon}{1+\varepsilon\cos\theta}} \right\rangle &= \left\langle \sqrt{1-(\lambda-\lambda\varepsilon)\left(1-\varepsilon\cos\theta\right)} \right\rangle \\
&= \left\langle \sqrt{1-\lambda+\lambda\varepsilon+\left(\lambda\varepsilon-\lambda\varepsilon^2\right)\cos\theta} \right\rangle \\
&= \left\langle \sqrt{1-\lambda+\lambda\varepsilon+\lambda\varepsilon\cos\theta} \right\rangle
\end{aligned}$$

$$\begin{aligned}
&\left\langle \sqrt{a+b\cos\theta} \right\rangle \\
a &\equiv 1-\lambda+\lambda\varepsilon \\
b &\equiv \lambda\varepsilon
\end{aligned}$$

and

$$\begin{aligned}
\left\langle \sqrt{a+b\cos\theta} \right\rangle &= \frac{1}{2\pi}\int_0^{2\pi} d\theta\sqrt{a+b\cos\theta} \\
&= \frac{1}{2\pi}\int_0^{\pi} d\theta\sqrt{a+b\cos\theta} + \frac{1}{2\pi}\int_{\pi}^{2\pi} d\theta\sqrt{a+b\cos\theta}
\end{aligned}$$

making in the second the substitution

$$\theta = \varphi + \pi$$

the second is

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\pi}^{2\pi} d\theta\sqrt{a+b\cos\theta} \\
=&\frac{1}{2\pi}\int_0^{\pi} d\varphi\sqrt{a+b\cos\left(\varphi+\pi\right)} \\
=&\frac{1}{2\pi}\int_0^{\pi} d\varphi\sqrt{a-b\cos\varphi}
\end{aligned}$$

We use (3.676), with the condition $a > b$

$$\frac{1}{2\pi}\int_0^{\pi} d\theta\sqrt{a+b\cos\theta} = \frac{1}{2\pi}2\sqrt{a+b}\mathbf{E}\left(\sqrt{\frac{2b}{a+b}}\right)$$

and

$$\frac{1}{2\pi}\int_0^{\pi} d\theta\sqrt{a-b\cos\theta} = \frac{1}{2\pi}2\sqrt{a-b}\mathbf{E}\left(\sqrt{\frac{-2b}{a-b}}\right)$$

Regarding the last expression, we use 19.5.2 of NIST

$$\mathbf{E}\left(\kappa\right) = \frac{\pi}{2}\sum_{m=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_m\left(\frac{1}{2}\right)_m}{m!\; m!}\kappa^{2m}$$

which shows that

$$\mathbf{E}\left(-\kappa\right) = \mathbf{E}\left(\kappa\right)$$

Then

$$\frac{1}{2\pi}\int_0^{\pi} d\theta\sqrt{a-b\cos\theta} = 2\sqrt{a-b}\mathbf{E}\left(\sqrt{\frac{2b}{a-b}}\right)$$

This means

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{\pi} d\theta\sqrt{a+b\cos\theta} + \frac{1}{2\pi}\int_{\pi}^{2\pi} d\theta\sqrt{a+b\cos\theta}\\
=\;&\frac{1}{2\pi}\int_0^{\pi} d\theta\sqrt{a+b\cos\theta} + \frac{1}{2\pi}\int_0^{\pi} d\theta\sqrt{a-b\cos\theta}\\
=\;&\frac{1}{2\pi}2\sqrt{a+b}\mathbf{E}\left(\sqrt{\frac{2b}{a+b}}\right) + \frac{1}{2\pi}2\sqrt{a-b}\mathbf{E}\left(\sqrt{\frac{2b}{a-b}}\right)
\end{aligned}$$

For the arguments

$$\begin{aligned}
\frac{2b}{a+b} &= \frac{2\lambda\varepsilon}{1-\lambda+\lambda\varepsilon+\lambda\varepsilon} = \frac{2\lambda\varepsilon}{1-\lambda+2\varepsilon} = \frac{1}{1-\lambda}\frac{2\lambda\varepsilon}{1+\frac{2}{1-\lambda}\varepsilon}\\
&\approx \frac{2\lambda\varepsilon}{1-\lambda}\left(1+\frac{2\varepsilon}{1-\lambda}\right) = \frac{2\lambda\varepsilon}{1-\lambda}
\end{aligned}$$

$$\frac{2b}{a-b} = \frac{2\lambda\varepsilon}{1-\lambda+\lambda\varepsilon-\lambda\varepsilon} = \frac{2\lambda\varepsilon}{1-\lambda}$$

The arguments are identical.

$$\begin{aligned}
&\left\langle\sqrt{a+b\cos\theta}\right\rangle\\
=\;&\frac{1}{2\pi}2\sqrt{a+b}\mathbf{E}\left(\sqrt{\frac{2b}{a+b}}\right) + \frac{1}{2\pi}2\sqrt{a-b}\mathbf{E}\left(\sqrt{\frac{2b}{a-b}}\right)\\
=\;&\frac{\sqrt{1-\lambda+\lambda\varepsilon+\lambda\varepsilon}}{\pi}\mathbf{E}\left(\frac{2\lambda\varepsilon}{1-\lambda}\right) + \frac{\sqrt{1-\lambda}}{\pi}\mathbf{E}\left(\frac{2\lambda\varepsilon}{1-\lambda}\right)\\
=\;&\frac{1}{\pi}\mathbf{E}\left(\frac{2\lambda\varepsilon}{1-\lambda}\right)\left(\sqrt{1-\lambda}\sqrt{1+\frac{2\lambda\varepsilon}{1-\lambda}}+\sqrt{1-\lambda}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sqrt{1-\lambda}}{\pi}\mathbf{E}\left(\frac{2\lambda\varepsilon}{1-\lambda}\right)\left(1+\frac{\lambda\varepsilon}{1-\lambda}+1\right) \\
&= \frac{\sqrt{1-\lambda}}{\pi}\left(2+\frac{\lambda\varepsilon}{1-\lambda}\right)\mathbf{E}\left(\frac{2\lambda\varepsilon}{1-\lambda}\right)
\end{aligned}
$$

Returning

$$
\begin{aligned}
& 1-\frac{3}{4}\frac{(1-\varepsilon)^{3/2}}{\sqrt{1+\varepsilon}}\int_0^1 \frac{\lambda d\lambda}{\left\langle\sqrt{1-\lambda\frac{1-\varepsilon}{1+\varepsilon\cos\theta}}\right\rangle} \\
&= 1-\frac{3}{4}\frac{(1-\varepsilon)^{3/2}}{\sqrt{1+\varepsilon}}\int_0^1 \frac{\lambda d\lambda}{\frac{\sqrt{1-\lambda}}{\pi}\left(2+\frac{\lambda\varepsilon}{1-\lambda}\right)\mathbf{E}\left(\frac{2\lambda\varepsilon}{1-\lambda}\right)} \\
&= 1-\frac{3}{4}\frac{(1-\varepsilon)^{3/2}}{\sqrt{1+\varepsilon}}\int_0^1 \frac{\lambda d\lambda}{\mathbf{E}\left(\frac{2\lambda\varepsilon}{1-\lambda}\right)}\frac{\pi}{\sqrt{1-\lambda}}\frac{1}{2}\left(1-\frac{2\lambda\varepsilon}{1-\lambda}\right)
\end{aligned}
$$

10.13 Collisionless trapping/detrapping of ions

The paper **Neoclassical Parallel Viscosity_LowCollisionality Shaing** invokes the process of trapping/detrapping of ions in a collisionless plasma where there are magnetic field perturbations. The effect is due to the loss of toroidal symmetry and of invariance. The banana orbits are not closed in the poloidal plane anymore. They are *wobbling* (clatina, tremura).

The *neoclassical parallel viscosity* is calculated in the low-collisional - banana regime by **Taguchi**.

11 Drift velocity for gyrokinetic calculations

Note

In **Wong Burrell**

$$
\begin{aligned}
\mathbf{v}_D &= \frac{1}{\Omega_0}\left(v_\parallel^2+\lambda\right)\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\ln B \\
&\quad +O\left(\varepsilon^2\right)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{v}_D &\approx \frac{1}{\Omega_0}\frac{v_\parallel^2+\lambda}{B}\,\mathbf{J}\cdot\boldsymbol{\nabla}\ln B \\
&\sim \varepsilon^2\frac{v_D}{r}
\end{aligned}
$$

End

12 Drift velocity derived in Beer, Jenko2008.

We start from the most common expression of the drift velocity.

The curvature and $\boldsymbol{\nabla} B$ drifts are included

$$\begin{aligned}\mathbf{v}_D &= \frac{1}{\Omega} v_\parallel^2\, \widehat{\mathbf{n}} \times [(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}] \quad \text{(curvature)} \\ &+ \frac{1}{\Omega} \widehat{\mathbf{n}} \times \mu \boldsymbol{\nabla} B \qquad\qquad \text{(grad-}B\text{)}\end{aligned}$$

Now the following three formulas are used to transform the curvature term, more precisely $[(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}]$. This will involve the gradient of pressure of the plasma, but after quantitative evaluation, such term will be shown to be small.

$$(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}} = (\boldsymbol{\nabla} \times \widehat{\mathbf{n}}) \times \widehat{\mathbf{n}}$$

$$0 = -\boldsymbol{\nabla} p + \mathbf{j} \times \mathbf{B}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j}$$

The first formula is an identity, simply resulting from the expansion of the double vector product in the RHS, as follows

$$(\boldsymbol{\nabla} \times \widehat{\mathbf{n}}) \times \widehat{\mathbf{n}} = -\widehat{\mathbf{n}} \times (\boldsymbol{\nabla} \times \widehat{\mathbf{n}}) = -\boldsymbol{\nabla} \left(\overset{\downarrow}{\widehat{\mathbf{n}}} \cdot \widehat{\mathbf{n}} \right) + \widehat{\mathbf{n}}\, (\boldsymbol{\nabla} \cdot \widehat{\mathbf{n}})$$

The LHS can be written

$$\begin{aligned} -\varepsilon_{ijk} \widehat{n}_j \left[\varepsilon_{klm} \frac{\partial}{\partial x^l} \widehat{n}_m \right] &= -\varepsilon_{ijk} \varepsilon_{klm}\, \widehat{n}_j \frac{\partial}{\partial x^l} \widehat{n}_m \\ -\varepsilon_{kij} \varepsilon_{klm}\, \widehat{n}_j \frac{\partial}{\partial x^l} \widehat{n}_m &= -\left(\delta_{il}\, \delta_{jm} - \delta_{im}\, \delta_{jl} \right) \widehat{n}_j \frac{\partial}{\partial x^l} \widehat{n}_m \\ &= -\widehat{n}_j \frac{\partial}{\partial x^i} \widehat{n}_j + \widehat{n}_j \frac{\partial}{\partial x^j} \widehat{n}_i \\ &= -\frac{\partial}{\partial x^i} \left(\frac{1}{2} \sum_j \widehat{n}_j^2 \right) + \left(\sum_j \widehat{n}_j \frac{\partial}{\partial x^j} \right) \widehat{n}_i \end{aligned}$$

and since $\sum_j \widehat{n}_j^2 = 1$ it results

$$(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}}$$

and verifies the identity

$$(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla})\, \widehat{\mathbf{n}} = (\boldsymbol{\nabla} \times \widehat{\mathbf{n}}) \times \widehat{\mathbf{n}}$$

We start from

$$\nabla \times \mathbf{B} = \nabla\times (B\widehat{\mathbf{n}}) = B\,(\nabla\times\widehat{\mathbf{n}}) + \nabla B \times \widehat{\mathbf{n}}$$

from where the rotational of the versor of magnetic direction is obtained

$$\begin{aligned}\nabla\times\widehat{\mathbf{n}} &= \frac{1}{B}\,(\nabla \times \mathbf{B} - \nabla B \times \widehat{\mathbf{n}}) \\ &= \frac{1}{B}\,(\mu_0 \mathbf{j} - \nabla B \times \widehat{\mathbf{n}})\end{aligned}$$

Returning to curvature

$$\begin{aligned}(\widehat{\mathbf{n}} \cdot \nabla)\,\widehat{\mathbf{n}} &= (\nabla\times\widehat{\mathbf{n}}) \times \widehat{\mathbf{n}} \\ &= \frac{1}{B}\,(\mu_0 \mathbf{j} - \nabla B \times \widehat{\mathbf{n}}) \times \widehat{\mathbf{n}} \\ &= \frac{1}{B^2}\mu_0 \mathbf{j} \times \mathbf{B} - \frac{1}{B}\,(\nabla B \times \widehat{\mathbf{n}}) \times \widehat{\mathbf{n}}\end{aligned}$$

The first term can be replaced using the gradient of the pressure

$$0 = -\nabla p + \mathbf{j} \times \mathbf{B}$$

and the second will expand the double product

$$\begin{aligned}(\widehat{\mathbf{n}} \cdot \nabla)\,\widehat{\mathbf{n}} &= \frac{1}{B^2}\mu_0\ \nabla p \\ &\quad + \frac{1}{B}\left[\nabla B\ \ (\widehat{\mathbf{n}} \cdot \widehat{\mathbf{n}}) - \widehat{\mathbf{n}}\ \ (\nabla B \cdot \widehat{\mathbf{n}})\right] \\ &= \frac{1}{B^2}\mu_0 \nabla p + \frac{1}{B}\nabla B - \frac{1}{B}\widehat{\mathbf{n}}\ \ (\nabla B \cdot \widehat{\mathbf{n}})\end{aligned}$$

This will be replaced in the expression of the drift velocity

$$\begin{aligned}\mathbf{v}_D &= \frac{1}{\Omega}v_\parallel^2\ \widehat{\mathbf{n}} \times [(\widehat{\mathbf{n}} \cdot \nabla)\,\widehat{\mathbf{n}}] + \frac{1}{\Omega}\widehat{\mathbf{n}} \times \mu \nabla B \\ &= \frac{1}{\Omega}v_\parallel^2\ \widehat{\mathbf{n}} \times \left\{\frac{1}{B^2}\mu_0 \nabla p + \frac{1}{B}\nabla B - \frac{1}{B}\widehat{\mathbf{n}}\ \ (\nabla B \cdot \widehat{\mathbf{n}})\right\} + \frac{1}{\Omega}\widehat{\mathbf{n}} \times \mu \nabla B\end{aligned}$$

This is the final expression of the particle's drift velocity.

Comment on approximations of this $\mathbf{v}_D$ expression
After vector product with $\widehat{\mathbf{n}}$, the third term in the paranthesis is zero
The second term in the paranthesis and the last term are combined

$$\frac{1}{\Omega}\frac{v_\parallel^2 + \frac{v_\perp^2}{2}}{B}\widehat{\mathbf{n}} \times \nabla B$$

and the final form is

$$\begin{aligned}
\mathbf{v}_D &= \frac{1}{\Omega}\frac{v_\parallel^2 + \frac{v_\perp^2}{2}}{B}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} B \\
&\quad + \frac{1}{\Omega}\frac{v_\parallel^2}{B^2}\mu_0 \widehat{\mathbf{n}} \times \boldsymbol{\nabla} p
\end{aligned}$$

When $\beta \ll 1$ the term with gradient of pressure can be neglected. In this case, since we have the expression

$$\underset{\text{curvature}}{(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla})\,\widehat{\mathbf{n}}} = \frac{1}{B^2}\mu_0 \boldsymbol{\nabla} p + \frac{1}{B}\boldsymbol{\nabla} B - \frac{1}{B}\widehat{\mathbf{n}}\ (\boldsymbol{\nabla} B \cdot \widehat{\mathbf{n}})$$

this gives an approximation

$$(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla})\,\widehat{\mathbf{n}} \approx \boldsymbol{\nabla}_\perp \ln B$$

Define

$$\varepsilon = \frac{r}{R}$$

Consider

$$\begin{aligned}
\frac{\partial}{\partial r}\varepsilon &= \frac{\partial}{\partial r}\frac{r}{R_0 h} = \frac{1}{R_0}\frac{\partial}{\partial r}\frac{r}{1 + \frac{r}{R}\cos\theta} \\
&= \frac{1}{R_0}\frac{1}{1+\varepsilon\cos\theta} - \frac{r}{R_0}\frac{1}{(1+\varepsilon\cos\theta)^2}\frac{\partial}{\partial r}\left(1 + \frac{r}{R}\cos\theta\right) \\
&= \frac{1}{R_0}\frac{1}{1+\varepsilon\cos\theta} - \frac{r}{R_0}\frac{1}{(1+\varepsilon\cos\theta)^2}\left[\frac{1}{R}\cos\theta - \frac{r\cos\theta}{R^2}\frac{\partial R}{\partial r}\right]
\end{aligned}$$

The last term contains

$$\begin{aligned}
\frac{\partial R}{\partial r} &= R_0\frac{\partial}{\partial r}\left(1 + \frac{r}{R}\cos\theta\right) \\
&= R_0\frac{1}{R}\cos\theta - R_0 r\cos\theta\frac{1}{R^2}\frac{\partial R}{\partial r}
\end{aligned}$$

which is an equation for $\frac{\partial R}{\partial r}$

$$\frac{\partial R}{\partial r}\left[1 + \frac{R_0 r\cos\theta}{R^2}\right] = \frac{R_0}{R}\cos\theta$$

$$
\begin{aligned}
\frac{\partial R}{\partial r}\left(1+\frac{R_0 r\cos\theta}{R_0^2\, h^2}\right) &= \frac{1}{h}\cos\theta \\
\frac{\partial R}{\partial r} &= \frac{\frac{1}{h}\cos\theta}{1+\frac{r\cos\theta}{R_0 h^2}} = \frac{\frac{1}{h}\cos\theta}{1+\frac{\varepsilon}{h}\cos\theta}
\end{aligned}
$$

Then the square paranthesis, after replacing $\partial R/\partial r$ becomes

$$
\begin{aligned}
&\left[\frac{1}{R}\cos\theta - \frac{r\cos\theta}{R^2}\frac{\partial R}{\partial r}\right] \\
&= \frac{1}{R}\cos\theta\left(1-\frac{r}{R}\frac{\frac{1}{h}\cos\theta}{1+\frac{\varepsilon}{h}\cos\theta}\right) \\
&= \frac{\cos\theta}{R}\left(1-\frac{\frac{\varepsilon}{h}\cos\theta}{1+\frac{\varepsilon}{h}\cos\theta}\right) = \frac{\cos\theta}{R}\frac{1}{1+\frac{\varepsilon}{h}\cos\theta}
\end{aligned}
$$

We can do an expansion in the formula

$$
\frac{\cos\theta}{R}\frac{1}{1+\frac{\varepsilon}{h}\cos\theta} \approx \frac{\cos\theta}{R}\left[1-\frac{\varepsilon}{h}\cos\theta\right]
$$

and obtain

$$
\left[\frac{1}{R}\cos\theta - \frac{r\cos\theta}{R^2}\frac{\partial R}{\partial r}\right] = \frac{\cos\theta}{R}\left[1-\frac{\varepsilon}{h}\cos\theta\right]
$$

Now return to the derivative of ε, with

$$
\begin{aligned}
\frac{\partial}{\partial r}\varepsilon &\approx \frac{1}{R_0}\frac{1}{1+\varepsilon\cos\theta} - \frac{r}{R_0}\frac{1}{(1+\varepsilon\cos\theta)^2}\frac{\cos\theta}{R}\left(1-\frac{\varepsilon}{h}\cos\theta\right) \\
&= \frac{1}{R_0}\frac{1}{1+\varepsilon\cos\theta}\left[1-\frac{r\cos\theta}{1+\varepsilon\cos\theta}\frac{1}{R}\left(1-\frac{\varepsilon}{h}\cos\theta\right)\right] \\
&= \frac{1}{R_0}\frac{1}{1+\varepsilon\cos\theta}\left(1-\frac{\varepsilon\cos\theta}{1+\varepsilon\cos\theta}+\frac{\varepsilon\cos\theta}{1+\varepsilon\cos\theta}\frac{\varepsilon}{h}\cos\theta\right)
\end{aligned}
$$

We will neglect the ε^2 term

$$
\begin{aligned}
\frac{\partial}{\partial r}\varepsilon &\approx \frac{1}{R_0}\frac{1}{1+\varepsilon\cos\theta}\frac{1}{1+\varepsilon\cos\theta} \\
&= \frac{1}{R_0}\frac{1}{h^2} \\
&= \frac{1}{Rh}
\end{aligned}
$$

Collecting the results

$$
\begin{aligned}
\boldsymbol{\nabla} B &= \boldsymbol{\nabla}\frac{B_0}{1+\varepsilon\cos\theta} = \boldsymbol{\nabla}\frac{B_0}{R/R_0} \\
&= B_0\left[\frac{\partial}{\partial r}\left(\frac{1}{1+\varepsilon\cos\theta}\right)\widehat{\mathbf{e}}_r + \frac{\partial}{r\partial\theta}\left(\frac{1}{1+\varepsilon\cos\theta}\right)\widehat{\mathbf{e}}_\theta + \frac{\partial}{\partial z}\left(\frac{1}{1+\varepsilon\cos\theta}\right)\widehat{\mathbf{e}}_z\right] \\
&\approx B_0\left[\left(-\frac{\left(\frac{\partial\varepsilon}{\partial r}\right)\cos\theta}{(1+\varepsilon\cos\theta)^2}\right)\widehat{\mathbf{e}}_r - \frac{1}{r}\frac{-\varepsilon\sin\theta}{(1+\varepsilon\cos\theta)^2}\widehat{\mathbf{e}}_\theta + \ 0\right]
\end{aligned}
$$

The direction $\widehat{\mathbf{e}}_r$ gives

$$
\begin{aligned}
-\frac{\left(\frac{\partial\varepsilon}{\partial r}\right)\cos\theta}{(1+\varepsilon\cos\theta)^2} &= -\frac{\frac{1}{Rh}\cos\theta}{(1+\varepsilon\cos\theta)^2} \\
&= \frac{1}{r}\frac{1}{1+\varepsilon\cos\theta}\left[-\frac{\frac{1}{h}\varepsilon\cos\theta}{1+\varepsilon\cos\theta}\right]
\end{aligned}
$$

and we notice that we can make again an expansion of $1/h$ at the numerator. The last paranthesis is

$$
-\frac{\frac{1}{h}\varepsilon\cos\theta}{1+\varepsilon\cos\theta} \approx -\frac{\varepsilon\cos\theta\,(1-\varepsilon\cos\theta)}{1+\varepsilon\cos\theta} = -\frac{\varepsilon\cos\theta}{1+\varepsilon\cos\theta}
$$

the r direction is

$$
\begin{aligned}
\left(-\frac{\left(\frac{\partial\varepsilon}{\partial r}\right)\cos\theta}{(1+\varepsilon\cos\theta)^2}\right) &= \frac{1}{r}\frac{1}{1+\varepsilon\cos\theta}\left(-\frac{\varepsilon\cos\theta}{1+\varepsilon\cos\theta}\right) \\
&= -\frac{1}{r}\frac{\varepsilon\cos\theta}{h^2}
\end{aligned}
$$

Remember the notation R and $\widehat{\mathbf{e}}_R$ means radius relative to the main axis and respectively the direction along the radius, at a location given by θ.

