1 Reference Plasma Ideas

Normalized β

$$\beta_N = \frac{\beta}{I/(aB_0)}$$

of the order ~ 3.5.

Highconfinement parameter

$$H = \frac{\tau}{\tau_{189b}}$$

Other parameters

$$\begin{aligned} \beta^* &= \frac{\sqrt{\langle p^2 \rangle}}{B^2/(2\mu_0)} \\ &\sim 15\% \text{ in advanced} \end{aligned}$$

1.1 A basic fact in the drift wave theory

Hirshman Molvig, dominski2015, Mahajan Ross
near the resonant surface the parallel wavelength is infinite

$$k_\parallel = \frac{nq_s + m}{Rq_s} \approx 0$$

(Mode Rational Surface in **dominski2015**, where $q_0 = -\frac{m}{n}$
Then the inequality

$$\left| \frac{\omega_r}{k_\parallel} \right| \ll v_{th,e}$$

is no more true, whatever high the thermal velocity of electrons can be, the phase velocity of the electric perturbation is higher. Then there is no extraction of energy by the wave from the electrons, the wave (electric perturbation) cannot grow.

Then there is adiabatic response of the electrons since the adiabatic response results from the very fast motion of electrons relative to the electric perturbation.

This has been solved by **Hirshman Molvig** who have considered scattering of the electron orbits around the resonant surface.

2 Reference Plasma Formulas

Many *Notes* in *plasma.*

The $s - \alpha$ equilibrium, uses

$$s = \frac{rq'}{q}$$

shear parameter

$$\alpha = -\frac{2Rq^2}{B_T^2}\frac{dp}{dr}$$

normalized pressure gradient parameter

2.1 Divergence of the electric velocity

From impurities. The divergence of the *electric flux* in the ion continuity equation produces poloidal mode coupling

$$\begin{aligned} & \boldsymbol{\nabla}\cdot(n\mathbf{V}_E) \\ = & \boldsymbol{\nabla}\cdot\left(n\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\right) = n\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\times\left(\frac{\widehat{\mathbf{n}}}{B}\right) \\ = & n\boldsymbol{\nabla}\phi\cdot\left[\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\ln B + \widehat{\mathbf{n}}\times(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla})\,\widehat{\mathbf{n}}\right] \end{aligned}$$

The two terms in the paranthesis: variation of the magnetic field and curvature. They produce neoclassical drifts of the ions. *But, no velocities as coefficients.*

Later in the same work

$$\begin{aligned} \mathbf{v}_{De} &= -\rho_s c_s\,\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\left(\frac{\widehat{\mathbf{n}}}{B}\right) \\ &\approx 2\rho_s c_s\frac{1}{R_0}\left(\widehat{\mathbf{e}}_\theta\ \cos\theta + \widehat{\mathbf{e}}_r\ \sin\theta\right) \end{aligned}$$

is the drift velocity of the electrons.

Formula

$$\frac{I}{B}\widehat{\mathbf{n}} - R^2\boldsymbol{\nabla}\varphi \sim \boldsymbol{\nabla}\psi\times\widehat{\mathbf{n}}$$

used for

$$\mathbf{v}_\perp = \frac{d\Phi_0}{d\psi}\left(\frac{I}{B}\widehat{\mathbf{n}} - R^2\boldsymbol{\nabla}\varphi\right)$$

3 Equations for plasma physics

Poloidal flux function

$$\psi = -RA_{\varphi}$$

such that

$$\frac{\partial \psi}{\partial t} = RE_{\varphi}$$

$$|\boldsymbol{\nabla}\psi| = 2\pi RB_{\theta}$$

Expressions.
Definition

$$|\boldsymbol{\nabla}\varphi \cdot (\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta)| = \frac{1}{\sqrt{g}}$$

where

$$\frac{\partial g}{\partial \varphi} = 0$$

$$\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta = \frac{R^2}{\sqrt{g}}\boldsymbol{\nabla}\varphi$$

An expression for g is

$$\sqrt{g} = \frac{1}{2\pi}\frac{qR}{B_T}$$

and further

$$(\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta) \cdot \mathbf{B} = B\frac{R}{\sqrt{g}}\left(\widehat{\mathbf{e}}_{\varphi} \cdot \widehat{\mathbf{e}}_{\|}\right)$$

$$\widehat{\mathbf{e}}_{\varphi} \cdot \widehat{\mathbf{e}}_{\|} = \frac{B_T}{B}$$

The quantity

$$\begin{aligned}
I &\equiv \sqrt{g}\,(\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta) \cdot \mathbf{B} \\
RB_{\varphi} &= \sqrt{g}2\pi RB_{\theta}\frac{1}{r}B_{\varphi} \\
\frac{r}{2\pi}\frac{1}{B_{\theta}} &= \sqrt{g} \\
\frac{r}{2\pi}\frac{1}{B_{\theta}}\frac{B_{\varphi}}{B_{\varphi}}\frac{R}{R} &= \sqrt{g} \\
\frac{1}{2\pi}\frac{qR}{B_{\varphi}} &= \sqrt{g}
\end{aligned}$$

only depends on ψ, the magnetic surface

$$\begin{aligned}
I &= I(\psi) \\
&= RB_T \quad \text{(for circular surfaces)}
\end{aligned}$$

(see **Hazeltine Hinton RMP** for the first occurence of I in the general formula for rotation).

And

$$\boldsymbol{\nabla}\theta \cdot \mathbf{B} = \boldsymbol{\nabla}\theta \cdot (\boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\psi)$$

Formulas

$$\begin{aligned} \nabla_{\|} B &= \frac{\varepsilon B}{qR} \sin\theta \\ \nabla_{\|} \ln B &= \frac{\varepsilon}{qR} \sin\theta \end{aligned}$$

from where one simply derives

$$\left\langle \left(\nabla_{\|} B\right)^2 \right\rangle = \frac{1}{2}\varepsilon^2 \left(\frac{1}{qR} B\right)^2$$

and

$$\begin{aligned} \boldsymbol{\kappa} &\equiv (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} = \boldsymbol{\nabla} \ln B \\ & \quad \text{curvature} \end{aligned}$$

For the coordinates

$$\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta = \nabla_{\|}\theta = \frac{1}{qR}$$

and

$$\left\langle \frac{B^2}{B_0^2} \right\rangle = \frac{1}{\sqrt{1-\varepsilon^2}}$$

$$\left\langle \frac{B_0^2}{B^2} \right\rangle = 1 + \frac{3\varepsilon^2}{2}$$

Formulas

$$\widehat{\mathbf{n}} \times \boldsymbol{\nabla}\psi = I\widehat{\mathbf{n}} - BR\widehat{\mathbf{e}}_{\varphi}$$

$$\begin{aligned} & (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} \cdot \widehat{\mathbf{e}}_{\varphi} R \\ = \; & \boldsymbol{\kappa} \cdot \widehat{\mathbf{e}}_{\varphi} R \quad \text{(toroidal projection of curvature)} \\ = \; & I \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} \left(\frac{1}{B}\right) \end{aligned}$$

And

$$\begin{aligned} \boldsymbol{\nabla}\cdot\widehat{\mathbf{n}} &= -\nabla_{\|} \ln B \\ &= -\frac{\varepsilon}{q} \frac{\sin\theta}{R} \end{aligned}$$

$$\widehat{q}^2 \equiv \frac{1}{2\left\langle B_\theta^2 \right\rangle} \left(\left\langle B_T^2 \right\rangle - \frac{1}{\left\langle \frac{1}{B_T^2} \right\rangle} \right)$$

$$\boldsymbol{\nabla}\psi \times \widehat{\mathbf{n}} \simeq I\widehat{\mathbf{e}}_\theta$$

$$\begin{aligned}
\mathbf{v}_D \cdot \boldsymbol{\nabla}\psi &= I v_\parallel \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\theta \frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{\Omega}\right) \\
&= I \frac{v_\parallel}{qR} \frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{\Omega}\right)
\end{aligned}$$

NOTE
In **electric field separatrix Kim**
It is neglected the variation in surface, i.e. $\sim \theta$, of p_i and of ϕ.
an approximation of the perpendicular velocity is

$$\begin{aligned}
\mathbf{u}_{i\perp} &= \frac{1}{B}\widehat{\mathbf{n}} \times \left(\frac{1}{n_i e}\boldsymbol{\nabla} p_i + \boldsymbol{\nabla}\phi\right) \\
&\approx \omega \left(-\widehat{\mathbf{n}}\ \frac{I}{B} + R\ \widehat{\mathbf{e}}_\varphi\right)
\end{aligned}$$

where

$$\omega = -\left(\frac{1}{n_i e}\frac{\partial p}{\partial\psi} + \frac{\partial\phi}{\partial\psi}\right)$$

The projection of the perpendicular ion velocity on the curvature vector is

$$\boldsymbol{\kappa} \cdot \mathbf{u}_{i\perp} = (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\ \widehat{\mathbf{n}}) \cdot \mathbf{u}_{i\perp} \approx \omega\ I\ \ (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \left(\frac{1}{B}\right)$$

And, after neglecting $\nabla_\parallel \omega$, it is obtained the divergence of the perpendicular velocity of the ions

$$\boldsymbol{\nabla} \cdot \mathbf{u}_{i\perp} \approx -2\ \omega\ I\ \ (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \left(\frac{1}{B}\right)$$

END note

Change of variable in integration

$$\int d^3v\ \ (...) \rightarrow \int d\mu d\epsilon d\zeta\ \frac{B}{|v_\parallel|}\ \ (...)$$

The Jacobian of the transformation

$$(v_x, v_y, v_z) \rightarrow (\mu, \varepsilon, \zeta)$$

is

$$J = \frac{B}{|v_\parallel|}$$

Shear length

$$L_s = \left(\frac{\widehat{s}}{qR}\right)^{-1}$$

$$\widehat{s} = \frac{rq'}{q}$$

Surface average

$$\begin{aligned}
\langle A \rangle &= \frac{\oint \frac{dl}{B} A(l)}{\oint \frac{dl}{B}} \\
&= \frac{\oint \frac{dl_\theta}{B_\theta} A(l)}{\oint \frac{dl_\theta}{B_\theta}} \\
&= \frac{\oint \frac{d\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta} A(\theta)}{\oint \frac{d\theta}{\mathbf{B}\cdot\boldsymbol{\nabla}\theta}}
\end{aligned}$$

The detailed form

$$\oint \frac{dl_\theta}{B_\theta}(...) = \frac{qR_0}{2\pi B_0}\sqrt{1-\varepsilon^2}\oint d\theta\,(1+\varepsilon\cos\theta)(...)$$

the *annihilator on the magnetic surfaces* is defined after the following averaging operator is defined

$$\langle A \rangle = \frac{1}{V'}\oint \frac{d\chi}{\boldsymbol{\nabla}\chi\cdot\mathbf{B}} A$$

where

$$(\psi,\chi,\varphi) \equiv \text{coordinates}$$

$$V' = \oint \frac{d\chi}{\boldsymbol{\nabla}\chi\cdot\mathbf{B}}$$

Variables related to the *trapping of particles*

$$\begin{aligned}
\xi &\equiv \frac{v_\parallel}{v} \\
\xi &= \sqrt{1-\lambda B}
\end{aligned}$$

where

$$\lambda = \frac{v_\perp^2}{v^2}\frac{1}{B}$$

$$\begin{aligned}\langle\xi\rangle &= \left\langle\frac{v_{\parallel}}{v}\right\rangle \\ &= \frac{2E\left(\kappa\right)}{\pi}\sqrt{\frac{2\varepsilon}{\kappa^{2}+\varepsilon}}\end{aligned}$$

$$\begin{aligned}\left\langle\xi^{2}\right\rangle &= \left\langle\left(\frac{v_{\parallel}}{v}\right)^{2}\right\rangle \\ &= \frac{2\varepsilon}{\kappa^{2}+\varepsilon}\end{aligned}$$

where

$$\kappa^{2}=\frac{2\varepsilon\lambda B_{0}}{1-\lambda B_{0}\left(1-\varepsilon\right)}$$

and

$$\kappa^{2}<1$$

For trapped

$$\tau_{Bounce}=\varepsilon^{-1/2}\frac{qR}{v_{th,i}}$$

The fraction of trapped particles

$$\begin{aligned}f_{c} &= \frac{3}{4}\left\langle B^{2}\right\rangle\int_{0}^{B_{\max}^{-1}}\frac{\lambda d\lambda}{\left\langle\left(1-\lambda B\right)^{1/2}\right\rangle} \\ f_{t} &= 1-f_{c}\end{aligned}$$

3.0.1 The equations from Wang Burrell 1982

$$\begin{aligned}\frac{dv_{\parallel}}{dt} &= -\frac{v_{\perp}^{2}}{2}\nabla_{\parallel}\ln B+v_{\parallel}\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}\ln B-\frac{e}{m}\nabla_{\parallel}\phi \\ \frac{d}{dt}\left(\frac{v_{\perp}^{2}}{2}\right) &= \frac{v_{\perp}^{2}}{2}v_{\parallel}\boldsymbol{\nabla}_{\parallel}\ln B+\frac{v_{\perp}^{2}}{2}\frac{-\boldsymbol{\nabla}\phi\times\widehat{\mathbf{n}}}{B}\cdot\boldsymbol{\nabla}\ln B\end{aligned}$$

(see application in **Novakovskii Liu Sagdeev Rosenbluth**).

