1 Introductory comments

In general the charged particles move in a magnetic field, either spontaneously created by themselves or externally imposed. The main characteristic is the Larmor gyration, with a radius ρ_s. The particles follow the magnetic field lines but due to geometry there is also a displacement (a drift) relative to the line, Δ, which can be substantially higher than ρ_s.

When there are collisions with frequency $\nu^{-1} \equiv \tau$, a scattering event move a charged particle from its initial line to a different position, at about ρ_s distance. A set of random such events will produce diffusion with a coefficient D given by the ratio of the squared typical displacement and the typical time, $D \sim \rho_s^2/\tau$. This is the *classical transport* and is small for the case of the strong magnetic field. However the drift, induced by the geometry of the magnetic field, can be large and a transport $D^{neo} \sim \Delta^2/\tau$ is much higher than D. This is the neoclassical transport.

Why is-it interesting? The transport processes $\leftarrow$ The turbulence $\leftarrow$ the instabilities

Fluid or kinetic?

If kinetic: any mechanism that prevent the particle to get in phase with the wave (of a perturbation) will exchange energy with that perturbation. The latter can give energy to the particle (decay) or tap energy of the particle (growth). We need to review these mechanisms. Therefore we need trajectories.

The geometry of the magnetic field.

The theory of particle motion in strong magnetic field (from **Berk Galeev** to general Hamiltonian). Banana and circulating. Figures. The kinetic point of view. Liouville theorem, Chapman Kolmogorov probabilities equation, Vlasov equation. The BBGKY hierarchy. Boltzmann equation.

The drift-kinetic equation.

The solution of the drift kinetic equation for: variation of the parameters in the magnetic surface, neoclassical equilibrium flows. Preparing for instability theory.

2 Field aligned coordinates

The paper **field aligned diamond cowley**

Comments in *particle equations of motion.*

Dominski2015 paper with **Jenko** using **GENE.**

Two sets of velocity variables. The one used is

$$(v_\parallel, \mu)$$

with integration over the full velocity space

$$\int_{all} d\mu dv_\parallel = \int_0^\infty d\mu \int_{-\infty}^\infty dv_\parallel$$

But in the case of trapped particles,

$$\int_{trapped} d\mu dv_{\|} = \int_0^{\infty} d\mu \int_{-v_{\|c}}^{+v_{\|c}} dv_{\|}$$

for critical parallel velocity

$$v_{\|c}(\mathbf{x}, \mu) = \sqrt{\frac{2\mu}{m_e}\left[B_{0\max}(x) - B(\mathbf{x})\right]}$$

where $B_{0\max}(x)$ is the maximum value of the magnetic field on the magnetic surface which is situated at the radial distance x. The point $\mathbf{x}$ from the argument of $B(\mathbf{x})$ is on this surface.

Distance from the resonant surface **Connor2007**

$$x = \frac{r - r_0}{nq\widehat{s}}$$

The Jacobian is

$$\frac{B_{0\|}^*}{m_j}$$

where

$$B_{0\|}^* = B_0 + \frac{m_j v_{\|}}{e_j}\left(\boldsymbol{\nabla}\times\widehat{\mathbf{n}}_0\right)\cdot\widehat{\mathbf{n}}_0$$

See also **Peeters** where $B_{\|}^*$ appears.

3 Polarization current Peeters

this is in *polarization.tex*.

The formulas for the field

$$B_{tor} = \frac{B_0}{1 + \varepsilon\cos\theta}$$

$$B_{pol} = \frac{B_{\theta 0}}{1 + \varepsilon\cos\theta}$$

and it is adopted

$$q(r) = q_0\left(1 + br^2\right)$$

Then

$$B_{\theta 0} = \frac{\varepsilon\ B_0}{q(r)\sqrt{1-\varepsilon^2}}$$

Connect the coordinates

$$(r, \theta)$$

with the Boozer coordinates

$$(\psi, \chi)$$

as

$$r = R_c \sqrt{1 - \frac{1+b}{b} \tanh^2 \left[\sqrt{b\,(1+b)}\ \ (a_0 - q_0 \psi) \right]}$$

$$\theta = 2 \arctan \left[\sqrt{\frac{1+r}{1-r}}\ \tan \left(\frac{\chi}{2} \right) \right]$$

where

$$a_0 = \frac{\arctan h \sqrt{b\,(b+1)}}{\sqrt{b\,(b+1)}}$$

The bootstrap current is calculated in the presence of the island with

$$j_{bs} = \left\langle en\ v_{i\parallel}\ B \right\rangle$$

4 Current ramp-up

In the paper **Stambaugh current diffusion DIII**.

The rise of the ohmic current

$$-\frac{1}{\mu_0} \iint d\mathbf{S} \cdot \mathbf{E} \times \mathbf{B} = \frac{\partial}{\partial t} \iiint dV\ \frac{B^2}{2\mu_0} + \iiint dV\ \eta J^2$$

The *internal inductance* l_i is defined as

$$\begin{aligned} \frac{1}{2} L_p I^2 &= \frac{1}{2} \left(\frac{\mu_0 R}{2} l_i \right) I^2 \\ &= \iiint dV\ \frac{B^2}{2\mu_0} \end{aligned}$$

under the assumption $R/a \to \infty$.

For example, for DIII, the internal inductance has a variation

$$l_i \sim 0.5 \text{ for flat current density}$$
$$l_i \sim 2 \text{ for very centrally peaked current density}$$

One can write a *circuit equation*

$$V\ I = \frac{\partial}{\partial t} \left(\frac{1}{2} L_p I^2 \right) + \iiint dV\ \eta J^2$$

The inductance (defined in Rossi thesis)

$$\begin{aligned} l_i &= \frac{\left\langle B_\theta^2 \right\rangle}{B_\theta^2\,(a)} \\ &= \frac{\int_0^a 2\pi\rho d\rho\ B_\theta^2\,(\rho)}{\pi a^2 B_\theta^2\,(a)} \end{aligned}$$

5 Geometry

5.1 Units

Standard gyro-Bohm units

$$\begin{aligned} a &\equiv \text{ minor radius of the tokamak} \\ &\quad \text{to be compared with } L_n = \left(\frac{d}{dr}\ln n\right)^{-1} \\ c_s &\equiv \text{sound speed } = 2T_e/m_i \\ \rho_s &\equiv \text{ Larmor radius at the sound speed,} \\ &\quad \text{to be compared with } L_n \end{aligned}$$

$$\Omega_{ci} = \frac{|e|\,B}{m_i} \equiv \text{cyclotronic frequency, higher than everything else}$$
$$\text{(except the plasma frequency } \omega_p\text{)}$$

5.2 Coordinates

The magnetic field

$$\mathbf{B} = \mathbf{B}_T + \mathbf{B}_p$$

$$\begin{aligned} \mathbf{B}_T &= F(\psi)\,\boldsymbol{\nabla}\varphi \quad \text{toroidal} \\ \mathbf{B}_p &= \boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi \quad \text{poloidal} \end{aligned}$$

$$2\pi\psi \equiv \text{poloidal flux}$$

NOTE

The sense of introducing $F(\psi)$: the real expression for the toroidal magnetic field is

$$\mathbf{B}_T = \frac{B_0}{h}\widehat{\mathbf{e}}_\varphi = B_0\frac{R_0}{R_0 h}\widehat{\mathbf{e}}_\varphi = B_0 R_0 \boldsymbol{\nabla}\varphi \sim F\boldsymbol{\nabla}\varphi$$

The function F depends only on the surface label, ψ.

Here however there is NO dependence of F, it is constant.

F is I.