We will use the geometric formula

$$
\widehat{\mathbf{e}}_R = \cos\theta\widehat{\mathbf{e}}_r - \sin\theta\widehat{\mathbf{e}}_\theta
$$

The equation becomes

$$
\begin{aligned}
\boldsymbol{\nabla} B &= B_0\frac{1}{(1+\varepsilon\cos\theta)^2}\left[-\varepsilon\cos\theta\frac{1}{r}\widehat{\mathbf{e}}_r + \frac{\varepsilon\sin\theta}{r}\ \widehat{\mathbf{e}}_\theta\right] \\
&= B_0\frac{\varepsilon}{rh^2}\left(-\cos\theta\widehat{\mathbf{e}}_r + \sin\theta\widehat{\mathbf{e}}_\theta\right)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\nabla} B &= \frac{1}{R}\left[B_0 \frac{-\cos\theta \widehat{\mathbf{e}}_r + \sin\theta \widehat{\mathbf{e}}_\theta}{h^2}\right] \\
&= \frac{-\widehat{\mathbf{e}}_R}{R}\left(\frac{B_0}{h^2}\right)
\end{aligned}
$$

and the variation of R is radial relative to the major axis.

$$
\boldsymbol{\nabla} B = B_0 \ \frac{1}{Rh^2}\left(-\cos\theta \widehat{\mathbf{e}}_r + \sin\theta \widehat{\mathbf{e}}_\theta\right)
$$

$$
\boldsymbol{\nabla} \ln B = \frac{-\cos\theta \widehat{\mathbf{e}}_r + \sin\theta \widehat{\mathbf{e}}_\theta}{h^2}
$$

$$
\boldsymbol{\nabla} \ln B = \frac{1}{h^2}\left(-\widehat{\mathbf{e}}_R\right)
$$

Therefore there remains NO variation of $1/R$ along the toroidal direction.

$$
(\boldsymbol{\nabla} B)_{toroidal} = 0
$$

Take the $\boldsymbol{\nabla} B$ projection on $\widehat{\mathbf{n}}$. The latter versor has θ and φ components.

$$
\widehat{\mathbf{n}} = \frac{B_\theta}{B}\widehat{\mathbf{e}}_\theta + \frac{B_\varphi}{B}\widehat{\mathbf{e}}_\varphi
$$

$$
\begin{aligned}
\boldsymbol{\nabla} B \cdot \widehat{\mathbf{n}} &= B_0 \ \frac{1}{Rh^2}\left(-\cos\theta \widehat{\mathbf{e}}_r + \sin\theta \widehat{\mathbf{e}}_\theta\right) \cdot \widehat{\mathbf{n}} \\
&= -B_0 \frac{1}{Rh^2}\cos\theta \ \left(\widehat{\mathbf{e}}_r \cdot \widehat{\mathbf{n}}\right) + B_0 \ \frac{1}{Rh^2} \ \sin\theta \left(\widehat{\mathbf{e}}_\theta \cdot \widehat{\mathbf{n}}\right)
\end{aligned}
$$

Taking

$$
\left(\widehat{\mathbf{e}}_\theta \cdot \widehat{\mathbf{n}}\right) = \frac{B_\theta}{B}
$$

$$
\nabla_{\|} B = \boldsymbol{\nabla} B \cdot \widehat{\mathbf{n}} = B_0 \ \frac{1}{Rh^2}\sin\theta \frac{B_\theta}{B} = \frac{B_\theta}{Rh}\sin\theta
$$

This means that a reasonable approximation is

$$
\begin{aligned}
\left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right) B &= \nabla_{\|} B \\
&= \frac{B_\theta}{Rh}\sin\theta
\end{aligned}
$$

Now we can calculate

$$
\begin{aligned}
\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}} &\approx \frac{1}{B}\boldsymbol{\nabla}B - \frac{1}{B}\widehat{\mathbf{n}}\ \left(\boldsymbol{\nabla}B\cdot\widehat{\mathbf{n}}\right) \\
&= \frac{1}{B}B_0\ \frac{1}{Rh^2}\left(-\cos\theta\widehat{\mathbf{e}}_r + \sin\theta\widehat{\mathbf{e}}_\theta\right) - \frac{1}{B}\widehat{\mathbf{n}}\left(\frac{B_\theta}{hR}\sin\theta\right) \\
&= \frac{1}{B}B_0\ \frac{1}{Rh^2}\left(-\cos\theta\widehat{\mathbf{e}}_r + \sin\theta\widehat{\mathbf{e}}_\theta\right) - \frac{1}{B}\frac{B_\theta}{hR}\sin\theta\left(\frac{B_\theta}{B}\widehat{\mathbf{e}}_\theta + \frac{B_\varphi}{B}\widehat{\mathbf{e}}_\varphi\right)
\end{aligned}
$$

or

$$
\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}} \approx \frac{-\cos\theta}{Rh}\widehat{\mathbf{e}}_r + \left(\frac{1}{Rh}\sin\theta - \left(\frac{B_\theta}{B}\right)^2\frac{1}{hR}\sin\theta\right)\widehat{\mathbf{e}}_\theta - \frac{B_\theta B_\varphi}{B^2}\frac{1}{hR}\sin\theta\widehat{\mathbf{e}}_\varphi
$$

Evaluation of the components

On r,

$$
\frac{-\cos\theta}{Rh}\widehat{\mathbf{e}}_r
$$

On θ,

$$
\left(\frac{1}{Rh}\sin\theta - \left(\frac{B_\theta}{B}\right)^2\frac{1}{hR}\sin\theta\right)\widehat{\mathbf{e}}_\theta \approx \frac{1}{Rh}\sin\theta\ \widehat{\mathbf{e}}_\theta
$$

On φ,

$$
\frac{B_\theta B_\varphi}{B^2}\frac{1}{hR}\sin\theta\widehat{\mathbf{e}}_\varphi
$$

(small ? since $B_\theta/B \ll 1$)

Then,

$$
\begin{aligned}
\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}} &\approx \frac{-\cos\theta}{Rh}\widehat{\mathbf{e}}_r + \frac{1}{Rh}\sin\theta\ \widehat{\mathbf{e}}_\theta + \frac{B_\theta B_\varphi}{B^2}\frac{1}{hR}\sin\theta\widehat{\mathbf{e}}_\varphi \\
&= \frac{1}{Rh}\left[-\cos\theta\widehat{\mathbf{e}}_r + \sin\theta\ \widehat{\mathbf{e}}_\theta\right] + \frac{B_\theta B_\varphi}{B^2}\frac{1}{hR}\sin\theta\widehat{\mathbf{e}}_\varphi
\end{aligned}
$$

and we recognize

$$
\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}} \approx \boldsymbol{\nabla}\ln B + \frac{B_\theta B_\varphi}{B^2}\frac{1}{hR}\sin\theta\widehat{\mathbf{e}}_\varphi
$$

$$
\left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}} \approx \boldsymbol{\nabla}_\perp\ln B
$$

which is $\approx -\frac{1}{Rh}\widehat{\mathbf{e}}_R$ with the versor along R directed from the major axis to the current point.

End comment

these formulas are used by **Jenko Hauf**.

This form is only to show the approximations that are doen to arrive to the well known form of the drift velocity,

$$\mathbf{v}_D = \frac{1}{\Omega}\frac{v_\parallel^2 + \frac{v_\perp^2}{2}}{B}\widehat{\mathbf{n}} \times \boldsymbol{\nabla} B$$

End comments.

We return to the form where the curvature $(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla})\,\widehat{\mathbf{n}}$ is not expanded

$$\mathbf{v}_D = \frac{1}{\Omega}v_\parallel^2 \;\widehat{\mathbf{n}} \times [(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla})\,\widehat{\mathbf{n}}] + \frac{1}{\Omega}\widehat{\mathbf{n}} \times \mu\boldsymbol{\nabla} B$$

and replace by extended formula the curvature part

$$\begin{aligned}(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla})\,\widehat{\mathbf{n}} &= \frac{1}{B^2}\mu_0 \mathbf{j} \times \mathbf{B} - \frac{1}{B}(\boldsymbol{\nabla} B \times \widehat{\mathbf{n}}) \times \widehat{\mathbf{n}} \\ &= \frac{1}{B^2}\mu_0 \mathbf{j} \times \mathbf{B} + \frac{1}{B}\boldsymbol{\nabla} B - \frac{1}{B}\widehat{\mathbf{n}}\;(\boldsymbol{\nabla} B \cdot \widehat{\mathbf{n}})\end{aligned}$$

replacing,

$$\mathbf{v}_D = \frac{1}{\Omega}v_\parallel^2 \;\widehat{\mathbf{n}} \times \left[\frac{1}{B^2}\mu_0 \mathbf{j} \times \mathbf{B} + \frac{1}{B}\boldsymbol{\nabla} B - \frac{1}{B}\widehat{\mathbf{n}}\;(\boldsymbol{\nabla} B \cdot \widehat{\mathbf{n}})\right] + \frac{1}{\Omega}\widehat{\mathbf{n}} \times \mu\boldsymbol{\nabla} B$$

The vector product of $\widehat{\mathbf{n}}$ with the last term is zero

Take separately the terms

The last term, containing μ is

$$\frac{1}{\Omega}\widehat{\mathbf{n}} \times \mu\boldsymbol{\nabla} B = \frac{m}{e}\frac{1}{B}\mu\;\widehat{\mathbf{n}} \times \boldsymbol{\nabla} B = \frac{m}{e}\frac{1}{B^2}\mu\;\mathbf{B} \times \boldsymbol{\nabla} B$$

The term with $v_\parallel^2$ is

$$\begin{aligned}&\frac{1}{\Omega}v_\parallel^2 \;\widehat{\mathbf{n}} \times \left[\frac{1}{B^2}\mu_0 \mathbf{j} \times \mathbf{B} + \frac{1}{B}\boldsymbol{\nabla} B\right] \\ =\;& \frac{1}{\Omega}v_\parallel^2 \left[\widehat{\mathbf{n}} \times \frac{1}{B}\boldsymbol{\nabla} B + \;\widehat{\mathbf{n}} \times \frac{1}{B^2}\mu_0 \mathbf{j} \times \mathbf{B}\right] \\ =\;& \frac{1}{\Omega}v_\parallel^2 \left\{\widehat{\mathbf{n}} \times \frac{1}{B}\boldsymbol{\nabla} B + \frac{1}{B^2}\mu_0\left[\mathbf{j}\,(\widehat{\mathbf{n}} \cdot \mathbf{B}) - \mathbf{B}\,(\widehat{\mathbf{n}} \cdot \mathbf{j})\right]\right\}\end{aligned}$$

where we replace

$$\mu_0 \mathbf{j} = \boldsymbol{\nabla} \times \mathbf{B}$$

just to conform to the standard form (**Jenko2008**), obtaining

$$\begin{aligned}
& \frac{1}{\Omega} v_\parallel^2 \left\{ \widehat{\mathbf{n}} \times \frac{1}{B} \boldsymbol{\nabla} B + \frac{1}{B^2} \mu_0 \left[\mathbf{j} \left(\widehat{\mathbf{n}} \cdot \mathbf{B} \right) - \mathbf{B} \left(\widehat{\mathbf{n}} \cdot \mathbf{j} \right) \right] \right\} \\
= & \frac{1}{\Omega} v_\parallel^2 \left\{ \widehat{\mathbf{n}} \times \frac{1}{B} \boldsymbol{\nabla} B + \frac{1}{B^2} \left[B \left(\boldsymbol{\nabla} \times \mathbf{B} \right) - \mathbf{B} \left(\frac{\mathbf{B}}{B} \cdot \left(\boldsymbol{\nabla} \times \mathbf{B} \right) \right) \right] \right\} \\
= & \frac{1}{\Omega} v_\parallel^2 \left\{ \widehat{\mathbf{n}} \times \frac{1}{B} \boldsymbol{\nabla} B + \frac{1}{B} \left(\boldsymbol{\nabla} \times \mathbf{B} \right) - \frac{\mathbf{B}}{B^2} \left[\frac{\mathbf{B}}{B} \cdot \left(\boldsymbol{\nabla} \times \mathbf{B} \right) \right] \right\}
\end{aligned}$$

We take out a factor

$$\frac{1}{B^2}$$

$$\frac{1}{\Omega} v_\parallel^2 \frac{1}{B^2} \left\{ \widehat{\mathbf{B}} \times \boldsymbol{\nabla} B + B \left(\boldsymbol{\nabla} \times \mathbf{B} \right) - \frac{\mathbf{B}}{B} \left[\mathbf{B} \cdot \left(\boldsymbol{\nabla} \times \mathbf{B} \right) \right] \right\}$$

or

$$\frac{m}{e} \frac{1}{B} v_\parallel^2 \frac{1}{B^2} \left\{ \widehat{\mathbf{B}} \times \boldsymbol{\nabla} B + B \left(\boldsymbol{\nabla} \times \mathbf{B} \right) - \frac{\mathbf{B}}{B} \left[\mathbf{B} \cdot \left(\boldsymbol{\nabla} \times \mathbf{B} \right) \right] \right\}$$

the full expression for the drift velocity is

$$\begin{aligned}
\mathbf{v}_D = & \frac{m}{e} \frac{1}{B^2} \mu \, \widehat{\mathbf{B}} \times \boldsymbol{\nabla} B \\
& + \frac{m}{e} \frac{1}{B} v_\parallel^2 \frac{1}{B^2} \left\{ \widehat{\mathbf{B}} \times \boldsymbol{\nabla} B + B \left(\boldsymbol{\nabla} \times \mathbf{B} \right) - \frac{\mathbf{B}}{B} \left[\mathbf{B} \cdot \left(\boldsymbol{\nabla} \times \mathbf{B} \right) \right] \right\}
\end{aligned}$$

This is the expression coming from the full drift velocity, in **Jenko2008**.

Note. In **Jenko2008** there is a factor

$$\frac{1}{q}$$

to this expression. This seems to an error, here by q it is understood e, the charge. The formula above already contains this factor $1/e$.
End

The last term, of the electric velocity

$$\frac{-\boldsymbol{\nabla} \phi^{eff} \times \mathbf{B}}{B^2}$$

The denominator in the expression of the particle guiding center velocity , $\mathbf{B}^*$, is derived according to Littlejohn, as reminded in the next Section.

13 Variational principle for guiding center equations Littlejohn

NOTE several coments are in **Hahm Fong**.

The variational equation

$$\delta \int L \, dt = 0$$

$$L = \frac{1}{\varepsilon}\mathbf{A}^* \cdot \frac{d\mathbf{X}}{dt} + \epsilon\mu\frac{d\zeta}{dt} - H$$

$$\begin{aligned} H &= \frac{1}{2}U^2 + \mu B + \Phi \\ & \text{hamiltonian} \end{aligned}$$

where $\mu = \frac{v_\perp^2}{2B}$, $\zeta \equiv$gyroangle, $\mathbf{A}$ is the magnetic potential.

The notation ϵ is more complicated.

It is

$$\epsilon = \frac{\rho_L}{a}$$

where ρ_L is the Larmor gyration radius and a is either the minor radius or the typical length of variation of the equilibrium, $\epsilon \sim L_n$ for example.

ϵ will be considered a small quantity and used for expansion.

Here

$$U \equiv \text{parallel velocity of the guiding center (usually } v_\parallel)$$

$$\begin{aligned} \frac{d\mu}{dt} &= 0 \\ \mu &\equiv \text{constant of motion} \end{aligned}$$

$$\frac{d\zeta}{dt} = \frac{B}{\epsilon}$$

The reason to use ϵ, a measure of weak spatial variation $\sim \rho/L$, is to introduce new variables $\mathbf{E}$ instead of the electric field, $\mathbf{E}^{phys} = \epsilon\ \mathbf{E}$. And electric potential $\phi^{phys} = \epsilon\ \phi$.

In addition, the time variation will be "expanded"

$$\begin{aligned} t &\rightarrow \tau \\ \tau &= \epsilon t \ \text{ slow time scale} \end{aligned}$$

The modified vector potential

$$\mathbf{A}^* = \mathbf{A} + \epsilon U \ \widehat{\mathbf{n}}$$

and for magnetic field

$$\mathbf{B}^* = \mathbf{B} + \epsilon U \ \boldsymbol{\nabla}\times\widehat{\mathbf{n}}$$

and for the electric field

$$\begin{aligned} \mathbf{E}^* &= -\frac{\partial \mathbf{A}^*}{\partial \tau} - \boldsymbol{\nabla}\Phi \\ &= \mathbf{E} - \epsilon U \ \frac{d\widehat{\mathbf{n}}}{d\tau} \end{aligned}$$

NOTE

This modification of the magnetic potential (and, as a consequence, of the magnetic and electric vectors) must be seen as the identification of the "drift magnetic surfaces" $\mathbf{A} \rightarrow \mathbf{A}^* = \mathbf{A} + \frac{v_{\parallel}\mathbf{B}}{\Omega}$. They are explained in **Hazeltine Hinton** Eq.(3.25) and in **Catto Kagan** (more recently).

The magnetic surfaces that correspond to

$$\begin{aligned} \mu &= \text{ct} \\ \epsilon &= \text{ct} \\ \psi^* &= \text{ct} \end{aligned}$$

are called *drift surfaces* [**Morozov Solovev**]

$$\mathbf{B}^* \cdot \boldsymbol{\nabla}\psi^* = 0$$

The guiding center motion is confined to this surface ψ^*.

$$\mathbf{v}_{guiding-center} = \frac{\mathbf{B}^*}{B} v_{\parallel}$$

$$\psi^* = \psi - \frac{I}{\Omega} v_{\parallel}$$

where $I \sim RB_T$.

END

The equation derived by Littlejohn

$$\mathbf{E}^* + \frac{1}{\varepsilon}\frac{d\mathbf{X}}{dt} \times \mathbf{B}^* = \frac{dU}{d\tau} \ \widehat{\mathbf{n}} + \mu \boldsymbol{\nabla} B$$

Since U is the parallel velocity

$$U = \widehat{\mathbf{n}} \cdot \frac{d\mathbf{X}}{dt}$$

the time variation of the position vector can be found from the two equations above

$$\frac{1}{\varepsilon}\frac{d\mathbf{X}}{dt} \times \mathbf{B}^* = \frac{dU}{d\tau}\widehat{\mathbf{n}} + (\mu \boldsymbol{\nabla} B - \mathbf{E}^*)$$

This is vector-multiplied by $\widehat{\mathbf{n}}$,

$$\begin{aligned}
\widehat{\mathbf{n}} \times \frac{1}{\varepsilon}\left(\frac{d\mathbf{X}}{dt} \times \mathbf{B}^*\right) &= \widehat{\mathbf{n}} \times (\mu \boldsymbol{\nabla} B - \mathbf{E}^*) \\
\frac{1}{\varepsilon}\frac{d\mathbf{X}}{dt}(\widehat{\mathbf{n}} \cdot \mathbf{B}^*) - \mathbf{B}^*\left(\widehat{\mathbf{n}} \cdot \frac{1}{\varepsilon}\frac{d\mathbf{X}}{dt}\right) &= \widehat{\mathbf{n}} \times (\mu \boldsymbol{\nabla} B - \mathbf{E}^*)
\end{aligned}$$

we now make explicit in the left hand side

$$(\widehat{\mathbf{n}} \cdot \mathbf{B}^*) = B_{\parallel}^*$$

and

$$-\mathbf{B}^*\left(\widehat{\mathbf{n}} \cdot \frac{1}{\varepsilon}\frac{d\mathbf{X}}{dt}\right) = -\mathbf{B}^*\frac{1}{\varepsilon}U$$

then

$$\begin{aligned}
\frac{1}{\epsilon}\frac{d\mathbf{X}}{dt}\ B_{\parallel}^* &= \frac{1}{\epsilon}U\ \mathbf{B}^* + \widehat{\mathbf{n}} \times (\mu \boldsymbol{\nabla} B - \mathbf{E}^*) \\
\frac{d\mathbf{X}}{dt} &= \frac{1}{B_{\parallel}^*}\left[U\ \mathbf{B}^* + \epsilon\ \widehat{\mathbf{n}} \times (\mu \boldsymbol{\nabla} B - \mathbf{E}^*)\right]
\end{aligned}$$

It is noted

$$B_{\parallel}^* = B + \epsilon U\ \widehat{\mathbf{n}} \cdot (\boldsymbol{\nabla} \times \widehat{\mathbf{n}})$$

which is obtained by

$$B_{\parallel}^* = \widehat{\mathbf{n}} \cdot \mathbf{B}^*$$

The formula to be used further is

$$\frac{d\mathbf{X}}{dt} = \frac{1}{B_{\parallel}^*}\left[U\ \mathbf{B}^* + \epsilon\ \widehat{\mathbf{n}} \times (\mu \boldsymbol{\nabla} B - \mathbf{E}^*)\right]$$

where we take

$$\mathbf{E}^* \equiv 0$$

then we have

$$\frac{d\mathbf{X}}{dt} = \frac{1}{B_{\parallel}^*}\left[U\ \mathbf{B}^* + \epsilon\ \widehat{\mathbf{n}} \times \mu \boldsymbol{\nabla} B\right]$$

Actually, the last term in the formula above must be written in the extended form, including all parts of the drift velocity (what is here is only the *magnetic drift*)

$$\begin{aligned} & \frac{1}{B_{\parallel}^{*}}\left[\epsilon\,\widehat{\mathbf{n}}\times\mu\boldsymbol{\nabla}B\right] \\ \rightarrow\; & \frac{1}{B_{\parallel}^{*}}\left[\mathbf{v}_{D}\right] \end{aligned}$$

and $\mathbf{v}_D$ is calculated above.