The order-one distribution function $\widehat{f_{1}}$ and the electrostatic potential ϕ_{1} is admitted with a *variation over the magnetic surface*

$$\phi=\phi_{0}\left(r\right)+\phi_{1}\left(r,\theta\right)$$

The equations of motion are

$$\frac{d\left(r\theta\right)}{dt}=v_{\parallel}\frac{B_{\theta}}{B_{T}}+\frac{1}{B_{0}}\frac{d\phi_{0}}{dr}$$

$$\frac{dr}{dt}=-\frac{1}{B_{0}}\frac{\partial\phi_{1}}{r\partial\theta}-\frac{1}{\Omega_{ci}}\left(\frac{v_{\perp}^{2}}{2}+v_{\parallel}^{2}\right)\frac{\sin\theta}{R}$$

$$\frac{dv_\parallel}{dt} = -\left(\frac{v_\perp^2}{2}\right)\frac{B_\theta}{B_T}\frac{\sin\theta}{R} + v_\parallel \frac{1}{B_0}\left(\frac{d\phi_0}{dr}\right)\frac{\sin\theta}{R} - \frac{e}{m}\frac{B_\theta}{B_T}\frac{\partial\phi_1}{r\partial\theta}$$

$$\frac{d}{dt}\left(\frac{v_\perp^2}{2}\right) = \left(\frac{v_\perp^2}{2}\right) v_\parallel \frac{B_\theta}{B_T}\frac{\sin\theta}{R} + \left(\frac{v_\perp^2}{2}\right)\frac{1}{B_0}\left(\frac{d\phi_0}{dr}\right)\frac{\sin\theta}{R}$$

3.0.2 Equations from Berk Galeev 1966.

The equations of the trajectories are

$$\frac{dr}{dt} = -\frac{1}{\Omega_{ci}}\frac{v_\perp^2/2 + v_\parallel^2}{R}\sin\theta$$

$$\frac{d(r\theta)}{dt} = -v_\parallel \frac{B_\theta}{B_T} - \frac{1}{\Omega_{ci}}\frac{v_\perp^2/2 + v_\parallel^2}{R}\cos\theta + \frac{1}{B_0}\frac{\partial\phi}{\partial r}$$

The "$-$ " sign comes from $-\frac{1}{\iota} = q$.

3.0.3 The equations according to Novakovskii Galeev Liu Sagdeev Hassam.

The equations are:

$$\frac{d\mathbf{r}}{dt} = v_\parallel \widehat{\mathbf{n}} + \mathbf{v}_E + \mathbf{v}_D$$

$$\begin{aligned}
\frac{dv_\parallel}{dt} &= \left(-\frac{v_\perp^2}{2}\widehat{\mathbf{n}} + v_\parallel \mathbf{v}_E\right)\cdot\boldsymbol{\nabla}\ln B \\
\frac{d}{dt}\left(\frac{v_\perp^2}{2}\right) &= \frac{v_\perp^2}{2}\left(\mathbf{v}_E + v_\parallel\widehat{\mathbf{n}}\right)\cdot\boldsymbol{\nabla}\ln B
\end{aligned}$$

The *drift velocity* is

$$\mathbf{v}_D = \frac{1}{\Omega_{ci}}\left(\frac{v_\perp^2}{2} + v_\parallel^2\right)\widehat{\mathbf{n}}\times\boldsymbol{\nabla}\ln B + \frac{1}{\Omega_{ci}}\widehat{\mathbf{n}}\times\frac{\partial\mathbf{v}_E}{\partial t}$$

These are used in the drift kinetic equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}f + \frac{dv_\parallel}{dt}\frac{\partial f}{\partial v_\parallel} + \frac{d\left(v_\perp^2/2\right)}{dt}\frac{\partial f}{\partial\left(v_\perp^2/2\right)} = St\left(f\right)$$

3.0.4 Drift-kinetic equation according to *Lebedev Diamond et al.*

The drift kinetic equation

$$\frac{\partial F}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} F + \frac{dU}{dt}\frac{\partial F}{\partial U} = St\left\{F\right\} + \frac{\delta F}{\delta t}$$

In the first order of the

$$\frac{\rho}{L}$$

expansion, the trajectories are

$$\frac{d\mathbf{X}}{dt} \equiv \mathbf{V} = \frac{1}{B}\left(\mathbf{B}U + \mathbf{V}_E + \mathbf{V}_d\right)$$

$$\frac{dU}{dt} = -\frac{\mu}{m}\nabla_{\|} B + \frac{U}{B}\mathbf{V}_E \cdot \boldsymbol{\nabla} B$$

where

$$\mathbf{V}_d = \frac{1}{\Omega_c}\widehat{\mathbf{n}} \times \frac{\mu B + mU^2}{m}\boldsymbol{\nabla} \ln B$$

$$\mu = \frac{mv_{\perp}^2}{2B}$$

Now it is introduced a reference velocity directed along the toroidal angle

$$\mathbf{V}_{\Phi} = -\frac{1}{B_T}\left(2\pi R B_T \frac{\partial \overline{\Phi}}{\partial \psi}\right)\left(R\boldsymbol{\nabla}\varphi\right)$$

where

$$\varphi \equiv \text{toroidal angle}$$

The system of reference is changed to this rotating frame as if the plasma would rotate as a rigid body

$$S \rightarrow S_{\mathbf{V}_{\Phi}}$$

3.0.5 The drift of particles according to Hahm Fong

The equations of motion, mentioned by **Fong Hahm** by quoting **Kadomtsev Pogutse**

$$\frac{dr}{dt} \approx -\frac{1}{\Omega}\left(\frac{v_{\perp}^2}{2} + v_{\|}^2\right)\frac{\sin\theta}{R_0}$$

(radial projection of the drift velocity $\mathbf{v}_D$)

$$\frac{d\theta}{dt} \approx \frac{v_{\|}}{qR_0} - \frac{1}{r}\frac{1}{\Omega}\left(\frac{v_{\perp}^2}{2} + v_{\|}^2\right)\frac{\cos\theta}{R_0}$$

(poloidal projection of the drift velocity $\mathbf{v}_D$ plus the projection of the *parallel* velocity $\mathbf{v}_{\|}$)

$$\frac{d\varphi}{dt} \approx \frac{v_{\|}}{R_0}$$

These equations are integrated as functions of θ (like in **Galeev Sagdeev**)

$$r(\theta) - r_0 = \pm \frac{v q_0}{\Omega \sqrt{\varepsilon}} \left[\frac{v_{\parallel 0}^2}{\varepsilon v_{\perp 0}^2} - 2 \sin^2 \left(\frac{\theta}{2} \right) \right]^{1/2}$$

where

$$\begin{aligned} r_0 &\equiv \text{radius of the surface to which the bounce points belong} \\ q_0 &= q(r_0) \\ v_{\parallel 0} &\equiv \text{velocity at the outer mid-plane} \\ v_{\perp 0} &\equiv \text{velocity at the outer mid-plane} \end{aligned}$$

Characteristic parameters of the motion

$$\Lambda_b = \frac{v q_0}{\Omega \sqrt{\varepsilon}} \equiv \text{banana radius}$$

$$\kappa^2 \equiv \frac{1}{2\varepsilon} \frac{v_{\parallel 0}^2}{v_{\perp 0}^2} \equiv \text{pitch angle}$$

Define the function $\varphi(\theta)$ by the equation

$$\varphi(\theta) = \arcsin \left[\frac{\sin\left(\frac{\theta}{2}\right)}{\kappa} \right]$$

The motion in toroidal direction (**Fong Hahm**)

$$\begin{aligned} \zeta(\theta) &= q_0 \theta \\ &\quad + \sqrt{2} \Lambda_b \left[\left(2 \frac{dq_0}{dr} + \frac{q_0}{r_0} \right) C_1 (\varphi(\theta), \kappa) \right. \\ &\quad \left. - \left(2 \frac{dq_0}{dr} \left(1 - \kappa^2\right) + \frac{q_0}{2 r_0} \right) C_2 (\varphi(\theta), \kappa) \right] \end{aligned}$$

where

$$C_1(\varphi(\theta), \kappa) = \begin{cases} E(\varphi, \kappa) + \mathbf{E}(\kappa) & \text{first half of orbit} \\ 3\mathbf{E}(\kappa) - E(\varphi, \kappa) & \text{second half of orbit} \end{cases}$$

$$C_2(\varphi(\theta), \kappa) = \begin{cases} F(\varphi, \kappa) + \mathbf{K}(\kappa) & \text{first half of orbit} \\ 3\mathbf{K}(\kappa) - F(\varphi, \kappa) & \text{second half of orbit} \end{cases}$$

3.1 Parallel velocity

The parallel velocity from **Galeev Sagdeev Liu Novakovskii**

$$v_{\parallel} = -\frac{B_{\varphi}}{B_{\theta}} V_E + 2\sigma \sqrt{\frac{\mu B_0}{m} \varepsilon S} \, \left(\kappa^2 - \sin^2 \frac{\theta}{2} \right)^{1/2}$$

where

$$\begin{aligned} \kappa^2 &= \frac{m \left(\Delta v_{\parallel} \right)^2}{4 \mu B_0 \varepsilon S} \\ \Delta v_{\parallel} &= v_{\parallel} \left(r_0, 0 \right) + \frac{B_{\varphi}}{B_{\theta}} V_E \left(r_0, t \right) \\ V_E \left(r, t \right) &= \frac{1}{B_0} \frac{\partial \Phi \left(r, t \right)}{\partial r} \end{aligned}$$

and the squeezing factor

$$S = 1 + \frac{\partial V_E}{\partial r} \frac{1}{\Omega_c} \left(\frac{B_{\varphi}}{B_{\theta}} \right)^2$$

3.2 Neoclassical drift

The approximation for the *neoclassical particle* drift velocity in **Stringer PRL**

$$\begin{aligned} \mathbf{v}_{Di} &= -\frac{T_i}{|e| \, B} \frac{1}{R} \widehat{\mathbf{e}}_z = -\frac{1}{\Omega_i} \frac{T_i / m_i}{R} \widehat{\mathbf{e}}_z \\ & \text{(vertical)} \end{aligned}$$

instead of

$$\mathbf{v}_{drift,j} = \frac{1}{\Omega_j} \frac{v_{\perp}^2 / 2 + v_{\parallel}^2}{R} \widehat{\mathbf{n}} \times \left(-\widehat{\mathbf{e}}_R \right)$$

Essnetially, the combination of squared velocities is replaced in terms of energy T_i.

The divergence of the flux associated with the particles' drifts

$$\boldsymbol{\nabla} \cdot \left(n \mathbf{v}_{Di} \right) = -\frac{1}{\Omega_i} \frac{T_i / m_i}{R} \frac{d n_0}{d r} \sin \theta$$

The electric velocity

$$\mathbf{v}_E = \frac{-\boldsymbol{\nabla} \phi \times \widehat{\mathbf{B}}}{B^2}$$

The divergence

$$\boldsymbol{\nabla} \cdot \left(\mathbf{v}_E \right) = -\frac{2}{R} v_{E\theta} \sin \theta$$

where it was assumed that $\boldsymbol{\nabla}\times\mathbf{B}\cdot\boldsymbol{\nabla}\phi=0$. the result comes from $\boldsymbol{\nabla}\left(\frac{1}{B^2}\right)$.

A geometrical equation

$$\nabla_{\parallel}\ln B=\frac{1}{qR}\varepsilon\sin\theta$$

and

$$\begin{aligned}\boldsymbol{\kappa} &= \left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}=\boldsymbol{\nabla}\ln B=-\frac{1}{R}\widehat{\mathbf{e}}_R \\ &\quad \text{curvature vector}\end{aligned}$$

In **Catto**

$$\begin{aligned}\left(\boldsymbol{\kappa}\times\widehat{\mathbf{n}}\right)\cdot\boldsymbol{\nabla}\psi &\simeq I\nabla\ln B \\ \left(\boldsymbol{\kappa}\times\widehat{\mathbf{n}}\right)\cdot\boldsymbol{\nabla}\theta &= -\frac{I\mathbf{B}\cdot\boldsymbol{\nabla}\theta}{2B^3}\frac{\partial}{\partial\psi}\left[B^2\left(1+\beta\right)\right]\end{aligned}$$

where I is introduced by the definition of $\mathbf{B}$:

$$\begin{aligned}\mathbf{B} &= I\boldsymbol{\nabla}\varphi+\boldsymbol{\nabla}\varphi\times\boldsymbol{\nabla}\psi \\ \boldsymbol{\kappa} &\equiv \left(\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\right)\widehat{\mathbf{n}}\ = \text{curvature} \\ \beta &\equiv \frac{p_e+p_i}{B^2/\left(2\mu_0\right)}\end{aligned}$$

and

$$\oint\frac{dl_\theta}{B_\theta}\left(...\right)=\frac{qR_0}{2\pi B_0}\sqrt{1-\varepsilon^2}\oint d\theta\left(1+\varepsilon\cos\theta\right)\left(...\right)$$

The mean guiding center motion parallel with the magnetic field is identical with the fluid velocity

Therefore equations for the fluid-plasma can be used. [see **Burrell Wang, Hazeltine Ware**].