In **Hazeltine Hinton**

$$I(\psi,\theta,\varphi) = \sqrt{g}\,(\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta)\cdot\mathbf{B}$$

where

$$\begin{aligned} g(\psi,\theta,\varphi) &= \frac{1}{|\boldsymbol{\nabla}\psi\cdot(\boldsymbol{\nabla}\theta\times\boldsymbol{\nabla}\varphi)|^2} \\ \text{and } \frac{\partial g}{\partial\varphi} &= 0 \end{aligned}$$

In Hamada coordinates

$$\text{and more, } g = \text{const} \quad \text{(Hamada)}$$

In general

$$\nabla\psi \text{ and } \nabla\theta \text{ are NOT orthogonal}$$
$$\text{(for example in } D\text{-shape plasma)}$$

Using usual definitions for *circular plasma* we have

$$g = \frac{1}{\left(RB_\theta \frac{1}{r}\frac{1}{R}\right)^2} = \left(\frac{r}{B_\theta}\right)^2$$

Such a B_θ would correspond to a radially uniform profile of $j(r)$, if g =const, since in this case

$$\begin{aligned} 2\pi r B_\theta(r) &= \mu_0 \int_0^r 2\pi r dr \; j(r) \sim \mu_0 2\pi \frac{r^2}{2} j_{ct} \\ B_\theta(r) &\sim r \times \left(\frac{\mu_0 j_{ct}}{2}\right) \\ \text{and } \frac{r}{B_\theta(r)} &= \text{ct} \end{aligned}$$

But in general (not Hamada) g is not constant.

Then, for *circular plasma*

$$\begin{aligned} I &= \sqrt{g}\left(\nabla\psi \times \nabla\theta\right) \cdot \mathbf{B} \\ &= \left(\frac{r}{B_\theta}\right) RB_\theta \frac{1}{r} B_T = RB_T \end{aligned}$$

But, the configuration is not so simple

$$\nabla\theta \neq \frac{1}{r}\widehat{\mathbf{e}}_\theta$$

because for a D-shaped plasma r is not relevant.

Also $|\nabla\psi| \neq RB_\theta$ in a D-shaped plasma.

This explains why

$$I \neq \text{const}$$

In **Hirshman Sigmar**

$$I = R^2 \mathbf{B} \cdot \nabla\varphi$$

and

$$|\nabla\varphi| = \frac{1}{R}$$

and

$$
\begin{aligned}
\mathbf{B} &= \frac{1}{2\pi}\frac{\partial\chi}{\partial\psi}\boldsymbol{\nabla}\varphi\times\boldsymbol{\nabla}\psi \quad \text{(poloidal)} \\
&\quad +I\boldsymbol{\nabla}\varphi \qquad\qquad\qquad \text{(toroidal)}
\end{aligned}
$$

where

$$
\begin{aligned}
\chi' &= \frac{\partial\chi}{\partial\psi} = 2\pi\sqrt{g}\ \mathbf{B}\cdot\boldsymbol{\nabla}\theta \\
&= \text{poloidal magnetic flux density}
\end{aligned}
$$

In **Hazeltine Hinton**

$$
I(\psi,\theta,\varphi) = \sqrt{g}\left(\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\theta\right)\cdot\mathbf{B}
$$

and the definitions

$$
\begin{aligned}
\psi(\mathbf{x}) &= \frac{1}{2\pi}\int d^3x\ \boldsymbol{\nabla}\theta\cdot\mathbf{B} \\
&\quad \text{poloidal flux function}
\end{aligned}
$$

$$
\begin{aligned}
\phi(\mathbf{x}) &= \frac{1}{2\pi}\int d^3x\ \boldsymbol{\nabla}\varphi\cdot\mathbf{B} \\
&\quad \text{toroidal flux function}
\end{aligned}
$$

$$
\begin{aligned}
V(\mathbf{x}) &= \int d^3x \\
&\quad \text{volume inside a magnetic surface}
\end{aligned}
$$

For an axisymmetric tokamak, the toroidal direction φ is irrelevant. It remain (ψ,θ) where

$$
\begin{aligned}
\psi &\equiv \text{poloidal magnetic flux} \\
&\sim \text{radial coordinate} \\
\theta &\equiv \text{poloidal angle}
\end{aligned}
$$

$$
\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial\psi}\left(\sqrt{g}\ \mathbf{A}\cdot\boldsymbol{\nabla}\psi\right) + \frac{\partial}{\partial\theta}\left(\sqrt{g}\ \mathbf{A}\cdot\boldsymbol{\nabla}\theta\right)\right]
$$

This equation is used in the expansion of the conditions of conservation of the fluxes of particles and of heat

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot\left(n_a\mathbf{u}_a\right) &= 0 \\
\boldsymbol{\nabla}\cdot\left(\mathbf{q}_a\right) &= 0
\end{aligned}
$$

in the first order in $\delta = \rho/L$.

One notes then that the ratio between the poloidal components of the flows and the poloidal magnetic field is a function of only the flux surface

$$\frac{\text{poloidal component of flux}}{\text{poloidal magnetic field}} \sim \text{function of } \psi$$

The one introduces new variables

$$\begin{aligned} U_{a\theta} &= \frac{\mathbf{u}_a \cdot \boldsymbol{\nabla}\theta}{\mathbf{B} \cdot \boldsymbol{\nabla}\theta} \\ Q_{a\theta} &= \frac{\mathbf{q}_a \cdot \boldsymbol{\nabla}\theta}{\mathbf{B} \cdot \boldsymbol{\nabla}\theta} \end{aligned}$$

See **Shaing** viscosity. here

$$\begin{aligned} U_{a\theta} &= \frac{u_{a\theta}}{B_\theta} \\ & \text{in the expression of the velocity} \\ & \text{we introduce } B_\theta \end{aligned}$$

Later it will become clear that this combination, although strange, has an advantage: *the ratio of the poloidal velocity and the poloidal magnetic field is a quantity that only depends on the magnetic surface, which means that it is function of r or ψ.*

If the coordinate along the line of the magnetic field is η,

$$\eta = l_{\|}$$

we have for the other coordinate

$$\theta = \frac{\eta}{qR} = \frac{l_{\|}}{qR} \tag{1}$$

The definition

$$|\boldsymbol{\nabla}\varphi \cdot (\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta)| = \frac{1}{\sqrt{g}}$$

$$\begin{aligned} I &\equiv \sqrt{g}\,(\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta) \cdot \mathbf{B} \\ RB_\varphi &= \sqrt{g} 2\pi R B_\theta \frac{1}{r} B_\varphi \\ \frac{r}{2\pi}\frac{1}{B_\theta} &= \sqrt{g} \\ \frac{r}{2\pi}\frac{1}{B_\theta}\frac{B_\varphi}{B_\varphi}\frac{R}{R} &= \sqrt{g} \\ \frac{1}{2\pi}\frac{qR}{B_\varphi} &= \sqrt{g} \end{aligned}$$

From **Catto spikes parallel current**

$$\nabla r \cdot (\nabla\theta \times \nabla\varphi) = \frac{1}{\sqrt{g}}$$

it results for this case

$$\frac{1}{rR} = \frac{1}{\sqrt{g}}$$

and

$$\begin{aligned} \nabla \cdot \mathbf{J} &= 0 \\ \mathbf{B} \cdot \nabla \left(\frac{j_{\|}}{B}\right) &= -\nabla_{\perp} \cdot \mathbf{j}_{\perp} \end{aligned}$$

NOTE the way to write this divergence, keeping a $1/B$ inside the operator and a $\mathbf{B}$ outside has in view the curviliniar system of coordinates and the occurence of the factor h. It is a simple consequence of $\nabla \cdot \mathbf{B} = 0$. **END.**

Derivative along the magnetic field

$$\mathbf{B} \cdot \nabla f = B_{\theta} \left(\frac{\partial}{r\partial\theta} + q\frac{\partial}{R\partial\varphi}\right) f$$

Since

$$\langle \mathbf{B} \cdot \nabla f \rangle = 0$$

one must have the *ambipolarity condition*

$$\langle \nabla_{\perp} \cdot \mathbf{j}_{\perp} \rangle = 0$$

5.3 Perpendicular on the magnetic field

Frequently there are vector products

$$\mathbf{a} \times \widehat{\mathbf{n}}$$

and this involves the expression

$$\left(\frac{I}{B}\widehat{\mathbf{n}} - R^2 \nabla\varphi\right)$$

See **Helander 3999**.

5.4 Magnetic field

From **Hazeltine Hinton**.