And the first term is

$$\begin{aligned} \frac{1}{B_{\parallel}^{*}}\left[U\ \mathbf{B}^{*}\right] & \rightarrow U\widehat{\mathbf{n}} \\ & \sim \mathbf{v}_{\parallel} \end{aligned}$$

This means to use as drift velocity (magnetic and curvature drifts)

$$\begin{aligned} & \frac{m}{e}\frac{1}{B^{2}}\mu\ \widehat{\mathbf{B}}\times\boldsymbol{\nabla}B \\ & +\frac{m}{e}\frac{1}{B}v_{\parallel}^{2}\ \frac{1}{B^{2}}\left\{\widehat{\mathbf{B}}\times\boldsymbol{\nabla}B+B\left(\boldsymbol{\nabla}\times\mathbf{B}\right)-\frac{\mathbf{B}}{B}\left[\mathbf{B}\cdot\left(\boldsymbol{\nabla}\times\mathbf{B}\right)\right]\right\} \end{aligned}$$

which is introduced in the expression of $\frac{d\mathbf{X}}{dt}$.

Before doing that, we must recall the explanations regarding the necessity to define the *drift surfaces* by modifying the poloidal flux function (surface function) ψ to ψ^*, as shown by **Morozov Solovev, Littlejohn, Hazeltine Hinton, Catto Kagan**.

It is found that

$$\mathbf{v}_{guiding-center}=\frac{\mathbf{B}^{*}}{B}v_{\parallel}$$

Then, everytime we will have a combination of parallel velocity $v_{\parallel}$ and of guiding-centre velocity, $\mathbf{v}_D$ there will be a coefficient

$$\frac{B}{B_{\parallel}^{*}}$$

in front of the guiding centre drift.

This is

$$\begin{aligned}
& \frac{d\mathbf{X}}{dt} \\
= \; & \mathbf{v}_{\|} + \frac{B}{B_{\|}^{*}} \left\{ \frac{m}{e} \frac{1}{B^{2}} \mu \, \widehat{\mathbf{B}} \times \boldsymbol{\nabla} B \right. \\
& + \frac{m}{e} \frac{1}{B} v_{\|}^{2} \, \frac{1}{B^{2}} \left(\widehat{\mathbf{B}} \times \boldsymbol{\nabla} B + B \left(\boldsymbol{\nabla} \times \mathbf{B} \right) - \frac{\mathbf{B}}{B} \left[\mathbf{B} \cdot \left(\boldsymbol{\nabla} \times \mathbf{B} \right) \right] \right) \\
& \left. + \text{electric drift} \right\}
\end{aligned}$$

This is the equation used by **Jenko2008**.

14 Clebsch variables and particle motion

This problem is related to the use of *field aligned coordinates* in the gyrokinetic calculations.

The motion of particles is essential.

The texts are: **field aligned Cowley Beer Hammet**, for gyrokinetic code.

The variables

$$\mathbf{B} = \boldsymbol{\nabla} \alpha \times \boldsymbol{\nabla} \psi$$

where

$$\alpha = \varphi - q(\psi)\,\theta - \nu(\varphi, \theta, \psi)$$

The vriable α is defined such that the magnetic lines are straight lines in the plane

$$(\alpha, \psi) \text{ - plane}$$

or

$$(\zeta, \psi) \text{ - plane}$$

where

$$\begin{aligned}
\zeta \quad = \quad & \varphi - \nu(\psi, \varphi, \theta) \\
& \text{new toroidal coordinate}
\end{aligned}$$

NOTE in **Hahm Fong** the first two terms $\varphi - q(\psi)\,\theta$ are called "perpendicular, but not radial". **END.**

The coordinates are

$$x = \frac{q_0}{B_0 r_0} (\psi - \psi_0)$$
$$y = -\frac{r_0}{q_0} (\alpha - \alpha_0)$$
$$z = \theta$$

For comparison **Jenko2008** uses

$$\mathbf{B} = \boldsymbol{\nabla}\beta \times \boldsymbol{\nabla}\psi$$

$$\beta = q(r)\chi - \varphi$$

15 Orbits near the magnetic axis

Paper by **Rome Peng**.

Paper **neoclassical center satake 2002**.
Lagrangian description of potato orbits.

Large orbit neoclassical transport.

16 Instabilities in the velocity space Berk Galeev

The paper by **Berk Galeev**.

Instabilities in the space of velocity can appear if there are gradients which arise from the geometry.

For example, at the edge of the tokamak, the particles may touch the limiter and get out from plasma. The particles have different velocities and the orbits depart from the magnetic surface in a measure which depends on this velocity: the spatial drift of the particle is determined by the perpendicular velocity. Those particles that have large radial excursions can touch the limiter. The loss of some particles will produce a hole in the velocity space. Also, an electric field in the plasma.

$$\begin{aligned} \mathbf{B} &= \frac{B_0}{h}\widehat{\mathbf{e}}_\varphi \\ h &= 1 + \frac{r}{R}\cos\theta \end{aligned}$$

The equations

$$
\begin{aligned}
\frac{dr}{dt} &= -\frac{1}{\Omega}\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{R}\sin\theta \\
\frac{rd\theta}{dt} &= -\frac{1}{\Omega}\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{R}\cos\theta - v_\parallel\frac{r}{qR}+\frac{1}{B}\frac{\partial\Phi}{\partial r}
\end{aligned}
$$

These two equations, combined,

$$
\frac{1}{\Omega}\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{R}\frac{d}{dr}\left(r\cos\theta\right) = -v_\parallel\frac{r}{qR}+\frac{1}{B}\frac{\partial\Phi}{\partial r}
$$

The two constants of motion

$$
\begin{aligned}
E &= \frac{1}{2}\left(v_\parallel^2+v_\perp^2\right)+\frac{e}{m}\Phi \\
\mu &= \frac{v_\perp^2}{2B}
\end{aligned}
$$

and

$$
v_\parallel = \pm\sqrt{2\left(E-\mu B-\frac{e}{m}\Phi\right)}
$$

From this expression for $v_\parallel$ we derive

$$
\begin{aligned}
\frac{e}{m}\frac{\partial\Phi}{\partial r} &\approx -v_\parallel\frac{dv_\parallel}{dr}+\frac{\mu B}{R}\frac{d}{dr}\left(r\cos\theta\right) \\
&\quad +O\left(\frac{r^2}{R^2}\right)
\end{aligned}
$$

This expression is replaced in the formula for the derivative of $r\cos\theta$ above,

$$
\frac{v_\parallel}{\Omega}\frac{1}{R}\frac{d}{dr}\left(r\cos\theta\right) = -\frac{r}{qR}-\frac{1}{\Omega}\frac{dv_\parallel}{dr}
$$

where

$$
R = R_0 h
$$

The hypothesis of **Berk Galeev** is that $v_\parallel$ has a weak variation with r and the RHS term can be combined with the derivative of $v_\parallel$

$$
\frac{d}{dr}\left[\frac{\Omega}{R}\int^r\frac{r}{q}dr+v_\parallel\left(1+\frac{r}{R}\cos\theta\right)\right]=0
$$

Then we have a new invariant

$$J \equiv \frac{\Omega}{R} \int^r \frac{r}{q} dr + v_\parallel \left(1 + \frac{r}{R}\cos\theta\right)$$

This invariant is taken for a reference point (r_0, θ_0) and is expanded in order 2 in the difference

$$r - r_0$$

is

$$\begin{aligned} & \frac{1}{2}\left[\Omega\ \Theta^2 - v_\parallel \frac{d\Theta}{dr} + \frac{d}{dr}\left(\frac{1}{B}\frac{\partial\Phi}{\partial r}\right)\right](r - r_0)^2 \\ & - \left(v_\parallel - \frac{1}{\Theta}\frac{1}{B}\frac{\partial\Phi}{\partial r}\right)\Theta\,(r - r_0) \\ & - \frac{1}{\Omega}\frac{\frac{v_\perp^2}{2} + v_\parallel^2}{R} r\,(\cos\theta - \cos\theta_0) \\ = \ & 0 \end{aligned}$$

where

$$\Theta \equiv \frac{r}{qR} \quad \text{at} \quad r = r_0$$

This formula introduces a quantity

$$\Delta v = v_\parallel - \frac{1}{\Theta} v_E$$

which is very close to 0 at equilibrium, since the polodal velocity v_E is projected onto the parallel direction with the factor $1/\Theta$ and compared with $v_\parallel$. The fact that Δv is close to zero results from the need to have as small as possible the poloidal rotation. The poloidal rotation induced by the radial gradient of the electrostatic potential Φ in the toroidal magnetic field B must be compensated by a displacement of an element of plasma along the magnetic line such that the *poloidal* displacement is cancelled.

If however this is not exactly realised and Δv is not zero, the second order algebraic equation leads to

$$r - r_0 = -\frac{1}{v_\parallel - \frac{1}{\Theta} v_E} \frac{1}{\Omega} \frac{\frac{v_\perp^2}{2} + v_\parallel^2}{R} r_0\,(\cos\theta - \cos\theta_0)$$

The formula allows to see that for

$$\Delta v > 0$$

(which means a certain projection in the poloidal direction) the *ions* travel in the clockwise direction. This means they travel on orbits that are *external* to the surface r_0.

For

$$\Delta v < 0$$

the *ions* travel in counter-clockwise direction. They travel on orbits that are *inside* the surface r_0.

The solution for any Δv is

$$r - r_0 = \frac{\Delta v \pm \sqrt{(\Delta v)^2 + 2r\left(\Omega - \frac{1}{\Theta}\frac{d\Theta}{dr}\frac{v_\parallel}{\Theta} + \frac{1}{\Theta^2}\frac{dv_E}{dr}\right)\left(\frac{1}{\Omega}\frac{\frac{v_\perp^2}{2} + v_\parallel^2}{R}\right)(\cos\theta - \cos\theta_0)}}{\Omega\Theta - v_\parallel \frac{1}{\Theta}\frac{d\Theta}{dr} + \frac{1}{\Theta}\frac{dv_E}{dr}}$$

where

$$v_E = \frac{1}{B}\frac{\partial \Phi}{\partial r}$$

From the square root we see that the particle is trapped if

$$(\Delta v)^2 < 4r\left(\Omega - \frac{1}{\Theta}\frac{d\Theta}{dr}\frac{v_\parallel}{\Theta} + \frac{1}{\Theta^2}\frac{dv_E}{dr}\right)\left(\frac{1}{\Omega}\frac{\frac{v_\perp^2}{2} + v_\parallel^2}{R}\right)$$

the maximum displacement for a barely *trapped* particle is

$$r - r_0 \approx 4\frac{1}{\Omega\Theta}\sqrt{\frac{v_\perp^2}{2B(r)}B_0\frac{r}{R}}$$

and occurs at $\theta = 0$.

The maximum displacement for a barely circulating particle is

$$r - r_0 \approx 2\frac{1}{\Omega\Theta}\sqrt{\frac{v_\perp^2}{2B(r)}B_0\frac{r}{R}}$$

at $\theta = \pi$.

The fluid approach.

It is intended to calculate the density variation with account taken of the trapped/circulating particles and their drifts.

The drift of a particle is simulated through a *gravitational* force acting in vertical direction

$$\mathbf{g} = \frac{v_{th}^2}{R/2} \widehat{\mathbf{e}}_{vert}$$

The *pressure* for species j is

$$P_j = n m_j v_{th,j}^2$$

The equation for the momentum

$$0 = -v_{th}^2 \boldsymbol{\nabla} n + n\mathbf{g} - n\,|e|\,\frac{\partial \Phi}{\partial r}\widehat{\mathbf{e}}_r + n\frac{|e|}{m}\mathbf{v}\times\mathbf{B}$$

and the continuity

$$\boldsymbol{\nabla}\cdot(n\mathbf{v}) = 0$$

Approximated conclusions

$$\begin{aligned} v_r &= 0 \\ \frac{d}{d\theta}(n v_\theta) &= 0 \ \rightarrow \ n v_\theta = j_\theta(r) \end{aligned}$$

Projecting the *momentum* equation (above) on the θ direction we have

$$\begin{aligned} n(r,\theta) &= n_0(r)\exp\left(2\frac{r}{R}\cos\theta\right) \\ &\quad \times \exp\left[-\frac{e}{m}\Phi(r)\frac{1}{v_{th}^2}\right] \end{aligned}$$

The second factor is an integration constant for the θ integration. The integration on θ of the equation for momentum actually involves the first two terms

$$-v_{th}^2 \boldsymbol{\nabla} n + n\mathbf{g}$$

Now the equation for *momentum* is projected on r.

$$\begin{aligned} \Omega\ n v_\theta &= \exp\left(2\frac{r}{R}\cos\theta - \frac{e}{m}\Phi\frac{1}{v_{th}^2}\right)\times v_{th}^2\frac{\partial n_0}{\partial r} \\ &\quad +\Omega\ n v_\varphi \frac{B_\theta}{B_\varphi} \end{aligned}$$

In the LHS we have

$$n v_\theta = j_\theta(r)$$

so the other current

$$nv_\varphi = j_\varphi$$

must cancel the θ dependence in the RHS. This comes from the exponetial. The term is expanded in r/R,

$$\begin{aligned} j_\theta(r) &= \frac{1}{\Omega} v_{th}^2 \frac{\partial n_0}{\partial r} \\ &\times \exp\left(-\frac{1}{v_{th}^2}\frac{e}{m}\Phi\right) \\ &\text{this is the diamagnetic current} \end{aligned}$$

$$\begin{aligned} j_\varphi &= 2q\frac{1}{\Omega} v_{th}^2 \frac{\partial n_0}{\partial r}\cos\theta \\ &\times \exp\left(-\frac{1}{v_{th}^2}\frac{e}{m}\Phi\right) \end{aligned}$$

This is the "rotational current" which is actually the Pfirsch Schluter current.

The distribution function for circulating and for trapped particles.
the invariants

$$E, \mu, J$$

The other variable is

$$\sigma \equiv sign\left(v_\parallel\right)$$

The *distribution function is any expression of these invariants.*
The orbit of a particle is determined by the variables (θ_0, r_0).
For $\Phi \equiv 0$ and for *untrapped* particles

$$f = F(E, \mu) \int_0^{2\pi} \frac{d\theta_0}{2\pi} \int_0^a dr_0 \; n(r_0) \; \delta(J - J_\sigma) \left| \frac{\partial J_\sigma}{\partial r_0} \right|$$

where

$$\begin{aligned} J_\sigma &= \frac{\Omega}{R}\int_0^{r_0} \frac{r}{q} dr \\ &+\sigma\sqrt{2\left(E - \mu B(r_0, \theta_0)\right)}\left(1 + \frac{r_0}{R}\cos\theta_0\right) \end{aligned}$$

The magnetic field

$$B(r, \theta) = B_0\left(1 - \frac{r}{R}\cos\theta\right)$$

and F is a Maxwelliana

$$F=\frac{1}{\left(2\pi v_{th}^{2}\right)^{3/2}}\exp\left(-\frac{E}{v_{th}^{2}}\right)$$

The distribution function for *trapped* particles must NOT contain the variable σ since this, for trapped particles is NOT an invariant.

It is adopted

$$f=\frac{1}{2}F\int_{-\frac{1}{2}\theta_m}^{\frac{1}{2}\theta_m}\frac{1}{\theta_m}d\theta_0\int_0^a dr_0\ n\left(r_0\right)\sum_{\sigma=\pm1}\delta\left(J-J_\sigma\right)\left|\frac{\partial J_\sigma}{\partial r_0}\right|$$

Here the angle

$$\theta_m=2\arccos\left[\frac{\mu B_0-E}{\mu B_0}\frac{R}{r_0}\right]$$

is the limit on θ of the orbit of the trapped particle characterized by the initial variables (θ_0, r_0).

To calculate the integrals one has to replace $n\left(r_0\right)$ with an expression in which occurs $n\left(r\right)$.

this is done by expanding $n\left(r_0\right)$ as

$$n\left(r_0\right)=n\left(r\right)-\left(r-r_0\right)\frac{dn}{dr}$$

and the condition

$$J=J_\sigma$$

determines $\left(r-r_0\right)$

$$\begin{aligned} r-r_0 &= \frac{1}{\Omega}\frac{qR}{r}\left\{-v_\parallel\left(1+\frac{r}{R}\cos\theta\right)\right.\\ &\left.+\sigma\left(1+\frac{r}{R}\cos\theta_0\right)\sqrt{2\left(E-\mu B_0+\mu B\frac{r}{R}\cos\theta_0\right)}\right\}\end{aligned}$$

Using this formula in the expression of the distribution function and taking the average

$$\begin{aligned}\overline{\left(r-r_0\right)} &= \int\frac{d\theta_0}{2\pi}\int dr_0\left(r-r_0\right)\ \delta\left(J-J_\sigma\right)\left|\frac{\partial J_\sigma}{\partial r}\right|\\ &= -\frac{q}{v_\parallel}\frac{\frac{v_\perp^2}{2}+v_\parallel^2}{\Omega}\cos\theta\end{aligned}$$

For the *circulating* but nearly trapped particles

$$\overline{(r-r_0)} = \frac{1}{\Omega}\frac{qR}{r}\left[-v_\parallel + \sigma\sqrt{2\mu B_0\frac{r}{R}} \times \int_0^{2\pi}\frac{d\theta_0}{2\pi}\sqrt{\frac{E-\mu B_0}{\mu B_0}R + \cos\theta_0}\right]$$

For the *trapped* particles one has to sum over σ

$$\overline{(r-r_0)} = -\frac{1}{\Omega}\frac{qR}{r}v_\parallel$$

The regions of *trapped* and *circulating* particles are separated by

$$\frac{E-\mu B_0}{\mu B_0} = \frac{r}{R}$$

At this point there is a discontinuity between the values of $\overline{(r-r_0)}$.

For the circulating particles the limit close to this point is

$$\begin{aligned}\overline{(r-r_0)} &= -\sigma\times 2\frac{q}{\Omega}\sqrt{\mu B\frac{R}{r}}\\ &\times\left[\left|\cos\left(\frac{\theta}{2}\right)\right| - \frac{2}{\pi}\right]\end{aligned}$$

and for trapped particles is

$$\begin{aligned}\overline{(r-r_0)} &= -sign\left(v_\parallel\right)\times 2\frac{q}{\Omega}\sqrt{\mu B\frac{R}{r}}\\ &\times\left|\cos\left(\frac{\theta}{2}\right)\right|\end{aligned}$$

In **Berk Galeev** it is illustrated the distribution function in the velocity space.

$$f = \left[n\left(r\right) - \frac{dn}{dr}\overline{(r-r_0)}\right]\ F$$

and the discontinuity at

$$v_c = 2\sqrt{\mu B_0\frac{r}{R}}$$

This is because the shape of the orbit changes at this limit.

17 Direct loss of ions. The loss region (Rome McAlees Callen Fowler 1976)

It is assumed that the two intersections of the fast-ion orbit with the equatorial plane occur at the minimum and maximum values of r . If one of this intersections occurs for $r > a$ the fast ion will intercept the limiter and it is considered to be in the **loss region**.

Fix a spatial position, for exemple: on the equatorial plane, at the distance

$$x = 0.3a$$

Fix the energy of the particles, for example

$$\varepsilon = 10 \; keV$$

for fast injected ions.

The loss region boundaries dependes now on the pitch angle and are defined by:

- **The pitch angle (nearest to 180°) which correspond to the untrapped-trapped particle boundary**. (*Note*: a pitch angle greater than $\pi/2$ means that the parallel component of the velocity is opposite to the direction of the magnetic field).

- **The smallest pitch angle $\left(v_\parallel/v\right)$ at which the banana width is large enough so that the orbit intersects the limiter**. *Note*: small pitch angle means that the direction of the particle velocity is close to the direction of the magnetic field and only very small transversal velocity is left (small magnetic moment $\mu \sim v_\perp^2$). This particle has small chance to be captured but, if it is, it has *very large* radial deviation, since iy is not far from the barely trapped particles, which are known to have largest radial deviation. Comming from small values of the pitch angle χ toward larger values, a particle will be captured at a certain value of χ and, precisely in this region it will have largest radial deviations. That is why one limit of the **loss region** must be searched for at *small values of the pitch angle.*

The banana width decreases with decreasing the particle energy.