3.3 Pfirsch Schluter

The key to the non-zero divergence of the poloidal flow (either *diamagnetic* or *neoclassical drifts*) is the geometric equation

$$\begin{aligned}\boldsymbol{\nabla}\cdot\left(\widehat{\mathbf{e}}_\theta\frac{B_0}{B}\right) &= \boldsymbol{\nabla}\cdot\left[\widehat{\mathbf{e}}_\theta\left(1+\varepsilon\cos\theta\right)\right] \\ &= \varepsilon\frac{\left(-2\sin\theta\right)}{r}\end{aligned}$$

The term from the equation of continuity is $\boldsymbol{\nabla}\cdot\left(n\mathbf{v}\right)$ where: $n\rightarrow n_0$ (zero approximation, constant in surface), $\mathbf{v}\rightarrow v_E^{(0)}\widehat{\mathbf{e}}_\theta$ and both are factored out from the differential operator and only remains the geometric expression. Then toroidality leads to the above result.

Starting from

$$\boldsymbol{\nabla}_{\perp} \cdot \mathbf{j}_{\perp} + \boldsymbol{\nabla}_{\parallel} \cdot \mathbf{j}_{\parallel} = 0$$

and replacing for the perpendicular current

- the *diamagnetic current*, or
- the *approximatted expression for the neoclassical drifts* (as in **Stringer PRL**)

we obtain the parallel current

$$J_{\parallel} = -\varepsilon \frac{2}{B_{\theta}} \frac{dp}{dr} \cos\theta$$

In the following it is sometimes assumed that B_0 *is function of* r. This must be eliminated for most of the applications.

The magnetic field in **Hassm Kulsrud** is

$$\mathbf{B} = \left(0, \frac{b\,(r)}{h}, \frac{B_0\,(r)}{h}\right)$$

and the magnitude of the magnetic field is

$$|\mathbf{B}| = \frac{B_0\,(r)}{h} \sqrt{1 + \frac{\varepsilon^2}{q^2}}$$

The perpendicular current

$$j_{\perp} = \frac{h}{B_0\,(r)} \frac{1}{\sqrt{1 + \frac{\varepsilon^2}{q^2}}} \frac{dp}{dr}$$

the parallel current

$$j_{\parallel} = \frac{q}{\varepsilon} \frac{1}{\sqrt{1 + \frac{\varepsilon^2}{q^2}}} h \left[-\frac{dB_0}{dr} \frac{1}{h^2} \left(1 + \frac{\varepsilon^2}{q^2}\right) - \frac{1}{B_0} \frac{dp}{dr} \right]$$

We introduce the notations

$$v_D \equiv -\eta \frac{1}{B_0} \frac{dp}{dr}$$

It is the *resistive classical flow.* The parameter

$$D \equiv -B_0 \frac{\frac{dB_0}{dr}}{\frac{dp}{dr}}$$

it should be zero. The average is

$$\langle f v_r \rangle = v_D \frac{1}{B_0} \left[D \langle f \rangle + \left(\frac{q}{\varepsilon} \right)^2 \left(\langle f h^2 \rangle - \frac{\langle f \rangle}{\langle \frac{1}{h^2} \rangle} \right) \right]$$

Variation of the magnetic field

$$\nabla_{\|} \ln B = \frac{\varepsilon}{q} \frac{\sin\theta}{R}$$

$$\begin{aligned} \boldsymbol{\kappa} &= (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} = \frac{\boldsymbol{\nabla} B}{B} = \boldsymbol{\nabla} \ln B \\ &= -\frac{1}{R} \widehat{\mathbf{e}}_R \text{ (curvature vector)} \end{aligned}$$

3.4 The factor in Pfirsch Schluter current

In **Hirshman neoclassical current**

The current

$$\mathbf{j} = \frac{1}{B} \widehat{\mathbf{n}} \times \boldsymbol{\nabla} p + j_{\|} \widehat{\mathbf{n}}$$

where

$$\mathbf{j}_{\perp} = \frac{1}{B} \widehat{\mathbf{n}} \times \boldsymbol{\nabla} p$$

We have

$$\mathbf{j} \cdot \mathbf{B}_p = \frac{B_p^2}{B^2} \frac{dp}{d\psi} F + j_{\|} \frac{B_p^2}{B}$$

Introduce for the left hand side the notation

$$\begin{aligned} K &\equiv \frac{\mathbf{j} \cdot \mathbf{B}_p}{B_p^2} \quad \text{(definition)} \\ &= \frac{j_p}{B_p} \text{ proportional with the poloidal velocity} \end{aligned}$$

and then divide by B_p^2 the equation. We can rewrite

$$j_{\|} = -\frac{F p'}{B} + K B$$

with $p' = dp/d\psi$. We **note** that $K = \mathbf{j} \cdot \widehat{\mathbf{e}}_\theta / B_\theta$ contains the projection of $\mathbf{j}$ on the poloidal direction. We can use this poloidal component to find what is its effect on the parallel direction. For this, we must multiply the poloidal component (part of K) with the angle B/B_θ,

$$j_\parallel^K = (\mathbf{j}\cdot\widehat{\mathbf{e}}_\theta) \times \frac{B}{B_\theta} = \frac{\mathbf{j}\cdot\widehat{\mathbf{e}}_\theta}{B_\theta} B = KB$$

This part must be added to the other component of parallel current, coming from $-Fp'/B$. the parallel current

$$j_\parallel = -\frac{Fp'}{B} + KB$$

We multiply by B

$$j_\parallel B = -Fp' + KB^2$$

and note that F, p', K are functions of only the surface. Then we take surface averaging

$$\left\langle j_\parallel B \right\rangle = -Fp' + K \left\langle B^2 \right\rangle$$

from where

$$K = \frac{Fp'}{\langle B^2 \rangle} + \frac{\langle \mathbf{j} \cdot \mathbf{B} \rangle}{\langle B^2 \rangle}$$

and this is replaced in the equation for $j_\parallel$,

$$\begin{aligned} j_\parallel &= -\frac{Fp'}{B} + \left(\frac{Fp'}{\langle B^2 \rangle} + \frac{\langle \mathbf{j} \cdot \mathbf{B} \rangle}{\langle B^2 \rangle} \right) B \\ &= -\frac{Fp'}{B} \left(1 - \frac{B^2}{\langle B^2 \rangle} \right) \\ &\quad + \frac{\langle \mathbf{j} \cdot \mathbf{B} \rangle}{\langle B^2 \rangle} B \end{aligned}$$

or

$$j_\parallel = j_{PS} + j_{neo}$$

The first term is the Pfirsch Schluter current

$$j_{PS} = -\frac{Fp'}{B} \left(1 - \frac{B^2}{\langle B^2 \rangle} \right)$$

This current exists in all collisional regimes.

$$j_{PS} = -I \frac{1}{B} \frac{dp}{d\psi} \left(1 - \frac{B^2}{\langle B^2 \rangle} \right)$$

The other term has zero divergence.

3.5 The plasma rotation (Hinton Wong)

(from *derivation drift kinetic*).

Used by **Fulop Helander** for impurity in rotating plasma.

After changing to the rotation referential, in the *rotating frame*

$$\mathbf{v}' = \mathbf{v} - \mathbf{u}_0 (\mathbf{x}, t)$$

where

$$\mathbf{u}_0 = \omega R \widehat{\mathbf{e}}_\varphi + F\mathbf{B}$$

and the new variables

$$(\mathbf{v}' \times \widehat{\mathbf{n}}) \cdot \boldsymbol{\nabla}\psi = v_\parallel \sin\zeta \ \ |\boldsymbol{\nabla}\psi|$$

$$\mathbf{v}' \ \mathbf{v}' : (\widehat{\mathbf{e}}_1\widehat{\mathbf{e}}_1 - \widehat{\mathbf{e}}_2\widehat{\mathbf{e}}_2) = v_\perp^2 \cos(2\zeta)$$

The drift velocity is

$$\begin{aligned} \mathbf{v}_D \quad = \quad & \frac{1}{\Omega}\widehat{\mathbf{n}} \times \Big(\mu \boldsymbol{\nabla} B + v_\parallel^2 (\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} + \frac{e}{m}\boldsymbol{\nabla}\Phi_0 \\ & -\omega^2 \mathbf{R} + 2\omega\widehat{\mathbf{e}}_z \times \widehat{\mathbf{n}} v_\parallel \Big) \end{aligned}$$

includes now the *centrifugal* drift and the *Coriolis* drift. The formulas

$$\widehat{\mathbf{n}} \times \boldsymbol{\nabla}\psi = I\widehat{\mathbf{n}} - BR\widehat{\mathbf{e}}_\varphi$$

$$(\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}) \widehat{\mathbf{n}} \cdot \widehat{\mathbf{e}}_\varphi R = I\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\left(\frac{1}{B}\right)$$

$$\omega\widehat{\mathbf{e}}_z \times \widehat{\mathbf{n}} \cdot \widehat{\mathbf{e}}_\varphi R = \omega\widehat{\mathbf{n}} \cdot \mathbf{R}$$

lead to

$$\mathbf{v}_D \cdot \boldsymbol{\nabla}\psi = \frac{m}{e} v_\parallel \widehat{\mathbf{n}} \cdot \boldsymbol{\nabla}\left(\frac{I v_\parallel}{B} + \omega R^2\right)$$

where

$$\boldsymbol{\nabla} \equiv \boldsymbol{\nabla}_{\varepsilon=ct,\mu=ct}$$

and

$$v_\parallel = \left\{ 2\left[\epsilon - \mu B - \frac{e}{m}\widetilde{\Phi}_0 + \frac{\omega^2 R^2}{2}\right]\right\}^{1/2}$$

Identities that involve the versors

$$-\widehat{\mathbf{e}}_2\widehat{\mathbf{e}}_2 : (\boldsymbol{\nabla} \ \widehat{\mathbf{n}}) = \widehat{\mathbf{e}}_1\widehat{\mathbf{e}}_1 : (\boldsymbol{\nabla} \ \widehat{\mathbf{n}}) + \frac{\nabla_\parallel B}{B}$$

$$\begin{aligned} \widehat{\mathbf{e}}_1 \cdot (\boldsymbol{\nabla} \ \widehat{\mathbf{n}}) \cdot \widehat{\mathbf{e}}_1 \quad & = \quad \widehat{\mathbf{e}}_2 \cdot (\boldsymbol{\nabla} \times \widehat{\mathbf{e}}_1) \\ & = \quad (\widehat{\mathbf{e}}_2 \cdot \widehat{\mathbf{e}}_\varphi) \widehat{\mathbf{e}}_\varphi \cdot (\boldsymbol{\nabla} \times \widehat{\mathbf{e}}_1) \end{aligned}$$

$$\widehat{\mathbf{e}}_\varphi \cdot (\nabla \times \widehat{\mathbf{e}}_1) = \frac{R\mathbf{B}\cdot\nabla\left|\nabla\psi\right|}{\left|\nabla\psi\right|^2}$$
$$\widehat{\mathbf{e}}_2 \cdot \widehat{\mathbf{e}}_\varphi = -\frac{\left|\nabla\psi\right|}{BR}$$

and

$$\begin{aligned}(\widehat{\mathbf{e}}_1\widehat{\mathbf{e}}_1 - \widehat{\mathbf{e}}_2\widehat{\mathbf{e}}_2) \quad &: \quad (\nabla\widehat{\mathbf{n}}) \\ &= \quad -\frac{B}{\left|\nabla\psi\right|^2}\widehat{\mathbf{n}}\cdot\nabla\left(\frac{\left|\nabla\psi\right|^2}{B}\right)\end{aligned}$$

Hinton Wong make the observation that the linearized ion-ion collision operator anihilates terms like

$$\left(a + bv_\parallel + cv^2\right) f_0$$

(due to the conservation in collisions of - respectively, number, momentum and energy).