For the coordinates

$$(\psi, \theta, \varphi)$$

$$\begin{aligned} g(\psi, \theta, \varphi) &= \det g_{ik} \\ &= \frac{1}{|\nabla\psi \cdot (\nabla\theta \times \nabla\varphi)|^2} \end{aligned}$$

the standard condition

$$\frac{dg}{d\varphi} = 0 \quad \text{(independence of the toroidal angle)}$$

and the stronger

$$g = \text{const} \quad \text{(Hamada)}$$

Other relations

$$\begin{aligned} \nabla\psi \times \nabla\theta &= \frac{1}{q} B_T \\ &= \frac{RB_\theta}{rB_T} B_T = \frac{B_\theta}{\varepsilon} \end{aligned}$$

actually we have for circular surfaces (a problem with the coefficient 2π)

$$\begin{aligned} |\nabla\psi| &= 2\pi R B_\theta \\ |\nabla\theta| &= \frac{1}{r} \\ |\nabla\varphi| &= \frac{1}{R} \end{aligned}$$

then

$$\sqrt{g} = \frac{1}{|\nabla\psi \cdot (\nabla\theta \times \nabla\varphi)|} = \frac{rR}{2\pi R B_\theta} = \frac{1}{2\pi} R \frac{rB_T}{RB_\theta} \frac{1}{B_T} = \frac{1}{2\pi} \frac{qR}{B_T}$$

or

$$\sqrt{g} = \frac{1}{2\pi} \frac{r}{B_\theta}$$

$$\sqrt{g} = \frac{1}{2\pi} \frac{qR}{B_T}$$

That will give

$$\begin{aligned} \sqrt{g} &= \frac{1}{2\pi} q R_0 h \frac{h}{B_0} \\ &\sim h^2 \end{aligned}$$

Below, after **Yushmanov** the determinant D is actually $\sqrt{g}$

$$D \equiv \frac{1}{2\pi}\frac{qR}{B_T} = \sqrt{g}$$

The scalar parameter,

$$I(\psi,\theta,\varphi) = \frac{1}{q}\sqrt{g}\mathbf{B}\cdot\mathbf{B}_T$$

Approximately

$$I \approx \frac{1}{q}B_T^2\frac{1}{2\pi}\frac{qR}{B_T} = \frac{1}{2\pi}RB_T$$

Three global quantities that may serve to the definition of the magnetic field lines and surfaces

$$\begin{aligned}
\psi(\mathbf{x}) &= \frac{1}{(2\pi)^2}\int d^3x\ \boldsymbol{\nabla}\theta\cdot\mathbf{B} \\
&\quad \text{poloidal flux function}
\end{aligned}$$

$$\begin{aligned}
\Phi(\mathbf{x}) &= \frac{1}{2\pi}\int d^3x\boldsymbol{\nabla}\varphi\cdot\mathbf{B} \\
&\quad \text{toroidal flux function}
\end{aligned}$$

$$\begin{aligned}
V(\mathbf{x}) &= \int d^3x \\
&\quad \text{volume inside a magnetic surface}
\end{aligned}$$

$$\begin{aligned}
q(\psi) &= \frac{1}{2\pi}\frac{d\Phi}{d\psi} \\
&= \text{safety factor} \\
&= \text{reciprocal of rotational transform} \\
&= \text{number of toroidal circuits} \\
&\quad \text{for one poloidal circuit}
\end{aligned}$$

another approach

$$\frac{\iota}{2\pi} = -q^{-1}$$

The system can be used to define

$$\mathbf{B}_\theta = \boldsymbol{\nabla}\varphi\times\boldsymbol{\nabla}\psi$$

$$\mathbf{B}_T = q\ \boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\theta$$

In the paper **current driven asymmetric fuelling Helander Fulop**

$$\nabla_{\|} B = \frac{\varepsilon B}{qR} \sin\theta$$

or

$$\nabla_{\|} \ln B = \frac{1}{qR} \varepsilon \sin\theta$$

(we write $1/(qR) \sim \nabla_{\|}$). The factor $\varepsilon \sin\theta$ is the harmonic variation along the line. If we write it like

$$\nabla_{\|} \ln B = \frac{\varepsilon}{q} \frac{\sin\theta}{R} = \Theta \frac{\sin\theta}{R}$$

it appears like the poloidal projection

$$\Theta = \frac{B_\theta}{B_\varphi}$$

of $\sin\theta / R$. The average that appears below

$$\begin{aligned}
\nabla_{\|} B &= B \frac{\varepsilon}{q} \frac{\sin\theta}{R} \\
\left\langle \left(\nabla_{\|} B\right)^2 \right\rangle &= \left\langle \left(B \frac{\varepsilon}{q} \frac{\sin\theta}{R} \right)^2 \right\rangle \\
&\approx \left(\frac{\varepsilon}{qR} B \right)^2 \left\langle (\sin\theta)^2 \right\rangle \\
&= \frac{1}{2} \left(\frac{\varepsilon}{qR} B \right)^2
\end{aligned}$$

NOTE

In **kinetic_th_flowing_plasma_hinton_hazeltine2005** several formulas

$$\boldsymbol{\nabla} \cdot \widehat{\mathbf{n}} = -\nabla_{\|} \ln B$$

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{V}_E &= -\mathbf{V}_E \cdot \boldsymbol{\nabla} \ln B \\
&\quad -\boldsymbol{\kappa} \cdot \mathbf{V}_E \\
&\quad -\frac{E_{\|}}{B} \widehat{\mathbf{n}} \cdot (\boldsymbol{\nabla} \times \widehat{\mathbf{n}})
\end{aligned}$$

See *rotation*.

END

The expression for the parallel variation of the magnetic field in tokamak, used by **Stacey** in **Neoclassical Poloidal**

$$\left\langle \left(\nabla_{\|} B\right)^2 \right\rangle$$

$$\begin{aligned}
\left\langle \left(\nabla_{\|} B\right)^2 \right\rangle &= \left\langle (\mathbf{n}\cdot\boldsymbol{\nabla} B)^2 \right\rangle \\
&= \frac{1}{2}\left(\frac{\varepsilon}{qR}\right)^2 B^2
\end{aligned}$$

where $\varepsilon = r/R$. Or:

$$\left\langle \left(\nabla_{\|} B\right)^2 \right\rangle = \frac{1}{2}\left(\frac{r}{R}\frac{RB_\theta}{rB_T}\frac{1}{R}\right)^2 B^2 \approx \frac{1}{2}\left(\frac{B_\theta}{R}\right)^2$$

Equivalently

$$\frac{1}{2}\varepsilon^2\left(\frac{B}{qR}\right)^2$$

with qR=connection length, $= k_{\|}^{-1}$. So what we have is:

$$\left\langle \left(\nabla_{\|} B\right)^2 \right\rangle = \frac{1}{2}\varepsilon^2\left(k_{\|}B\right)^2$$

So the special thing is that the averaging process introduces a ε^2.

The parallel gradient of the magnetic field magnitude is small, $\sim \varepsilon$. It is however at the basis of the neoclassical processes.

In the paper **analytic viscosity expressions for plasma viscosity Shaing Hsu Wakatani**

$$\begin{aligned}
&\left\langle (\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla} B)^2 \right\rangle \\
= \;& \frac{2}{\langle \mathbf{B}\cdot\boldsymbol{\nabla}\theta\rangle}\sum_{m=1}^{\infty}\left[\langle \sin(m\theta)\;\; \widehat{\mathbf{n}}\cdot\boldsymbol{\nabla} B\rangle\;\; \langle \sin(m\theta)\;\; (\mathbf{B}\cdot\boldsymbol{\nabla}\theta)\;\; \widehat{\mathbf{n}}\cdot\boldsymbol{\nabla} B\rangle\right. \\
&\qquad\left. +\langle \cos(m\theta)\;\; \widehat{\mathbf{n}}\cdot\boldsymbol{\nabla} B\rangle\;\; \langle \cos(m\theta)\;\; (\mathbf{B}\cdot\boldsymbol{\nabla}\theta)\;\; \widehat{\mathbf{n}}\cdot\boldsymbol{\nabla} B\rangle\right]
\end{aligned}$$

The object that must be taken care of is

$$\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta$$

which is a *function of only* ψ . For circular surfaces

$$\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\theta = \widehat{\mathbf{n}}\cdot\widehat{\mathbf{e}}_\theta\frac{1}{r} = \frac{B_\theta}{B_T}\frac{1}{r} = \frac{1}{qR}$$

which is NOT a function of ψ. The safety factor

$$q = q(\psi) = \frac{m}{n}$$

but

$$R = R_0 + r\cos\theta = R_0(1+\varepsilon\cos\theta) = R_0 h$$

which depends on θ.