Then the loss region will vanish as the parameter:

$$
\begin{aligned}
P &= \frac{qv}{\Omega a} \\
&= \frac{rB_T}{RB_\theta}\frac{m}{ZeB}\frac{v}{a} \approx \frac{mv}{Ze}\frac{r}{a}\frac{1}{R}\frac{1}{\mu_0 I_p/(2\pi r)} = \frac{2\pi}{\mu_0 Ze}\frac{r^2}{a}\frac{1}{R}\frac{mv}{I_p} \\
&\approx \frac{2\pi}{\mu_0 Ze}\frac{1}{A}\frac{mv}{I_p}
\end{aligned}
$$

if we approximate

$$
\frac{r^2}{Ra} \approx \frac{1}{A}
$$

which means that it is calculated at $r = a$ where

$$
A = \frac{R}{a}
$$

The definition in **Rome et al** is

P is $\frac{1}{A}$ times$\frac{\rho_\theta}{a}$ at $r = a$

Here I_p is the total plasma current (with which we calculate the poloidal magnetic field at the plasma border) and μ_0 is the vacuum permeability.

To determine **the loss-region boundaries**.

The fast-ion is produced at a point with the major radius R_s. Then the magnetic field is

$$
B \approx B_0 \frac{R_0}{R_s}
$$

We calculate the invariants of the particle motion.

Magnetic moment:

$$
\mu = \frac{mv_\perp^2}{2B} = \frac{mv_\perp^2 R_s}{2R_0 B_0}
$$

where $v_\perp^2 = v^2 \sin^2 \chi$.

Energy (in the absence of the radial electric field):

$$
\epsilon = \frac{m}{2}\left(v_\perp^2 + v_\parallel^2\right)
$$

Now **Rome et al** define

$$
\begin{aligned}
R_0 \equiv \; & \text{the major radius position} \\
& \text{where the production of the new ion} \\
& \text{has taken place}
\end{aligned}
$$

$$\begin{aligned} \chi &\equiv \text{the pitch angle, } \cos\chi = \frac{v_\parallel}{v} \\ &\quad \text{at the place of born, } R_B \end{aligned}$$

Canonical angular momentum.

First one evaluates approximately the toroidal momentum

$$\begin{aligned} mRv_\varphi &\approx mRv_\parallel \\ &= mv\sqrt{R\left(R - R_B \sin^2\chi\right)} \end{aligned}$$

$$-Ze\psi + mRv_\varphi = const$$

where ψ is the poloidal magnetic flux which in the circular surface approximation is

$$\psi = -R_0 A_\varphi$$

where A_φ is the magnetic potential due to the plasma current.

Taking a model for the current density

$$j = j_0\left[1 - \left(\frac{r}{a}\right)^n\right]^p$$

we obtain the expression for the magnetic potential

$$\psi \equiv A_\varphi = \frac{\mu_0 I_p}{2\pi} \frac{\sum_{j=0}^{p} \frac{(-1)^j \left(\frac{r}{a}\right)^{nj+2}}{(nj+2)^2} \frac{p!}{j!} \frac{1}{(p-j)!}}{\sum_{j=0}^{p} \frac{(-1)^j}{(nj+2)} \frac{p!}{j!} \frac{1}{(p-j)!}}$$

which is denoted

$$A_\varphi = \frac{\mu_0 I}{2\pi} F(r)$$

where $F(r)$ is the sum of powers of r. This expression allows to represent ψ as

$$\psi = -R_0 \frac{\mu_0 I}{2\pi} F(r)$$

and to re-write the expression of the *canonical angular momentum*

$$F(r) \pm P\left[R\left(R - R_s \sin^2\chi\right)\right]^{1/2} = const \qquad (5)$$

This looks reasonable: we have

$$\begin{aligned} &F(r) \quad \text{instead of } \psi(r) \\ &P \text{ instead of } \varepsilon\frac{\rho_\theta}{a} \end{aligned}$$

The coefficients that come from $I_p/(2\pi)$ occur similarly since they are connected with B_θ or q, the same.

The choice of the signs is:

- the **minus sign** must be used when

$$\mathbf{v} \cdot \mathbf{J} > 0$$

  this means that the initial velocity of the particle is in the same direcation with the current. Since $\mathbf{J}$ is assumed $\parallel$ $\mathbf{B}$ it results that the helicity of the lines is directed as a rotation to the right, when looking from back along the magnetic field. Then the velocity parallel with $\mathbf{B}$ means that the vertical drift for the ions will constrain the banana to stay *entirely* *inside* the magnetic surface.

- the **plus sign** must be used when

$$\mathbf{v} \cdot \mathbf{J} < 0$$

  and this means that the velocity is opposite to $\mathbf{B} \parallel \mathbf{J}$. And, the poloidal projection of the displacement of the ion along the magnetic line is counter-clockwise on the surface. Combined with the vetical ion drift this produces a banana which is *entirely outside* the magnetic surface. These particles are called *counter-streaming.*

The **largest deviation of an ion** from its corresponding flux surface occurs when the ion **is barely trapped**. A barely trapped orbit is *tangent to the equatorial plane at a point defined as the* **pinch point**. At this point **the vertical component of the drift motion of the ion** equals and cancels **the vertical component of the ion motion due to the motion along the magnetic field line**.

$$\begin{aligned} v_{Di} &= \frac{1}{\Omega_{ci} R}\left(\frac{v_\perp^2}{2} + v_\parallel^2\right) \text{ (there is no more projection, all is vertical)} \\ v_{vert} &= v_\parallel \frac{B_\theta}{B_T} \end{aligned}$$

In cartesian coordinates the **pinch point** is given by

$$\begin{aligned} y &= 0 \text{ on the equatorial plane} \\ \frac{dy}{dx} &= 0 \text{ no vertical velocity} \end{aligned}$$

Now the idea of calculus is

to obtain the expression of dy/dx from the conservation of the angular momentum (5) for $y=0$ and so to obtain its position x_p and R_p:

$$R_p = \frac{R_s \sin^2\chi}{2}\left[1+\sqrt{\frac{\left[F'(x_p)\right]^2}{\left[F'(x_p)\right]^2 - P}}\right]$$

for the barely trapped particles, evaluate equation (5) at the **banana tip**;

for the barely trapped particles, evaluate equation (5) at the **pinch point** and equalize these expressions

$$F(x_p) + P\left[R_p\left(R_p - R_s \sin^2\chi\right)\right]^{1/2} = F(r_s) + PR_s\left|\cos\chi\right|$$

or

$$\left|\cos\chi\right| = \frac{-R_B K + \left\{R_B (A+1)\left[K2 - P^2(1-r_B)^2\right]\right\}^{1/2}}{PR_B(1-r_B)}$$

where

$$K \equiv F(1) - F(r_B)$$

The lowest energy portion of the loss region occurs for $v_\parallel < 0$.

Particle losses from counter-injection are expected to be higher than those for co-injection.

This is because the counter-injection ions need to scatter through less velocity space to reach the loss region.

The tip of the loss region always occurs for $v_\parallel < 0$.

The following is similar to NBI.

See *NBI*.

The equation for the distribution function for fast ions

$$\frac{df}{dt} = \frac{1}{\tau_s} C(f)$$

where

$$\tau_s = 3\pi^{3/2} \frac{\varepsilon_0^2}{Z_{fast}^2 e^4} m_{fast} m_e \frac{1}{\ln\Lambda} \frac{v_e^3}{n_e}$$

is the Spitzer slowing down time, $\tau \sim \frac{T^{3/2}}{n}$.

Detailed form of the collision operator

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{1}{\tau_s}\left\{\left(-\frac{\tau_s}{\tau_{cx}}+3\right) f+\frac{\partial f}{\partial v}\left(-\frac{eE^*}{m_{fast}}\xi\tau_s+\frac{v^3+v_c^3}{v^2}\right)\right. \\
&+\frac{1}{2}\frac{1}{v^2}\frac{\partial^2}{\partial v^2}\left[v^2\left(v_e^2\frac{m_e}{m_{fast}}+\frac{v_c^3 v_i^2}{v^3}\frac{m_i}{m_{fast}}\right) f\right] \\
&+\frac{1}{2}\frac{m_i}{m_{fast}}\frac{\langle Z\rangle}{[Z]}\frac{v_c^3}{v^3}\frac{\partial}{\partial\xi}\left[\left(1-\xi^2\right)\frac{\partial f}{\partial\xi}\right] \\
&\left.-\frac{eE^*}{m_f}\tau_s\frac{1}{v}\left(1-\xi^2\right)\frac{\partial f}{\partial\xi}\right\} \\
&+S\left(v,\xi,t\right)
\end{aligned}
$$

where the velocity v_c is that which corresponds to the energy

$$
\begin{aligned}
E_c &= \left(\frac{3\sqrt{\pi}}{4}\right)^{2/3}\left(\frac{m_H}{m_e}\right)^{1/3} kT_e \left(\frac{m_{fast}}{m_H}\right)^{1/3}\left(\frac{m_{fast}}{m_i}\right)^{2/3}[Z]^{2/3} \\
&= 14.8\ kT_e \left(\frac{m_{fast}}{m_H}\right)^{1/3}\left(\frac{m_{fast}}{m_i}\right)^{2/3}\ [Z]^{2/3}
\end{aligned}
$$

The velocity v_c separates in velocity space the electron slowing down from the ion slowing down and pitch angle scatterin on background ions.

$$
[Z] \equiv \frac{\sum_j \frac{m_i}{m_j} n_j Z_j^2}{\sum_j n_j Z_j}
$$

$$
\langle Z\rangle \equiv \frac{\sum_j n_j Z_j^2}{\sum_j n_j Z_j}
$$

$$\tau_{cx} \equiv \text{ charge exchange time}$$

$$E^* \equiv \text{neoclassical effective electric field}$$

$$
\begin{aligned}
\xi &\equiv \frac{v_\parallel}{v} = \cos\chi \\
\text{evaluated at } \theta &= 0
\end{aligned}
$$

the solution for a source that is a δ function of v_0 and of ξ_0.

$$\begin{aligned} f(v,\xi,t) &= \mathrm{H}\left[t-\frac{\tau_s}{3}\ln\left(\frac{v_0^3+v_c^3}{v^3+v_c^3}\right)\right] \\ &\quad\times\frac{\tau_s}{2\pi\left(v^3+v_c^3\right)}\left[\frac{v_0^3+v_c^3}{v^3+v_c^3}\right]^{-\frac{\tau_s}{3\tau_{cx}}} \\ &\quad\times\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)P_n(\xi)\,P_n(\xi_0) \\ &\qquad\times\left[\frac{v_0^3+v_c^3}{v^3+v_c^3}\frac{v^3}{v_0^3}\right]^{\frac{m_i n(n+1)}{6\ m_{fast}}\frac{\langle Z\rangle}{[Z]}}\mathrm{H}(v_0-v) \end{aligned}$$

where

$$\mathrm{H}\equiv\text{Heaviside function}$$

The pitch angle scattering is controlled by the factor in the square brackets, which is rised to the power depending on Z.

For

$$v<v_0$$

as requested by the last Θ function (and as normal since all velocities of the fast ions must be lower than the velocity of the initial beam), the factor is

$$\frac{v_0^3+v_c^3}{v^3+v_c^3}\frac{v^3}{v_0^3}<1$$

subunitary and rised to a higher power will vanish the contribution to the last summation.

Then if

$$\langle Z\rangle\gg 1$$

the terms other than $n=0$ in the summation will be very small. The term $n=0$ on the other hand is *isotropic.*

This shows that, for high presence of heavy impurities the NBI fast ions will get a distribution function that is more close to isotropic than directed along the initial beam.

The conditions

$$\begin{aligned} v_0 &\gg v_c \ \text{and} \\ \langle Z\rangle &\gg 1 \end{aligned}$$

makes the loss to be very important before the slowing down of the fast ions.

NOTE remember that P_n appear in the formulas due to the Rosenbluth potentials that contain $|\mathbf{r}-\mathbf{r}'|$, as in electromagnetism. For the same reason the operator of pitch angle scattering becomes similar to the equation for P_n. **END.**

In the paper **Rome McAlees Callen Fowler** the examination of the case

$$Z_\eta \rightarrow \langle Z \rangle = 15.8$$

shows that only 6% of the energy of the beam goes to the plasma ions.

The pitch angle scattering is very high.

17.1 The equations of Nemov PoP6 1999

The *second adiabatic (*longitudinal) invariant

$$J_\parallel = \oint v_\parallel dl$$

This can vary on a magnetic surface due to the position of a banana on the surface. The mass (factor) is not there. The magnetic field is also absent.

The equations

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}_D + v_\parallel \widehat{\mathbf{n}}$$

$$\mathbf{v}_D = \frac{e\rho_\parallel}{mD}\left\{\boldsymbol{\nabla}\times\left(\rho_\parallel \mathbf{B}\right) - \widehat{\mathbf{n}}\left[\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\times\left(\rho_\parallel \mathbf{B}\right)\right]\right\}$$

$$\frac{dv_\parallel}{dt} = -\left(\frac{1}{2}\mu\boldsymbol{\nabla}B - \frac{e}{m}\mathbf{E}\right)\cdot\left(\widehat{\mathbf{n}} + \rho_\parallel \boldsymbol{\nabla}\times\widehat{\mathbf{n}}\right)\frac{1}{D}$$

where

$$\begin{aligned}\mu &\equiv J_\perp = \text{transversal adiabatic invariant (magnetic moment)} \\ &= \frac{v_\perp^2}{B}\end{aligned}$$

(one would expect a factor 2 at the denominator): $\mu = v_\perp^2/(2B)$, eventually the factor m)

$$\rho_\parallel = \frac{mv_\parallel}{eB}$$

$$D = 1 + \rho_\parallel \frac{\widehat{\mathbf{n}}\cdot(\boldsymbol{\nabla}\times\mathbf{B})}{B}$$

or

$$D = 1 + \rho_{\|} \frac{\mu_0 j_{\|}}{B}$$

Note the drift velocity in **Asymmetries Rosenbluth Hazeltine,** as

$$v_D = \frac{m}{e} v_{\|} \frac{\partial}{r \partial \theta} \left(\frac{v_{\|}}{B} \right)$$

17.2 The polarization motion of particles (Novakovskii Galeev Liu Sagdeev Hassam)

The paper discusses the poloidal damping due to magnetic pumping in the *plateau* regime.

This is also in *Notes density enhanced confinement.*

And in *polarization*.tex, in *plasma, general.*

It is considered a fast temporal variation of the radial electric field.

This is accompanied by a change in the neoclassical prolarization.

For the barely circulating ions, it is possible to calculate the radial polarization current (**Novakovski**). It is assumed that the radial electric field has a time variation which can be linearized

$$E_r = E_{r0} + \left(\frac{\partial E_r}{\partial t} \right) t$$

which means

$$v_E = v_{E0} + \left(\frac{\partial v_E}{\partial t} \right) t \quad \text{(in the poloidal direction)}$$

The particles that are considered are

$$\begin{array}{rcl} & & \text{circulating ions} \\ \text{with } v_{\|} & \ll & v_{\perp} \end{array}$$

The equation for the poloidal motion

$$r \frac{d\theta}{dt} = v_E + v_{\|} \frac{B_\theta}{B_T}$$

where

$$\frac{B_\theta}{B_T} = \frac{\varepsilon}{q} \equiv \Theta$$

is the factor that projects the parallel direction on the poloidal direction. Integrating

$$r\theta(t) = r\theta_0 + \left(v_{E0} + v_\parallel \frac{B_\theta}{B_T}\right) t + \frac{1}{2}\left(\frac{\partial v_E}{\partial t}\right) t^2$$

The radial velocity is the radial component of the *drift* of the guiding center

$$v_r = \mathbf{v}_D \cdot \widehat{\mathbf{e}}_r = v_D \sin\theta$$

$$v_D = \frac{1}{\Omega} \frac{v_\perp^2/2 + v_\parallel^2}{R}$$

and since we assume very small *parallel* velocity, $v_\parallel \ll v_\perp$,

$$v_D \approx \frac{1}{\Omega}\frac{1}{R}\frac{v_\perp^2}{2}\sin\theta = \frac{1}{\Omega}\frac{1}{R}\frac{mv_\perp^2}{2B}\frac{B}{m}\sin\theta = \frac{1}{\Omega R}\frac{\mu B}{m}\sin\theta$$

and

$$v_r(t) = \frac{1}{\Omega R}\frac{\mu B}{m}\sin[\theta(t)]$$

and the average over a *bounce* is

$$\begin{aligned}\langle v_r \rangle_{bounce} &= \frac{1}{2}\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} dt\ v_r(t) \\ &= \frac{1}{\Omega R}\frac{\mu B}{m}\frac{1}{2}\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} dt\ \sin[\theta(t)]\end{aligned}$$

Now we expand the function $\sin\theta$ for small argument

$$\begin{aligned}\sin[\theta(t)] &= \sin\left[\theta_0 + \frac{1}{r}\left(v_{E0} + v_\parallel\frac{B_\theta}{B_T}\right)t + \frac{1}{2}\left(\frac{\partial v_E}{\partial t}\right)t^2\right] \\ &\approx \sin\theta_0 \\ &\quad + \cos\theta_0\left[\frac{1}{r}\left(v_{E0} + v_\parallel\frac{B_\theta}{B_T}\right)t + \frac{1}{r}\frac{1}{2}\left(\frac{\partial v_E}{\partial t}\right)t^2\right]\end{aligned}$$

and the integrations over the bounce period gives

$$\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} dt\ \sin\theta_0 = 0$$

$$\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} dt\ \cos\theta_0\frac{1}{r}\left(v_{E0} + v_\parallel\frac{B_\theta}{B_T}\right)t = \cos\theta_0\frac{1}{r}\left(v_{E0} + v_\parallel\frac{B_\theta}{B_T}\right)\frac{1}{\tau}\left[\frac{\tau^2}{2}\right]_{-\tau/2}^{\tau/2} = 0$$

$$\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} dt\ \cos\theta_0 \frac{1}{r}\frac{1}{2}\left(\frac{\partial v_E}{\partial t}\right) t^2 = \frac{1}{24}\cos\theta_0 \frac{1}{r}\left(\frac{\partial v_E}{\partial t}\right)\tau^2$$

The radial velocity averaged over a bounce is at this moment

$$\begin{aligned}\langle v_r\rangle_{bounce} &= \frac{1}{\Omega R}\frac{\mu B}{m}\frac{1}{2}\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} dt\ \sin\left[\theta\left(t\right)\right] \\ &= \frac{1}{\Omega R}\frac{\mu B}{m}\frac{1}{48}\cos\theta_0\frac{1}{r}\left(\frac{\partial v_E}{\partial t}\right)\tau^2\end{aligned}$$

Now we take

$$\begin{aligned}\tau &\equiv \text{bounce time}=\frac{\text{connection length } 2\pi qR}{\text{parallel velocity } v_\parallel} \\ &= \frac{2\pi qR}{\left|v_\parallel\right|}\end{aligned}$$

It is necessary to define the regime by few parameters.

$$\begin{aligned}\widehat{\nu} &= \frac{r\ \nu_{ii}}{\Theta v_{th,i}} \\ &= \frac{\text{freq. of ion collisions}}{\text{freq. of ion transit with poloidal velocity } \Theta v_{th,i} \text{ on poloidal circle}}\end{aligned}$$

the same formula is written

$$\begin{aligned}\widehat{\nu} &= \frac{r\ \nu_{ii}}{\Theta v_{th,i}} \\ &= \frac{r\ \nu_{ii}}{\frac{\varepsilon}{q}v_{th,i}} \\ &= \frac{\nu_{ii}}{v_{th,i}/\left(qR\right)}\end{aligned}$$

The parallel velocity of the ions is taken at the limit where the effective ion collision frequency is equal with the parallel transit frequency

$$\nu_{eff} = \frac{v_\parallel}{qR}$$

where by definition

$$\nu_{eff} \overset{def}{=} \nu_{ii}\frac{v_{i,th}^2}{v_\parallel^2}$$

We combine the two expression of ν_{eff} and further use the expression for $\widehat{\nu}$

$$\begin{aligned}
\nu_{ii}\frac{v_{i,th}^2}{v_\parallel^2} &= \frac{v_\parallel}{qR} \quad \text{or} \\
\frac{\nu_{ii}}{v_{i,th}/\left(qR\right)} &= \frac{v_\parallel^3}{v_{i,th}^3} \\
\widehat{\nu} &= \frac{v_\parallel^3}{v_{i,th}^3}
\end{aligned}$$

from where we derive

$$\frac{v_\parallel}{v_{i,th}} = \widehat{\nu}^{1/3}$$

and we replace $v_\parallel$ with the expression in terms of thermal ion velocity and the effective collision parameter $\widehat{\nu}$

$$v_\parallel = v_{i,th}\widehat{\nu}^{1/3}$$

Then the square of the bounce time τ is

$$\begin{aligned}
\tau^2 &= \frac{\left(2\pi\right)^2 q^2 R^2}{v_\parallel^2} \\
&= \frac{\left(2\pi\right)^2 q^2 R^2}{v_{i,th}^2}\widehat{\nu}^{-2/3}
\end{aligned}$$

and

$$\begin{aligned}
\left\langle v_r \right\rangle_{bounce} &= \frac{1}{\Omega R}\frac{\mu B}{m}\frac{1}{48}\cos\theta_0 \frac{1}{r}\left(\frac{\partial v_E}{\partial t}\right)\tau^2 \\
&= \frac{1}{\Omega R}\frac{\mu B}{m}\frac{1}{48}\cos\theta_0 \frac{1}{r}\left(\frac{\partial v_E}{\partial t}\right)\frac{\left(2\pi\right)^2 q^2 R^2}{v_{i,th}^2}\widehat{\nu}^{-2/3} \\
&= \frac{\left(2\pi\right)^2}{48}\frac{q^2}{\varepsilon\Omega}\frac{\mu B}{m}\cos\theta_0\left(\frac{\partial v_E}{\partial t}\right)\widehat{\nu}^{-2/3}
\end{aligned}$$

To calculate the radial current two steps are necessary:

- take the fraction of the particles that have this regime

- integrate over the positions θ_0. Actually, the parameter θ_0 appears in the magnitude of the magnetic field $B = B_0\left(1-\varepsilon\cos\theta_0\right)$ and this, in turn, appears in the expression of the magnetic momentum $\mu = v_\perp^2/\left(2B\right)$. Then to integrate over the Maxwell distribution of the variable $v_\perp$ we can equivalently integrate over the variable θ_0 for fixed μ.