The formula used by **Hirshman Sigmar Clarke** (see *impurities.tex*)

$$\begin{aligned}&\frac{1}{n_k}\sum_{\sigma=\pm}\iint 2\pi B\ d\mu\ dw\ \frac{\mu B}{\frac{T_k}{m_k}}\ F_{M,k}\left(\frac{1}{\overline{\left|v_\parallel\right|}} - \frac{1}{\left|v_\parallel\right|}\right) \\ = \quad &-1.46\ \varepsilon^{1/2} \\ &+O\left(\varepsilon\right)\end{aligned}$$

4 Astrophysics: motion of particles

From **Hazeltine Ware**

See *plasma general rotation*.

$$\frac{d\epsilon}{dt} = \mathbf{F}\cdot\mathbf{v} - \mathbf{v}\cdot\left(\mathbf{v}\cdot\nabla\right)\mathbf{V}$$
$$B\frac{d\mu}{dt} = \mu\frac{dB}{dt} - u\mathbf{s}\cdot\frac{d\widehat{\mathbf{n}}}{dt} + \mathbf{F}\cdot\mathbf{s} - \mathbf{s}\cdot\left(\mathbf{v}\cdot\nabla\right)\mathbf{V}$$
$$\frac{d\zeta}{dt} = \Omega + \widehat{\mathbf{e}}_\perp\cdot\frac{d\widehat{\mathbf{e}}_n}{dt} + \frac{\widehat{\boldsymbol{\rho}}}{s}\cdot\left[u\frac{d\widehat{\mathbf{e}}_\perp}{dt} - \mathbf{F} + \left(\mathbf{v}\cdot\nabla\right)\mathbf{V}\right]$$

where

$$\widehat{\boldsymbol{\rho}} = \widehat{\mathbf{e}}_n\sin\zeta + \widehat{\mathbf{e}}_\perp\cos\zeta$$

where

$$\begin{aligned}\mathbf{V} \quad &\equiv \quad \text{center of mass velocity} \\ \mathbf{v} \quad &\equiv \quad \text{particle velocity in the referntial of } \mathbf{V} \\ \mathbf{u} \quad &\equiv \quad \text{parallel component of } \mathbf{v} \\ \mathbf{s} \quad &\equiv \quad \mathbf{v}_\perp \text{ perpendicular component of } \mathbf{v}\end{aligned}$$

5 Change of variables

The volume

$$\int d^3v = \int d\epsilon d\lambda\ \epsilon\ \frac{B}{|v_\parallel|}$$

and (**Rewoldt Tang Frieman**)

$$\int d^3v = \frac{\pi}{2}\left(\frac{2}{m_j}\right)^{3/2}\sum_{\sigma_\parallel}\int_0^\infty d\epsilon\ \epsilon^{1/2}\int_0^{h(\theta)} d\Lambda\frac{1}{h(\theta)}\frac{1}{\sqrt{1-\frac{\Lambda}{h(\theta)}}}$$

Chang bootstrap pedestal. Volume

$$\int d^3v = 2\pi v_{th}^3\ B\sum_{\sigma=\pm 1}\int_0^\infty d\epsilon\ \epsilon\ \int_0^{\lambda_{\max}}\sigma\frac{d\lambda}{|v_\parallel|}$$

and

$$\begin{aligned} v_\perp &= \sqrt{2\epsilon\lambda B} \\ v_\parallel &= \sqrt{2\left[\epsilon\left(1-\lambda B\right)-e_j\delta\phi\right]} \end{aligned}$$

but with particular definitions

$$\begin{aligned} \epsilon &= \frac{m}{2}\left(\frac{v}{v_{th}}\right)^2 + e_j\frac{\delta\phi}{eT} \\ \lambda &= \frac{\mu}{\epsilon} = \frac{v_\perp^2}{v^2}\frac{1}{B} \end{aligned}$$

Volume

$$d^3v = \sum_{\sigma=\pm 1}\frac{2\pi B}{|v_\parallel|}\epsilon\ d\epsilon\ d\lambda$$

The *invariants* of the particle motion are

$$\begin{aligned} \epsilon &= \frac{v^2}{2} \\ \lambda &= \frac{v_\perp^2}{v^2}\frac{1}{B} \end{aligned}$$

The measure

$$d^3v = \sum_{\sigma=\pm 1}\frac{2\pi B}{|v_\parallel|}\epsilon\ d\epsilon\ d\lambda$$

The change of variables to the invariants (w, λ) transforms also the derivative

$$\frac{\partial}{\partial v_\perp^2} = \frac{v_\parallel^2}{v^4} \frac{1}{B} \frac{\partial}{\partial \lambda} + \frac{1}{2} \frac{\partial}{\partial \epsilon}$$

Very small λ means very small $v_\perp$ which means large $v_\parallel$, which means *passing* particles.

The domains of variation of the variable λ are

$$\begin{aligned} 0 &< \lambda < 1 - \varepsilon \\ & \quad \text{circulating particles} \\ & \quad v_\perp \text{ is very small} \end{aligned}$$

and

$$\begin{aligned} 1 - \varepsilon &< \lambda < 1 + \varepsilon \\ & \quad \text{trapped} \end{aligned}$$

$$\int d^3 v = 2\pi v_{th}^3 \; B \sum_{\sigma = \pm 1} \int_0^\infty d\epsilon \; \epsilon \; \int_0^{\lambda_{\max}} \sigma \frac{d\lambda}{|v_\parallel|}$$

6 NOTES 1

In **helander**

$$\begin{aligned} \nu_{ee} \quad \equiv \quad & \text{the rate at which the collisions drive} \\ & \text{the distribution towards a Maxwellian} \end{aligned}$$

7 Notes of the form of the drift-kinetic equation

The various forms.

7.1 From Shaing Dominguez resonance viscosity

The drift kinetic equation is

$$\left(\mathbf{v}_\parallel + \mathbf{v}_D\right) \cdot \boldsymbol{\nabla} f + \dot{w} \frac{\partial f}{\partial w} = C$$

7.2 From Galeev Sagdeev Liu Novakovskii.

The drift kinetic equation

$$\left(\frac{B_\theta}{B_\varphi}v_\parallel + V_E\right)\frac{\partial f_1}{r\partial\theta} - \frac{\mu}{m}\nabla_\parallel B\ \frac{\partial f_1}{\partial v_\parallel} = C\left(f_0\right)$$

The last term in LHS accounts for the energy change for a particle when it moves along the line to a stronger magnetic field.

Normally this motion (particle along the line with variation of the strength of the magnetic field) should not involve any change in the total energy of the particle

$$w = \frac{v^2}{2} = \text{const}$$

and the two components of the velocity are changing periodically their mgnitudes.

7.3 From Helander ECRH

The equation

$$\left(\mathbf{v}_\parallel + \mathbf{v}_D\right)\cdot\boldsymbol{\nabla} f = C\left(f\right) + Q^{ECRH}\left(f\right)$$

It is solved to calculate the *bootstrap* current in the presence of ECRH. Finally only the trapped particle distribution is needed to find the bootstrap.

7.4 The *alpha* particle distribution, from Hsu Shaing Gormley Sigmar

For *bootstrap* sustained by alphas.

$$\begin{aligned}
& v_\parallel\nabla_\parallel\left(\overline{f}_\alpha + I\frac{v_\parallel}{\Omega_\alpha}\frac{\partial}{\partial\psi}f_\alpha^{(0)} - v_\parallel V_{\parallel i}^*\frac{\partial}{\partial w}f_\alpha^{(0)}\right) \\
& +\frac{Z_\alpha e}{m_\alpha}v_\parallel E_\parallel\frac{\partial}{\partial w}f_\alpha^{(0)} \\
= & \ C_\alpha\left(\overline{f}_\alpha\right) \\
& +\frac{S}{4\pi v^2}\delta\left(v - v_0\right)
\end{aligned}$$

where

$$\begin{aligned}
V_{\parallel i}^* = & -\frac{I}{n_i m_i\Omega_i}\frac{\partial}{\partial\psi}p_i \quad (\parallel - \text{projected diamagnetic}) \\
& +K_i B
\end{aligned}$$

7.5 From Connor1973

The equation

$$v_{\|}\frac{\partial \widehat{f}_j}{\partial l_{\|}} + v_{Dj,r}\frac{\partial F_{Mj}}{\partial r} + \frac{Z_j e\ E}{m_j}v_{\|}\frac{\partial F_{M,j}}{\partial \epsilon} = C\left(\widehat{f}_j\right)$$

where $f_j = F_{M,j} + \widehat{f}_j$.

7.6 From Helander3999, impurities

For bulk ions

$$v_{\|}\nabla_{\|}\left(f_i^{(1)} + I\frac{v_{\|}}{\Omega_i}\frac{\partial f_{i0}}{\partial \psi}\right) + e\nabla_{\|}\widetilde{\Phi}\frac{v_{\|}}{T_i}f_{i0} = C_i\left(f_{i1}\right)$$

7.7 From Hirshman Sigmar Clarke

for the species a is

$$\begin{aligned} & \frac{\partial f_a}{\partial t} + \left(v_{\|}\widehat{\mathbf{n}} + \mathbf{v}_{d,a}\right)\cdot\boldsymbol{\nabla} f_a + \left[\mu\frac{\partial B}{\partial t} + \frac{e_a}{m_a}\left(v_{\|}E_{\|}^{(A)} + \frac{\partial \phi}{\partial t}\right)\right]\frac{\partial f_a}{\partial \epsilon} \\ = & \sum_b C_{ab}\left(f_a, f_b\right) \end{aligned}$$

7.8 From Helander Fulop

The drift-kinetic equation for species a based on **Hinton Wong** (in *rotation*).
The variation on surface $\widetilde{\Phi}$ is essential.
Due to *centrifugal* force.

$$\begin{aligned} & v_{\|}\nabla_{\|}f_a + v_{\|}\ e_a\left(-\nabla_{\|}\widetilde{\Phi}\right)\frac{\partial f_a}{\partial H} - C\left(f\right) \\ = & v_{\|}f_{a0}\sum_{j=1}^{3}A_{aj}\nabla_{\|}\alpha_{aj} \end{aligned}$$

(**Hinton Wong**)

$$f_a^{(0)} = \frac{n_a\left(\psi\right)}{\left(\sqrt{\pi}v_{th,a}\right)^3}\exp\left(-\frac{H}{T_a}\right)$$

The *thermodynamic forces* are

$$
\begin{aligned}
A_{a1} &= \frac{n_a'}{n_a} + \frac{T_a'}{T_a} \\
A_{a2} &= \frac{T_a'}{T_a} \\
A_{a3} &= \frac{\omega'}{\omega}
\end{aligned}
$$

The *fluxes* are

$$
\begin{aligned}
\alpha_{a1} &= \frac{m_a}{e_a}\left(\frac{Iv_{\|}}{B} + \omega R^2\right) \\
\alpha_{a2} &= \left(\frac{H}{T_a} - \frac{5}{2}\right)\alpha_{a1} \\
\alpha_{a3} &= \frac{m_a^2 \omega}{2e_a T_a}\left[\left(\frac{Iv_{\|}}{B} + \omega R^2\right) + \mu \frac{|\boldsymbol{\nabla}\psi|^2}{m_a B}\right]
\end{aligned}
$$

8 Forms of the ion velocity

From **Hsu Gromley alpha bootstrap**

$$
\begin{aligned}
\mathbf{V}_i &= K_i \mathbf{B} + \omega_i R^2 \boldsymbol{\nabla}\varphi \\
& \qquad (\text{parallel}) + (\text{toroidal})
\end{aligned}
$$

$$
\begin{aligned}
V_{\|i}^* &= -\frac{I}{n_i m_i \Omega_i}\frac{\partial}{\partial \psi} p_i \quad (\| - \text{projected diamagnetic}) \\
& \quad + K_i B
\end{aligned}
$$

From **Hirshman neoclassic current.**

$$
j_{\|} = -F\frac{p'}{B} + KB
$$

where

$$
K = -F' \frac{1}{4\pi}
$$

From **Helander3999**

$$
\begin{aligned}
V_{Z\|} &= -\frac{I}{B}\frac{d\Phi_0}{d\psi} + \frac{K_Z(\psi)\, B}{n_Z} \\
Z &\equiv \text{impurity}
\end{aligned}
$$

9 Notes on neoclassical correction

Basic fact in the *neoclassic* and in the *wave*.