The average can be written

$$
\begin{aligned}
&\left\langle \left(\nabla_{\|} B\right)^2 \right\rangle \\
= \quad & \frac{2}{\left\langle B \frac{1}{qR} \right\rangle} \sum_{m=1}^{\infty} \left[\left\langle \sin(m\theta) \ \nabla_{\|} B \right\rangle \ \left\langle \sin(m\theta) \ \frac{B}{qR} \ \nabla_{\|} B \right\rangle \right. \\
& \left. + \left\langle \cos(m\theta) \ \nabla_{\|} B \right\rangle \ \left\langle \cos(m\theta) \ \frac{B}{qR} \ \nabla_{\|} B \right\rangle \right]
\end{aligned}
$$

But, for this case we have

$$B = \frac{B_0}{h}$$

$$\frac{1}{qR} = \frac{1}{qR_0 h}$$

then

$$\frac{B}{qR} = \frac{B_0}{qR_0} \frac{1}{h^2}$$

and

$$
\begin{aligned}
\left\langle \frac{B}{qR} \right\rangle &= \frac{B_0}{qR_0} \left\langle \frac{1}{h^2} \right\rangle = \frac{B_0}{qR_0} \left\langle \frac{1}{1 + 2\varepsilon\cos\theta + \varepsilon^2 \cos^2\theta} \right\rangle \\
&\approx \frac{B_0}{qR_0} \left\langle 1 - 2\varepsilon \cos\theta - \varepsilon^2 \cos^2\theta \right\rangle
\end{aligned}
$$

Wong Burrell. A formula

$$(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} = (\boldsymbol{\nabla} \times \widehat{\mathbf{n}}) \times \widehat{\mathbf{n}}$$

$$(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} = -\frac{1}{B^2} \mu_0 \boldsymbol{\nabla} p + \frac{1}{B} \boldsymbol{\nabla} B - \frac{1}{B} \widehat{\mathbf{n}} \ (\boldsymbol{\nabla} B \cdot \widehat{\mathbf{n}})$$

$$
\begin{aligned}
(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} &\approx \frac{\boldsymbol{\nabla} B}{B} \\
&= \boldsymbol{\nabla} \ln B
\end{aligned}
$$

used in **Wong Burrell**.

This is actually the curvature of the magnetic field line

$$\boldsymbol{\kappa} = (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} = -\frac{1}{R} \widehat{\mathbf{e}}_R$$

And

$$\boldsymbol{\nabla} \cdot \widehat{\mathbf{n}} = -\nabla_{\|} \ln B$$

$$
\begin{aligned}
\boldsymbol{\nabla} \times \widehat{\mathbf{n}} &= \widehat{\mathbf{n}} \ (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \times \widehat{\mathbf{n}}) \\
&\quad + \widehat{\mathbf{n}} \times \boldsymbol{\kappa}
\end{aligned}
$$

(**kinetic th flowing Hinton Hazeltine**).

Systems of coordinates for magnetic field lines From **Hirschamn Sigmar.**

$$\sqrt{g} = \frac{1}{|(\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta) \cdot \boldsymbol{\nabla}\varphi|}$$

$$\begin{aligned} \mathbf{B}_\varphi &= \sqrt{g}\,(\mathbf{B} \cdot \boldsymbol{\nabla}\varphi)\ \ (\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta) \\ &= I\boldsymbol{\nabla}\varphi \end{aligned}$$

where

$$\begin{aligned} I &= R^2\ \ (\mathbf{B} \cdot \boldsymbol{\nabla}\varphi) \\ &\approx RB_\varphi \end{aligned}$$

The poloidal magnetic field flux density

$$\frac{\partial\chi}{\partial\psi} = 2\pi\sqrt{g}\ \ (\mathbf{B} \cdot \boldsymbol{\nabla}\theta)$$

Then

$$\mathbf{B} = \frac{1}{2\pi}\frac{\partial\chi}{\partial\psi}\boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi + I\boldsymbol{\nabla}\varphi$$

Expression 1 Absolute magnitude of the magnetic field

$$\begin{aligned} B &= \frac{B_0}{h} \\ &\approx B_0\,(1 - \varepsilon\cos\theta) \\ &= B_0\left(1 - \varepsilon\cos\left(\frac{l_\parallel}{qR}\right)\right) \end{aligned}$$

where

$$k_0 = \frac{1}{qR}$$

appears as a wavelength.

Expression 2 Clebsch representation for **B**.

Used by ([**?**])

$$\mathbf{B} = \boldsymbol{\nabla}\phi_0 \times \boldsymbol{\nabla}\psi$$

Expression 3 Is used by ([**?**]).

$$\mathbf{B} = B_0 \widehat{\mathbf{e}}_\varphi + B_\perp \widehat{\mathbf{e}}_\chi$$

where the orthonormal coordinates are

$$\psi, \chi, \varphi$$

and:

- $2\pi\psi$ is the flux of the poloidal field, B_θ; It is a *radial coordinate*.
- in the absence of a plasma current, χ would be the magnetic potential; it is a poloidal coordinate; it is like an *angle*. But it contains the poloidal magnetic field which makes that the ratio (see below)

  $$\frac{d\chi}{B_\theta^2} \sim \frac{r d\theta}{B_\theta} \sim \frac{dl_\parallel}{B}$$

  and it is used for averaging over magnetic surfaces, $\langle A \rangle$.
- φ is the toroidal angle;

  There is the relation

  $$\widehat{\mathbf{e}}_\psi = \widehat{\mathbf{e}}_\chi \times \widehat{\mathbf{e}}_\varphi$$

  A possible connection with the annihilator

  $$\oint \frac{d\chi}{B_\perp^2} (...)$$

used by **Rutherford** in the treatment **1970** of the drift-kinetic equation. **Frieman** uses the Hamada coordinates for the magnetic field, where χ occurs too.

Expression 4 Used by [**?**]:

$$\mathbf{B} = \frac{I(\psi)}{R} \widehat{\mathbf{e}}_\phi + \frac{1}{R} \widehat{\mathbf{e}}_\phi \times \boldsymbol{\nabla}\psi$$

Actually an approximate expression for $I(\psi)$ is, according to **Connor MHD stability 1998**,

$$I(\psi) \simeq R B_\varphi$$

Expression 5 Used by **Rosenbluth Hazeltine Hinton 1972**

$$\begin{aligned} B_r &= 0 \\ B_\theta &= \frac{b(r)}{1 + (r/R)\cos\theta} \\ B_\varphi &= \frac{B_0}{1 + (r/R)\cos\theta} \end{aligned}$$

$$B \equiv |\mathbf{B}| \approx \frac{B_0}{h}$$

and the classical notation

$$h \equiv 1 + (r/R) \cos \theta$$

Expression 6 Used by various (**Galeev Sagdeev Liu Novakovskii**)

$$\mathbf{B} \equiv \left(0, \frac{\varepsilon}{q} \frac{B_0}{h}, \frac{B_0}{h} \right)$$

where

$$h \equiv 1 + \varepsilon \cos \theta$$

and

$$\Theta(r) \equiv \frac{\varepsilon}{q} = \frac{B_\theta}{B_\varphi} \ll 1$$

Expression 7 The paper by **Rosenbluth Lee Hazeltine**

The line

$$ds^2 = h_r^2 dr^2 + h_\theta^2 d\theta^2 + h_\varphi^2 d\varphi^2$$

with

$$\begin{aligned} h_r &= 1 + \varepsilon X(r) \cos \theta + ... \\ h_\theta &= r [1 + \varepsilon Y(r) \cos \theta] + ... \\ h_\varphi &= 1 + \varepsilon \cos \theta + ... \end{aligned}$$

the higher terms are of order ε^2.