The fraction of particles is

$$\frac{v_{\parallel}}{v_{i,th}}$$

and this is

$$\text{fraction of particles} \sim \widehat{\nu}^{1/3}$$

When we multiply the average radial velocity by this factor

$$\begin{aligned} & \widehat{\nu}^{1/3} \ \times \ \widehat{\nu}^{-2/3} \\ = \ & \widehat{\nu}^{-1/3} \end{aligned}$$

we get a dependence of the effective collisional parameter as $\widehat{\nu}^{-1/3}$ which will be found in the final expression.

The Maxwellian in velocity space is

$$\begin{aligned} f_M &= N \exp\left(-\frac{mv^2}{2T}\right) \\ &= N \exp\left(-\frac{m\left(v_{\parallel}^2 + v_{\perp}^2\right)}{2T}\right) \sim N \exp\left(-\frac{mv_{\perp}^2}{2T}\right) \\ &= N \exp\left(-\frac{\mu B}{T}\right) = N \exp\left(-\frac{\mu B_0}{T\left(1+\varepsilon\cos\theta\right)}\right) \\ &\approx N \exp\left[-\frac{\mu B_0}{T}\left(1-\varepsilon\cos\theta\right)\right] \end{aligned}$$

We use this velocity integration to suppress the indeterminancy given by the presence of θ_0 in the radial current.

$$\frac{\partial v_E}{\partial t} = \frac{1}{B}\left(\frac{\partial E}{\partial t}\right)$$

The radial electric current induced by the time variation of the radial electric field is

$$\langle j_r \rangle \approx \left(1 + q^2 + \widehat{\nu}^{-1/3} q^2\right) \frac{m}{B^2}\left(\frac{\partial E}{\partial t}\right)$$

In this formula, 1 is the standard polarization term. The second term is the neoclassical polarization term due to ions with comparable parallel and perpendicular velocities, $v_{\parallel} \approx v_{\perp}$.

The *neoclassical* polarization radial current due to radial excursions of the banana (*trapped* particles) is

$$j_r^{bananas} \approx \varepsilon^{3/2} \frac{c^2}{v_{A\theta}^2}\left(\frac{\partial E_r}{\partial t}\right)$$

The equations are

$$\frac{d\mathbf{r}}{dt} = v_\parallel \widehat{\mathbf{n}} + \mathbf{v}_E + \mathbf{v}_D$$

$$\begin{aligned}
\frac{dv_\parallel}{dt} &= \left(-\frac{v_\perp^2}{2} \widehat{\mathbf{n}} + v_\parallel \mathbf{v}_E \right) \cdot \boldsymbol{\nabla} \ln B \\
\frac{d}{dt}\left(\frac{v_\perp^2}{2} \right) &= \frac{v_\perp^2}{2} \left(v_\parallel \widehat{\mathbf{n}} + \mathbf{v}_E \right) \cdot \boldsymbol{\nabla} \ln B
\end{aligned}$$

The *drift velocity* is

$$\mathbf{v}_D = \frac{1}{\Omega_{ci}} \left(\frac{v_\perp^2}{2} + v_\parallel^2 \right) \widehat{\mathbf{n}} \times \boldsymbol{\nabla} \ln B + \frac{1}{\Omega_{ci}} \widehat{\mathbf{n}} \times \frac{\partial \mathbf{v}_E}{\partial t}$$

Comparing with previous expressions of the drift velocity v_D we note that there is an additional term, which gives the effect of the fast time variation of the radial electric field, like in transitions with rapid change of toroidal and/or poloidal rotation velocity.

We note however that the time variation of the electric drift velocity has the following effect on the drift:

we suppose that

$$\frac{\partial E_r}{\partial t} \sim \widehat{\mathbf{e}}_r$$

exists due to the polarization effect related to the forced increase of the poloidal velocity

$$\frac{\partial v_\theta}{\partial t} \rightarrow \frac{\partial E_r}{\partial t}$$

Then $\mathbf{v}_E$ will increase in two possible directions

$$\frac{\partial \mathbf{v}_E}{\partial t} \sim \frac{1}{B} \left(\frac{\partial E_r}{\partial t} \widehat{\mathbf{e}}_r \times \mathbf{B}_\theta + \frac{\partial E_r}{\partial t} \widehat{\mathbf{e}}_r \times \mathbf{B}_T \right)$$

Then the terms mentioned by **Novakovskii** is

$$\begin{aligned}
\frac{1}{\Omega_{ci}} \widehat{\mathbf{n}} \times \frac{\partial \mathbf{v}_E}{\partial t} &\sim \frac{1}{\Omega_{ci}} \widehat{\mathbf{n}} \times \left(\frac{\partial E_r}{\partial t} \widehat{\mathbf{e}}_r \times \mathbf{B}_\theta \right) \quad \text{almost zero} \\
&\quad + \frac{1}{\Omega_{ci}} \widehat{\mathbf{n}} \times \left(\frac{\partial E_r}{\partial t} \widehat{\mathbf{e}}_r \times \mathbf{B}_T \right) \quad \text{radial}
\end{aligned}$$

Therefore *none* of these contributions is aligned along the toroidal direction, giving a *drift* of the particle population in the toroidal direction.

It seems that a treatment based on the equations of motion of the particles governed by the invariants

$$E, \mu$$

cannot give us a drift of the bananas in the toroidal direction.

Here again the drift-kinetic equation is

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} f + \frac{dv_{\|}}{dt}\frac{\partial f}{\partial v_{\|}} + \frac{d\left(v_{\perp}^2/2\right)}{dt}\frac{\partial f}{\partial\left(v_{\perp}^2/2\right)} = St\left(f\right)$$

The paper of **Novakovskii** wants to solve the problem of decay of poloidal rotation in the *plateau* regime.

Then the Drift-Kinetic equation is solved by perturbations.

Zero + order 1 + order 2 are necessary.

In the zeroth order

$$\left(v_{\|}\frac{B_{\theta}}{B_T} + v_E\right)\frac{\partial f_0}{r\partial\theta} - St\left(f_0\right) = 0$$

which gives a Maxwellian function *possibly* shifted in the parallel direction by a velocity U_0.

$$f_0 = \left(1 - \frac{mv_{\|}U_0}{T}\right) f_M$$

for

$$f_M = \frac{n}{\left(2\pi T/m\right)^{3/2}} \exp\left[-\frac{m\left(v_{\|}^2 + v_{\perp}^2\right)}{2T}\right]$$

NOTE

We remark the combination

$$\begin{aligned} & v_{\|}\frac{B_{\theta}}{B_T} + v_E \\ \sim\ & \Theta\left(v_{\|} + \frac{v_E}{\Theta}\right) \\ \approx\ & 0 \quad \text{(since the paranthesis is } \sim 0\text{)} \end{aligned}$$

The combination $v_{\|}\frac{B_{\theta}}{B_T} + v_E$ is the *poloidal velocity.*

It is composed of the *projection* of the parallel velocity plus the poloidal velocity due to the radia electric field.

The first term $\left(v_{\|}\frac{B_{\theta}}{B_T} + v_E\right)\frac{\partial f_0}{r\partial\theta}$ is the convection of the distribution function f_0 in the poloidal direction.

It is balanced by collisions.

END

Nothing at this moment suggests there can be a velocity U in the *parallel* direction, *i.e.* along the magnetic field lines. But the equation for f_0 allows it and since we know it can exist, it is introduced at this point.

Note that the velocity along the magnetic field lines comes from a shift in the parallel *particle* velocity, as

$$\begin{aligned} & -\frac{\left(v_\parallel - U_0\right)^2}{2T/m} \\ = & -\frac{v_\parallel^2}{2T/m} - \frac{2v_\parallel U_0}{2T/m} - \frac{U_0^2}{2T/m} \end{aligned}$$

and the last term is much less than 1 since the flow with velocity U_0 is slower than the *thermal* velocity, $U_0 \ll v_{th}$.

Then a substitution is made for the distribution function to extract a rigid body rotation

$$\begin{aligned} f &= f_0 + \varepsilon \left(\frac{mv_\parallel U_0}{T}\right) f_M \\ &\quad + \widetilde{f} \end{aligned}$$

Then the Drift-kinetic equation to order ε^2 gives

$$\begin{aligned} & \frac{\partial \widetilde{f}}{\partial t} + \left(v_E + v_\parallel \frac{B_\theta}{B_T}\right) \frac{\partial \widetilde{f}}{r\partial \theta} - St\left(\widetilde{f}\right) \\ = & \frac{\sin\theta}{R} \frac{m\left(v_\parallel^2 + \frac{v_\perp^2}{2}\right)}{T} W f_M \end{aligned}$$

where W is a velocity in the poloidal direction

$$\begin{aligned} W &\equiv v_E + v_{*n} + U_0 \frac{B_\theta}{B_T} \\ &\quad + v_{*T} \left[\frac{m\left(v_\parallel^2 + v_\perp^2\right)}{2T} - \frac{3}{2}\right] \end{aligned}$$

$$\begin{aligned} v_{*n} &\equiv \frac{T}{eB} \frac{d}{dr} \ln n \\ v_{*T} &\equiv \frac{T}{eB} \frac{d}{dr} \ln T \end{aligned}$$

COMMENT

Then W is

$$\begin{aligned} W &= \text{poloidal velocity due to the radial electric field} \\ &\quad + \text{diamagnetic-density velocity (poloidal)} \\ &\quad + \text{diamagnetic-temperature velocity (poloidal)} \\ &\quad + \text{poloidal projection of the flow velocity } U_0 \end{aligned}$$

The term $\frac{\sin\theta}{R}\frac{m\left(v_\parallel^2+\frac{v_\perp^2}{2}\right)}{T}Wf_M$ comes from

$$\mathbf{v}\cdot\boldsymbol{\nabla}f_M$$

and reflects the convective variation of the Maxwellian with the flow velocity that exists in the plasma. The space variation of the Maxwellian f_M is *RADIAL* and the operator $\boldsymbol{\nabla}$ will be reduced to radial derivative

$$\boldsymbol{\nabla}\rightarrow\frac{\partial}{\partial r}=\frac{d}{dr}$$

Then *who is the plasma velocity taht will take advantage of this radial variation of the equilibrium distribution function*? It is the particle's drift velocty $\mathbf{v}_D$ which will act along the minor radius v_{Dr}.

The term is actually

$$v_{Dr}\frac{df_M}{dr}$$

This will become the inhomogeneous term that drives a variation of the distribution function, asking therefore for the existence of a f_1.

But what can the correction do to balance this *radial convective variation of the equilibrium distribution Maxwellian function* ?

The correction f_1 actually has variation in the magnetic surface.

It will again be question of a *convective variation*, which means that there is a poloidal velocity that will advect the function f_1 along its variation.

This poloidal advection of the correction $f_1(\theta)$ will compensate for the *radial* variation of the equilibrium distribution function.

To **comment** further, we note in **Rosenbluth Hazeltine Hinton 1972** the equation

$$v_{Dr}\frac{\partial f_0}{\partial r}+v_\parallel\frac{B_\theta}{B}f_0\frac{\partial\widehat{f}}{r\partial\theta}+|e|\,E_\parallel v_\parallel\frac{\partial f_0}{\partial\epsilon}=C\left(f\right)$$

where

$$f=f_0\left(1+\widehat{f}\right)$$

We recognize the same picture:

- the zero-order distribution function f_0

END COMMENT

The collision operator is adopted as

$$St(f) = -\nu_{eff} f$$

$$\nu_{eff} = \frac{v_{th}^2}{v_{\parallel}^2}\nu_c$$

Regarding the application of this analysis to the case of a **fast time variation** of the radial electric field (for fast transients of the poloidal or toroidal rotation) the range of validity is established by **Novakovskii et al** by choosing

$$\frac{v_{Th}}{qR} \approx \nu_{eff} \gg \frac{\partial}{\partial t}$$

which means: the frequency of the *bounce* of the trapped particle, $v_{Th}/(qR)$, comparable with the frequency of collisions ν_{eff} is much higher than the frequency associated to the variation of the radial electric field, $\partial/\partial t$. Then during the variation of the radial electric field, the trapped particle makes many bounces.

Then a new small parameter has been identified and the distribution function can be expanded in a series. The distribution function is only the *correction* to the shifted Maxwellian, *i.e.* the function $\widetilde{f}$ and the series is

$$\widetilde{f} = f_1 + f_2 + ...$$

Separately and related this time to the spatial variation of the distribution function, it is *considered* the variation in the magnetic surface, *i.e.* the dependence of the distribution functions f_i of the *poloidal angle* θ:

$$f_i = \sum_{\sigma=\pm 1} f_{i\sigma} \exp(i\sigma\theta)$$

Then we get the solution for the first order correction $f_{1\sigma}$ as

$$f_{1\sigma} = -\frac{\varepsilon \frac{\left(v_{\perp}^2/2+v_{\parallel}^2\right)}{2T/m} W}{v_{\parallel}(B_\theta/B_T) + v_E - \iota\sigma\nu_{eff}} f_M$$

where

$$\iota = -\frac{1}{q}$$

$$\sigma = \pm 1$$

Using the first order in the small parameter

$$\frac{\partial/\partial t}{v_{Th}/(qR)} \ll 1$$

and the ordering

$$\begin{aligned} v_E &\ll v_{Th}\frac{B_\theta}{B_T} \\ v_E &\ll v_{th}\Theta \end{aligned}$$

the second order contribution to the distribution function $\widetilde{f}$ is obtained from the differential equation

$$\frac{\partial f_1}{\partial t} + v_\parallel \frac{B_\theta}{B_T}\frac{\partial f_2}{r\partial\theta} = -\nu_{eff} f_2$$

(we **note** that $v_\parallel \frac{B_\theta}{B_T} = v_\theta$) from which a solution is obtained

$$f_{2\sigma} = -\iota \frac{\varepsilon\sigma r \frac{\left(v_\perp^2/2+v_\parallel^2\right)}{2T/m}}{\left[v_\parallel (B_\theta/B_T) - \iota\sigma r\nu_{eff}\right]^2} f_M \frac{\partial v_E}{\partial t}$$

COMMENT

The second order correction is obtained from the balance with the *time variation* of the first order variation, which means

$$f_2 \sim \frac{\partial f_1}{\partial t}$$

This is because the expression for the first oorder correction f_1 contains the factor W which was derived from the radial variation of the equilibrium distribution function $f_0 \sim f_M$.

The factor W contains the *electric potential* ϕ that has radial variation

$$\phi = \phi(r)$$

BUT it also has a time variation

$$\phi = \phi(r,t)$$

since the decay of poloidal rotation consists of the chage of the radial electric field that produces the torque.

$$\begin{array}{rcl} & & \text{torque to stop poloidal rotation} \\ & \downarrow & \\ E_r & = & E_r(t) \\ & \sim & \text{decay of } v_\theta = \dfrac{E_r}{B_T} \end{array}$$

Here it is substituted

$$\nu_{eff} \approx \frac{v_{Th}^2}{v_\parallel^2} \nu_i$$

and the second order contribution to the distribution function becomes (omitting the term with $\cos\theta$)

$$f_2 = \frac{\left(\frac{v_\parallel}{v_{Th}}\right)^6 - \widehat{\nu}^2}{\left[\left(\frac{v_\parallel}{v_{Th}}\right)^6 + \widehat{\nu}^2\right]^2} \sin\theta \ \frac{\varepsilon r}{\left(v_{Th}\frac{B_\theta}{B_T}\right)^2} \frac{\left(v_\perp^2/2 + v_\parallel^2\right)}{T/m} \left(\frac{v_\parallel}{v_{Th}}\right)^4 f_M \ \frac{\partial v_E}{\partial t}$$

where it is noted later

$$\frac{v_\parallel}{v_{Th}} \equiv x$$

and

$$\widehat{\nu} \equiv \frac{r}{v_{Th}\frac{B_\theta}{B_T}} \nu_i$$

which is related to the standard neoclassical collisional parameter ν_* by

$$\begin{array}{rcl} \widehat{\nu} & = & \varepsilon^{3/2} \nu_* \\ & = & \text{plateau collisionality parameter} \end{array}$$

In order to calculate the magnetic damping of the poloidal rotation it is necessary to start from the radial electric current which on a magnetic surface must have the average equal to zero

$$\langle j_r \rangle = 0$$

The radial fluxes are considered

$$\langle n V_r \rangle = \frac{1}{2\pi} \int_0^{2\pi} d^3v d\theta \ v_r \left(1 + \varepsilon\cos\theta\right) \ f$$

where

$$d^3v = 2\pi dv_\parallel d\left(\frac{v_\perp^2}{2}\right)$$

The radial component of the *particle drift* velocity is

$$v_r = -\frac{1}{\Omega_c}\left(\frac{v_\perp^2}{2} + v_\parallel^2\right)\frac{\sin\theta}{R} + \frac{1}{\Omega_c}\frac{\partial v_E}{\partial t}$$

In the integral for the radial particle fluxes one substitutes the *ion* distribution function

$$f = f_0 + f_1 + f_2 + ...$$

the expansion in the small parameter representing the ratio between the characteristic frequency of the variation of the radial electric field and the bounce frequency.

$$\begin{aligned} & \int d^3v d\theta \; \left[f_0 \frac{\partial v_E}{\partial t} - f_2\left(\frac{v_\perp^2}{2} + v_\parallel^2\right)\frac{\sin\theta}{R}\right] \\ = & \int d^3v d\theta \; f_1\left(\frac{v_\perp^2}{2} + v_\parallel^2\right)\frac{\sin\theta}{R} \end{aligned}$$

If the plateau collision parameter is small

$$\widehat{\nu} \ll 1$$

then the distribution function in order 1 can be approximated

$$f_1 \approx -\pi q \frac{\frac{v_\perp^2}{2} + v_\parallel^2}{T/m} W \; f_M \; \delta\left(v_\parallel\right)\sin\theta$$

NOTE that the most important contribution to the distribution function, *i.e.* f_1 comes from the *barely trapped ions*. **End**.

Then the equation for the poloidal velocity becomes

$$\left(1 + q^2\Lambda\right)\frac{\partial v_E}{\partial t} = -\nu_{MP}\left(v_E + v_{*n} + U_0\frac{B_\theta}{B_T} + \frac{3}{2}v_{*T}\right)$$

where the rate of magnetic pumping damping is

$$\nu_{MP} = \sqrt{\frac{\pi}{2}}\frac{qv_{Th}}{R}$$

and

$$\Lambda \equiv \frac{3}{2} + \Xi\widehat{\nu}^{-1/3}$$

$$\Xi \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \; x^4 \frac{x^6 - 1}{\left(x^6 + 1\right)^2}\exp\left(-\widehat{\nu}^{2/3}\frac{x^2}{2}\right)$$

17.3 Annihilators, in the drift kinetic equations

In **Hirshman Sigmar p1112**.

The equation

$$v_{\parallel}\nabla_{\parallel}\left[\overline{f}_a^{(1)(0)}+\frac{2\pi}{\partial\chi/\partial\psi}\frac{I}{\Omega_a}v_{\parallel}\frac{\partial f_{Ma}}{\partial\psi}\right]=0$$

Here both expansions are present

- expansion in $\delta=\rho_\theta/L$ and it is (1) and
- expansion in ν_*/ω_b (collisional) and it is (0).

This is the equation in order *zero* for the *collisionality.*

The next order in expansion for collisionality

$$v_{\parallel}\nabla_{\parallel}\overline{f}_a^{(1)(1)}+e_aE_{\parallel}v_{\parallel}\frac{\partial f_{Ma}}{\partial\epsilon}=C_a\left[\overline{f}_a^{(1)(0)},\overline{f}_b^{(1)(0)}\right]$$

This equation offers a *constraint* if it is used the periodicity

$$\oint\frac{dl_{\parallel}}{v_{\parallel}}(...)=0$$

$$\begin{aligned}&e_a\oint dl_{\parallel}\ E_{\parallel}\frac{\partial f_{Ma}}{\partial\epsilon}-\oint\frac{dl_{\parallel}}{v_{\parallel}}C_a\left[-\frac{2\pi}{\partial\chi/\partial\psi}\frac{I}{\Omega_a}v_{\parallel}\frac{\partial f_{Ma}}{\partial\epsilon}\ ,\ -\frac{2\pi}{\partial\chi/\partial\psi}\frac{I}{\Omega_b}v_{\parallel}\frac{\partial f_{Mb}}{\partial\epsilon}\right]\\
=\ &\oint\frac{dl_{\parallel}}{v_{\parallel}}C\left[g_a,g_b\right]\end{aligned}$$

where g_a is a perturbation of $f_a^{(1)(0)}$ beyond the neoclassical one, $-\frac{2\pi}{\partial\chi/\partial\psi}\frac{I}{\Omega_a}v_{\parallel}\frac{\partial f_{Ma}}{\partial\epsilon}$.