A good basic text is **collisional diffusion non-axisymmetric Frieman**

The basic content of the first order drift-kinetic equation is

$$v_{\parallel}\nabla_{\parallel}f^{(1)} + \mathbf{v}_D \cdot \boldsymbol{\nabla} f^{(0)} = 0$$

The short expression for this would be

parallel advection of the perturbation is balanced by the radial advection of the equilibrium

NOTE

Since the variation in the parallel direction of $f^{(1)}$ is actually the variation in θ-angle poloidal direction - projected on parallel, it is possible to integrate this equation if one represents the drift velocity as

$$\mathbf{v}_D = -v_{\parallel}\widehat{\mathbf{n}} \times \boldsymbol{\nabla}\left(\frac{v_{\parallel}}{\Omega_c}\right)$$

The expression that is useful is

$$\mathbf{v}_D \cdot \boldsymbol{\nabla}\psi = v_{\parallel}\nabla_{\parallel}\left(I\frac{v_{\parallel}}{\Omega_c}\right)$$

since it will multiply

$$\frac{\partial f^{(0)}}{\partial \psi}$$

which is the assumption

$$\begin{aligned} f_0 &\sim \text{function of surface coordinate, } \psi \\ \text{then } \boldsymbol{\nabla} f_0 &= \frac{\partial f_0}{\partial \psi}\boldsymbol{\nabla}\psi \end{aligned}$$

Here

$$\frac{\partial}{\partial l_{\parallel}} = \frac{1}{qR}\frac{\partial}{\partial \theta}$$

But here one actually MUST include collisions

$$f^{(-1)} \sim \frac{1}{\nu}$$

and *energetic* effects, like

$$v_{\parallel}\ e\left(-\nabla_{\parallel}\widetilde{\Phi}\right)\frac{\partial f^{(0)}}{\partial \epsilon}$$

An example is **Rosenbluth Hazeltine Hinton 1972**

$$f = f_0 + \widehat{f}\ f_0$$

$$
\begin{aligned}
& v_\parallel \frac{B_\theta}{B} \frac{\partial \widehat{f}}{r\partial\theta}\ f_0 \quad \left(\text{parallel convection } v_\parallel \nabla_\parallel \text{ of the perturbed } f\right) \\
& + v_D|_r \ \frac{\partial f_0}{\partial r} \quad \text{(drift convection of the equilibrium, only radial)} \\
& + v_\parallel \ eE_\parallel \frac{\partial f_0}{\partial \epsilon} \quad \left(\text{energetic effect of the neoclassic } \widetilde{\Phi} \text{ or of the wave}\right) \\
= \quad & C(f) \qquad \text{(collision)}
\end{aligned}
$$

After substituting the Maxwellian distribution

$$
f_M \left\{ v_D|_r \ \left[A_1 + A_2(\epsilon - e\Phi)\right] + v_\parallel \left(\frac{B_\theta}{B} \frac{\partial \widehat{f}}{r\partial\theta} - \frac{eE_\parallel}{T} \right) \right\} = C(f)
$$

where the *forces* are

$$
A_{1a} = \frac{d}{dr} \ln n_a - \frac{3}{2} \frac{d}{dr} \ln T_a + \frac{e_a}{T_a} \frac{d\Phi}{dr}
$$

$$
A_{2a} = \frac{1}{T} \frac{d}{dr} \ln T
$$

$$
a \equiv e, i
$$

(see further: the electric term is extracted by shifting the distribution function with the Spitzer function

$$
-v_\parallel \ eE_\parallel \frac{1}{T_e} f_M = C\left(v_\parallel E_\parallel \ f^{Spitzer}\right)
$$

and

$$
f \rightarrow f - v_\parallel E_\parallel \ f^{Spitzer}
$$

)

In **Helander 3999** the function is

$$
\begin{aligned}
f_i^{(1)} \quad = \quad & -\frac{I}{\Omega_i} v_\parallel \frac{\partial f_i^{(0)}}{\partial \psi} \quad \text{(neoclassic drift)} \\
& -\frac{e\Phi}{T_i} f_i^{(0)} \quad \text{(Boltzmann)} \\
& + g_i(\epsilon_0, \mu, \psi, \sigma)
\end{aligned}
$$

10 Notes on correction due to electrostatics on θ

This is necessary when there is *variation of potential on the magnetic surface* $\widetilde{\Phi}$ and looks line

$$\begin{aligned} f_1^{(1)} &= -I\frac{v_\parallel}{\Omega_i}\frac{\partial f_i^{(0)}}{\partial \psi} \quad \left(\text{neoclassics, } \rho_\theta \frac{\partial f}{\partial r}\right) \\ &\quad -\frac{e\widetilde{\Phi}}{T_i} f_i^{(0)} \quad \left(\text{energy effect along the line, from } \widetilde{\Phi}\right) \\ &\qquad\qquad\qquad\qquad \text{(Boltzmann factor)} \\ &\quad +h_i\left(\epsilon,\mu,\psi,\sigma\right) \quad \text{(collisional effects)} \end{aligned}$$

The particles are accelerated along the line because they traverse $\widetilde{\Phi}(\theta)$. They move in the velocity space due to this acceleration.

11 Notes on the average of C

From **Hirshman Sigmar Clarke** (ions plus impurities, see *plasma, general, impurities*)

$$v_\parallel \nabla_\parallel \left(f_{a1} + I\frac{v_\parallel}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial \psi}\right) = v_\parallel \frac{e_a E^{(A)}}{T_a} f_{a0} + \sum_b C_{ab}\left(f_{a1}, f_{b1}\right)$$

The constraint of periodicity is

$$\left\langle \frac{B}{v_\parallel}\left(v_\parallel \frac{e_a E^{(A)}}{T_a} f_{a0} + \sum_b C_{ab}\left(f_{a1}, f_{b1}\right)\right)\right\rangle = 0$$

Note

In **Helander ECRH** the bootstrap current is found from the distribution function $f^{(1)}$ that verifies

$$\left\langle \frac{B}{v_\parallel} C(f)\right\rangle = 0$$

The neoclassical correction exists for electrons (small) for ions and for NBI fast ions.

Helander ECRH

Equation for g is

$$\left\langle \frac{\partial}{\partial \lambda}\lambda v_\parallel \frac{\partial}{\partial \lambda}\left(g - I\frac{v_\parallel}{\Omega}\frac{\partial f_0}{\partial \psi}\right)\right\rangle = 0$$

This formula takes the following form in **Rutherford collisional diffusion 1970**

$$\begin{aligned}\frac{\partial}{\partial\mu}\left(\mu\frac{\partial g_{1e}^{(0)}}{\partial\mu}\oint\frac{v_{\parallel}\,d\chi}{B_{\perp}^{2}}\right)+2RB_{T}\frac{1}{\left(\frac{e}{m_{e}}\right)}\frac{1}{n}\frac{\partial n}{\partial\psi}\oint\frac{d\chi}{B_{\perp}^{2}} &= 0\\ \text{particles} &= \text{circulating}\end{aligned}$$

where (using a change of notation here $B_{\perp}^{Ruth}\equiv B_{\theta}^{here}$)

$$\begin{aligned}\mathbf{B}_{\theta} &= \nabla\psi\times\nabla\varphi\ \text{ poloidal component}\\ &= \nabla\chi\ \text{ derived from a potential }\chi\end{aligned}$$

The annihilator is

$$\oint\frac{dl}{B}$$

and in axisymmetric system

$$\oint\frac{d\chi}{B_{\theta}^{2}}$$

12 Notes on gyration/banana versus grad-p flows

NOTE

In **Hirshman Sigmar Clarke**

$$\begin{aligned}f_{a1} &= -I\frac{v_{\parallel}}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial\psi}\\ &\quad+g_{a}\left(\epsilon,\mu,\psi\right)\end{aligned}$$

where

$$RB_{T}\ \frac{v_{\parallel}}{eB/m}\frac{1}{RB_{\theta}}\frac{\partial f_{a0}}{\partial r}=\frac{v_{\parallel}}{\Omega_{a\theta}}\frac{\partial f_{a0}}{\partial r}=\rho_{a\theta}\frac{\partial f_{a0}}{\partial r}$$

$$\begin{aligned}-I\frac{v_{\parallel}}{\Omega_{ca}}\frac{\partial f_{a0}}{\partial\psi} &\equiv \text{diamagnetic response of the species }a\\ g_{a} &\equiv \text{collisional response of the species }a\end{aligned}$$

This is the equivalent of the *diamagnetic unbalance of flows* in the poloidal plane.

In that case, the fluxes are produced by the Larmor gyration of the particles.

In that case the final flow is poloidal and results from the combination of radial gradient and the *toroidal* magnetic field

$$\frac{1}{B_{T}}\times\frac{\partial f}{\partial r}$$

This would suggest that for the toroidal diamagnetic flows we have to do with fluxes of trapped particles (since they are similar to Larmor gyration). We would expect that only part of the density participates, the *trapped* particles. Actually the gradient of the equilibrium pressure is here combined with B_θ field. And the direction is toroidal, but it is again due to the

$$\text{(radial gradient of pressure)} \times \text{(magnetic field } B_\theta \text{)}$$

The similarity with the poloidal diamagnetic flow and looking for parallel:

$$\text{(Larmor gyration)} \rightarrow \text{(trapped bananas)}$$

is NOT relevant.

NO particle is excluded from the calculation of the "toroidal diamagnetic" flow (*i.e.* the untrapped particles are NOT excluded).

In an area which is common for

- NBI fast ions
- bootstrap current
- impurity friction

it is located a problem of discerning the role of

1. diamagnetic flow, seen as *unbalanced fluxes*
   - Larmor gyration for poloidal flow AND
   - banana-trapped for toroidal flow
2. radial gradient of pressure, which is a force, combined with toroidal magnetic field respectively poloidal magnetic field $\mathbf{F} \times \mathbf{B}$.

It is seen that the second is sufficient to get a parallel flow (which is attributed to the gradient of pressure of all particles NOT ONLY trapped ones and the poloidal magnetic field)

Further, it is noted that this flow is different for different species.

Then there is collisional *friction*.

Hirshman Sigmar Clarke for impurities.

Hsu Catto Sigmar for NBI fast ions isotropic.

Helander 3999 for ion-impurity parallel friction.

The friction tends to equalize the flows and on a slow time scale this modifies the gradients.

13 Notes on $(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}$ in stationary states

In **hose like Rosenbluth**

In **Helander 3999**

$$\begin{aligned} m_Z n_Z\ \widehat{\mathbf{n}}\cdot\left[(\mathbf{v}_Z\cdot\boldsymbol{\nabla})\,\mathbf{v}_Z\right] &= -\nabla_{\|}p_Z-\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_Z \\ &\quad -n_Z\ Ze\ \nabla_{\|}\Phi^{(1)} \\ &\quad +R_{Z\|} \end{aligned}$$

but

$$\frac{m_Z n_Z\ \widehat{\mathbf{n}}\cdot\left[(\mathbf{v}_Z\cdot\boldsymbol{\nabla})\,\mathbf{v}_Z\right]}{R_{Z\|}}\sim\frac{\delta}{Z\widehat{\nu}_{ii}}$$

Most contributions are overwhelmed by the *ion-impurity energy equilibration.*

Then it remains

$$n_Z\ Ze\ \nabla_{\|}\Phi^{(1)}+T_i\nabla_{\|}n_Z=R_{Z\|}$$

14 Notes on collisional friction, pitch-angle for fast ions (Gaffey)

The subject is for

- NBI fast ions
- impurities Z relative to background ions.

For fast ions (**Hsu Catto Sigmar**)

- in the interval from high v_0 (at born) and crictical v_c, there is *slowing down* and is dominated by *electrons*.
- there is a critical value v_c where the electron drag becomes equal to the ion drag
- this interval was called *banana collapse* because the reduction of the energy of new fast ions leads to reduction of the width of the banana; during the reduction of the energy of the fast ions, the process becomes dominated by *pitch angle* deflection.
- the *pitch angle scattering* of the fast ions by the background ions, at around v_b.

- picth angle by background ions is very important close to the separatrix *trapped/circulating.*

From **Gaffey**.
The derivatives of the relative velocity. This is

$$g = |\mathbf{g}| = |\mathbf{v}_i - \mathbf{v}_j|$$

$$\frac{\partial g}{\partial \mathbf{v}_i} = \frac{\mathbf{g}}{g}$$

$$\begin{aligned}
\frac{\partial^2 g}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} &= \frac{1}{g^3} \left(g^2 \, \mathbf{I} - \mathbf{g} \, \mathbf{g} \right) \\
&\equiv \boldsymbol{\omega}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{v}_i} \cdot \boldsymbol{\omega} &= \frac{\partial}{\partial \mathbf{v}_i} \left[\frac{1}{g^3} \left(g^2 \, \mathbf{I} - \mathbf{g} \, \mathbf{g} \right) \right] \\
&= -2 \frac{\mathbf{g}}{g^3}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2}{\partial \mathbf{v}_i \partial \mathbf{v}_i} : \boldsymbol{\omega} &= \frac{\partial}{\partial \mathbf{v}_i} \cdot \left(-2 \frac{\mathbf{g}}{g^3} \right) \\
&= 0
\end{aligned}$$

Define

$$x_{ij} \equiv \frac{v_i}{v_{th,j}}$$

$$\frac{\partial x_{ij}}{\partial \mathbf{v}_i} = \frac{1}{v_{th,j}} \frac{\partial v_i}{\partial \mathbf{v}_i} = \frac{1}{v_{th,j}} \frac{\mathbf{v}_i}{v_i}$$

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{v}_i} \cdot \frac{\partial x_{ij}}{\partial \mathbf{v}_i} &= \frac{1}{v_{th,j}} \frac{\partial}{\partial \mathbf{v}} \cdot \frac{\partial}{\partial \mathbf{v}_i} v_i \\
&= \frac{1}{v_{th,j}} \frac{2}{v_i}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 x_{ij}}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} &= \frac{1}{v_{th,j}} \frac{\partial^2 v_i}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} \\
&= \frac{1}{v_{th,j}} \left(\frac{1}{v_i} \mathbf{I} - \frac{\mathbf{v}_i \, \mathbf{v}_i}{v_i^3} \right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 x_{ij}}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} &= \frac{1}{v_{th,j}} \frac{\partial}{\partial \mathbf{v}_i} \cdot \left(\frac{1}{v_i} \mathbf{I} - \frac{\mathbf{v}_i \, \mathbf{v}_i}{v_i^3} \right) \\
&= -\frac{2}{v_{th,j}} \frac{\mathbf{v}_i}{v_i^3}
\end{aligned}$$

$$\begin{aligned}\frac{\partial x_{ij}}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 x_{ij}}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} &= \frac{1}{v_{th,j}} \frac{\mathbf{v}_i}{v_i} \cdot \left(\frac{1}{v_i} \mathbf{I} - \frac{\mathbf{v}_i \ \mathbf{v}_i}{v_i^3} \right) \\ &= 0\end{aligned}$$