NOTE that the values for circular geometry are

$$\begin{aligned} h_r^C &= 1 \\ h_\theta^C &= r \\ h_\varphi^C &= R_0 + r \cos \theta \end{aligned}$$

END.

In the following the collection of formulas from **Hassam Drake** must be carefully taken since they consider $B_0(r)$, *i.e.* a function of r.

The magnetic field is

$$B_r = 0$$

$$\begin{aligned} B_\theta &= \frac{b(r)}{h_r h_\varphi} \\ &= \frac{b(r)}{1 + \varepsilon \cos \theta} \end{aligned}$$

$$B_\varphi = \frac{B(r)}{1+\varepsilon\cos\theta}$$

NOTE that in circular it is

$$\begin{aligned} B_r &= 0 \\ B_\theta &= \frac{b(r)}{h} \\ B_\varphi &= \frac{B(r)}{h} \end{aligned}$$

and this is expressed in the following way in **Hassam Kulsrud**

$$\begin{aligned} B_r &= 0 \\ B_\theta &= \Theta(r)\frac{B_0}{h} = \frac{\varepsilon}{q}\frac{B_0}{h} \\ B_\varphi &= \frac{B_0}{h} \end{aligned}$$

This shows that

$$b(r) = \frac{\varepsilon}{q}B_0$$

For later use we note that

$$B(r)^{HintonLeeRosenbluth} = B_0(r)^{HassamDrake}$$

END.

and

$$J_r = 0$$

$$J_\theta = -\frac{1}{h_r h_\varphi}\frac{dB}{dr}$$

$$J_\varphi = -\frac{B}{bh_\varphi}\frac{dB}{dr} - \frac{h_\varphi}{b}\frac{dp}{dr}$$

here we note that

$$\begin{aligned} h_\varphi &\approx h = 1+\varepsilon\cos\theta \\ &= \frac{R}{R_0} \end{aligned}$$

and

$$J_\theta = -\frac{1}{1+\varepsilon\cos\theta}\frac{dB}{dr}$$

$$J_\varphi = -\frac{B}{b(r)}\frac{dB}{dr}\frac{1}{1+\varepsilon\cos\theta} - \frac{1}{b}\frac{dp}{dr}(1+\varepsilon\cos\theta)$$

NOTE

For comparison, in **Hassam Kulsrud** we have

$$j_r = 0$$

$$j_\theta = -\frac{1}{1+\varepsilon\cos\theta}\frac{dB_0(r)}{dr} \quad \text{the same as in HLR}$$

$$j_\perp = \frac{1}{B_0}\frac{h}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}\frac{dp}{dr} \quad \text{see below}$$

We remark that the perpendicular current $j_\perp$ comes from

$$\mathbf{j}\times\mathbf{B} = -\boldsymbol{\nabla}p$$

where

$$\begin{aligned}\mathbf{B} &= B_\theta\widehat{\mathbf{e}}_\theta + B_\varphi\widehat{\mathbf{e}}_\varphi \\ &= \left[\frac{\varepsilon}{q}\frac{B_0(r)}{h}\right]\widehat{\mathbf{e}}_\theta + \left[\frac{B_0(r)}{h}\right]\widehat{\mathbf{e}}_\varphi\end{aligned}$$

from where we have

$$\begin{aligned}|\mathbf{B}| &= B \\ &= \sqrt{B_\theta^2 + B_\varphi^2} = \frac{B_0(r)}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}}\end{aligned}$$

and

$$\begin{aligned}\mathbf{j}\times\mathbf{B} &= j_\perp B\left(-\widehat{\mathbf{e}}_r\right) \\ &= j_\perp\frac{B_0(r)}{h}\sqrt{1+\frac{\varepsilon^2}{q^2}}\left(-\widehat{\mathbf{e}}_r\right) \\ &= -\frac{dp}{dr}\widehat{\mathbf{e}}_r\end{aligned}$$

from where

$$j_\perp = \frac{h}{B_0(r)}\frac{1}{\sqrt{1+\frac{\varepsilon^2}{q^2}}}\frac{dp}{dr}$$

It is OK

Magnetic field with ripple The expression

$$B = B_0\left(1-\varepsilon\cos\theta - \delta\cos N\varphi\right)$$

where

$$\delta(r,\theta) \approx \delta(r)\exp\left(-\alpha\theta^2\right)$$

with $\alpha \sim 0.5$.

Note the decay which is Gaussian-like in the *poloidal angle* θ. The maximum of effect is at equatorial plane, $\theta = 0$.

Magnetic field with ripple, expression used by Boozer ([?]) The formula used by **Boozer**

$$B = B_0(\psi)\left(1 + 2\varepsilon\sin^2\left(\frac{\theta}{2}\right) + \delta\sin(N\phi)\right)$$

where $\varepsilon = r/R$ and the ripple is

$$\delta = \delta(\psi, \theta)$$

Since in his paper on ripple he uses

$$\mathbf{B} = \boldsymbol{\nabla}\phi_0 \times \boldsymbol{\nabla}\psi$$

with

$$\phi = \phi_0 + q(\psi)\theta$$

it results

$$\frac{dB}{d\phi_0} = N\delta B_0\cos(N\phi_0 + Nq\theta)$$

Shaing Sabbagh Chu In the paper **neoclassical toroidal viscosity** (bananas that are shaken stochastically) the perturbed magnetic field is used

$$B \equiv \text{magnitude of magnetic field}$$

$$\begin{aligned} B &= B_0(1 - \varepsilon\cos\theta) \\ &\quad - B_0\sum_n \left[A_n(\theta)\cos(n\zeta_0) + B_n\sin(n\zeta_n)\right] \end{aligned}$$

where

$$\zeta_0 \equiv q\theta - \varphi$$

Banana wobbling Helander Connor Hastie The expression is used for trapping - detrapping process.

Geometry of magnetic field and of the flow This is from **Yushmanov PoP1(1994)1583**

The coordinates are

$$\psi, \theta, \varphi$$

It is defined the *determinant of the metric tensor*

$$D \equiv \frac{1}{2\pi}\frac{qR}{B_T}$$

or

$$D = \frac{1}{2\pi}\frac{rB_T}{RB_p}\frac{R}{B_T} = \frac{1}{2\pi}\frac{r}{B_p} = \frac{1}{2\pi}\frac{r}{\frac{b(r)}{1+\varepsilon\cos\theta}}$$

Note that this is

$$|\nabla\varphi\cdot(\nabla\psi\times\nabla\theta)| = \frac{1}{\sqrt{g}}$$

and after few lines it results

$$\begin{aligned} I &\equiv \sqrt{g}\,(\nabla\psi\times\nabla\theta)\cdot\mathbf{B} \\ RB_\varphi &= \sqrt{g}2\pi RB_\theta\frac{1}{r}B_\varphi \\ \frac{r}{2\pi}\frac{1}{B_\theta} &= \sqrt{g} \\ \frac{r}{2\pi}\frac{1}{B_\theta}\frac{B_\varphi}{B_\varphi}\frac{R}{R} &= \sqrt{g} \\ \frac{1}{2\pi}\frac{qR}{B_\varphi} &= \sqrt{g} \end{aligned}$$

then the notation has changed

$$D \equiv \sqrt{g}$$

After **Ware Wiley**

$$\frac{B_p}{B} \sim \frac{r}{R}$$

(because $q \sim 1$). This gives an estimation for the magnitude of the poloidal magnetic field, B_θ.