The guiding center drift is

$$v_g=\frac{1}{\Omega_c}\frac{\mu B_0+\left[\frac{v_E}{r/(qR)}\right]^2}{R}$$

$$v_E=(-)\frac{1}{B}\frac{d\Phi}{dr}$$

This is the guiding center *drift* velocity (usual notation is $v_g\to v_D$). The sign (*i.e.* directions) must be such that the equality is verified approximately (cancellation of the poloidal effective displacement)

$$\frac{v_E}{r/(qR)}-v_{\parallel}\approx0$$

with

$$\frac{r}{qR} = \frac{B_\theta}{B_T} \ll 1 \text{ factor of projection from parallel to poloidal}$$

The drift becomes

$$\left(v_D \equiv\right) \; v_g = \frac{1}{\Omega_c} \frac{\frac{v_\perp^2}{2}\frac{B_0}{B} + v_\parallel^2}{R}$$

From the condition of existence of the real (not complex) deviation $r - r_0$,

$$(\Delta v)^2 + 2 r_0 \Omega_c v_g (\cos\theta - 1) \geq 0$$

we have that the particles with

$$\begin{aligned} (\Delta v)^2 \quad &< \quad 4 r_0 \Omega_c v_g \\ & \quad\;\; \text{are trapped} \end{aligned}$$

The deviation from the magnetic surface, of a barely *trapped* particle is

$$\Delta r_{trap} (\theta = 0) = 4 \frac{1}{\Omega_c} \frac{qR}{r} \sqrt{\mu B_0 + \left(\frac{v_E}{r/(qR)} \right)^2}$$

The barely *circulating* particles have deviation *half* of the trapped ones

$$\Delta r_{untrap} \sim \frac{1}{2} \Delta r_{trap}$$

The equation of particle displacement

$$\begin{aligned} \frac{r d\theta}{dt} &= -sign \left[\Delta v (r, \theta) \right] \\ & \quad \times \frac{r}{qR} \sqrt{\varepsilon \left[v^2 + \left(\frac{v_E}{r/(qR)} \right)^2 \right]} \\ & \quad \times \sqrt{2\kappa^2 - 1 + \cos\theta} \end{aligned}$$

where

$$\begin{aligned} v^2 &= \frac{2}{m} \left[E - e\Phi (r) \right] \\ & \quad \text{only kinetic energy} \end{aligned}$$

$$
\begin{aligned}
2\kappa^2 &= \frac{1}{\varepsilon}\frac{\left[\Delta v\left(r_0,\theta=0\right)\right]^2}{v^2+\left(\frac{v_E}{r/(qR)}\right)^2} \\
&\text{(notation)}
\end{aligned}
$$

$$
\varepsilon = \frac{r}{R}
$$

Note alternative definitions, $\varepsilon = r/R_0$. **End**.

The time of bounce

$$
\begin{aligned}
\tau_{bounce} &= \frac{4r}{\sqrt{\varepsilon\left[\left(\frac{r}{qR}v\right)^2+v_E^2\right]}} \\
&\quad\times\int_0^{\theta_0}\frac{d\theta}{\sqrt{2\left(\kappa^2-\sin^2\left(\frac{\theta}{2}\right)\right)}} \\
&= \frac{4r\sqrt{2}}{\frac{r}{qR}\sqrt{\varepsilon\left[v^2+\left(\frac{v_E}{r/(qR)}\right)^2\right]}}K\left(\kappa\right)
\end{aligned}
$$

A new parameter, θ_0 is a root of the expression under square root $2\left(\kappa^2-\sin^2\left(\frac{\theta}{2}\right)\right)=0$. It is the maximum poloidal angle attained by the banana.

18 Applications and alternative formulations

In **Galeev Sagdeev Liu Novakovskii Spontaneous poloidal** the expression is

$$
\begin{aligned}
J &= m\left(\frac{eB_0}{m}\right)\int_0^r\frac{B_\theta}{B_\varphi}dr \\
&\quad -mv_{\parallel}\left(1+\varepsilon\cos\theta\right)
\end{aligned}
$$

(note the opposite sign compared with above J). Since

$$
\begin{aligned}
B_\varphi &= \frac{B_0}{h} \\
B_0 &\equiv \text{const} \\
&= \text{magnetic field on the axis } r=0
\end{aligned}
$$

we have

$$J = |e| \int_0^r B_\theta h dr - m v_\parallel h$$

the same as above.

For a magnetic field

$$\mathbf{B} = \frac{B_\theta}{B_\varphi} \frac{B_0}{1 + \varepsilon \cos\theta} \widehat{\mathbf{e}}_\theta + \frac{B_0}{1 + \varepsilon \cos\theta} \widehat{\mathbf{e}}_\varphi$$

They use systematically

$$\begin{aligned} \Theta(r) &\equiv \frac{B_\theta}{B_\varphi} = \frac{r}{qR} = \frac{\varepsilon}{q} \\ &\ll 1 \end{aligned}$$

The time of relaxation of the distribution function of trapped particles due to *collisions* is

$$\begin{aligned} \tau_R^{trapped} &= \frac{\varepsilon}{\nu} \\ \text{where } \nu &\equiv \text{collision frequency} \end{aligned}$$

The time of bounce of trapped ions

$$\begin{aligned} \tau_{Bounce} &= \frac{1}{\Theta} \frac{r}{v_{th,i}\sqrt{\varepsilon}} \\ &= \frac{q}{\varepsilon} \frac{r}{R} \frac{R}{v_{th,i}} \frac{1}{\sqrt{\varepsilon}} \\ &= \varepsilon^{-1/2} \frac{qR}{v_{th,i}} \end{aligned}$$

This shows that the average length of a banana is $r/[\Theta(r)] = r/(B_\theta/B_\varphi) \gg r$.

The bounce time is calculated using an estimation of the parallel velocity of a trapped particle

$$\begin{aligned} &\text{velocity trapped particle} \\ \sim\ & v_{th,i}\sqrt{\varepsilon} \end{aligned}$$

it is natural, since the *parallel* velocity must be *small* for trapped particles.

The parallel velocity of a trapped particle is

$$v_\parallel = \sigma \sqrt{2 \left(E - e\Phi(r,t) - \frac{\mu B(r,\theta)}{m} \right)}$$

with the sign

$$\sigma = \pm 1$$

In the paper **Adiabatic invariants PhysPlasmas 6** the equations of motion of particles are

$$\frac{d\mathbf{r}}{dt} = v_{\parallel}\widehat{\mathbf{n}} + \mathbf{v}_D$$

(**note** the absence of $\mathbf{v}_E$)

$$\mathbf{v}_D = \frac{e}{m}\frac{\rho_{\parallel}}{1+\rho_{\parallel}\frac{\mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{B})}{B^2}}\left(\boldsymbol{\nabla}\times\left(\rho_{\parallel}\mathbf{B}\right) + \frac{\mathbf{B}\left[\mathbf{B}\cdot\boldsymbol{\nabla}\times\left(\rho_{\parallel}\mathbf{B}\right)\right]}{B^2}\right)$$

$$\frac{dv_{\parallel}}{dt} = -\left(\frac{1}{2}J_{\perp}\boldsymbol{\nabla}B - \frac{e}{m}\mathbf{E}\right)\cdot\frac{\left(\widehat{\mathbf{n}} + \rho_{\parallel}\boldsymbol{\nabla}\times\widehat{\mathbf{n}}\right)}{1+\rho_{\parallel}\frac{\mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{B})}{B^2}}$$

where

$$\begin{aligned}\rho_{\parallel} &= \frac{mv_{\parallel}}{eB} \\ &= \frac{v_{\parallel}}{\Omega}\end{aligned}$$

and the transverse adiabatic invariant is

$$J_{\perp} = \frac{v_{\perp}^2}{B}$$

which is the magnetic moment

$$J_{\perp} \to \mu = \frac{v_{\perp}^2}{2B}$$

The magnetic surface is defined as

$$\psi = \text{const}$$

and the coordinates are

$$(\psi, \theta_0, \varphi)$$

with metric tensor

$$g_{jk} \text{ and } g = \det\left(g_{jk}\right)$$

A magnetic field line is defined as the intersection of the surfaces

$$\begin{aligned}\psi &= \text{const} \quad \text{(a magnetic surface)} \\ \theta_0 &= \text{const} \quad \text{(label of a magnetic field line)}\end{aligned}$$

The (third) adiabatic invariant $J_{\|}$ varies over the surface, and the equation of its change is

$$\frac{\partial J_{\|}}{\partial \theta_0} = \frac{eB}{m} \sqrt{\frac{g}{g_{33}}} \delta\psi$$

where $\delta\psi$ is the variation of the surface function ψ when the trapped particle makes a bounce, in a one-bounce period τ_b. It is obtained integrating the approximative form of the first equation of motion written as

$$\frac{d\psi}{dt} = \mathbf{v}_d \cdot \boldsymbol{\nabla}\psi$$

Now we have

$$\begin{aligned} \psi &= \iint\limits_{\Sigma} d\mathbf{s} \cdot \mathbf{B} \\ &= 2\pi R A_\varphi \end{aligned}$$

Leaving the factor 2π for a normalization of the surface integral in the definition of ψ we note that

$$\psi = RA_\varphi$$

and this allows us to express the longitudinal invariant as

$$\psi + \frac{Rmv_\varphi}{e} = \text{const}$$

the other invariant

$$\varepsilon = \frac{v^2}{2} + \frac{e\phi}{m}$$

The distribution function solution of the *drift-kinetic* equation can be expressed in terms of the two invariants

$$f = f\left(\varepsilon, \psi + \frac{Rmv_\varphi}{e}\right)$$

From **Rewoldt Tang Frieman**, with their notation

$$\begin{aligned} P_\varphi &\approx m_i R \left(v_\varphi - \Omega_{\theta j} r\right) \\ &= m_j R_0 \left(v_{\|} - \Omega_{\theta j}^{(0)} r\right) \end{aligned}$$

From **Minardi adiabatic invariants.** The position is of the *particle* not of the *guiding center*.

$$\begin{aligned} m_i \frac{d^2\mathbf{x}}{dt^2} &= |e| \frac{d\mathbf{x}}{dt} \times \mathbf{B} && \text{Lorentz force } |e|\, v \times B \\ &\quad -\mu \boldsymbol{\nabla} B && \text{grad-B} \\ &\quad + |e|\, \mathbf{E} && \text{acceleration in electric field} \end{aligned}$$

The Hamiltonian for the guiding center

$$
\begin{aligned}
H &= \frac{1}{2} g^{ik} \left(p_i - |e| A_i\right)\left(p_k - |e| A_k\right) \\
&\quad +\mu B
\end{aligned}
$$

and the equations

$$
\begin{aligned}
\frac{dp_i}{dt} &= -\left(\frac{\partial H}{\partial x^i}\right)_{p_i} \\
\frac{dx^i}{dt} &= \left(\frac{\partial H}{\partial p_i}\right)_{x_i}
\end{aligned}
$$

The equations for the motion of particles, to be used in the drift-kinetic equation by **Rozhansky Tendler 1992**

$$
\mathbf{V}_{cj} = \text{guiding center velocity for species } j
$$

$$
\begin{aligned}
V_0 &\equiv \frac{1}{B_0}\frac{d\Phi_0}{dr} \\
&\equiv V_E
\end{aligned}
$$

$$
V_E \sim \text{ poloidal}
$$

We **note** that in this form the velocity V_0 is the velocity induced by the radial electric field $E_r = -\frac{d\Phi_0}{dr}$ in combination with the *toroidal* magnetic field $V_E \sim \frac{E \times B}{B^2} \sim \frac{E_r}{B_T} \sim v_\theta$.

$$
\begin{aligned}
\mathbf{V}_j &= -\frac{1}{\frac{e_j B}{m_j}} \frac{\left(\frac{V_{\perp j}^2}{2} + V_{\parallel j}^2\right)}{R} \left(\widehat{\mathbf{e}}_\theta \cos\theta + \widehat{\mathbf{e}}_r \sin\theta\right) \quad \text{vertical particle drift velocity} \\
&\quad + \frac{-\boldsymbol{\nabla}\phi \times \widehat{\mathbf{n}}}{B} \quad \text{drift due to the electric field} \\
&\quad + V_{\parallel j} \widehat{\mathbf{e}}_z \quad \text{parallel velocity} \\
&\quad + u_r \quad \text{radial velocity due to turbulence (anomalous transport)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{dV_{\parallel j}}{dt} &= \frac{\varepsilon}{q}\left(-\frac{e_j}{m_j}\frac{\partial\phi}{r\partial\theta}\right) \\
&\quad -\frac{\varepsilon}{q} V_{\perp j}^2 \frac{\varepsilon \sin\theta}{2r} \\
&\quad + V_E V_\parallel \frac{\varepsilon \sin\theta}{r}
\end{aligned}
$$

(See also **Novakovskii Liu Sagdeev Rosenbluth** in *drift kinetic solutions.*)

The first term in the RHS is the result of the variation of electrostatic potential $\phi(r,\theta)$ in the magnetic surface. This generetes an electric field $E_\theta = -\frac{\partial\phi}{r\partial\theta}$ which is further projected in the parallel $\|$ direction by ε/q.

The perpendicular velocity

$$\begin{aligned}\frac{dV_{\perp j}}{dt} &= V_{\|j}V_{\perp j}\frac{\varepsilon}{q}\frac{\varepsilon\sin\theta}{2r} \\ &+V_E V_{\perp j}\frac{\varepsilon\sin\theta}{2r}\end{aligned}$$

we see the combination

$$\left(V_{\|j}\frac{\varepsilon}{q}+V_E\right)$$

NOTE these can be found also in **Stringer 1991**. **END.**

We **note** that

$$\frac{\varepsilon}{q}=\frac{B_\theta}{B_T}\equiv\Theta(r)\ll 1$$

therefore we have *poloidal projection* of the respective terms if they are parallel, or, *parallel projection* if the term is poloidal.

Note the last two equations, for

$$\frac{d\left(v_\perp^2/2\right)}{dt}\quad,\quad\frac{dv_\|}{dt}$$

are identical with the equations from **Burrell Wong**

End.

The equation is

$$\frac{\partial f_j}{\partial t}+\mathbf{V}_j\cdot\boldsymbol{\nabla}f_j+\frac{dV_{\|j}}{dt}\left(\frac{\partial f_j}{\partial V_{\|j}}\right)+\frac{dV_{\perp j}}{dt}\left(\frac{\partial f_j}{\partial V_{\perp j}}\right)=St\left(f_j\right)$$

NOTE that in **Stringer 1991** one takes $\left(v_\|,v_\perp^2\right)$. **END.**

In **Novakovskii, Galeev Sagdeev Liu** it is written the system of equations for the particle's motion, space coordinates and velocity components (they follow **Wong Burrell**)

$$\frac{d\mathbf{r}}{dt}=v_\|\widehat{\mathbf{b}}+\mathbf{v}_E+\mathbf{v}_D$$

$$\begin{aligned}
\frac{dv_{\parallel}}{dt} &= \left(-\frac{v_{\perp}^2}{2}\widehat{\mathbf{b}} + v_{\parallel}\mathbf{v}_E\right)\cdot\boldsymbol{\nabla}\ln B \\
\frac{d}{dt}\left(\frac{v_{\perp}^2}{2}\right) &= \frac{v_{\perp}^2}{2}\left(\mathbf{v}_E + v_{\parallel}\widehat{\mathbf{b}}\right)\cdot\boldsymbol{\nabla}\ln B
\end{aligned}$$

NOTE the occurence of two scalar product = projections

$$\widehat{\mathbf{b}}\cdot\boldsymbol{\nabla}\ln B = \frac{\nabla_{\parallel}B}{B}$$

and

$$\begin{aligned}
& \mathbf{v}_E\cdot\boldsymbol{\nabla}\ln B \\
&= \frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{b}}}{B}\cdot\boldsymbol{\nabla}\ln B \\
&= \left(\frac{1}{B}\frac{d\phi}{dr}\widehat{\mathbf{e}}_r\times\widehat{\mathbf{b}}\right)\cdot\boldsymbol{\nabla}\ln B \\
&= \frac{1}{B}\frac{d\phi}{dr}\widehat{\mathbf{e}}_{\theta}\cdot\widehat{\mathbf{e}}_R\frac{1}{R} \\
&= \frac{1}{B}\phi_0'\frac{\sin\theta}{R}
\end{aligned}$$

End.

The drift velocity contains also a term with the time derivative of the electric drift, due to the assumption that the radial electric field has time variation, generating the polarization effect

$$\begin{aligned}
\mathbf{v}_D &= \frac{1}{\Omega_{ci}}\left(\frac{v_{\perp}^2}{2} + v_{\parallel}^2\right)\widehat{\mathbf{b}}\times\boldsymbol{\nabla}\ln B \\
&\quad -\frac{1}{\Omega_{ci}}\frac{\partial\mathbf{v}_E}{\partial t}\times\widehat{\mathbf{b}} \quad \text{(this is polarization drift)}
\end{aligned}$$

Note the last term in the drift velocity: it is significant in the case that the electric field varies very fastly, as is the case in the transitions like reversal of the toroidal rotation. See also **Robertson Hinton 1983**.

The distribution function results from the equation

$$\begin{aligned}
& \frac{\partial f}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}f \\
& +\frac{dv_{\parallel}}{dt}\frac{\partial f}{\partial v_{\parallel}} + \frac{d\left(v_{\perp}^2/2\right)}{dt}\frac{\partial f}{\partial\left(v_{\perp}^2/2\right)} \\
&= St\left(f\right)
\end{aligned}$$

Note that in **Rutherford1970** two series are used, $f = f_M + f^{(1)} + f^{(2)} + ...$ (purely neoclassic) and $g = g^{(1)} + g^{(2)} + ...$

The annihilator is here simply a request of *periodicity* after dividing by $v_{\|}$.

19 The orbit of high energy ions from NBI

19.1 Trapped particles and NBI Cordey

NF16 (1976) 3 slowing down of fast NBI due to trapping.

The following formulas are from **Cordey** and can be found in *plasma, general, NBI.*

Special notations

$$\lambda = \frac{v_\perp^2}{v^2}\frac{B_0}{B}$$

but

$$\begin{aligned} B_0 &\equiv \text{magnetic field} \\ &\quad\ \text{at the outermost point} \\ \theta &= 0 \end{aligned}$$

B_0 is $B_{\max}$ in **Beers** et al. $B_0 = \frac{B^{axis}}{1-\varepsilon}$, (?) and

$$B = B_0\frac{1-\varepsilon\cos\theta}{1-\varepsilon}$$

Def

$$\begin{aligned} \xi &= \sqrt{1-\lambda} \\ &= \sqrt{1-\frac{\mu B_0}{\epsilon}} \\ &\quad\ \text{this is an invariant} \end{aligned}$$

$$\left|v_{\|}\right| = v\sqrt{1-\frac{B}{B_0}\left(1-\xi^2\right)}$$

The new averages are

$$\begin{aligned} \left\langle\frac{v_{\|}}{v}\right\rangle &= \frac{1}{2\pi\sqrt{B_0}}\oint\sqrt{\xi^2 B-(B-B_0)}\,d\theta \\ &\quad\ \text{for circulating particles} \end{aligned}$$

$$
\begin{aligned}
\left\langle \frac{v_\parallel}{v} \right\rangle &= \frac{1}{2\pi\sqrt{B_0}} \int_A^B \sqrt{\xi^2 B - (B - B_0)}\, d\theta \\
& \text{for trapped particles between } A \text{ and } B
\end{aligned}
$$

These "averages" are integrations along the trajectories:

- full circulating trajectory, all θ's.
- Respectively between the two limits θ's of the banana.

Here the magnetic field can be introduced

$$
\begin{aligned}
B &= B_0 \frac{1 - \varepsilon \cos\theta}{1 - \varepsilon} \quad (?) \\
(?) &\equiv \text{different definition of } \theta \text{ and } B \\
& \quad\ \text{relative to other papers}
\end{aligned}
$$

NOTE that B_0 is the magnetic field at $\theta = 0$, *i.e.* at the farthest point on the equatorial plane. Then

$$
\begin{aligned}
\left\langle \frac{v_\parallel}{v} \right\rangle &= \frac{2\xi}{\pi} E\left(\frac{2\varepsilon}{\xi^2}\right) \\
& \text{circulating ions} \\
\xi^2 &> 2\varepsilon
\end{aligned}
$$

$$
\begin{aligned}
\left\langle \frac{v_\parallel}{v} \right\rangle &= \frac{2\sqrt{2\varepsilon}}{\pi} \left[E\left(\frac{\xi^2}{2\varepsilon}\right) - \left(1 - \frac{\xi^2}{2\varepsilon}\right) K\left(\frac{\xi^2}{2\varepsilon}\right) \right] \\
& \text{trapped particles} \\
\xi^2 &< 2\varepsilon
\end{aligned}
$$

and

$$
\begin{aligned}
\left\langle \frac{v}{v_\parallel} \right\rangle &= \frac{2}{\pi} K\left(\frac{2\varepsilon}{\xi^2}\right) \\
& \text{circulating ions} \\
\xi^2 &> 2\varepsilon
\end{aligned}
$$

$$
\begin{aligned}
\left\langle \frac{v}{v_\parallel} \right\rangle &= \frac{2}{\pi \xi_{bound}} K\left(\frac{\xi^2}{2\varepsilon}\right) \\
& \text{trapped ions} \\
\xi^2 &< 2\varepsilon
\end{aligned}
$$

The expressions only depend on $\xi = \sqrt{1 - \frac{\mu B_0}{\epsilon}}$ which is an invariant of the orbit.

The boundary in velocity space between trapped and circulating particles is

$$\xi_{boundary} = \sqrt{2\varepsilon}$$

which makes

$$\frac{\xi^2}{2\varepsilon} \approx 1$$

the arguments of the elliptic functions is close to 1.

The equation for f_0 (resulted from the consistency condition - periodicity on θ - of the first order f_1) needs a condition at boundary

$$\begin{aligned} & \frac{\left\langle \frac{v_\parallel}{v} \right\rangle}{\xi} \frac{\partial f}{\partial \xi}\bigg|_{\xi^+_{boundary}} - \frac{\left\langle \frac{v_\parallel}{v} \right\rangle}{\xi} \frac{\partial f}{\partial \xi}\bigg|_{-\xi^-_{boundary}} \\ = & \ 2 \frac{\left\langle \frac{v_\parallel}{v} \right\rangle}{\xi} \frac{\partial f}{\partial \xi}\bigg|_{\xi^-_{boundary}} \end{aligned}$$

This is the conservation of the flux of particles between the circulating and trapped regions in velocity space.