We this preparation one will try to give explicit form to an important expression in the collision operator

$$\frac{\partial}{\partial \mathbf{v}_i} \cdot \left[\frac{\partial f_b}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 v_i}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} \right]$$

is calculated introducing the variables in the velocity space

$$(\xi, \theta, \varphi)$$

$$\begin{aligned}\xi &= \cos\theta \\ &= \frac{\mathbf{v}_i}{v_i} \cdot \widehat{\mathbf{e}}_B \\ &= \frac{v_\parallel}{v} \\ & \quad \text{we do not write "species"} - i\end{aligned}$$

then

$$\begin{aligned}\frac{\partial \xi}{\partial \mathbf{v}_i} &= \frac{\partial}{\partial \mathbf{v}_i} \frac{\mathbf{v}_i}{v_i} \cdot \widehat{\mathbf{e}}_B \\ &= \frac{\partial^2 v}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} \cdot \widehat{\mathbf{e}}_B\end{aligned}$$

It is assumed that the distribution function f_b does not depend on the angle φ.

$$\frac{\partial f_b}{\partial \varphi} = 0$$

$$\frac{\partial f_b}{\partial \mathbf{v}_i} = \frac{\partial f_b}{\partial v} \frac{\partial v}{\partial \mathbf{v}_i} + \frac{\partial f_b}{\partial \xi} \frac{\partial \xi}{\partial \mathbf{v}_i}$$

We multiply this formal equality by

$$\frac{\partial^2 v}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i}$$

It results

$$\begin{aligned}\frac{\partial f_b}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 v}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} &= \frac{\partial f_b}{\partial v} \frac{\partial v}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 v}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} \\ &\quad + \frac{\partial f_b}{\partial \xi} \frac{\partial \xi}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 v}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i}\end{aligned}$$

we take into account, for the second term

$$\frac{\partial \xi}{\partial \mathbf{v}_i} = \frac{\partial^2 v}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} \cdot \widehat{\mathbf{e}}_B$$

and for the first term

$$\begin{aligned}
\frac{\partial x_{ij}}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 x_{ij}}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} &= \frac{1}{v_{th,j}} \frac{\mathbf{v}_i}{v_i} \cdot \left(\frac{1}{v_i} \mathbf{I} - \frac{\mathbf{v}_i \ \mathbf{v}_i}{v_i^3} \right) \\
&= 0
\end{aligned}$$

where we can take the particular case, for clarity

$$\begin{aligned}
\mathbf{v}_j &= 0 \\
x_{ij} &= |\mathbf{v}_i| = v_i
\end{aligned}$$

Then the first term in the RHS of the equation is zero and we only have the second one

$$\begin{aligned}
\frac{\partial f_b}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 v_i}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} &= \frac{\partial f_b}{\partial \xi} \frac{\partial \xi}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 v_i}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} \\
&= \frac{\partial f_b}{\partial \xi} \left(\frac{\partial^2 v_i}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} \cdot \widehat{\mathbf{e}}_B \right) \cdot \frac{\partial^2 v_i}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} \\
&= \frac{\partial f_b}{\partial \xi} \widehat{\mathbf{e}}_B \cdot \frac{1}{v_i} \frac{\partial^2 v_i}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial}{\partial \mathbf{v}} \cdot \left[\frac{\partial f_b}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 v}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} \right] \\
= & \left[\frac{\partial^2 f_b}{\partial \xi^2} \widehat{\mathbf{e}}_B \cdot \frac{1}{v_i} \frac{\partial^2 v_i}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} + \frac{1}{v} \frac{\partial f_b}{\partial \xi} \frac{\partial}{\partial \mathbf{v}} \right] \cdot \frac{\partial^2 v_i}{\partial \mathbf{v}_i \, \partial \mathbf{v}_i} \cdot \widehat{\mathbf{e}}_B \\
= & \left(1 - \xi^2\right) \frac{1}{v^3} \frac{\partial^2 f_b}{\partial \xi^2} - \frac{2\xi}{v^3} \frac{\partial f_b}{\partial \xi} \\
= & \frac{1}{v^3} \frac{\partial}{\partial \xi} \left[\left(1 - \xi^2\right) \frac{\partial f_b}{\partial \xi} \right]
\end{aligned}$$

This is the pitch angle scattering.

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16 Notes on Spitzer problem

Distribution function with the Spitzer function

$$-v_\parallel \ e E_\parallel \frac{1}{T_e} f_M = C \left(v_\parallel E_\parallel \ f^{Spitzer} \right)$$

and

$$f \to f - v_{\parallel} E_{\parallel} \; f^{Spitzer}$$

In **Hazeltine Ware electrostatic trapping** we find

$$\langle C_0 \left(f_s \right) \rangle_{\theta} = -v\xi \; eE_0 \; \frac{1}{T} F_{Me}$$

where $\xi = v_{\parallel}/v$.

The solution

$$\begin{aligned} f_s &= -\tau_e \; v_{th,e} \xi \; eE_0 \frac{1}{T_e} F_{Me} \\ &\quad \times S_e \left(\frac{v}{v_{th,e}} \right) \end{aligned}$$

with S_e calculated by Spitzer.

The collision operator

$$C_0 \left(f_e \right) = \left(\frac{3\sqrt{\pi}}{8} \right) \frac{1}{\tau_e} \frac{n_e Z_{eff}}{\overline{n}_e} \frac{v_{th,e}^3}{v^3} \; \frac{\partial}{\partial \xi} \left(1 - \xi^2 \right) \frac{\partial f_e}{\partial \xi}$$

$$\begin{aligned} \tau_e &= \frac{3}{16\sqrt{\pi}} \frac{m_e^2}{e^4} \frac{1}{\ln \Lambda} \frac{v_{th,e}^3}{\overline{n}_e} \\ Z_{eff} &= \frac{n_0 + Z^2 n_Z}{n_e} \end{aligned}$$

Here $C_0 \left(f_e \right)$, the pitch angle scattering operator, is the first part of the collision operator, the second part is related to shifted Maxwellian and is due to the average motion of the plasma ions

$$\begin{aligned} u_d &= \frac{n_i V_{\parallel i} + Z^2 n_Z V_{\parallel Z}}{n_i + Z^2 n_Z} \\ &\quad \text{approx the velocity of all ions} \end{aligned}$$

17 Notes on parallel current (bootstrap)

In **Helander ECRH**

The current

$$\begin{aligned} \langle j_{elect} \rangle &= -e \left\langle \int d^3 v \; f_1 v_{\parallel} \right\rangle \\ &= 2\pi e \sum_{\sigma = \pm 1} \int_0^{\infty} w dw \; \int_0^{1-\varepsilon} \; d\lambda \; B_0 \; \left(\lambda \frac{\partial g}{\partial \lambda} + I \frac{\langle |v_{\parallel}| \rangle}{\Omega} \frac{\partial f_0}{\partial \psi} \right) \end{aligned}$$

here we have used the measure

$$d^3v = \sum_{\sigma=\pm 1} \frac{2\pi B}{|v_\parallel|} w\ dw\ d\lambda$$

18 Notes on the inertia $\left(1+2q^2\right)$ in the poloidal rotation

18.1 From Hirshman NF18 ambipolarity paradox.

The equation of momentum conservation projected on the parallel direction

$$\sum_j m_j n_j \left\langle \mathbf{B} \cdot \frac{\partial \mathbf{u}_j}{\partial t} \right\rangle = -\sum_j \langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Pi}_j \rangle$$

Now we need an expression for the fluid velocity u_j,

$$\begin{aligned} \mathbf{u}_j &= \left(\frac{\mathbf{u}_j \cdot \mathbf{B}_\theta}{B_\theta^2} \right) \mathbf{B} \\ &\quad + \frac{R^2 \boldsymbol{\nabla}\varphi}{\langle R^2 \rangle} \left(\left\langle R^2 \boldsymbol{\nabla}\varphi \cdot \mathbf{u}_j \right\rangle \ - \ \frac{\mathbf{u}_j \cdot \mathbf{B}_\theta}{B_\theta^2} \langle R B_T \rangle \right) \end{aligned}$$

It has been introduced the polodal projection of the velocity, which after projection is scaled by B_θ to remove the θ dependence and produce a function (velocity) that depends on only ψ.

$$\frac{\mathbf{u}_j \cdot \mathbf{B}_\theta}{B_\theta^2} = K_j(\psi)$$

(usual notation for the poloidal part of the velocity. **Hsu Gromley Shaing bootstrap** α)

Then

$$\begin{aligned} & \sum_j m_j n_j \frac{\partial}{\partial t} \left[\left(1 + 2\widehat{q}^2\right) \ \langle \mathbf{u}_j \cdot \mathbf{B}_\theta \rangle + \langle R\ B_T \rangle \frac{\left\langle R^2 \boldsymbol{\nabla}\varphi \cdot \mathbf{u}_j \right\rangle}{\langle R^2 \rangle} \right] \\ = & \ -\sum_j \langle \mathbf{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\Pi}_j \rangle \end{aligned}$$

where

$$\widehat{q}^2 \equiv \frac{1}{2\langle B_\theta^2 \rangle} \left(\left\langle B_T^2 \right\rangle - \frac{1}{\left\langle \frac{1}{B_T^2} \right\rangle} \right)$$

This reduces to

$$\widehat{q}^2 \rightarrow q^2$$

for small ε.

18.2 This is Spontaneous spin-up (Hassam Drake) in *Equilibrium flows.*

The equations.

The continuity

$$\frac{\partial n}{\partial t}+\boldsymbol{\nabla}\cdot\left(n\mathbf{u}_{\perp}\right)+\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{nu_{\parallel}}{B}\right)=S-\frac{1}{r}\frac{\partial}{\partial r}\left(r\Gamma_{r}\right)$$

This equation is important for the derivation of the expression for the Pfirsch Schluter current.

This is because it introduces the *divergence of the flux of particles*, of the flow. This is where the *geometrical* poloidal compression and dilation will enter the dynamics. In the term $\boldsymbol{\nabla}\cdot\left[\widehat{\mathbf{e}}_{\theta}\left(1+\varepsilon\cos\theta\right)\right]$.

The momentum for all plasma (the mass is taken m_i), isothermal

$$nm_{i}\left[\frac{\partial\mathbf{u}}{\partial t}+\left(\mathbf{u}\cdot\boldsymbol{\nabla}\right)\mathbf{u}\right]=-T\boldsymbol{\nabla}n+\mathbf{j}\times\mathbf{B}-m_{i}S\mathbf{u}$$

The current conservation is essential in connecting the perpendicular current (diamagnetic) with the parallel current (Pfirsch Schluter)

$$\boldsymbol{\nabla}\cdot\mathbf{j}=0$$

The Ohm's law. Here, without the resistivity. This means that the radial velocity v_r will be attributed to another reason for which the charge neutrality cannot be fully ensured by the parallel currents. The reason may be the *Landau damping* which acts when the collisionality is low. This appears in **Stringer** where a kinetic treatment allows to calculate the variation of the density and of the potential on the magnetic surface, by integrating over the velocity space the distribution functions of electron and ions and imposing neutrality. During the integration, one has to traverse the singularity $v_{\parallel}-\frac{\varepsilon}{q}v_{\theta}^{E}=0$.

$$-\boldsymbol{\nabla}\phi+\mathbf{u}\times\mathbf{B}=0$$

The magnetic field is

$$\mathbf{B}=\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\varphi+I\left(\psi\right)\boldsymbol{\nabla}\varphi$$

Relative to the work **Hassam Kulsrud** here it is assumed that the electrons and ions are *isothermal*.

$$S\left(r,\theta\right)\equiv\text{particle source}$$

It is interesting to note how the *source* extracts from the momentum a part which is proportional with $m_i\mathbf{u}$ through S.

The radial flux

$$\begin{aligned} \Gamma_r &= \langle\langle \widetilde{n}\widetilde{v}_r \rangle\rangle \\ &= -D(r,\theta)\frac{\partial n}{\partial r} \end{aligned}$$

An object of study is the *circulation.*

It is a similar object with the contour integral of velocity or surface integral of the vorticity.

This is obtained taking the projection in the *parallel* direction of the equation of momentum conservation. It is interesting that the variation of the density in the parallel direction (for isothermal plasma) gives the pressure that opposes to the geometrical (also called *inertial*) advection of the flow, $\mathbf{B}\cdot(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}$, which can be static. The imbalance gives a nonzero $\frac{\partial}{\partial t}(\mathbf{u}\cdot\mathbf{B})$.

$$\begin{aligned} \frac{\partial}{\partial t}(\mathbf{u}\cdot\mathbf{B})+\mathbf{B}\cdot(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} &= -c_s^2\mathbf{B}\cdot\boldsymbol{\nabla}\ln n \quad \text{(parallel pressure)} \\ & -\frac{S}{n}\mathbf{u}\cdot\mathbf{B} \quad \text{(external source of momentum)} \end{aligned}$$

The poloidal component of the equation for the plasma momentum

$$\mathbf{B}_{pol}\cdot\left(nm_i\frac{d\mathbf{u}}{dt}+T\boldsymbol{\nabla}n\right)=\frac{B_\varphi}{R}\mathbf{j}\cdot\boldsymbol{\nabla}\psi \quad \text{(radial current)}$$

We note that

$$\mathbf{B}_{pol}=\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\varphi$$

and the product $\mathbf{j}\cdot\boldsymbol{\nabla}\psi$ extracts the radial current.