Then

$$\begin{aligned} \text{at } q &\sim 1 \\ D &= \frac{1}{2\pi}\frac{r}{(r/R)\,B} = \frac{1}{2\pi}\frac{R}{B} \end{aligned}$$

and

$$B = \frac{B_0}{R/R_0} = \frac{B_0}{h}$$

then the scaling

$$\begin{aligned} D_{q=1} &\approx \frac{1}{2\pi}\frac{R}{B} = \frac{1}{2\pi}\frac{R}{B_0}\frac{R}{R_0} \\ &\sim \text{const}\times R^2 \end{aligned}$$

The metric coefficients are

$$\begin{aligned} h_r &= D\frac{|\nabla\psi|}{R} \\ &= qR\frac{B_{pol}}{B_T} \\ &= r\ \ (?) \end{aligned}$$

This allows to identify

$$
\begin{aligned}
|\boldsymbol{\nabla}\psi| &= \frac{R}{D} qR \frac{B_{pol}}{B_T} \\
&= \frac{qR^2}{\frac{1}{2\pi}\frac{qR}{B_T}} \frac{B_{pol}}{B_T} = 2\pi R B_{pol}
\end{aligned}
$$

$$
|\boldsymbol{\nabla}\psi| = 2\pi R B_{pol}
$$

NOTE

Sometimes without 2π,

$$
|\boldsymbol{\nabla}\psi| = R B_\theta
$$

END

The physical unit of ψ is m^2/s like *diffusion* coefficient or like the *stream-function.*

$$
\begin{aligned}
h_\theta &= D \frac{|\boldsymbol{\nabla}\theta|}{R} \\
&= \frac{1}{2\pi}\frac{qR}{B_T}\frac{|\boldsymbol{\nabla}\theta|}{R} = \frac{1}{2\pi}\frac{qR}{B_T}\frac{1}{R}\frac{1}{r} \\
&= \frac{1}{2\pi}\frac{rB_T}{RB_{pol}}\frac{R}{B_T r R} = \frac{1}{2\pi R B_{pol}} \\
&= \frac{1}{|\boldsymbol{\nabla}\psi|}
\end{aligned}
$$

$$
\begin{aligned}
h_\varphi &= -D^2 \left(\boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}\theta\right) \frac{1}{R^2} \left(\widehat{\mathbf{e}}_r \cdot \widehat{\mathbf{e}}_\theta\right) \\
&= -\left(\frac{1}{2\pi}\frac{qR}{B_T}\right)^2 \left[\left(2\pi R B_{pol}\right)\left(\frac{1}{r}\right)\right] \frac{1}{R^2} \left(\widehat{\mathbf{e}}_r \cdot \widehat{\mathbf{e}}_\theta\right) \\
&= -\left(\frac{1}{2\pi}\right)^2 \frac{r^2 B_T^2}{R^2 B_{pol}^2} \frac{R^2}{B_T^2} \left(2\pi R B_{pol}\right)\left(\frac{1}{r}\right) \frac{1}{R^2} \left(\widehat{\mathbf{e}}_r \cdot \widehat{\mathbf{e}}_\theta\right) \\
&= -\frac{1}{2\pi}\frac{r}{B_{pol}}\frac{1}{R} \left(\widehat{\mathbf{e}}_r \cdot \widehat{\mathbf{e}}_\theta\right) \\
&= -\frac{1}{2\pi}\frac{q}{B_T} \left(\widehat{\mathbf{e}}_r \cdot \widehat{\mathbf{e}}_\theta\right) \\
&\approx 0 \text{ in orthogonal system}
\end{aligned}
$$

Formulas From the paper **Neoclassical toroidal and poloidal rotation Kim Diamond Groebner** the following formulas should be retained

$$
B \simeq B_{tor} = \frac{B_0}{1 + \varepsilon \cos\theta}
$$

and from **Novakovski**

$$B_\theta = \frac{\varepsilon}{q}\frac{B_0}{h}$$

$$\oint \frac{dl_\theta}{B_\theta}(...) = \frac{qR_0}{2\pi B_0}\sqrt{1-\varepsilon^2}\oint d\theta\,(1+\varepsilon\cos\theta)\,(...)$$

where we have

$$dl_\theta = r d\theta$$

and

$$\frac{dl_\theta}{B_\theta} = \frac{r d\theta}{B_\theta} = \frac{dl_\parallel}{B}$$

NOTE
In **Stringer 1991**

$$dS = rR_0\,(1+\varepsilon\cos\theta)\,d\theta d\varphi$$

This is similar to what is above
END

Flux surface average

$$\begin{aligned}\langle A\rangle &= \frac{\oint_{line}\frac{dl}{B}A(l)}{\oint_{line}\frac{dl}{B}} \\ &= \frac{\oint_{poloidal}\frac{dl_\theta}{B_\theta}A(l)}{\oint_{poloidal}\frac{dl_\theta}{B_\theta}} \\ &= \frac{\oint\frac{d\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}A(\theta)}{\oint\frac{d\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}}\end{aligned}$$

In working with drift kinetic equation at the first order in neoclassical parameter (**Cordey, Hsu Catto Sigmar,** etc.)
the solubility condition for the equation

$$v_\parallel\nabla_\parallel\left(\overline{f}_1 + \frac{I}{\Omega}v_\parallel\left.\frac{\partial f_0}{\partial\psi}\right|_{\epsilon=\mathrm{const}}\right) = \overline{C}\left(\overline{f}_1\right)$$

We **note** that this is the usual neoclassical correction, in order 1 in ρ_θ/L_n.
This is, after averaging over the surface (since this removes the left hand side, being the *annihilator*)

$$\left\langle\frac{B}{v_\parallel}\overline{C}\left(\overline{f}_1\right)\right\rangle = 0$$

The average $\langle\rangle$ will lead to periodicity after first multiplication by B. We have

$$B\times| \quad \nabla_{\parallel}(...) = C$$

but

$$\begin{aligned}
dl_{\parallel} &= dl_{\theta}\frac{B_{\theta}}{B_T} = rd\theta\frac{B_{\theta}}{B_T} = R\left(\frac{rB_{\theta}}{RB_T}\right)d\theta = qR\ d\theta \\
\frac{1}{dl_{\parallel}} &= \frac{1}{qR}\frac{1}{d\theta}
\end{aligned}$$

$$\begin{aligned}
\nabla_{\parallel} &= \frac{\partial}{\partial l_{\parallel}} = \frac{1}{qR}\frac{\partial}{\partial\theta} \quad \text{and} \\
B &= \frac{B_0}{h}
\end{aligned}$$

so that

$$B\nabla_{\parallel}() \to \frac{B_0}{h}\frac{1}{\frac{rB_T}{RB_{\theta}}R}\frac{\partial}{\partial\theta}() = B_{\theta}\frac{\partial}{r\partial\theta}()$$

Now, we have to recall the definition of the average operation

$$\langle A\rangle = \frac{\oint \frac{d\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}A(\theta)}{\oint\frac{d\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}} = \frac{\oint\frac{rd\theta}{B_{\theta}}A(\theta)}{\oint\frac{rd\theta}{B_{\theta}}} = \frac{\frac{1}{b(r)}\oint rd\theta\ h\ A(\theta)}{\frac{1}{b(r)}\oint rd\theta\ h}$$

and

$$\begin{aligned}
\left\langle B\nabla_{\parallel}()\right\rangle &= \left\langle B_{\theta}\frac{\partial}{r\partial\theta}()\right\rangle \\
&= \frac{\oint rd\theta\ h\ B_{\theta}\frac{\partial}{r\partial\theta}()}{\oint rd\theta\ h} \\
&= \frac{b(r)\oint rd\theta\ h\ \frac{1}{h}\frac{\partial}{r\partial\theta}()}{\oint rd\theta\ h} = \frac{b(r)}{\oint rd\theta\ h}()_{\theta=0}^{\theta=2\pi} \\
&= 0 \quad \text{for periodic}
\end{aligned}$$

This is the reason for which we must multiply $(...)$ with B before taking the surface average, if we intend to exploit poloidal periodicity.