In the trapped region there should be symmetry relative to the sign of $v_\parallel$,

$$f(-\xi) = f(\xi)$$

The derivative

$$\frac{\partial f}{\partial \xi} \text{ is discontinuous at } \xi_{boundary}$$

(see **Berk Galeev, Galeev Sagdeev, Peeters)**

Close to the transition

$$\left\langle \frac{v}{v_\parallel} \right\rangle \text{ has logarithmic singularity}$$

and

$$\frac{\partial}{\partial \xi} \left\langle \frac{v_\parallel}{v} \right\rangle \quad \text{is singular}$$

and

$$\tau_{bounce} \to \infty$$

since the trapped particles become circulating.

The equation obtained through expansion in $\frac{\tau_{bounce}}{\tau_s} < 1$ cannot be used. *the width of the transition layer in velocity space is*

$$\begin{aligned}\delta\xi &\sim \sqrt{\frac{\tau_{bounce}}{\tau_s}} \\ &\sim 10^{-3}\end{aligned}$$

The slowing down time is much longer than the bounce on banana orbit.

See **Peeters bootstrap, additional**.

19.2 The loss of particles at the edge Rome McAlees Callen Fowler

See also *above*.

The paper adopts

$$j(r) = j_0 \left[1 - \left(\frac{r}{a}\right)^n\right]^p$$

It is defined a parameter

$$P \equiv \frac{qv}{\Omega_c a}$$

The invariants

$$\epsilon = \frac{mv^2}{2}$$

$$\mu = \frac{mv_\perp^2}{2B}$$

$$\cos\chi = \frac{v_\parallel}{v}$$

and the magnetic field

$$B = B_0 \frac{R_0}{R}$$

then

$$\begin{aligned}\mu &= \left(\frac{mv^2}{2} - \frac{mv_\parallel^2}{2}\right)\frac{R}{B_0 R_0} \\ &= \frac{m}{2}v^2\left(1 - \frac{v_\parallel^2}{v^2}\right)\frac{R}{B_0 R_0} \\ &= \frac{m}{2}v^2 \sin^2\chi \frac{R}{B_0 R_0}\end{aligned}$$

This invariant is calculated in two points

$$\mu = \frac{m}{2} v^2 \sin^2 \chi \frac{R}{B_0 R_0} = \frac{m}{2} v^2 \sin^2 \chi_B \frac{R_B}{B_0 R_0}$$

which is written

$$\begin{aligned}
0 &= v^2 \sin^2 \chi \ R - v^2 \sin^2 \chi_B \ R_B \\
0 &= v^2 \left(1 - \cos^2 \chi\right) \ R - v^2 \sin^2 \chi_B \ R_B \\
0 &= \left(v^2 - v_\parallel^2\right) \ R - v^2 \sin^2 \chi_B \ R_B
\end{aligned}$$

$$v_\parallel^2 \ R = v^2 \ R - v^2 \sin^2 \chi_B \ R_B$$

we multiply with R

$$v_\parallel^2 R^2 = v^2 R \left(R - R_B \sin^2 \chi_B\right)$$

$$v_\parallel R = v \sqrt{R \left(R - R_B \sin^2 \chi_B\right)}$$

This expression $Rv_\parallel$ is necessary as part of the *longitudinal invariant*

$$\begin{aligned}
mRv_\varphi &\approx mRv_\parallel \\
&= mv \sqrt{R \left(R - R_B \sin^2 \chi_B\right)}
\end{aligned}$$

the longitudinal invariant is

$$-Ze\psi + mRv_\varphi = \text{const}$$

The part with the magnetic potential

$$\psi = -R_0 A_\varphi$$

The expression of the magnetic potential A_φ is obtained using the Ampere equation for the current density that has been adopted in the general form above, with two indices n and p.

$$\begin{aligned}
\iint \mathbf{dS} \cdot \ \mathrm{curl}\, \mathbf{B} &= \mu_0 \iint dS \ j(r) \\
\oint dl \ B_\theta(r) &= 2\pi B_\theta(r) = \mu_0 \int_0^r 2\pi r dr \ j(r) \\
B_\theta(r) &= \frac{1}{2\pi r} \mu_0 \int_0^r 2\pi r dr \ j(r)
\end{aligned}$$

and

$$B_\theta(a) = \frac{\mu_0 I_p}{2\pi a}$$

but

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

then

$$B_\theta = \frac{1}{r}\frac{\partial}{\partial r}(r A_\varphi)$$

$$\begin{aligned} A_\varphi(r) &= \frac{\mu_0 I_p}{2\pi} \frac{\sum_{j=0}^{p} \frac{(-1)^j \left(\frac{r}{a}\right)^{nj+2}}{(nj+2)} \frac{p!}{j!} \frac{1}{(p-j)!}}{\sum_{j=0}^{p} \frac{(-1)^j}{(nj+2)} \frac{p!}{j!} \frac{1}{(p-j)!}} \\ &= \frac{\mu_0 I_p}{2\pi} F(r) \end{aligned}$$

In this expression it has been defined a new *nondimensional* function

$$F(r)$$

With this expression for $A_\varphi(r)$ we return to the longitudinal invariant

$$ZeR_0 \frac{\mu_0 I_p}{2\pi} F(r) + mv\sqrt{R\left(R - R_B \sin^2 \chi_B\right)} = \text{const}$$

$$F(r) \mp \frac{2\pi m v}{ZeR_0 \mu_0 I_p} \sqrt{R\left(R - R_B \sin^2 \chi_B\right)} = \text{const}$$

We can use the parameter

$$P \equiv \frac{qv}{\Omega_c a}$$

and we replace

$$q = \frac{r B_\varphi}{R B_\theta}$$

where

$$B_\theta(r) = \frac{\mu_0 I_p}{2\pi a}$$

The boundary value of the safety factor is

$$\begin{aligned} q(a) &= \frac{a B_\varphi}{R B_\theta} = \frac{a B_\varphi}{R \mu_0 I_p} 2\pi a \\ &= \frac{2\pi a^2 B_\varphi}{R \mu_0 I_p} \end{aligned}$$

$$
\begin{aligned}
P &= \frac{qv}{\frac{ZeB}{m}a} = v\frac{2\pi a^2 B_\varphi}{R\mu_0 I_p}\frac{m}{ZeBa} \\
&\approx \frac{2\pi a m}{\mu_0 ZeRI_p}v
\end{aligned}
$$

It is introduced the ratio

$$A \equiv \frac{R}{a}$$

and

$$P = \frac{2\pi m v}{\mu_0 ZeAI_p}$$

The *longitudinal invariant* has provided the expression

$$
\begin{aligned}
F(r) \mp \frac{2\pi m v}{ZeR_0\mu_0 I_p}\sqrt{R\left(R - R_B \sin^2\chi_B\right)} &= \text{const} \\
F(r) \mp \frac{P}{a}\sqrt{R\left(R - R_B \sin^2\chi_B\right)} &= \text{const}
\end{aligned}
$$

If all lengths are normalized to a,

$$R \rightarrow \frac{R}{a}$$

the equation above becomes

$$F(r) \mp P\sqrt{R\left(R - R_B \sin^2\chi_B\right)} = \text{const}$$

For the signs

$$
\begin{array}{lll}
\text{sign } - & \text{is taken for} & \mathbf{v}\cdot\mathbf{J} > 0 \\
\text{sign } + & \text{is taken for} & \mathbf{v}\cdot\mathbf{J} < 0
\end{array}
$$

The largest displacement is for the barely trapped particles.

19.2.1 The limit in the velocity space of the barely trapped particles

A barely trapped particle is tangent to the equatorial plane at a point defined as *pinch point.*

This point is special, in the sense that the vertical component of the particle's drift velocity and the vertical component of the motion of the particle along the magnetic field line cancel.

If the coordinates are (x, y) where

$$\begin{aligned} x &= R + r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

the *pinch* point corresponds to

$$\begin{aligned} \frac{dy}{dx} &= 0 \\ y &= 0 \end{aligned}$$

Then one returns to the equation that we have derived for the *longitudinal invariant*

$$F(r) \mp P\sqrt{R\left(R - R_B \sin^2 \chi_B\right)} = \text{const}$$

and takes the derivative dy/dx to zero. It results an implicit equation for the pinch point, x_p,

$$R_p = \frac{R_B \sin^2 \chi}{2}\left[1 + \sqrt{\frac{\left[F'(x_p)\right]^2}{\left[F'(x_p)\right]^2 - P^2}}\right]$$

The *longitudinal invariant* can be used for two different points of the orbit

- the point where the parallel velocity is zero

$$v_{\parallel} = 0$$

- the *pinch* point, as defined above

Then

$$\begin{aligned} & F(x_p) + P\sqrt{R\left(R - R_B \sin^2 \chi\right)} \\ = \; & F(r_B) + P R_B \left|\cos\chi\right| \end{aligned}$$

The procedure of identifying the largest bananas (for barely trapped particles) consists of

- solve the equation which expresses R_p and find the pitch angle χ.

- take this pitch angle χ in the equation which results from this equality (written above) between the *longitudinal invariant* calculated in the pinch point x_p and at the tip $v_{\parallel} = 0$ of the banana; this is to obtain a relationship between x_p (the pitch point position) and the energy, given by P.

Then

$$f(x_p, P) = 0$$

becomes the connection that allows to find in the velocity space the coordinates χ, at a given energy P, of the limit-bananas, the barely trapped particles ("fattest bananas").

19.2.2 The limit in the velocity space of the orbits that are tangent to the limiter

The other side of the loss region in the velocity space is defined as the pitch angle χ of theose orbits that are tangent to the limiter at $x = a$.

$$\begin{aligned} & F(1) - P\sqrt{(A+1)\left(A+1-R_B \sin^2 \chi\right)} \\ = \;& F(r_B) + P R_B \left|\cos \chi\right| \end{aligned}$$

The equation is solved analytically for $|\cos \chi|$,

$$|\cos \chi| = \frac{-R_B K + \sqrt{R_B (A+1)\left[K^2 - P^2 (1-r_B)^2\right]}}{P R_B (1-r_B)}$$

where

$$K \equiv F(1) - F(r_B)$$

19.2.3 The Vertex of the loss region in the velocity space

This point corresponds to the *energy* such that the barely trapped particles has orbit that is tangent to the limiter (both above conditions are met).

20 Particle equations Belli Candy

The field

$$\mathbf{B} = \boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi + I(\psi)\,\boldsymbol{\nabla}\psi$$
$$B_{tor} = \frac{I(\psi)}{R}$$
$$B_{pol} = \frac{|\boldsymbol{\nabla}\psi|}{R}$$

and

$$\psi \equiv \frac{\text{poloidal flux}}{2\pi}$$

The Jacobian

$$\begin{aligned} J_\psi &= \frac{\partial(x,y,z)}{\partial(\psi,\theta,\varphi)} \\ &= \frac{1}{(\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta)\cdot\boldsymbol{\nabla}\varphi} > 0 \end{aligned}$$

and when expresed in terms of r,

$$\begin{aligned} J_r &= \frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} \\ &= \frac{1}{(\boldsymbol{\nabla}r \times \boldsymbol{\nabla}\theta)\cdot\boldsymbol{\nabla}\varphi} = \frac{\partial\psi}{\partial r}J_\psi \end{aligned}$$

The flux surface average

$$\langle f\rangle = \frac{1}{\frac{\partial V}{\partial r}}\oint d\theta d\varphi\ J_r\ f$$
$$V'(r) = \frac{\partial V}{\partial r} = \oint d\theta d\varphi\ J_r$$

The volume inside a flux surface is

$$\begin{aligned} dV &= dn\ dS \\ &= dr d\theta d\varphi\ J_r \end{aligned}$$

The average

$$\begin{aligned} \int dV\ \boldsymbol{\nabla}\cdot\mathbf{A} &= \oint dS\ \mathbf{A}\cdot\mathbf{n} \\ &= \oint \frac{dS}{|\boldsymbol{\nabla}r|}\mathbf{A}\cdot\boldsymbol{\nabla}r \\ &= \frac{dV}{dr}\ \langle \mathbf{A}\cdot\boldsymbol{\nabla}r\rangle \end{aligned}$$

21 Surface - and bounce - averaging

21.0.4 Surface averages

Hazeltine Hinton.

Flux surface average is defined such as to produce *annihilation* of parallel gradients, using the periodicity of the torus.

$$\langle \mathbf{B} \cdot \boldsymbol{\nabla} F \rangle = 0$$

It may be defined as the volume-average with the volume being an infinitesimal shell between two magnetic surfaces

$$\langle F \rangle = \frac{\int_{\delta V} d^3x \ F}{\int_{\delta V} d^3x}$$

or

$$\langle F \rangle = \frac{d\psi}{dV} \int \frac{dS}{|\boldsymbol{\nabla}\psi|} F$$

In [**?**] it is used the operator of surface averaging:

$$\langle A \rangle \equiv \frac{\oint \frac{d\theta}{\mathbf{B} \cdot \boldsymbol{\nabla}\theta} A}{\oint \frac{d\theta}{\mathbf{B} \cdot \boldsymbol{\nabla}\theta}}$$

Different other formulas

$$\begin{aligned}
\langle A \rangle &= \frac{\oint \frac{d\theta}{\mathbf{B} \cdot \boldsymbol{\nabla}\theta} \ A}{\oint \frac{d\theta}{\mathbf{B} \cdot \boldsymbol{\nabla}\theta}} \\
&= \frac{\oint \sqrt{g} d\theta \ A}{\oint \sqrt{g} d\theta} \\
&= \frac{\oint \frac{dl_p}{B_p} \ A}{\oint \frac{dl_p}{B_p}}
\end{aligned}$$

where g is the Jacobian of the transformation from the cylindrical coordinates to flux coordinates, B_p is the poloidal field, dl_p is the element of length along the line.

Also in comments to the papers on **bootstrap** and on **NBI**.

Flux surface average

$$
\begin{aligned}
\langle A \rangle &= \frac{\oint_{line} \frac{dl}{B} A(l)}{\oint_{line} \frac{dl}{B}} \\
&= \frac{\oint_{poloidal} \frac{dl_\theta}{B_\theta} A(l)}{\oint_{poloidal} \frac{dl_\theta}{B_\theta}} \\
&= \frac{\oint \frac{d\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta} A(\theta)}{\oint \frac{d\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}}
\end{aligned}
$$

In working with drift kinetic equation at the first order in neoclassical parameter (**Cordey** and other papers by **Hsu Catto Sigmar,** etc. on *fast ions* and **Helander ECRH** on *bootstrap*)

the solubility condition for the equation

$$
v_\parallel \nabla_\parallel \left(\overline{f}_1 + \frac{I}{\Omega} v_\parallel \left. \frac{\partial f_0}{\partial \psi} \right|_{\epsilon=\text{const}} \right) = \overline{C}\left(\overline{f}_1\right)
$$

[We **note** that this is the usual neoclassical correction, in order 1 in ρ_θ / L_n] is, after averaging over the surface (since this removes the left hand side, being the *annihilator*)

$$
\left\langle \frac{B}{v_\parallel} \overline{C}\left(\overline{f}_1\right) \right\rangle = 0
$$

How it works

The averaging $\langle\rangle$ will lead to periodicity after first multiplication by B. And this is what we want since periodicity on θ is naturally assumed.

We have

$$
B \times | \quad \nabla_\parallel (...) = C
$$

but

$$
\begin{aligned}
dl_\parallel &= dl_\theta \frac{B_\theta}{B_T} = r d\theta \frac{B_\theta}{B_T} = R \left(\frac{r B_\theta}{R B_T} \right) d\theta = qR \ d\theta \\
\frac{1}{dl_\parallel} &= \frac{1}{qR} \frac{1}{d\theta}
\end{aligned}
$$

$$\begin{aligned}\nabla_{\parallel} &= \frac{\partial}{\partial l_{\parallel}} = \frac{1}{qR}\frac{\partial}{\partial\theta} \quad \text{and} \\ B &= \frac{B_0}{h}\end{aligned}$$

so that

$$B\nabla_{\parallel}\left(\right) \rightarrow \frac{B_0}{h}\frac{1}{\frac{rB_T}{RB_\theta}R}\frac{\partial}{\partial\theta}\left(\right) = B_\theta\frac{\partial}{r\partial\theta}\left(\right)$$

Now, we have to recall the definition of the surface averaging operation

$$\langle A\rangle = \frac{\oint\frac{d\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}A\left(\theta\right)}{\oint\frac{d\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}} = \frac{\oint\frac{rd\theta}{B_\theta}A\left(\theta\right)}{\oint\frac{rd\theta}{B_\theta}} = \frac{\frac{1}{b(r)}\oint rd\theta\ h\ A\left(\theta\right)}{\frac{1}{b(r)}\oint rd\theta\ h}$$

and

$$\begin{aligned}\left\langle B\nabla_{\parallel}\left(\right)\right\rangle &= \left\langle B_\theta\frac{\partial}{r\partial\theta}\left(\right)\right\rangle \\ &= \frac{\oint rd\theta\ h\ B_\theta\frac{\partial}{r\partial\theta}\left(\right)}{\oint rd\theta\ h} \\ &= \frac{b\left(r\right)\oint rd\theta\ h\ \frac{1}{h}\frac{\partial}{r\partial\theta}\left(\right)}{\oint rd\theta\ h} = \frac{b\left(r\right)}{\oint rd\theta\ h}\left(\right)_{\theta=0}^{\theta=2\pi} \\ &= 0 \quad \text{for periodic}\end{aligned}$$

This is the reason for which we must multiply (...) with B before taking the surface average, if we intend to exploit poloidal periodicity.

In **Hazeltine Hinton Rosenbluth 1973** the *annihilator on the magnetic surfaces* is defined after the following averaging operator is defined

$$\langle A\rangle = \frac{1}{V'}\oint\frac{d\chi\ A}{\boldsymbol{\nabla}\chi\cdot\mathbf{B}}$$

where

$$(\psi,\chi,\varphi) \equiv \text{coordinates}$$

$$V' = \oint\frac{d\chi}{\boldsymbol{\nabla}\chi\cdot\mathbf{B}}$$

is the derivative of volume inside the magnetic surface ψ to the coordinate ψ:

$$V' = \frac{dV}{d\psi} = \oint \frac{d\chi}{\boldsymbol{\nabla}\chi \cdot \mathbf{B}}$$

The expression inside the integral operator can be seen as a generalization of the case of circular symmetry where we can adopt

$$\chi \rightarrow \theta$$

and from this it results

$$\boldsymbol{\nabla}\chi \cdot \mathbf{B} \rightarrow \boldsymbol{\nabla}\theta \cdot \mathbf{B} = \frac{1}{r}\widehat{\mathbf{e}}_\theta \cdot \widehat{\mathbf{n}} B$$

and we have

$$\begin{aligned} \widehat{\mathbf{e}}_\theta \cdot \widehat{\mathbf{n}} &= \frac{B_\theta}{B} \\ &\equiv \Theta \quad \text{(mostly Russian notation)} \\ &\ll 1 \end{aligned}$$

with

$$B = |\mathbf{B}| \approx \frac{B_0}{h} = B_T$$

then

$$\boldsymbol{\nabla}\theta \cdot \mathbf{B} = \frac{1}{r}\widehat{\mathbf{e}}_\theta \cdot \widehat{\mathbf{n}} B = \frac{B}{r}\frac{B_\theta}{B} \approx B\frac{1}{R}\frac{RB_\theta}{rB_T} = B\frac{1}{qR}$$

From this equality we recognize

$$\begin{aligned} \boldsymbol{\nabla}\theta \cdot \mathbf{B} &= B\frac{1}{qR} \\ B\left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right)\theta &= B\frac{1}{qR} \\ B\nabla_{\|}\theta &= B\frac{1}{qR} \end{aligned}$$

or

$$\nabla_{\|}\theta = \frac{1}{qR}$$

This is the *connection length*. Also, the shift along the poloidal direction for the advancement along a magnetic line is

$$\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta = \frac{1}{qR}$$

expressed as

$$\frac{d\theta}{dl_{\parallel}} = \frac{1}{qR}$$

which is the small angle made by the magnetic line relative to the toroidal direction

$$\frac{rd\theta}{dl_{\parallel}} = \frac{r}{qR} = \frac{\varepsilon}{q}$$

Then we can re-express

$$\begin{aligned} V' &= \frac{dV}{d\psi} = \oint \frac{d\chi}{\boldsymbol{\nabla}\chi \cdot \mathbf{B}} \\ &\rightarrow \oint \frac{d\theta}{B\frac{1}{qR}} = \oint \frac{qR}{B} d\theta = \oint \frac{qR}{B}\frac{dl_{\parallel}}{qR} \\ &= \oint \frac{dl_{\parallel}}{B} \end{aligned}$$

We easily recognize the ratio $dl_{\parallel}/B$ as

$$\begin{aligned} \frac{dl_{\parallel}}{B} &= \frac{qRd\theta}{B} \\ &= \frac{rB_T}{RB_\theta}\frac{R}{B}d\theta \\ &\approx \frac{rd\theta}{B_\theta} \quad \text{(another operator of averaging on surface)} \end{aligned}$$

The averaging operator is

$$\langle A \rangle = \frac{1}{V'}\oint \frac{d\chi}{\boldsymbol{\nabla}\chi \cdot \mathbf{B}} A \rightarrow \frac{\oint \frac{dl_{\parallel}}{B} A}{\oint \frac{dl_{\parallel}}{B}}$$

(**NOTE** the **Kulikovski** surfaces in astrophysics).