Now comes the constraint that will provide the third equation:

the total radial current traversing a magnetic surface must be zero

Integrated over a magnetic surface the zero-divergence of the current density leads to

$$\int_{flux\_surf}\mathbf{ds}\cdot\mathbf{j}=\int\frac{ds}{|\boldsymbol{\nabla}\psi|}(\mathbf{j}\cdot\boldsymbol{\nabla}\psi)=0$$

where

$$\begin{aligned} \mathbf{ds} &= ds\,\widehat{\mathbf{e}}_r \\ &= 2\pi Rrd\theta\,\widehat{\mathbf{e}}_r \end{aligned}$$

then

$$\int_{flux\_surf}\frac{ds}{|\boldsymbol{\nabla}\psi|}R^2\mathbf{B}_{pol}\cdot\left(nm_i\frac{d\mathbf{u}}{dt}+T\boldsymbol{\nabla}n\right)=0$$

This equation, derived from current conservation

$$\boldsymbol{\nabla}\cdot\mathbf{j}=0$$

will be used to derive the *time variation of the poloidal velocity.*

It is assumed first that the plasma velocity is smaller than the sound velocity. The equilibrium, in zero order

$$\frac{\partial n_0}{\partial t} = 0$$

the equilibrium density is constant in time

$$\mathbf{u}_{\perp 0} \cdot \boldsymbol{\nabla} n_0 = 0$$

The perpendicular advection of the equilibrium density is zero: the density does not vary along *perpendicular* direction.

$$0 = T\mathbf{B}_0 \cdot \boldsymbol{\nabla} n_0$$

The equilibrium density does not vary parallel with the magnetic field line

$$\int d\theta \, \frac{1}{r} \frac{\partial n_0}{\partial \theta} = 0$$

If there is a poloidal variation of the density along the poloidal direction, the periodicity must be taken into account.

$$0 = -\boldsymbol{\nabla}\phi_0 + \mathbf{u}_0 \times \mathbf{B}_0$$

The Ohm's law without *resistivity.*

This means that the lowest order density is constant on the surfaces

$$n_0(r)$$

and the velocity which is perpendicular on the magnetic field is contained in the magnetic surface. It is the electric velocity

$$\begin{aligned} \mathbf{u}_{\perp 0} &= V_E \widehat{\mathbf{e}}_\theta \\ &= \frac{1}{B} \frac{d\phi_0}{dr} \widehat{\mathbf{e}}_\theta \end{aligned}$$

This velocity V_E is poloidal.

the first order in ε will reveal the presence of a perturbation of the density on the magnetic surface, n_1.

Also we will have to work with the parallel velocity $u_{\|}$.

The parallel velocity is a response to the nonzero divergence of the diamagnetic flow, it is therefore Pfirsch-Schluter flow and current

$$\begin{aligned} & \frac{\partial n_1}{\partial t} + V_E \frac{\partial n_1}{r \partial \theta} + n_0 V_E \left(-2\varepsilon \frac{\sin\theta}{r} \right) \\ & + n_0 \nabla_{\|} u_{\|} \\ = \; & S - \frac{1}{r} \frac{\partial}{\partial r} (r \Gamma_r) \end{aligned}$$

We see here that the poloidal rotation velocity V_E is very active: it carries the perturbation of the density on the surface

$$V_E \frac{\partial n_1}{r \partial \theta}$$

However we know that the poloidal velocity V_E must contain actually the diamagnetic velocity, $nv_{Dia} = 1/(m\Omega)\, dp/dr$. And this is equal with $-\boldsymbol{\nabla}\phi$, and from here we can introduce the symbol V_E. To understand the third term we should remember

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \left[\widehat{\mathbf{e}}_\theta \left(1+\varepsilon \cos\theta\right)\right] &= \frac{1}{r\left(R_0 + r\cos\theta\right)} \frac{\partial}{\partial\theta}\left(\left(R_0 + r\cos\theta\right)\left(1+\varepsilon\cos\theta\right)\right) \\
&= \varepsilon \frac{(-2\sin\theta)}{r}
\end{aligned}$$

The factor $h = 1 + \varepsilon \cos\theta$ comes from the magnitude of the magnetic field $B = B_0/h$. The divergence is calculated for the poloidal flow resulting from the electric velocity V_E that carries the density $n_0 + n_1$. Both quantities do not have variation in this order but the *geometry* is essential.

Termenul $\nabla_{\|} u_{\|}$

The parallel momentum

$$n_0 m_i \left(\frac{\partial u_{\|}}{\partial t} + V_E \frac{\partial u_{\|}}{r\partial\theta}\right) = -T \nabla_{\|} n_1$$

Note that it is here that the *parallel viscosity* $\mathbf{\Pi}$ should appear to introduce the *magnetic damping*. Shaing, etc. **End.**

The parallel gradient is

$$\nabla_{\|} = \frac{1}{qR}\frac{\partial}{\partial\theta}$$

Finally the condition

$$\int d\theta \; B_{\theta 0} \frac{\partial n_1}{r\partial\theta} = 0$$

Since $B_{\theta 0}$ is actually constant on the surface and is taken out the integral the condition is trivially satisfied in this order.

The condition satisfied trivially at the first order must be recalculated in higher order, *i.e.* two, ε^2.

The equation to be used is

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{j} &= 0 \\
\text{or, the integral form} \quad \int_{flux_surf} \mathbf{ds} \cdot \mathbf{j} &= \mathbf{0}
\end{aligned}$$

$$\int_{flux\_surf} \frac{ds}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol} \cdot \left(n m_i \frac{d\mathbf{u}}{dt} + T \boldsymbol{\nabla} n \right) = 0$$

derived from the condition of zero-divergence of the current.

The part

$$\int_{flux\_surf} \frac{ds}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol} \cdot (T \boldsymbol{\nabla} n)$$

will be calculated as

$$\begin{array}{rcl} R^2 & \approx & 1 \\ \dfrac{ds}{|\boldsymbol{\nabla}\psi|} & \sim & \text{order } 1 \\ \mathbf{B}_{pol} \cdot \boldsymbol{\nabla} n & \sim & \text{order } 1 \end{array}$$

An approximation

$$\mathbf{B}_{pol} \cdot \boldsymbol{\nabla} n \approx B_\theta \frac{\partial n_1}{r \partial \theta}$$

and

$$|\boldsymbol{\nabla}\psi| = 2\pi R B_\theta$$

In the product

$$\mathbf{B}_{pol} \cdot \left(n m_i \frac{d\mathbf{u}}{dt} \right)$$

we only retain

$$\mathbf{B}_{pol} \cdot \left(n m_i \frac{\partial \mathbf{u}}{\partial t} \right)$$

since $\mathbf{B}_{pol} \cdot (\mathbf{u} \cdot \boldsymbol{\nabla}) \mathbf{u}$ is of higher order. This term will provide the time variation of the *poloidal velocity* $V_E(r,t)$.

We also have

$$ds = 2\pi R r d\theta$$

The integration of the first part is

$$\begin{array}{rl} & \displaystyle\int_{flux\_surf} \frac{ds}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol} \cdot \left(n m_i \frac{d\mathbf{u}}{dt} \right) \\ \approx & \displaystyle\int_{flux\_surf} \frac{2\pi R r d\theta}{2\pi R B_\theta} B_\theta n m_i \frac{\partial V_E}{\partial t} \\ = & (2\pi)\, r n m_i \dfrac{\partial V_E}{\partial t} \end{array}$$

the integration of the second term

$$\begin{array}{rl} & \displaystyle\int_{flux\_surf} \frac{ds}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol} \cdot (T \boldsymbol{\nabla} n) \\ = & T \displaystyle\int_{flux\_surf} \frac{ds}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol} \cdot \boldsymbol{\nabla} \left[n_0(r) + n_1(r,\theta) \right] \\ = & T \displaystyle\int_{flux\_surf} \frac{ds}{|\boldsymbol{\nabla}\psi|} R^2 \mathbf{B}_{pol} \cdot \boldsymbol{\nabla} n_1(r,\theta) \end{array}$$

we make an integration by parts and take into account the periodicity

$$\begin{aligned}
& []_0^{2\pi} - T\int_{flux_surf} n_1 \boldsymbol{\nabla}\cdot\left(\frac{ds}{|\boldsymbol{\nabla}\psi|}R^2\mathbf{B}_{pol}\right) \\
= & -T\int_{flux_surf} n_1 \boldsymbol{\nabla}\cdot\left(\frac{2\pi Rr d\theta}{2\pi RB_\theta}R^2 B_\theta\widehat{\mathbf{e}}_\theta\right)
\end{aligned}$$

we must find

$$\begin{aligned}
& \int_{flux_surf}\frac{ds}{|\boldsymbol{\nabla}\psi|}R^2\mathbf{B}_{pol}\cdot(T\boldsymbol{\nabla}n) \\
= & -T\int_{flux_surf} n_1\boldsymbol{\nabla}\cdot\left(\frac{ds}{|\boldsymbol{\nabla}\psi|}R^2\mathbf{B}_{pol}\right) \qquad \text{(integration by parts)} \\
= & -Tr\int n_1 d\theta\boldsymbol{\nabla}\cdot\left(h\widehat{\mathbf{e}}_\theta\right) \\
= & -Tr\int d\theta\left(\varepsilon\frac{-2\sin\theta}{r}\right)n_1
\end{aligned}$$

Then

$$\int_{flux\_surf}\frac{ds}{|\boldsymbol{\nabla}\psi|}R^2\mathbf{B}_{pol}\cdot\left(nm_i\frac{d\mathbf{u}}{dt}+T\boldsymbol{\nabla}n\right)=0$$

$$\begin{aligned}
(2\pi)\,rnm_i\frac{\partial V_E}{\partial t}+\left\{-Tr\int d\theta\left(\varepsilon\frac{-2\sin\theta}{r}\right)n_1\right\} &= 0 \\
(2\pi)\,rnm_i\frac{\partial V_E}{\partial t} &= -2T\varepsilon\int d\theta\sin\theta\ n_1 \\
\frac{\partial V_E}{\partial t} &= -\frac{1}{r}\varepsilon\frac{c_s^2}{n_0}\int\frac{d\theta}{2\pi}\sin\theta\ n_1
\end{aligned}$$

We must underline, the equation

$$\frac{\partial V_E}{\partial t}=-\frac{1}{r}\varepsilon\frac{c_s^2}{n_0}\int\frac{d\theta}{2\pi}\sin\theta\ n_1$$

comes from the zero total current crossing the surface ψ.

It means that the gradient of pressure (which is a force) is balanced by the time variation (acceleration) of the momentum (rotation).

New notation

$$N\equiv\frac{n_1}{n_0}$$

the equations

the density conservation, with advection and sources

$$\begin{aligned}
& \frac{\partial N}{\partial t}+V_E\frac{\partial N}{r\partial\theta}+V_E\left(-2\varepsilon\frac{\sin\theta}{r}\right) \\
& +\nabla_\parallel u_\parallel \\
= & \frac{S}{n_0}-\frac{1}{n_0}\frac{1}{r}\frac{\partial}{\partial r}\left(r\Gamma_r\right)
\end{aligned}$$

The balance of parallel momentum, with parallel gradient of pressure and variation of the parallel velocity, including the advection by poloidal rotation

$$\frac{\partial u_{\parallel}}{\partial t}+V_E\frac{\partial u_{\parallel}}{r\partial\theta}=-c_s^2\nabla_{\parallel}N$$

The constraint that the total *current* that traverses a magnetic surface is zero

$$\frac{\partial V_E}{\partial t}=c_s^2\int\frac{d\theta}{2\pi}N\left(-2\varepsilon\frac{\sin\theta}{r}\right)$$

NOTE

Let us stop to make a comparison between this (**Hassam Drake**) system prepared for the spontaneous spin-up and the **Stringer PRL** system.

We note that the *time variation* in the equation for

- the density, $\partial n_1/\partial t$, and
- the velocity

  $$nm_iV_{E\theta}^{(0)}\frac{\partial v_{i\parallel}}{r\partial\theta}=-\left(T_e+T_i\right)\frac{\varepsilon}{q}\frac{\partial n_1}{r\partial\theta}$$

  this equation shows the balance of momentum carried by the "static advected" velocity (*i.e.* space variation of the velocity, $(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}$) with the *pressure*. The projection is made on poloidal direction.

is *absent* at **Stringer**.

Since Hassam Drake work for spin-up driven by external source (poloidally asymmetric) the explicit time variation must be retained.

The main term however in the formulation Hassam Drake is still $V_E\frac{\partial u_{\parallel}}{r\partial\theta}$ (later $\widehat{u}_E\frac{\partial\widetilde{u}_{\parallel}}{r\partial\theta}$) which is the same as in Stringer. This term will be the main part of the expansion around the equilibrium static state.