In **Hazeltine Hinton Rosenbluth 1973** the *annihilator on the magnetic surfaces* is defined after the following averaging operator is defined

$$\langle A\rangle = \frac{1}{V'}\oint\frac{d\chi\ A}{\boldsymbol{\nabla}\chi\cdot\mathbf{B}}$$

where

$$(\psi, \chi, \varphi) \equiv \text{coordinates}$$

$$V' = \oint \frac{d\chi}{\boldsymbol{\nabla}\chi \cdot \mathbf{B}}$$

is the derivative of volume inside the magnetic surface ψ to the coordinate ψ:

$$V' = \frac{dV}{d\psi} = \oint \frac{d\chi}{\boldsymbol{\nabla}\chi \cdot \mathbf{B}}$$

The expression inside the integral operator can be seen as a generalization of the case of circular symmetry where we can adopt

$$\chi \rightarrow \theta$$

and from this it results

$$\boldsymbol{\nabla}\chi \cdot \mathbf{B} \rightarrow \boldsymbol{\nabla}\theta \cdot \mathbf{B} = \frac{1}{r}\widehat{\mathbf{e}}_\theta \cdot \widehat{\mathbf{n}}B$$

and we have

$$\begin{aligned} \widehat{\mathbf{e}}_\theta \cdot \widehat{\mathbf{n}} &= \frac{B_\theta}{B} \\ &\equiv \Theta \quad \text{(mostly Russian notation)} \\ &\ll 1 \end{aligned}$$

with

$$B = |\mathbf{B}| \approx \frac{B_0}{h} = B_T$$

then

$$\boldsymbol{\nabla}\theta \cdot \mathbf{B} = \frac{1}{r}\widehat{\mathbf{e}}_\theta \cdot \widehat{\mathbf{n}}B = \frac{B}{r}\frac{B_\theta}{B} \approx B\frac{1}{R}\frac{RB_\theta}{rB_T} = B\frac{1}{qR}$$

From this equality we recognize

$$\begin{aligned} \boldsymbol{\nabla}\theta \cdot \mathbf{B} &= B\frac{1}{qR} \\ B\left(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\right)\theta &= B\frac{1}{qR} \\ B\nabla_\parallel \theta &= B\frac{1}{qR} \end{aligned}$$

or

$$\nabla_\parallel \theta = \frac{1}{qR}$$

This is the *connection length*. Also, the shift along the poloidal direction for the advancement along a magnetic line is

$$\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta = \frac{1}{qR}$$

expressed as

$$\frac{d\theta}{dl_{\|}} = \frac{1}{qR}$$

which is the small angle made by the magnetic line relative to the toroidal direction

$$\frac{rd\theta}{dl_{\|}} = \frac{r}{qR} = \frac{\varepsilon}{q}$$

Then we can re-express

$$\begin{aligned}
V' &= \frac{dV}{d\psi} = \oint \frac{d\chi}{\boldsymbol{\nabla}\chi \cdot \mathbf{B}} \\
&\rightarrow \oint \frac{d\theta}{B\frac{1}{qR}} = \oint \frac{qR}{B} d\theta = \oint \frac{qR}{B} \frac{dl_{\|}}{qR} \\
&= \oint \frac{dl_{\|}}{B}
\end{aligned}$$

We easily recognize the ratio $dl_{\|}/B$ as

$$\begin{aligned}
\frac{dl_{\|}}{B} &= \frac{qRd\theta}{B} \\
&= \frac{rB_T}{RB_\theta} \frac{R}{B} d\theta \\
&\approx \frac{rd\theta}{B_\theta} \quad \text{(another operator of averaging on surface)}
\end{aligned}$$

The averaging operator is

$$\langle A \rangle = \frac{1}{V'} \oint \frac{d\chi}{\boldsymbol{\nabla}\chi \cdot \mathbf{B}} A \rightarrow \frac{\oint \frac{dl_{\|}}{B} A}{\oint \frac{dl_{\|}}{B}}$$

(**NOTE** the **Kulikovski** surfaces in astrophysics).

Together with the *annihilator* in the magnetic surface, defined by using the averaging operator acting on *scalar* functions A, **Hazeltine Hinton Rosenbluth 1973** introduce an average of the divergence of a vector

$$\langle \boldsymbol{\nabla} \cdot \mathbf{A} \rangle = \frac{1}{V'} \frac{d}{d\psi} \left(V' \langle \mathbf{A} \cdot \boldsymbol{\nabla}\psi \rangle \right)$$

One notices that when

$$\mathbf{A} \perp \boldsymbol{\nabla}\psi \ \ , \ \mathbf{A} \cdot \boldsymbol{\nabla}\psi = 0$$

then

$$\langle \boldsymbol{\nabla} \cdot \mathbf{A} \rangle = 0$$

as a simple expression of periodicity on the magnetic surface.

As usual, the scalar projection of $\mathbf{A}$ on the vector transversal to the surface $\boldsymbol{\nabla}\psi$ is averaged using

$$\begin{aligned}\langle \mathbf{A}\cdot\boldsymbol{\nabla}\psi\rangle &= \frac{1}{V'}\oint\frac{d\chi}{\boldsymbol{\nabla}\chi\cdot\mathbf{B}}\mathbf{A}\cdot\boldsymbol{\nabla}\psi \\ &\rightarrow \frac{\oint\frac{dl_{\parallel}}{B}\left(\mathbf{A}\cdot\boldsymbol{\nabla}\psi\right)}{\oint\frac{dl_{\parallel}}{B}}\end{aligned}$$

NOTE

We can take

$$B\approx\frac{B_0}{h}$$

$$\oint\frac{dl_{\parallel}}{B}=\oint\frac{dl_{\parallel}}{B_0/h}$$

but

$$dl_{\parallel}=qRd\theta$$

$$\oint\frac{dl_{\parallel}}{B}=\frac{1}{B_0}\oint hdl_{\parallel}=\frac{1}{B_0}\oint hqRd\theta$$

The flux surface average in some papers is (?)

$$\langle A\rangle=\oint\frac{d\theta}{2\pi}hA$$

where

$$h=1+\frac{r}{R_0}\cos\theta$$

Some averages that will be necessary in the determination of the paralle fluxes (see **Hazeltine Hinton review**) for the Pfirsch Schluter regime.

$$\left\langle\frac{B^2}{B_0^2}\right\rangle=\frac{1}{\sqrt{1-\varepsilon^2}}$$

$$\left\langle\frac{B_0^2}{B^2}\right\rangle=1+\frac{3\varepsilon^2}{2}$$

In **Hassam Kulsrud** these are

$$\left\langle\frac{1}{h^2}\right\rangle\quad\text{and}\quad\left\langle h^2\right\rangle$$

and occur in the equations averaged on surfaces (the symbols α's).

Stringer In the paper **Rosenbluth Lee Hazeltine** and in **Stringer**

$$B_r = 0$$

$$\begin{aligned} B_\theta &= \frac{b(r)}{h_r h_\varphi} \\ &= \frac{b(r)}{1+\varepsilon\cos\theta} \end{aligned}$$

$$B_\varphi = \frac{B(r)}{1+\varepsilon\cos\theta}$$

It may seem strange comparing with **Rosenbluth Hazeltine Hinton** and **Novakovski** where

$$B_\varphi = \frac{B_0}{1+\varepsilon\cos\theta}$$

But it is question of the absolute magntitude that replaces the constant B_0.

In **Rosenbluth Lee Hazeltine** it is written

$$B' \sim p' \sim b^2 \sim \frac{a^2}{R^2}$$

5.5 Velocity space

In the paper **rewoldttangfrieman** the integration over the velocity space

$$\begin{aligned} & \int d^3 v_j \ (...) \\ = \ & \frac{\pi}{2}\left(\frac{2}{m_j}\right)^{3/2} \sum_\sigma \int_0^\infty d\epsilon \ \epsilon^{1/2} \int_0^{h(\theta)} d\Lambda \ \frac{1}{h(\theta)\sqrt{1-\frac{\Lambda}{h(\theta)}}} \end{aligned}$$

with

$$\epsilon \equiv \frac{1}{2}\left(v_\parallel^2 + v_\perp^2\right) \equiv \text{energy without electric potential}$$

$$\begin{aligned} \Lambda &= \frac{\mu B_0}{\epsilon} \\ &= \frac{v_\perp^2}{2B}\frac{B_0}{v^2/2} = \frac{v_\perp^2}{v^2}\frac{B_0}{\left(\frac{B_0}{h(\theta)}\right)} \\ &= \frac{v_\perp^2}{v^2} h(\theta) \end{aligned}$$

This is λ.

$$v_\parallel = \sigma \ v \sqrt{1-\frac{\Lambda}{h(\theta)}}$$

Trapped particle turning points

$$\theta_0 = \arccos\left(\frac{\Lambda - 1}{\varepsilon_0}\right)$$

where

$$\varepsilon_0 \equiv \frac{r}{R_0}$$

The single particle constants of motion are

$$\epsilon \equiv \text{energy per mass} = v^2/2$$

$$\begin{aligned}\Lambda &= \frac{\mu B_0}{\epsilon} \\ &\equiv \lambda \text{ (classical notation)}\end{aligned}$$

and the *toroidal angular momentum*

$$\begin{aligned}P_\varphi &\approx m_j R\left(v_\varphi - \Omega_{\theta j}\ r\right) \\ &= m_j R\left(v_\parallel - \Omega^0_{\theta j}\ r\right)\end{aligned}$$

for

$$\begin{aligned}\Omega_{\theta j} &= \frac{e_j B_\theta}{m_j} \\ \text{and } \Omega^0_{\theta j} &= \frac{e_j B^0_\theta}{m_j}\end{aligned}$$

The magnetic field is

$$\mathbf{B} = \frac{B_0}{h(\theta)}\widehat{\mathbf{e}}_\varphi + \frac{B^0_\theta(r)}{h(\theta)}\widehat{\mathbf{e}}_\theta$$

The poloidal field has a factor that is only dependent on the minor radius r, it is constant of circular magnetic surfaces; and it is divided with the toroidicity modulation $h(\theta)$.