21.0.5 Bounce averages

The formula used in ([**?**]) for the **bounce average** on banana orbits

$$\langle A \rangle = \frac{\oint \frac{d\theta}{v_{\parallel}\widehat{\mathbf{b}}\cdot\boldsymbol{\nabla}\theta} A}{\oint \frac{d\theta}{v_{\parallel}\widehat{\mathbf{b}}\cdot\boldsymbol{\nabla}\theta}}$$

The formula used by [**?**] is the same.

We can see that:

$$dl_{\|} = qRd\theta$$

and

$$\frac{d\theta}{\widehat{\mathbf{b}} \cdot \boldsymbol{\nabla}\theta} = \frac{1}{(1/r)\,\widehat{\mathbf{e}}_\theta \cdot \widehat{\mathbf{b}}} d\theta$$

but

$$\begin{aligned} \widehat{\mathbf{e}}_\theta \cdot \widehat{\mathbf{b}} &= \frac{B_\theta}{B_\varphi} = \frac{r}{qR} \\ &= \frac{\varepsilon}{q} \\ &\equiv \Theta \quad \text{(frequent Russian's notation)} << 1 \end{aligned}$$

Then

$$\frac{1}{(1/r)\,\widehat{\mathbf{e}}_\theta \cdot \widehat{\mathbf{b}}} d\theta = \frac{r}{r/\left(qR\right)} d\theta = qRd\theta$$

and

$$\frac{d\theta}{\widehat{\mathbf{b}} \cdot \boldsymbol{\nabla}\theta} = qRd\theta$$

so that

$$\begin{aligned} \frac{dl_{\|}}{v_{\|}} &= \frac{1}{v_{\|}} qRd\theta \\ &= \frac{d\theta}{v_{\|}/\left(qR\right)} \end{aligned}$$

where

$$\frac{v_{\|}}{qR} \equiv \left(\tau_{\|}\right)^{-1} = \text{(time to travel a connection length)}^{-1}$$

The time for bounce is

$$T = \oint \frac{dl_{\|}}{v_{\|}} = \oint \frac{1}{v_{\|}} qRd\theta$$

and the **bounce average of the quantity** A is

$$\left\langle A \right\rangle_b = \frac{1}{T} \oint A \frac{1}{v_{\|}} qRd\theta = \frac{1}{T} \oint Ad\tau_{\|}\left(\zeta\right)$$

where $d\tau_{\|}\left(\zeta\right)$ is the interval of time on the orbit.

In the case of an up-down asymmetric plasma, the gradient of the toroidal rotation velocity can drive a radial heat flux. This is an off-diagonal component in the linear relation between fluxes and forces.

From **Relaxation poloidal rotation Diamond Smolyakov Yushmanov** the divergence of the electric field velocity $\mathbf{v}_E$ in a space-varying magnetic field

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{v}_E &\approx -2\mathbf{v}_E\cdot\boldsymbol{\nabla}\ln B \\ &= -2\sin\theta\frac{v_E}{R}\end{aligned}$$

This is because the modulus B changes in space, along the magnetic line, and gives a non-zero divergence of the electric velocity, $\boldsymbol{\nabla}\cdot\mathbf{v}_E\neq 0$.

$$\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{E}\times\mathbf{B}}{B^2}\right)=\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{B})\frac{1}{B^2}+(\mathbf{E}\times\mathbf{B})\cdot\left(-\frac{2}{B^3}\right)\boldsymbol{\nabla}B$$

and

$$\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{B})=\mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{E})-\mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{B})$$

$$\mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{E})=\mathbf{B}\cdot\left(-\frac{\partial\mathbf{B}}{\partial t}\right)=0 \text{ the time variation of } \mathbf{B} \text{ appears at } \varepsilon^3$$

$$\mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{B})=\mathbf{E}\cdot\mu_0\mathbf{J}=0 \text{ since the electric field here is radial}$$

Both these terms are *zero* and we have

$$\begin{aligned}\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{E}\times\mathbf{B}}{B^2}\right) &= (\mathbf{E}\times\mathbf{B})\cdot\left(-\frac{2}{B^3}\right)\boldsymbol{\nabla}B \\ &= \mathbf{v}_E\cdot\left(-\frac{2}{B}\boldsymbol{\nabla}B\right) \\ &= -2\mathbf{v}_E\cdot\boldsymbol{\nabla}\ln B\end{aligned}$$

The absolute value of the magnetic field is

$$B=\frac{B_0}{1+\varepsilon\cos\theta}$$

$$\begin{aligned}\boldsymbol{\nabla}B &= \left(\widehat{\mathbf{e}}_r\frac{\partial}{\partial r}+\widehat{\mathbf{e}}_\theta\frac{\partial}{r\partial\theta}+\widehat{\mathbf{e}}_\varphi\frac{1}{1+r\cos\theta}\frac{\partial}{\partial\varphi}\right)\frac{B_0}{1+\varepsilon\cos\theta} \\ &\approx \widehat{\mathbf{e}}_r\left[\frac{B_0}{(1+\varepsilon\cos\theta)^2}\left(-\frac{1}{R}\cos\theta\right)\right]+\frac{B_0}{(1+\varepsilon\cos\theta)^2}\widehat{\mathbf{e}}_\theta\frac{\partial}{r\partial\theta}(1+\varepsilon\cos\theta) \\ &\approx B_0(1-2\varepsilon\cos\theta)\left[\left(-\frac{1}{R}\cos\theta\right)\widehat{\mathbf{e}}_r+\left(-\frac{1}{R}\sin\theta\right)\widehat{\mathbf{e}}_\theta\right] \\ &= -\frac{B_0}{R}\widehat{\mathbf{e}}_R(1-2\varepsilon\cos\theta)\end{aligned}$$

This is expected, since the gradient of the scalar B (magnitude of the magnetic field) is directed towards the major symmetry axis of the torus.

This is the *curvature* drift.

We **note** that

$$\boldsymbol{\nabla} \ln B = \frac{\boldsymbol{\nabla} B}{B} \approx -\frac{1}{R}\widehat{\mathbf{e}}_R$$

NOTE. the average of the square of the parallel gradient of the magnetic field has been calculated above. It is

$$\left\langle \left(\nabla_{\|} B\right)^2 \right\rangle = \frac{1}{2}\left(\frac{B_\theta}{R}\right)^2$$

and the result is similar to the expression above

$$\begin{aligned}\left\langle (\boldsymbol{\nabla} B)^2 \right\rangle &= \left\langle \left(\frac{B_0}{R}\sin\theta \widehat{\mathbf{e}}_\theta\right)^2 \right\rangle \sim \left\langle (\sin\theta)^2 \right\rangle \frac{B_0^2}{R_0^2} \\ &= \frac{1}{2}\left(\frac{B_0}{R_0}\right)^2\end{aligned}$$

Now we see the difference

$$\begin{aligned}\left\langle \left(\nabla_{\|} B\right)^2 \right\rangle &= \frac{1}{2}\left(\frac{B_\theta}{R}\right)^2 \rightarrow \text{magnetic mirror} \\ \left\langle (\boldsymbol{\nabla} B)^2 \right\rangle &= \frac{1}{2}\left(\frac{B_0}{R_0}\right)^2 \rightarrow \text{curvature}\end{aligned}$$

We see that the gradient (vector) of the magnitude (scalar) of the magnetic field $\boldsymbol{\nabla} B$ is much bigger than the parallel gradient (scalar) of the magnitude (scalar) of the magnetic field, $\nabla_{\|} B$, with a factor B_0/B_θ. The gradient of the magnitude reflects the *curvature* of the field. The parallel gradient is toroidicity-induced magnetic mirror.

END.

Then the divergence of the electric velocity, due to the variation of the absolute magnitude of the magnetic field in the torus, is

$$\begin{aligned}\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{E}\times\mathbf{B}}{B^2}\right) &= -2\mathbf{v}_E \cdot \boldsymbol{\nabla} \ln B \\ &= -2\mathbf{v}_E \cdot \frac{1}{B}\frac{B_0}{R}\sin\theta\widehat{\mathbf{e}}_\theta\end{aligned}$$

The electric drift is directed along the poloidal circle

$$\mathbf{v}_E \cdot \widehat{\mathbf{e}}_\theta = v_E$$

because the electric drift is due to *radial* electric field E_r and to the main magnetic field, toroidal B_φ.

Then

$$\nabla \cdot \mathbf{v}_E \approx -2\frac{v_E}{R}\sin\theta$$

after taking $B \approx B_0$.

Balance of fluid pressure, forces and sources

The α-momentum of the Fokker Planck equation is:

$$\mathbf{0} = -\nabla \cdot \mathbf{P}_\alpha^{ai} + \mathbf{F}_\alpha^{ai} + \mathbf{S}_\alpha^{ai}$$

This is the steady state equilibrium of the

- $\alpha = 2 :$ momentum (force equation)
- $\alpha = 3 :$ heat (heat equation)

Now we take the **parallel** component and perform the **surface averaging** of the equations.

The steady-state flux surface averaged **parallel force balance** equations are

$$\left\langle \mathbf{B} \cdot \nabla \cdot \mathbf{P}_\alpha^{ai} \right\rangle = \left\langle \mathbf{F}_\alpha^{ai} \cdot \mathbf{B} \right\rangle + S_{\|,\alpha}^{ai} \quad , \ \alpha = 1, 2, 3$$

where

- $\mathbf{B}$ is the toroidal magnetic field,
- a denotes the *izotope*
- i denotes the *charge state*
- α denotes the order of the odd velocity moment,
  - $\alpha = 1$ is the **momentum balance**
  - $\alpha = 2$ is the heat balance
  - $\alpha = 3$ is rarely used
- $\nabla \cdot \mathbf{P}_\alpha^{ai}$ are the viscous forces,
- $\mathbf{F}_\alpha^{ai}$ are friction forces
- parallel momentum and heat flux **source** and **sink** terms

  $$S_{\|,\alpha}^{ai} = S_{E\|,\alpha}^{ai} + S_{NBI\|,\alpha}^{ai} + S_{cx\|,\alpha}^{ai} + S_{an\|,\alpha}^{ai}$$

  from

- parallel electric field
- neutral beam injection
- charge exchange
- anomalous processes

21.1 Formulas

From the paper **Bootstrap Hsu Shaing Sigmar** the formula for circular geometry

$$\frac{I}{B_0}\frac{\partial}{\partial\psi} \simeq \frac{1}{B_{pol}}\frac{\partial}{\partial r}$$

where

$$\begin{aligned} I &\equiv \mathbf{B}\cdot R^2\boldsymbol{\nabla}\varphi \\ &= \mathbf{B}\cdot R\widehat{\mathbf{e}}_\varphi \\ &= RB_{tor} \end{aligned}$$

$$\boldsymbol{\nabla}\psi = RB_{pol}$$

(in other notation it is $\boldsymbol{\nabla}\psi = 2\pi RB_{pol}$)

$$|\boldsymbol{\nabla}\psi|\frac{\partial}{\partial\psi} = \frac{\partial}{\partial r}$$

Then

$$\begin{aligned} \frac{I}{B_0}\frac{\partial}{\partial\psi} &= \frac{RB_T}{B_0}\frac{1}{|\boldsymbol{\nabla}\psi|}|\boldsymbol{\nabla}\psi|\frac{\partial}{\partial\psi} = \frac{RB_T}{B_0}\frac{1}{RB_{pol}}\frac{\partial}{\partial r} \\ &\approx \frac{1}{B_{pol}}\frac{\partial}{\partial r} \end{aligned}$$

after taking $B_T/B_0 \sim 1$.

The expression for the Pfirsch-Schluter current (from **Connor MHD stability PoP5 1998 2687 ELM**)

$$J_\parallel^{PS} = -\frac{Ip'}{B}\left(1-\frac{B^2}{\langle B^2\rangle}\right)$$

where

$$I \equiv RB_\varphi$$

and

$$\langle ... \rangle = \frac{\oint \frac{dl}{B_\theta} (...)}{\oint \frac{dl}{B_\theta}}$$

It is parallel with the magnetic field on the outboard side and antiparallel on the inward side.

In the paper **Multiple equilibria poloidal rotation Ware Wiley** it is expressed the Pressure tensor

$$\mathbf{P} = \sum_j \left(P_j + \boldsymbol{\pi}_j + n_j m_j \mathbf{V}_j \mathbf{V}_j \right)$$

where

$$P_j \equiv \int d^3v \; m_j \left(\mathbf{v} - \mathbf{V}_j \right) \left(\mathbf{v} - \mathbf{V}_j \right) \widetilde{f}_j^{\zeta}$$

$$\boldsymbol{\pi}_j \equiv \int d^3v \; m_j \left(\mathbf{v} - \mathbf{V}_j \right) \left(\mathbf{v} - \mathbf{V}_j \right) \widetilde{f}_j^{\zeta}$$

The function $\overline{f}_j^{\zeta}$ is the distribution function f_j that has been averaged over the gyrophase angle: it is the solution of the *drift-kinetic* equation.
The function $\widetilde{f}_j^{\zeta}$ is the part of the distribution function f_j which depends on the gyrophase angle.

The divergence of a vectorial quantity like $\widetilde{n}\overline{V}_{\parallel}$ in the coordinate system (r, θ, φ) will contain a term

$$... + B_\theta \frac{\partial}{r\partial\theta} \left(\frac{\widetilde{n}\overline{V}_{\parallel}}{B} \right)$$

since

$$B \approx \frac{B_0}{1 + \varepsilon \cos\theta} = \frac{B_0}{h}$$

this will introduce the *metric factor* h in the derivation

$$... + \frac{B_\theta}{B_0} \frac{\partial}{r\partial\theta} \left(h \widetilde{n}\overline{V}_{\parallel} \right)$$

It looks similar to

$$\boldsymbol{\nabla} \cdot \mathbf{J}_{\parallel} = \boldsymbol{\nabla} \cdot \left(\mathbf{J} \cdot \frac{\mathbf{B}}{B} \right) = \mathbf{B} \cdot \boldsymbol{\nabla} \left(\frac{J_{\parallel}}{B} \right)$$

The equation of continuity for the species j to first order in $\rho_{i\theta}/L$ (**Ware Wiley**)

$$\begin{aligned}
& \frac{\partial \widetilde{n}_j}{\partial t} - \frac{E_r}{B}\frac{\partial \widetilde{n}_j}{r\partial \theta} + B_\theta \frac{\partial}{r\partial\theta}\left(\frac{\widetilde{n}_j \overline{V}_{j\parallel}}{B}\right) \\
& -\frac{2\sin\theta}{R}\frac{1}{Z_j e B}\left(\overline{p}_j - \overline{n}_j Z_j e \overline{E}_r\right) \\
= \; & 0
\end{aligned}$$

In the paper **bootstrap current ecrh Helander Hastie Connor** we find

$$\frac{\partial}{\partial v_\perp^2} = \frac{v_\parallel^2}{v^4}\frac{1}{B}\frac{\partial}{\partial\lambda} + \frac{1}{2}\frac{\partial}{\partial w}$$

where

$$\begin{aligned}
\lambda &\equiv \frac{v_\perp^2}{v^2}\frac{1}{B} \\
w &= \frac{1}{2}v^2
\end{aligned}$$

and

$$d^3 v = \sum_{\sigma=\pm 1} \frac{2\pi B}{v_\parallel} w \; dw \; d\lambda$$

22

23 Comments on Galeev Sagdeev theory

23.1 From Galeev Sagdeev vol.7

Some notations

$$\Theta = \frac{\varepsilon}{q} = \frac{B_\theta}{B_T} \ll 1$$

used to make projections from parallel to poloidal directions.

$$\mu = \frac{m v_\perp^2}{2B_0}$$

$$
\begin{aligned}
v_{\parallel} &= \sigma\sqrt{\frac{2}{m}\left[\epsilon - e\Phi_0 - \mu B_T(r,\theta)\right]} \\
\sigma &= \pm 1
\end{aligned}
$$

The equations

$$
\begin{aligned}
\frac{dr}{dt} &= -\frac{1}{\Omega_c}\frac{\frac{\mu B_0}{m}+v_{\parallel}^2}{R}\sin\theta \\
\frac{rd\theta}{dt} &= \frac{B_\theta}{B_T}v_{\parallel} + \frac{1}{B_T}\frac{d\Phi_0}{dr} - \frac{1}{\Omega_c}\frac{\frac{\mu B_0}{m}+v_{\parallel}^2}{R}\cos\theta
\end{aligned}
$$

NOTE in **Galeev Sagdeev** the term coming from the drift velocity is considered as being composed of

- diamagnetic velocity
- curvature drift velocity

and this is explained in a different work, where the spatial invariant $p + \frac{B^2}{2\mu_0}$ =const is derived. The $\boldsymbol{\nabla} B$ part of the particle drift is shown in fluid treatment to be identical with the $\boldsymbol{\nabla} p$ drift, *i.e.* with the diamagnetic drift. This is however a purely formal identification of the constraint that one needs a current with non-zero perpendicular component, such as to obtain $0 = -\boldsymbol{\nabla} p + \mathbf{j}\times\mathbf{B}$ balance.

23.2 Derivation of the equations Galeev Sagdeev 1968 from the invariant (mostly repetition)

The equations are

$$
\frac{dr}{dt} = -\frac{1}{\Omega_c}\frac{\frac{\mu B}{m}+v_{\parallel}^2}{R}\sin\theta
$$

$$
\frac{rd\theta}{dt} = -\frac{1}{\Omega_c}\frac{\frac{\mu B_0}{m}+v_{\parallel}^2}{R}\cos\theta + \frac{1}{B_T}\frac{d\Phi}{dr} - \Theta v_{\parallel}
$$

where

$$
\mu \equiv \frac{mv_{\perp}^2}{2B_T}
$$

Note that the drift velocity is expressed in terms of μ and $v_{\parallel}$.

$$
v_{\parallel} = \sigma\left\{\frac{2}{m}\left[\epsilon - e\Phi(r) - \mu B_T(r,\theta)\right]\right\}^{1/2}
$$

As shown previously (**Berk Galeev**) this leads to the identification of the invariant

$$J = \Omega_c \int_0^r \Theta \ dr + v_{\|} \left(1 + \varepsilon \cos\theta\right)$$

where

$$\Theta \equiv \frac{B_\theta}{B_T} = \frac{\varepsilon}{q} \ll 1$$

One takes a reference point.

This can be:

$$\begin{array}{l} \left(r_0, \theta = 0\right) \\ \text{reference point} \end{array}$$

Then, expand the invariant J in small radial deviation of the trajectory relative to the reference point

$$r - r_0$$

$$\frac{1}{2}\Omega_c \left(\frac{B_\theta}{B_T}\right)^2 \left(r - r_0\right)^2 + \left(\delta v_{\|}\right) \left(\frac{B_\theta}{B_T}\right) \left(r - r_0\right) + v_D r_0 \left(1 - \cos\theta\right) = 0$$

where

$$\delta v_{\|} = v_{\|}\left(r_0, 0\right) + \frac{v_0}{B_\theta / B_T}$$

and

$$v_0 \equiv v_E \quad \text{poloidal}$$

$$v_D = \frac{1}{\Omega_c} \frac{\frac{\mu B_0}{m} + v_{\|}^2}{R}$$

What is obtained

$$r - r_0 = \frac{1}{\Omega_c} \frac{B_T}{B_\theta} \left\{ -\left(\delta v_{\|}\right) \pm \left[\left(\delta v_{\|}\right)^2 + 2 r_0 \Omega_c v_D \left(\cos\theta - 1\right)\right]^{1/2} \right\}$$

where

$$\begin{aligned} & \delta v\left(r_0, \theta = 0\right) \\ = \ & v_{\|}\left(r_0, \theta = 0\right) - v_E \frac{B_T}{B_\theta} \\ = \ & \text{the parallel velocity of reference} \end{aligned}$$

If $\delta v_{\|}$ is large, an approximation is possible

$$r - r_0 \approx \frac{1}{\Omega_c} \frac{\frac{\mu B_0}{m} + v_{\|}^2}{R} \frac{1}{\delta v_{\|}} r_0 \left(\cos\theta - 1\right)$$

This quantity δv should be small.

23.2.1 Equations of motion for the guiding centre Galeev Sagdeev.

Considering the projection on the (r, θ) plane of the drift velocity $\mathbf{v}_D$ and replacing $v_\perp^2/2 = \mu B_0/m$ (if we include the mass m in the definition of the magnetic momentum μ), we have

$$\frac{dr}{dt} = -\frac{\frac{\mu B_0}{m} + v_\parallel^2}{\Omega} \frac{\sin\theta}{R_0} \tag{6}$$

$$\frac{r d\theta}{dt} = \frac{\varepsilon}{q} v_\parallel - \frac{\frac{\mu B_0}{m} + v_\parallel^2}{\Omega R_0} \cos\theta + \frac{1}{B_z} \frac{d\Phi_0}{dr} \tag{7}$$

Here we have

$$\frac{\varepsilon}{q} v_\parallel = \frac{B_\theta}{B_T} v_\parallel \sim v_\theta$$

is the projection of the parallel velocity on the poloidal direction.

The equations are

$$\begin{aligned}
\frac{dr}{dt} &= -\frac{v_\perp^2/2 + v_\parallel^2}{\Omega R_0} \sin\theta \\
\frac{r d\theta}{dt} &= \frac{B_\theta}{B_T} v_\parallel + \frac{1}{B_z}\frac{d\Phi_0}{dr} - \frac{v_\perp^2/2 + v_\parallel^2}{\Omega R_0} \cos\theta
\end{aligned}$$

There is a toroidal drift of the particles.

This results from the average over the *bounce*. [**Note** the similarity with the diamagnetic flow which results from averaging over the gyromotion. But, a *force* was necessary, i.e. $(-\boldsymbol{\nabla} p)$. **End.**]