The equilibrium static state at Hassam Drake is

$$\begin{aligned}\nabla_{\parallel}\widetilde{u}_{\parallel}&=\widetilde{F}\\ \nabla_{\parallel}\widetilde{N}&=0\end{aligned}$$

and the expansion introduces new, small, quantities

$$\widehat{u}_E\ ,\ \widehat{u}_{\parallel}\ ,\ \widehat{N}$$

with the system

$$\begin{aligned}-2\varepsilon\widehat{u}_E\frac{\sin\theta}{r}+\nabla_{\parallel}\widehat{u}_{\parallel}&=0\quad\text{(continuity)}\\ \frac{\partial\widehat{u}_{\parallel}}{\partial t}+\widehat{u}_E\frac{\partial\widetilde{u}_{\parallel}}{r\partial\theta}&=-c_s^2\nabla_{\parallel}\widehat{N}\qquad\text{(parallel force balance)}\\ \frac{\partial\widehat{u}_E}{\partial t}&=-\frac{2\varepsilon c_s^2}{r}\int\frac{d\theta}{2\pi}\widehat{N}\sin\theta\ \ \text{(zero total current crossing }\psi)\end{aligned}$$

See the explanations below.
END

The functions that must be determined

$$N\left(r,\theta,t\right) \quad , \quad u_{\parallel}\left(r,\theta,t\right) \quad , \quad V_E\left(r,t\right)$$

The global balance is obtained by integrating over the surface $\int \frac{d\theta}{2\pi}\left(...\right)$.

$$\begin{aligned}
\frac{\partial \overline{N}}{\partial t} &= \frac{\overline{S}}{n_0} - \frac{1}{n_0}\frac{\partial}{r\partial\theta}\left(r\overline{\Gamma}_r\right) \\
\frac{\partial \overline{u}_{\parallel}}{\partial t} &= 0
\end{aligned}$$

After introducing the average over surfaces, the new variables are the differences that have variations in the surfaces

$$\widetilde{f} = f - \overline{f}$$

The source in the surface is

$$F \equiv \frac{S - \frac{1}{r}\frac{\partial}{\partial r}\left(r\Gamma_r\right)}{n_0}$$

The state that is taken as reference is the absence of the poloidal rotation

$$V_E^{ref} = 0$$

and this reduces the equations to

$$\begin{aligned}
N^{ref} &= \overline{N} \\
\text{i.e.} \quad \widetilde{N}^{ref} &= 0 \quad \text{(no variation of the density in the surface)}
\end{aligned}$$

$$\nabla_{\parallel}\widetilde{u}_{\parallel}^{ref} = \widetilde{F}$$

$$\nabla_{\parallel}\widetilde{N}^{ref} = 0 \;\; \text{from where } \widetilde{N} = 0$$

The variation of the parallel velocity along the magnetic line (equivalently, in the magnetic surface) is obtained in terms of the source

$$\begin{aligned}
\nabla_{\parallel}\widetilde{u}_{\parallel}^{ref} &= \widetilde{F} \\
\frac{1}{qR}\frac{\partial}{\partial\theta}\widetilde{u}_{\parallel}^{ref} &= \widetilde{F} \\
\widetilde{u}_{\parallel}^{ref} &= qR\int d\theta' \; \widetilde{F}
\end{aligned}$$

Consider a perturbation of this reference state

$$\begin{aligned} V_E &= V_E^{ref} + \widehat{V}_E \\ u_\parallel &= \widetilde{u}_\parallel^{ref} + \widehat{u}_\parallel \\ N &= \overline{N} + \widetilde{N}^{ref} + \widehat{N} \end{aligned}$$

This will induce a time variation of the poloidal (electric) velocity and of the density N and of the parallel velocity.

However the time variation is assumed to be slower than the sound speed

$$\frac{\partial}{\partial t} \ll \frac{c_s}{qR}$$

The time variation for N is neglected and the equation for density becomes a balance of flows

$$\widehat{V}_E \left(-2\varepsilon \frac{\sin\theta}{r} \right) + \nabla_\parallel \widehat{u}_\parallel = 0$$

Further we have a momentum balance equation along the parallel direction: *the gradient of pressure along the line generates rotation (time variation of parallel velocity)*

$$\frac{\partial \widehat{u}_\parallel}{\partial t} + \widehat{V}_E \frac{\partial \widetilde{u}_\parallel^{ref}}{r\partial\theta} = -c_s^2 \nabla_\parallel \widehat{N}$$

$$\frac{\partial \widehat{V}_E}{\partial t} = c_s^2 \int \frac{d\theta}{2\pi} \widehat{N} \left(-2\varepsilon \frac{\sin\theta}{r} \right)$$

Note the preservation of the poloidal derivative of the *reference* parallel velocity in the second equation. This reference value of the paralel velocity is fixed by the radial flux and the source of particles. It exists only because these sources and fluxes are *NOT constant on the poloidal circumference.*

This set of equations can be integrated.

The operator that must be made explicit is

$$\nabla_\parallel = \frac{1}{qR} \frac{\partial}{\partial\theta}$$

Then, since $\widehat{V}_E$ is constant on magnetic surfaces, the first equation is

$$\begin{aligned} \widehat{V}_E \left(-2\varepsilon \frac{\sin\theta}{r} \right) + \nabla_\parallel \widehat{u}_\parallel &= 0 \quad \text{or} \\ \frac{1}{qR} \frac{\partial}{\partial\theta} \widehat{u}_\parallel &= \widehat{V}_E \left(2\varepsilon \frac{\sin\theta}{r} \right) \\ \widehat{u}_\parallel &= -2qR\varepsilon \frac{\cos\theta}{r} \widehat{V}_E \end{aligned}$$

This can be expressed;

if there is a poloidal velocity V_E then an - inevitable - parallel velocity $\widehat{u}_{\parallel}$ exists, and this is modulated as $\cos\theta$

The parallel velocity so calculated $\widehat{u}_{\parallel}$ is introduced in the second equation, which is an equation for the *momentum*, equivalently for the *parallel* velocity

$$\begin{aligned}\frac{\partial}{\partial t}\left[-2qR\varepsilon\frac{\cos\theta}{r}\widehat{V}_E\right]+\widehat{V}_E\frac{\partial\widetilde{u}_{\parallel}^{ref}}{r\partial\theta} &= -c_s^2\nabla_{\parallel}\widehat{N} \\ &= -c_s^2\frac{1}{qR}\frac{\partial\widehat{N}}{\partial\theta}\end{aligned}$$

Here there was a gradient of the pressure along the magnetic field line, $\boldsymbol{\nabla}_{\parallel}p = T\nabla_{\parallel}\widehat{N}$.

From T one gets c_s^2.

This can be expressed as

if there is a variation in the surface of the pressure $\nabla_{\parallel}\widehat{p}$, which means that there is a parallel variation of the density $\widehat{N}(\theta)$, then this acts as a SOURCE for the rotation, and there is a time variation of the parallel velocity.

First it is multiplied by (qR) from the denominator of the term with θ variation of the density (equivalently the *parallel* veriation of density) and then is integrated up to a current θ

$$\begin{aligned}-c_s^2\widehat{N}(\theta) &= -2(qR)^2\varepsilon\frac{1}{r}\frac{\partial\widehat{V}_E}{\partial t}\int^{\theta}d\theta'\cos\theta' \\ &\quad+\widehat{V}_E\frac{qR}{r}\widetilde{u}_{\parallel}^{ref}\end{aligned}$$

we ignore for the moment the constant of integration which should be a function of surface.

This $\widehat{N}(\theta)$ is introduced in the equation for the time variation of $\widehat{V}_E$, the third equation

$$\begin{aligned}\frac{\partial\widehat{V}_E}{\partial t} &= c_s^2\int\frac{d\theta}{2\pi}\widehat{N}\left(-2\varepsilon\frac{\sin\theta}{r}\right) \\ &= \int\frac{d\theta}{2\pi}\left(2\varepsilon\frac{\sin\theta}{r}\right)\left\{-2(qR)^2\varepsilon\frac{1}{r}\frac{\partial\widehat{V}_E}{\partial t}\int^{\theta}d\theta'\cos\theta'+\widehat{V}_E\frac{qR}{r}\widetilde{u}_{\parallel}^{ref}\right\} \\ &= -4(qR)^2\varepsilon^2\frac{\partial\widehat{V}_E}{\partial t}\frac{1}{r^2}\int\frac{d\theta}{2\pi}\sin\theta\sin\theta \\ &\quad+2\varepsilon\frac{qR}{r^2}\widehat{V}_E\int\frac{d\theta}{2\pi}\sin\theta\widetilde{u}_{\parallel}^{ref}\end{aligned}$$

The first term in RHS

$$-4(qR)^2\varepsilon^2\left(\frac{\partial\widehat{V}_E}{\partial t}\right)\frac{1}{r^2}\int\frac{d\theta}{2\pi}\sin\theta\sin\theta = -4q^2R^2\frac{r^2}{R^2}\left(\frac{\partial\widehat{V}_E}{\partial t}\right)\frac{1}{r^2}\frac{1}{2} = -2q^2\left(\frac{\partial\widehat{V}_E}{\partial t}\right)$$

and the second

$$
\begin{aligned}
2\varepsilon\frac{qR}{r^2}\widehat{V}_E &= 2\frac{r}{R}q\frac{R}{r}\frac{1}{r}\widehat{V}_E \\
&= \frac{2q}{r}\widehat{V}_E
\end{aligned}
$$

$$
\left(1+2q^2\right)\frac{\partial\widehat{V}_E}{\partial t} = \frac{2q}{r}\widehat{V}_E\int\frac{d\theta}{2\pi}\sin\theta\widetilde{u}_{\|}^{ref}
$$

This is the way one finds the INERTIA factor

$$
\left(1+2q^2\right)
$$

of the plasma in the poloidal rotation.

In this equation we replace the reference state for the parallel velocity, which is fixed by the source and the flux, both these contributions being retained with their variation along the poloidal direction

$$
\left(1+2q^2\right)\frac{\partial\widehat{V}_E}{\partial t} = \widehat{V}_E \quad \times\frac{1}{\varepsilon^2}2q^2\left[\frac{1}{n_0}\int\frac{d\theta}{2\pi}S\cos\theta - \frac{1}{n_0}\frac{1}{r}\frac{\partial}{\partial r}\left(r\int\frac{d\theta}{2\pi}\Gamma_r\cos\theta\right)\right]
$$

we can easily recognize that an integration by parts have been made in the right hand side.

NOTE
How is generated this *inertia factor* $1+2q^2$.
We have seen that the first integration

$$
\widehat{u}_{\|} = -2qR\varepsilon\frac{\cos\theta}{r}\widehat{V}_E
$$

actually obtains the Pfirsch Schluter parallel current, when a poloidal flow is given by $\widehat{V}_E$. This PS flow has a coefficient q, as usual.

The second equation has the RHS term $-c_s^2\frac{1}{qR}\frac{\partial\widehat{N}}{\partial\theta}$ and this introduces the second q factor. The actual derivative was of the pressure along the line and was requested by momentm forces along the line

$$
\frac{\partial}{\partial l_{\|}}\widehat{N}(\theta) = \frac{1}{qR}\frac{\partial}{\partial\theta}\widehat{N}(\theta)
$$

They now multiply the term $\partial\widehat{u}_{\|}/\partial t$.

The enhancement of the *radial diffusion* with a factor q^2, the known characteristics of the Pfirsch Schluter "regime" has the same origin. We note however that it is not yet clear what means PS regime.
END

19 Formulas

The volume in velocity space.

The element of angle in velocity space (**Hirshman Sigmar Clarke**)

$$\begin{aligned} u_{a1}(v) &= \frac{1}{f_{a0}} \frac{3}{4\pi} \int v_{\|} f_{a1} d\Omega \\ d\Omega &\equiv \text{solid angle in velocity space} \end{aligned}$$

$$d\Omega = \pi \sum_{\sigma=\pm 1} \frac{B d\mu}{\epsilon} \frac{1}{\left| v_{\|} \right| / v}$$

$$\frac{3}{4\pi} \int d\Omega \; v_{\|} \; \Theta \left[V_{\|} \right] = f_c \; v^2 \frac{1}{h}$$

The proportions of circulating and trapped particles

$$\begin{aligned} f_c &= \frac{3}{4} \int_0^{\lambda_c} \frac{\lambda d\lambda}{\left\langle \sqrt{1 - \frac{\lambda}{h}} \right\rangle} \\ f_T &= 1 - f_c \\ &\approx 1.46 \times \sqrt{\varepsilon} \end{aligned}$$

(**Hinton Oberman** in **Connor 1973**).

The *divergence* of the pressure tensor (**Landau Lifshitz**), in **Rozhansky Tendler**.

$$(\boldsymbol{\nabla} \cdot \boldsymbol{\pi})_l = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^k} \left(\sqrt{|g|} \pi_l^k \right) - \frac{1}{2} \frac{\partial g_{km}}{\partial x^l} \pi^{km}$$

where the metric tensor has the components

$$\begin{aligned} g_{11} &= 1 \\ g_{22} &= r^2 \\ g_{33} &= (1 + \varepsilon \cos\theta)^2 = h^2 \end{aligned}$$

By the way

$$\begin{aligned} g &= \det(g_{ij}) \\ &= 1 \times r^2 \times (1 + \varepsilon \cos\theta)^2 \\ &= r^2 h^2 \end{aligned}$$

and

$$\sqrt{g} = rh$$