Change of variable in integration

$$\int d^3v\ (...) \rightarrow \int d\mu d\epsilon d\zeta\ \frac{B}{\left|v_\parallel\right|}\ (...)$$

The Jacobian of the transformation

$$(v_x, v_y, v_z) \rightarrow (\mu, \varepsilon, \zeta)$$

is

$$Jac = \frac{B}{\left|v_\parallel\right|}$$

Formulas form **Helander Hastie Connor bootstrap ecrh** (we use ξ instead of χ)

$$\begin{aligned}\xi &\equiv \frac{v_{\|}}{v} \\ \xi &= \sqrt{1-\lambda B}\end{aligned}$$

where

$$\lambda = \frac{v_{\perp}^2}{v^2}\frac{1}{B}$$

NOTE that this is

$$\lambda = \frac{v_{\perp}^2}{v^2}\frac{1}{B} = \frac{2\mu}{v^2} = \frac{\mu}{\epsilon} \quad \text{(invariant of the trajectory)}$$

and this definition of λ is different of that previously given $\mu B_0/\epsilon$.
END

Another important parameter

$$\xi \equiv \frac{v_{\|}}{v}$$

The equations are (see **Cordey**)

$$\begin{aligned}\langle\xi\rangle &= \left\langle \frac{v_{\|}}{v} \right\rangle \\ &= \frac{2E(\kappa)}{\pi}\sqrt{\frac{2\varepsilon}{\kappa^2+\varepsilon}}\end{aligned}$$

$$\begin{aligned}\left\langle\xi^2\right\rangle &= \left\langle \left(\frac{v_{\|}}{v}\right)^2 \right\rangle \\ &= \frac{2\varepsilon}{\kappa^2+\varepsilon}\end{aligned}$$

where

$$\kappa^2 = \frac{2\varepsilon\lambda B_0}{1-\lambda B_0(1-\varepsilon)}$$

and

$$\kappa^2 < 1$$

In **Hazeltine Hinton Rosenbluth 1973**

$$B_{pol} = \left|\frac{\boldsymbol{\nabla}\psi}{R}\right|$$

which is $|\boldsymbol{\nabla}\psi| = RB_{\theta}$.

Consider

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial \psi} |\boldsymbol{\nabla} \psi| = R B_\theta \frac{\partial}{\partial \psi}$$

or

$$\frac{\partial}{\partial \psi} = \frac{1}{R B_\theta} \frac{\partial}{\partial r}$$

Another example, from **Mollen Fulop**

The parameters are normalized to the *impurity ion* temperature

$$\widetilde{\mathcal{E}} = \frac{\mathcal{E}}{T_z}$$

$$\widetilde{\mu} = \frac{\mu}{T_z}$$

and these are the new variables in the velocity space.

$$\int d^3 v = \pi v_{th,z}^3 \int_{\frac{Ze\Phi_E}{T_z}}^{\infty} d\widetilde{\mathcal{E}} \int_0^{\frac{\widetilde{\mathcal{E}} - Ze\Phi_z}{B}} d\widetilde{\mu} \frac{B}{\sqrt{\widetilde{\mathcal{E}} - \widetilde{\mu} B - \frac{Ze\Phi_z}{T_z}}}$$

6 Parameters

From the thesis **Kinetic theory plasma rotation**

the collisionality parameter

$$\gamma_* = \nu \frac{1}{\omega_{bounce}} \frac{B}{\delta B}$$

where

$$\delta B = B_{\max} - B_{\min}$$

The *plateau* regime is characterized by

$$\gamma_* < \left(\frac{B}{\delta B} \right)^{3/2}$$

$$\approx \begin{array}{c} \text{particle transit time} \\ \text{or } \gtrsim \\ \text{collison time} \end{array}$$

From **Anomalous momentum camenen**.

Signs of rotation relative to the magnetic field

$$\begin{aligned} s_B &= sign\left(\mathbf{B} \cdot \boldsymbol{\nabla} \varphi \right) = \pm 1 \\ \text{where } \varphi &\equiv \text{toroidal angle counterclockise from above tok.} \end{aligned}$$

$$s_j = sign\left(\mathbf{j} \cdot \boldsymbol{\nabla}\varphi\right) = \pm 1$$

$$\begin{aligned} s_{\widehat{s}} &= sign\left(\widehat{s}\right) = \pm 1 \\ \text{sign of magnetic shear } \widehat{s} &= \frac{rq'}{q} \end{aligned}$$

$$s_\gamma = sign\left(\gamma_E\right) = -sign\left(\boldsymbol{\nabla}E_r \cdot \widehat{\mathbf{e}}_r\right) = \pm 1$$

For purely toroidal rotation there is cancellation between the poloidal projection of the parallel flow and the electric field $E \times B$ velocity

$$\gamma_E - s_B s_j \frac{\varepsilon}{q} u'_\parallel = 0$$

OBE this is the classical expression

$$v_E - \Theta v_\parallel \approx 0$$

that requires the *Principal Value* and the *residuum* in the integration of $f_j^{(1)}$ to find $n_j^{(1)}$ in **Stringer**, and in **Rozhansky Tendler**.

In **Hassam Kulsrud** flow in tokamak plasma.

The displacement of a particle is done along the tube which is parallel with the magnetic field line.

But the tube itself moves such that the point moves only toroidally.

7 Range of experimental data

The Table from **NumericalSol_DriftKinetic_Santarius Hinton 1980_PFL000537.pdf**.

Parameter	Explanation	Value
m_0	poloidal mode number	6
l	toroidal mode number	2
q	safety factor	2.95

note the small difference, which means it is close to the resonant surface

$$q \sim \frac{m}{l}$$

ε	inverse aspect ratio	0.1
η_e	$\frac{d\ln T_e}{d\ln n_e} = \frac{L_n}{L_{T_e}}$	1
T_e	electron temperature	$1500\ (eV)$
T_i	ion temperature	$500\ (eV)$
B_0	magnetic field	$4\ (T)$
n	density	$5 \times 10^{13}\ \left(cm^{-3}\right)$
R_0	major radius	$150\ (cm)$

Table of parameters from the same work.

$v_{Th,e}$	electron thermal velocity	2.3×10^{9} (cm/s)	2.3×10^{7} (m/s)
$v_{Th,i}$	ion thermal velocity	3.09×10^{7} (cm/s)	3.09×10^{5} (m/s)
Ω_{ce}	electron cyclotron frequency	7.04×10^{11} $\left(s^{-1}\right)$	
ω_{be}	electron bounce frequency	1.64×10^{6} $\left(s^{-1}\right)$	

L_n	density scale length	a =minor radius
Λ	Coulomb logarithm	16.8
λ_{De}	electron Debye length	4.07×10^{-3} (cm)
$\nu_{ei}\left(v_{Th,e}\right)$	electron-ion collision frequency at the thermal electron velocity $v = v_{Th,e}$	5.56×10^{4} $\left(s^{-1}\right)$

Other data is in files on Transport code.